Introduction to Computer and Human Vision
Exercise Set 1
Fourier Transform and Convolution

Submission: individually
Due date: Monday, Nov 27$^{\text{th}}$, 2016

1. Prove: You can choose whether to prove for the continuous or the discrete case (1D is sufficient).

(a) The Convolution Theorem: $\mathfrak{F}\{f \star g\} = F \cdot G$ (where $F = \mathfrak{F}\{f\}$ and $G = \mathfrak{F}\{g\}$). (in the discrete case you may have a scale factor corresponding to the size of the image).

(b) The convolution is commutative: $f \star g = g \star f$.

(c) Show that: $f \star (\alpha g + \beta h) = \alpha(f \star g) + \beta(f \star h)$, where α and β are scalars.

(d) The convolution is shift invariant: $f(x) \star g(x - d) = (f \star g)(x - d)$.

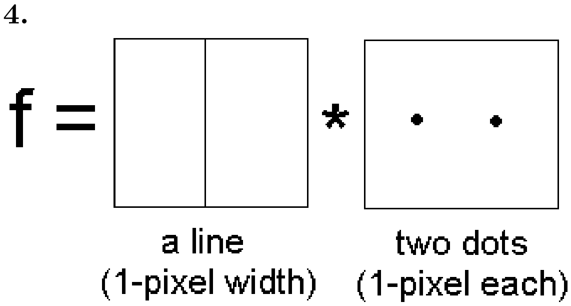
2. What is $\frac{d}{dx}\{f(x) \star g(x)\}$?
Hint: use the convolution theorem.

3. Prove: You can choose whether to prove for the continuous or the discrete case (1D is sufficient).

(a) If f is a symmetric function, then its Fourier transform $F = \mathfrak{F}\{f\}$ is also symmetric.

(b) If f is symmetric and real, then F is also symmetric and real.

4.



(a) What does the image f look like?

(b) What does its Fourier transform F look like? (The magnitude $|F|$ only.) Explain why.

5. Let $F(u,v)$ be the Fourier transform of an $M \times N$ image $f(x,y)$. Let $g(x,y)$ be an image of dimensions $(2M) \times (2N)$ whose Fourier transform $G(u,v)$ is defined as follows:

$$G(u,v) = \begin{cases} F(u,v) & \text{if } 0 \le u < M \text{and } 0 \le v < N \\ 0 & \text{otherwise} \end{cases}$$

What does the image $g(x,y)$ look like in terms of $f(x,y)$? Show mathematically, and explain the result (one sentence, it suffices to explain the values at even coordinates).

6.

Let $f(x,y)$ be an $M \times N$ image, and let $F(u,v)$ be its Fourier transform. Let $g(x,y)$ be an image of dimensions $(2M) \times (2N)$, whose Fourier transform $G(u,v)$ is generated from $F(u,v)$ by inserting a row of Zeros (0's) between every two rows of F, and a column of Zeros (0's) between every two columns of F. What does the image $g(x,y)$ look like in terms of $f(x,y)$? Show mathematically and explain the result (one sentence). Remember, $g(x,y)$ is of size $(2M) \times (2N)$.

7.

(a) Is it possible to discretely sample the function $f_1(x) = \sin(\alpha x)$ without losing any information? (i.e., can we sample $f_1(x)$ discretely, and be able to reconstruct the continuous $f_1(x)$ back from its discrete samples?) If the answer is Yes - then what is the maximal allowed distance between the samples? If the answer is No - explain why.

(b) Same question as above, but for

$$f_2(x) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

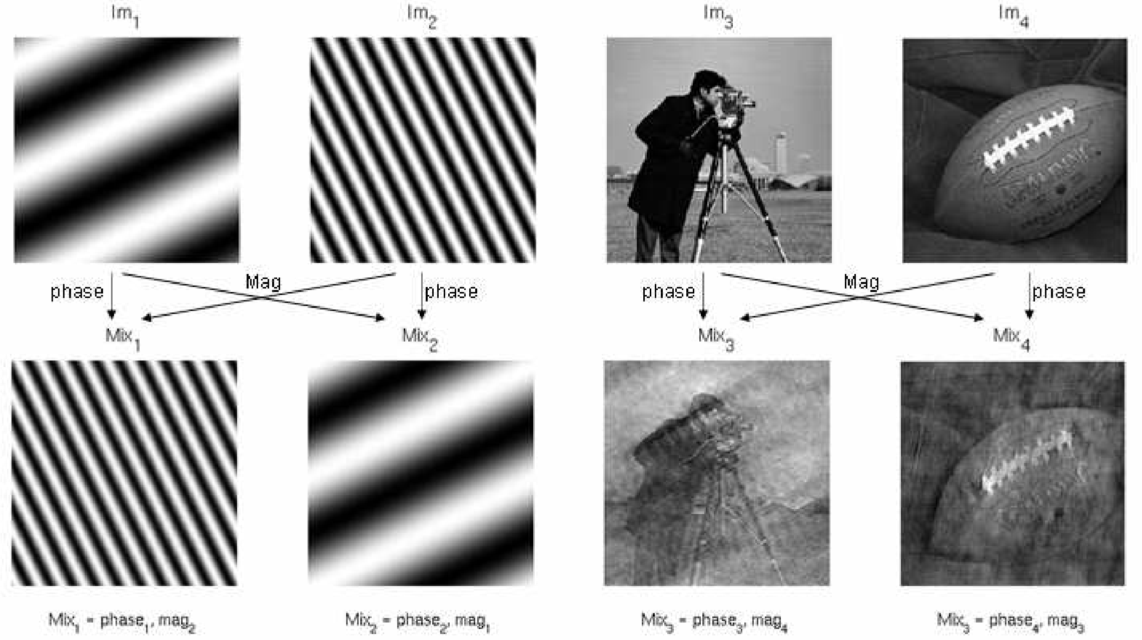
(c) Same question as above, now for

$$f_3(x) = f_1(x) \star f_2(x)$$

8. Exploring the importance of magnitude vs. phase: We took pairs of images and mixed their magnitude and phase in the following manner:

$$\begin{aligned} Mix_1 &= \mathfrak{F}^{-1}\left\{ |\mathfrak{F}\{Im_2\}| e^{i \cdot \measuredangle(\mathfrak{F}\{Im_1\})} \right\} \\ Mix_2 &= \mathfrak{F}^{-1}\left\{ |\mathfrak{F}\{Im_1\}| e^{i \cdot \measuredangle(\mathfrak{F}\{Im_2\})} \right\} \\ Mix_3 &= \mathfrak{F}^{-1}\left\{ |\mathfrak{F}\{Im_4\}| e^{i \cdot \measuredangle(\mathfrak{F}\{Im_3\})} \right\} \\ Mix_4 &= \mathfrak{F}^{-1}\left\{ |\mathfrak{F}\{Im_3\}| e^{i \cdot \measuredangle(\mathfrak{F}\{Im_4\})} \right\} \end{aligned}$$

where $|\mathfrak{F}\{Im\}|$ is the magnitude of the Fourier transform of Im, and $\measuredangle(\mathfrak{F}\{Im1\})$ is its phase.



(a) In the first mixed image pair (Mix_1 and Mix_2) which input image are they more similar to - the one from which the phase was taken, or the one from which the magnitude was taken? In what way is the output image different form the most similar input image? Explain why.

(b) Same question as the above, but for the second mixed image pair (Mix_3 and Mix_4). Explain why the importance of phase vs. magnitude in this case is reversed?

9.

Let $g(x)$ be a scaled version of $f(x)$, i.e.:

$$g(x) = f(sx)$$

where s is a positive scalar. What does the Fourier transform G of g look like (in terms of the Fourier transform F of f)?