Gaussian Mixture Model

Jacob Goldberger

Generative model:

\[ X \sim \text{p}(X = j) = \alpha_j \quad j = 1, \ldots, k \]
\[ Y | X = j \sim \mathcal{N}(\mu_j, \Sigma_j) \]

The distribution of a random variable \( Y \in \mathbb{R}^d \) is a mixture of \( k \) Gaussians if:

\[
f(Y = y | \theta) = \sum_{j=1}^{k} \alpha_j \frac{1}{\sqrt{(2\pi)^d |\Sigma_j|}} \exp\left\{ -\frac{1}{2}(y - \mu_j)^T \Sigma_j^{-1} (y - \mu_j) \right\}
\]

such that \( \theta = \{\alpha_j, \mu_j, \Sigma_j\}_{j=1}^{k} \) consists of:

- \( \alpha_j > 0 \quad j = 1, \ldots, k \quad \sum_{j=1}^{k} \alpha_j = 1 \)
- \( \mu_j \in \mathbb{R}^d \) and \( \Sigma_j \) is a \( d \times d \) positive definite matrix \( j = 1, \ldots, k \)

Likelihood of \( y_1, \ldots, y_n \):

\[
f(y_1, \ldots, y_n | \theta) = \prod_{t=1}^{n} \sum_{j=1}^{k} \alpha_j f(y_t | \mu_j, \Sigma_j)
\]

EM iteration:

- Expectation step:
  \[
w_{tj} = p(x_t = j | y_t) = \frac{\alpha_j f(y_t | \mu_j, \Sigma_j)}{\sum_{i=1}^{k} \alpha_i f(y_t | \mu_i, \Sigma_i)}
  \]

- Maximization step:
  \[
  \hat{\alpha}_j \leftarrow \frac{1}{n} \sum_{t=1}^{n} w_{tj} \\
  \hat{\mu}_j \leftarrow \frac{\sum_{t=1}^{n} w_{tj} y_t}{\sum_{t=1}^{n} w_{tj}} \\
  \hat{\Sigma}_j \leftarrow \frac{\sum_{t=1}^{n} w_{tj}(y_t - \hat{\mu}_j)(y_t - \hat{\mu}_j)^T}{\sum_{t=1}^{n} w_{tj}}
  \]