Exercise 5: Motion, Segmentation, and Illumination

Q1:
Let the spatial derivatives of the image intensity be denoted by $I_x$, $I_y$ and the temporal derivative by $I_t$. At each location in the image, we assume the following Brightness Constancy Constraint:

$$I_x u + I_y v + I_t = 0$$

where $[u, v]^t$ is the velocity (or displacement) at the pixel $[x, y]^t$. Lucas and Kanade assumed that within an image patch $W$ around each pixel, the displacement vector $[u, v]^t$ is constant. They derived a set of 2 equations in the 2 unknowns $[u, v]^T$ that best fits the measurements of $I_x$, $I_y$, $I_t$ within this patch, in the least-squares sense (which we also derived in class). Write this set of two equations, and describe the degenerate cases (cases where the solution is not unique), both algebraically, as well geometrically (i.e., provide an explanation of what geometric settings within an image patch would give rise to such singular cases).

(Hint: You might find the Cauchy-Schwartz inequality usefull here.)

Q2:
Let $P_i = (X_i, Y_i, Z_i)^T$, $i = 1, ..., N$ be a set of 3D points in space with image projections $p_i = (x_i, y_i)^T = f \frac{P_i}{Z_i}$, where $f$ is the focal length of the camera. In the general case of a rigid camera motion

$$P_i' = RP_i + t$$

where $R$ is the rotational component and $t$ is the translational component of camera motion, with projections $p_i' = f \frac{P_i'}{Z_i}$.

(a) Show that in the case of pure translation (i.e., the camera does not rotate), the induced displacements of all image points form a radial flow field emerging from the focus-of-expansion (FOE) of the camera $e = f \frac{t}{1Z}$ (this is also the epipole), where $t = (t_X, t_Y, t_Z)^T$.

(Hint: A radial displacement field has the form: $p_i' - e = \alpha_i (p_i - e)$, where $\alpha_i$ is a point-dependent scalar.)

(b) What happens if $t_Z = 0$?

Q3:
A bilaterally symmetric object is a surface $S$ such that for every point $P = (X, Y, Z) \in S$ also $Q = (-X, Y, Z) \in S$. An image of such an object shows the two symmetric portions of the object, and so it in fact contains two pictures of the same shape. In this question you are asked to show that epipolar lines relate the symmetric portions.

(a) Suppose a bilaterally symmetric object is observed by a camera at a general position (namely, after some rotation and translation). Show that
in such an image there exist two sets of epipolar lines on which symmetric points lie. In other words, let \( P \) and \( Q \) denote the 3D locations of \( P \) and \( Q \) relative to the camera, and let \( p \) and \( q \) denote the projected locations in the image. Then, there exists a \( 3 \times 3 \) matrix \( F \) such that \( p^TFq = 0 \) for all pairs of symmetric points.

(Hint: Show that an affine transformation in 3D (linear + translation) relates \( P \) and \( Q \), and derive \( F \) from this.)

(b) Where are the epipoles?

Q4:

An image containing \( n \) edgels (edge elements) is given, where each edgel is described by a triplet \((x_i, y_i, \theta_i)\), \( 1 \leq i \leq n \), denoting the position and orientation of the edgel. Suppose you are told that the edgel are tangents to 5 circles. Describe methods to derive the parameters of these circles using:

(a) RANSAC

(b) Hough transform

Explain each step of the two methods and derive its complexity.

Q5:

(a) Given a surface \( z(x, y) \) (so every point on the surface is given by \((x, y, z(x, y))\)), write an expression for the surface normal.

(b) Suppose following a photometric stereo algorithm we recover the correct surface normals for \( z(x, y) \), \((n_x, n_y, n_z)^T\), but only up to the following linear transformation (the ambiguity matrix):

\[
\begin{pmatrix}
1 & 0 & -\alpha \\
0 & 1 & -\beta \\
0 & 0 & 1
\end{pmatrix}
\]

Show that these recovered normals form an integrable surface. (In other words, call the surface determined by the recovered normals \( z'(x, y) \); show that \( z'_{xy} = z'_{yx} \). Subscripts denote partial differentiation.) In addition, write an expression for \( z' \) (by integrating the relevant components of the surface normals).

(c) Repeat this process for normals transformed by:

\[
\begin{pmatrix}
2 & 0 & -\alpha \\
0 & 1 & -\beta \\
0 & 0 & 1
\end{pmatrix}
\]

Do these normals form an integrable surface?