ALGEBRAIC AND ANALYTIC GEOMETRY
OVER THE FIELD OF ONE ELEMENT

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Chapter II. Analytic geometry

§1. Banach $\mathbf{F}_1$-algebras

1.1. Banach $\mathbf{F}_1$-algebras.

1.1.1. Definition. A *Banach* $\mathbf{F}_1$*-algebra* is an $\mathbf{F}_1$-algebra A provided with a *Banach norm*, i.e., a function $||\ || : A \to \mathbf{R}_+$ possessing the following two properties:

(1) $||f|| = 0$ if and only if $f = 0$;

(2) $||fg|| \le ||f|| \cdot ||g||$ for all $f, g \in A$.

For example, every $\mathbf{F}_1$-algebra A can be provided with the *trivial norm* $||\ ||_0$, i.e., the norm with $||f||_0 = 1$ for all nonzero $f \in A$. Banach $\mathbf{F}_1$-algebras form a category with respect to *bounded homomorphisms*, i.e., homomorphisms of $\mathbf{F}_1$-algebras $\varphi : A \to B$ for which there exists a constant $C > 0$ with $||\varphi(f)|| \le C||f||$ for all $f \in A$. Notice that, given a second Banach norm $||\ ||'$ on a Banach $\mathbf{F}_1$-algebra A, the identity map $(A, ||\ ||) \to (A||\ ||')$ is an isomorphism if an only if the norms $||\ ||$ and $||\ ||'$ are *equivalent*, i.e., there exist $C, C' > 0$ with $C||f|| \le ||f||' \le C'||f||$ for all $f \in A$. Notice that any Banach $\mathbf{F}_1$-algebra A admits an equivalent norm $||\ ||'$ with $||1_A||' = 1$. Namely, it is given by the formula $||f||' = \sup \frac{||fg||}{||g||}$, where the supremum is taken over all nonzero elements $g \in A$.

Notice also that, given a bounded homomorphism of Banach $\mathbf{F}_1$-algebras $\varphi : A \to B$, the norm on B admits an equivalent norm $||\ ||'$ with respect to which the canonical homomorphism is *contracting*, i.e., such that $||\varphi(f)||' \le ||f||$ for all $f \in A$. Indeed, it is defined by $||g||' = \inf\{||f|| \cdot ||h||\}$, where the infimum is taken over all representations of $g \in B$ in the form $g = \varphi(f)h$ with $f \in A$ and $h \in B$.

1.1.2. Examples. (i) A Banach $\mathbf{F}_1$-algebra K is said to be a *real valuation* $\mathbf{F}_1$*-field* if it is an $\mathbf{F}_1$-field and its norm is multiplicative. In this case $|K| = \{|\lambda| \big| \lambda \in K\}$ is an $\mathbf{F}_1$-subfield of

$\mathbf{R}_+$. One has $K/K^{**} \xrightarrow{\sim} |K|$, where K^{**} is the subgroup of K^* consisting of the elements λ with $|\lambda| = 1$. Any algebraic extension L of K is also a real valuation $\mathbf{F}_1$-field with respect to the unique real valuation which extends that of K. (If $g \in L^*$ and $g^n = f \in K^*$, then $|g| = |f|^{\frac{1}{n}}$.) Notice that the subgroup K^{**} lies in the set $K^\circ = \{f \in K \big| |f| \le 1\}$, which is a Banach $\mathbf{F}_1$-subalgebra of K (provided with the induced Banach norm). The set $K^{\circ\circ} = \{f \in K \big| |f| < 1\}$ is the unique maximal Zariski ideal of K°, whose complement is the group K^{**}. The *residue* $\mathbf{F}_1$*-field* of K is the quotient $\widetilde{K}_1 = K^\circ/K^{\circ\circ}$. Notice that $K^{**} \xrightarrow{\sim} \widetilde{K}_1^*$.

(ii) The multiplicative monoid $A^\cdot$ of any commutative Banach ring A with unity can be considered as a Banach $\mathbf{F}_1$-algebra. If $A = k$ is a (usual) field complete with respect to a real valuation, then $k^\cdot$ is a real valuation $\mathbf{F}_1$-field. If k is non-Archimedean, then $(k^\cdot)^\circ = k^\circ$, $(k^\cdot)^{\circ\circ} = k^{\circ\circ}$ and $(\widetilde{k}^\cdot_1)^* = (k^\circ)^*$. If $k = \mathbf{C}$ is the field of complex numbers, then $(\mathbf{C}^\cdot)^\circ$ and $(k^\cdot)^{\circ\circ}$ are the closed and open unit discs, respectively, and $(\widetilde{\mathbf{C}}^\cdot_1)^*$ is the group of complex numbers of length one.

(iii) For a Banach $\mathbf{F}_1$-algebra A and a tuple of positive numbers $(r_i)_{i\in I}$, the A-algebras $A[T_i]_{i\in I}$ and $A[T_i, T_i^{-1}]_{i\in I}$, provided with the norm

$$||fT_{i_1}^{\nu_{i_1}} \dots T_{i_n}^{\nu_{i_n}}|| = ||f|| r_{i_1}^{\nu_{i_1}} \dots r_{i_n}^{\nu_{i_n}} \ ,$$

are Banach $\mathbf{F}_1$-algebras, denoted by $A\{r_i^{-1}T_i\}_{i\in I}$ and $A\{r_i^{-1}T_i, r_iT_i^{-1}\}_{i\in I}$, respectively. If the norm on A is multiplicative (i.e., $||fg|| = ||f|| \cdot ||g||$), then so are the norms on $A\{r_i^{-1}T_i\}_{i\in I}$ and $A\{r_i^{-1}T_i, r_iT_i^{-1}\}_{i\in I}$. If K is a real valuation $\mathbf{F}_1$-field, then so is $K\{r_i^{-1}T_i, r_iT_i^{-1}\}_{i\in I}$. For example, $\mathbf{Z}^\cdot \xrightarrow{\sim} \mathbf{F}^\cdot_3\{p_n^{-1}T_n\}_{n\ge 1}$ for the ring of integers $\mathbf{Z}$ provided with the archimedean absolute value $|\ |_\infty$, where p_n is the n-th prime number (see Example I.1.1.3(iii)).

Given an ideal $E \subset A \times A$ on a Banach $\mathbf{F}_1$-algebra A, the quotient $\mathbf{F}_1$-algebra A/E is provided with the following real valued function: $||\overline{f}|| = \inf\limits_{f\in\overline{f}} ||f||$. This function possesses the property (2) (i.e., it is a *seminorm*), and it possesses the property (1) if and only if the ideal E is *closed*, i.e., it satisfies the condition that the infimum of the Banach norm on elements of any equivalence class, which does not contain zero, is positive. For example, the ideal $E_{\mathbf{a}}$ associated with a Zariski ideal $\mathbf{a} \subset A$ is always closed, and so the quotient $A/\mathbf{a}$ is a Banach $\mathbf{F}_1$-algebra. The kernel $\mathrm{Ker}(\varphi)$ of a bounded homomorphism of Banach $\mathbf{F}_1$-algebras $\varphi : A \to B$ is also always closed. The *closure* $\overline{E}$ of an ideal $E \subset A \times A$ is the minimal closed ideal that contains E. One has $\overline{E} = E \cup (\mathbf{a}_{\overline{E}} \times \mathbf{a}_{\overline{E}})$, where $\mathbf{a}_{\overline{E}}$ is the Zariski ideal of all elements $f \in A$ with the property that there is a sequence of elements $f_1, f_2, \dots$ equivalent to f and such that $f_n \to 0$ as $n \to \infty$. For example, if A is Zariski Noetherian (e.g., finitely generated over a real valuation $\mathbf{F}_1$-field), then the closure of any finitely generated ideal is finitely generated. The *closed ideal generated by a subset* $S \subset A \times A$ is the minimal closed ideal that contains S or, equivalently, the closure of the ideal generated by S.

1.1.3. Examples (cf. [Ber1, Example 1.1.1(v)]). (i) For a Banach $\mathbf{F}_1$-algebra A, a number $r > 0$ and an element $f \in A$, let E be the ideal of $A\{r^{-1}T\}$ generated by the pair (T, f). The ideal E consists of the pairs (aT^m, bT^n) with $af^m = bf^n$. Indeed, it suffices to verify that the set E' of such pairs is an ideal. If $(aT^m, bT^n), (cT^k, dT^l) \in E'$ and $bT^n = cT^k$, then $k = n$ and $c = b$. We also have $af^m = bf^n = cf^k = df^l$ and, therefore, $(aT^m, dT^l) \in E'$, i.e., $E' = E$. It follows that $A \widetilde{\to} A\{r^{-1}T\}/E : a \mapsto \overline{a}$ and, therefore, $||\overline{a}|| = \inf\{||b||r^n\}$, where the infimum is taken over all representations $a = bf^n$ with $b \in A$ and $n \geq 0$. *We claim that* $||\overline{a}|| = 0$ *if and only if there exist integers* $0 < n_1 < n_2 < \dots$ *and elements* $b_1, b_2, \dots$ *with* $a = b_k f^{n_k}$ *and* $||b_k||r^{n_k} \to 0$ *as* $k \to \infty$. Indeed, the converse implication is clear. Suppose that $||\overline{a}|| = 0$. Then there exist sequences of elements $b_1, b_2, \dots \in A$ and of positive integers $n_1, n_2, \dots$ with $a = b_k f^{n_k}$ and $||b_k||r^{n_k} \to 0$ as $k \to \infty$. If the sequence $n_1, n_2, \dots$ is bounded, we get $||a|| \leq (||b_k||r^{n_k}) \cdot \frac{||f^{n_k}||}{r^{n_k}} \to 0$ as $k \to \infty$, i.e., $a = 0$. If the sequence is unbounded, we can replace it by a strictly increasing subsequence. The claim implies that the closure of E is the trivial ideal of $A\{r^{-1}T\}$ if and only if f is invertible and there is a sequence $0 < n_1 < n_2 < \dots$ with $||f^{-n_k}||r^{n_k} \to 0$ as $k \to \infty$.

(ii) Let E be the ideal of $A\{r^{-1}T\}$ generated by the pair $(fT, 1)$. Then E coincides with the set $E' = \{(aT^m, bT^n) \big| af^{n+p} = bf^{m+p}$ for some $p \geq 0\}$. Indeed, if $(aT^m, bT^n) \in E'$, then $aT^m \sim af^{n+p}T^{n+p}T^m = bf^{m+p}T^{m+p}T^n \sim bT^n$, i.e., $(aT^m, bT^n) \in E$ and $E' \subset E$. If $(aT^m, bT^n), (cT^k, dT^l) \in E'$ and $bT^n = cT^k$, then $k = n$, $c = b$, $af^{n+p} = bf^{m+p}$ and $cf^{l+q} = df^{k+q}$ for some $p, q \geq 0$. We also have $af^{l+n+p+q} = bf^{m+p}f^{l+q} = cf^{l+q}f^{m+p} = df^{k+q}f^{m+p} = df^{m+n+p+q}$, i.e., $(aT^m, dT^l) \in E'$ and, therefore, $E' = E$. It follows that $A_f \widetilde{\to} A\{r^{-1}T\}/E : \frac{a}{f^m} \mapsto \overline{\frac{a}{f^m}}$ and, therefore, $||\overline{\frac{a}{f^m}}|| = \inf\{||b||r^n\}$, where the infimum is taken over all representations $\frac{a}{f^m} = \frac{b}{f^n}$ in A_f with $b \in A$ and $n \geq 0$. In particular, the closure of E is the trivial ideal of $A\{r^{-1}T\}$ if and only if there exist exist sequences of positive integers $n_1, n_2, \dots$ and of elements $b_1, b_2, \dots \in A$ with $\frac{b_k}{f^{n_k}} = 1$ and $||b_k||r^{n_k} \to 0$ as $k \to \infty$.

1.1.4. Examples. (i) Let I be a finite idempotent $\mathbf{F}_1$-subalgebra of a Banach $\mathbf{F}_1$-algebra A. Then the ideal F of A generated by an ideal E of I is always closed. Indeed, since the intersection of closed ideals is a closed ideal, Lemma 1.4.1(iii) reduces the situation to the case $E = \Pi_e$ and $F = F_e$. Furthermore, since the Zariski ideal $\mathfrak{p}_e A$ is closed, we can replace A by $A/\mathfrak{p}_e A$ and assume that $\mathfrak{p}_e = 0$. This means that e is a unique maximal idempotent in $\check{I}$, and in this case, $F_e = \{(a, b) \big| ae = be\}$. If now $(a, a_n) \in F_e$ and $||a_n|| \to 0$ as $n \to \infty$, then $ae = 0$, i.e., $(a, 0) \in F_e$.

(ii) Let $\mathfrak{p}$ be a Zariski prime ideal of a Banach $\mathbf{F}_1$-algebra A, and assume that, for a Zariski ideal $\mathbf{a} \subset \mathfrak{p}$, the ideal $E = \mathrm{Ker}(A \to A_{\mathfrak{p}}/\mathbf{a}A_{\mathfrak{p}})$ is finitely generated (see Corollary 1.6.5(ii)). Then E is closed. Indeed, let $(f_1, g_1), \dots, (f_n, g_n)$ be a finite system of generators of E, and assume that $f_i \notin \mathbf{a}_E$ (resp. $f_i \in \mathbf{a}_E$) for $1 \le i \le m$ (resp. $m+1 \le i \le n$). For each $1 \le i \le m$, we take an element $h_i \notin \mathfrak{p}$ with $f_i h_i = g_i h_i$, and set $h = h_1 \cdot \ldots \cdot h_m$. Then $h \notin \mathfrak{p}$ and $fh = gh$ for all $(f, g) \in E \backslash (\mathbf{a}_E \times \mathbf{a}_E)$. Assume now that there is an element $f \in A \backslash \mathbf{a}_E$ which admits a sequence of elements $f_1, f_2, \dots$ with $(f, f_i) \in E$ and $f_i \to 0$ as $i \to \infty$. Then $fh = f_i h \to 0$ as $i \to \infty$. It follows that $fh = 0$, which is a contradiction. ∎

1.1.5. Remark. One could define a Banach norm on an $\mathbf{F}_1$-algebra A as a function $|| \; || : A \to \mathbf{R}_+$ with the properties (1) as in Definition 1.1.1 and the following version of the property (2): there exists a constant $C > 0$ such that $||fg|| \le C||f|| \cdot ||g||$ for all $f, g \in A$. For such a function $|| \; ||$, the function $|| \; ||' : A \to \mathbf{R}_+$, defined by $||f||' = \sup\limits_{h \ne 0} \frac{||fh||}{||h||}$, possesses both properties of Definition 1.1.1 and is equivalent to $|| \; ||$ (i.e., there are constants $C', C'' > 0$ such that $C'||f|| \le ||f||' \le C''||f||$ for all $f \in A$). By the way, for the new norm one has $||1||' = 1$.

1.2. Banach modules over a Banach $\mathbf{F}_1$-algebra.

Let A be a Banach $\mathbf{F}_1$-algebra.

1.2.1. Definition. A *Banach A-module* is an A-module M provided with a *Banach norm* i.e., a function $|| \; || : M \to \mathbf{R}_+$ possessing the following properties:

(1) $||m|| = 0$ if and only $m = 0$, and

(2) $||fm|| \le ||f|| \cdot ||m||$ for all $f \in A$ and $m \in M$.

Banach A-modules form a category with respect to bounded A-homomorphisms. This category has an inner Hom-functor, i.e., for any pair of Banach A-modules M and N the set $\mathrm{Hom}_A(M, N)$ of bounded A-homomorphisms $f : M \to N$ has the structure of a Banach A-module with respect to the Banach norm $||f|| = \sup\limits_{m \ne 0} \frac{||f(m)||}{||m||}$. (If $M = N = A$, the isomorphism $A \widetilde{\to} \mathrm{Hom}_A(A, A)$ provides A with the Banach norm from the end of the first paragraph in §1.1).

A *Banach A-algebra* is a Banach $\mathbf{F}_1$-algebra B which is also a Banach A-module. In particular, the map $A \to B : f \mapsto f \cdot 1_B$ is a bounded homomorphism of Banach $\mathbf{F}_1$-algebras. Conversely, given a bounded homomorphism of Banach $\mathbf{F}_1$-algebras $\varphi : A \to B$, B can be provided with the structure of a Banach A-module. Namely, the formula $||g||' = \inf\{||f|| \cdot ||h||\}$, where the infimum is taken over all representations of $g \in B$ in the form $g = \varphi(f)h$ with $f \in A$ and $h \in B$, provides B with an equivalent Banach norm with the property $||\varphi(f)g||' \le ||f|| \cdot ||g||'$.

For a real valuation $\mathbf{F}_1$-field K, a *real valuation K-field* is a real valuation $\mathbf{F}_1$-field K' which is a Banach K-module, i.e., it is provided with an isometric homomorphism $K \to K'$. For example, $|K|$ is a valuation K-field.

As in §1.1, one introduces the notions of a *closed A-submodule* $E \subset M \times M$ and of *the closure of an A-submodule*. For a closed A-submodule E, the quotient A/E is again a Banach A-module. Of course, the A-submodule E_N associated with a Zariski A-submodule $N \subset M$ is always closed.

Given Banach A-modules M, N and P, an A-bilinear homomorphism $\varphi : M \times N \to P$ is said to be *bounded* if there exists a constant $C > 0$ with $\varphi(m, n) \leq C||m|| \cdot ||n||$ for all $(m, n) \in M \times N$. The *complete tensor product* of M and N over A is a Banach A-module $M\widehat{\otimes}_A N$ provided with a bounded A-bilinear homomorphism $M \times N \to M\widehat{\otimes}_A N$ such that, for any bounded A-bilinear homomorphism $\varphi : M \times N \to P$, there exists a unique bounded homomorphism of A-modules $M\widehat{\otimes}_A N \to P$ which is compatible with φ. The complete tensor product is unique up to a unique isomorphism, and it is constructed as follows. The tensor product $M \otimes_A N$ is provided with the following *seminorm* (i.e., a function that possesses the property (2)): $||x|| = \inf\{||f|| \cdot ||g||\}$, where the infimum is taken over all representations of x in the form $f \otimes g$. Then $M\widehat{\otimes}_A N$ is the quotient of $M \otimes_A N$ by the Zariski A-submodule consisting of the elements x with $||x|| = 0$. If B is a Banach A-algebra, then $M\widehat{\otimes}_A B$ is a Banach B-module. If B and C are Banach A-algebras, then so is $B\widehat{\otimes}_A C$. Notice that if K is a real valuation $\mathbf{F}_1$-field and K' and K'' are real valuation K-fields, then $K' \otimes_K K''$ is again a real valuation K-field.

A bounded homomorphism of Banach A-modules $\varphi : M \to N$ is said to be *admissible* if the bijective bounded homomorphism $M/\mathrm{Ker}(\varphi) \to \mathrm{Im}(\varphi)$ is an isomorphism of Banach A-modules. Notice that, given admissible epimorphisms of Banach A-modules $M \to M'$ and $N \to N'$, the induced map $M\widehat{\otimes}_A N \to M'\widehat{\otimes}_A N'$ is an admissible epimorphism. Notice also that for a Banach A-algebra B the multiplication homomorphism gives rise to an admissible epimorphism $B\widehat{\otimes}_A B \to B$.

Furthermore, given a family Banach A-modules $\{M_i\}_{i\in I}$, their direct sum $\oplus_{i\in I} M_i$ provided with the evident Banach norm is a Banach A-module. In particular, for every set I, the free A-module $A^{(I)}$ is a Banach A-module. A Banach A-module M is said to be *finitely generated* (resp. *finite*) if there is an admissible epimorphism $A^{(n)} \to M$ (resp. such that its kernel is finitely generated). Notice that the full subcategory of finitely generated Banach A-modules is preserved under the complete tensor product. A Banach A-algebra B is said to be *finite* if it is finite as a Banach A-module.

1.2.2. Proposition. *Let A be a Banach $\mathbf{F}_1$-algebra. Then*

(i) the forgetful functor from the category of finitely generated Banach A-modules to that of

finitely generated A-modules is fully faithful;

(ii) every finitely generated A-module M has a unique minimal Zariski A-submodule N such that the quotient M/N has the structure of a finitely generated Banach A-module.

1.2.3. Lemma. *Any A-homomorphism $M \to N$ from a finitely generated Banach A-module M to a Banach A-module N is bounded.*

Proof. If $A^{(n)} \to M$ is an admissible epimorphism, then it suffices to verify that the composition map $\varphi : A^{(n)} = \oplus_{i=1}^n Ae_i \to N$ is bounded. Let $C = \max\{||\varphi(e_i)||\}$. Then for an element $f = ae_i \in A^{(n)}$, one has $||\varphi(f)|| \leq ||a|| \cdot ||\varphi(e_i)|| \leq C||f||$. ∎

Proof of Proposition 1.2.2. The statement (i) follows from Lemma 1.2.3.

(ii) Consider an arbitrary epimorphism of A-modules $A^{(n)} \to M$. If E is its kernel, the closure $\overline{E}$ has the form $E \cup (K \times K)$, where K is the Zariski A-submodule of $A^{(n)}$. (It consists of all elements b for which there exists a sequence of elements $b_1, b_2, \dots$ with $(b, b_n) \in E$ and $||b_n|| \to 0$ as $n \to \infty$.) If N is the image of K in M, then the quotient M/N has the structure of a Banach A-module. Any other epimorphism $A^{(n')} \to M$ (with kernel E' and similar Zariski A-submodules K' and N') can be represented as a composition of a homomorphism $\psi : A^{(n')} \to A^{(n)}$ with the previous epimorphism. It follows that $\psi(E') \subset E$. By (i), ψ is bounded and, therefore, $\psi(K') \subset K$. It follows that $N' \subset N$. By symmetry, the converse inclusion also holds, i.e., $N = N'$. ∎

The forgetful functor of Lemma 1.2.2(i) is not essentially surjective for the simple reason that an arbitrary Banach $\mathbf{F}_1$-algebra A may have ideals which are not closed. Here is a case when a finite A-algebra has a structure of a finite Banach A-algebra.

1.2.4. Lemma. *Let A be a Banach $\mathbf{F}_1$-algebra, and let B be an A-algebra which is finitely generated (resp. finite) as an A-module. Assume that (1) the canonical homomorphism $A \to B$ is injective, (2) B has no zero divisors, and (3) there exists $m \geq 1$ such that $g^m \in A$ for all $g \in B$. Then B has a structure of a Banach A-algebra which is a finitely generated (resp. finite) Banach A-module.*

Proof. Step 1. *The kernel* $\mathrm{Ker}(\varphi)$ *of any surjective A-homomorphism $\varphi : A^{(n)} \to B$ is closed.* Indeed, we may assume that $g_i = \varphi(e_i) \neq 0$ for all $1 \leq i \leq n$, and set $f_i = g_i^m \in A$. Assume that for a nonzero element $ae_i \in A^{(n)}$ there exists a sequence of elements $b_k e_{j_k} \in A^{(n)}$ with $(ae_i, b_k e_{j_k}) \in \mathrm{Ker}(\varphi)$ and $||b_k|| \to 0$ as $k \to \infty$. Replacing the sequence by a subsequence, we may assume that $j_k = j$ for all $k \geq 1$. We have $ag_i = b_k g_j$ for all $k \geq 1$. It follows that $a^m f_i = b_k^m f_j$. Since $||b_k|| \to 0$, we get $a^m f_i = 0$, which is a contradiction. Thus, B has the structure of a finitely generated Banach A-module induced by the homomorphism φ. If B is a finite A-module, then taking such a homomorphism φ whose kernel is a finitely generated A-submodule we provide B with the structure of a finite Banach A-module.

Step 2. *The finitely generated A-module $B \otimes_A B$ possesses the properties (1)-(3).* Indeed, the composition of the map $A \to B \otimes_A B : f \mapsto f \otimes 1$ with the multiplication homomorphism $B \otimes_A B \to B$ coincides with the canonical homomorphism $A \to B$. Since the latter is injective, (1) follows. Furthermore, since B has no zero divisors, the same is true for the tensor product $B \otimes B$. The tensor product $B \otimes_A B$ is the quotient of $B \otimes B$ by the A-submodule generated by pairs of the form $(fg \otimes h, g \otimes fh)$ with $f \in A$ and $g, h \in B$. Again, since B has no zero divisors and the homomorphism $A \to B$ is injective, the latter A-submodule does not contain pairs of the form $(g \otimes h, 0)$ for nonzero $g, h \in B$, and (2) follows. Finally, one has $(g \otimes h)^m = g^m \otimes h^m \in A$, i.e., (3) is true.

Step 3. Consider the surjective A-homomorphism $A^{(n^2)} \to B \otimes_A B : e_{ij} \mapsto g_i \otimes g_j$, $1 \le i, j \le n$. By Steps 1 and 2, its kernel is closed, and so it induces a structure of a finite Banach A-module on $B \otimes_A B$. Notice that $||g \otimes h|| \le ||g|| \cdot ||h||$ for all $g, h \in B$. By Lemma 1.2.2, the multiplication homomorphism $B \otimes_A B \to B$ is bounded, and so there exists $C > 0$ such that $||gh|| \le C||g \otimes h|| \le C||g|| \cdot ||h||$ for all $g, h \in B$. By Remark 1.1.5, $|| \ ||$ is equivalent to a Banach norm on B. ∎

1.2.5. Examples. (i) For a Banach A-module M and a tuple of positive numbers $(r_1, \dots, r_n)$ the $A[T_1, \dots, T_n]$-module $M[T_1, \dots, T_n]$ provided with the norm

$$||T_1^{\nu_1} \cdot \ldots \cdot T_n^{\nu_n} m|| = r_1^{\nu_1} \cdot \ldots \cdot r_n^{\nu_n} ||m||$$

is a Banach $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$-module. It is denoted by $M\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$.

(ii) For a Banach A-module M, a number $r > 0$ and an element $f \in A$, the $A\{r^{-1}T\}$-submodule E of $M\{r^{-1}T\}$ generated by the pairs (Tm, fm) for $m \in M$ coincides with the set of pairs of the form $(T^k m, T^l n)$ with $f^k m = f^l n$ (see Example 1.1.3(i)). It follows that $M \stackrel{\sim}{\to} M\{r^{-1}T\}/E : m \mapsto \overline{m}$ and, therefore, $||m|| = \inf\{||n|| r^k\}$, where the infimum is taken over all representations $m = f^k n$ with $k \ge 0$ and $n \in M$. As in Example 1.1.3(i), one shows that $||\overline{m}|| = 0$ if and only if there a sequence of integers $0 < k_1 < k_2 < \ldots$ and of elements $n_1, n_2, \ldots$ with $m = f^{k_i} n_i$ and $||n_i|| r^{k_i} \to 0$ as $i \to \infty$.

(iii) The $A\{r^{-1}T\}$-submodule E of $M\{r^{-1}T\}$ generated by the pairs (fTm, m) for $m \in M$ coincides with the set of pairs of the form $(T^k m, T^l n)$ with $f^{k+p} m = f^{l+p} n$ for some $p \ge 0$ (see Example 1.1.3(ii)). It follows that $M_f \stackrel{\sim}{\to} M\{r^{-1}T\}/E : \frac{m}{f^k} \mapsto \overline{\frac{m}{f^k}}$ and, therefore, $||\frac{m}{f^k}|| = \inf\{||n|| r^l\}$, where the infimum is taken over all representations $\frac{m}{f^k} = \frac{n}{f^l}$ in M_f with $n \in M$ and $l \ge 0$.

1.3. The spectrum of a Banach $\mathbf{F}_1$-algebra.

1.3.1. Definition. The *spectrum* $\mathcal{M}(A)$ of a Banach $\mathbf{F}_1$-algebra A is the set of all bounded homomorphisms of $\mathbf{F}_1$-algebras $|\ |: A \to \mathbf{R}_+$.

Notice that for such a homomorphism one has $|f| \le ||f||$. For example, if A is trivial, $\mathcal{M}(A)$ is empty. If K is a real valuation $\mathbf{F}_1$-field, then $\mathcal{M}(K)$ consists of one point (that corresponds to the norm of K). If the norm on K is nontrivial, the spectrum $\mathcal{M}(K^\circ)$ of the Banach $\mathbf{F}_1$-algebra K° (see Example 1.1.2(i)) coincides with $[0,1]$. (A canonical map $[0,1] \to \mathcal{M}(K^\circ)$ takes $t \in]0,1]$ to the norm $|\ |^{\frac{1}{t}}$, and 0 to the seminorm which is induced by the trivial norm on $K^\circ/K^{\circ\circ}$.)

There is a canonical map $\mathcal{M}(A) \to \mathrm{Zspec}(A)$ that takes a point $x \in \mathcal{M}(A)$ to the Zariski kernel $\mathfrak{p}_x = \mathrm{Zker}(|\ |_x)$ of the corresponding bounded homomorphism $|\ |_x : A \to \mathbf{R}_+$. Such a point x gives rise to a norm on the field $\kappa(\mathfrak{p}_x)$, which will be denoted by $\mathcal{H}(x)$, i.e., the point x gives rise to a bounded homomorphism $\chi_x : A \to \mathcal{H}(x)$ to the real valuation $\mathbf{F}_1$-field $\mathcal{H}(x)$. The image of an element $f \in A$ under χ_x is denoted by $f(x)$. The spectrum $\mathcal{M}(A)$ is provided with the weakest topology with respect to which all real valued functions of the form $x \mapsto |f(x)|$ are continuous.

There is also a continuous map $\mathcal{M}(A) \to \mathrm{Spec}(A)$, that takes a point $x \in \mathcal{M}(A)$ to the kernel $\Pi_x = \mathrm{Ker}(|\ |_x)$. It is compatible with the above map $\mathcal{M}(A) \to \mathrm{Zspec}(A)$. The fraction field of A/Π_x is denoted by $\mathcal{G}(x)$. It coincides with the quotient of $\mathcal{H}(x)$ by the kernel of $|\ |_x$, and is embedded in the $\mathbf{F}_1$-field $\mathbf{R}_+$.

1.3.2. Proposition. *The spectrum $\mathcal{M}(A)$ of a nontrivial Banach $\mathbf{F}_1$-algebra A is a nonempty compact space.*

Proof. *Nonemptyness* (cf. the proof of [Ber1, Theorem 1.2.1]). Replacing A by the quotient $A/\mathbf{m}_A$, we may assume that A is an $\mathbf{F}_1$-field. Let S be the set of nonzero bounded seminorm on A. It is nonempty since the norm of A belongs to S, and it is partially ordered with respect to the ordering for which $|\ |' \le |\ |''$ if $|f|' \le |f|''$ for all $f \in A$. This ordering satisfies the conditions of Zorn's Lemma and, therefore, there exist minimal elements in S. *We claim that any minimal element of S (which is a bounded norm $|\ |$) is multiplicative.*

Suppose that there exists an element $f \in A$ with $|f^n| < |f|^n$ for some $n > 1$. Let $r = |f^n|^{\frac{1}{n}}$. We claim that the closure E of the ideal of $A\{r^{-1}T\}$ generated by the pair (T, f) is nontrivial. Indeed, by Example 1.1.3(i), it suffices to show that, for any sequence $0 < i_1 < i_k < \dots$, the sequence $|f^{-i_k}|r^{i_k}$ does not tend to zero. If $i_k = pn + q$ with $0 \le q \le n-1$, then $|f^{i_k}| \le |f^n|^p |f^q|$ and

$$|f^{-i_k}|r^{i_k} \ge |f^{i_k}|^{-1}|f^n|^{p+\frac{q}{n}} \ge \frac{|f^n|^{\frac{q}{n}}}{|f^q|}\ .$$

Hence, $|f^{-i_k}|r^{i_k} \geq \varepsilon > 0$ for all $k \geq 1$, where ε is the minimum of the n positive numbers on the right hand side of the above inequality. It follows that the norm on the nontrivial quotient $A\{r^{-1}T\}/E$ gives rise to a bounded seminorm on A whose value at f is at most $r < |f|$. Thus, $|f^n| = |f|^n$ for all $f \in A$ and $n \geq 1$.

Now suppose that there exists a nonzero element $f \in A$ with $|f|^{-1} < |f^{-1}|$. Let $r = |f^{-1}|^{-1}$. Then the closure E of the ideal of $A\{r^{-1}T\}$ generated by the pair (T, f) is nontrivial. (This again follows from Example 1.1.3(i) since $|f^{-n}|r^n = |f^{-1}|^n \cdot |f^{-1}|^{-n} = 1$.) It follows that the norm on the nontrivial quotient $A\{r^{-1}T\}/E$ gives rise to a bounded seminorm on A whose value at f is at most $r < |f|$. Thus, for any two nonzero elements $f, g \in A$, we have $|fg|^{-1} = |(fg)^{-1}| \leq |f^{-1}| \cdot |g^{-1}| = |f|^{-1}|g|^{-1}$ and, therefore, $|fg| = |f| \cdot |g|$, i.e., the norm $|\ |$ is multiplicative.

Compactness. By the Tichonov theorem, the direct product $\prod[0, ||f||]$, taken over all nonzero elements of A, is a compact space. The canonical map $\mathcal{M}(A) \to \prod[0, ||f||]$ that takes a bounded multiplicative seminorm $|\ |$ to $(|f|)$ identifies the former with a closed subset of the latter and, therefore, $\mathcal{M}(A)$ is a compact space. ■

1.3.3. Corollary. *An element f of a Banach $\mathbf{F}_1$-algebra A is invertible if and only if $f(x) \neq 0$ for all $x \in \mathcal{M}(A)$.*

Proof. The direct implication is trivial. Suppose f is not invertible. Then the Zariski ideal $\mathbf{a}$ generated by f does not coincide with A, i.e., $A/\mathbf{a}$ is a nontrivial Banach $\mathbf{F}_1$-algebra. Since the spectrum $\mathcal{M}(A/\mathbf{a})$ is nonempty, any point of it gives rise to a point $x \in \mathcal{M}(A)$ with $f(x) = 0$. ■

The *spectral radius* of an element $f \in A$ is the number

$$\rho(f) = \lim_{n\to\infty} \sqrt[n]{||f^n||} = \inf_n \sqrt[n]{||f^n||}\ .$$

(The existence of the limit and its equality with the infimum is well known.) Notice that the function $f \mapsto \rho(f)$ is a bounded seminorm on A.

1.3.4. Corollary. *For any element $f \in A$, one has*

$$\rho(f) = \max_{x\in\mathcal{M}(A)} |f(x)|\ .$$

Proof. That the right hand side is at most the left hand side is trivial. To verify the reverse inequality, it suffices to show that if $|f(x)| < r$ for all $x \in \mathcal{M}(A)$, then $\rho(f) < r$.

Consider the Banach $\mathbf{F}_1$-algebra $B = A\{rT\}$. Since $||T|| = r^{-1}$, then $|T(x)| \leq r^{-1}$ for all $x \in \mathcal{M}(B)$ and, therefore, $|(fT)(x)| < 1$ for all $x \in \mathcal{M}(B)$. Let E be the ideal of B generated by the pair $(fT, 1)$. If the closure $\overline{E}$ of E is nontrivial, then there exists a point $x \in \mathcal{M}(B/\overline{E}) \subset \mathcal{M}(B)$ with $|(fT)(x)| = 1$, which is impossible. Thus, $\overline{E}$ is the trivial ideal of B. By Example 1.1.3(ii), there exist sequences of positive numbers $n_1, n_2, \dots$ and of elements $b_1, b_2, \dots \in A$ with $\frac{b_k}{f^{n_k}} = 1$ and $||b_k||r^{-n_k} \to 0$ as $k \to \infty$. Let k be a big enough integer such that $||b_k||r^{-n_k} = \alpha < 1$, and let p_k be a positive integer with $b_k f^{p_k} = f^{p_k + n_k}$. If $n = n_k$, it follows that $f^{p+n} = b_k f^p$ for all $p \geq p_k$ and, therefore, $||f^{p+n}||r^{-(p+n)} \leq \alpha ||f^p||r^{-p}$ for all $p \geq p_k$. If $p \geq p_k$, then for any integer $l \geq 1$ we get $||f^{p+ln}||r^{-(p+ln)} \leq \alpha^l ||f^p||r^{-p}$. It follows that $||f^{p+ln}||r^{-(p+ln)} \to 0$ as $l \to \infty$ and, therefore, $\rho(f) < r$. ∎

For a compact topological space X, let $\mathcal{C}(X)$ denote the $\mathbf{F}_1$-algebra of continuous functions $X \to \mathbf{R}_+$ provided with the supremum norm. There is an evident continuous embedding $X \to \mathcal{M}(\mathcal{C}(X))$ but, if X contains at least two different points x_1 and x_2, this map is not a bijection. Indeed, the following bounded multiplicative homomorphism $\chi : \mathcal{C}(X) \to \mathbf{R}_+$ does not come from X: $\chi(f) = 0$, if $f(x_1) = 0$, and $\chi(f) = f(x_2)$, otherwise.

If X is the spectrum $\mathcal{M}(A)$ of a Banach $\mathbf{F}_1$-algebra A, then there is a canonical bounded homomorphism of Banach $\mathbf{F}_1$-algebras $\hat{} : A \to \mathcal{C}(X)$, called the *Gelfand transform*. In particular, its kernel, which consists of the pairs (f, g) with $|f(x)| = |g(x)|$ for all $x \in X$, is a closed ideal of A. Corollary 1.3.4 means that the Gelfand transform is isometric with respect to the spectral norm, i.e., $\rho(f) = ||\widehat{f}||$ for all $f \in A$. The image of the Gelfand transform is denoted by $\widehat{A}$, and its coimage, i.e., the quotient of A by the kernel will be denoted by $|A|$. The canonical bounded homomorphism $|A| \to \widehat{A}$ is a bijection, but is not an isomorphism in general (see Remark 1.3.11 and Corollary 8.3.3).

1.3.5. Examples. (i) Let A be a finite idempotent $\mathbf{F}_1$-algebra. It is a Banach $\mathbf{F}_1$-algebra with respect to the trivial norm which coincides with the spectral norm. (In fact any Banach norm on A is equivalent to the trivial norm.) Since $\kappa(\mathfrak{p}) = \mathbf{F}_1$ for any Zariski prime ideal $\mathfrak{p} \subset A$, the canonical map $\mathcal{M}(A) \to \mathrm{Zspec}(A) : x \mapsto \mathrm{Zker}(|\ |_x)$ is a bijection and, therefore, the canonical map $\mathcal{M}(A) \to \mathrm{Spec}(A)$ is a bijection. Since the latter map is continuous and $\mathrm{Spec}(A)$ is discrete, it follows that $\mathcal{M}(A)$ is discrete.

(ii) If A is a commutative Banach ring, there is a canonical continuous map $\mathcal{M}(A) \to \mathcal{M}(A^{\cdot})$ which identifies $\mathcal{M}(A)$ with the closed subset of $\mathcal{M}(A^{\cdot})$ consisting of the seminorms possessing the property $|f + g| \leq |f| + |g|$ for all $f, g \in A$.

(iii) If $A = K\{r_i^{-1}T_i\}$, where K is a real valuation field, then $\mathcal{M}(A) \xrightarrow{\sim} \prod_{i \in I} [0, r_i]$. For example, $\mathcal{M}(\mathbf{Z}^{\cdot}) \xrightarrow{\sim} \prod_{n=1}^{\infty} [0, p_n]$ (see Example 1.1.2(iii)).

1.3.6. Lemma. *Let $\varphi : A \to B$ be a bounded homomorphism of Banach $\mathbf{F}_1$-algebras, Σ the image of the induced map $\mathcal{M}(B) \to \mathcal{M}(A)$, and $A_{(\Sigma)}$ the localization of A with respect to the set of all elements of A that do not vanish at any point of Σ. Then*

(i) if the induced homomorphism $A_{(\Sigma)} \to B$ is surjective, the map $\mathcal{M}(B) \to \mathcal{M}(A)$ is injective;

(ii) if φ is an admissible epimorphism, then the image of $\mathcal{M}(B)$ coincides with the subset $\{x \in \mathcal{M}(A) \big| |f(x)| = |g(x)| \text{ for all } (f,g) \in \mathrm{Ker}(\varphi)\}$;

(iii) if φ is a bijection and an isometry with respect to the spectral norm, then $\mathcal{M}(B) \stackrel{\sim}{\to} \mathcal{M}(A)$.

Proof. The statements (i) and (iii) are trivial. As for (ii), it is clear that the image of $\mathcal{M}(B)$ is contained in the subset considered. Assume that a point $x \in \mathcal{M}(A)$ is such that $|f(x)| = |g(x)|$ for all $(f,g) \in \mathrm{Ker}(\varphi)$. It is then clear that the bounded homomorphism $| \ |_x : A \to \mathbf{R}_+$ goes through a homomorphism $| \ | : B \to \mathbf{R}_+$, and we have to verify that the latter is bounded. Since φ is an admissible epimorphism, there exists its section $\sigma : B \to A$ and a constant $C > 0$ such that $||g|| \le C||\sigma(g)||$ for all $g \in B$. We have $|g| = |\sigma(g)| \le ||\sigma(g)|| \le C||g||$ for all $g \in B$, and the required fact follows. ∎

1.3.7. Corollary. *For a Banach $\mathbf{F}_1$-algebra A, one has $\mathcal{M}(\widehat{A}) \stackrel{\sim}{\to} \mathcal{M}(|A|) \stackrel{\sim}{\to} \mathcal{M}(A)$.*

Proof. The bijectivity of the second map follows from Lemma 1.3.6(ii), and that of the first map follows from (iii). ∎

Let X be the spectrum $\mathcal{M}(A)$ of a Banach $\mathbf{F}_1$-algebra, and let $\mathfrak{p}$ be a Zariski prime ideal of A. By Lemma 1.4.1(ii), the canonical map $\mathcal{M}(A/\mathfrak{p}) \to X$ identifies $\mathcal{M}(A/\mathfrak{p})$ with the closed subset $X_{\mathfrak{p}} = \{x \in X \big| \mathrm{Zker}(| \ |_x) \subset \mathfrak{p}\}$. One can define as follows a retraction map $\tau_{\mathfrak{p}} : X \to X_{\mathfrak{p}}$: $|f(\tau_{\mathfrak{p}}(x)| = |f(x)|$, if $f \not\in \mathfrak{p}$, and $|f(x)| = 0$, if $f \in \mathfrak{p}$. This retraction map is compatible with the retraction maps $\mathrm{Zspec}(A) \to \mathrm{Zspec}(A/\mathfrak{p})$ and $\mathrm{Spec}(A) \to \mathrm{Spec}(A/\mathfrak{p})$, introduced in §I.1.2 and denoted in the same way.

1.3.8. Lemma. *For Banach A-algebras B and C, there is a canonical homeomorphism*

$$\mathcal{M}(B\widehat{\otimes}_A C) \to \mathcal{M}(B) \times_{\mathcal{M}(A)} \mathcal{M}(C) \ .$$

Proof. If $y \in \mathcal{M}(B)$ and $z \in \mathcal{M}(C)$ are points over a point $x \in \mathcal{M}(A)$, then the preimage of the point (y,z) under the map considered is $\mathcal{M}(\mathcal{H}(y)\widehat{\otimes}_{\mathcal{H}(x)}\mathcal{H}(z))$. But as we already noticed in §1.2, the complete tensor product of two real valuation $\mathbf{F}_1$-fields over a real valuation $\mathbf{F}_1$-field is again a real valuation $\mathbf{F}_1$-field. ∎

1.3.9. Lemma. *Let $A \to B$ be a bounded injective homomorphism of Banach $\mathbf{F}_1$-algebras* *such that it is isometric with respect to the spectral norm and B is integral over A. Then the canonical map $\mathcal{M}(B) \to \mathcal{M}(A)$ is surjective.*

Proof. (**Proverit'!!!**) Let $x \in \mathcal{M}(A)$ and $\mathfrak{p} = \mathrm{Zker}(|\ |_x)$. By Proposition I.2.5.4(i), there exists a Zariski prime ideal $\mathfrak{q}$ of B with $\mathfrak{q} \cap A = \mathfrak{p}$. By the assumption, the $\mathbf{F}_1$-field $\kappa(\mathfrak{q})$ is algebraic over $\kappa(\mathfrak{p}) = \mathcal{H}(x)$ and, therefore, its real valuation extends in a unique way to a real valuation on $\kappa(\mathfrak{q})$. It remains to verify that the induced seminorm $|\ | : B \to \kappa(\mathfrak{q}) \to \mathbf{R}_+$ is bounded, i.e., $|g| \le \rho(g)$ for all $g \in B$. If $g \in \mathfrak{q}$, this is trivial. Assume that $g \notin \mathfrak{q}$, and take an equation $g^m = fg^n$ for some $m > n \ge 0$ and $f \in A$. For every point $y \in \mathcal{M}(B)$, we have $|g(y)|^m = |f(y)| \cdot |g(y)|^n$ and, therefore, $|f(y)| = |g(y)|^{m-n}$. It follows that $\rho(f) = \rho(g)^{m-n}$. We get $|g| = |f|^{\frac{1}{m-n}} = |f(x)|^{\frac{1}{m-n}} \le \rho(f)^{\frac{1}{m-n}} = \rho(g)$. ■

1.3.10. Lemma. *Let A be a Banach $\mathbf{F}_1$-field, and let E be an ideal of A that corresponds to a subgroup $G \subset A^*$. Then the following are equivalent:*

(a) E is closed;

(b) $\rho(g) \ge 1$ for all $g \in G$;

(c) there exists a point $x \in \mathcal{M}(A)$ with $|g(x)| = 1$ for all $g \in G$.

Moreover, in this case one has $\mathcal{M}(A/E) \stackrel{\sim}{\to} \{x \in \mathcal{M}(A) \big| |g(x)| = 1$ for all $g \in G\}$.

Proof. If (a) is true, then the validity of (c) and of the last statement follows from Lemma 1.2.5(ii). The implication (c)$\Longrightarrow$(b) is trivial. Suppose (b) is true. Then $||g|| \ge 1$ for all $g \in G$. Given an element $f \in A^*$, one has $||g|| \le ||fg|| \cdot ||f^{-1}||$ and, therefore, $||fg|| \ge ||g|| \cdot ||f^{-1}||^{-1} \ge ||f^{-1}||^{-1}$. The latter means that the quotient seminorm on A/E is nonzero, i.e., E is closed and (a) is true. ■

1.3.11. Remark. Let $A = \mathbf{F}_1[T]$ be provided with the following Banach norm: $||T^n|| = n+1$ for $n \ge 0$. Then $|A| = A$. but, since $\rho(T^n) = 1$ for all $n \ge 0$, $\widehat{A}$ is A provided with the trivial norm. It follows that the canonical map $|A| \to \widehat{A}$ is not an isomorphism (there is no a constant $C > 0$ with $||T^n|| \le C\rho(T^n)$ for all $n \ge 0$).

1.4. K-Banach spaces and Banach K-algebras. Let K be a real valuation $\mathbf{F}_1$-field. Banach K-modules are said to be *K-Banach spaces*. If M is a K-Banach space, then $||\lambda m|| = |\lambda| \cdot ||m||$ for all $\lambda \in K$ and $m \in M$. Indeed, one has $||\lambda m|| \le |\lambda| \cdot ||m||$. On the other hand, if $\lambda \ne 0$, then $||m|| = ||\lambda^{-1}(\lambda m)|| \le |\lambda|^{-1}||\lambda m||$ and, therefore, $||\lambda m|| = |\lambda| \cdot ||m||$. It follows that $M \otimes_K N \stackrel{\sim}{\to} M \widehat{\otimes}_K N$ for all K-Banach spaces M and N. If K' is a real valuation K-field then, for any K-Banach space M, $M \otimes_K K'$ is a K'-Banach space. For example, the quotient M/K^{**} of M under the action of the group K^{**} is a $|K|$-Banach space. A *Banach K-algebra* is a Banach

$\mathbf{F}_1$-algebra A which has the structure of a Banach K-module. For a Banach K-algebra A, the quotient A/K^{**} is a Banach $|K|$-algebra.

The category of K-Banach space admits direct sums, which coincide with the direct sums of the underlying K-vector spaces. For example, the canonical decomposition of a K-Banach space M into a direct sum of cyclic K-vector spaces gives rise to a similar decomposition of M into a direct sum of cyclic K-Banach spaces. Notice that the stabilizers of nonzero elements of M are subgroups of K^{**}. A *K-Banach subspace* of a Banach space M is a K-vector subspace of M which is closed as a Banach K-submodule.

A K-Banach space is said to be *free* if it is free as a K-vector space (or, equivalently, the stabilizers of all nonzero elements are trivial). For example, if the real valuation homomorphism $|\ | : K \to \mathbf{R}_+$ is injective (i.e., $K^{**} = \{1\}$), then all K-Banach spaces are free and all their Banach K-submodules are K-Banach subspaces. Notice that in this case any closed ideal E of a finitely generated Banach K-algebra A is finitely generated. Indeed, A/E is free as a K-Banach space and, therefore, E is a K-ideal. By Proposition I.1.5.4, A is K-Noetherian and, therefore, E is finitely generated.

1.4.1. Lemma. *Let K be a real valuation $\mathbf{F}_1$-field, and let A be a finitely generated Banach K-algebra. Then*

(i) for every Zariski prime ideal $\mathfrak{p} \subset A$ and every Zariski ideal $\mathbf{a} \subset \mathfrak{p}$, the ideal $E = \mathrm{Ker}(A \to A_\mathfrak{p}/\mathbf{a}A_\mathfrak{p})$ is closed; in particular, $\Pi_\mathfrak{p}$ is a closed ideal;

(ii) the radical $\mathbf{r}(E)$ of any closed ideal $E \subset A \times A$ is closed and, in particular, the nilradical $\mathbf{n}(A)$ is closed.

Proof. (i) Suppose first that the valuation homomorphism $|\ | : K \to \mathbf{R}_+$ is injective. *We claim that in this case E is a K-ideal.* Indeed, assume that $(f, \lambda f) \in E$ for some $f \notin \mathbf{a}_E$ and $\lambda \in K^*$. Then there exists $h \notin \mathfrak{p}$ with $fh = \lambda fh$. Since A is free as a K-vector space and $fh \neq 0$, we get $\lambda = 1$, i.e., the claim is true. Since A is K-Noetherian, the K-ideal E is finitely generated, and the required fact follows from Example 1.1.4(ii). In the general case, consider the isometric epimorphism $A \to \overline{A} = A/K^{**} : f \mapsto \overline{f}$, and assume that $(f, f_n) \in E$ and $||f_n|| \to 0$ as $n \to \infty$. Then $(\overline{f}, \overline{f}_n) \in \overline{E} = \mathrm{Ker}(\overline{A} \to \overline{A}_{\overline{\mathfrak{p}}}/\overline{\mathbf{a}}\overline{A}_{\overline{\mathfrak{p}}})$, where $\overline{\mathfrak{p}}$ and $\overline{\mathbf{a}}$ are the images of $\mathfrak{p}$ and $\mathbf{a}$ in $\overline{A}$, and $||\overline{f}_n|| \to 0$ as $n \to \infty$. By the previous case, $\overline{f} = 0$ and, therefore, $f = 0$.

(ii) Replacing A by A/E, the statement is reduced to the particular case. Since $\mathbf{n}(A) = \bigcap_{i=1}^n \Pi_{\mathfrak{p}_i}$ for some Zariski prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset A$ (see §I.2.3), the required fact follows. ∎

Let k be a non-Archimedean field. For an $\mathbf{F}_1$-Banach space M, let $k\{M\}$ denote the k-Banach space consisting of infinite sums $F = \sum_{m \in \check{M}} \lambda_m m$ with $\lambda_m \in k$ and such that $||\lambda_m|| \cdot ||m|| \to 0$ with respect to the filter of complements of finite subsets of $\check{M} = M \backslash \{0\}$, and provided with the norm $||F|| = \max_{m \in \check{M}} ||\lambda_m|| \cdot ||m||$. Notice that the canonical map $M \to k\{M\}$ is isometric. If M is a Banach A-module, then $k\{M\}$ is a Banach $k\{A\}$-module. Given a family of Banach A-modules $\{M_i\}_{i \in I}$, there is a canonical isometric isomorphism of Banach $k\{A\}$-modules

$$k\{\oplus_{i \in I} M_i\} \widetilde{\to} \oplus_{i \in I} k\{M_i\} ,$$

where the right hand side is a direct sum in the category of Banach $k\{A\}$-modules. It is easy to see that, if $M \to N$ is an admissible homomorphism (resp. monomorphism, resp. epimorphism) of Banach A-modules, then so is the corresponding homomorphism of Banach $k\{A\}$-modules $k\{M\} \to k\{N\}$. Furthermore, for Banach A-modules M and N, there is an isomorphism of Banach $k\{A\}$-modules $k\{M \widehat{\otimes}_A N\} \widetilde{\to} k\{M\} \widehat{\otimes}_{k\{A\}} k\{N\}$.

1.4.2. Lemma. *There is a canonical continuous map $\tau : \mathcal{M}(k\{A\}) \to \mathcal{M}(A)$, and the preimage $\tau^{-1}(x)$ of a point $x \in \mathcal{M}(A)$ is the space $\mathcal{M}(k\{\mathcal{H}(x)\})$.*

Proof. Recall that $\mathcal{M}(k\{A\})$ is the space of all bounded multiplicative seminorms on $k\{A\}$. The restriction of such a seminorm to A is a bounded homomorphism of $\mathbf{F}_1$-algebras $A \to \mathbf{R}_+$, and in this way we get the map τ. A point $x \in \mathcal{M}(A)$ gives rise to a bounded homomorphism of Banach k-algebras $k\{A\} \to k\{\mathcal{H}(x)\}$. Since $\mathcal{H}(x)$ is a real valuation $\mathbf{F}_1$-field, the restriction of any bounded multiplicative seminorm on $k\{\mathcal{H}(x)\}$ to $\mathcal{H}(x)$ coincides with the canonical norm on it, and so the image of $\mathcal{M}(k\{\mathcal{H}(x)\})$ is contained in $\tau^{-1}(x)$. Furthermore, since the k-subalgebra of $k\{\mathcal{H}(x)\}$, generated by the image of the localization of A with respect to the multiplicative system of all $f \in k\{A\}$ with $f(x) \neq 0$, is dense, the induced map $\mathcal{M}(k\{\mathcal{H}(x)\}) \to \tau^{-1}(x)$ is injective. Finally, any point $y \in \tau^{-1}(x)$ gives rise to a bounded multiplicative seminorm on $k\{\mathcal{H}(x)\}$ and, therefore, $\mathcal{M}(k\{\mathcal{H}(x)\}) \widetilde{\to} \tau^{-1}(x)$. ∎

1.4.3. Lemma. *Let K is a real valuation $\mathbf{F}_1$-field, and k a non-Archimedean field. Then*

(i) if the group K^* is torsion free, the norm on $k\{K\}$ is multiplicative;

(ii) if the orders of torsion elements of K^* are prime to the characteristic of the residue field $\widetilde{k}$ of k, the norm on $k\{K\}$ is power multiplicative.

Proof. First of all, we notice that every nonzero element $F = \sum_{f \in K} \lambda_f f \in k\{K\}$ can be represented in the form $F_1 + F_2$, where F_1 is the finite sum of elements $\lambda_f f$ with $|\lambda_f| \cdot |f| = ||F||$, and the norm of every summand in $F_2 = F - F_1$ is strictly less than $||F||$. It follows that it suffices to verify the required facts only for the finite sums F_1. We can therefore replace K by the $\mathbf{F}_1$-subfield whose multiplicative group is generated by the elements that have nonzero entry in such a finite sum F_1 for a finite number of elements. Thus, we may assume that the group K^* is finitely generated.

(i) Since the finitely generated group K^* has not torsion, it is a free abelian group. Let $f_1, \dots, f_n$ be free generators of K^*, and set $r_i = |f_i|$ for $1 \le i \le n$. Then there is an isometric isomorphism $k\{K\} \stackrel{\sim}{\to} k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, r_1T_1^{-1}, \dots, r_nT_n^{-1}\}$, and the norm of the latter is multiplicative.

(ii) By (i), it suffices to verify the following fact. Let $\mathcal{A}$ be a Banach k-algebra with power multiplicative norm, and n an integer $n \ge 1$ with $|n| = 1$ (in k). Then the norm of the Banach algebra $\mathcal{B} = \mathcal{A}[T]/(T^n - 1)$, defined by $||f|| = \max\limits_{0 \le i \le n-1} ||a_i||$ at $f = \sum_{i=0}^{n-1} a_i T^i \in \mathcal{B}$, is power multiplicative. To show this, we may increase the field k and assume that it contains all n-th roots of unity. By the assumption, $||a|| = \max\{|a(x)|\}$ for every $a \in \mathcal{A}$, where the maximum is taken over all points $x \in \mathcal{M}(\mathcal{A})$. Let π denotes the canonical surjective map $\mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$. To verify the required fact, it suffices to show that, for any any point $x \in \mathcal{M}(\mathcal{A})$, one has

$$\max_{y \in \pi^{-1}(x)} |f(y)| = \max_{0 \le i \le n-1} |a_i(x)| \ .$$

The latter follows from the following simple fact. Given elements $a_0, \dots, a_{n-1}$ of a non-Archimedean field K which contains a primitive n-th root of unity ζ and in which $|n| = 1$, one has

$$\max_{0 \le j \le n-1} |\sum_{i=0}^{n-1} a_i \zeta^{ij}| = \max_{0 \le i \le n-1} |a_i| \ .$$

Indeed, that the left hand side is at most than the ride hand one is trivial, and the converse inequality follows from the fact that the determinant of the Vandermonde matrix $(\zeta^{ij})_{0 \le i,j \le n-1}$ is equal to $\prod_{0 \le i < j \le n-1} (\zeta^j - \zeta^i)$, and, by the assumption $|n| = 1$, the norm of the latter in k is equal to one. ∎

1.4.4. Corollary. *Let A be a Banach $\mathbf{F}_1$-algebra, and assume that for every Zariski prime ideal $\mathfrak{p} \subset A$ the group $\kappa(\mathfrak{p})^*$ has no torsion. Then, for any non-Archimedean field k, each fiber $\tau^{-1}(x)$ of the canonical map $\tau : \mathcal{M}(k\{A\}) \to \mathcal{M}(A)$ has a unique maximal point (denoted by $\sigma(x)$), and the map $\mathcal{M}(A) \to \mathcal{M}(k\{A\}) : x \mapsto \sigma(x)$ is continuous.*

Proof. By Lemma 1.4.2, one has $\tau^{-1}(x) \stackrel{\sim}{\to} \mathcal{M}(k\{\mathcal{H}(x)\})$, and, by the assumption and Lemma 1.4.3(i), the fiber $\tau^{-1}(x)$ has a unique maximal point. ∎

1.5. Twisted products of Banach $\mathbf{F}_1$-algebras. Let K be a real valuation $\mathbf{F}_1$-field.

1.5.1. Definition. A twisted datum $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$ is said to be a *twisted datum of Banach K-algebras* if it satisfies the following conditions:

(1) the set I is finite;

(2) each A_i is a Banach K-algebra;

(3) if $i \le j$, the quasi-homomorphism ν_{ij} is a bounded K-linear map and induces an admissible epimorphism $A_i \to A_j/\mathbf{a}_{ji}$.

Let $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$ be a twisted datum of Banach K-algebras. The twisted product $A = \prod_I^\nu A_i$ is provided with the Banach norm which is induced by the supremum norm via the canonical embedding $\nu : A \hookrightarrow \prod_{i \in I} A_i$.

1.5.2. Lemma. *There is a constant $C > 0$ such that, for every $i \in I$ and every $a \in A_i$, there exist $j \le i$ and $b \in \mathbf{a}^{(j)}$ with $\nu_{ji}(b) = a$ and $||b|| \le C||a||$.*

Proof. Let $C > 0$ be a constant with the property that, for every $k \le l$ and every $c \in A_l \backslash \mathbf{a}_{lk}$, there exists $d \in A_k$ with $||d|| \le C||c||$. Suppose that $a \notin \mathbf{a}^{(i)}$. Then there exist $j = j_n < \dots < j_0 = i$ in I and $c_k \in A_{j_k}$ for $0 \le k \le n$ such that $c_0 = a$, $c_k = \nu_{j_{k+1}j_k}(c_{j_{k+1}}) \notin \mathbf{a}_{j_k j_{k+1}}$ and $||c_{j_{k+1}}|| \le D||c_{j_k}||$ for $0 \le k \le n-1$, and $b = c_j \in \mathbf{a}^{(j)}$. Then $\nu_{ji}(b) = a$ and $||b|| \le C^n||a||$. Since $n+1$ does not exceed the number of elements in I, the required fact follows. ∎

1.5.3. Corollary. *For every subset $J \subset I$ which is preserved under the infumum operation, the canonical map $p_J : A \to A_J = \prod_J^\nu A_i$ of Corollary I.3.1.3 is an admissible epimorphism.*

Proof. It suffices to consider the case when J is a one element subset. Let $C > 0$ be a constant with the property of Lemma 1.5.2, and let $D > 0$ be a constant with the property that, for every $i \le j$ and every $a \in A_i$, one has $||\nu_{ij}(a)|| \le D||a||$. Furthermore, let $J = \{i\}$ and $a \in A_i$. If $a \in \mathbf{a}^{(i)}$, let $c = (c_j) \in A$ be the element of A with $c_j = \nu_{ij}(a)$, if $i \le j$, and $c_j = 0$, otherwise. One then has $p_i(c) = a$ and $||c|| = \max\{||c_j||\} \le D||a||$. Suppose now that $a \notin \mathbf{a}^{(i)}$. By Lemma 1.5.2, there exist $j < i$ and $b \in \mathbf{a}^{(j)}$ with $\nu_{ji}(b) = a$ and $||b|| \le C||a||$. If now $c = (c_k) \in A$ is the element of A with $c_k = \nu_{jk}(b)$, if $j \le k$, and $c_k = 0$, otherwise, then $p_i(c) = a$ and $||c|| \le D||b|| \le CD||a||$. ∎

We set $X = \mathcal{M}(A)$ and, for $i \in I$, $X_i = \mathcal{M}(A)$. Corollary 1.5.3 implies that the canonical map $X_i \to X$ is injective.

1.5.4. Proposition. *(i) For every point $x \in X$, there exists a unique minimal $i \in I$ with $x \in X_i$ and, in particular, $X = \bigcup_{i \in I} X_i$;*

(ii) if $i \le j$, then $X_i \cap X_j = \mathcal{M}(A_i/E_{ij}) = \mathcal{M}(A_j/\mathbf{a}_{ji})$;

(iii) $X_i \subset X_j$ (resp. $X_j \subset X_i$) if and only if $E_{ij} \subset \mathbf{n}(A_i)$ (resp. $\mathbf{a}_{ji} \subset \mathrm{z}\mathbf{n}(A_j)$).

Proof. (i) One has $x \in X_i$ if and only if $E_i = \mathrm{Ker}(p_i) \subset \Pi = \mathrm{Ker}(|\ |_x)$. The latter implies that $\mathbf{a}_i = \mathrm{Zker}(p_i) \subset \mathfrak{p} = \mathrm{Zker}(|\ |_x)$. By Proposition I.3.3.2, the set of $i \in I$ with $\mathbf{a}_i \subset \mathfrak{p}$ has a unique minimal element $i = i_{\mathfrak{p}}$, and one has $E_i = \mathrm{Ker}(p_i) \subset \Pi_{\mathfrak{p}}$. Since $\mathbf{a}_\Pi = \mathfrak{p}$, we get $E_i \subset \Pi$, i.e., $x \in X_i$. ∎

A *morphism of twisted data of Banach K-algebras* $f : \{I', A_{i'}, \nu_{i'j'}, \mathbf{a}_{j'i'}\} \to \{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$ is a morphism of twisted data in the sense of Definition I.3.2.2 such that, for every $i \in I$, the homomorphism $f_i : A_{i'} \to A_i$ is a bounded homomorphism of Banach K-algebras. Such a morphism is called a *quasi-isomorphism* if it is a quasi-isomorphism in the sense of Definition I.3.2.4 with the stronger property (1): for every $i \in I$, the homomorphism $f_i : A_{i'} \to A_i$ is an admissible epimorphism. It follows easily from §I.3.2 that twisted data of Banach K-algebras form a category, that the correspondence $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\} \mapsto A = \prod_I^\nu A_i$ is a funtor from it to the category of Banach K-algebras, and that quasi-isomorphisms are precisely the morphisms that go to isomorphisms under that functor.

Furthermore, let A be a Banach $\mathbf{F}_1$-algebra. A twisted datum of A-modules $\{I, M_i, \nu_{ij}, N_{ji}\}$ (see §I.3.6) is said to be a *twisted datum of Banach A-modules* if, for every $i \in I$, M_i is a Banach A-module and, for every pair $i \le j$ in I, the quasi-homomorphism ν_{ij} is bounded and induces an admissible epimorphism $M_i \to M_j/N_{ji}$. For such a twisted datum, the twisted product $M = \prod_I^\nu M_i$ is a Banach A-module with respect to the supremum norm via the canonical embedding $\nu : M \hookrightarrow \prod_{i \in I} M_i$.

1.5.5. Remark. By the construction of the twisted product $A = \prod_I^\nu A_i$ (in the proof of Proposition I.3.1.2), A is the union of all $\mathbf{a}^{(i)}$'s taken over $i \in I$ and glued along their zeros. The function $|| \ ||' : A \to \mathbf{R}_+$, defined by $||0||' = 0$ and $||a||' = ||a_i||$ for $a = (a_j)_{j \in I} \in \mathbf{a}^{(i)} \backslash \{0\}$, is of the type considered in Remark 1.1.5, i.e., it is equivalent to the Banach norm $|| \ ||$, and there is a constant $C > 0$ such that $||ab||' \le C||a||' \cdot ||b||'$ for all $a, b \in A$.

§2. K-affinoid algebras

2.1. Definition of a K-affinoid algebra. Let K be a real valuation $\mathbf{F}_1$-field.

2.1.1. Definition. A *K-affinoid algebra* is a Banach K-algebra A for which there exists an admissible epimorphism $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$. A K-affinoid algebra for which such an epimorphism can be found with $r_1 = \dots = r_n = 1$ is said to be *strictly K-affinoid.*

2.1.2. Example. Every real valuation K-field K' with finitely generated cokernel of the canonical homomorphism $K^* \to K'^*$ is a K-affinoid algebra. Indeed, represent the cokernel as a direct sum $\mathbf{Z}^m \oplus (\oplus_{i=m+1}^n \mathbf{Z}/d_i\mathbf{Z})$, take representatives $f_1, \dots, f_n$ of the canonical generators of the direct summands, and set $r_i = |f_i|$ for $1 \le i \le n$. Then K' is isomorphic to a quotient of $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, r_1S_1, \dots, r_nS_n\}$. For example, the valuation K-field $\mathcal{H}(x)$ of any point x of the spectrum of a K-affinoid algebra A is a K-affinoid algebra.

The following lemma lists properties of (strictly) K-affinoid algebras that easily follow from previous results.

2.1.3. Proposition. *Let A and B be a (strictly) K-affinoid algebras. Then*

(i) the quotient of $A\{r^{-1}T_1, \dots, r_n^{-1}T_n\}$ by a closed ideal is a (strict if $r_1 = \dots = r_n = 1$) K-affinoid algebra (called an A-affinoid algebra);

(ii) given bounded homomorphisms of (strictly) K-affinoid algebras $A \to B$ and $A \to C$, $B\widehat{\otimes}_A C$ is a (strictly) K-affinoid algebra;

(iii) given a real valuation K-field K', $A \otimes_K K'$ is a (strictly) K'-affinoid algebra;

(iv) A/K^{**} and $|A|$ are finitely presented (strictly) $|K|$-affinoid algebra;

(v) if A is finitely presented over K, then there exist a real valuation $\mathbf{F}_1$-subfield $K' \subset K$ with finitely generated group K'^* and a (strictly) K'-affinoid K'-subalgebra $A' \subset A$ such that $A' \otimes_{K'} K \xrightarrow{\sim} A$ (an isometric isomorphism);

(vi) if A and B are finitely presented over K, then for any bounded homomorphism $\varphi : A \to B$ there exist a real valuation $\mathbf{F}_1$-subfield $K' \subset K$ and K'-affinoid K'-subalgebras $A' \subset A$ and $B' \subset B$ with the properties (v) and such that φ is induced by a bounded homomorphism of K'-affinoid algebras $A' \to B'$.

Proof. The statements (i)-(iii) are trivial.

(iv) Since the canonical map $K\{r^{-1}T\} \to |K|\{r^{-1}T\}$ is isometric, it follows that, for any admissible epimorphism $K\{r^{-1}T\} \to A$, the induced epimorphism $|K|\{r^{-1}T\} \to A/K^{**}$ is also admissible, and so the quotient A/K^{**} is a (strictly) $|K|$-affinoid algebra. Thus, we can replace K by $|K|$ and A by A/K^{**}, and we may assume that $K = |K|$. Then $|A| = A/E$, where the closed ideal E consists of the pairs (f, g) with $|f(x)| = |g(x)|$ for all $x \in \mathcal{M}(A)$, i.e., $|A|$ is also (strictly) K-affinoid. It remains to notice that both K-algebras are free K-modules and, therefore, they are finitely presented over K, by Proposition 1.6.2.

(v) Assume that A is the quotient of $K\{r^{-1}T\} = K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ by a finitely generated ideal E, and let K' be the $\mathbf{F}_1$-subfield of K which is generated by coefficients of all term components from a finite set of generators of E. Let also E' be the ideal of $K'\{r^{-1}T\}$ generated by the same system of generators. Then E' is the intersection of E with $K'\{r^{-1}T\}$ and, therefore, it is closed in the latter. It follows that, for $A' = K'\{r^{-1}T\}/E'$, there is an isometric isomorphism $A' \otimes_{K'} K \xrightarrow{\sim} A$.

(vi) We do the same procedure as in (v) but choose the valuation subfield $K' \subset K$ big enough so that the properties (v) hold for both A and B and the image of a finite system of generators of A' lies in B'. ∎

Proposition 2.1.3 allows one to reduce investigation of K-affinoid algebras to the case $K = \mathbf{F}_1$. Namely, the canonical homomorphism $A \to A/K^{**}$ is isometric, and this allows one to reduce situation considered to the case when $K \widetilde{\to} |K|$ and, in particular, when a K-affinoid algebra A is finitely presented. In this case, one can find an $\mathbf{F}_1$-subfield $K' \subset K$ with finitely generated group K'^* and a K'-affinoid subalgebra $A' \subset A$ with $A' \otimes_{K'} K \widetilde{\to} A$, and this allows one to reduce the situation to the case when the group K^* is finitely generated. In this case, A can be viewed as an $\mathbf{F}_1$-affinoid algebra. Here is an example.

2.1.4. Corollary. *Every finitely presented K-affinoid algebra A admits a primary decomposition $\Delta(A) = \bigcap_{i=1}^n E_i$ with finitely generated closed ideals E_i.*

Proof. Let K' be an $\mathbf{F}_1$-subfield of K with finitely generated group K'^*, and A' a K'-affinoid subalgebra of A with $A' \otimes_{K'} K \widetilde{\to} A$. Then A' is a finitely generated $\mathbf{F}_1$-algebra and, therefore, it admits a primary decomposition $\Delta(A') = \bigcap_{i=1}^n E'_i$ with $E'_i = \mathrm{Ker}(A' \to A'_{\mathfrak{p}'_i}/\mathbf{a}'_i A'_{\mathfrak{p}'_i})$, where $\mathfrak{p}'_i$ is a Zariski prime ideal in A and $\mathbf{a}'_i$ is a Zariski ideal in $\mathfrak{p}'_i$. Proposition I.2.6.6 implies that the ideal E_i of A generated by E'_i coincides with $\mathrm{Ker}(A \to A_{\mathfrak{p}_i} \to A_{\mathfrak{p}}/\mathbf{a}_i A_{\mathfrak{p}_i})$, where $\mathfrak{p}_i = \mathfrak{p}'_i A$ and $\mathbf{a}_i = \mathbf{a}'_i A$, and one has $\Delta(A) = \bigcap_{i=1}^n E_i$. Since the ideals E_i are finitely generated, Example 1.1.4(ii) implies that they are closed in A. ∎

2.2. Basic properties of K-affinoid algebras. We are now going to establish properties of K-affinoid algebras which are analogous to properties of k-affinoid algebras over a non-Archimedean field k and are, in fact, deduced from those of the latter. Namely, reducing a situation considered to the case of an $\mathbf{F}_1$-affinoid algebra, we notice that, for such A and any non-Archimedean field k, the Banach $\mathbf{F}_1$-algebra $k\{A\}$ is k-affinoid and the canonical map $A \to k\{A\}$ is isometric. Here is an example.

2.2.1. Proposition. *Let A be a K-affinoid algebra. Then, for any non-nilpotent element $f \in A$, there exists a constant $C > 0$ such that $||f^n|| \le C\rho(f)^n$ for all $n \ge 1$; in particular, $\rho(f) > 0$.*

Proof. Using the remark from the previous subsection, the situation is reduced to the case when A is an $\mathbf{F}_1$-affinoid algebra. Since for any non-Archimedean field k the canonical map $A \to k\{A\}$ is isometric, the required statement follows from [Ber1, Proposition 2.1.4(i)]. ∎

It will be convenient to us to have a special term for the class of Banach $\mathbf{F}_1$-algebras with the property of Proposition 2.2.1.

2.2.2. Definition. A Banach $\mathbf{F}_1$-algebra A is said to be *quasi-affinoid* if, for any non-nilpotent element $f \in A$, there exist a constant $C > 0$ such that $||f^n|| \le C\rho(f)^n$ for all $n \ge 1$.

Besides K-affinoid algebras, the main example of a quasi-affinoid $\mathbf{F}_1$-algebra, which will be used later, is the following one. Let $K \to k$ be an isometric homomorphism from K to a non-Archimedean field k (e.g., $K = k^{\cdot}$). Then any Banach k-algebra A can be viewed as a Banach K-algebra and, if A is k-affinoid, then $A^{\cdot}$ is a quasi-affinoid $\mathbf{F}_1$-algebra (by the fact used in the proof of Proposition 2.2.1).

2.2.3. Corollary. *Let $\varphi : A \to B$ be a bounded homomorphism from a K-affinoid algebra A to a quasi-affinoid $\mathbf{F}_1$-algebra B. Let $f_1, \dots, f_n \in B$, and let $r_1, \dots, r_n$ be positive numbers with $r_i \ge \rho(f_i)$, $1 \le i \le n$. Then there exists a unique bounded homomorphism $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B$ extending φ and sending T_i to f_i.* ∎

2.2.4. Corollary. *Let A be a K-affinoid algebra, and let $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A : T_i \mapsto f_i$ be an admissible epimorphism. If f_i is not nilpotent, let $s_i = \rho(f_i)$ and, if f_i is nilpotent, let s_i be an arbitrary positive number. Then the homomorphism $K\{s_1^{-1}T_1, \dots, s_n^{-1}T_n\} \to A : T_i \mapsto f_i$ is an admissible epimorphism.*

Proof. That the homomorphism considered is bounded follows from Corollary 2.2.3. Suppose that $s_i > r_i$ for $1 \le i \le m$, and $s_i \le r_i$ for $m+1 \le i \le n$. (The elements f_i for $1 \le i \le m$ are of course nilpotent.) Let $d \ge 0$ be such that $f_i^{d+1} = 0$ for all $1 \le i \le m$. There exists a constant $C > 0$ such that every nonzero element $f \in A$ has a preimage λT^μ in $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ with $|\lambda| r^\mu \le C||f||$. We have

$$s^\mu \le r^\mu \left(\frac{s_1}{r_1}\right)^{\mu_1} \cdot \ldots \cdot \left(\frac{s_m}{r_m}\right)^{\mu_m}$$

Since $\mu_i \le d$ for $1 \le i \le m$, all possible multiple of r^μ on the right hand side are at most a constant $C' > 0$. We get $|\lambda| s^\mu \le CC'||f||$, and the required statement follows. ∎

2.2.5. Corollary. *Let A be a K-affinoid algebra. Then*

(i) the map $\mathcal{M}(A) \to \mathrm{Zspec}(A) : x \mapsto \mathrm{Zker}(|\ |_x)$ is surjective;

(ii) the following properties of a pair (f, g) of elements of A are equivalent:

(a) $f(x) = g(x)$ for all $x \in \mathcal{M}(A)$;

(b) $(f, g) \in \mathbf{n}(A)$;

(iii) $I_A \xrightarrow{\sim} I_{\widehat{A}}$.

Proof. (i) It suffices to verify that, for any Zariski prime ideal $\mathfrak{p} \subset A$, the zero ideal of $A/\mathfrak{p}$ lies in the image of the similar map $\mathcal{M}(A/\mathfrak{p}) \to \mathrm{Zspec}(A/\mathfrak{p})$. Thus, replacing A by $A/\mathfrak{p}$, we may assume that A has no zero divisors. Let $f_1, \dots, f_n$ be nonzero generators of A. By Proposition 2.2.1, for the element $f = f_1 \cdot \ldots \cdot f_n$ one has $\rho(f) > 0$ and, therefore, there exists a point $x \in \mathcal{M}(A)$ with $f(x) \neq 0$. This implies that $\mathrm{Zker}(|\ |_x) = 0$.

The statement (ii) follows straightforwardly from (i) and Corollary I.2.1.5.

(iii) If $e, f \in I_A$ then, for any point $x \in \mathcal{M}(A)$, $f(x)$ is either 0 or 1 and, therefore, the equality $|e(x)| = |f(x)|$ is equivalent to the equality $e(x) = f(x)$. Thus, if the images of e and g in $I_{\widehat{A}}$ are equal, then $(e, f) \in \mathbf{n}(A)$ and, therefore, $e = f$. Conversely, let $f \in A$ be an element whose image in $\widehat{A}$ is an idempotent. It follows that $f^2(x) = f(x)$ for all points $x \in \mathcal{M}(A)$, and the statement (ii) implies that $(f^2, f) \in \mathbf{n}(A)$, i.e., the image of f in $A/\mathbf{n}(A)$ is an idempotent. Lemma I.2.1.7 implies that there exists an idempotent $e \in I_A$ whose image in $\widehat{A}$ is f. ∎

2.2.6. Proposition. *A K-affinoid algebra A is strictly K-affinoid if and only if $\rho(f) \in \sqrt{|K|}$ for all $f \in A$.*

Proof. To prove the required property of A, we may assume, by Proposition 2.1.3(v), that the group K^* is finitely generated. Then A can be viewed as an $\mathbf{F}_1$-affinoid algebra. Let $r_1, \dots, r_n$ be a maximal set of numbers from the group $|K^*|$ which are linearly independent over $\mathbf{Q}$. Take an arbitrary field k provided with the trivial valuation, and denote by k_r the non-Archimedean field of formal Laurent series $\sum_{\mu \in \mathbf{Z}^n} a_\mu T^\mu$ with $a_\mu \in k$ and $|a_\mu| r^\mu \to 0$ as $|\mu| \to \infty$ (see [Ber1, §2.1]). Then the k_r-affinoid algebra $k_r\{A\}$ is strictly k_r-affinoid, and it follows from [Ber1, Corollary 2.1.6] that $\rho(f) \in \sqrt{|k_r|} = \sqrt{|K|}$ for all elements $f \in k_r\{A\}$. In particular, $\rho(f) \in \sqrt{|K|}$ for all $f \in A$.

Conversely, assume that $\rho(f) \in \sqrt{|K|}$ for all $f \in A$. By Corollary 2.2.4, it suffices to show that the K-affinoid algebra $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is strictly K-affinoid if $r_i \in \sqrt{|K^*|}$ for all $1 \le i \le n$. If $|K^*| = \{1\}$, the required fact is trivial. Assume therefore that $|K^*| \neq \{1\}$. For every $1 \le i \le n$, there exist an integer $d_i \ge 1$ and an element $\alpha_i \in K^*$ with $r_i^{d_i} = |\alpha_i|$. This gives rise to an isometric embedding $A = K\{S_1, \dots, S_n\} \to B = K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} : S_i \mapsto \frac{T^{d_i}}{\alpha_i}$. Notice that every element of B can be represented in a unique way in the form fT^μ with $\mu_i \le d_i - 1$ for all $1 \le i \le n$, and that $||fg|| = ||f|| \cdot ||g||$ for all $f \in A$ and $g \in B$. Since $|K^*| \neq \{1\}$, we can find, for every $1 \le i \le n$, an element $\beta_i \in K^*$ with $|\beta_i| \ge r_i$. It follows that the bounded surjective homomorphism $C = A\{V_1, \dots, V_n\} \to B : V_i \mapsto \frac{T_i}{\beta_i}$ is admissible and, therefore, B is a strictly K-affinoid algebra. ∎

2.2.7. Proposition. *Let A be a K-affinoid algebra. Then any bounded A-homomorphism be-*

tween finitely generated Banach A-modules is admissible. In particular, every Zariski A-submodule of a finitely generated Banach A-module is a finitely generated Banach A-module.

Proof. As above, the situation is easily reduced to the case when A is an $\mathbf{F}_1$-affinoid algebra. Let k be a non-Archimedean field with nontrivial valuation. Then $k\{A\}$ is a k-affinoid algebra and, for every finitely generated Banach A-module M, $k\{M\}$ is a finitely generated Banach $k\{A\}$-module. By [BGR, 5.7.3/5] (see also [Ber1, 2.1.10]), any bounded homomorphism between finitely generated Banach $k\{A\}$-modules is admissible. Let now $\varphi : M \to N$ be a bounded homomorphism between finitely generated Banach A-modules. Since the induced homomorphism $k\{M\} \to k\{N\}$ is admissible, there is a constant $C > 0$ such that every element G from the image of the latter homomorphism has a preimage F with $||F|| \le C||G||$. For $G = n \in \varphi(M)$, take $G = \sum \lambda_m m$ as above. It follows that $|\sum \lambda_m| = 1$ and $\max |\lambda_m| \cdot ||m|| \le C||n||$, where the sum and the maximum are taken over all $m \in M$ with $\varphi(m) = n$. The equality implies that there exists m with $\varphi(m) = n$ and $|\lambda_m| \ge 1$, and the inequality implies that for this m one has $||m|| \le \frac{C}{|\lambda_m|}||n|| \le C||n||$. ∎

2.2.8. Corollary. *For any pair of finite Banach A-modules M and N, $\mathrm{Hom}_A(M, N)$ is a finite Banach A-module.*

Proof. Take an admissible epimorphism $A^{(m)} \to M$. Then the canonical homomorphism of Banach A-modules $\mathrm{Hom}(M, N) \to \mathrm{Hom}(A^{(m)}, N)$ is an admissible monomorphism. It follows that $\mathrm{Hom}(M, N)$ is a Zariski Banach A-submodule of the finitely generated Banach A-module $\mathrm{Hom}(A^{(m)}, N) = N^{(m)}$, and the required fact follows from Proposition 2.2.7. ∎

2.2.9. Proposition *(see Lemma 2.3.6). Let $\varphi : A \to B$ be a bounded homomorphism from a (strictly) K-affinoid algebra A to a Banach K-algebra B. If B is a finitely generated Banach A-module, then it is a (strictly) K-affinoid algebra.*

Proof. Let $A^{(n)} \to B : e_i \mapsto g_i$ be an admissible epimorphism, and let r_i be positive numbers with $r_i \ge \rho(g_i)$. Then the bounded epimorphism $A\{r^{-1}T\} = A\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to B : T_i \mapsto g_i$ is admissible since its composition with the bounded homomorphism $A^{(n)} \to A\{r^{-1}T\} : e_i \mapsto T_i$ is admissible. This implies that B is K-affinoid. Assume now that A is strictly K-affinoid. If the valuation on K is nontrivial, we can find the above numbers r_i in $|K^*|$, and so B is strictly K-affinoid. Suppose therefore that the valuation on K is trivial. Then the spectral norm of any non-nilpotent element of A is one. Let g be a non-nilpotent element of B. Then $g^m = fg^n$ for some $m > n \ge 0$ and a non-nilpotent element $f \in A$. If $g(y) \ne 0$ for a point $y \in \mathcal{M}(B)$, then $|g(y)|^{m-n} = |f(y)| \le 1$, i.e., $\rho(g) \le 1$. This means that the above admissible epimorphism is well defined for the numbers $r_i = 1$, i.e., B is again strictly K-affinoid. ∎

2.2.10. Corollary. *Let A be an integral (strictly) K-affinoid algebra. Then the integral closure of A in any finite extension of its fraction field has the structure of a (strictly) K-affinoid algebra.*

Proof. By Lemma I.2.5.7, B is a finitely generated A-module and, by Lemma 1.2.4, it has the structure of a Banach A-algebra which is a finitely generated Banach A-module. The required fact now follows from Lemma 2.2.9. ∎

2.2.11. Remarks. By [BGR, 3.7.3/6] (see also [Ber1, 2.1.10]), if k is a non-Archimedean field and $\mathcal{A}$ is a k-affinoid algebra, then $M \otimes_{\mathcal{A}} N \stackrel{\sim}{\to} M \widehat{\otimes}_{\mathcal{A}} N$ and $M \otimes_{\mathcal{A}} \mathcal{B} \stackrel{\sim}{\to} M \widehat{\otimes}_{\mathcal{A}} \mathcal{B}$ for any finite Banach A-modules M and N and any $\mathcal{A}$-affinoid algebra $\mathcal{B}$. The corresponding facts are not true for K-affinoid algebras. Indeed, assume that $|K^*| \neq \{1\}$, and let A be the K-affinoid field $K\{T_1, T_2\}/E$, where E is the closed ideal generated by the pair (T_1T_2, λ) for some $\lambda \in K^*$ with $|\lambda| < 1$. If f_1 and f_2 are the images of T_1 and T_2 in A, then $\rho(f_1) = \rho(f_2) = 1$ and $\rho(f_1^{-1}) = \rho(f_2^{-1}) = |\lambda|^{-1} > 1$. By Lemma 1.3.10, the ideals E_1 and E_2 that correspond to the subgroups of A^* generated by f_1 and f_2, respectively, are closed, and so $B_1 = A/E_1$ and $B_2 = A/E_2$ are finite Banach A-algebras (also K-affinoid fields). The tensor product $B = B_1 \otimes_A B_2$ is the quotient A/E, where E corresponds to the subgroup of A^* generated by the elements f_1 and f_2 (and so B^* is the quotient of K^* by the subgroup generated by the element λ). On the other hand, $f_1 f_2 = \lambda$ and $|\lambda| < 1$, the ideal E is not closed and, therefore, $B_1 \widehat{\otimes}_A B_2 = 0$.

2.3. The spectral norm of K-affinoid algebras.

2.3.1. Proposition. *Let A be a reduced K-affinoid algebra. Then the Banach norm on A is equivalent to the spectral norm.*

The statement is not true for the class of Zariski reduced K-affinoid algebras (see Remark 2.3.8).

Proof. The situation is easily reduced to the case when $K = \mathbf{F}_1$, i.e., A is an $\mathbf{F}_1$-affinoid algebra. By [BGR, 6.2.4/1] (see also [Ber1, 2.1.4(ii)]), it suffices to show that for some non-Archimedean field k the k-affinoid algebra $k\{A\}$ has no nilpotent elements. Furthermore, since A embeds in a direct product $B = \prod_{i=1}^n A_i$ of integral $\mathbf{F}_1$-affinoid algebras, it suffices to show that the Banach k-algebra $k\{B\}$ has no nilpotent elements. *We claim that this is true if the characteristic of k is prime to the orders of torsion elements of all groups F_i^*, where F_i is the fraction $\mathbf{F}_1$-field of A_i.*

For a nonempty subset $I \subset \{1, \dots, n\}$, let B_I denote the subset of elements $(a_1, \dots, a_n) \in B$ with either $a_i \neq 0$ precisely for $i \in I$, or $a_i = 0$ for all $1 \leq i \leq n$. It is a sub-semigroup of B and an $\mathbf{F}_1$-algebra isomorphic to the tensor product $\otimes_{i \in I} A_i$. One also has $B_I \cdot B_J = B_{I \cap J}$ if one sets $B_\emptyset = \{0\}$. We consider each B_I as a Banach $\mathbf{F}_1$-algebra provided with the Banach norm induced from B. Then every element $f \in B$ has a unique representation in the form $\sum_I f_I$ with $f_I \in B_I$. If $f \neq 0$ and I is a minimal subset of $\{1, \dots, n\}$ with $f_I \neq 0$, then $(f^l)_I = (f_I)^l$ for all $l \geq 1$. Thus, it suffices to verify that the Banach k-algebra $k\{B_I\}$ has no nilpotent elements.

Since each map $X_i = \mathcal{M}(A_i) \to \mathrm{Zspec}(A_i)$ is surjective, there exists a point $x_i \in X_i$ with $\mathrm{Zker}(|\ |_{x_i}) = 0$. Its valuation $\mathbf{F}_1$-field $\mathcal{H}(x_i)$ coincides with the $\mathbf{F}_1$-field F_i. The space $\mathcal{M}(B_I)$ coincides with the direct product $\prod_{i \in I} X_i$, and for the point $y = (x_i)_{i \in I}$ one also has $\mathrm{Zker}(|\ |_y) = 0$. Since B_I is an integral domain, the valuation $\mathbf{F}_1$-field $\mathcal{H}(y)$ coincides with the fraction $\mathbf{F}_1$-field of B_I, which naturally embeds in the tensor product $\otimes_{i \in I} F_i$ (it is also an $\mathbf{F}_1$-field). Thus, there is an injective bounded homomorphism of Banach k-algebras $k\{B_I\} \hookrightarrow k\{\mathcal{H}(y)\}$. The latter has no nilpotent elements, by Lemma 1.4.3(ii). ∎

2.3.2. Corollary. *Let $\varphi : A \to B$ be a bijective bounded homomorphism between reduced Banach K-algebras which is an isometry with respect to the spectral norm, and assume that A is K-affinoid and the Banach norm on B coincides with the spectral norm. Then φ is an isomorphism.*

Proof. By Proposition 2.3.1, the Banach norm on A is equivalent to the spectral norm, and the statement follows. ∎

2.3.3. Corollary. *Let A be a K-affinoid algebra. Then the canonical map $|A| \to \widehat{A}$ is an isomorphism of $|K|$-Banach algebras and, in particular, $\widehat{A}$ is a $|K|$-affinoid algebra.*

Proof. The statement follows from Corollaries 1.3.7 and 2.3.2. ∎

2.3.4. Corollary. *Let A be a quasi-integral K-affinoid algebra. Then the Banach norm on A is equivalent to a Banach norm with respect to which $||f|| = \rho(f)$ for all $f \notin \mathrm{z}\mathbf{n}(A)$.*

Proof. Let $||\ ||'$ be the function $A \to \mathbf{R}_+$ defined by $||f||' = \rho(f)$ for $f \notin \mathrm{z}\mathbf{n}(A)$ and $||f||' = \inf\{\rho(g)||h||\}$ for $f \in \mathrm{z}\mathbf{n}(A)$, where the infimum is taken over all representations $f = gh$ with $g \notin \mathrm{z}\mathbf{n}(A)$ and $h \in \mathrm{z}\mathbf{n}(A)$. *We claim that $||\ ||'$ is a Banach norm equivalent to $||\ ||$.* Indeed, the canonical map $A \to A' = A/\mathrm{z}\mathbf{n}(A)$ is an admissible epimorphism. Since A' is reduced, its norm is equivalent to the spectral norm and, therefore, there exists $C > 0$ with $||f|| \leq C\rho(f)$ for all $f \notin \mathrm{z}\mathbf{n}(A)$. Furthermore, if $f \in \mathrm{z}\mathbf{n}(A)$, then $||f||' \leq ||f||$ since $f = 1 \cdot f$. If $f = gh$, where $g \notin \mathrm{z}\mathbf{n}(A)$ and $h \in \mathrm{z}\mathbf{n}(A)$, then $||f|| \leq ||g|| \cdot ||h|| \leq C\rho(g)||h||$. It follows that $||f|| \leq C||f||$ for all $f \in \mathrm{z}\mathbf{n}(A)$. Finally, we have to verify that $||f_1 f_2||' \leq ||f_1||' \cdot ||f_2||'$. If at least one of the elements f_1 or f_2 is not nilpotent, this is clear. Suppose that $f_1, f_2 \in \mathrm{z}\mathbf{n}(A)$. If $f_1 = g_1 h_1$ and $f_2 = g_2 h_2$ are representations as above, then $f_1 f_2 = (g_1 g_2) \cdot (h_1 h_2)$ and $g_1 g_2 \notin \mathrm{z}\mathbf{n}(A)$ and, therefore, $||f_1 f_2||' \le \rho(g_1 g_2)||h_1 h_2||$. The required property follows. ∎

2.3.5. Proposition. *Let $\varphi : A \to B$ be a bijective bounded homomorphism of K-affinoid algebras which is an isometry with respect to the spectral norm. Then φ is an isomorphism.*

2.3.6. Lemma. *Let $\varphi : A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B : T_i \mapsto g_i$ be an admissible epimorphism of K-affinoid algebras. Assume that, for every $1 \le i \le n$, either g_i is nilpotent, or g_i is non-nilpotent and $g_i^{m_i} = g_i^{k_i}\varphi(f_i)$ for some $m_i > k_i \ge 0$ and $f_i \in A$ with $\rho(g_i\varphi(f_i)) = \rho(g_i)\rho(f_i)$. Then*

(i) B is a finitely generated Banach A-module;

(ii) for any $s_1 \ge r_1, \dots, s_n \ge r_n$ the epimorphism $A\{s_1^{-1}T_1, \dots, s_n^{-1}T_n\} \to B : T_i \mapsto g_i$ is admissible.

Notice that, if g_i is not nilpotent, then one has $\rho(\varphi(f_i)) = \rho(f_i) = \rho(g_i)^{m_i - k_i}$.

Proof. Suppose that g_i is nilpotent precisely for $1 \le i \le l$. For such i, let m_i be such that $g_i^{m_i} = 0$, and set $f_i = 0$ and $k_i = 0$. Let M be the Zariski Banach A-submodule of $A\{r^{-1}T\}$ generated by the monomials T^μ with $\mu_i \le m_i - 1$ for all $1 \le i \le n$. Let M be the Zariski Banach A-submodule of $A\{r^{-1}T\}$ generated by the monomials T^μ with $\mu_i \le m_i - 1$ for all $1 \le i \le n$. *We claim that the induced map $M \to B$ is an admissible epimorphism.*

Indeed, by Proposition 2.2.1, we can find a constant $C_1 > 0$ such that, for every $l + 1 \le i \le n$ and every $\nu \ge 0$, one has $||f_i^\nu|| \le C_1 \rho(f_i)^\nu$. We can also find a constant $C_2 > 0$ such that, for any nonzero element $g \in B$, there exists a term $aT^\alpha \in A\{r^{-1}T\}$ in the preimage of g with $||aT^\alpha|| = ||a||r^\alpha \le C_2||g||$. Our purpose is to show that such a preimage (with a different constant) can be found with $\alpha_i \le m_i - 1$ for all $1 \le i \le n$. (Notice that $\alpha_i \le m_i - 1$ for all $1 \le i \le l$ since g is nonzero.) For this it suffices to show that, if $\alpha_i \ge m_i$ for some i, such a preimage can be found with a strictly smaller power of T_i. Let $\alpha = \mu + m\nu$, where $\mu, \nu, m = (m_1, \dots, m_n) \in \mathbf{Z}_+^n$ and $\mu_i \le m_i - 1$ for all $1 \le i \le n$. Then the image of the term $af^\nu T^{\mu + k\nu}$ in B is g, where $k = (k_1, \dots, k_n)$. Since $||f^\nu|| \le C_2^l \rho(f)^\nu = C_2^l \rho(g)^{(m-k)\nu} \le C_2^l r^{(m-k)\nu}$, one has $||af^\nu T^{\mu+k\nu}|| \le ||a|| \cdot ||f^\nu|| r^{\mu + k\nu} \le C_2^l ||a|| r^{(m-k)\nu} r^{\mu + k\nu} = C_2^l ||a|| r^\alpha \le C_1 C_2^l ||g||$. The claim and (i) follow.

To prove (ii), we let M' denote the Zariski Banach A-submodule of $A\{s^{-1}T\}$ generated by the monomials T^μ with $\mu_i \le m_i - 1$ for all $1 \le i \le n$. The canonical bijection of finitely generated A-modules $M \to M'$ is an isomorphism of Banach A-modules. Since $M \to B$ is an admissible epimorphism, then $M' \to B$ is an admissible epimorphism. It follows that $A\{s^{-1}T\} \to B$ is an admissible epimorphism. ∎

Recall that, for an $\mathbf{F}_1$-algebra A and an integer $n \geq 1$, we denoted by A^n the $\mathbf{F}_1$-subalgebra of A consisting of elements of the form f^n for $f \in A$ (see the end of §I.2.4). Recall also that, given such A, there exists $d \geq 1$ such that for every n divisible by d the $\mathbf{F}_1$-algebra A^n is reduced (see Lemma I.2.7.2).

2.3.7. Lemma. *Let A be an $\mathbf{F}_1$-affinoid algebra, and let $n \geq 1$ be an integer such that the $\mathbf{F}_1$-subalgebra A^n is reduced. Then*

(i) the $\mathbf{F}_1$-subalgebra A^n provided with the induced Banach norm is $\mathbf{F}_1$-affinoid;

(ii) A is a finitely generated Banach A^n-module.

Proof. Step 1. Let first $n \geq 1$ be arbitrary. Then the Banach $\mathbf{F}_1$-algebra $A_n = A$, provided with the norm $||f||_n = ||f||^n$, is $\mathbf{F}_1$-affinoid. Indeed, if $\mathbf{F}_1\{r^{-1}T\} \to A$ is an admissible epimorphism, then the induced epimorphism $\mathbf{F}_1\{r^{-n}T\} \to A$ (it coincides with the previous one) is also admissible since there is an evident equality $\mathbf{F}_1\{r^{-1}T\}_n = \mathbf{F}_1\{r^{-n}T\}$. Furthermore, the correspondence $|\ | \mapsto |\ |^n$ gives rise to a homeomorphism $\mathcal{M}(A) \widetilde{\to} \mathcal{M}(A_n)$. It follows that for any element $f \in A$ one has $\rho_n(f) = \rho(f)^n$, where $\rho_n(f)$ is the spectral radius of f with respect to A_n. The correspondence $f \mapsto f^n$ induces a bounded surjective homomorphism of Banach $\mathbf{F}_1$-algebras $\varphi : A_n \to A^n$ and, therefore, a bounded bijective homomorphism $\psi : A_n/\mathrm{Ker}(\varphi) \to A^n$. Notice that the quotient $A_n/\mathrm{Ker}(\varphi)$ is an $\mathbf{F}_1$-affinoid algebra, and there are canonical homeomorphisms $\mathcal{M}(A^n) \widetilde{\to} \mathcal{M}(A_n/\mathrm{Ker}(\varphi)) \widetilde{\to} \mathcal{M}(A_n)$. The latter implies that the homomorphism ψ is isometric with respect to the spectral norms.

Step 2. Assume that n is such that A^n is reduced. In particular, the $\mathbf{F}_1$-affinoid algebra $A_n/\mathrm{Ker}(\varphi)$ is reduced. By Proposition 2.3.1, the Banach norm on it is equivalent to the spectral norm. Together with boundness of ψ, this implies that there is a constant $C > 0$ such that $||f^n|| \leq C\rho(f)^n$ for all elements $f \in A$. Since the spectral norm of f^n with respect to Banach $\mathbf{F}_1$-algebra A^n is equal to $\rho(f)^n$, it follows that the bijection ψ is an isomorphism of Banach $\mathbf{F}_1$-algebras and, in particular, A^n is a $\mathbf{F}_1$-affinoid algebra (i.e., (i) is true), and its Banach norm is equivalent to the spectral norm. The statement (ii) now follows from Lemma 2.3.6 applied to any admissible epimorphism $A^n\{r^{-1}T\} \to A$. ∎

Proof of Proposition 2.3.5. The situation is easily reduced to the case when $K = \mathbf{F}_1$. By Lemmas I.2.7.2 and 2.3.7, we can find an integer $n \geq 1$ such that both A^n and B^n are reduced $\mathbf{F}_1$-affinoid algebras, and A and B are finitely generated Banach modules over A^n and B^n, respectively. By the assumption and Proposition 2.3.1, the induced bijection $A^n \to B^n$ is an isomorphism of $\mathbf{F}_1$-affinoid algebras. The required fact now follows from Lemma 1.2.2(i). ∎

2.3.8. Remark. Let A be the $\mathbf{F}_1$-affinoid algebra which is the quotient of $\mathbf{F}_1\{r^{-1}T_1, T_2, T_3\}$, $r > 1$, by the ideal generated by the pairs (T_1T_3, T_3^2), (T_2T_3, T_3^2) and (T_2^2, T_3^2). If t_i denotes the image of T_i in A, then every element of $A\backslash\{0\}$ is of one of the following forms t_1^n, $t_1^nt_2$, or t_3^n with $n \geq 0$, and one has $||t_1^n|| = ||t_1^nt_2|| = r^n$ and $||t_3^n|| = 1$. But since $(t_1^nt_2)^2 = t_3^n$, one has $\rho(t_1^nt_2) = 1$. Thus, the Banach norm on A is not equivalent to the spectral norm.

2.4. Twisted products of K-affinoid algebras.

2.4.1. Proposition. *Given a twisted datum of (strictly) K-affinoid algebras $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$, the Banach K-algebra $A = \prod_I^\nu A_i$ is (strictly) K-affinoid.*

Proof. We can find sufficiently large $m \geq n \geq 1$ such that, for every $i \in I$, there are an admissible epimorphism $\pi_i : B_i = K\{r_{i1}^{-1}T_{i1}, \ldots, r_{in}^{-1}T_{in}\} \to A_i$ and nonzero elements $f_{i1}, \ldots, f_{im} \in A_i$ with the following properties: if $\mathbf{a}^{(i)} \neq A_i$, they generate the Zariski ideal $\mathbf{a}^{(i)}$, and, if $\mathbf{a}^{(i)} = A_i$, they contain among themselves all of the elements $\pi_i(T_{ik})$, $1 \leq k \leq n$. We may assume that the Banach norm on each A_i coincides with the quotient norm with respect to π_i and, in particular, $||\pi_i(b)|| \leq ||b||$ for all $b \in B_i$. Furthermore, suppose that the elements $f_{i1}, \ldots, f_{im_i}$ are nilpotent in A_i and $f_{im_i+1}, \ldots, f_{im}$ are not. Let g_{ik} be a preimage of f_{ik} in B_i. If $1 \leq k \leq m_i$, let t_{ik} be a positive number which is at least the spectral radius of f_{ik} in A and, if $m_i + 1 \leq k \leq m$, let $t_{ik} = ||g_{ik}||$. Then $\rho_A(f_{ik}) \leq t_{ik}$ for all $i \in I$ and $1 \leq k \leq m$ and, by Corollary 2.2.3, there is a bounded homomorphism of Banach K-algebras

$$B = K\{t_{ik}^{-1}S_{ik}\}_{i\in I, 1\leq k\leq m} \xrightarrow{\pi} A : S_{ik} \mapsto f_{ik} \ .$$

We claim that π is an admissible epimorphism. Indeed, that it is an epimorphism easily follows from the construction (see the proof of Proposition I.3.1.6), and it suffices to verify the following fact: there exists $C > 0$ such that every $a \in \mathbf{a}^{(i)}$ has a preimage $b \in B$ with $||b|| \leq C||a||$ (see Remark 1.5.5). For this we take a positive constant C with the following properties for every $i \in I$: (1) for every element $a \in A_i$, there exists an element $b \in B_i$ with $\pi_i(b) = a_i$ and $||b|| \leq C||a||$; and (2) for every element $a \in A_i$, there exist $j \leq i$ and $a' \in \mathbf{a}^{(j)}$ with $\nu_{ji}(a') = a$ and $||a'|| \leq C||a||$ (see Lemma 1.5.2).

Suppose first that $\mathbf{a}^{(i)} = A_i$. By (1), there exists an element $b \in B_i$ with $||b|| \leq C||a||$. Since such B_i can be considered as a subalgebra of B, the required fact follows. This allows us to verify the required fact in the general case by induction. Namely, suppose that $\mathbf{a}^{(i)} \neq A_i$ and that the required fact is true with a constant $C' > 0$ for all $j < i$. Then we can increase the constant C' so that for every $j < i$ and every element $a \in \mathbf{a}^{(j)}$ there exists an element $b \in B$ with $\pi(b) = a$ and

$||b|| \leq C'||a||$. Let also l is the maximal integer for which there are $i \in I$ and $1 \leq k \leq m_i$ with $f^l_{ik} \neq 0$.

Given a nonzero element $a \in \mathbf{a}^{(i)}$, take an element $b \in B_i$ with $\pi_i(b) = a$ and $||b|| \leq C||a||$ as above. Since the elements $g_{i1}, \dots, g_{im}$ generate the Zariski preimage of $\mathbf{a}^{(i)}$ modulo the Zariski kernel of π_i, it follows that the element b has the form $\lambda b' g_{i1}^{\nu_{i1}} \cdot \ldots \cdot g_{im}^{\nu_{im}}$ with $\lambda \in K^*$, $\nu_{ik} \in \mathbf{Z}_+$ and $b' \in B_i \backslash \mathbf{b}_i$. Notice that $\nu_{ik} \leq l$ for $1 \leq k \leq m_i$ and, since the norm on B_i is multiplicative, one has $||b|| = |\lambda| \cdot ||b'|| \cdot ||g_{i1}||^{\nu_{i1}} \cdot \ldots \cdot ||g_{im}||^{\nu_{im}}$. It follows that $a = \lambda a' f_{i1}^{\nu_{i1}} \cdot \ldots \cdot f_{im}^{\nu_{im}}$, where $a' = \pi_i(b') \in A_i \backslash \mathbf{a}^{(i)}$. Since $||a'|| \leq ||b'||$, we get an inequality

$$|\lambda| \cdot ||a'|| t_{i1}^{\nu_{i1}} \cdot \ldots \cdot t_{in}^{\nu_{in}} \cdot C'' \leq C||a|| \ .$$

where

$$C'' = \min\{\prod_{k=1}^{m_i} \left(\frac{||g_{ik}||}{t_{ik}}\right)^{\nu_{ik}} \big| i \in I,\ \nu_{ik} \leq l\} \ .$$

Furthermore, (2) implies that there exist $j < i$ and $a'' \in \mathbf{a}^{(j)}$ with $\nu_{ji}(a'') = a'$ and $||a''|| \leq C||a'||$. By the induction assumption, there exists an element $b'' \in B$ with $\pi(b'') = a''$ and $||b''|| \leq C'||a''||$. It follows that $||b''|| \leq CC'||a'||$. If now $c = \lambda b'' S_{i1}^{\nu_{i1}} \cdot \ldots \cdot S_{im}^{\nu_{im}}$, then $\pi(c) = a$, and we have

$$||c|| = |\lambda| \cdot ||b''|| t_{i1}^{\nu_{i1}} \cdot \ldots \cdot t_{im}^{\nu_{im}} \leq \frac{C^2 C'}{C''} ||a|| \ ,$$

If all A_i's are strictly K-affinoid, the admissible epimorphisms π_i can be found with $r_{ik} \in \sqrt{|K^*|}$. In this case the numbers t_{ik} for $m_i \leq k \leq m$ lie in $\sqrt{|K^*|}$ and for $1 \leq k \leq m_i$ can be chosen in $\sqrt{|K^*|}$, and so A is strictly K-affinoid. ∎

2.4.2. Example. Let I be a finite idempotent $\mathbf{F}_1$-subalgebra of a (strictly) $\mathbf{F}_1$-affinoid algebra. By Example 1.1.4(i), for every nonzero idempotent $e \in \check{I}$ the ideal F_e of A generated by the prime ideal Π_e of I is closed and, in particular, the quotient $A^{(e)} = A/F_e$ is a (strictly) $\mathbf{F}_1$-affinoid algebra. *We claim that the bijective bounded homomorphism $A \to \prod_{\check{I}}^{\nu} A^{(e)}$ (from Example I.3.1.6) is an isomorphism of Banach $\mathbf{F}_1$-algebras.* Indeed, by Proposition 1.5.4, one has $\mathcal{M}(A) = \coprod_{e \in \check{I}} \mathcal{M}(A^{(e)})$ and, therefore, the homomorphism considered is isometric with respect to the spectral norm, and Proposition 2.3.5 implies that it is an isomorphism. Proposition I.3.2.5 implies that the functor $A \mapsto \{\check{I}_A, A^{(e)}, \nu_{ef}\}$ from the category of K-affinoid algebras to that of twisted data of Banach K-algebras is fully faithful.

Let K-$\mathcal{A}ff^\circ$ denote the category of K-affinoid algebras (it is opposite to the category of K-affinoid spaces K-$\mathcal{A}ff$ which will be introduced in §6), and let K-$\mathcal{A}ftw$ denote the category of twisted data of K-affinoid algebras. By Proposition 2.4.1, the twisted product construction gives rise to a functor $K\text{-}\mathcal{A}ftw \to K\text{-}\mathcal{A}ff^\circ$. Furthermore, let $K\text{-}\mathcal{A}ff^{f,\circ}$ denote the full subcategory of finitely presented K-affinoid algebras, and let $K\text{-}\mathcal{A}ftw^{qi}$ denote the full subcategory of $K\text{-}\mathcal{A}ftw$ consisting of twisted data of finitely presented quasi-integral K-affinoid algebras. Recall that, for such a twisted datum $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$, there is a canonical map $I \to \mathrm{Spec}(A)$, where $A = \prod_I^\nu A_i$. Let $K\text{-}\mathcal{A}ftw_r^{qi}$ denote the full subcategory of $K\text{-}\mathcal{A}ftw^{qi}$ consisting of the twisted data for which the above map is injective. Finally, let Q_K and $Q_{K,r}$ denote the families of quasi-isomorphisms in $K\text{-}\mathcal{A}ftw^{qi}$ and $K\text{-}\mathcal{A}ftw_r^{qi}$, respectively. Theorem I.3.4.3(i) easily implies that the systems Q_K and $Q_{K,r}$ admit calculus of right fractions.

2.4.2. Theorem. *There are equivalences of categories*

$$K\text{-}\mathcal{A}ftw_r^{qi}[Q_{K,r}^{-1}] \stackrel{\sim}{\to} K\text{-}\mathcal{A}ftw^{qi}[Q_K^{-1}] \stackrel{\sim}{\to} K\text{-}\mathcal{A}ff^{f,\circ} \ .$$

Of course, the similar equivalences take place for the categories of finitely presented *strictly* K-affinoid algebras.

Proof. We only verify that the functor $K\text{-}\mathcal{A}ftw_r^{qi} \to K\text{-}\mathcal{A}ff$ is essentially surjective.

Let A be a finitely presented K-affinoid algebra. Since it is decomposable, it follows from §I.3.4 that there are a finite partially ordered set I with infimum operation, an injective map $I \to \mathrm{Zspec}(A) : i \mapsto \mathfrak{p}_i$ that commutes with the infimum operation and, for every $i \in I$, a $\mathfrak{p}_i$ primary ideal $\mathbf{a}_i$ such that $\mathbf{a}_{\inf(i,j)} \subset \mathbf{a}_i \cup \mathbf{a}_j$ and $\Delta(A) = \bigcap_{i\in I} E_i$, where E_i is the $\mathfrak{p}_i$-primary ideal $\mathrm{Ker}(A \to A_\mathfrak{p}/\mathbf{a}A_\mathfrak{p})$. By the proof of Corollary 2.1.4, the ideals E_i are finitely generated and closed, and so the quotients $A_i = A/E_i$ are finitely presented K-affinoid algebras. Furthermore, if $i \le j$, the quasi-homomorphism $\nu_{ij} : A_i \to A_j$ is contracting (with respect to the quotient norms) and, if $\mathbf{a}_{ji}$ is the Zariski ideal of A_j which is the image of $\mathbf{a}_i$ under the admissible epimorphism $p_j : A \to A_j$, then ν_{ij} induces an admissible epimorphism. Thus, the tuple $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$ is a quasi-integral twisted datum of finitely presented K-affinoid algebras.

We already know that the canonical map $A \to B = \prod_I^\nu A_i$ is an isomorphism of K-algebras, and it is easy to see that it is bounded. That it is an isomorphism of K-affinoid algebras follows from Proposition 2.3.5. ∎

Notice that, if A is reduced, one can find the above data with $\mathbf{a}_i = \mathfrak{p}_i$ for all $i \in I$. In this case, all of the K-affinoid algebras are integral. Thus, every reduced K-affinoid algebra is a twisted product of integral K-affinoid algebras.

2.5. A description of the kernel of the homomorphism $A \to \widehat{A}$.

2.5.1. Proposition. *The following properties of a pair (f, g) of elements of an irreducible K-affinoid algebra A are equivalent:*

(a) $f^n = g^n h$ for some $h \in A$;

(b) there exists $C > 0$ with $|f(x)| \le C|g(x)|$ for all $x \in \mathcal{M}(A)$.

Recall that A is called irreducible if its spectrum $\mathrm{Spec}(A)$ is an irreducible topological space or, equivalently, its nilradical $\mathbf{n}(A)$ is a prime ideal (see Lemma I.3.5.9). If A is integral, (a) is equivalent to the property that, in the case when g is nonzero, the element $\frac{f}{g}$ of the fraction $\mathbf{F}_1$-field of A is integral over A.

The main ingredient of the proof of Proposition 2.5.1 is Lütkebohmert's Riemann Extension Theorem ([Lu, Theorem 1.6], [Ber1, Proposition 3.3.14]). An elementary proof in the particular case when $A \xrightarrow{\sim} \widehat{A}$ will be given in §4.3.

2.5.2. Lemma. *Let A be a normal $\mathbf{F}_1$-affinoid algebra, and assume that the multiplicative group F^* of the fraction field F of A has no torsion. Then, for any non-Archimedean field k, the k-affinoid algebra $k\{A\}$ is normal.*

Proof. First of all, we notice that, if k' is a bigger non-Archimedean field, the canonical homomorphism $k\{A\} \to k'\{A\}$ is faithfully flat and, therefore, it suffices to verify it for a sufficiently large field k. We may therefore assume that $\rho(f) \in |k|$ for all $f \in A$, i.e., the k-affinoid algebra $k\{A\}$ is strictly k-affinoid.

The assumptions means that $\check{A}$ is a saturated lattice in the torsion free group F^*. It follows that the k-algebra $k[A]$ is normal (e.g., see [Ful, p. 29-30]). By [Ber1, 3.4.3], the analytification $\mathcal{X}^{\mathrm{an}}$ of the affine scheme $\mathcal{X} = \mathrm{Spec}(k[A])$ is normal. Since $\mathcal{M}(k\{A\})$ is an affinoid domain in $\mathcal{X}^{\mathrm{an}}$, it follows that $k\{A\}$ is normal. ∎

Proof of Proposition 2.5.1. The implication (a)$\Longrightarrow$(b) is trivial. To verify the implication (b)$\Longrightarrow$(a), we may assume that A is integral. Indeed, if the required fact is true in the latter case, then $(f^n, g^n h) \in \mathbf{n}(A)$ for some $n \ge 1$ and $h \in A$ and, therefore, there exists $m \ge 1$ with $f^{mn} = g^{mn} h^m$. Thus, we assume that A is integral. Furthermore, we may assume, by Proposition 2.1.3(v), that the multiplicative group F^* of the fraction $\mathbf{F}_1$-field F of A is finitely generated, and so we may consider A as an integral $\mathbf{F}_1$-affinoid algebra.

Step 1. *If A satisfies the assumptions of Lemma 2.5.2, then $f = gh$ for some $h \in A$.* Indeed, we may assume that the elements f and g are nonzero, and we can consider them as elements of the normal k-affinoid algebra $k\{A\}$. It follows that $|f(x)| \le C|g(x)|$ for all for all points of the normal k-affinoid space $X = \mathcal{M}(k\{A\})$. Let Y be the Zariski closed subset $\{x \in X \big| g(x) = 0\}$.

Then the function $\frac{f}{g}$ is analytic and bounded in the complement of Y. By Lütkebohmert's Riemann Extension Theorem, this function has a unique analytic continuation to X, i.e., $f = gH$ for some $H \in k\{A\}$. This easily implies that $f = gh$ for some $h \in A$.

Step 2. *The required fact is true if the group F^* has no torsion.* Indeed, let B be the integral closure of A in F. By Corollary 2.2.10, it is also an $\mathbf{F}_1$-affinoid algebra. By Step 1, given a pair (f, g) as in (b), there exists an element $h \in B$ with $f = gh$. One has $h^n = u \in A^*$ for some $n \geq 1$ and, therefore, $f^n = g^n u$.

Step 3. *The required is is true in the general case.* Indeed, let E be the ideal of A consisting of the pairs (f, g) with $f^n = g^n$ for some $n \geq 1$. Since the Banach norm on A is equivalent to the spectral norm, the ideal E is closed. It is easy to see that the $\mathbf{F}_1$-affinoid algebra $B = A/E$ satisfies the assumption of Step 2. Thus, given a pair (f, g) as in (c), there exist $n \geq 1$ and $h \in A$ with $(f^n, g^n h) \in E$. It follows that there exists $m \geq 1$ with $f^{mn} = g^{mn} h^m$, and we are done. ∎

2.5.3. Corollary. *The following properties of a pair (f, g) of elements of an irreducible K-affinoid algebra A are equivalent:*

(a) $f^n = g^n h$ for some $h \in A^*$ and $n \geq 1$;

(b) there exist $0 < C' < C''$ with $C'|f(x)| \leq |g(x)| \leq C''|f(x)|$ for all $x \in \mathcal{M}(A)$.

In particular, if A^{**} denotes the subgroup of elements $f \in A^*$ with the property that $|f(x)| = 1$ for all $x \in \mathcal{M}(A)$, then $\mathrm{Ker}(A \longrightarrow \widehat{A}) = \{(f, g) \big| f^n = g^n h \text{ for some } n \geq 1 \text{ and } h \in A^{**}\}$.

Proof. The implication (a)$\Longrightarrow$(b) is trivial. Let (f, g) be a pair of non-nilpotent elements of A with the property (b). Assume first that A is integral. By Proposition 2.5.1, there exist elements $u, v \in A$ and integers $m, n \geq 1$ such that $f^m = g^m u$ and $g^n = f^n v$. It follows that $f^{mn} = f^{mn} u^n v^m$. Since A is integral, the latter equality implies that $u^n v^m = 1$ and, therefore, $u, v \in A^{**}$. If A is arbitrary, then $(f^n, g^n h) \in \mathbf{n}(A)$ for some $n \geq 1$ and $h \in A^{**}$. It follows that $f^{mn} = g^{mn} h^m$ for some $m \geq 1$. ∎

2.5.4. Corollary. *If A is an irreducible K-affinoid algebra, then $\widehat{A}$ is an integral domain.*

Proof. Suppose that $\widehat{f} \cdot \widehat{h} = \widehat{g} \cdot \widehat{h}$ for some $f, g, h \in A$ with $\widehat{h} \neq 0$. Corollary 2.5.3 implies that $f^n h = g^n h u$ for some $u \in A^{**}$. Since h is not nilpotent and the nilradical of A is a prime ideal, it follows that $f^n = g^n u$ and, therefore, $\widehat{f} = \widehat{g}$. ∎

2.5.5. Corollary. *Let A be a K-affinoid algebra, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the Zariski prime ideals of A such that* $\mathrm{Spec}(A_i)$ *for $A_i = A/\Pi_{\mathfrak{p}_i}$ are the irreducible components of* $\mathrm{Spec}(A)$. *Then the ideal* $\mathrm{Ker}(A \longrightarrow \widehat{A})$ *consists of the pairs (f, g) for which there exists $m \geq 1$ such that, for every $1 \leq i \leq n$, one has $f_i^m = g_i^m h$ with $h \in A_i^{**}$, where f_i is the image of f in A_i.* ∎

2.5.6. Remark. The converse implication in Corollary 2.5.4 is not true (see Remark 4.3.7).

§**3.** R_S-**polytopes**

Let R be an $\mathbf{F}_1$-subfield of $\mathbf{R}$. An *R-affinoid polytope* is a subset V of $\mathbf{R}_+^n$ defined by a finite number of equalities of the form $f(t) = g(t)$ with $f, g \in R[T_1, \dots, T_n]$ and the inequalities $t_i \le r_i$ for all $1 \le i \le n$, $r_i > 0$. One associates to V the Banach R-algebra $A_{V/R}$ of the restrictions of functions from $R[T_1, \dots, T_n]$ to V provided with the supremum norm. An *R-polytopal algebra* is a Banach R-algebra isomorphic $A_{V/R}$ for some R-affinoid polytope V. Before studying these objects, we consider polytopes of a more general form.

A *semiring* is a set provided with two binary operations, addition and multiplication, so that with respect to each of them it is a commutative monoid with the neutral elements denoted by 0 and 1 and $x(y+z) = xy + xz$. For example, every (commutative) ring is a semiring, and the same set $\mathbf{R}_+$ is a sub-semiring of $\mathbf{R}$. Furthermore, for a semiring S, an *S-module* is a commutative monoid M provided with an action of S possessing the usual properties of modules over a ring.

Let S be a sub-semiring of $\mathbf{R}$. We consider $\mathbf{R}_+^*$ as an S-module under the action $(s, t) \mapsto t^s$. If $S \subset \mathbf{R}_+$, then the same formula (with $0^0 = 1$ and $0^s = 0$ for $s > 0$) defines the structure of an S-module on $\mathbf{R}_+$. Let R be an S-submodule of $\mathbf{R}_+$. If $0 \in R$, we always assume that $S \subset \mathbf{R}_+$, and we set $\check{R} = R \backslash \{0\}$. If $0 \notin R$ (resp. $0 \in R$), we denote by $A^n(R_S)$ the set of functions on $(\mathbf{R}_+^*)^n$ (resp. $\mathbf{R}_+^n$) of the form $(t_1, \dots, t_n) \mapsto r t_1^{s_1} \cdot \ldots \cdot t_n^{s_n}$ with $r \in R$ and $s_1, \dots, s_n \in S$. Notice that each function from $A^n(R_S)$ is continuous on $(\mathbf{R}_+^*)^n$ (resp. $\mathbf{R}_+^n$), and the set $A^n(R_S)$ is preserved by multiplication and provided with an action of S. If $0 \in R$, $A^n(R_S)$ is an $\mathbf{F}_1$-algebra.

If $0 \notin R$ (resp. $0 \in R$), a *generalized R_S-polytope in* $(\mathbf{R}_+^*)^n$ (resp. $\mathbf{R}_+^n$) is a subset V of $(\mathbf{R}_+^*)^n$ (resp. $\mathbf{R}_+^n$) defined by a finite system of inequalities of the form $f(t) \le g(t)$ with $f, g \in A^n(R_S)$. If V is compact, it is said to be an *R_S-polytope in* $(\mathbf{R}_+^*)^n$ (resp. $\mathbf{R}_+^n$). If all of the above inequalities are in fact equalities, V is said to be an *R_S-affine subspace of* $(\mathbf{R}_+^*)^n$ (resp. $\mathbf{R}_+^n$). If, in addition to the latter, $R = \{1\}$ (resp. $R = \{0, 1\}$), then V is said to be an *S-vector subspace of* $(\mathbf{R}_+^*)^n$ (resp. $\mathbf{R}_+^n$).

In what follows, we often denote the spaces $\mathbf{R}_+^n$ and $(\mathbf{R}_+^*)^n$ by W and $\check{W}$. More generally, for $I \subset \{1, \dots, n\}$ and $V \subset W$, we set $V_I = \{t = (t_1, \dots, t_n) \in V \big| t_i = 0 \text{ for all } i \notin I\}$ and $\check{V}_I = \{t \in V_I \big| t_i \ne 0 \text{ for all } i \in I\}$, and we denote by $\mathcal{I}(V)$ the set of all $I \subset \{1, \dots, n\}$ with $\check{V}_I \ne \emptyset$. If $m = \#I$, then $W_I \xrightarrow{\sim} \mathbf{R}_+^m$ and $\check{W}_I \xrightarrow{\sim} (\mathbf{R}_+^*)^m$. We denote by τ_I the composition of the canonical projection $W \to W_I$ with the canonical embedding $W_I \to W$.

3.1. Generalized R_S-polytopes for $0 \notin R$: first properties. If $S = \mathbf{R}$ and $R = \mathbf{R}^*_+$, the notion of a generalized R_S-polytope coincides with that of an $\mathcal{H}$-polyhedron from [Zie, §1.1], and will be called here a *generalized polytope in* $(\mathbf{R}^*_+)^n$. If the latter is compact, it is called a *polytope in* $(\mathbf{R}^*_+)^n$. We recall some basic facts on generalized polytopes from [Zie, §§1-2].

First of all, the Minkowski product of two sets $U, V \subset (\mathbf{R}^*_+)^n$ is the set

$$U \cdot V = \{x \cdot y \big| x \in U,\ y \in V\}.$$

Furthermore, given a finite set $X = \{x_1, \dots, x_k\} \subset (\mathbf{R}^*_+)^n$, one defines the *conical hull* $\mathrm{cone}(X)$ (resp. the *convex hull* $\mathrm{conv}(X)$) as the set of all vectors of the form $x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$ with $\lambda_1, \dots, \lambda_k \geq 0$ (resp. and $\sum_{i=1}^k \lambda_i = 1$). The theorem of Motzkin states that a set $P \subset (\mathbf{R}^*_+)^n$ is a generalized polytope if and only if there exist finite sets $X, Y \subset (\mathbf{R}^*_+)^n$ with $P = \mathrm{conv}(X) \cdot \mathrm{cone}(Y)$ (see [Zie, Theorem 1.2]).

The set $\mathrm{cone}(Y)$ coincides with the set of all $y \in (\mathbf{R}^*_+)^n$ such that $x \cdot y^t \in P$ for all $x \in P$ and $t \geq 0$ and, in particular, it is uniquely determined by P. It is called the *recession cone* of P and denoted by $\mathrm{rec}(P)$. If P is nonempty and defined by inequalities $f(t) \leq g(t)$, then $\mathrm{rec}(P)$ is defined by the inequalities $\overline{f}(t) \leq \overline{g}(t)$, where for a function $f = rt_1^{s_1} \cdot \ldots \cdot t_n^{s_n}$ with $r \neq 0$ one sets $\overline{f} = t_1^{s_1} \cdot \ldots \cdot t_n^{s_n}$ (see [Zie, §1.5]). For example, a subset $P \subset (\mathbf{R}^*_+)^n$ is a polytope if and only if it is the convex hull of a finite set of points. For a polytope P one has $\mathrm{rec}(P) = \{1\}$. For an affine subspace P, $\mathrm{rec}(P)$ is a vector subspace of $(\mathbf{R}^*_+)^n$, and it is the image of P under any shift $x \mapsto xy^{-1}$ with a fixed point $y \in P$. Our first goal is to find when a generalized polytope is a generalized R_S-polytope.

Recall that a *face* of a generalized polytope $P \subset (\mathbf{R}^*_+)^n$ is P itself, or a nonempty subset which is the intersection of P with a hyperplane such that P lies in one half of the space $(\mathbf{R}^*_+)^n$ divided by hyperplane. Every face of P is defined by the same inequalities as P but with some of them turned to equalities and, in particular, it is a generalized polytope. The set of faces $\mathrm{face}(P)$ of P is finite and, if two faces intersect, their intersection is a face. Zero dimensional faces are called *vertices* and their set is denoted by $\mathrm{ver}(P)$. The *cell* of a face is the complement of the union of all strictly smaller faces. The generalized polytope P is a disjoint union of all of its cells. The inclusion partial ordering on the set of faces $\mathrm{face}(P)$ gives rise to a partial ordering on the set of cells $\mathrm{cell}(P)$. The maximal cell of P is an open subset in the affine subspace of $(\mathbf{R}^*_+)^n$ generated by P. If P is a polytope, then P is the convex hull of its set of vertices $\mathrm{ver}(P)$, and for every face $F \subset P$ one has $\mathrm{ver}(F) = F \bigcap \mathrm{ver}(P)$. It is easy to see that all faces of a (generalized) R_S-polytope are (generalized) R_S-polytopes.

Let $\overline{S}$ denote the subfield of $\mathbf{R}$ generated by S, and let $\overline{R}$ denote the $\overline{S}$-vector subspace of $\mathbf{R}_+^*$ generated by R. A point of $(\mathbf{R}_+^*)^n$ is said to be an *R-point* (resp. *$\overline{R}$-point*) if all of its coordinates are contained in R (resp. $\overline{R}$). For example, if $R = \{1\}$, $(1, \dots, 1)$ is the only $\overline{R}$-point. A line in $(\mathbf{R}_+^*)^n$ is said to be *$\overline{S}$-rational* if there exist $s_1, \dots, s_n \in \overline{S}$ such that, for some (and therefore every) pair of distinct points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of the line, one has $\frac{x_i}{y_i} = t^{s_i}$ with $t \in \mathbf{R}_+^*$, $1 \le i \le n$. Notice that replacing t by $t^{\frac{1}{s}}$ for some nonzero $s \in S$, one can achieve that $s_1, \dots, s_n \in S$, i.e., such a line can also be called *S-rational*.

3.1.1. Proposition. *A generalized polytope $P \subset (\mathbf{R}_+^*)^n$ is a generalized R_S-polytope if and only if it can be represented in the form* $\mathrm{conv}(X) \cdot \mathrm{cone}(Y)$ *with the following properties:*

(1) every vertex of $\mathrm{conv}(X)$ *is an $\overline{R}$-point;*

(2) every edge of $\mathrm{conv}(X)$ *is $\overline{S}$-rational;*

(3) every line that connects a point of Y with 1 *is $\overline{S}$-rational.*

Proof. Let us say that a subset $P \subset (\mathbf{R}_+^*)^n$ is a generalized $\overline{R}_{\overline{S}}$-polytope if it is a generalized $\overline{R}_{\overline{S}}^{\mathbf{R}_+^*}$-polytope. To prove the lemma, it suffices to show that a generalized polytope is a generalized $\overline{R}_{\overline{S}}$-polytope if and only if it can be represented in the form $\mathrm{conv}(X) \cdot \mathrm{cone}(Y)$ with the properties (1)-(3). In particular, we may assume that S is a field. If $R = \{1\}$ or P is a polytope, the statement is an easy consequence of linear algebra. We may therefore assume that $R \neq \{1\}$, and P is different from $(\mathbf{R}_+^*)^n$ and not a polytope .

Assume first that P is a generalized R_S-polytope. It can be defined by a finite system of inequalities $t_1^{s_{i1}} \cdot \ldots \cdot t_n^{s_{in}} \le r_i$ with $r_i \ge 1$, $1 \le i \le m$. If $r_i = 1$ for all $1 \le i \le m$, then P is a generalized $\{1\}_S$-polytope, and we can use the first of the previous cases. Assume therefore that $r_i > 1$ for $1 \le i \le k$ with $1 \le k \le m$ and $r_i = 1$ for $k+1 \le i \le m$. Consider the generalized $\{1\}_S$-polytope $\widetilde{P} \subset (\mathbf{R}_+^*)^{n+k}$ defined by the inequalities $t_1^{s_{i1}} \cdot \ldots \cdot t_n^{s_{in}} \le t_{n+i}$ and $t_{n+i} \ge 1$, $1 \le i \le k$. Notice that $P = \{(t_1, \dots, t_n) \big| (t_1, \dots, t_n, r_1, \dots, r_k) \in \widetilde{P}\}$. By the first of the previous case, one has $\widetilde{P} = \mathrm{cone}(\widetilde{Y})$, where $\widetilde{Y}$ is a finite subset of $(\mathbf{R}_+^*)^{n+k}$ such that every line that connects a point from $\widetilde{Y}$ with 1 is S-rational. Let $Y' = \{y \in \widetilde{Y} \big| y_{n+i} = 1$ for all $1 \le i \le k\}$ and $Y'' = \widetilde{Y} \backslash Y'$. If $Y = \{y = (y_1, \dots, y_n) \big| (y_1, \dots, y_n, 1, \dots, 1) \in Y'\}$, and $V = \{(t_1, \dots, t_n) \big| (t_1, \dots, t_n, r_1, \dots, r_k) \in \mathrm{cone}(Y'')\}$, then one evidently has $P = V \cdot \mathrm{cone}(Y)$. *We claim that V is a polytope.* Indeed, let $V' = \{y \in \mathrm{cone}(Y'') \big| y_{n+i} = r_i$ for all $1 \le i \le k\}$. Then the recession cone of V' is the subcone of $\mathrm{cone}(Y'')$ defined by the equalities $y_{n+i} = 1$ for all $1 \le i \le k$ (see [Zie, 1.12]). It follows that $\mathrm{rec}(V') = \{1\}$ and, therefore, V is a polytope. The second of the previous cases implies that V possesses the properties (1) and (2).

Conversely, assume that P is represented in the form $\mathrm{conv}(X) \cdot \mathrm{cone}(Y)$ with $\mathrm{conv}(X)$ and $\mathrm{cone}(Y)$ possessing the properties (1)-(3). Then the polytope $\mathrm{conv}(X)$ can be defined by a finite system of inequalities $t_1^{s_{i1}} \cdot \ldots \cdot t_n^{s_{in}} \le r_i$ with $r_i \ge 1$, $1 \le i \le m$. Let $r_i > 1$ for $1 \le i \le k$ and $r_i = 1$ for $k+1 \le i \le m$. Consider the cone C in $(\mathbf{R}_+^*)^{n+k}$ defined by the inequalities $t_1^{s_{i1}} \cdot \ldots \cdot t_n^{s_{in}} \le t_{n+i}$ and $t_{n+i} \ge 1$, $1 \le i \le k$. Notice that $\mathrm{conv}(X) = \{(x_1, \ldots, x_n) \big| (x_1, \ldots, x_n, r_1, \ldots, r_n) \in C\}$. One has $C = \mathrm{cone}(Y'')$ for a finite subset $Y'' \subset (\mathbf{R}_+^*)^{n+k}$ such that every line that connects a point of Y'' with 1 is S-rational. Let $\widetilde{Y} = Y' \bigcup Y''$, where Y' is the set of all $(y_1, \ldots, y_n, 1, \ldots, 1)$ with $(y_1, \ldots, y_n) \in Y$, and let $\widetilde{P} = \mathrm{cone}(\widetilde{Y})$. Then $P = \{(t_1, \ldots, t_n) \big| (t_1, \ldots, t_n, r_1, \ldots, r_k) \in \widetilde{P}\}$. Since $\widetilde{P}$ is a generalized R_S-polytope, then so is P. ∎

3.1.2. Corollary. *Let P be a generalized R_S-polytope in $(\mathbf{R}_+^*)^n$. Then*

(i) the image of P under any map $f = (f_1, \ldots, f_m) : P \to (\mathbf{R}_+^*)^m$ with $f_i \in A^n(R_S)$ is a generalized R_S-polytope in $(\mathbf{R}_+^*)^m$;

(ii) if $R \ne \{1\}$, the subset of $\overline{R}$-points is dense in P;

(iii) the affine subspace of $(\mathbf{R}_+^*)^n$ generated by P is an R_S-affine subspace. ∎

Notice that the recession cone $\mathrm{rec}(V)$ of a generalized R_S-polytope V is an $\{1\}_S$-polytope. If V is an R_S-affine subspace, $\mathrm{rec}(V)$ is an S-vector subspace L, and V is a principal homogeneous space for L.

3.2. Generalized R_S-polytopes for $0 \notin R$: closure in $\mathbf{R}_+^n$. For a subset $I \subset \{1, \ldots, n\}$, we set $C^I = \{(t_1, \ldots, t_n) \in \check{W} \big| t_i = 1 \text{ for all } i \in I \text{ and } t_i < 1 \text{ for all } i \notin I\}$. Notice that $C^I \cdot C^J = C^{I \cap J}$.

3.2.1. Proposition. *Let P be a generalized R_S-polytope in $\check{W}$, Q the closure $\overline{P}$ of P in W, and I a subset of $\{1, \ldots, n\}$. Then*

(i) $\check{Q}_I \ne \emptyset$ if and only if $\mathrm{rec}(P) \bigcap C^I \ne \emptyset$;

(ii) if $\check{Q}_I \ne \emptyset$, then $Q_I = \tau_I(Q)$ and $\check{Q}_I = \tau_I(P)$.

Proof. Assume first that a point $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{rec}(P)$ is such that $\alpha_i = 1$ for all $i \in I$ and $\alpha_i < 1$ for all $i \notin I$. Then for every $x \in Q$ the point $x \cdot \alpha^t$ with $t \ge 0$ lies in P and tends to the point $\tau_I(x)$ as $t \to \infty$. It follows that $\tau_I(Q) \subset Q_I$ and $\tau_I(P) \subset \check{Q}_I$.

Let I be a maximal proper subset of $\{1, \ldots, n\}$ with $\check{Q}_I \ne \emptyset$, and let Σ be a compact subset of $\mathrm{rec}(P) \backslash \{1\}$ with the property that for every $y \in \mathrm{rec}(P) \backslash \{1\}$ there exist unique $\beta \in \Sigma$ and $t > 0$ with $y = \beta^t$. (For example, we can provide $(\mathbf{R}_+^*)^n$ with an Euclidean space structure and take the intersection of $\mathrm{rec}(P)$ with the unit sphere of a positive radius with center at one.) Given a point $x \in \check{Q}_I$, we can find two sequences of points $v_k \in \mathrm{conv}(X)$, $\beta_k \in \Sigma$ and a sequence of positive numbers t_k with $t_i \to \infty$ and $v_k \cdot \beta_k^{t_k} \to x$ as $k \to \infty$. Since the sets $\mathrm{conv}(X)$ and Σ are compact, we can replace the sequences of points with convergent subsequences so that $v_k \to u \in \mathrm{conv}(X)$ and $\beta_k \to \alpha \in \Sigma$. Since $u \cdot \beta_k^{t_k} \to x \in \check{W}_I$ and $\beta_{k,i} \to \alpha_i$ for all $1 \le i \le n$, it follows that $\alpha_i \le 1$ for all $1 \le i \le n$ and $\alpha_i = 1$ for all $i \in I$. If $J = \{i \big| \alpha_i = 1\}$, the set $\check{Q}_J$ is nonempty because it contains $\tau_J(P)$. Since $\alpha \ne 1$, J is a proper subset of $\{1, \dots, n\}$. It follows that $J = I$, and therefore $\tau_I(Q) \subset Q_I$ and $\tau_I(P) \subset \check{Q}_I$. Since $\tau_I(P)$ is a generalized polytope in $\check{W}_I$, it is closed in $\check{W}_I$, and since $\tau_I(u \cdot \beta_k^{t_k}) \to x$ it follows that $x \in \tau_I(P)$, i.e., $\check{Q}_I = \tau_I(P)$. Notice that, if $x \in Q_J$ for a subset $J \subset I$, there is a sequence of points $v_k \in P$ with $v_k \to x$ as $k \to \infty$. Then $\tau_I(v_k) \to x$ and, therefore, $Q_J = \overline{\check{Q}_I} \bigcap W_J$. In particular, $Q_I = \overline{\check{Q}_I}$. Since $\tau_I(Q) \subset Q_I$ and $\check{Q}_I = \tau_I(P)$, it follows that $Q = \tau_I(Q)$.

To verify the same properties for an arbitrary proper subset $I \subset \{1, \dots, n\}$ with $\check{Q}_I \ne \emptyset$, we may assume that there is a strictly bigger proper subset $J \subset \{1, \dots, n\}$ for which $\check{Q}_J = \emptyset$ and the same properties hold. Let $\beta \in \mathrm{rec}(P)$ with $\beta_i = 1$ for $i \in J$ and $\beta_i < 1$ for $i \notin J$. By the above remark, $Q_I = \overline{\check{Q}_J} \bigcap W_I$. The space W_I has smaller dimension than W. By induction, there exists a point $\gamma \in \mathrm{rec}(P)$ with $\gamma_i = 1$ for $i \in I$ and $\gamma_i < 1$ for $i \in J \backslash I$. Then for a sufficiently large $m \ge 1$, the point $\alpha = \beta^m \cdot \gamma \in \mathrm{rec}(P)$ satisfies the condition $\alpha_i = 1$ for $i \in I$ and $\alpha_i < 1$ for $i \notin I$. ■

3.2.2. Corollary. *In the situation of Proposition 3.2.1, the following is true:*

(i) $\check{Q}_I$ *is a generalized* R_S*-polytope in* $\check{W}_I$*;*

(ii) if $\check{Q}_I \ne \emptyset$ *and* $J \subset I$*, then* $Q_J = \overline{\check{Q}_I} \bigcap W_J$*;*

(iii) if $\check{Q}_I \ne \emptyset$ *and* $\check{Q}_J \ne \emptyset$*, then* $\tau_J(\check{Q}_I) = \check{Q}_{I \cap J}$ *and* $Q_{I \cap J} = Q_I \bigcap Q_J$*.* ■

3.2.3. Lemma. *In the situation of Proposition 3.2.1, the following are equivalent:*

(a) the set Q *is compact;*

(b) P *is contained in a set of the form* $\{t \in \check{W} \big| t_i \le r_i$ *for* $1 \le i \le n\}$, $r_1, \dots, r_n > 0$*;*

(c) for all points $y \in \mathrm{rec}(P)$*, one has* $y_i \le 1$, $1 \le i \le n$*.*

Furthermore, in this case $\mathrm{ver}(P)$ *is a unique minimal set* X *with* $P = \mathrm{conv}(X) \cdot \mathrm{rec}(P)$*.*

Proof. The equivalence (a)$\Longleftrightarrow$(b) is trivial. To prove other implications, let us represent P in the form $\mathrm{conv}(X) \cdot \mathrm{rec}(P)$, where X is a finite subset of $\check{W}$. Then every point from P has the form $x \cdot y$ with $x \in \mathrm{conv}(X)$ and $y \in \mathrm{rec}(P)$. If (c) is true, then for every $1 \le i \le n$, one has $(x \cdot y)_i = x_i y_i \le x_i$ and, therefore, (b) is true. On the other hand, assume that there exists a point $y \in \mathrm{rec}(P)$ with $y_i > 1$ for some $1 \le i \le n$. Then for every $t \ge 0$ one has $x \cdot y^t \in P$ and $(x \cdot y^t)_i = x_i y_i^t \to \infty$ as $t \to \infty$, which contradicts (b).

Assume now that P possesses the equivalent properties (a)-(c). It is clear that $\mathrm{rec}(P)$ lies in any finite set X with $P = \mathrm{conv}(X) \cdot \mathrm{rec}(P)$. It is also clear that P is the convex hull of $\mathrm{ver}(P)$ if P is a polytope in $(\mathbf{R}_+^*)^n$. In general, suppose that $\dim(P) \geq 2$ and that the equality $P = \mathrm{conv}(\mathrm{ver}(P)) \cdot \mathrm{rec}(P)$ holds for generalized polytopes of strictly smaller dimension. Then all points from the boundary of P lie in the set on the right hand side. Let now $x = (x_1, \dots, x_n)$ be a point from the interior of P. Then x lies in the intersection of P with the affine subspace defined by the equation $t_1 \cdot \ldots \cdot t_n = x_1 \cdot \ldots \cdot x_n$. The latter intersection is a polytope in $(\mathbf{R}_+^*)^n$, and so x lies in the convex hull of the set of vertices of that polytope. Since all of them lie in the boundary of P, the induction hypothesis implies the required equality. ∎

3.2.4. Proposition. *The following properties of an R_S-affine subspace $V \subset (\mathbf{R}_+^*)^n$ are equivalent:*

(a) V is closed in $\mathbf{R}_+^n$;

(b) there exist positive elements $s_1, \dots, s_n \in S$ and $r \in R$ such that all points of V satisfy the equality $t_1^{s_1} \cdot \ldots \cdot t_n^{s_n} = r$.

Proof. The implication (b)$\Longrightarrow$(a) is trivial. Assume that V is closed in $\mathbf{R}_+^n$. By Proposition 3.2.1(i), this means that the recession cone $\mathrm{rec}(V)$, which is an S-vector subspace of $(\mathbf{R}_+^*)^n$, possesses the property that the intersection of the S-vector subspace $\mathrm{rec}(V)$ with the quadrant $\{t \in (\mathbf{R}_+^*)^n \big| t_i \leq 1 \text{ for all } 1 \leq i \leq n\}$ is the point of origin $(1, \dots, 1)$. Let L be the image of $\mathrm{rec}(P)$ under the isomorphism $-\log : (\mathbf{R}_+^*)^n \stackrel{\sim}{\to} \mathbf{R}_+^n$ that take a point $(t_1, \dots, t_n) \in (\mathbf{R}_+^*)^n$ to $(-\log(t_1), \dots, -\log(t_n)) \in \mathbf{R}_+^n$. It is a linear subspace of the Euclidean space $\mathbf{R}^n$ with the property $L \cap \mathbf{R}_+^n = 0$, and it is defined by linear equations of the form $(\mu_1 - \nu_1)x_1 + \ldots + (\mu_n - \nu_n)x_n = 0$ for each equality $pt_1^{\mu_1} \cdot \ldots \cdot t_n^{\mu_n} = qt_1^{\nu_1} \cdot \ldots \cdot t_n^{\nu_n}$ among those that define V. We now need the following lemma.

3.2.5. Lemma. *Let L be a linear subspace of the Euclidean space $\mathbf{R}^n$. Then $L \cap \mathbf{R}_+^n = 0$ if and only if $L^\perp \cap (\mathbf{R}_+^*)^n \neq \emptyset$.*

Proof. The converse implication is trivial, and so assume that $L \cap \mathbf{R}_+^n = 0$. The statement is evidently true for $n = 1$. Suppose that $n \geq 2$ and that it is true for $n - 1$. If $L \subset \mathbf{R}^{n-1} \subset \mathbf{R}^n$ and $(x_1, \dots, x_{n-1}, 0) \in L^\perp \cap (\mathbf{R}_+^*)^{n-1}$, then for any $x_n > 0$ the vector $(x_1, \dots, x_{n-1}, x_n)$ lies in $L^\perp \cap (\mathbf{R}_+^*)^n$. Assume therefore that L is not contained in $\mathbf{R}^{n-1}$. Since $L \cap \mathbf{R}_+^n = 0$, then $L = (L \cap \mathbf{R}^{n-1}) \oplus \mathbf{R}v$ for some nonzero $v \in L$. Let π denote the canonical projection $\mathbf{R}^n \to \mathbf{R}^{n-1}$ to the first $n-1$ coordinates, and consider the two cases: (1) $\pi(L) \cap \mathbf{R}_+^{n-1} \neq 0$, and (2) $\pi(L) \cap \mathbf{R}_+^{n-1} = 0$.

(1) We may assume that $\pi(v) \in \mathbf{R}_+^{n-1}$, i.e., if $v = (v_1, \dots, v_n)$, then $v_1, \dots, v_{n-1} \geq 0$ (and not all of them are zero) and $v_n < 0$. By induction, there is $x = (x_1, \dots, x_{n-1}, 0) \in (L \cap \mathbf{R}^{n-1}) \cap$

$(\mathbf{R}_+^*)^{n-1}$. Then the numbers $\langle x,v\rangle$ and $x_n = -\frac{\langle x,v\rangle}{v_n}$ are positive. It follows that the vector $(x_1,\dots,x_{n-1},x_n)$ lies in $L^\perp\cap(\mathbf{R}_+^*)^n$.

(2) By induction, there is $x=(x_1,\dots,x_{n-1},0)\in\pi(L)^\perp\cap(\mathbf{R}_+^*)^{n-1}$. Since $\pi(L)\neq L\cap\mathbf{R}^{n-1}$, there is $y\in(L\cap\mathbf{R}^{n-1})^\perp\backslash\pi(L)^\perp$. Then $x+ay\in(L\cap\mathbf{R}^{n-1})^\perp\cap(\mathbf{R}_+^*)^{n-1}$ for any sufficiently small $a\in\mathbf{R}$. *We claim that there exists $x_n>0$ such that, for $x'=(x_1,\dots,x_{n-1},x_n)$ and for some small $a\in\mathbf{R}$, one has $z=x'+ay\in L^\perp\cap(\mathbf{R}_+^*)^n$.* Indeed, by the construction, any such z lies in $(L\cap\mathbf{R}^{n-1})^\perp$, and one has $\langle z,v\rangle = x_nv_n+a\langle y,v\rangle$. Let a be a sufficiently small number such that the sign of $a\langle y,v\rangle$ is opposite to the sign of v_n. Then the number $x_n=-\frac{a\langle y,v\rangle}{v_n}$ is positive, and the corresponding vector z is orthogonal to v and lies in $(\mathbf{R}_+^*)^n$. ∎

By Lemma 3.2.5, one has $L^\perp\cap(\mathbf{R}_+^*)^n\neq\emptyset$. Since L is defined by linear equations with coefficients in S, then so its orthogonal complement $L^\perp$, and we can therefore find a vector with positive coordinates $(s_1,\dots,s_n)$ in S such that $s_1x_1+\ldots+s_nx_n=0$ for all points $(x_1,\dots,x_n)\in L$. It follows that all points of the recession cone $\mathrm{rec}(V)$ satisfy the equation $t_1^{s_1}\cdot\ldots\cdot t_n^{s_n}=1$. Let $(r_1,\dots,r_n)$ be an $\overline{R}$-point of V. Multiplying all s_i's by a positive element of S, we can achieve inclusion $r=r_1^{s_1}\cdot\ldots\cdot r_n^{s_n}\in R$. Then all points of V satisfy the equality $t_1^{s_1}\cdot\ldots\cdot t_n^{s_n}=r$. ∎

3.3. Generalized R_S-polytopes for $0\in R$. Let $\check{R}=R\backslash\{0\}$.

3.3.1. Theorem. *A closed subset $V\subset\mathbf{R}_+^n$ is a generalized R_S-polytope (resp. an R_S-affine subspace) if and only if it possesses the following properties:*

(1) $\mathcal{I}(V)$ is preserved under intersections;

(2) for every $I\subset J$ in $\mathcal{I}(V)$, $\tau_I(\check{V}_J)\subset\check{V}_I$;

(3) for every $I\in\mathcal{I}(V)$, $\check{V}_I$ is a generalized $\check{R}_S$-polytope (resp. an $\check{R}_S$-affine subspace) in $\check{W}_I$.

For an element $f=rt_1^{s_1}\cdot\ldots\cdot t_n^{s_n}\in A^n(R_S)$, we set $c(f)=\{i\big|s_i\neq 0\}$ (if $r=0$, $c(f)=\emptyset$).

3.3.2. Lemma. *Given two subsets $I\subset J\subset\{1,\dots,n\}$ and a generalized $\check{R}_S$-polytope (resp. an $\check{R}_S$-affine subspace) P in $\check{W}_J$, assume that for its closure Q in $\mathbf{R}_+^n$ one has $\check{Q}_I=\emptyset$. Then there exist functions $f,g\in A^n(R_S)$ with $c(f)=I$, $c(g)\subset J$ and $c(g)\not\subset I$ such that all points of P satisfy the inequality $f(t)\leq g(t)$ (resp. the equality $f(t)=g(t)$).*

Notice that the above inequality is not satisfied at any point of $\check{W}_I$.

Proof. Of course, we may assume that $J=\{1,\dots,n\}$ and P is nonempty. First of all, we deduce the statement for affine subspaces from that for generalized polytopes. Thus, suppose P is an $\check{R}_S$-affine subspace of $(\mathbf{R}_+^*)^n$, and let f and g be elements of $A^n(R_S)$ with $c(f)=I$, $c(g)\subset J$

and $c(g) \not\subset I$ such that all points of P satisfy the inequality $f(t) \le g(t)$. This means that the affine subspace P lies in one half of the space $(\mathbf{R}_+^*)^n$ divided by the hyperplane $L = \{t \big| f(t) = g(t)\}$. Let x and y be $\overline{R}$-rational points of P and L, respectively. Then the shift $t \mapsto tx^{-1}y$ takes the point x to y and, therefore, the image of P is contained in L. It follows that $f(tx^{-1}y) = g(tx^{-1}y)$ for all points $t \in P$. One has $f(tx^{-1}y) = \frac{r_1}{r_2} f(t)$ and $g(tx^{-1}y) = \frac{s_1}{s_2} g(t)$ for $r_1, r_2, s_1, s_2 \in \{\alpha \in \mathbf{R}_+^* \big| \alpha^m \in \check{R}$ for some $m \ge 1\}$. Replacing f and g by f^m and g^m for some $m \ge 1\}$, we may assume that $r_1, r_2, s_1, s_2 \in \check{R}$. Then the functions $f' = r_1 s_2 f$ and $g' = r_2 s_1 g$ are contained in $A^n(R_S)$, satisfy the conditions $c(f') = I$, $c(g') \subset J$ and $c(g') \not\subset I$, and one has $f'(t) = g'(t)$ for all points $t \in P$.

By Proposition 3.2.1(i), one has $\mathrm{rec}(P) \bigcap C^I = \emptyset$. It suffices to find two functions f and g of the form $t_1^{s_1} \cdot \ldots \cdot t_n^{s_n}$ such that (a) all points of C^I satisfy the inequality $f(t) > g(t)$, and (b) all points of $\mathrm{rec}(P)$ satisfy the inequality $f(t) \le g(t)$. Indeed, assume that such functions f and g exist. We may assume that $c(f) \bigcap c(g) = \emptyset$. It is easy to see that the property (a) is equivalent to the following: $c(f) \subset I$ and $c(g) \not\subset I$. By Lemma 3.1.1, P can be represented in the form $V \cdot \mathrm{rec}(P)$, where V is a polytope in $\check{W}$ all of whose vertices are $\overline{R}$-points. The restriction of the function $\frac{g(t)}{f(t)}$ to P takes its minimum r at a vertex of V and, in particular, $r \in \overline{R}$. Replacing the functions f and g by their powers, we may assume that $r = \frac{r_1}{r_2}$ with $r_1, r_2 \in \check{R}$. Then the inequality $r_1 f(t) \le r_2 g(t)$ is satisfied on P but, since $c(f) \subset I$ and $c(g) \not\subset I$, it is not satisfied at any point of $\check{W}_I$. It remains to multiply both functions f and g by some of the variables to achieve the equality $c(f) = I$.

To find the above functions f and g, it is more convenient to consider the additive vector space $\mathbf{R}^n$ (instead of $(\mathbf{R}_+^*)^n$), and we can replace S by the subfield $\overline{S}$ of $\mathbf{R}$ generated by it. Our problem is the following. Let $0 \le m \le n-1$, $C = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \big| x_i = 0$ for $1 \le i \le m$ and $x_i < 0$ for $m+1 \le i \le n\}$, and let P be the convex hull of a finite set of S-rational vectors in $\mathbf{R}^n$ such that $P \bigcap C = \emptyset$. Then there exists an S-rational linear functional ℓ on $\mathbf{R}^n$ such that $\ell(x) < 0 \le \ell(y)$ for all $x \in C$ and $y \in P$. If we provide $\mathbf{R}^n$ with the usual scalar product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, then the latter is equivalent to the fact that there exists an S-rational vector $z \in \mathbf{R}^n$ such that $\langle z, x \rangle < 0 \le \langle z, y \rangle$ for all $x \in C$ and $y \in P$.

For a subset $V \subset \mathbf{R}^n$, let V' denote its polar set $\{z \in \mathbf{R}^n \big| \langle z, x \rangle \le 0\}$, and let $V^\perp$ denote its orthogonal complement. We have to find an S-rational vector $z \in (C' \backslash C^\perp) \bigcap (-P)'$. Notice that $C' = D'$ and $C^\perp = D^\perp$, where $D = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \big| x_i = 0$ for $1 \le i \le m$ and $x_i \le 0$ for $m+1 \le i \le n\}$. Since P and D are the convex hulls of finite sets of S-rational vectors, then so are D', P', $D - P$ and $(D-P)'$. Since $(C' \backslash C^\perp) \bigcap (-P)' = (D-P)' \backslash D^\perp$, it follows that to prove that the latter set contains an S-rational vector, it suffices to show that it is nonempty.

Assume that $(D-P)' \subset D^\perp$. Then $(D^\perp)' \subset (D-P)'' = D - P$. But $(D^\perp)'$ coincides with the vector subspace $E = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \big| x_i = 0 \text{ for } 1 \leq i \leq m\}$, and we get $E \subset D - P$. Since $D \subset E$, it follows that $E \subset D - (P \bigcap E)$. To show that the latter is impossible, we can replace $\mathbf{R}^n$ by E, i.e., we may assume that $m = 0$.

Let V^I denote the relative interior of a subset $V \subset \mathbf{R}^n$. Notice that the convex hull of a finite set of vectors coincides with the closure of its relative interior. One has

$$(D - P)^I = (\overline{D^I} - \overline{P^I})^I = (\overline{D^I - P^I})^I = D^I - P^I \ .$$

Thus, if $D - P = \mathbf{R}^n$, then $D^I - P^I = \mathbf{R}^n$. Since $D^I = C$, it follows that $C - P = \mathbf{R}^n$, which is a contradiction because $P \bigcap C = \emptyset$. (The above reasoning is borrowed from [StWi, Ch. 3].) ∎

Proof of Theorem 3.3.1. (a)$\Longrightarrow$(b). It is clear that, for every subset $I \subset \{1, \ldots, n\}$, $\check{V}_I$ is a generalized $\check{R}_S$-polytope. Assume that for some subsets $I, J \subset \{1, \ldots, n\}$ one has $\check{V}_I \neq \emptyset$ and $\check{V}_J \neq \emptyset$. To show that $\tau_{I \cap J}(\check{V}_I) \subset \check{V}_{I \cap J}$, we have to verify that every inequality $f(t) \leq g(t)$ with $f, g \in A^n(R_S)$ which is satisfied at V is also satisfied at $\tau_{I \cap J}(\check{V}_I)$. Let $f = aT^\mu$ and $g = bT^\nu$, where $a, b \in \mathbf{R}_+$ and $\mu, \nu \in S^n$. If $a = 0$, then the inequality is satisfied everywhere, and so we may assume that $a = 1$. If there exists $i \notin I \bigcap J$ with $\mu_i \neq 0$, then the inequality is satisfied at $W_{I \cap J}$, and so we may assume that $\mu_i = 0$ for all $i \notin I \bigcap J$. If there exists $i \notin I$ (resp. $i \notin J$) with $\nu_i \neq 0$, then the inequality is not satisfied at $\check{V}_I$ (resp. $\check{V}_J$) and, therefore, $\nu_i = 0$ for all $i \notin I \bigcap J$. It follows that the validity of the inequality at $\check{V}_I$ is equivalent to its validity at $\tau_{I \cap J}(\check{V}_I)$, and the inclusion $\tau_{I \cap J}(\check{V}_I) \subset \check{V}_{I \cap J}$ follows.

(b)$\Longrightarrow$(a). To show that V is a generalized R_S-polytope (resp. an R_S-affine subspace), we associate with each subset of $\{1, \ldots, n\}$ a finite family of inequalities (resp. equalities) so that all of them together define the set V. The inequalities depend on the type of a subset. Namely, let A be the family of subsets $I \subset \{1, \ldots, n\}$ with $\check{V}_I \neq \emptyset$, B the family of subsets $I \notin A$ for which there exists $J \in A$ with $I \subset J$, and C the family of subsets $I \subset \{1, \ldots, n\}$ not in $A \bigcup C$.

A. For $I \in A$, let $(f_k, g_k)_k$ be a finite system of functions from $A^n(R_S)$ such that the inequalities $f_k \leq g_k$ (resp. the equalities $f_k = g_k$) define $\check{V}_I$ in $\check{W}_I$. We associate with I the following inequalities (resp. equality)

$$(\prod_{i \in I} t_{j_i}) f_k \leq (\prod_{i \in I} t_{j_i}) g_k \quad \left(\text{resp. } (\prod_{i \in I} t_{j_i}) f_k = (\prod_{i \in I} t_{j_i}) g_k\right)$$

for every system of elements $j_i \in I \backslash \{i\}$. Notice that this system of inequalities (resp. equalities) defines the set $\tau_I^{-1}(\check{V}_I \bigcup \bigcup_{i \in I} W_{I \backslash \{i\}})$, and we claim that the latter contains V. Indeed, let $J \in A$. If $J \supset I$, then $\tau_I(\check{V}_J) \subset \check{V}_I$. If $J \not\supset I$, then $J \cap I \subset I \backslash \{i\}$ for some $i \in I$ and, therefore, $\tau_I(\check{V}_J) \subset W_{I \backslash \{i\}}$.

B. For $I \in B$, let J be the minimal subset of $\{1, \ldots, n\}$ which is contained in A and contains I. By Lemma 3.3.2, there exist functions $f, g \in A^n(R_S)$ with $c(f) = I$, $c(g) \subset J$ and $c(g) \not\subset I$ such that the inequality $f \leq g$ (resp. the equality $f = g$) is satisfied at $\check{V}_J$. We claim that it is satisfied at V. Indeed, since $c(f), c(g) \subset J$, the validity of the inequality (resp. equality) at a point is equivalent to its validity at the image of the point under the projection τ_J. Let $J' \in A$. If $J' \supset J$, then $\tau_J(\check{V}_{J'}) \subset \check{V}_J$. Assume that $J' \not\supset J$. Then $J' \cap J \neq I$ and, therefore, the function f is equal to zero at the set $\check{W}_{J' \cap J}$ that contains $\tau_J(\check{V}_{J'})$. Notice that the above inequality (resp. equality) is not satisfied at any point of $\check{W}_I$.

C. If $I \in C$, we associate with it the equality

$$\prod_{i \in I} t_i = 0 \ .$$

This equality is evidently satisfied at V, and is not satisfied at any point of $\check{W}_I$.

Thus, all of the above inequalities (resp. equalities) define the set V. ∎

3.3.3. Corollary. *Let V be a generalized R_S-polytope in $\mathbf{R}^n_+$, and Q the image of V under the map $f = (f_1, \ldots, f_m) : V \to \mathbf{R}^m_+$ with $f_i \in A^n(R_S)$. Then, for every subset $J \subset \{1, \ldots, m\}$, $\check{Q}_J$ is a generalized $\check{R}_S$-polytope and, for every pair of subsets $I, J \subset \{1, \ldots, m\}$ with $\check{Q}_J \neq \emptyset$, one has $\tau_J(\check{Q}_I) \subset \check{Q}_{I \cap J}$. In particular, if the map f is proper (e.g., V is an R_S-polytope), then Q is a generalized R_S-polytope.*

Proof. We may assume that all of the functions f_i are nonzero. Then for a subset $I \subset \{1, \ldots, n\}$ one has $f(\check{W}_I) \subset \check{W}'_J$, where $W' = \mathbf{R}^m_+$ and J is the subset of $j \in \{1, \ldots, m\}$ with $c(f_j) \subset I$. Let A, B and C be the families of subsets of $\{1, \ldots, n\}$ introduced in the proof of Theorem 3.3.1. For a subset $I \subset \{1, \ldots, m\}$, we set $\widetilde{I} = \bigcup_{i \in I} c(f_i)$ and, if $\widetilde{I} \in A \bigcup B$, we denote by $\overline{I}$ the minimal subset from A that contains $\widetilde{I}$. It follows that $\check{Q}_I \neq \emptyset$ if and only if either $\widetilde{I} \in C$, or $\widetilde{I} \in A \bigcup B$ and $c(f_i) \not\subset \overline{I}$ for all $i \notin \overline{I}$, and in this case one has $\check{Q}_I = f(\check{V}_{\overline{I}})$. In particular, $\check{Q}_I$ is a generalized $\check{R}_S$-polytope in $\check{W}'_I$. Let now $I, J \subset \{1, \ldots, m\}$ be such that $\check{Q}_I \neq \emptyset$ and $\check{Q}_J \neq \emptyset$. We have to verify that $\tau_J(\check{Q}_I) \subset \check{Q}_{I \cap J}$. The left hand side is $\tau_J(f(\check{V}_{\overline{I}}))$, and the right hand side is $f(\check{V}_{\overline{I \cap J}})$. Since $\widetilde{I \bigcap J} \subset \widetilde{I} \bigcap \widetilde{J}$, it follows that $\widetilde{I \bigcap J} \in A \bigcup B$ and $\overline{I \bigcap J} \subset \overline{I} \bigcap \overline{J}$. Moreover, for every $i \notin I \bigcap J$ either $i \notin I$, or $i \notin J$. In both cases, $c(f_i) \not\subset \overline{I} \bigcap \overline{J}$ and, in particular, $c(f_i) \not\subset \overline{I \bigcap J}$, i.e., $\check{Q}_{I \cap J} \neq \emptyset$. It follows also that $\tau_J(f(\check{V}_{\overline{I}})) = f(\tau_{\overline{I \cap J}}(\check{V}_{\overline{I}}))$. Since $\tau_{\overline{I \cap J}}(\check{V}_{\overline{I}}) \subset \check{V}_{\overline{I \cap J}}$, the required inclusion follows. ∎

3.3.4. Corollary. *If P is a generalized $\check{R}_S$-polytope in (resp. an $\check{R}_S$-affine subspace of) $\check{W}_I$, its closure $\overline{P}$ and the set $\overline{P} \backslash P$ are connected generalized R_S-polytopes in (resp. R_S-affine subspaces* *of)* $\mathbf{R}_+^n$.

Proof. The statement follows from Corollary 3.2.2 and Theorem 3.3.1. ∎

3.3.5. Corollary. *Let V be a generalized R_S-polytope in (resp. an $\check{R}_S$-affine subspace of) $\mathbf{R}_+^n$, and let U be the subset $U \subset \mathbf{R}_+^n$ such that, for every $I \subset \{1,\dots,n\}$, $\check{U}_I$ is the recession cone of $\check{V}_I$. Then U is a generalized $\{0,1\}_S$-polytope in (resp. an S-vector subspace of) $\mathbf{R}_+^n$.*

Proof. It suffices to verify that U is closed in $\mathbf{R}_+^n$, i.e., $\overline{\check{U}_I} \subset U$ for all $I \subset \{1,\dots,n\}$ with $\mathrm{rec}(\check{V}_I) \neq \emptyset$. For this we may assume that $I = \{1,\dots,n\}$, i.e., we have to verify that $\overline{\check{U}} \subset U$. ∎

3.3.6. Definition. In the situation of Corollary 3.3.5, the generalized $\{0,1\}_S$-polytope (resp. the S-vector subspace) U is said to be the *recession cone* of V and denoted by $\mathrm{rec}(V)$.

3.4. Irreducible and connected components of a generalized R_S-polytope. For a generalized R_S-polytope V in $\mathbf{R}_+^n$, let $\mathcal{I}(V)$ denote the set of all subsets $I \subset \{1,\dots,n\}$ with $\check{V}_I \neq \emptyset$. Theorem 3.3.1 implies that $\mathcal{I}(V)$ is preserved under intersections and, in particular, it has a unique minimal element with respect to partial ordering by inclusion. Notice that, if I is the minimal element of $\mathcal{I}(V)$, then $\check{V}_I = \overline{\check{V}}_I$.

We introduce a weaker partial ordering on the set $\mathcal{I}(V)$ as follows: $I \leq J$ if $\check{V}_I \subset \overline{\check{V}}_J$. (Of course, one then has $I \subset J$.) An *irreducible component* of V is a subpolytope of the form $\overline{\check{V}_I}$, where I is a maximal element with respect to $\leq$. Any V is a finite union of its irreducible components, and it is called *irreducible* if it has only one irreducible component.

3.4.1. Lemma. *If V is irreducible, then the partial ordering $\leq$ on $\mathcal{I}(V)$ coincides with the ordering by inclusion.*

Proof. We may assume that the maximal element of $\mathcal{I}(V)$ coincides with $\{1,\dots,n\}$. Let I, J be elements of $\mathcal{I}(V)$ with $I \subset J$. Since $\check{V}_I \subset \overline{\check{V}}$, Proposition 3.2.1 implies that there is an element $y = (y_1,\dots,y_n) \in \mathrm{rec}(V)$ with $y_i = 1$ for $i \in I$ and $y_i < 1$ for $i \notin I$. Since $\check{V}_J \subset \overline{\check{V}}$, it follows that $\tau_J(y)$ is an element of $\mathrm{rec}(\check{V}_J)$ with similar properties and, therefore, $\check{V}_I \subset \overline{\check{V}}_J$, by the same Proposition 3.2.1. ∎

We say that a generalized R_S-polytope is *quasi-irreducible* if the partial ordering $\leq$ on $\mathcal{I}(V)$ coincides with the ordering by inclusion.

3.4.2. Lemma. *The following properties of generalized R_S-polytope V are equivalent:*

(a) V is quasi-irreducible;

(b) for every $I \in \mathcal{I}(V)$, one has $V_I = \overline{\check{V}_I}$.

Proof. (a)$\Longrightarrow$(b). For every $I \in \mathcal{I}(V)$, V_I is a union of all $\check{V}_J$ with $J \in \mathcal{I}(V)$ and $J \subset I$. Since for every such J one has $J \leq I$, it follows that $\check{V}_J \subset \overline{\check{V}_I}$, i.e., $V_I = \overline{\check{V}_I}$.

(b)$\Longrightarrow$(a). If $J \subset I$ for $I, J \in \mathcal{I}(V)$, then $V_J \subset V_I = \overline{\check{V}_I}$, i.e., $J \leq I$. ∎

3.4.3. Lemma. *Every connected component of a generalized R_S-polytope (resp. an R_S-affine subspace) is a generalized R_S-polytope (resp. an R_S-affine subspace).*

Proof. Let U be a connected component of $V \subset \mathbf{R}_+^n$. Then for every subset $I \subset \{1, \ldots, n\}$ one has either $\check{U}_I = \emptyset$ or $\check{U}_I = \check{V}_I$ and, in particular, $\check{U}_I$ is a generalized $\check{R}_S$-polytope in (resp. an $\check{R}_S$-affine subspace of) $\check{W}_I$. By Theorem 3.3.1, we have to show that, given subsets I and J with $\check{U}_I \neq \emptyset$ and $\check{U}_J \neq \emptyset$, one has $\tau_J(\check{U}_I) \subset \check{U}_{I\cap J}$. For this we notice that the projection $\tau_J(U)$ is a connected subset of V_J that contains $\check{U}_J = \check{V}_J$. It follows that $\tau_J(U)$ lies in U, i.e., it coincides with U_J. Since $\check{V}_{I\cap J}$ contains the nonempty set $\tau_J(\check{U}_I)$, it follows that it coincides with $\check{U}_{I\cap J}$ and, in particular, $\tau_J(\check{U}_I) \subset \check{U}_{I\cap J}$. ∎

We now introduce a related partial ordering on the set $\mathcal{I}(V)$: $I \preceq J$ if there are sets $I = I_1 \subset I_2 \subset \ldots \subset I_m = J$ such that $\overline{\check{V}}_{I_j} \cap \overline{\check{V}}_{I_{j+1}} \neq \emptyset$ for all $1 \leq j \leq m-1$. Connected components of the partially ordered set $(\mathcal{I}(X), \preceq)$ correspond to connected components of X, i.e., $\pi_0(\mathcal{I}(V)) \xrightarrow{\sim} \pi_0(V)$. If U is a connected component of V, then the restriction of the partial ordering $\preceq_V$ to $\mathcal{I}(U)$ coincides with $\preceq_U$ and, since the set $\mathcal{I}(U)$ is preserved under intersection, there is a unique minimal element of $\mathcal{I}(U)$ with respect to the usual inclusion; it will be denoted by I_U.

3.4.4. Lemma. *Let U be a connected component of a generalized R_S-polytope V. Then*

(i) I_U is a unique minimal element of $\mathcal{I}(U)$ with respect to the partial ordering $\preceq$;

(ii) if U' is a connected component of V with $I_U \subset J$ for some $J \in \mathcal{I}(U')$, then $I_U \subset I_{U'}$.

Proof. (i) We may assume that V is connected, and we have to show that $I_V \preceq J$ for every $J \in \mathcal{I}(V)$. The latter is verified by induction on the cardinality $\#\mathcal{I}(V)$ of $\mathcal{I}(V)$. Assume that $\#\mathcal{I}(V) \geq 2$ and that the required fact is true for connected generalized R_S-polytopes U with $\#\mathcal{I}(U) < \#\mathcal{I}(V)$. Let J be a maximal element of $(\mathcal{I}(V), \preceq)$. From Theorem 3.3.1 it follows that the set $U = V \backslash \check{V}_J$ is a generalized R_S-polytope, and Corollary 3.3.4 implies that U is connected. By induction, the required fact is true for U and, therefore, it is also true for V.

(ii) It suffices to show that, for any subset $I \subset J$ with $\overline{\check{U}'_J} \cap \check{W}_I \neq \emptyset$, one has $I_U \subset I$. By Proposition 3.2.1(i), there exists an element $y = (y_1, \ldots, y_n) \in \mathrm{rec}(\check{U}'_J)$ with $y_i = 1$ for $i \in I$ and $y_i < 1$ for $i \in J \backslash I$ (and $y_i = 0$ for $i \notin J$). One has $x \cdot y^t \in \check{U}'_J$ for any point $x \in \check{U}'_J$ and any $t \in \mathbf{R}_+^*$. It follows that $\tau_{I_U}(x) \cdot \tau_{I_U}(y)^t \in \check{U}_{I_U}$. Since the set $\check{U}_{I_U}$ is closed in W_{I_Y}, the latter is possible only if $I_U \cap (J \backslash I) = \emptyset$, i.e., $I_U \subset I$. ∎

3.4.5. Corollary. *Given connected components U' and U'' of V, there exists a connected component U that contains $\check{V}_{I'\cap I''}$ for all pairs of subsets $I' \in \mathcal{I}(U')$ and $I'' \in \mathcal{I}(U'')$.*

Proof. Let V' be the connected component of V that contains $\check{V}_{I'\cap I''}$, and let V'' be the connected component of V that contains $\check{V}_{J'\cap J''}$ for $J' = I_{U'}$ and $J'' = I_{U''}$. Since $I_{V''} \subset J' \cap J'' \subset I' \cap I'' \in \mathcal{I}(V')$, Lemma 3.4.4(ii) implies that $I_{V''} \subset I_{V'}$. On the other hand, since $I_{V'} \subset I' \in \mathcal{I}(U')$ and $I_{V'} \subset I'' \in \mathcal{I}(U'')$, it follows that $I_{V'} \subset I_{U'} \cap I_{U''} = J' \cap J'' \in \mathcal{I}(V'')$ and, therefore, $I_{V'} \subset I_{V''}$. Thus, $V' = V''$. ∎

We introduce a partial ordering on the finite set $\pi_0(V)$ of connected components of a generalized R_S-polytope V as follows: $U' \leq U''$ if $I_{U'} \subset I_{U''}$. Notice that the partially ordered set $\pi_0(V)$ possesses the property that for any pair $U', U'' \in \pi_0(V)$ there is a well defined connected component $\inf(U', U'')$, which is given by Corollary 3.4.5.

3.4.6. Corollary. *Given a subset $\mathcal{U} \subset \pi_0(V)$, the union $\bigcup_{U\in\mathcal{U}} U$ is a generalized R_S-polytope in $\mathbf{R}^n_+$ if and only if, for any pair $U', U'' \in \mathcal{U}$, one has $\inf(U', U'') \in \mathcal{U}$.*

Proof. The statement follows straightforwardly from Theorem 3.3.1 and Corollary 3.4.6. ∎

3.5. The convex hull of a subset in $\mathbf{R}^n_+$. For a pair of points $x, y \in \mathbf{R}^n_+$, let $\ell_{x,y}$ denote the intersection of all polytopes that contain both x and y. It is easy to see that $\ell_{x,y}$ is a polytope which is describes as follows. Suppose that $x \in \check{W}_I$ and $y \in \check{W}_J$ (i.e., $\mathcal{I}(x) = \{I\}$ and $\mathcal{I}(y) = \{J\}$). If $I = J$, then $\ell_{x,y}$ is the interval in $\check{W}_I$ that connects x and y. If $I \subset J \neq I$, then $\ell_{x,y} = \{y\} \cup \ell_{x,y'}$, where $y' = \tau_I(y)$, and $\ell_{x,y'} \leq y$ in $\pi_0(\ell_{x,y})$. If none of the inclusions $I \subset J$ and $J \subset I$ holds, then $\ell_{x,y} = \{x\} \cup \{y\} \cup \ell_{x',y'}$, where $x' = \tau_{I\cap J}(x)$ and $y' = \tau_{I\cap J}(y)$, and $\ell_{x',y'} = \inf(x, y)$ in $\pi_0(\ell_{x,y})$.

3.5.1. Definition. A subset $\Sigma \subset \mathbf{R}^n_+$ is said to be *convex* if it contains $\ell_{x,y}$ for any pair of points $x, y \in \Sigma$. The *convex hull* $\mathrm{conv}(\Sigma)$ of Σ is the minimal convex subset that contains Σ.

For example, if $\Sigma \subset (\mathbf{R}^*_+)^n$, then $\mathrm{conv}(\Sigma)$ is the usual convex hull of Σ in $(\mathbf{R}^*_+)^n$.

3.5.2. Proposition. *The convex hull P of the union of R_S-polytopes $P^1, \ldots, P^m$ in $\mathbf{R}^n_+$ is an R_S-polytope.*

Proof. Consider first the case when $P^i = \overline{\check{P}^i}$ for all $1 \leq i \leq m$. Then $\check{P}^i = \mathrm{conv}(X_i) \cdot \mathrm{cone}(Y_i)$ for finite subsets $X_i, Y_i \subset (\mathbf{R}^*_+)^n$, and P is the closure of $Q = \mathrm{conv}(\bigcup_{i=1}^m X_i) \cdot \mathrm{cone}(\bigcup_{i=1}^m Y_i)$ in $\mathbf{R}^n_+$, i.e., it is a polytope. That it is an R_S-polytope follows from Proposition 3.1.1. *We claim that a subset $I \subset \{1, \ldots, n\}$ belongs to $\mathcal{I}(P)$ (i.e., $\check{P}_I \neq \emptyset$) if and only if there exist $1 \leq i_1, \ldots, i_l \leq m$ and, for every $1 \leq k \leq l$, a set $I_k \in \mathcal{I}(P^{i_k})$ such that $I = I_1 \cap \ldots \cap I_l$.* Indeed, by Proposition 3.2.1, $I \in \mathcal{I}(P)$ if and only if there exists $(t_1, \ldots, t_n) \in \mathrm{rec}(P)$ with $t_i = 1$ for $i \in I$ and $t_i < 1$ for $i \notin I$.

By the above argument, one has $\mathrm{rec}(P) = \mathrm{rec}(P^1) \cdot \ldots \cdot \mathrm{rec}(P^m)$ and, since all P^k's are compact, one has $t_i \leq 1$ for all $1 \leq i \leq n$ and all $(t_1, \ldots, t_n) \in \mathrm{rec}(P^k)$. This easily implies the claim.

In the general case, let $\mathcal{I}$ be the minimal subset of $\mathcal{P}(\{1, \ldots, n\})$ which is preserved under intersections and contains all of the sets $\mathcal{I}(P^i)$. For $I \in \mathcal{I}$, we denote by Q_I the convex hull in $\check{W}_I$ of the union of the sets $\tau_I(\check{P}^i_J)$ taken over all $1 \leq i \leq m$ and $J \in \mathcal{I}(P^i)$ with $I \subset J$. It is an $\check{R}_S$-polytope in $\check{W}_I$. The collection $\{Q_I\}_{I \in \mathcal{I}}$ possesses the property (b) of Theorem 3.3.1, and *we claim that the union $P = \bigcup_{I \in \mathcal{I}} Q_I$ is a closed subset of $\mathbf{R}^n_+$.* Indeed, it suffices to verify that the closure $\overline{Q_I}$ of each Q_I for $I \in \mathcal{I}$ lies is Q. By Proposition 3.2.1, the closure $\overline{Q_I}$ is the union of Q_I with the projections $\tau_K(Q_I)$ taken over all $K \in \mathcal{I}(\overline{Q_I})$, and so it suffices to verify that every set $\tau_K(Q_I)$ lies in Q. For this we notice that, by the above claim, K is the intersection of sets from $\mathcal{I}(P^i)$'s, i.e., $K \in \mathcal{I}$. Since for such K one has $\tau_K(Q_I) \subset Q_K$, the claim follows.

By Theorem 3.2.1, P is an R_S-polytope. Since the convex hull considered should contain all of the sets Q_I, it follows that it coincides with P. ∎

3.5.3. Proposition. *If P^1 and P^2 are connected (resp. irreducible) polytopes and, for any pair $I \in \mathcal{I}(P^1)$ and $J \in \mathcal{I}(P^2)$, none of the inclusions $I \subset J$ and $J \subset I$ holds, then P is a disjoint union of P^1, P^2 and a nonempty connected (resp. irreducible) polytope which is* $\inf(P^1, P^2)$ *in* $\pi_0(P)$.

Proof. It follows from the construction and Proposition 3.5.2 that P is a disjoint union of P^1, P^2 and a nonempty polytope P', and it suffices to verify that if P^1 and P^2 are irreducible, then so is P'. Suppose $P^1 = \overline{\check{P}^1_{I^1}}$ and $P^2 = \overline{\check{P}^2_{I^2}}$. Then every set J from $\mathcal{I}$ (see the proof of Proposition 3.5.2) has the form $J^1 \cap J^2$ for some $J^1 \in \mathcal{I}(P^1)$ and $J^2 \in \mathcal{I}(P^2)$, and Proposition 3.2.1 easily implies that $Q_J \subset \overline{Q_I}$, where $I = I^1 \cap I^2$. The required fact follows. ∎

3.6. A partial ordering and a transitive relation on an R_S-polytope. Let V be a generalized R_S-polytope in $\mathbf{R}^n_+$. We introduce a partial ordering $\leq$ on the set of points of V as follows: $x \leq y$ if $x = \tau_I(y)$ for some $I \in \mathcal{I}(V)$. If the latter is true for some I, it is also true for the set I with $x \in \check{V}_I$. Notice that the map $V \to \mathcal{I}(V)$ that takes a point x to the set I with $x \in \check{V}_I$ commutes with the partial orderings on both sets. Notice also that any pair of points $x, y \in V$, for which there exists a point $z \in V$ with $x \leq z$ and $y \leq z$, admits the infimum $\inf(x, y)$; namely, if $x \in \check{V}_I$ and $y \in \check{V}_J$, then $\inf(x, y) = \tau_{I \cap J}(z)$.

Furthermore, let $x < y$, and suppose that $x \in \check{V}_I$ and $y \in \check{V}_J$. We write $x \ll y$ if, for any subset $K \in \mathcal{I}(V)$ with $I \subset K \subset J$ and $I \neq K$, one has $x \notin \overline{\check{V}_K}$.

3.6.1. Lemma. *The relation $\ll$ is transitive.*

Proof. Let $x \ll y \ll z$, $x \in \check{V}_I$, $y \in \check{V}_J$, $z \in \check{V}_K$, and suppose that there is a subset $L \in \mathcal{I}(V)$ with $I \subset L \subset K$, $I \neq L$ and $x \in \overline{\check{V}_L}$. ∎

For a point $x \in V$, we set $V^{(\geq x)} = \{y \in V \big| x \leq y\}$. It is a subpolytope of V since $V^{(\geq x)} = \{y \in V \big| t_i(y) = t_i(x)$ for all $i \in I \in \mathcal{I}(V)$ with $x \in \check{V}_I\}$. We write $x \ll y$ if the points x and y lie in different connected components of the set $V^{(\geq x)}$ (and, in particular, $x < y$).

3.5.1. Lemma. *If $x \ll y \leq z$, then $x \ll z$ and, in particular, the relation $\ll$ is transitive.*

Proof. Suppose that the points x and z lie in one connected component of $V^{(\geq x)}$, and let $x \in \check{V}_I$, $y \in \check{V}_J$ and $z \in \check{V}_K$. Replacing V by $V^{(\geq x)}$, we may assume that I is the minimal element of $\mathcal{I}(V)$ and that $V_I = \{x\}$. The assumption and Lemma 3.4.4(i) imply that $I \preceq K$, i.e., there are sets $I = I_1 \subset I_2 \subset \ldots \subset I_m = K$ such that $\overline{\check{V}}_{I_j} \cap \overline{\check{V}}_{I_{j+1}} \neq \emptyset$ for all $1 \leq j \leq m-1$. It follows that the points $x \in \tau_J(\overline{\check{V}_I})$ and $y = \tau_J(z) \in \tau_J(\overline{\check{V}_K})$ lie in one connected component of V, which is a contradiction. ∎

§4. R-affinoid polytopes and R-polytopal algebras

4.1. R-affinoid polytopes and associated R-algebras. Let R be an $\mathbf{F}_1$-subfield of $\mathbf{R}_+$. $R_{\mathbf{Z}_+}$-affine and $R^*_{\mathbf{Z}_+}$-affine subspaces of $\mathbf{R}^n_+$ and $(\mathbf{R}^*_+)^n$ will be called *R-affine* and *R^*-affine* subspaces, respectively. (Since R is an $\mathbf{F}_1$-field, we use the notation R^* instead of $\check{R}$.)

4.1.1. Definition. An *R-affinoid polytope in $\mathbf{R}^n_+$* is an $(\mathbf{R}_+)_{\mathbf{Z}_+}$-polytope V which can be represented as the intersection of an R-affine subspace with a set of the form $\{t \in \mathbf{R}^n_+ \big| t_i \leq r_i$ for all $1 \leq i \leq n\}$, $r_i > 0$.

4.1.2. Proposition. *Let V be an R-affinoid polytope in $\mathbf{R}^n_+$, which is the intersection of an R-affine subspace with the set $P = \{t \in \mathbf{R}^n_+ \big| t_i \leq r_i$ for all $1 \leq i \leq n\}$, and, for $I \in \mathcal{I}(V)$, let $\check{L}_I$ be the R^*-affine subspace of $\check{W}_I$ generated by $\check{V}_I$. Then*

*(i) $L = \bigcup \check{L}_I$ is an R-affine subspace of $\mathbf{R}^n_+$ (the R-*affine subspace generated by V*) and, in particular, V is the intersection of this R-affine subspace with P;*

(ii) for every $I \in \mathcal{I}(V)$, one has $\overline{\check{V}_I} = \overline{\check{L}_I} \cap P$ and, in particular, $\overline{\check{V}_I}$ is an R-affinoid polytope.

Proof. (i) To apply Theorem 3.3.1, we have to verify that the set L is closed in $\mathbf{R}^n_+$. It suffices to show that, for any $J \in \mathcal{I}(V)$, the closure of $\check{L}_J$ is contained in L. For this we can replace V by V_J, and so we may assume that $J = \{1, \ldots, n\}$. *We claim that, for every subset $I \subset \{1, \ldots, n\}$, one has* $\mathrm{rec}(\check{V}) \cap C^I = \mathrm{rec}(\check{L}) \cap C^I$. Indeed, let $z \in \mathrm{rec}(\check{L}) \cap C^I$, i.e., $z_i = 1$ for $i \in I$ and $z_i < 1$

for $i \notin I$, and $yz^t \in \check{L}$ for all $y \in \check{L}$ and $t \geq 1$. If $y \in \check{V}$, then $(yz^t)_i = y_i \leq r_i$ for $i \in I$ and $(yz^t)_i = y_i z_i^t < y_i \leq r_i$ for $i \notin I$ and $t \geq 1$. Since $\check{V} = \check{L} \cap P$, we get $yz^t \in \check{V}$ for all $t \geq 1$, and the claim follows. Let now x be a point from the closure of $\check{L}$ which lies in $\check{W}_I$ for a smaller subset $I \subset \{1, \dots, n\}$. By Proposition 3.2.1, one has $x = \tau_I(y)$ for some $y \in \check{L}$, and there exists a point $z \in \mathrm{rec}(\check{L}) \cap C^I = \mathrm{rec}(\check{V}) \cap C^I$. In particular, $\check{V}_I \neq \emptyset$. Furthermore, since $\check{L}$ is generated by $\check{V}$, it follows that $y = a^s b^{1-s}$ for some points $a, b \in \check{V}$ and a number $s \in \mathbf{R}_+$. Since $\tau_I(a), \tau_I(b) \in \check{V}_I$, it follows that the point $x = \tau_I(a)^s \tau_I(b)^{1-s}$ lies in $\check{L}_I$.

(ii) Let x be a point from $\overline{\check{L}_I} \cap P$ which is contained in $\check{W}_J$ for some $J \subset I$. This means that $x = \tau_J(y)$ for $y \in \check{L}_I$. As in (i), we can find a point $z \in \mathrm{rec}(\check{V}_I)$ with $z_i = 1$ for $i \in J$ and $z_i < 1$ for $i \in I \backslash J$. One has $yz^t \in \check{L}_I$ for all $t \geq 1$. We see that, if t is big enough, the point yz^t is contained in $\check{V}_I = \{t \in \check{L}_I \big| t_i \leq r_i \text{ for all } i \in I\}$. It follows that $x \in \overline{\check{V}_I}$. ∎

Notice that, for any R-affine subspace $L \subset \mathbf{R}_+^n$, there exist $r'_1, \dots, r'_n > 0$ such that, if $r_i > r'_i$ for $1 \leq i \leq n$, then L is generated by the R-affinoid polytope $L \cap \{t \in \mathbf{R}_+^n \big| t_i \leq r_i \text{ for } 1 \leq i \leq n\}$.

For an R-affinoid polytope V in $\mathbf{R}_+^n$, let $A_{V/R}$ denote the $\mathbf{F}_1$-algebra of continuous functions $V \to \mathbf{R}_+$ which are the restrictions of functions from $A^n(R_{\mathbf{Z}_+}) = R[T_1, \dots, T_n]$. It is a Banach R-algebra with respect to the supremum norm $||f|| = \max\limits_{x \in V} f(x)$. One evidently has $\rho(f) = ||f||$ for all $f \in A_{V/R}$ and, in particular, there is an isometric isomorphism $A_{V/R} \stackrel{\sim}{\to} \widehat{A}_{V/R}$. If R' is an $\mathbf{F}_1$-subfield of R that contains the coefficients of the terms from the equalities that define the R-affine space generated by V, then V is also an R'-affinoid polytope, and one has $A_{V/R'} \otimes_{R'} R \stackrel{\sim}{\to} A_{V/R}$. Since one can find such R' with finitely generated group R'^*, the R-algebra $A_{V/R}$ is finitely presented (see Proposition I.1.6.1). For example, if V is the R-affinoid polytope defined only by the inequalities $t_i \leq r_i$ with $r_i > 0$ for all $1 \leq i \leq n$, then $A_{V/R}$ is isometrically isomorphic to the Banach R-algebra $R\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$.

4.1.3. Proposition. *Let V be an R-affinoid polytope in $\mathbf{R}_+^n$. Then*

(i) the canonical map $V \to \mathcal{M}(A_{V/R})$ is a homeomorphism;

(ii) $A_{V/R}$ is an R-affinoid algebra with $A_{V/R} \stackrel{\sim}{\to} \widehat{A}_{V/R}$;

(iii) $A_{V/R}$ is strictly R-affinoid if and only if V can be represented as the intersection of an R-affine subspace with a set $\{t \in \mathbf{R}_+^n \big| t_i \leq r_i \text{ for } 1 \leq i \leq n\}$ with all $r_i \in \sqrt{R}$.

Proof. (i) It suffices to verify that every bounded homomorphism $\chi : A_{V/R} \to \mathbf{R}_+$ is of the form $\chi(f) = f(x)$ for some point $x \in V$. Let $x = (x_1, \dots, x_n)$ be the point of $\mathbf{R}_+^n$ with $x_i = \chi(t_i)$, where t_i is the restriction of the i-th coordinate function. Suppose V is defined by equalities $f_j = g_j$ and inequalities $t_i \leq r_i$, where $f_j, g_j \in R[T_1, \dots, T_n]$, $r_i > 0$, $1 \leq j \leq m$ and $1 \leq i \leq n$. One has

$f_j(x) = g_j(x)$ since χ is a homomorphism, and $t_i(x) \le r_i$ since χ is bounded. It follows that $x \in V$. Since $\chi(f) = f(x)$ for all $f \in R[T_1, \dots, T_n]$, the required fact follows.

(ii) Assume that V is the intersection of an R-affine subspace with a closed subset $U = \{t \in \mathbf{R}_+^n \big| t_i \le r_i \text{ for all } 1 \le i \le n\}$. *We claim that the canonical map* $\varphi : A_{U/R} = R\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A_{V/R}$ *is an admissible epimorphism.* Indeed, this map induces a bounded bijection of Banach R-algebras $A_{U/R}/\mathrm{Ker}(\varphi) \to A_{V/R}$ which, in its turn, induces a homeomorphism between their spectra, It follows that this bijection is isometric with respect to the spectral norm, and the claim follows from Proposition 2.4.5. The isomorphism $A_{V/R} \widetilde{\to} \widehat{A}_{V/R}$ is evident.

(iii) We change the representation of V as the intersection of an R-affine subspace with a closed subset U as in (ii) in the following way. If the coordinate function t_i is identically zero at V, we replace the inequality $t_i \le r_i$ by the equality $t_i = 0$ and the inequality $t_i \le 1$. Otherwise, we replace r_i by the maximal value of t_i at V, which is $\rho(t_i)$. Thus, if $A_{V/R}$ is strictly R-affinoid, then $r_i \in \sqrt{|R|}$, by Proposition 2.2.7. Conversely, if $r_i \in \sqrt{|R|}$ for all $1 \le i \le n$, the same proposition implies that $R\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is strictly R-affinoid and, therefore, so is $A_{V/R}$ since it is a quotient of the latter. ∎

An R-affinoid polytope possessing the equivalent properties of Proposition 4.1.3(iii) is said to be *strictly R-affinoid*. It follows from Proposition 4.1.2(ii) that, if V is strictly R-affinoid, then so is $\overline{\check{V}_I}$ for every $I \in \mathcal{I}(V)$. An R-polytopal algebra is said to be *strictly R-polytopal* if it is isomorphic to an algebra of the form $A_{V/R}$ for a strictly R-affinoid polytope V.

4.1.4. Corollary. *For an R-affinoid polytope V in $\mathbf{R}_+^n$, let R_V denote the $\mathbf{F}_1$-subfield of $\mathbf{R}_+$ generated by the spectral norms $\rho(f)$ of elements $f \in A_{V/R}$. Then*

(i) the quotient group R_V^*/R^* is finitely generated;

(ii) V is strictly R-affinoid if and only if the group R_V^*/R^* is torsion (and therefore finite).

Proof. We may assume that $V \ne \{(0, \dots, 0)\}$. Let V be the intersection of an R-affine subspace with the closed subset $\{t \in \mathbf{R}_+^n \big| t_i \le r_i \text{ for } 1 \le i \le n\}$.

(ii) If the group R_V^*/R^* is torsion, then any strictly R_V-affinoid polytope is automatically strictly R-affinoid. Conversely, assume that V is strictly R-affinoid, i.e., the above representation of V can be found with $r_i \in \sqrt{R^*}$. In this case V is an $R_{\mathbf{Z}_+}$-polytope and, by Proposition 3.1.1, for every nonempty element $I \in \mathcal{I}(V)$ the coordinates of vertices of the generalized polytopes $\check{V}_I$ are contained $\sqrt{R^*}$. Since any bounded linear function on $\check{V}_I$ takes its maximum at a vertex, it follows that the quotient group R_V^*/R^* is finite.

(i) Let $\widetilde{R}$ denote the R-subfield of $\mathbf{R}_+$ generated by the numbers r_i. Then V is a strictly

$\widetilde{R}$-affinoid polytope and, by the fact already established, the quotient group $\widetilde{R}_V^*/\widetilde{R}^*$ is finite. Since the quotient group $\widetilde{R}^*/R^*$ is finitely generated, the required statement follows. ∎

Let V be an R-affinoid polytope in $\mathbf{R}_+^n$, and let D be the $\{0,1\}$-vector subspace of $\mathbf{R}_+^n$ that corresponds to the R-affine space $L \subset \mathbf{R}_+^n$ generated by V. Then the recession cone $U = \mathrm{rec}(V)$ (see Definition 3.3.6) coincides with the set $U = \{t \in D \big| t_i \leq 1 \text{ for all } 1 \leq i \leq n\}$ and, in particular, U is a $\{0,1\}$-affinoid polytope in $\mathbf{R}_+^n$. Let A_U denote the corresponding $\{0,1\}$-polytopal algebra $A_{U/\{0,1\}}$, and let $\overline{A}_V$ denote the quotient $A_{V/R}/R^*$ provided with the trivial norm.

4.1.5. Proposition. *There is a canonical isomorphism of Banach* $\mathbf{F}_1$*-algebras* $\widehat{\overline{A}}_V \widetilde{\to} A_U$.

Proof. Let E be the kernel of the canonical surjective homomorphism $R[T_1, \ldots, T_n] \to A_V : f \mapsto f\big|_V$. If for elements $f, g \in \mathbf{F}_1[T_1, \ldots, T_n]$ one has $(af, bg) \in E$ with $a, b \in R^*$, then $f\big|_U = g\big|_U$. This means that the canonical surjective homomorphism $\mathbf{F}_1[T_1, \ldots, T_n] \to A_U$ goes through the surjective homomorphism $\overline{A}_V \to A_U$. That the latter is an admissible epimorphism is clear and, in particular, $U = \mathcal{M}(A_U)$ is embedded in $\mathcal{M}(\overline{A}_V)$. Thus, to prove the required fact, it suffices to show that $U \widetilde{\to} \mathcal{M}(\overline{A}_V)$.

Suppose that $x \in \mathcal{M}(\overline{A}_V) \cap \check{W}_I$ for some $I \subset \{1, \ldots, n\}$. If $I \in \mathcal{I}(V)$, then clearly $x \in \mathrm{rec}(\check{V}_I)$. Assume therefore that $I \notin \mathcal{I}(V)$, i.e., $\check{V}_I = \emptyset$. If $V_I = \emptyset$, then the Zariski ideal of A_V generated by t_i (the image of T_i in A) for some $i \notin I$ is trivial, i.e., the element t_i is invertible in A_V. Then it is also invertible in $\overline{A}_V$, which contradicts the inclusion $x \in \check{W}_I$. If $V_I \neq \emptyset$, then the function $f = \prod_{i \in I} t_i$ is equal to zero at all points of V. This implies that $(f, 0) \in E$, which again contradicts the inclusion $x \in \check{W}_I$. ∎

For an R-affine subspace $L \subset \mathbf{R}_+^n$, let $A_{L/R}$ denote the $\mathbf{F}_1$-algebra of continuous functions $L \to \mathbf{R}_+$ which are the restrictions of functions from $A^n(R_{\mathbf{Z}_+}) = R[T_1, \ldots, T_n]$. It is a reduced finitely generated R-algebra which is free as an R-vector space. If L is generated by an R-affinoid polytope $V \subset \mathbf{R}_+^n$, then there is a canonical isomorphism of R-algebras $A_{L/R} \widetilde{\to} A_{V/R}$. For a finitely generated R-algebra A, we set $\mathcal{X} = \mathrm{Spec}(A)$ and denote by $\mathcal{X}^{\mathrm{an}}$ the set of all homomorphisms of R-algebras $|\ | : A \to \mathbf{R}_+$ provided with the weakest topology with respect to which all functions $\mathcal{X}^{\mathrm{an}} \to \mathbf{R}_+$ of the form $|\ | \mapsto |f|$ with $f \in A$ are continuous. If $A = A_{L/R}$, there is a canonical homeomorphism $L \widetilde{\to} \mathcal{X}^{\mathrm{an}}$. If A is arbitrary, then any system of generators $f_1, \ldots, f_m$ of A over R gives rise to a continuous map $\mathcal{X}^{\mathrm{an}} \to \mathbf{R}_+^m$ that identifies $\mathcal{X}^{\mathrm{an}}$ with an R-affine subspace of $\mathbf{R}_+^m$.

4.1.6. Proposition. *Let P be an $(\mathbf{R}_+)_{\mathbf{Z}_+}$-polytope in $\mathbf{R}_+^m$. Then there exists a $\{0,1\}$-affinoid polytope V in $\mathbf{R}_+^{m+n}$ such that the canonical projection $\mathbf{R}_+^{m+n} \to \mathbf{R}_+^m : (t_1, \ldots, t_{m+n}) \mapsto (t_1, \ldots, t_m)$ induces a surjective map with connected fibers $V \to P$.*

Proof. The polytope P is defined by a finite set $C(P)$ of inequalities of the form $t_1^{\mu_1} \cdot \ldots \cdot t_m^{\mu_k} \le rt_1^{\nu_1} \cdot \ldots \cdot t_m^{\nu_m}$. We may assume that the set $C(P)$ includes the inequalities $t_i \le r_i$ with $r_i \ne 0$ for all $1 \le i \le m$, and let $S(P)$ be the subset of the inequalities from $C(P)$ which are not of the latter form and in which either both sides are not identically equal at P or $r \ne 1$. If n is the cardinality of $S(P)$, we define a polytope V in $\mathbf{R}_+^{m+n}$ by the inequalities from $C(P)\backslash S(P)$ and, instead of every inequality $t_1^{\mu_1} \cdot \ldots \cdot t_m^{\mu_k} \le rt_1^{\nu_1} \cdot \ldots \cdot t_m^{\nu_m}$ from $S(P)$, the equality $t_1^{\mu_1} \cdot \ldots \cdot t_m^{\mu_k} = t_1^{\nu_1} \cdot \ldots \cdot t_m^{\nu_m} s$ and the inequality $t_{m+1} \le r$, where s is the corresponding additional variable in $\mathbf{R}_+^{m+n}$. Then V is a $\{0,1\}$-affinoid polytope, and it is easy to see that the canonical projection $\mathbf{R}_+^{m+n} \to \mathbf{R}_+^m$ induces a surjective map $V \to P$. It remain to verify that the fibers of the latter map are connected. For this notice that V is the result of the n-th step of the following construction. Given an inequality from $S(P)$ as above, let Q be the polytope in $\mathbf{R}_+^{m+1}$ by the other inequalities from $C(S)$ and the equality $t_1^{\mu_1} \cdot \ldots \cdot t_m^{\mu_k} = t_1^{\nu_1} \cdot \ldots \cdot t_m^{\nu_m} s$ and the inequality $t_{m+1} \le r$. Since $|S(Q)| < |S(P)|$, it suffices to verify that the fibers of the canonical projection $\varphi : Q \to P$ are connected.

Let V^* be the open subset $\{(t_1, \ldots, t_m) \in V \big| t_i \ne 0$ for all $1 \le i \le m$ with $\nu_i \ne 0\}$. *We claim that* $\varphi^{-1}(V^*) \widetilde{\to} V^*$ *and* $\varphi^{-1}(t) \widetilde{\to} [0, r]$ *for all points* $t \in V\backslash V^*$. Indeed, the first bijection is trivial. If $t \in V\backslash V^*$, there exists $1 \le i \le m$ with $\nu_i \ne 0$ and $t_i = 0$ and, therefore, $t_1^{\mu_1} \cdot \ldots \cdot t_m^{\mu_m} = 0$. Since all other inequalities from $C(S)$ do not contain the variable s, it follows that all points $(t_1, \ldots, t_m, s)$ with $s \le r$ belong to Q, i.e., $\varphi^{-1}(t) \widetilde{\to} [0, r]$. ∎

4.2. Comparison of properties of V and $A_{V/R}$. In this subsection, V is an R-affinoid polytope in $\mathbf{R}_+^n$. Since the $\mathbf{F}_1$-field R is fixed, we use the notation A_V instead of $A_{V/R}$.

Consider the canonical continuous map $V = \mathcal{M}(A_V) \to \mathrm{Zspec}(A_V)$ that takes a point $x \in V$ to the Zariski prime ideal $\mathrm{Zker}(|\ |_x)$. For a Zariski prime ideal $\mathfrak{p} \subset A_V$, we set $V_\mathfrak{p} = \{x \in V \big| \mathfrak{p} \subset \mathrm{Zker}(|\ |_x)\}$ and $\check{V}_\mathfrak{p} = \{x \in \mathcal{M}(A_V) \big| \mathfrak{p} = \mathrm{Zker}(|\ |_x)\}$. The former is the homeomorphic image of the canonical map $\mathcal{M}(A_V/\mathfrak{p}) \to V = \mathcal{M}(A_V)$, and the latter is the preimage of $\mathfrak{p}$ under the above map $V \to \mathrm{Zspec}(A_V)$.

4.2.1. Lemma. *(i) The correspondence* $\mathfrak{p} \mapsto J = J_\mathfrak{p} = \{i \big| t_i \notin \mathfrak{p}\}$ *gives rise to an isomorphism of partially ordered sets* $\mathrm{Zspec}(A_V) \widetilde{\to} \mathcal{I}(V)$;

(ii) one has $V_\mathfrak{p} = V_J$ *and* $\check{V}_\mathfrak{p} = \check{V}_J$;

(iii) the projection $\tau_J : V \to V_J$ *coincides with the retraction* $\tau_\mathfrak{p} : V \to V_\mathfrak{p}$.

Proof. Let f be the product of t_i's with $i \in J_\mathfrak{p}$. Then $f \notin \mathfrak{p}$ and, in particular, the spectral radius of the image f in $A_V/\mathfrak{p}$ is positive. This implies that there exists a point $x \in \mathcal{M}(A_V/\mathfrak{p}) = V_\mathfrak{p}$ with $f(x) \ne 0$ and, therefore, $t_i(x) \ne 0$ for all $i \in J_\mathfrak{p}$. The latter means that $x \in \check{V}_{J_\mathfrak{p}}$ and, in particular, $J_{\mathfrak{p}} \in \mathcal{I}(V)$. It follows also that $x \in \check{V}_{\mathfrak{p}}$ if and only if $t(x) \neq 0$ for all $i \in J_{\mathfrak{p}}$, i.e., $\check{V}_{\wp} = \check{V}_{J_{\mathfrak{p}}}$. The remaining equality in (ii) and the statement (iii) are now trivial. Let now $J \in \mathcal{I}(V)$, and let $x \in \check{V}_J$. Then the Zariski kernel of χ_x is the Zariski prime ideal $\mathfrak{p}$ generated by the elements t_i with $i \notin J$. It follows that $J = J_{\mathfrak{p}}$. ∎

Since $V_{\mathfrak{p}} = V_{J_{\mathfrak{p}}}$, it follows that $V_{\mathfrak{p}}$ is an R-affinoid polytope in $\mathbf{R}_+^n$.

4.2.2. Corollary. *The canonical map $A_V \to A_{V_{\mathfrak{p}}}$ gives rise to an isometric isomorphism of Banach R-algebras $A_V/\mathfrak{p} \widetilde{\to} A_{V_{\mathfrak{p}}}$.*

Proof. The map considered is evidently surjective. Furthermore, if $f \notin \mathfrak{p}$, then $f(x) = f(\tau_{\mathfrak{p}}(x))$ for all points $x \in V$ and, in particular, that the supremum norm of f is achieved at $V_{\mathfrak{p}}$. This implies that the map considered is isometric. Finally, assume that, for a pair of nonzero elements $f, g \in A_V$, one has $f(x) = g(x)$ for all $x \in V_{\mathfrak{p}}$. Then for each point $x \in X$ one has $f(x) = f(\tau_{\mathfrak{p}}(x)) = g(\tau_{\mathfrak{p}}(x)) = g(x)$. It follows that $f = g$, i.e., the map considered is injective. ∎

4.2.3. Corollary. *The following are equivalent:*

(a) the R-algebra A_V has no zero divisors;

(b) the set $\mathcal{I}(V)$ has a unique maximal element.

Proof. The statement follows from Corollary I.2.2.3(ii) and Lemma 4.2.1(i). ∎

4.2.4. Proposition. *The following are equivalent:*

(a) the R-algebra A_V is an $\mathbf{F}_1$-field;

(b) the set $\mathcal{I}(V)$ consists of only one element, i.e., $V \subset \check{W}_I$ for some $I \subset \{1, \dots, n\}$;

(c) there exist a subset $I \subset \{1, \dots, n\}$ and positive integers $\{\mu_i\}_{i \in I}$ such that all points of V satisfy the equalities $\prod_{i \in I} t_i^{\mu_i} = p$ with $p \in R^*$ and $t_i = 0$ for $i \notin I$.

Proof. The equivalence (a)$\Longleftrightarrow$(b) and the implication (c)$\Longrightarrow$(b) are trivial. To prove the implication (b)$\Longrightarrow$(c), we may assume that $I = \{1, \dots, n\}$, i.e., $V \subset (\mathbf{R}_+^*)^n$. Assume V is defined by equalities $p_j f_j = q_j g_j$ for $1 \leq j \leq m$ and inequalities $t_i \leq r_i$ for $1 \leq i \leq n$, where f_j and g_j are monomials in $t_1, \dots, t_n$, $p_j \in R$ and $r_i > 0$. Notice that, since $V \subset (\mathbf{R}_+^*)^n$, then $p_j, q_j > 0$ for all $1 \leq j \leq m$. One has $V = \{t \in P \big| t_i \leq r_i \text{ for } 1 \leq i \leq n\}$, where P is the affine subspace of $(\mathbf{R}_+^*)^n$ defined by the equalities $p_j f_j = q_j g_j$ for $1 \leq j \leq m$, and so the assumption implies that P is closed $(\mathbf{R}_+^*)^n$. The required property (c) now follows from Proposition 3.2.4. ∎

Let I_V denote the idempotent $\mathbf{F}_1$-subalgebra I_{A_V} of A_V. By Corollary I.2.1.6, I_V is finite and, therefore, any Banach norm on it (and, in particular, that induced from A_V) is equivalent to the trivial one.

4.2.5. Proposition. *(i) Given a nonzero idempotent $e \in A_V$, there is a unique minimal connected component U_e with $e\big|_{U_e} = 1$, and, for a connected component U', $e\big|_{U'} = 1$ if and only if $U_e \le U'$;*

(ii) the map $e \mapsto U_e$ induces an isomorphism of partially ordered sets $\check{I}_V = I_V \backslash \{0\} \xrightarrow{\sim} \pi_0(V)$.

Proof. (i) The idempotent is equal to 0 or 1 at each point of V. Since $\{x \in V \big| e(x) = 0\} = \{x \in V \big| e(x) < 1\}$, it follows that e is identically equal to 0 or 1 at every connected component of V. Furthermore, let e be represented by a monomial $pt_1^{\mu_1} \cdot \ldots \cdot t_n^{\mu_n}$ with $p \in R^*$, and set $I = \{i \big| \mu_i \ge 1\}$. From the above remark it follows that, given a connected component U of V, $e\big|_U = 1$, if $I \subset I_U$, and $e\big|_U = 0$, if $I \not\subset I_U$. The required facts now follow from Lemma 3.4.2.

(ii) The statement (i) straightforwardly implies that the map considered is injective and, if $e \le f$, then $U_e \le U_f$. Conversely, let U be a connected component of V, and set $I = I_U$. Since $\check{V}_I = \check{U}_I$ is compact, Proposition 4.2.4 implies that there is a monomial t^μ with $\mu_i \ge 1$ if and only if $i \in I$ and such that all points from $\check{V}_I$ satisfy the equality $t^\mu = p \in R^*$. If e_U denotes the restriction of $p^{-1} t^\mu$ to V, Lemma 3.4.2 implies that, for a connected component U' of V, $e_U\big|_{U'} = 1$ if and only if $U \le U'$. It follows that the idempotent e_U is really determined by U and $U_{e_U} = U$. In particular, the map considered is surjective. It remains to show that, if $U \le U'$, then $e_U \le e_{U'}$. By the characterization e_U, one has $e_U\big|_{U'} = 1$, and we see that the product $e_U e_{U'}$ possesses the property that characterizes $e_{U'}$. It follows that $e_U e_{U'} = e_{U'}$, i.e., $e_U \le e_{U'}$. ∎

4.2.6. Proposition. *The following properties of V are equivalent:*

(a) A_V is an integral domain;

(b) V is irreducible;

(c) if $f\big|_{\mathcal{U}} = g\big|_{\mathcal{U}}$ for $f, g \in A_V$ and a nonempty open subset $\mathcal{U} \subset V$, then $f = g$.

Furthermore, in this case the dimension of $\check{V}$ is equal to the (rational) rank of the quotient group F^*/R^*, where F is the fraction $\mathbf{F}_1$-field of A_V.

Notice that the group F^* is torsion free, but the quotient F^*/R^* may have torsion.

Proof. (a)$\Longrightarrow$(b). By Corollary 4.2.3, the set $\mathcal{I}(V)$ has a unique element maximal with respect to the inclusion relation, and we may assume that it coincides with $\{1, \ldots, n\}$, i.e., $\check{V} \ne \emptyset$. Assume that $U = \overline{\check{V}}$ does not coincide with V. If $V = L \cap P$, where L is the R-affine subspace of $\mathbf{R}_+^n$ generated by V and $P = \{t \in \mathbf{R}_+^n \big| t_i \le r_i \text{ for } 1 \le i \le n\}$. Then $U = \overline{\check{L}} \cap P$, by Proposition 4.1.2. This implies that the R-affine subspace $\overline{\check{L}}$ does not coincide with L and, therefore, there exists a pair (F, G) of terms in $R\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ which are equal at U but are not equal at all V. If f and g are the images of F and G in A_V, and h is the product of the images of all coordinate functions, we get $fh = gh$. Since A_V is an integral domain, it follows that $f = g$, which is a contradiction.

(b)$\Longrightarrow$(c). We may assume that $V = \overline{\check{V}}$. Suppose that $f\big|_{\mathcal{U}} = g\big|_{\mathcal{U}}$ for $f, g \in A_V$ and a nonempty open subset $\mathcal{U} \subset V$. Since $V = \overline{\check{V}}$, the open subset $\mathcal{U} \subset V$ has nonempty intersection with $\check{V}$. It follows that $f\big|_{\check{V}} = g\big|_{\check{V}}$ and, therefore, $f = g$.

(c)$\Longrightarrow$(a). Suppose that $fh = gh$ for elements of A_V with nonzero h. Then $\mathcal{U} = \{x \in V \big| h(x) \neq 0\}$ is a nonempty open subset of V, and one has $f\big|_{\mathcal{U}} = g\big|_{\mathcal{U}}$. It follows that $f = g$.

To prove the last statement, we again may assume that $I = \{1, \ldots, n\}$. Let $\check{L}$ be the R-affine subspace generated by $\check{V}$. It has the same dimension as $\check{V}$, and the group F^* coincides with the group of restrictions of functions of the form $t = (t_1, \ldots, t_n) \mapsto rt^\mu = rt_1^{\mu_1} \cdot \ldots \cdot t_n^{\mu_n}$ to $\check{L}$ with $r \in R^*$ and $\mu_i \in \mathbf{Z}$. If $\check{L}$ is defined by equalities $t^{\mu^{(j)}} = r_j$ for $r_j \in R^*$, $\mu^{(j)} \in \mathbf{Z}^n$ and $1 \leq j \leq m$, then the dimension of $\check{L}$ is equal to the rank l of the integral matrix $(\mu_i^{(j)})_{1 \leq i \leq n, 1 \leq j \leq m}$. After a permutation of the coordinates we may assume that the projection to the first l-coordinates $(\mathbf{R}_+^*)^n \to (\mathbf{R}_+^*)^l$ gives rise to an isomorphism $\check{L} \xrightarrow{\sim} (\mathbf{R}_+^*)^l$. It follows that the rank of the group F^*/R^* is at least l. It follows also that each t_j with $l + 1 \leq j \leq m$ is expressed in the form $rt_1^{\nu_1} \cdot \ldots \cdot t_l^{\nu_l}$ with $r \in R^*$ and $\nu_i \in \mathbf{Q}$ for $1 \leq i \leq l$. This implies that the rank of F^*/R^* is at most l, i.e., they are equal. ∎

Let V be an irreducible R-affinoid polytope in $\mathbf{R}_+^n$, and assume for simplicity that $\check{V} \neq \emptyset$, i.e., $V = \overline{\check{V}}$. The recession cone $U = \mathrm{rec}(V)$ is a strictly $\{0,1\}$-affinoid polytope in $\mathbf{R}_+^n$ with $\mathcal{I}(U) = \mathcal{I}(V)$, and $\check{U}$ is the recession cone $\mathrm{rec}(\check{V})$ of V. By Proposition 4.1.8, there is an isomorphism of strictly $\{0,1\}$-polytopal algebras $\widehat{\widetilde{A}}_V \xrightarrow{\sim} A_U$. The later is an integral finitely generated $\mathbf{F}_1$-algebra. In §I.1.3 we associated with A_U a convex rational polyhedral cone C in $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, where N is the multiplicative group of the fraction field of A_U written additively. Notice that since $A_U^* = \{1\}$, the cone is strictly convex, i.e., $C \cap (-C) = \{0\}$. Recall also that, by Lemma I.1.3.3, there is a canonical isomorphism of partially ordered sets $\mathrm{Zspec}(A_U) \xrightarrow{\sim} \mathrm{face}(C)$. Let $N_{\mathbf{R}}^*$ be the dual space of $N_{\mathbf{R}}$, and $C^\vee$ the strictly convex rational polyhedral cone $\{v \in N_{\mathbf{R}}^* \big| \langle v, u\rangle \geq 0\}$. It is easy to see that the map that takes a point $v \in C^\vee$ to the homomorphism of $\mathbf{F}_1$-algebras $\chi_v : A_U \to [0,1]$ with $\chi_v(a) = e^{-\langle v, a\rangle}$ for $a \in \check{A}_U$ gives rise to an isomorphism of strictly convex cones $C^\vee \xrightarrow{\sim} \check{U}$.

Turning back to the ambient space $\mathbf{R}_+^n$, we see that there is a bijection $\mathcal{I}(U) \xrightarrow{\sim} \mathrm{face}(\check{U})$ that reverses the partial orderings on both sets and takes an element $I \in \mathcal{I}(U)$ to the face $F_I = \{(\alpha_1, \ldots, \alpha_n) \in \check{U} \big| \alpha_i = 1 \text{ for } i \in I\}$.

4.2.7. Corollary. *In the above situation, let $V^{(x)}$ be the fiber of the canonical projection $\tau_I : V \to V_I$ over a point $x \in \check{V}_I$. Then:*

(i) $V^{(x)}$ is an irreducible $\mathbf{R}_+$-affinoid polytope;

(ii) the recession cone of $V^{(x)}$ is canonically isomorphic to F_I;

(iii) $\dim(\check{V}^{(x)}) = \dim(\check{F}_I)$.

Recall that, by Corollary I.1.3.5, $\dim(\check{F}_I) = m$, where m is maximal for which there is a strictly increasing sequence $I_0 = I \subset I_1 \subset \ldots \subset I_m = \{0, \ldots, n\}$ of elements of $\mathcal{I}(V)$.

Proof. The statement (i) and (ii) are trivial. To prove (iii), we notice that I is the minimal element of $\mathcal{I}(V^{(x)})$, and $(V^{(x)})_I = \{x\}$. Proposition 4.2.6 implies that $A^*_{V^{(x)}/\mathbf{R}_+} = \mathbf{R}^*_+$ and that $\dim(V^{(x)})$ and $\dim(\check{F}_I)$ are equal to the Krull dimensions of $A_{V^{(x)}/\mathbf{R}_+}$ and $A_{F_I/\mathbf{R}_+}$, respectively. Propositions 4.1.6(ii) and 4.1.8 then imply that both dimensions are equal. ∎

4.2.7. Proposition. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be Zariski prime ideals of A_V. Then*

(i) there is an isomorphism of Banach R-algebras $A_V/\Pi_\mathfrak{p} \xrightarrow{\sim} A_{\overline{\check{V}_\mathfrak{p}}}$ and, in particular, $A_V/\Pi_\mathfrak{p}$ is an R-polytopal algebra;

(ii) the dimension of $\check{V}_\mathfrak{p}$ is equal to the rank of the quotient group $\kappa(\mathfrak{p})^*/R^*$;

(iii) $\check{V}_\mathfrak{p} \subset \overline{\check{V}_\mathfrak{q}}$ if and only if $\Pi_\mathfrak{q} \subset \Pi_\mathfrak{p}$; in particular, there is a canonical bijection between the set of minimal prime ideals of A_V and the set of irreducible components of V.

Proof. (i) First of all, if $(f, g) \in \Pi_\mathfrak{p} \backslash (\mathfrak{p} \times \mathfrak{p})$, then there is $h \notin \mathfrak{p}$ with $fh = gh$. It follows that $f(x) = g(x)$ for all $x \in \check{V}_\mathfrak{p}$ and, therefore, the homomorphism considered is well defined and surjective. *We claim that it is bijective.* Indeed, assume that $f(x) = g(x)$ for all $x \in \overline{\check{V}_\mathfrak{p}}$. If the functions f and g do not lie in $\mathfrak{p}$, they are expressible in the coordinate functions t_i for $i \in J = J_\mathfrak{p}$ (see Lemma 4.2.1). If h is the product of all coordinate functions t_i with $i \notin J$ (i.e., $t_i \in \mathfrak{p}$), then $fh = gh$. Indeed, let $x \in \check{V}_I$. If $J \not\subset I$, then $h(x) = 0$. If $J \subset I$, then $f(x) = f(\tau_\mathfrak{p}(x)) = g(\tau_\mathfrak{p}(x)) = g(x)$. It follows that $(fh)(x) = (gh)(x)$ for all $x \in V$ and, therefore, $fh = gh$, and the claim follows. Furthermore, the image of $\mathcal{M}(A_V/\Pi_\mathfrak{p})$ in V contains $\overline{\check{V}_\mathfrak{p}}$. By Proposition 4.2.6, it is irreducible. Since it is contained in $V_\mathfrak{p}$, it follows that $\mathcal{M}(A_V/\Pi_\mathfrak{p}) \xrightarrow{\sim} \overline{\check{V}_\mathfrak{p}}$. The required statement now follows from Corollary 2.4.2.

(ii) and (iii) follow from (i) and Proposition 4.2.6. ∎

4.2.8. Corollary. *Let $\mathcal{I}$ be a subset of $\mathcal{I}(V)$ which is preserved under intersection and contains all $I \in \mathcal{I}(V)$ for which $\overline{\check{V}_I}$ is an irreducible component of V. Then there is an isomorphism of Banach R-algebras $A_V \xrightarrow{\sim} \prod^\nu_\mathcal{I} A_{\overline{\check{V}_I}}$.* ∎

4.2.9. Corollary. *The following properties of V are equivalent:*

(a) A_V is quasi-irreducible;

(b) V is quasi-irreducible;

(c) if $f\big|_{\mathcal{U}} = g\big|_{\mathcal{U}} \neq 0$ for $f, g \in A_V$ and a nonempty open subset $\mathcal{U} \subset V$, then $f = g$.

Proof. (a)$\Longrightarrow$(b). Given a Zariski prime ideal $\mathfrak{p} \subset A_V$, one has $A_V/\mathfrak{p} \xrightarrow{\sim} A_{V_{\mathfrak{p}}}$, by Corollary 4.2.2, and $A_V/\Pi_{\mathfrak{p}} \xrightarrow{\sim} A_{\overline{\check{V}_{\mathfrak{p}}}}$, by Proposition 4.2.7(i). Since $\Pi_{\mathfrak{p}} = \Delta(A_V) \cup (\mathfrak{p} \times \mathfrak{p})$, then $A_V/\mathfrak{p} \xrightarrow{\sim} A_V/\Pi_{\mathfrak{p}}$ and, therefore, $V_{\mathfrak{p}} = \overline{\check{V}_{\mathfrak{p}}}$, i.e., V is quasi-irreducible, by Lemma 3.4.2.

(b)$\Longrightarrow$(c). We may assume that $\mathcal{U} \subset \overline{\check{V}_{\mathfrak{p}}}$ for an irreducible component $\overline{\check{V}_{\mathfrak{p}}}$ of V. Then $f\big|_{\overline{\check{V}_{\mathfrak{p}}}} = g\big|_{\overline{\check{V}_{\mathfrak{p}}}}$ and $f, g \notin \mathfrak{p}$. Since $V_{\mathfrak{p}} = \overline{\check{V}_{\mathfrak{p}}}$, it follows that $f = g$ in $A_V/\mathfrak{p}$ and, therefore, $f = g$ in A_V.

(c)$\Longrightarrow$(a). It suffices to verify that $\Pi_{\mathfrak{p}} = \Delta(A_V) \cup (\mathfrak{p} \times \mathfrak{p})$ only for Zariski prime ideals $\mathfrak{p}$ for which $\overline{\check{V}_{\mathfrak{p}}}$ is an irreducible component of V. Suppose that $(f, g) \in \Pi_{\mathfrak{p}}$ for $f, g \notin \mathfrak{p}$. Then $fh = gh$ for some $h \notin \mathfrak{p}$ and, therefore, $f\big|_{\overline{\check{V}_{\mathfrak{p}}}} = g\big|_{\overline{\check{V}_{\mathfrak{p}}}}$. Since $\check{V}_{\mathfrak{p}}$ contains a nonempty open subset of V, it follows that $f = g$. ∎

4.3. Faces and cells of an R-affinoid polytope. A *face* of an R-affinoid polytope $V \subset \mathbf{R}_+^n$ is a nonempty closed subset F of the form $V_f = \{t \in V \big| f(t) = \rho(f)\}$ with $f \in A_{V/R}$. For example, V itself is a face since it coincides with V_1. The set of faces face(V) of V is not changed if V is considered as an R'-affinoid polytope for an $\mathbf{F}_1$-subfield $R \subset R' \subset \mathbf{R}_+$. Notice that V_f is an R'-affinoid polytope where R' is the R-subfield of $\mathbf{R}_+$ generated by the number $\rho(f)$ and, in particular, if V is strictly R-affinoid, then so are all faces of V. Because of this, we again use the notation A_V instead of $A_{V/R}$.

The intersection $V_f \cap V_g$ is nonempty if and only if $\rho(fg) = \rho(f)\rho(g)$, and in this case it coincides with the face V_{fg}. In particular, any nonempty intersection of two faces is a face. Notice also that, if F is a face of V, then any face of V which is contained in F is a face of F (see also Lemma 4.3.4). The set of faces of V is denoted by face(V). The following lemma implies that this set is finite.

4.3.1. Lemma. *(i) Given a face F of V, $\check{F}_I$ is a face of $\check{V}_I$ for every $I \in \mathcal{I}(F)$;*

(ii) given $I \in \mathcal{I}(V)$, for every face U of $\check{V}_I$ there exists a face F of V with $\check{F}_I = U$.

Proof. The statement (i) is trivial, and to verify (ii) recall that any face of a polytope in an affine space is defined by the same inequalities but with some of them turned to equalities. Suppose V is represented as the intersection of an R-affine subspace of $\mathbf{R}_+^n$ with a set $\{t \in \mathbf{R}_+^n \big| t_i \leq r_i$ for $1 \leq i \leq n\}$ for $r_i > 0$. Then there is a subset $J \subset I$ such that $U = \{t \in \check{V}_I \big| t_i = r_i$ for $i \in J\}$. Then $F = \{t \in V \big| \prod_{i \in J} t_i = \prod_{i \in J} r_i\}$ is a face of V with $\check{F}_I = U$. ∎

4.3.2. Corollary. *If V is irreducible, then* face$(V) \xrightarrow{\sim}$ face$(\check{V})$. ∎

The subset of one point faces (the *vertices*) of V is denoted by $\mathrm{ver}(V)$.

4.3.3. Proposition. *(i) The set* $\mathrm{ver}(V)$ *is the* Shilov boundary *of A_V, i.e. a unique minimal subset of V at which each function from A_V achieves its maximum;*

(ii) if $\overline{\check{V}_{I_1}}, \dots, \overline{\check{V}_{I_m}}$ are the irreducible components of V, then

$$\mathrm{ver}(V) = \bigcup_{j=1}^{m} \left(\mathrm{ver}(\check{V}_{I_j}) \backslash \bigcup_{k \neq j} \tau_{I_j}(\check{V}_{I_k}) \right) .$$

Proof. Let Γ denote the set on the right hand side of (ii). First of all, *we claim that for every point $x_0 \in \Gamma$ there is a function $f \in A_V$ that achieves its maximal precisely at x_0.* Indeed, suppose $x_0 \in \mathrm{ver}(\check{V}_{I_j})$. Then there exists a function $f \in A_V$ in the variables t_i for $i \in I_j$ such that $f(x) \le r$ for all $x \in \check{V}_{I_j}$ and some $r > 0$ and $\{x_0\} = \{x \in \check{V}_{I_j} \big| f(x) = r\}$. Furthermore, if $x \in \overline{\check{V}_{I_j}} \backslash \check{V}_{I_j}$, then $x = \tau_I(y)$ for some proper subset $I \subset I_j$ and a point $y \in \check{V}_{I_j}$, and there exists a point $z \in \mathrm{rec}(\check{V}_{I_j})$ with $z_i = 1$ for $i \in I$ and $z_i < 1$ for $i \notin I$. It follows that $f(x) = \lim\limits_{s \to \infty} f(yz^s) < r$. Finally, if $x \in V \backslash \overline{\check{V}_{I_j}}$, then $x \in \check{V}_{I_k}$ for some $k \neq j$ and, therefore, $f(x) = f(\tau_{I_j}(x)) < r$. Thus, the claim follows and, in particular, $\Gamma \subset \mathrm{ver}(V)$. To prove the lemma, it suffices to show that every nonzero function $f \in A_V$ achieves its maximum at Γ. It is clear that it achieves its maximum at a point $x \in \mathrm{ver}(\check{V}_{I_j})$ for some $1 \le j \le m$. If $x \in \tau_{I_j}(\check{V}_{I_k})$ for some $k \neq j$, then $f(y) = f(x)$ for every point $y \in \check{V}_{I_k}$ with $\tau_{I_J}(y) = x$. It follows that f achieves its maximum at a point from $\mathrm{ver}(\check{V}_{I_k})$. Since $I_j \subset I_k$, we can continue this process and find a point from Γ at which f achieves its maximum. ∎

4.3.4. Corollary. *An R-affinoid polytope V is strictly R-affinoid if and only if all vertices of V have coordinates in $\sqrt{R}$.* ∎

4.3.5. Proposition. *If F is a face of V, then any face of F is a face of V.*

4.3.6. Lemma. *Let $F = V_g$ be a face of V. Then for every function $f \in A_V$ with $f\big|_F \neq 0$ there exists $k_0 \ge 1$ such that $V_{fg^k} \subset F$ for all $k > k_0$.*

Proof. Let $y_1, \dots, y_m$ be the vertices of V outside F. Then $g(y_i) < \rho(g)$ for all $1 \le i \le m$. It follows that there exists $k_0 \ge 1$ such that, for every $k \ge k_0$, one has

$$(fg^k)(y_i) = f(y_i) \left(\frac{g(y_i)}{\rho(g)} \right)^k \rho(g)^k < \rho_F(f)\rho(g)^k = \rho_F(fg^k) ,$$

where $\rho_F(f) = \max\limits_{x \in F} f(x)$. Since the function fg^k takes its maximum at $\mathrm{ver}(V)$, it is achieved at $\mathrm{ver}(V) \cap F$ and, in particular, the right hand side of the above inequality is equal to $\rho(fg^k)$. *We claim that $V_{fg^k} \subset F$ for all $k > k_0$.* Indeed, if $x \notin F$, then $(fg^{k-1})(x) \le \rho(fg^{k-1}) = \rho_F(fg^{k-1})$ and, therefore, $(fg^k)(x) < \rho(fg^k)$, which implies the claim. ∎

Proof of Proposition 4.3.5. Let U be a face of F, and let R' be the $\mathbf{F}_1$-subfield of $\mathbf{R}_+$ generated by R and $\rho(g)$. Then there exists a function $f' \in A_{F/R'}$ with $U = \{x \in F \big| f'(x) = \rho_F(f')\}$. The function f' is the restriction of a function from $A_{V/R'}$. The latter has the form rf for some $r \in R'$ and $f \in A_V = A_{V/R}$ and, therefore, one has $U = \{x \in F \big| f(x) = \rho_F(f)\}$. *We claim that $U = V_{fg^k}$ for all $k > k_0$, where k_0 is the number provided for the function f, by Lemma 4.3.6.* Indeed, the inclusions $U \subset V_{fg^k} \subset F$ are clear. If $x \in F \backslash U$, then $f(x) < \rho(f)$ and $g(x) = \rho(g)$ and, therefore, $x \notin V_{fg^k}$. ∎

The *cell of a face* F is the subset $\mathring{F}$ consisting of the points $x \in F$ for which F is the minimal face that contains x, i.e., $\mathring{F}$ is the complement of the union of all strictly smaller faces. It is an open subset of F. A *cell of an R-affinoid polytope* V is a cell of a face of V. Notice that V is a finite disjoint union of all of its cells.

4.3.7. Proposition. *The cell $\mathring{F}$ of a face F is always nonempty and connected, and it lies in the minimal connected component of F.*

4.3.7. Lemma. *(i) If $\mathring{F} \cap \check{V}_I \neq \emptyset$, then $\mathring{F} \cap \check{V}_I$ is the cell of the face $\check{F}_I$ of $\check{V}_I$;*

(ii) if C is a cell of $\check{V}_I$ and F is the minimal face of V that contains it, then $C = \mathring{F} \cap \check{V}_I$.

Proof. (i) Suppose a point $x \in \mathring{F} \cap \check{V}_I$ lies in the cell $\mathring{U}$ of a face U of $\check{V}_I$. Since $\check{F}_I$ is a face of $\check{V}_I$ that contains the point x, it follows that $U \subset \check{F}_I$. By Lemma 4.3.1(ii), there exists a face F' of V with $U = \check{F}'_I$, and we may assume that F' is the minimal face of V with the latter property. In particular, $F' \subset F$. On the other hand, since F is the minimal face of V that contains the point x, it follows that $F \subset F'$, i.e., $F = F'$ and $U = \check{F}_I$.

(ii) Since $\check{F}_I$ is a face of $\check{V}_I$, it follows that, for any point $x \in C$, F is the minimal face of V that contains x, i.e., $C \subset \mathring{F} \cap \check{V}_I$. It follows also that $C = \mathring{U}$ for $U = \check{F}_I$. The required equality now follows from (i). ∎

Proof of Proposition 4.3.7. Proposition 4.3.5 reduces the situation to the case $F = V$. Let I be the minimal element of $\mathcal{I}(V)$ (with respect to the inclusion relation), and let x be a point from the cell of the maximal face of $V_I = \check{V}_I$ (which is V_I itself). *We claim that any face F of V that contains the point x coincides with V.* Indeed, F must contain V_I. But $F = \{y \in V \big| f(y) = \rho(f)\}$ for some nonzero function $f \in A_V$. Since $V_I \subset F$, it follows that f is expressed in the variables t_i for $i \in I$. Then for any point $y \in V$ one has $f(y) = f(\tau_I(y)) = \rho(f)$, i.e., the function f is a constant and $F = V$. Thus, $\mathring{V}$ is nonempty. that it is contained in the minimal connected component of V follows from Proposition 4.2.5.

Suppose now that the cell $\mathring{V}$ is not connected, and let J be a minimal subset of $\{1, \ldots, n\}$ with the property that the intersection $\mathring{V} \cap \check{V}_J$ is nonempty and contained in a connected component of $\mathring{V}$ different from that of $\check{V}_I$. Since $\mathring{V}$ is contained in the minimal connected component of V, it follows that $\check{V}_J$ is not compact and, in particular, there exists a point $z \in \mathrm{rec}(\check{V}_J)$ with $z_i = 1$ for $i \in L$ and $z_i < 1$ for $i \in J \backslash L$, where L is a proper subset of J that contains I. If $y \in \check{V}_J$, then for any function $f \in A_V$ and any $s \geq 0$ one has $f(yz^s) \leq f(y)$. If, in addition, y lies in $\mathring{V} \cap \check{V}_J$, then $f(y) < \rho(f)$ for any nonconstant function f on V and, therefore, all points yz^s and their limit $\tau_L(y)$ lie in $\mathring{V}$. the latter contradicts the assumption on the minimality of J. ∎

Thus, if $\mathrm{cell}(V)$ denotes the set of cells, then the correspondence $F \mapsto \mathring{F}$ gives rise to a bijection $\mathrm{face}(V) \xrightarrow{\sim} \mathrm{cell}(V)$. The inclusion partial ordering on the former set defines a partial ordering on the latter set. Notice that $\mathrm{ver}(V)$ is precisely the subset of minimal elements of $\mathrm{cell}(V)$, but a one point cell is not necessarily a vertex. If V is irreducible, then $\mathrm{cell}(V) \xrightarrow{\sim} \mathrm{cell}(\check{V})$.

4.4. A property of R-affinoid polytopes. For an R-affinoid polytope $V \subset \mathbf{R}_+^n$, we introduce a partial ordering on the algebra A_V as follows: $f \leq g$ if $f(x) \leq g(x)$ for all $x \in V$. Notice that, if $f \leq g$, then $fh \leq gh$ for any function $g \in A_V$.

4.4.1. Proposition. *Any set of functions in A_V, which tends to zero with respect to the filter of complements of finite subsets, has a finite number of maximal elements (with respect to $\leq$).*

Notice that any such set is at most countable.

Proof. Let $\mathcal{F} = \{f_k\}_{k \geq 1}$ be such a set, and consider an admissible epimorphism

$$R\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to A_V \ .$$

We can find a map σ in the opposite direction which is a section of φ and possesses the property that there exists a constant $C > 0$ such that $\rho(\sigma(f)) \leq C\rho(f)$ for all $f \in A_V$. It follows that the functions $\{\sigma(f_k)\}_{k \geq 1}$ tend to zero as $k \to \infty$. Thus, it suffices to verify the required fact for the R-affinoid polytope $V = [0, r_1] \times \ldots \times [0, r_n]$ with $A_V = R\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$. Notice that, for $f = at^\mu \in A_V$, one has $\rho(f) = ar^\mu$. If, for $\mu \in \mathbf{Z}_+^n$, the set contains a nonzero function of the form at^μ, there is such a function with the maximal coefficient a, and we can remove from $\mathcal{F}$ all other functions of the same form. We may therefore assume that, if $f_k = at^\mu$ and $f_l = bt^\mu$, then $k = l$. We may also assume that $\mathcal{F}$ does not contain constants. Furthermore, let $f_k = a_k t^{\mu^{(k)}}$, and fix a monomial order on the set of monomials in $T_1, \ldots, T_n$ as in §I.1.4. After a permutation of the set $\mathcal{F}$, we may assume that $T^{\mu^{(1)}} < T^{\mu^{(2)}} < \ldots$.

Let l be maximal with the property that $\rho(f_l) \geq \rho(f_k)$ for all $k \geq 1$. *We claim that, for every $k \neq l$ with $\mu_i^{(k)} \geq \mu_i^{(l)}$ for all $1 \leq i \leq n$, one has $f_k \leq f_l$.* Indeed, the assumption implies that $T^{\mu^{(l)}} < T^{\mu^{(k)}}$, i.e., $k > l$ and, therefore, $a_l r^{\mu^{(l)}} > a_k r^{\mu^{(k)}}$. It follows that $a_l x^{\mu^{(l)}} \geq a_k x^{\mu^{(k)}}$ for all $x \in [0, r_1] \times \ldots \times [0, r_n]$, and the claim follows. Notice that the claim immediately implies the required fact for $n = 1$.

Assume that $n \geq 2$ and the required fact is true for $n-1$. By the above claim we already know that $f_k \leq f_l$ for all $k \neq l$ with $\mu_i^{(k)} \geq \mu_i^{(l)}$ for $1 \leq i \leq n$. Given $1 \leq i \leq n$ and $0 \leq j \leq \mu_i^{(l)} - 1$, let $\mathcal{F}_{i,j}$ denote the set of all functions f in the variables $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ for which $f t_i^j \in \mathcal{F}$. The functions from $\mathcal{F}_{i,j}$ tend to zero with respect to the filter of complements of finite subsets and, by induction, the set $m(\mathcal{F}_{i,j})$ of maximal elements of $\mathcal{F}_{i,j}$ is finite. It follows that the set of maximal elements of $\mathcal{F}$ is contained in $\{f_l\} \cup \bigcup m(\mathcal{F}_{i,j}) t_i^j$, where the second union is taken over $1 \leq i \leq n$ and $0 \leq j \leq \mu_i^{(l)} - 1$, and, therefore, it is finite. ∎

§5. Further properties of K-affinoid algebras

Let K be a valuation $\mathbf{F}_1$-field, and A a (strictly) K-affinoid algebra. Any admissible epimorphism $K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to A$ gives rise to a homeomorphism between the spectrum $X = \mathcal{M}(A)$ of A and a (strictly) $|K|$-affinoid polytope in $\mathbf{R}_+^n$. In this section we deduce properties of K-affinoid algebras from those of $|K|$-affinoid polytopes.

5.1. The $|K|$-polytopal algebra associated to a K-affinoid algebra. Let A be a (strictly) K-affinoid algebra.

5.1.1. Proposition. *(i) $\widehat{A}$ is a (strictly) $|K|$-polytopal algebra;*

(ii) there is an isomorphism of partially ordered sets $\mathrm{Zspec}(\widehat{A}) \xrightarrow{\sim} \mathrm{Zspec}(A) : \widehat{\mathfrak{p}} \mapsto \mathfrak{p}$*;*

(iii) for every Zariski prime ideal $\mathfrak{p} \subset A$, one has $\widehat{A/\mathfrak{p}} \xrightarrow{\sim} \widehat{A}/\widehat{\mathfrak{p}}$.

Proof. The above admissible epimorphism gives rise to a homeomorphism between $\mathcal{M}(A)$ and a $|K|$-affinoid polytope V in $\mathbf{R}_+^n$ and a bounded bijective homomorphism $\widehat{A} \to A_{V/|K|}$. Since it is isometric with respect to the spectral norm, it is an isomorphism, by Corollary 2.4.2. The statement (ii) follows from the facts that the maps $\mathcal{M}(A) \to \mathrm{Zspec}(A)$ and $\mathcal{M}(\widehat{A}) \to \mathrm{Zspec}(\widehat{A})$ are surjective (Corollary 2.2.4) and $\mathcal{M}(\widehat{A}) \xrightarrow{\sim} \mathcal{M}(A)$. The statement (iii) is trivial. ∎

5.1.2. Corollary. *Let R be an $\mathbf{F}_1$-subfield of $\mathbf{R}_+$. An R-affinoid algebra A is R-polytopal if and only if $A \xrightarrow{\sim} \widehat{A}$.* ∎

5.1.3. Corollary. *In the situation of Proposition 5.1.1, the following properties in (i) and (ii) are equivalent:*

(i) (a) $\widehat{A}$ has no zero divisors;

(b) $A/\mathrm{z}\mathbf{n}(A)$ has no zero divisors.

(ii) (a) $\widehat{A}$ is an $\mathbf{F}_1$-field;

(b) $A/\mathrm{z}\mathbf{n}(A)$ is an $\mathbf{F}_1$-field;

(c) A is a local artinian $\mathbf{F}_1$-algebra. ∎

We say that the space $X = \mathcal{M}(A)$ is *irreducible* (resp. *quasi-irreducible*) if it possesses this property as a $|K|$-affinoid polytope or, equivalently, the $|K|$-polytopal algebra is irreducible (resp. quasi-irreducible). The set of irreducible components of X will be denoted by $\mathrm{Irr}(X)$.

For a finitely generated K-algebra B, we set $\mathcal{Y} = \mathrm{Spec}(B)$ and denote by $\mathcal{Y}^{\mathrm{an}}$ the set of all homomorphisms of $\mathbf{F}_1$-algebras $|\ | : B \to \mathbf{R}_+$ that extend the valuation on K. We provide $\mathcal{Y}^{\mathrm{an}}$ with the weakest topology with respect to which all functions $\mathcal{Y}^{\mathrm{an}} \to \mathbf{R}_+$ of the form $|\ | \mapsto |f|$ with $f \in B$ are continuous. Any system of generators $f_1, \ldots, f_m$ of B over K gives rise to a continuous map $\mathcal{Y}^{\mathrm{an}} \to \mathbf{R}_+^m$ that identifies $\mathcal{Y}^{\mathrm{an}}$ with an $|K|$-affine subspace of $\mathbf{R}_+^m$. Notice that there is a continuous map $\mathcal{Y}^{\mathrm{an}} \to \mathcal{Y} : y \mapsto \mathbf{y}$ that takes a point $y \in \mathcal{Y}^{\mathrm{an}}$, that corresponds to a homomorphism $|\ |_y : B \to \mathbf{R}_+$, to the point $\mathbf{y} \in \mathcal{Y}$, that corresponds to the prime ideal $\mathrm{Ker}(|\ |_y)$.

For example, if $\mathcal{Y} = \mathrm{Spec}(\widehat{A})$, then the above admissible epimorphism gives rise to a homeomorphism between $\mathcal{Y}^{\mathrm{an}}$ and the $|K|$-affine subspace of $\mathbf{R}_+^n$ generated by the associated $|K|$-affinoid polytope X (see the end of §4.1). The $|K|$-affine subspace $\mathcal{X}^{\mathrm{an}}$ for $\mathcal{X} = \mathrm{Spec}(A)$ may be bigger (see Remark 5.1.6(i)).

5.1.4. Proposition. *The canonical maps $X \to \mathcal{Y}^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ give rise to isomorphisms of partially ordered sets $\check{I}_A \xrightarrow{\sim} \pi_0(\mathcal{X}) \xrightarrow{\sim} \pi_0(\mathcal{X}^{\mathrm{an}}) \xrightarrow{\sim} \pi_0(\mathcal{Y}^{\mathrm{an}}) \xrightarrow{\sim} \pi_0(X)$.*

Proof. The statement follows from Propositions I.3.5.1 and 4.2.5. ∎

For a Zariski prime ideal $\mathfrak{p} \subset A$, we set $X_{\mathfrak{p}} = \{x \in X \big| f(x) = 0 \text{ for all } f \in \mathfrak{p}\}$, $\check{X}_{\mathfrak{p}} = \{x \in X_{\mathfrak{p}} \big| f(x) \neq 0 \text{ for all } f \notin \mathfrak{p}\}$, and $X^{(\mathfrak{p})} = \overline{\check{X}_{\mathfrak{p}}}$.

5.1.5. Proposition. *(i) There is an isomorphism of $|K|$-polytopal algebras $\widehat{A/\Pi_{\mathfrak{p}}} \xrightarrow{\sim} \widehat{A}/\Pi_{\widehat{\mathfrak{p}}}$ (see Proposition 5.1.1(ii));*

(ii) $\mathcal{M}(A/\Pi_{\mathfrak{p}}) \xrightarrow{\sim} X^{(\mathfrak{p})}$.

Proof. By Corollary 2.5.4, the $|K|$-polytopal algebra $\widehat{A/\Pi_{\mathfrak{p}}}$ is an integral domain. It follows that its spectrum, which coincides with that of $A/\Pi_{\mathfrak{p}}$, is irreducible. Since it contains $\check{X}_{\mathfrak{p}}$ and is contained in $X_{\mathfrak{p}}$, we get (ii). Furthermore, by Proposition 4.2.7(i), the spectrum of the $|K|$-polytopal algebra $\widehat{A}/\Pi_{\widehat{\mathfrak{p}}}$ also coincides with $\overline{\check{X}_{\mathfrak{p}}}$. It follows that the homomorphism of (ii) is bijective, and the required isomorphism follows. ∎

5.1.6. Remark. (i) Let $A = \mathbf{F}_1\{T, T^{-1}\}$. Then $X = \{(1,0)\} \in \mathbf{R}_+^2$ and $\mathcal{X}^{\rm an}$ is the affine line $\{(1,t)\big|0 \le t < \infty\} \subset \mathbf{R}_+^2$. Notice also that $\widehat{A} = \mathbf{F}_1$, i.e., for $\mathcal{Y} = \mathrm{Spec}(\widehat{A})$ one has $\mathcal{Y}^{\rm an} = \{(1,0)\} = X$.

(ii) If, for two Zariski prime ideals $\mathfrak{p}, \mathfrak{q} \subset A$, $\Pi_{\mathfrak{q}} \subset \Pi_{\mathfrak{p}}$ then, of course, $\check{X}_{\mathfrak{p}} \subset \overline{\check{X}_{\mathfrak{q}}}$, but the converse implication is not true in general. Indeed, let A be the $\mathbf{F}_1$-affinoid algebra which is the quotient of $\mathbf{F}_1\{T_1, T_1^{-1}, T_2\}$ by the ideal generated by the pair (T_1T_2, T_2), and let f and g be the images of T_1 and T_2 in A. The spectrum $X = \mathcal{M}(A)$ is naturally identified with the interval $\{(1,t)\big|0 \le t \le 1\} \subset \mathbf{R}_+^2$. The only Zariski prime ideals of A are the zero ideal 0 and the maximal ideal $\mathbf{m} = A \backslash A^*$. (Notice that A^* is the cyclic group generated by the element f.) One has $\check{X}_0 = \{(1,t)\big|0 < t \le 1\}$ and $\check{X}_{\mathbf{m}} = X_{\mathbf{m}} = \{(1,0)\}$. In particular, $\check{X}_{\mathbf{m}} \subset \overline{\check{X}_0}$. On the other hand, the prime ideal $\Pi_{\mathbf{m}} = \Delta(A) \cup (\mathbf{m} \times \mathbf{m})$ does not contain $\Pi_0 = \Delta(A) \cup \{(f^m, f^n)\}_{m,n \in \mathbf{Z}}$. By the way, in this example $\widehat{A} \stackrel{\sim}{\to} \mathbf{F}_1[T_2]$ is an integral domain, but A is not irreducible (see Corollary 2.5.4). One also has $\mathcal{X}^{\rm an} = \{(1,t)\big|0 \le t \le \infty\} \cup \{(t,0)\big|0 < t < \infty\}$.

5.2. Finite Banach modules over a K-affinoid algebra. Let A be a K-affinoid algebra, and let M be a finite Banach A-module.

5.2.1. Theorem. *Given elements $f \in A$ and $m \in M$, one of the following is true:*

(1) $f^k m = 0$ for some $k \ge 1$ (i.e., $f \in \mathrm{z}\mathbf{r}(0:m)$);

(2) there exist a unique positive number r such that, for some positive constants $C' < C''$ and for all $k \ge 1$, one has $C' r^k \le ||f^k m|| \le C'' r^k$.

Furthermore, if A is strictly K-affinoid, the number r from (2) belongs to $\sqrt{|K^*|}$.

5.2.2. Definition. The *spectral radius $\rho_m(f)$ of an element $f \in A$ with respect to an element $m \in M$* is zero in the case (1) and the number r in the case (2).

5.2.3. Lemma. *The finite Banach A-module M has a finite chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \ldots \subset N_k = M$ which are finite Banach A-modules such that each quotient N_i/N_{i-1} is isomorphic to a Banach A-module of the form A/Π, where Π is a closed prime ideal of A.*

Proof. By Proposition I.2.7.1(iii), if we disregard the Banach structure, such a chain exists. Then each Zariski A-submodule is finitely generated and, by Proposition 2.2.8, it is a finite Banach A-module. By the proof of Corollary 2.4.3, the Zariski A-submodule N_1 is generated by an element

$m \in M$ such that $\Pi = \mathrm{ann}(m)$ is a prime ideal of A. The homomorphism of A-modules $A \to M$: $f \mapsto fm$ is bounded and, therefore, the prime ideal Π which is its kernel Π is closed. That the bijection $A/\Pi \to N_1$ is an isomorphism of Banach A-modules follows from Proposition 2.2.9. ∎

5.2.4. Lemma. *Assume that the K-affinoid algebra A is integral. Then for any pair of nonzero elements $f, g \in A$ there exist positive constants $C' < C''$ such that $C'\rho(f)^k \leq ||f^k g|| \leq C''\rho(f)^k$ for all $k \geq 1$ (i.e., $\rho_g(f) = \rho(f)$).*

Proof. By Proposition 2.4.1, we may assume that the Banach norm on A coincides with the spectral norm. Then $||f^k g|| = \rho(f^k g) \leq \rho(g)\rho(f)^k$, and so it suffices to prove the existence of $C > 0$ with $\rho(f^k g) \geq C\rho(f)^k$ for all $k \geq 1$. Let $x_1, \ldots, x_n$ be the points of the Shilov boundary of A. Since A is integral, every nonzero element of A has a nonzero value at every point x_i (see Proposition 4.3.2) and, therefore, there is a positive constant C with $|g(x_i)| \geq C$ for all $1 \leq i \leq n$. It follows that $\rho(f^k g) = \max\limits_{1 \leq i \leq n} |(f^k g)(x_i)| \geq C \max\limits_{1 \leq i \leq n} |f(x_i)|^k = C\rho(f)^k$. ∎

Proof of Proposition 5.2.1. Suppose that $f \notin \mathrm{z}\mathbf{r}(0 : m)$, and consider a chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \ldots \subset N_k = M$ provided by Lemma 5.2.3. Then there are $i, l \geq 1$ such that $f^j m \in N_i \backslash N_{i-1}$ for all $j \geq l$. Let n be an element from $N_i \backslash N_{i-1}$ such that $A/\Pi \xrightarrow{\sim} N_i/N_{i-1}$: $g \mapsto gn$. In particular, since the norm on $B = A/\Pi$ is equivalent to the spectral norm, there are positive constants $C' < C''$ such that $C'\rho_B(g) \leq ||gn|| \leq C''\rho_B(g)$ for all $g \in A$ with $(g, 0) \notin \Pi$. If $f^l m = gn$, then $f^j m = f^{j-l} gn$ for all $j \geq l$. It follows that $C'\rho_B(f^{j-l} g) \leq ||f^j m|| \leq C''\rho_B(f^{j-l} g)$ for all $j \geq l$, and the required fact follows from Lemma 5.2.4. As for the last statement, it suffices to notice that, if A is strictly K-affinoid, then so is the quotient of A by any closed ideal. ∎

5.3. The reduction of K-affinoid algebras. Let A be a Banach $\mathbf{F}_1$-algebra. For $r \in \mathbf{R}_+^*$, we set $\widetilde{A}_r = \{f \in A \big| \rho(f) = r\} \cup \{0\}$ and, for $r, s \in \mathbf{R}_+^*$ we define as follows a map $m : \widetilde{A}_r \times \widetilde{A}_s \subset \widetilde{A}_{rs}$: for $f \in \widetilde{A}_r$ and $g \in \widetilde{A}_s$, $m(f, g) = fg$, if $\rho(fg) = rs$, and $m(f, g) = 0$, otherwise. Then the direct sum $\widetilde{A} = \oplus_{r \in \mathbf{R}_+^*} \widetilde{A}_r$ is an $\mathbf{F}_1$-algebra, called the *reduction* of A. Notice that $\widetilde{A}_1$ is an $\mathbf{F}_1$-subalgebra of $\widetilde{A}$. Notice also that $\widetilde{A}$ is Zariski reduced and that there is a canonical isomorphism $\widetilde{A/\mathbf{n}(A)} \xrightarrow{\sim} \widetilde{A}/\mathbf{n}(\widetilde{A})$. If A is quasi-integral, then $\widetilde{A}$ is reduced.

There is a map $A \to \widetilde{A} : f \mapsto \widetilde{f}$ that takes an element $f \in A$ to zero, if $\rho(f) = 0$, and to the corresponding element of $\widetilde{A}_r$, if $r = \rho(f) > 0$. If the equality $\rho(fg) = 0$ implies that either $f = 0$ or $g = 0$, then the above map is an isomorphism of $\mathbf{F}_1$-algebras. For example, this is so for any finite idempotent $\mathbf{F}_1$-algebra A, and for any Banach $\mathbf{F}_1$-algebra whose norm is multiplicative. The correspondences $A \mapsto \widetilde{A}$ and $A \mapsto \widetilde{A}_1$ are functorial on A.

There are canonical maps $\pi : \mathcal{M}(A) \to \mathrm{Zspec}(\widetilde{A})$ and $\pi_1 : \mathcal{M}(A) \to \mathrm{Zspec}(\widetilde{A}_1)$, called the *reduction maps*, that take a point $x \in \mathcal{M}(A)$ to the Zariski kernels of the induced homomorphisms $\widetilde{A} \to \widetilde{\mathcal{H}(x)} = \mathcal{H}(x)$ and $\widetilde{A}_1 \to \widetilde{\mathcal{H}(x)}_1$, respectively. The latter coincide with the Zariski kernels of the induced homomorphisms $\widetilde{A} \to \widetilde{\mathbf{R}}_+ = \mathbf{R}_+$ and $\widetilde{A} \to (\widetilde{\mathbf{R}}_+)_1 = \{0, 1\}$, respectively. For example, the set of nonzero elements of the Zariski prime ideal $\pi(x)$ (resp. $\pi_1(x)$) consists of elements $\widetilde{f} \in \widetilde{A}$ with $|f(x)| < \rho(f)$ (resp. $|f(x)| < \rho(f) = 1$). Notice, that the map π_1 is the composition of the map π with the canonical projection $\mathrm{Zspec}(\widetilde{A}) \to \mathrm{Zspec}(\widetilde{A}_1)$.

Let now K be a valuation $\mathbf{F}_1$-field. As we already mentioned, one has $K \stackrel{\sim}{\to} \widetilde{K}$. One also has $\widetilde{K}_1 = K^1 \cup \{0\}$. Applying the above construction to a K-affinoid algebra A, we get a K-algebra $\widetilde{A}$ and a $\widetilde{K}_1$-algebra $\widetilde{A}_1$. Notice that there is a canonical injective homomorphism of K-algebras $\widetilde{A}_1 \otimes_{\widetilde{K}_1} K \to \widetilde{A}$, which is bijective if $\rho(A) \subset |K|$. Notice also that, if $A = A' \otimes_{K'} K$ as in Proposition 2.1.3(v), then $\widetilde{A} = \widetilde{A}' \otimes_{K'} K$.

5.3.1. Proposition. *The K-algebra $\widetilde{A}$ is finitely presented.*

Proof. By the above remark, it suffices to show that $\widetilde{A}$ is finitely generated. Let us fix an admissible epimorphism $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$. It gives rise to a homeomorphism of the spectrum $\mathcal{M}(A)$ with an $|K|$-affinoid polytope V in $\mathbf{R}_+^n$. For a monomial $F \in \mathbf{F}_1[T_1, \dots, T_n]$ whose image f in A is not identically zero at V, let $\sigma(F)$ denote the set of all vertices $x \in \mathrm{ver}(V)$ with $f(x) = \rho(f)$. Notice that, for such elements F and G, one has $\rho(fg) = \rho(f)\rho(g)$ if and only if $\sigma(F) \cap \sigma(G) \neq \emptyset$, and in this case the intersection coincides with $\sigma(FG)$ (here g is the image of G in A). Notice also that the number of decompositions of a nonconstant monomial $F = T_1^{\mu_1} \cdot \ldots \cdot T_n^{\mu_n}$ as an ordered product of nonconstant monomials $G \cdot H$ is equal to $\lambda(F) = (\mu_1 + 1) \cdot \ldots \cdot (\mu_n + 1) - 2$. Let N be the number of pairs (σ_1, σ_2) of nonempty subsets of $\mathrm{ver}(V)$ with empty intersection, and let Σ be the finite set of nonzero elements of A which are the images to V of nonconstant monomials F with $\lambda(F) \leq N$. *We claim that $\widetilde{A}$ is generated over K by the elements $\widetilde{f}$ for $f \in \Sigma$.* Indeed, let f be a nonzero element of A which is, up to an element of K^*, the image of a monomial $F = T_1^{\mu_1} \cdot \ldots \cdot T_n^{\mu_n}$, and assume that $\lambda(F) > N$. Then there is a decomposition of F as a product $G \cdot H$ of two monomials with $\sigma(G) \cap \sigma(H) \neq \emptyset$. It follows that for the corresponding elements $g, h \in A$ one has $f = gh$ and $\rho(f) = \rho(g)\rho(h)$, i.e., $\widetilde{f} = \widetilde{g}\widetilde{h}$ in $\widetilde{A}$. It remains to notice that $\lambda(G), \lambda(H) < \lambda(F)$. ∎

5.3.2. Corollary. *If A is strictly K-affinoid, then the monomorphism $\widetilde{A}_1 \otimes_{\widetilde{K}_1} K \to \widetilde{A}$ is finite.*

Proof. By Corollary 4.1.4(ii), there exists $m \geq 1$ such that $\rho(f)^m \in |K|$ for all $f \in A$. This implies that the homomorphism considered is integral. It is then finite because the K-algebra $\widetilde{A}$ is finitely generated. ∎

Let $X = \mathcal{M}(A)$, and consider the reduction map $\pi : X \to \mathrm{Zspec}(\widetilde{A})$. For points $x, y \in X$, one has $\pi(x) = \pi(y)$ if and only if, for any element $f \in A$, the inequality $|f(x)| < \rho(f)$ is equivalent to the inequality $|f(y)| < \rho(f)$. It follows that, if the preimage of a Zariski prime ideal of $\widetilde{A}$ is nonempty, it is a cell of X.

5.3.3. Proposition. *The reduction map induces a bijection* $\mathrm{cell}(X) \widetilde{\to} \mathrm{Zspec}(\widetilde{A})$ *which reverses the partial ordering on both sets.*

For example, the preimage of the maximal Zariski ideal of $\widetilde{A}$ is a unique maximal cell of X, and the preimages of the minimal Zariski ideals of $\widetilde{A}$ are the vertices of X.

Let $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$ be a quasi-integral twisted datum of K-affinoid algebras that represents A and, in particular, $A \widetilde{\to} \prod_I^\nu A_i$. For every pair $i \le j$ in I, the quasi-homomorphism ν_{ij} induces a homomorphism of reduced K-algebras $\widetilde{\nu}_{ij} : \widetilde{A}_i \to \widetilde{A}_j$. Let $\widetilde{\mathbf{a}}_{ji}$ be the image of the Zariski ideal $\mathbf{a}_{ji}$ under the map $A_j \to \widetilde{A}_j : f \mapsto \widetilde{f}$. It is a Zariski ideal of $\widetilde{A}_j$. The following statement is easily verified.

5.3.4. Lemma. $\{I, \widetilde{A}_i, \widetilde{\nu}_{ij}, \widetilde{\mathbf{a}}_{ji}\}$ *is a twisted datum, and there is an isomorphism of K-algebras* $\widetilde{A}/\mathbf{n}(\widetilde{A}) \widetilde{\to} \prod_I^\nu \widetilde{A}_i$. ∎

Proof of Proposition 5.3.3. It suffices to show that the map $\mathrm{cell}(X) \to \mathrm{Zspec}(\widetilde{A})$ is surjective. By Lemma 5.3.4 and Corollary I.3.3.3, the situation is easily reduced to the case when A is integral. To prove the required fact, we may also increase the $\mathbf{F}_1$-field K and assume that $\rho(A) \subset |K|$ and, in particular, that A is strictly K-affinoid. Let $\mathfrak{p}$ is a Zariski prime ideal of $\widetilde{A}$. Since $\widetilde{A}$ is finitely generated over K, there exists an element $f \in A$ such that $\mathfrak{p}$ is the maximal Zariski ideal of $\widetilde{A}$ that does not contain the element $\widetilde{f}$. Multiplying f by an element of K^*, we may assume that $\rho(f) = 1$. *We claim that* $\pi^{-1}(\mathfrak{p}) = \overset{\circ}{V}$ *for the face* $V = X_f$. Indeed, V is the rational domain $\{x \in X \big| |f(x)| \ge 1\}$. As a K-algebra, A_V coincides with the localization A_f.

5.3.5. Lemma. *In the above situation, the following is true:*

(i) the ideal E of $A\{T\}$ generated by the pair $(fT, 1)$ is closed and, in particular, the K-affinoid algebra $B = A\{T\}/E$ is isomorphic (as a K-algebra) to the localization A_f;

(ii) the canonical homomorphism $A \to B$ gives rise to a homeomorphism $\mathcal{M}(B) \widetilde{\to} V$ and an isomorphism $\widetilde{A}_{\widetilde{f}} \widetilde{\to} \widetilde{B}$.

Proof. (i) Notice that the homomorphism $A_f \to A\{T\} : \frac{a}{f^m} \mapsto aT^m$ gives rise to an isomorphism of K-algebras $A_f \stackrel{\sim}{\to} A\{T\}/E$. Suppose that the quotient norm of an element $\frac{a}{f^m}$ is zero. By Example 1.1.3(ii), there exist sequences of elements $b_1, b_2, \ldots \in A$ and of positive integers $n_1, n_2, \ldots$ with $\frac{a}{f^m} = \frac{b_k}{f^{n_k}}$ and $||b_k|| \to 0$ as $k \to \infty$. Since A is integral, it follows that $||af^{n_k}|| \to 0$ as $k \to \infty$ and, therefore, the restriction of a to $V = X_f$ is zero and, in particular, a is zero at the vertices of X in X_f. This implies that $a = 0$.

(ii) The first statement is trivial, and so we have to verify bijectivity of the homomorphism $\widetilde{A}_{\widetilde{f}} \to \widetilde{B}$.

Injectivity. Suppose that the images of nonzero elements $\frac{\widetilde{g}}{\widetilde{f^m}}, \frac{\widetilde{h}}{\widetilde{f^n}} \in \widetilde{A}_{\widetilde{f}}$ in $\widetilde{B}$ coincide. Then the images of $\widetilde{g}\widetilde{f}^n$ and $\widetilde{h}\widetilde{f}^m$ in $\widetilde{B}$ coincide and, therefore, the images of the elements gf^n and hf^m in B coincide. Since $B = A_f$ and A is integral, it follows that $gf^n = hf^m$ and, therefore, $\frac{\widetilde{g}}{\widetilde{f^m}} = \frac{\widetilde{h}}{\widetilde{f^n}}$.

Surjectivity. Let $\widetilde{(\frac{g}{f^m})}$ be a nonzero element of $\widetilde{B}_r$. Then $\rho_V(g) = r$. By Lemma 4.3.6, there exists $k_0 \geq 1$ such that $X_{f^k g} \subset V = X_f$ for all $k > k_0$. It follows that $\rho_V(g) = \rho(f^k g)$ for all $k > k_0$ and, therefore, $\widetilde{(\frac{g}{f^m})}$ is the image of the element $\frac{\widetilde{f^k}\widetilde{g}}{\widetilde{f^{m+k}}}$ from $\widetilde{A}_{\widetilde{f}}$. ∎

By Lemma 5.3.5, we can replace X by V, and so we may assume that $\mathfrak{p}$ is the maximal Zariski ideal of $\widetilde{A}$. In this case $\pi^{-1}(\mathfrak{p}) = \mathring{X}$. Indeed, let $g \in A$ be such that $\widetilde{g}$ is nonzero and non-invertible in $\widetilde{A}$. If $|g(x)| = \rho(g)$ for some point $x \in \mathring{X}$, then the same equality holds for all points from X, i.e., $\widetilde{g}$ is invertible in $\widetilde{A}$, which is a contradiction. ∎

5.3.6. Corollary. *(i)* $\mathrm{Zspec}(\widetilde{\widetilde{A}}) \stackrel{\sim}{\to} \mathrm{Zspec}(\widetilde{A})$*;*

(ii) if A is strictly K-affinoid, then the reduction map $\pi_1 : \mathcal{M}(A) \to \mathrm{Zspec}(\widetilde{A}_1)$ is surjective. ∎

5.3.7. Corollary. *The following properties of non-nilpotent elements $f, g \in A$ are equivalent:*

(a) $X_f \subset X_g$;

(b) $f^n = gh$ for some $n \geq 1$ and $h \in A$ with $\rho(gh) = \rho(f)\rho(h)$.

Proof. (b)$\Longrightarrow$(a). If $x \in X_f$, then $\rho(f)^n = f(x)^n = g(x)h(x) \leq \rho(g)\rho(h) = \rho(gh) = \rho(f)^n$. It follows that $g(x) = \rho(g)$ and $h(x) = \rho(h)$ and, therefore, $x \in X_g$.

(a)$\Longrightarrow$(b). Let $\mathfrak{p}$ be the Zariski prime ideal of $\widetilde{A}$ whose preimage under the reduction map is $\mathring{X}_f$. From Proposition 5.3.1 it follows that there exist elements $u_1, \ldots, u_n \in \widetilde{A}\backslash\mathfrak{p}$ such that the monoid $\widetilde{A}\backslash\mathfrak{p}$ is generated by them and K^*. Notice that $\widetilde{f}, \widetilde{g} \in \widetilde{A}\backslash\mathfrak{p}$.

Step 1. *For every $1 \leq i \leq n$, there exists $m \geq 1$ such that $\widetilde{f}^m = u_i h$ with $h \in A\backslash\mathfrak{p}$.* Indeed, assume this is not true precisely for $k+1 \leq i \leq n$ with $k \leq n-1$, and let M be the submonoid of $\widetilde{A}\backslash\mathfrak{p}$ generated by $u_1, \ldots, u_k$ and K^*. Then $u_{k+1}, \ldots, u_n \notin M$ and, in particular, the Zariski prime ideal $\mathfrak{q} = \widetilde{A}\backslash M$ is strictly larger than $\mathfrak{p}$. Proposition 5.3.3 implies that, if h is the element of A with

$\widetilde{h} = u_1 \cdot \ldots \cdot u_k$, then the face X_h is strictly smaller that X_f and, in particular, $X_h \cap \mathring{X}_f = \emptyset$. Since $u_{k+1} \in \mathfrak{q}$, it follows that $v(x) < \rho(v)$ for any point $x \in \mathring{X}_f$ and the element $v \in A$ with $\widetilde{v} = u_{k+1}$. This contradicts the fact that the preimage of $\mathfrak{p}$ under the reduction map is $\mathring{X}_f$.

Step 2. *There exists $m \geq 1$ such that $\widetilde{f}^m = \lambda u^\mu$ for some $\lambda \in K^*$ and $\mu_1, \ldots, \mu_n \geq 1$.* Indeed, by Step 1, for every $1 \leq i \leq n$ there exists $m_i \geq 1$ such that $\widetilde{f}^{m_i} = u_i \cdot (\lambda_i u^\nu)$ for some $m_i \geq 1$, $\lambda_i \in K^*$ and $\nu \in \mathbf{Z}_+^n$. The product of these equalities gives the required claim.

Step 3. One has $\widetilde{g} = \alpha u^\nu$ for some $\alpha \in K^*$ and $\nu \in \mathbf{Z}_+^n$. Let $k \geq 1$ be such that $k\mu_i \geq \nu_i$ for all $1 \leq i \leq n$, where μ is from Step 2. Then we get $f^{km} = gh$, where $h = \lambda^k \alpha^{-1} u^{k\mu - \nu} \in A \backslash \mathfrak{p}$. It is clear that $\rho(gh) = \rho(g)\rho(h)$, and so (b) is true. ∎

5.3.8. Proposition. *Given a bounded homomorphism of K-affinoid algebras $\varphi : A \to B$, the following are equivalent:*

(a) B is a finite Banach A-algebra;

(b) B is integral over $\varphi(A)$;

(c) $\widetilde{B}$ is a finite $\widetilde{A}$-algebra;

(d) $\widetilde{B}$ is integral over $\widetilde{\varphi}(\widetilde{A})$.

Proof. The implications (a)$\Longrightarrow$(b) and (c)$\Longrightarrow$(d) are trivial, and (d)$\Longrightarrow$(c) follows from Corollary I.2.5.4 and the fact that $\widetilde{B}$ is finitely generated over K (Proposition 5.3.1).

(b)$\Longrightarrow$(a). The above argument shows that B is a finitely generated A-module.

To prove other implications, it suffices to consider the case when the group K^* is finitely generated and, therefore, we may assume that A and B are $\mathbf{F}_1$-affinoid algebras.

(a)$\Longrightarrow$(c). By Proposition 2.2.8, there is an isomorphism of Banach K-algebras $A/\mathrm{Ker}(\varphi) \xrightarrow{\sim} \varphi(A)$. Thus, to verify the property (c), it suffices to consider the following two cases: (1) φ is injective and the norm on A is induced from that of B, and (2) φ is an admissible epimorphism.

(1) Since φ is isometric with respect to the spectral norm, the induced map $\widetilde{A} \to \widetilde{B}$ is also injective. By Proposition I.2.6.1, all elements of B are integral over A, i.e., for every $g \in B$ there exist $m > n \geq 0$ and $f \in A$ with $g^m = fg^n$. *We claim that $\rho(fg) = \rho(f)\rho(g)$.* Indeed, if g is nilpotent, or $n = 0$, the claim is trivial, and so assume that $\rho(g) > 0$ and $n \geq 1$. Let x be a point from X_f. Then $g(x)^m = f(x)g(x)^n = \rho(f)g(x)^n$ and, therefore, $\rho(f) = g(x)^{m-n} \leq \rho(g)^{m-n}$. It follows that $\rho(g)^m \leq \rho(fg)\rho(g)^{n-1} \leq \rho(f)\rho(g)^n \leq \rho(g)^m$, i.e., the inequalities are in fact equalities and, in particular, we get the claim. The claim implies that $\widetilde{g}^m = \widetilde{f}\widetilde{g}^n$, i.e., all elements of $\widetilde{B}$ are integral over $\widetilde{A}$.

(2) First of all, we notice that it suffices to verify the required fact in the case when A and

B are reduced. Indeed, assume this is true. First of all, if $C = \mathbf{F}_1\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$ is an admissible epimorphism, then we may replace A by C. Furthermore, since $\widetilde{B}/\mathbf{n}(\widetilde{B}) \widetilde{\to} \widetilde{B/\mathbf{n}(B)}$, the assumption implies that for every element $g \in B$ there exist $m > n \geq 0$ and $f \in A$ with $(\widetilde{g}^m, \widetilde{\varphi}(\widetilde{f})\widetilde{g}^m) \in \mathbf{n}(\widetilde{B})$. In its turn, the latter implies that $\widetilde{g}^{mi} = \widetilde{\varphi}(\widetilde{f})^i\widetilde{g}^{ni}$ for large enough i, i.e., $\widetilde{g}$ is integral over $\widetilde{\varphi}(\widetilde{A})$. Thus, assume that A and B are reduced.

5.3.9. Lemma. *Let A be a reduced $\mathbf{F}_1$-affinoid algebra, and let k be a non-Archimedean field. Assume that the order of any torsion element of $\kappa(\mathfrak{p})^*$ for each minimal prime ideal $\Pi_\mathfrak{p}$ of A is prime to the characteristic of the residue field of k. Then for any element $F = \sum_{f \in \check{A}} \lambda_f f \in k\{A\}$ one has*

$$\rho(F) = \max_{f \in \check{A}} |\lambda_f|\rho(f) \ .$$

The statement is not true without the assumption that A is reduced. Indeed, if $(f, g) \in \mathbf{n}(A)$, then $f(x) = g(x)$ for all $x \in \mathcal{M}(A)$ and, therefore, $(f - g)(y) = 0$ for all $y \in \mathcal{M}(k\{A\})$, i.e., $\rho(f - g) = 0$.

Proof. Step 1. *The statement is true if A is an integral domain.* Indeed, by Lemma 1.4.2, one has

$$\rho(F) = \max_{x \in \mathcal{M}(A)} \rho(F_x) \ ,$$

where F_x is the image of F in $k\{\mathcal{H}(x)\}$. Since the quotient $A/\mathfrak{p}_x$ is also an integral domain, the canonical map $A/\mathfrak{p}_x \to \mathcal{H}(x)$ is injective. The assumption and Lemma 1.4.3(ii) imply that the Banach norm on $k\{\mathcal{H}(x)\}$ coincides with the spectral norm and, therefore, $\rho(F_x) = \max_{f \in \check{A}} |\lambda_f| \cdot |f(x)|$. The claim follows.

Step 2. *The statement is true in the general case.* Indeed, let $\{I, A_i, \nu_{ij}, \mathbf{a}_{ji}\}$ be a twisted datum of integral $\mathbf{F}_1$-affinoid algebras that represents A. One has $F = \sum_{i \in I} F_i$, where $F_i = \sum_{f \in \mathbf{a}^{(i)}} \lambda_f f$. For every $i \in I$, the element F_i and its powers can be considered as elements $k\{A_i\}$, and it follows from Step 1 that the required fact is true for F_i. To verify it for F, we can withdraw from F all summands F_j with $\rho(F_j) < \max_{i \in I} \rho(F_i)$, i.e., we may assume that $\rho(F_i) = \rho(F_j)$ for all pairs $i, j \in I$ with nonzero F_i and F_j. Notice that the supremum k of two elements $i, j \in I$ exists, then the product $F_i \cdot F_j$ is of the form $\sum_{f \in \mathbf{a}^{(k)}} \lambda_f f$ and, if it does not exists, then $F_i \cdot F_j = 0$. It follows that, if i is a minimal element of I with nonzero F_i, then $(F^n)_i = F_i^n$ and, in particular, $||F^n|| \geq ||F_i^n||$ for all $n \geq 1$. Thus, $\rho(F) = \lim_{n \to \infty} \sqrt[n]{||F^n||} \geq \rho(F_i)$, and the required fact follows. ∎

Let k be a field with trivial valuation satisfying the assumptions of Lemma 5.3.9 for A and B. Then $k\{\widetilde{A}\} \widetilde{\to} \widetilde{k\{A\}}$ and $k\{\widetilde{B}\} \widetilde{\to} \widetilde{k\{B\}}$. Since φ is an admissible epimorphism, then so is the induced map $\widetilde{\varphi} : k\{A\} \to k\{B\}$. By [Tem, Proposition I.3.1(iii)], the induced homomorphism $k\{\widetilde{A}\} \to k\{\widetilde{B}\}$ is finite. It follows that, for every element $g \in B$, $\widetilde{g}$ satisfies an equation $\widetilde{g}^m + \widetilde{\varphi}(\widetilde{F}_1)\widetilde{g}^{m-1} + \ldots + \widetilde{\varphi}(\widetilde{F}_m) = 0$ with $F_i \in k\{A\}$. This implies that there exists $f \in A$ with $\widetilde{g}^m = \widetilde{\varphi}(\widetilde{f})\widetilde{g}^l$ for some $0 \le l < m$, i.e., $\widetilde{g}$ is integral over $\widetilde{\varphi}(\widetilde{A})$.

(c)$\Longrightarrow$(a). Let $C = \mathbf{F}_1\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to A$ be an admissible epimorphism. By the implication already proved, $\widetilde{A}$ is a finite $\widetilde{C}$-algebra and, therefore, $\widetilde{B}$ is a finite $\widetilde{C}$-algebra. Thus, to prove the required fact, we can replace A by C, and so we may assume that A is reduced, i.e., the homomorphism φ satisfies the assumption (1) of Lemma 2.4.6. To verify validity of the assumption (2), it suffices to show that, for every non-nilpotent element $g \in B$, one has $g^m = g^n\varphi(f)$ for some $f \in A$ and $m > n \ge 0$ with $\rho(\varphi(f)g) = \rho(\varphi(f))\rho(g)$. The assumption implies that $\widetilde{g}^m = \widetilde{\varphi}(\widetilde{f})\widetilde{g}^n$ for some $m > n \ge 0$ and $f \in A$. It follows that $g^m = g^n\varphi(f)$, $\rho(\varphi(f)g) = \rho(\varphi(f))\rho(g)$, and $\rho(\varphi(f)) = \rho(f)$. Thus, the assumption (2) is satisfied, and the property (a) follows from Lemma 2.4.6. ∎

5.3.10. Corollary. *Let A be a K-affinoid algebra, and B and C are A-affinoid algebras. Then the canonical homomorphism $\widetilde{B\widehat{\otimes}_A C} \to \widetilde{B} \otimes_{\widetilde{A}} \widetilde{C}$ is finite.*

Proof. Take an admissible epimorphism $A\{r^{-1}T\} = A\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to C$. By Proposition 5.3.8, the induced homomorphism $\widetilde{A}[r^{-1}T] \to \widetilde{C}$ is finite, and so is the homomorphism $\widetilde{B}[r^{-1}T] = \widetilde{B} \otimes_{\widetilde{A}} \widetilde{A}[r^{-1}T] \to \widetilde{B} \otimes_{\widetilde{A}} \widetilde{C}$. On the other hand, the same epimorphism gives rise to an admissible epimorphism $B\{r^{-1}T\} = B\widehat{\otimes}_A A\{r^{-1}T\} \to B\widehat{\otimes}_A C$. By Proposition 5.3.8, the latter induces a finite homomorphism $\widetilde{B}[r^{-1}T] \to \widetilde{B\widehat{\otimes}_A C}$, which is compatible with the above finite homomorphism $B\{r^{-1}T\} \to \widetilde{B} \otimes_{\widetilde{A}} \widetilde{C}$. This implies required fact. ∎

§6. K-affinoid spaces

6.1. K-affinoid spaces and affinoid domains. Let K be a real valuation $\mathbf{F}_1$-field. The category K-$\mathcal{A}ff$ of *K-affinoid* (resp. *st*-K-$\mathcal{A}ff$ of *strictly K-affinoid*) *spaces* is, by definition, the category opposite to that of K-affinoid (resp. strictly K-affinoid) algebras. For brevity, the K-affinoid space that corresponds to an K-algebra A will be mentioned by its spectrum $X = \mathcal{M}(A)$, and the morphism of K-affinoid spaces that corresponds to a bounded homomorphism of K-algebras will be mentioned by the induced map of their spectra $Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$. The categories K-$\mathcal{A}ff$ and *st*-K-$\mathcal{A}ff$ admit fiber products which correspond to complete tensor products of K-affinoid algebras. (Recall that, by Lemma 1.3.8, the forgetful functor to the category of topological spaces commutes with fiber products.) If K' is a valuation $\mathbf{F}_1$-field over K, there is a *ground field* *extension functor* $K\text{-}\mathcal{A}ff \to K'\text{-}\mathcal{A}ff$ that takes $X = \mathcal{M}(A)$ to $X \widehat{\otimes} K' = \mathcal{M}(A \widehat{\otimes} K')$. Notice that the canonical map $X \widehat{\otimes} K' \to X$ is a homeomorphism.

6.1.1. Definition. (i) A K-affinoid space $X = \mathcal{M}(A)$ is said to be *integral* (resp. *quasi-integral*, resp. *finitely presented*, resp. *reduced*, resp. *Zariski reduced*, resp. *artinian*, resp. *local artinian*) if the K-affinoid algebra A possesses the corresponding property.

(ii) A morphism of K-affinoid spaces $\varphi : Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ is said to be a *finite morphism* (resp. *a closed immersion*) if the homomorphism $A \to B$ makes B a finite Banach A-algebra (resp. is surjective and admissible).

(iii) A closed subset P of a (strictly) K-affinoid space $X = \mathcal{M}(A)$ is said to be a *(strictly) rational polytope* if it can be defined by a finite number of inequalities of the form $|f(x)| \le r|g(x)|$ with $f, g \in A$ and $r \in \mathbf{R}_+$ (resp. $r \in |K|$).

If we fix an admissible epimorphism $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$ that gives rise to a homeomorphism between $X = \mathcal{M}(A)$ and a $|K|$-affinoid polytope in $\mathbf{R}_+^n$, then rational polytopes in X are precisely $(\mathbf{R}_+)_{\mathbf{Z}_+}$-polytopes in $\mathbf{R}_+^n$ which lie in X. It follows easily that the image of a (strictly) rational polytope under a morphism of (strictly) K-affinoid spaces is a (strictly) rational polytope.

6.1.2. Lemma. *Given a (strictly) rational polytope P in a (strictly) K-affinoid space X, there exists a morphism of (strictly) K-affinoid spaces $\varphi : Y \to X$ whose image coincides with P and all of the fibers are connected.*

Proof. Let $X = \mathcal{M}(A)$, and let P be defined by inequalities $|f_i(x)| \le r_i|g_i(x)|$, $1 \le i \le m$, and suppose that $r_i \ne 0$ precisely for $1 \le i \le n$. If $B = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}/E$, where E is the closed ideal generated by the pairs (f_i, g_iT_i) for $1 \le i \le n$ and $(f_i, 0)$ for $n+1 \le i \le m$, then for the morphism $\varphi : Y = \mathcal{M}(A) \to X$ one has $\varphi(Y) = P$ (see the proof of Proposition 4.1.6). ∎

Here is an important example of a rational polytope.

6.1.3. Definition. A closed subset V of a K-affinoid space $X = \mathcal{M}(A)$ is said to be an *affinoid domain* if there is a homomorphism of K-affinoid algebras $A \to A_V$ such that

(1) the image of $\mathcal{M}(A_V)$ in X lies in V;

(2) any homomorphism of K-affinoid algebras $A \to B$ such that the image of $\mathcal{M}(B)$ in X lies in V goes through a unique homomorphism of K-affinoid algebras $A_V \to B$.

It is clear that for a subset V with the above properties the homomorphism $A \to A_V$ is unique up to a unique isomorphism.

6.1.4. Lemma. *Let V be an affinoid subdomain of a K-affinoid space $X = \mathcal{M}(A)$. Then* *(i) the induced map $\mathcal{M}(A_V) \to V$ is bijective and, for every point $y \in \mathcal{M}(A_V)$ with the image $x \in X$, there is an isometric isomorphism $\mathcal{H}(x) = \kappa(\mathfrak{p}_x) \xrightarrow{\sim} \mathcal{H}(y) = \kappa(\mathfrak{p}_y)$;*

(ii) the induced map $\mathrm{Zspec}(A_V) \to \mathrm{Zspec}(A)$ *is injective, and the partial ordering on* $\mathrm{Zspec}(A_V)$ *coincides with the restriction of that on* $\mathrm{Zspec}(A)$.

Proof. (i) Let x be a point from V. Since the valuation $\mathbf{F}_1$-field $\mathcal{H}(x) = \mathcal{H}(\mathfrak{p}_x)$ is a K-affinoid algebra, we can apply the property (2) to the canonical homomorphism $A \to \mathcal{H}(x)$. It follows that the later goes through a unique homomorphism of K-affinoid algebras $A_V \to \mathcal{H}(x)$, which corresponds to a point $y \in \mathcal{M}(A_V)$ whose image in X is x. It follows also that $\mathcal{H}(x) \xrightarrow{\sim} \mathcal{H}(y)$ and, in particular, y is a unique preimage of x.

(ii) Injectivity easily follows from (i). Suppose that, for two Zariski prime ideals $\mathfrak{q}_1, \mathfrak{q}_2 \subset A_V$, one has $\mathfrak{p}_1 \subset \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are their preimages in A. Then the preimage of the ideal $\mathfrak{p}_2 = \mathfrak{p}_1 \cup \mathfrak{p}_2$ coincides with $\mathfrak{q}_1 \cup \mathfrak{q}_2$. The injectivity implies that $\mathfrak{q}_1 \cup \mathfrak{q}_2 = \mathfrak{q}_2$, i.e., $\mathfrak{q}_1 \subset \mathfrak{q}_2$. ∎

Notice that, if V and W are affinoid domains in $X = \mathcal{M}(A)$, then $V \cap W$ is an affinoid domain in X which corresponds to the homomorphism $A \to A_V \widehat{\otimes}_A A_W$. Furthermore, if V is an affinoid domain in X, then any affinoid subdomain of V is an affinoid domain in X, and V is an affinoid domain in any bigger affinoid subdomain of X. Notice also that the preimage of an affinoid domain V under a morphism of K-affinoid spaces $Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ is an affinoid domain that corresponds to the homomorphism $B \to B \widehat{\otimes}_A A_V$.

6.1.5. Definition. A morphism of K-affinoid spaces $\varphi : Y \to X$ is said to be an *affinoid domain embedding* or, for brevity, an *ad-embedding* if, for any morphism of K-affinoid spaces $\psi : Z \to X$ with $\psi(Z) \subset \varphi(Y)$, there is a unique morphism $\chi : Z \to Y$ with $\psi = \varphi \circ \chi$.

If $\varphi : Y \to X$ is an *ad*-embedding, then $\varphi(Y)$ is an affinoid domain in X. The correspondence $\varphi \mapsto \varphi(Y)$ gives rise to a bijection between the set of equivalence classes of *ad*-embeddings in X and the set of affinoid domains in X. We shall denote by K-$\mathcal{A}ff^{ad}$ the subcategory of K-$\mathcal{A}ff$ with the same family of objects and with *ad*-embeddings as morphisms.

We now consider examples of affinoid domains. Let $X = \mathcal{M}(A)$ be a K-affinoid space.

6.1.6. Lemma. *Given tuples $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_n)$ of elements of A and tuples of positive numbers $p = (p_1, \dots, p_m)$ and $q = (q_1, \dots, q_n)$, the following is true:*

(i) the subset $X(p^{-1}f, qg^{-1}) = \{x \in X \big| |f_i(x)| \le p_i,\ |g_j(x)| \ge q_j\}$ is an affinoid domain (called Laurent*) that corresponds to the homomorphism*

$$A \to A\{p^{-1}f, qg^{-1}\} = A\{p_1^{-1}T_1, \dots, p_m^{-1}T_m, q_1S_1, \dots, q_nS_n\}/E \ ,$$

where E is the closed ideal generated by the pairs (T_i, f_i) and $(g_jS_j, 1)$;

(ii) if $g = g_1 \cdot \ldots \cdot g_n$, the canonical homomorphism $A_g \to A_V$ is surjective, and its kernel coincides with the Zariski kernel.

Proof. (i) That the image of the spectrum of $A\{p^{-1}f, qg^{-1}\}$ is contained in $X(p^{-1}f, qg^{-1})$ is easy. Let $\varphi : A \to B$ be a bounded homomorphism to an *arbitrary quasi-affinoid* algebra B (see Definition 2.2.2) such that the image of $Y = \mathcal{M}(B)$ in X is contained in $X(p^{-1}f, qg^{-1})$. This means that $|(\varphi f_i)(y)| \le p_i$ and $|(\varphi g_j)(y) \ge q_j$ for all $y \in Y$. The former inequalities imply that $\rho(\varphi f_i) \le p_i$, and the latter inequalities imply that the elements $\varphi(g_j)$ are invertible in B and $\rho((\varphi g_j)^{-1}) \le q_j$. Thus, by Corollary 2.2.3, the homomorphism $\varphi : A \to B$ can be extended in a unique way to a bounded homomorphism $A\{p^{-1}T, qS\} \to B$ that takes T_i to $\varphi(g_j)$ and S_j to $\varphi(g_j)^{-1}$. The ideal E lies in the kernel of the latter, and so it gives rise to a bounded homomorphism $A\{p^{-1}f, qg^{-1}\} \to B$.

(ii) The ideal E is the closure of the ideal E' generated by the pairs (T_i, f_i) and $(g_jS_j, 1)$. It follows that $E = E' \cup (\mathbf{a}_E \times \mathbf{a}_E)$ (see §1.1). Since the canonical homomorphism $A \to A\{p^{-1}f, qg^{-1}\}/E'$ induces an isomorphism of K-algebras $A_g \xrightarrow{\sim} A\{p^{-1}f, qg^{-1}\}/E'$, the required fact follows. ∎

Notice that every point of X has a fundamental system of compact neighborhoods consisting of Laurent domains. If $n = 0$ in Lemma 6.1.6, the affinoid domain is called *Weierstrass* and denoted by $X(p^{-1}f)$. If V is a Weierstrass domain, the canonical homomorphism $A \to A_V$ is surjective, and its kernel coincides with the Zariski kernel.

Here is an example of a K-affinoid space in which any rational subpolytope is an affinoid subdomain (see also Corollary 6.2.3).

6.1.7. Lemma. *Assume that X is a local artinian K-affinoid space. Then*

(i) every rational subpolytope V of X is a Weierstrass domain;

(ii) if V is nonempty, the kernel of the homomorphism $A \to A_V$ is a Zariski ideal in $\mathrm{z}\mathbf{n}(A)$ and, in particular, V is also a local artinian K-affinoid space.

Proof. (i) An admissible epimorphism $K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to A$ gives rise to a presentation of X in the form of an $(\mathbf{R}_+)_{\mathbf{Z}_+}$-polytope in $\check{W}_I \subset \mathbf{R}_+^n$ for some $I \subset \{1, \ldots, n\}$. Any nonempty subpolytope V of the same type is defined by a finite number of inequalities $f(t) \le rg(t)$, where f and g are monoms in T_i for $i \in I$, which are invertible at X, and $r \in \mathbf{R}_+^*$. The restrictions of f and g to X define invertible elements of A, and so the above inequality can be written in the form $|(\frac{f}{g})(x)| \le r$. This implies that V is a Weierstrass domain.

(ii) If $V = X(p^{-1}f)$, the quotient of $A\{r^{-1}T\}$ by the ideal generated by the pairs (T_i, f_i) for

$1 \le i \le m$ is isomorphic to A, i.e., is a local artinian $\mathbf{F}_1$-algebra. It follows that, if the quotient seminorm of one of its invertible elements is zero, then it is zero identically. ∎

6.1.8. Lemma. *Given similar tuples $f = (f_1, \dots, f_m)$ and $p = (p_1, \dots, p_m)$, an element $g \in A$ and a number $q > 0$, the following is true*

(i) the subset $X(p^{-1}\frac{f}{g}, qg^{-1}) = \{x \in X \big| |f_i(x)| \le p_i|g(x)|, |g(x)| \ge q\}$ is an affinoid domain (called rational*), and it corresponds to the homomorphism*

$$A \to A\{p^{-1}\frac{f}{g}, qg^{-1}\} = A\{p_1^{-1}T_1, \dots, p_m^{-1}T_m, qS\}/E \ ,$$

where E is the closed ideal generated by the pairs (gT_i, f_i) and $(gS, 1)$;

(ii) the canonical homomorphism $A_g \to A_V$ is surjective, and its kernel coincides with the Zariski kernel.

Proof. Both statements are verified in the same way as Lemma 6.1.6 (and for homomorphisms $A \to B$ to arbitrary quasi-affinoid algebras B). ∎

If in Lemma 6.1.8 $g = 1$ and $q = 1$, we again get a Weierstrass domain. Notice that, for a rational domain V, the canonical homomorphism $A_{(V)} \to A_V$ is surjective. (If V is defined as above, then the homomorphism $A_g \to A_V$ is surjective.) Notice that the preimage of a Weierstrass (resp. Laurent, resp. rational) subdomain under a morphism of K-affinoid spaces is an affinoid domain of the same type.

6.1.9. Lemma. *Let V be a rational subdomain of an K-affinoid space $X = \mathcal{M}(A)$. Then*

(i) if V' is a Weierstrass (resp. rational) subdomains of X, then $V \cap V'$ is a Weierstrass (resp. rational) subdomain of X;

(ii) if U is a Weierstrass (resp. rational) subdomain of V, then U is a Weierstrass (resp. rational) subdomain of X.

Proof. (i) The statement is trivial for Weierstrass (and Laurent) domains. Let V and V' be rational domains in an K-affinoid space $X = \mathcal{M}(A)$, i.e., $V = \{x \in X \big| |g(x)| \ge q,\ |f_i(x)| \le p_i|g(x)|,\ 1 \le i \le m\}$ and $V' = \{x \in X \big| |g'(x)| \ge q',\ |f'_j(x)| \le p'_j|g'(x)|,\ 1 \le j \le n\}$. Then the intersection $V \cap V'$ is defined by the inequalities $|(gg')(x)| \ge qq'$, $|g(x)| \le (q')^{-1}|(gg')(x)|$, $|g'(x)| \le q^{-1}|(gg')(x)|$, $|(f_i g')(x)| \le p_i|(gg')(x)|$, and $|(f'_j g)(x)| \le p'_j|(gg')(x)|$, i.e., it is a rational subdomain of X.

(ii) If V is Weierstrass, the canonical homomorphism $A \to A_V$ is surjective, and the required fact easily follows. Assume that V is rational and defined as in (i). Then the map $A_g \to A_V$ is surjective, and so $U = \{x \in V \big| |(\frac{f'_j}{g^k})(x)| \le p'_j|(\frac{g'}{g^k})(x)|,\ |(\frac{g'}{g^k})(x)| \ge q',\ 1 \le j \le n\}$ for some

$f'_1, \dots, f'_m, g' \in A$ and $k \geq 0$. The latter can be defined by the following inequalities in X: $|(gg')(x)| \geq q^{k+1}q'$, $|g'(x)| \leq q^{-1}|(gg')(x)|$, $|g^{k+1}(x)| \leq (q')^{-1}|(gg')(x)|$, $|(f_i g')(x)| \leq p_i|(gg')(x)|$, and $|(f'_j g)(x)| \leq p'_j|(gg')(x)|$, i.e., it is a rational subdomain of X. ∎

6.1.10. Corollary. *Any Laurent domain is also a rational domain.* ∎

6.1.11. Lemma. *Let E be an ideal of the idempotent subalgebra I_A of A. Then the subset*

$$X(E) = \{x \in X \big| |e(x)| = |f(x)| \text{ for all } (e, f) \in E\}$$

is an affinoid domain (called idempotent*), and it corresponds to the homomorphism $A \to A\{E\} = A/F$, where F is the ideal of A generated by E (it is closed by Example 1.1.4(i)).*

Proof. Let $\varphi : A \to B$ be a bounded homomorphism to an *arbitrary* Banach $\mathbf{F}_1$-algebra B such that the image of $Y = \mathcal{M}(B)$ in X is contained in $X(E)$. If, for $(e, f) \in E$, e' and f' denote the images of e and f in B, then for every point $y \in \mathcal{M}(B)$ one has $|e'(y)| = |f'(y)|$. Since $e'(y)$ and $f'(y)$ are idempotents in the $\mathbf{F}_1$-field $\mathcal{H}(y)$, it follows that they are equal either to 0, or to 1. Thus, $e'(y) = f'(y)$ for all points $y \in \mathcal{M}(B)$. Corollary 2.2.5 implies that $e' = f'$. It follows that the homomorphism $\varphi : A \to B$ goes through a bounded homomorphism $A\{E\} \to B$. ∎

Notice that an idempotent domain is an open-closed subset of X, i.e., a union of connected components.

6.1.12. Proposition. *Let V be an open-closed subset of X, i.e., $V = \bigcup_{U \in \Sigma} U$ for some subset $\Sigma \subset \pi_0(X)$. Then the following are equivalent:*

(a) V is an affinoid domain in X,

(b) for any pair $U', U'' \in \Sigma$, one has $\inf(U', U'') \in \Sigma$.

(c) V is an idempotent domain.

Proof. The implication (c)$\Longrightarrow$(a) is trivial, and (a)$\Longrightarrow$(b) follows from Proposition 4.2.5.

(b)$\Longrightarrow$(c). If Y is the spectrum of the idempotent subalgebra I_A of A, then, by Proposition 6.2.5, there is an isomorphism of partially ordered sets $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$. Recall that Y is a discrete set, $\pi_0(Y) = Y$, and there is an isomorphism of partially ordered sets $Y \xrightarrow{\sim} I_A \backslash \{0\}$ that takes a point $y \in Y$ to the maximal idempotent e with $e(y) = 1$. Let P be the subset of $I_A \backslash \{0\}$ that corresponds to the set Σ. By Lemma I.1.3.5(i), for $E_P = \bigcap_{e \in P} \Pi_e$, the image of $\mathcal{M}(A/E_P)$ in $Y \xrightarrow{\sim} I_A \backslash \{0\}$ is precisely the set P. It follows that $V = X(E_P)$. ∎

6.2. A description of affinoid domains in quasi-integral affinoid spaces. Let $X = \mathcal{M}(A)$ be an integral K-affinoid space. We fix an admissible epimorphism $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to$

$A : T_i \mapsto f_i$ with $f_i \neq 0$ for all $1 \le i \le n$. Then X is identified with an $|K|$-affinoid polytope in $\mathbf{R}_+^n$ such that $X = \overline{\check{X}}$, and $\mathrm{Zspec}(A)$ is identified with the partially ordered set $\mathcal{I}(X)$. Notice that any affinoid domain in X is a rational polytope. For a rational polytope $V \subset X$, we set $\mathcal{I}'(V) = \mathcal{I}(V)\backslash\{\{1, \dots, n\}\}$ and $\langle V \rangle = \bigcap_{I \in \mathcal{I}'(V)} \tau_I^{-1}(V_I)$ (τ_I is the canonical projection $X \to X_I$). Notice that $V \backslash \check{V} = \bigcup_{I \in \mathcal{I}'(V)} V_I \subset V \subset \langle V \rangle$.

6.2.1. Theorem. *In the above situation, the following properties of a rational polytope $V \subset X$ are equivalent:*

(a) V is an affinoid domain;

(b) V is a rational domain;

(c) for every $I \in \mathcal{I}(V)$, V contains a neighborhood of V_I in $\tau_I^{-1}(V_I)$;

(d) $V = \overline{\check{V}}$, and $\mathrm{rec}(\check{V})$ is a face of $\mathrm{rec}(\check{X})$.

Furthermore, in this case V the following is true:

(1) if $I \in \mathcal{I}(V)$ and $I \subset J$, then $J \in \mathcal{I}(V)$;

(2) V is a Weierstrass domain if and only if $\mathrm{rec}(\check{V}) = \mathrm{rec}(\check{X})$ or, equivalently, $\mathcal{I}(V) = \mathcal{I}(X)$.

Proof. The implication (b)$\Longrightarrow$(a) is trivial. To prove the implications (a)$\Longrightarrow$(d)$\Longrightarrow$(c)$\Longrightarrow$(b) and the last statement, we can replace A by $\widehat{A}$, and so we may assume that $K \stackrel{\sim}{\to} |K|$ and A is a K-polytopal algebra.

(a)$\Longrightarrow$(d). Every face of $\mathrm{rec}(\check{X})$ has the form $\mathrm{rec}_I(\check{X}) = \{(\alpha_1, \dots, \alpha_n) \in \mathrm{rec}(\check{X}) \big| \alpha_i = 1$ for $i \in I\}$ with $I \in \mathcal{I}(X)$, and the interior of such a face is the set $\mathring{\mathrm{rec}}_I(\check{X}) = \{(\alpha_1, \dots, \alpha_n) \in \mathrm{rec}_I(\check{X}) \big| \alpha_i < 1$ for $i \notin I\}$. Assume that, for some $I \in \mathcal{I}(V)$, there is a point $(\alpha_1, \dots, \alpha_n) \in \mathring{\mathrm{rec}}_I(\check{X}) \backslash \mathrm{rec}(\check{V})$. We may assume that it represents a rational direction, i.e., $\alpha_i = \alpha^{k_i}$ for $i \notin I$, where k_i are positive integers and $0 < \alpha < 1$. (Recall that $\alpha_i = 1$ for $i \in I$, and so we may set $k_i = 0$ for $i \in I$.) Take a point $x = (x_1, \dots, x_n) \in \check{V}_I$ (i.e., $x_i \neq 0$ for $i \in I$ and $x_i = 0$ for $i \notin I$) and a point $y = (y_1, \dots, y_n) \in \check{X}$ with $y_i = x_i$ for $i \in I$. By Proposition 3.1.1, we may assume that all y_i lie in a bigger $\mathbf{F}_1$-subfield $K \subset K' \subset \mathbf{R}_+$ with finite quotient group K'^*/K^*. The ray $\check{L} = \{y_t = (y_1 t^{k_1}, \dots, y_n t^{k_n})\}_{0 < t \le 1}$ lies in $\check{X} \backslash \check{V}$ and $y_t \to x$ as $t \to 0$. Consider the homomorphism $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B = K'\{T\}$ that takes T_i to $y_i T^{k_i}$. This homomorphism is clearly bounded, and the induced map $Y = \mathcal{M}(B) \to \mathcal{M}(K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\})$ is injective and its image coincides with the set $L = \check{L} \cup \{x\}$. Since $A \stackrel{\sim}{\to} \widehat{A}$ and $L \subset X$, it follows that the above homomorphism goes through a bounded homomorphism $A \to B$. Since B is a K-affinoid algebra, it follows that the preimage of the affinoid domain V in $Y \stackrel{\sim}{\to} L$ is an affinoid domain U that coincides with the point x. *We claim that the latter is impossible.*

Indeed, let g be the product of the images of the elements f_i in B, and let h be the image of g in B_U. Since $h(x) = 0$ and $\mathcal{M}(B_U) = \{x\}$, it follows that $h^m = 0$ for some $m \geq 1$. But if $\mathbf{b}$ denote the Zariski ideal of B generated by g, then $\mathbf{b}^{m+1} \neq \mathbf{b}^m$. Indeed, if $\mathbf{b}^{m+1} = \mathbf{b}^m$, then $g^m = g^{m+1}u$ for some $u \in B$. Since $g(y_t) \neq 0$ and $g(y_t) \to 0$ as $t \to 0$, it follows that $u(y_t) \to \infty$ as $t \to 0$, which is a contradiction. Hence, we get a bounded homomorphism $B \to C = B/\mathbf{b}^{m+1}$ such that the image of $\mathcal{M}(C)$ in Y coincides with $U = \{x\}$. It follows that the latter homomorphism goes through a bounded homomorphism $B_U \to C$, which is impossible since $h^m = 0$ but the m-th power of its image in C is not zero.

Thus, $\mathrm{rec}_I(\check{X}) \subset \mathrm{rec}(\check{V})$ for all $I \in \mathcal{I}(V)$. Since $\mathrm{rec}(\check{V}) \subset \mathrm{rec}(\check{X})$, it follows that $\mathrm{rec}(\check{V}) = \mathrm{rec}_I(\check{X})$, where I is the minimal element of $\mathcal{I}(V)$. It follows also that $V = \overline{\check{V}}$.

(d)$\Longrightarrow$(c). Replacing X by $\langle V \rangle$, we may assume that $\mathcal{I}(X) = \mathcal{I}(V)$ and $X_I = V_I$ for all $I \in \mathcal{I}(X)$, and that $\mathrm{rec}(\check{X}) = \mathrm{rec}(\check{V})$. Notice that maximal elements of $\mathcal{I}'(X)$ correspond to one dimensional faces of $\mathrm{rec}(\check{X})$ and, for every I maximal in $\mathcal{I}'(X)$, the fibers of the canonical projection $\check{X} \to \check{X}_I$ are rays (see Lemma 4.3.9). Let us fix $\alpha_I \in \check{\mathrm{rec}}_I(X)$ for every such I, and let $x_1, \ldots, x_m$ be the vertices of $\check{X}$. Then there exists $t_0 > 0$ such that $x_i \alpha_I^t \in \check{V}$ for all I as above, $1 \leq i \leq m$, and $t \geq t_0$. Since $\check{X} = \mathrm{conv}(\mathrm{ver}(\check{X})) \cdot \mathrm{conv}(\{\alpha_I\})$, it follows that the set of points of the form $\prod_{i=1}^m x^{\lambda_i} \cdot \prod_I \alpha_I^{t_I}$ is a neighborhood of $X \backslash \check{X}$ in X, where $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^m \lambda_i = 1$, $t_I \geq 0$, and there exists J with $t_J \geq t_0$. The above point is equal to $\prod_{i=1}^m (x_i \alpha_J^t)^{\lambda_i} \cdot \prod_{I \neq J} \alpha_I^{t_I}$. Since the first product lies in V, by the construction, and the second one lies in $\mathrm{rec}(\check{X}) = \mathrm{rec}(\check{V})$, it follows that the point lies in V.

(c)$\Longrightarrow$(b). The required implication is evidently true if $n = 1$, and so assume that $n \geq 2$ and that the implication is true in the cases when the dimension of the ambient space is strictly less than n. We notice that, for any $I \in \mathcal{I}(X)$, the projection $\tau_I : X \to X_I$ is a morphism of K-affinoid spaces (that corresponds to the canonical isometric homomorphism $A_I \to A$, where A_I is the quotient of A by the Zariski prime ideal generated by f_i for $i \notin I$). It follows that the preimage of a rational subdomain of X_I under τ_I is a rational domain in X. If $I \in \mathcal{I}'(V)$, then the property (c) holds for the pair (X_I, V_I) and, by the induction hypothesis, V_I is a rational domain in X_I. It follows that the set $Y = \langle V \rangle$ is a rational domain in X that contains V. Moreover, one has $\mathcal{I}(V) = \mathcal{I}(Y)$, $\mathrm{rec}(\check{V}) = \mathrm{rec}(\check{Y})$, and $V_I = Y_I$ for all $I \in \mathcal{I}'(Y)$. A representation of Y as a rational domain, gives rise to an embedding of Y into $\mathbf{R}_+^m$ with $m \geq n$ such that its image is a K-affinoid polytope there. Thus, replacing X by that image, we may assume that $\mathcal{I}(V) = \mathcal{I}(X)$, $\mathrm{rec}(\check{V}) = \mathrm{rec}(\check{X})$, and $V_I = X_I$ for all $I \in \mathcal{I}'(X)$, and so V is a Weierstrass domain in X, by the following lemma.

6.2.2. Lemma. *In the situation of Theorem 6.2.1, suppose that $I(V) = I(X)$, $\mathrm{rec}(\check{V}) = \mathrm{rec}(\check{X})$, and $V_I = X_I$ for all $I \in I'(X)$. Then V is defined by a finite number of inequalities of the form $|f(x)| \le r$ with $r > 0$ and $f \in A$ such that $f\big|_{X_I} = 0$ for all $I \in I'(X)$. In particular, V is a Weierstrass domain in X.*

Proof. As at the beginning of the proof of the theorem, we may assume that $K \stackrel{\sim}{\to} |K|$ and A is an K-polytopal algebra. The rational polytope V is defined in X by a finite number of inequalities of the form $f(x) \le rg(x)$ for $f, g \in A\backslash\{0\}$ and $r > 0$. By the implication (d)$\Longrightarrow$(c) already verified, V contains an open neighborhood $\mathcal{U}$ of $X\backslash\check{X}$ in X and $X\backslash\mathcal{U}$ is a compact subset of $\check{X}$, it follows that for any such pair (f, g) there exists $C > 0$ with $f(x) \le Cg(x)$ for all $x \in X$. Proposition 2.5.1 implies that $f^m = g^m h$ for some $m \ge 1$ and $h \in A\backslash\{0\}$. It follows that the inequality $f(x) \le rg(x)$ is equivalent to the inequality $h(x) \le r$. If $h\big|_{V_I} \ne 0$ for some $I \in I'(X)$, then h comes from a function in A_I and, therefore, $h(x) = h(\tau_I(x))$ for all points $x \in X$. Since $X_I \subset V$, the inequality $h(x) \le r$ holds for all points of X, i.e., it can be removed. The required fact follows. ∎

It remains to verify the last statement. Suppose first that V is a Weierstrass domain, i.e., $V = \{x \in X \big| f_i(x) \le r_i \text{ for } 1 \le i \le m\}$, where $f_1, \dots, f_m \in A$ and $r_1, \dots, r_m > 0$. If I is the minimal element of $\mathcal{I}(X)$, then for any point $x \in V$ the point $\tau_I(x)$ satisfies the same inequalities, i.e., $\tau_I(V) \subset V$ and, in particular, $I \in \mathcal{I}(V)$. Since $\mathrm{rec}(\check{V})$ is a face of $\mathrm{rec}(\check{X})$, it follows that $\mathrm{rec}(\check{V}) = \mathrm{rec}(\check{X})$. Conversely, suppose that $\mathrm{rec}(\check{V}) = \mathrm{rec}(\check{X})$. If $X = \check{X}$, the required fact follows from Lemma 6.1.7(i), and so assume that $X \ne \check{X}$ and that the required fact is true in dimensions of the ambient space less than n. Then, for every $I \in \mathcal{I}'(X)$, V_I is a Weierstrass domain in X_I. It follows that $\langle V \rangle$ is a Weierstrass domain in V. The proof of the implication (c)$\Longrightarrow$(b) shows that V is a Weierstrass domain in $\langle V \rangle$, and so V is a Weierstrass domain in X. ∎

6.2.3. Corollary. *Let $X = \mathcal{M}(A)$ be a quasi-integral K-affinoid space, and V a nonempty affinoid domain in X. Then the following properties of an open affine subscheme $\mathcal{V}$ of $\mathcal{X} = \mathrm{Spec}(A)$ are equivalent:*

(a) $\mathcal{V}$ is the preimage of $\mathrm{Zspec}(A_V)$ with respect to the map $\mathcal{X} \to \mathrm{Zspec}(A)$;

(b) $\mathcal{V}^{\mathrm{an}}$ contains V and the homomorphism $A_{\mathcal{V}} \to A_V$ is surjective;

(c) $\mathcal{V}$ is a unique minimal open affine subscheme of $\mathcal{X}$ such that $\mathcal{V}^{\mathrm{an}}$ contains V.

Furthermore, in this case the following is true:

(1) the kernel of the homomorphism $A_{\mathcal{V}} \to A_V$ is a Zariski ideal in $\mathrm{z}\mathbf{n}(A_{\mathcal{V}})$;

(2) V is a quasi-integral rational domain;

(3) for a Zariski ideal $\mathbf{a} \subset A$, $V \cap \mathcal{M}(A/\mathbf{a}) \neq \emptyset$ if and only if $\mathcal{V} \cap \mathrm{Spec}(A/\mathbf{a}) \neq \emptyset$;

(4) V is a Weierstrass domain if and only if $\mathcal{V} = \mathcal{X}$.

Proof. Theorem 6.2.1 implies that V is a rational domain, i.e., $V = \{x \in X \big| |f_i(x)| \leq p_i,\ |g(x)| \geq q\}$ with $f_1, \ldots, f_n, g \in A$ and $p_1, \ldots, p_n, q > 0$. Then $g \notin \mathrm{z}\mathbf{n}(A)$ since V is nonempty, and $A_V = B/\overline{E}$, where $B = A\{p_1^{-1}T_1, \ldots, p_m^{-1}T_m, qS\}$ and $\overline{E}$ is the closure of the ideal E generated by the pairs (gT_i, f_i) and $(gS, 1)$. *We claim that the principal open subset $\mathcal{U} = D(g) = \mathrm{Spec}(A_g)$ possesses the properties (a)-(c) and (1)-(2).* Indeed, it is clear that $\mathcal{U}^{\mathrm{an}}$ contains V and $A_{\mathcal{U}} = B/E$. The latter implies that the homomorphism $A_{\mathcal{U}} \to A_V$ is surjective and, in particular, $\mathcal{U}$ possesses the property (b). Furthermore, since A is quasi-integral, it suffices to verify the property (1) for a smaller affinoid domain. We may therefore assume that $V = \{x\}$ for a point $x \in \check{X}$. In this case, $A_{\mathcal{U}}$ is the localization of A with respect to the complement of the Zariski prime ideal $\mathrm{z}\mathbf{n}(A_{\mathcal{U}})$, i.e., a local artinian $\mathbf{F}_1$-algebra. It follows that any nontrivial Zariski ideal of $A_{\mathcal{U}}$ is contained in its nilradical $\mathrm{z}\mathbf{n}(A_{\mathcal{U}})$, and the property (1) follows (see the proof of Lemma 6.1.7(ii)). The property (1) implies (2) and also the property (a), i.e., $\mathcal{U}$ is the preimage of $\mathrm{Zspec}(A_V)$ with respect to the map $\mathcal{X} \to \mathrm{Zspec}(A)$. To verify the property (c), we may assume that X is reduced, i.e., A is integral. In this case the homomorphism $A_{\mathcal{U}} \to A_V$ is bijective and, therefore, $\mathcal{V}$ is a unique minimal open affine subscheme of $\mathcal{X}$ such that $\mathcal{U}^{\mathrm{an}}$ contains V. Thus, the claim is true. It implies the implications (c)$\Longleftrightarrow$(a)$\Longrightarrow$(b).

Suppose that an open affine subscheme $\mathcal{V}$ possesses the property (b). Then $\mathcal{V} \supset \mathcal{U}$. This implies that the kernel of the surjective homomorphism $A_{\mathcal{V}} \to A_V$ is an ideal in $\mathrm{z}\mathbf{n}(A_{\mathcal{V}})$ and, in particular, $A_{\mathcal{U}}/\mathrm{z}\mathbf{n}(A_{\mathcal{U}}) \stackrel{\sim}{\to} A_V/\mathrm{z}\mathbf{n}(A_V)$. Since f is invertible in A_V, it follows that it is also invertible in $A_{\mathcal{V}}$, i.e., $\mathcal{V} = \mathcal{U}$.

Finally, the property (3) implies that $\mathcal{V} \cap \mathrm{Spec}(A/\mathbf{a})$ is the preimage of $\mathrm{Zspec}(A_V) \cap \mathrm{Zspec}(A/\mathbf{a})$. Since the map $V \to \mathrm{Zspec}(A_V)$ is surjective, the latter intersection is nonempty if and only if $V \cap \mathcal{M}(A_{\mathbf{a}}) \neq \emptyset$, and (3) follows. It remains to show that, if $\mathcal{U} = \mathcal{X}$, then V is a Weierstrass domain in X. By the above construction, the equality $\mathcal{U} = \mathcal{X}$ implies that $g \in A^*$ and, therefore, V can be represented in the form $\{x \in X \big| |\frac{f_i}{g}| \leq p_i,\ |\frac{1}{g}| \leq \frac{1}{q}\}$, i.e., it is a Weierstrass domain. ∎

6.2.4. Corollary. *In the situation of Corollary 6.2.3, the following is true:*

(i) if X is integral, then V is also integral and $A_{\mathcal{V}} \stackrel{\sim}{\to} A_V$;

(ii) if A is a K-polytopal algebra (in particular, $K \stackrel{\sim}{\to} |K|$), then A_V is K-polytopal if and only if $\dim(\check{V}) = \dim(\check{X})$;

(iii) if for some nontrivial Zariski ideal $\mathbf{a} \subset A$, $V \cap \mathcal{M}(A/\mathbf{a})$ is a nonempty Weierstrass domain

in $\mathcal{M}(A/\mathbf{a})$, *then* V *is a Weierstrass domain in* X.

Proof. The statement (i) follows directly from Corollary 6.2.3.

(ii) Suppose that A is a K-polytopal algebra, i.e., $A \xrightarrow{\sim} \widehat{A}$. If A_V is also K-polytopal then, by Proposition 4.3.1, the dimensions $\dim(\check{X})$ and $\dim(\check{V})$ are equal to the ranks of the quotient groups F^*/K^* and F_V^*/K^*, where F and F_V are the fraction $\mathbf{F}_1$-fields of A and A_V, respectively. Since $F \xrightarrow{\sim} F_V$, we get $\dim(\check{V}) = \dim(\check{X})$. Conversely, if $\dim(\check{V}) = \dim(\check{X})$, then the interior of V in X is nonempty, and Proposition 4.3.1 implies that the canonical homomorphism $A \to \widehat{A}_V$ is injective. If $(\alpha, \beta) \in \mathrm{Ker}(A_V \to \widehat{A}_V)$ for $\alpha = \frac{u}{g^m}$ and $\beta = \frac{v}{g^m}$ with $u, v \in A$ and $m \geq 0$ (we use the notation from the proof of Corollary 6.2.4), then $u(x) = v(x)$ for all $x \in V$. The previous remark implies that $u = v$, i.e., $A_V \xrightarrow{\sim} \widehat{A}_V$. If A_V is K-polytopal, then $A_V \xrightarrow{\sim} \widehat{A}_V$.

(iii) We know that V is a rational domain in X, i.e., $V = \{x \in X \big| |f_i(x)| \leq p_i|g(x)|, |g(x)| \geq q\}$ for some $f_1, \dots, f_n, g \in A$ and $p_1, \dots, p_n, q > 0$, and the associated open affine subscheme of $\mathcal{X}$ is the principal open subset $D(g)$. It follows that $V \cap \mathcal{M}(A/\mathbf{a}) = \{x \in \mathcal{M}(A/\mathbf{a}) \big| |f_i(x)| \leq p_i|g(x)|, |g(x)| \geq q\}$, and the associated open affine subscheme of $\mathrm{Spec}(A/\mathbf{a})$ is the intersection $D(g) \cap \mathrm{Spec}(A/\mathbf{a})$. The assumption implies that the latter coincides with $\mathrm{Spec}(A/\mathbf{a})$ and, therefore, g is invertible in $A/\mathbf{a}$. It follows that g is invertible in A. ∎

6.2.5. Corollary. *Every affinoid domain in an irreducible* K*-affinoid space is an irreducible rational domain.* ∎

6.2.6. Corollary. *In the situation of Theorem 6.2.1, if* $V \supset X_I$ *for some* $I \in \mathcal{I}(V)$*, then* V *is a Weierstrass domain and a neighborhood of* X_I *in* X.

Proof. We may assume that $I \in \mathcal{I}'(V)$. The statement is evidently true if $n = 1$. Suppose that $n \geq 2$ and that the statement is true if the dimension of the ambient space is strictly less than n. Then, for every $J \in \mathcal{I}'(X)$, V_J contains $X_{I \cap J}$. The induction hypothesis implies that V_J is a neighborhood of $X_{I \cap J}$ and a Weierstrass domain in X_J. It follows that $\tau_J^{-1}(V_J)$ is a neighborhood of $\tau_J^{-1}(X_{I \cap J})$ and a Weierstrass domain in X. Since $\tau_J^{-1}(X_{I \cap J}) \supset X_I$, it follows that $\langle V \rangle = \bigcap_J \tau_J^{-1}(V_J)$ is a neighborhood of X_I and a Weierstrass domain in X. By Theorem 6.2.1, V is a Weierstrass domain and a neighborhood of X_I in $\langle V \rangle$, and the required fact follows. ∎

6.3. A description of affinoid domains in arbitrary affinoid spaces. Let $X = \mathcal{M}(A)$ be a K-affinoid space. Recall that, for a Zariski prime ideal $\mathfrak{p} \in \mathrm{Zspec}(A)$, we set $X_{\mathfrak{p}} = \{x \in X \big| f(x) = 0 \text{ for all } f \in \mathfrak{p}\}$, $\check{X}_{\mathfrak{p}} = \{x \in X_{\mathfrak{p}} \big| f(x) \neq 0 \text{ for all } f \notin \mathfrak{p}\}$, and $X^{(\mathfrak{p})} = \overline{\check{X}_{\mathfrak{p}}}$. By Proposition 5.1.5(ii), one has $X^{(\mathfrak{p})} = \mathcal{M}(A/\Pi_{\mathfrak{p}})$. Given Zariski prime ideals $\mathfrak{p} \subset \mathfrak{q}$, the canonical injective isometric homomorphism $A/\mathfrak{q} \to A/\mathfrak{p}$ induces a bounded homomorphism $A/\Pi_{\mathfrak{q}} \to A/\Pi_{\mathfrak{p}}$ which, in its turn, gives rise to continuous maps $\tau_{\mathfrak{p}\mathfrak{q}} : \check{X}_{\mathfrak{p}} \to \check{X}_{\mathfrak{q}}$ and $X^{(\mathfrak{p})} \to X^{(\mathfrak{q})}$. For a subset $U \subset X$, we set $U_{\mathfrak{p}} = U \cap X_{\mathfrak{p}}$, $\check{U}_{\mathfrak{p}} = U \cap \check{X}_{\mathfrak{p}}$ and $U^{(\mathfrak{p})} = U \cap X^{(\mathfrak{p})}$. We also set $\mathcal{I}(U) = \{\mathfrak{p} \in \mathrm{Zspec}(A) \big| \check{U}_{\mathfrak{p}} \neq \emptyset\}$. (Notice that the latter notation is consistent with that from the previous subsection since, for the set $\mathcal{I}(X)$ introduced there, there is a canonical bijection $\mathcal{I}(X) \overset{\sim}{\to} \mathrm{Zspec}(A)$.)

6.3.1. Theorem *The following properties of a nonempty subset $U \subset X$ are equivalent:*

(a) U is an affinoid domain;

(b) (b.1) for every $\mathfrak{p} \in \mathcal{I}(U)$, $U^{(\mathfrak{p})}$ is a rational domain in $X^{(\mathfrak{p})}$;

(b.2) for any pair $\mathfrak{p}, \mathfrak{q} \in \mathcal{I}(U)$, one has $\mathfrak{p} \cup \mathfrak{q} \in \mathcal{I}(U)$;

(b.3) if $\mathfrak{p} \subset \mathfrak{q}$ in (b.2), then $\tau_{\mathfrak{p}\mathfrak{q}}(\check{U}_{\mathfrak{p}}) \subset \check{U}_{\mathfrak{q}}$.

Furthermore, in this situation the following is true:

(i) the homomorphism $A \to A_U$ possesses the property 6.1.3(2) for bounded homomorphisms to arbitrary quasi-affinoid algebras B;

(ii) if U is connected, then

(ii.1) U is a rational domain;

(ii.2) U is a Weierstrass domain if and only if $U \cap X_{\mathbf{m}} \neq \emptyset$ (i.e., $\mathbf{m}_A \in \mathcal{I}(U)$);

(ii.3) given a pair $\mathfrak{p} \subset \mathfrak{q}$ in $\mathcal{I}(U)$, one has $\mathfrak{r} \in \mathcal{I}(U)$ for all $\mathfrak{r} \in \mathcal{I}(U)$ with $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$;

(iii) if X is connected and $U \cap X_{\mathbf{m}} \neq \emptyset$, then U is connected.

Proof. Step 1. *If U is an affinoid domain, the properties (b.1)-(b.3) hold.* Indeed, the property (b.1) holds, by Theorem 6.2.1, (b.2) holds because $\mathcal{I}(U)$ is the image of $\mathrm{Zspec}(A_U)$ in $\mathrm{Zspec}(A)$, and (b.3) holds because the maps $\tau_{\mathfrak{p}\mathfrak{q}} : X^{(\mathfrak{p})} \to X^{(\mathfrak{q})}$ is consistent with the corresponding map on the K-affinoid space U.

On the contrary, suppose that the properties (b.1)-(b.3) hold for U. Our first aim (Steps 2-7) is to show that, if U is connected, it possesses the properties (ii.1)-(ii.3). Thus, suppose that U is connected.

Step 2. *The exists a connected Weierstrass domain domain W with $U \subset W$ and $U_{\mathbf{m}} = W_{\mathbf{m}}$* (where $U_{\mathbf{m}} = U \cap X_{\mathbf{m}}$ and $W_{\mathbf{m}} = W \cap X_{\mathbf{m}}$). Indeed, since $X_{\mathbf{m}}$ is a local artinian K-affinoid space, Lemma 6.1.7 implies that $U_{\mathbf{m}}$ is a Weierstrass domain in $X_{\mathbf{m}}$, i.e., $U_{\mathbf{m}} = \{x \in X_{\mathbf{m}} \big| |f_l(x)| \leq r_l\}$ with $f_1, \ldots, f_n \in A^*$ and $r_1, \ldots, r_n > 0$. If W is the Weierstrass domain $\{x \in X \big| |f_l(x)| \leq r_l\}$, then $W_{\mathbf{m}} = W \cap X_{\mathbf{m}}$ coincides with $U_{\mathbf{m}}$, and property (b.3) implies that W contains U. It remains to notice that the latter properties also hold for the minimal connected component of W (which is also a Weierstrass domain in X).

Step 3. Suppose we are given a Zariski prime ideal $\mathfrak{p} \in \mathcal{I}(U)$ with $\mathfrak{p} \neq \mathbf{m}$ and a connected

Weierstrass domain W with $U \subset W$ and $U_{\mathfrak{q}} = W_{\mathfrak{q}}$ for all Zariski prime ideals $\mathfrak{q} \in \mathcal{I}(W)$ with $\mathfrak{q} \supset \mathfrak{p}$ and $\mathfrak{q} \neq \mathfrak{p}$. *Then there exists a Weierstrass domain $U \subset W' \subset W$ with $U_{\mathfrak{q}} = W'_{\mathfrak{q}}$ for all Zariski prime ideals $\mathfrak{q} \in \mathcal{I}(W')$ with $\mathfrak{q} \supset \mathfrak{p}$.* Indeed, we may replace X by W and assume that $W = X$. By (b.1), $U^{(\mathfrak{p})}$ is a rational domain in $X^{(\mathfrak{p})}$. We apply Theorem 6.2.1 and Lemma 6.2.2 to $U^{(\mathfrak{p})}$ and $X^{(\mathfrak{p})}$. The above assumption implies that $\mathcal{I}(U^{(\mathfrak{p})}) = \mathcal{I}(X^{(\mathfrak{p})})$ and, therefore, $U^{(\mathfrak{p})}$ is a Weierstrass domain in $X^{(\mathfrak{p})}$ defined by a finite number of inequalities of the form $|f(x)| \leq r$ with $r > 0$ and $f \in A^{(\mathfrak{p})} = A/\Pi_{\mathfrak{p}}$ such that $f\big|_{X_{\mathfrak{q}}} = 0$ for all Zariski prime ideals $\mathfrak{q} \supset \mathfrak{p}$ different from $\mathfrak{p}$. We may view f as an element of A, and the latter property implies that f lies in the intersection of all Zariski prime ideals $\mathfrak{q} \supset \mathfrak{p}$ different from $\mathfrak{p}$. It follows that the Weierstrass domain W' defined by the same inequalities on the whole space X possesses the required property. It remains to replace W' by its minimal connected component.

Step 4. By Step 3, there exists a connected Weierstrass domain $U \subset W \subset X$ such that $U^{(\mathfrak{p})} = W^{(\mathfrak{p})}$ for all Zariski prime ideals $\mathfrak{p} \in \mathcal{I}(W)$ that contain some $\mathfrak{q} \in \mathcal{I}(U)$. *We claim that $U = W$.* Indeed, suppose that $U \neq W$. Then there exists $\mathfrak{p} \in \mathcal{I}(W)$ that does not contain any Zariski prime ideals from $\mathcal{I}(U)$. We may assume that $\mathfrak{p}$ is maximal with the latter property. Since W is connected and $W \cap X_{\mathbf{m}} \neq U \cap X_{\mathbf{m}} = \emptyset$, there exists a Zariski prime ideal $\mathfrak{q} \in \mathcal{I}(W)$ with $\mathfrak{q} \supset \mathfrak{p}$ and $\mathfrak{q} \not\supset \mathfrak{p}$ such that $W^{(\mathfrak{p})} \cap W^{(\mathfrak{q})} \neq \emptyset$. By the maximality of $\mathfrak{p}$, it follows that $\mathfrak{q} \in \mathcal{I}(U)$ and, since $U^{(\mathfrak{q})} = W^{(\mathfrak{q})}$, it follows that $\mathcal{U}^{(\mathfrak{p})} \neq \emptyset$, which is a contradiction.

Thus, if $U \cap X_{\mathbf{m}} \neq \emptyset$, then U is a Weierstrass domain. (Notice that the property (b.2) was not used so far.)

Step 5. Let U be just connected, but the equality $U \cap X_{\mathbf{m}} \neq \emptyset$ is not assumed. *Then U is a rational domain.* Indeed, by the property (b.2), there exist $\mathfrak{p} \in \mathcal{I}(U)$ maximal among Zariski prime ideals in $\mathcal{I}(U)$. Let f be an element from $A \backslash \mathfrak{p}$ which lies in all Zariski prime ideals $\mathfrak{q} \supset \mathfrak{p}$ with $\mathfrak{q} \neq \mathfrak{p}$. Since U is compact, there exists $r > 0$ with $|f(x)| \geq r$ for all $x \in U$. We may therefore replace X by the Laurent domain $\{x \in X \big| |f(x)| \geq r\}$. In this case, the element f becomes invertible in A and, therefore, $\mathfrak{p}$ becomes the maximal Zariski ideal of A. In particular, the intersection $U \cap X_{\mathbf{m}}$ is nonempty and, by Step 4, U is a Weierstrass domain in X.

Step 6. *In the situation of Step 5, given a pair $\mathfrak{p} \subset \mathfrak{q}$ in $\mathcal{I}(U)$, one has $\mathfrak{r} \in \mathcal{I}(U)$ for all $\mathfrak{r} \in \mathcal{I}(U)$ with $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$.* Indeed, by Step 5, it suffices to verify the required property for an arbitrary rational domain and, therefore, it suffices to consider the following two cases (with $f \in A$ and $r > 0$).

(1) $U = \{x \in X \big| |f(x)| \geq r\}$. Since $U^{(\mathfrak{q})} \neq \emptyset$, then $f \notin \mathfrak{q}$. It follows that the inequality $|f(x)| \geq r$ holds at every point from $\tau_{\mathfrak{p}\mathfrak{r}}(U^{(\mathfrak{p})})$ and, therefore, $U^{(\mathfrak{r})} \neq \emptyset$, i.e., $\mathfrak{r} \in \mathcal{I}(U)$.

(2) $U = \{x \in X \big| |f(x)| \le r\}$. If $f \in \mathfrak{r}$, then $X^{(\mathfrak{r})} \subset U$. If $f \notin \mathfrak{r}$ then, as in (1), the inequality $|f(x)| \le r$ holds at every point from $\tau_{\mathfrak{p}\mathfrak{r}}(U^{(\mathfrak{p})})$ and, therefore, $U^{(\mathfrak{r})} \ne \emptyset$. In both cases, $\mathfrak{r} \in \mathcal{I}(U)$.

Step 7. *In the situation of Step 5, if U is a Weierstrass domain, then $U \cap X_{\mathbf{m}} \ne \emptyset$.* Indeed, suppose $U = \{x \in X \big| |f_i(x)| \le r_i$ for all $1 \le i \le n\}$, and let $y \in U^{(\mathfrak{p})}$ for some $\mathfrak{p} \in \mathcal{I}(U)$ with $\mathfrak{p} \ne \mathbf{m} = \mathbf{m}_A$. If $f_i \in \mathbf{m}$, then the inequality $|f_i(x)| \le r_i$ holds at every point from $X_{\mathbf{m}}$. Otherwise, this inequality holds at the point $\tau_{\mathfrak{p}\mathbf{m}}(y)$ and, therefore, $\tau_{\mathfrak{p}\mathbf{m}}(y) \in U \cap X_{\mathbf{m}}$.

Step 8. *If X is connected and $U \cap X_{\mathbf{m}} \ne \emptyset$, then U is connected.* Indeed, let V be the connected component of U with $V_{\mathbf{m}} = U_{\mathbf{m}}$, and suppose that there exists a Zariski prime ideal $\mathfrak{p} \in \mathcal{I}(U) \backslash \mathcal{I}(V)$. Since X is connected, there is a strictly decreasing sequence of Zariski prime ideals $\mathfrak{p}_0 = \mathbf{m} \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_n = \mathfrak{p}$ with $X^{(\mathfrak{p}_i)} \cap X^{(\mathfrak{p}_{i+1})} \ne \emptyset$ for all $0 \le i \le n-1$. Let i be maximal with the property that $\mathfrak{p}_i \in \mathcal{I}(V)$. Since $\mathbf{m} \in \mathcal{I}(V)$ and $\mathfrak{p} \notin \mathcal{I}(V)$, then $0 \le i \le n-1$. One has $\tau_{\mathfrak{p}\mathfrak{p}_i}(X^{(\mathfrak{p})}) \subset \tau_{\mathfrak{p}_{i+1}\mathfrak{p}_i}(X^{(\mathfrak{p}_{i+1})}) \subset X^{(\mathfrak{p}_i)} \cap X^{(\mathfrak{p}_{i+1})}$. By the property (b.3), $\tau_{\mathfrak{p}\mathfrak{p}_i}(U^{(\mathfrak{p})}) \subset U^{(\mathfrak{p}_i)} = V^{(\mathfrak{p}_i)}$ and, therefore, $V \cap X^{(\mathfrak{p}_{i+1})} \ne \emptyset$. By the property (b.1), the latter is a rational domain in $X^{(\mathfrak{p}_{i+1})}$, and Theorem 6.2.1 implies that $V^{(\mathfrak{p}_{i+1})} \ne \emptyset$, i.e., $\mathfrak{p}_{i+1} \in \mathcal{I})(V)$, which contradicts the maximality of i.

It remains to prove the implication (b)$\Longrightarrow$(a) and the property (i). We already know that all of the connected components V of U are rational domains, and we are going to show that the K-affinoid algebras A_V form a twisted datum of K-affinoid algebras whose twisted product defines an affinoid domain structure on U.

Step 9. We define a partial ordering on the set $\pi_0(U)$ as follows: $V \le W$ if $\mathfrak{p}^{(V)} \supset \mathfrak{p}^{(W)}$, where $\mathfrak{p}^{(V)}$ is maximal among the Zariski prime ideals from $\mathcal{I}(U)$ (i.e., $\mathfrak{p}^{(V)}$ is the minimal element of $\mathcal{I}(U)$). *We claim that this partial ordering admits the infimum operation.* Indeed, if $V_1, V_2 \in \pi_0(U)$, the property (b.2) implies that $\mathfrak{q} = \mathfrak{p}^{(V_1)} \cup \mathfrak{p}^{(V_2)} \in \mathcal{I}(U)$. Let V be the connected component of U that contains $U^{(\mathfrak{q})}$. Since $\mathfrak{p}^{(V)} \supset \mathfrak{q}$, it follows that $V \le V_1, V_2$. To verify the equality $V = \inf(V_1, V_2)$, we have to verify the following fact: given connected components V and W of U, if $\mathfrak{p}^{(W)} \supset \mathfrak{q}$ for some $\mathfrak{q} \in \mathcal{I}(V)$, then $\mathfrak{p}^{(W)} \supset \mathfrak{p}^{(V)}$. Suppose the latter is not true, and let $\mathfrak{q} \in \mathcal{I}(V)$ be maximal among the Zariski prime ideals from $\mathcal{I}(U)$ with $\mathfrak{q} \subset \mathfrak{p}^{(W)}$. Since $\mathfrak{q} \ne \mathfrak{p}^{(V)}$ and V is connected, there exists a Zariski prime ideal $\mathfrak{r} \in \mathcal{I}(V)$ with $\mathfrak{q} \subset \mathfrak{r}$, $\mathfrak{q} \ne \mathfrak{r}$ and $\check{V}_{\mathfrak{r}} \cap V^{(\mathfrak{q})} \ne \emptyset$. One has $\tau_{\mathfrak{q}\mathfrak{p}^{(W)}}(V^{(\mathfrak{q})}) \subset W^{(\mathfrak{p}^{(W)})}$ and $\tau_{\mathfrak{q}\mathfrak{p}^{(W)}}(\check{V}_{\mathfrak{r}} \cap V^{(\mathfrak{q})}) \subset \check{U}_{\mathfrak{r} \cup \mathfrak{p}^{(W)}}$. Since $W^{(\mathfrak{p}^{(W)})} = \check{W}_{\mathfrak{p}^{(W)}}$, it follows that $\mathfrak{r} \cup \mathfrak{p}^{(W)} = \mathfrak{p}^{(W)}$, which is a contradiction.

Step 10. *For every connected component V of U, there exists a rational domain $Y \subset X$ such that V is the minimal connected component of Y and $U \cap Y$ is the union of the connected components W of U with $V \le W$.* Indeed, for each Zariski prime ideal $\mathfrak{q} \subset A$ which does not lie in $\mathfrak{p}$, take an element $f_{\mathfrak{q}} \in \mathfrak{q} \backslash \mathfrak{p}$ and denote by f the product of such elements $f_{\mathfrak{q}}$. Then for any $r > 0$ the Laurent domain $Y_r = \{x \in X \big| |f(x)| \geq r\}$ has empty intersection with every connected component W of U with $V \not\leq W$. On the other hand, if r is sufficiently small, Y_r contains all of the connected components W of U with $V \leq W$. Replacing X by such Y_r, we may assume that V is the minimal connected component of U and $U \cap X_{\mathbf{m}} \neq \emptyset$. Let X' be the minimal connected component of X. By Step 8, V is the only connected component of U that lies in X'. By Step 4, V is a Weierstrass domain in X', i.e., $V = \{x' \in X' \big| |f_i(x')| \leq r_i \text{ for all } 1 \leq i \leq n\}$ with $f_1, \dots, f_n \in A_{X'}$. The K-affinoid algebra A is the twisted product of the K-affinoid algebras $A_{X''}$ of connected components X'' of X. We can therefore consider the elements f_i as elements of A. By (b.3), the set U lies in the Weierstrass domain $Y = \{x \in X \big| |f_i(x)| \leq r_i \text{ for all } 1 \leq i \leq n\}$, which possesses the required properties.

Step 11. Let V and W be connected components of U with $U < W$, and let Y be a rational domain in X with the properties of Step 10 for U. then Y contains W, and W lies in a connected component Y' of Y different from U. It follows that there is an associated bounded quasi-homomorphism $A_V \to A_{Y'}$ whose composition with the restriction homomorphism $A_{Y'} \to A_W$ gives a bounded quasi-homomorphism $\nu_{VW} : A_V \to A_W$. It is easy to verify that the system of quasi-homomorphisms ν_{VW} possesses the properties of Definition I.3.1.1, and so we get a disconnected twisted datum of K-affinoid algebras $\{\pi_0(U), A_V, \nu_{VW}\}$.

Step 12. We set $A_U = \prod^{\nu}_{\pi_0(U)} A_V$. It is clear that the canonical bounded homomorphism $A \to A_U$ induces a homeomorphism $\mathcal{M}(A_U) \widetilde{\to} U$. Let $\varphi : A \to B$ be a bounded homomorphism to an arbitrary quasi-affinoid algebra B for which the image of $Y = \mathcal{M}(B)$ in X lies in U. That the required fact is true if U is a rational domain or an idempotent domain is verified in the proof of Lemmas 6.1.6 and 6.1.11. The assumption (b) implies that the required fact is true if U is connected. Suppose that U is not connected and that the required fact is true for subsets of X possessing the property (b) and having strictly smaller number of connected components. By Step 10, we can replace X by a rational domain and assume that U contains the minimal connected component of X. Let J be the image of I_A in B. By Examples 2.4.2 and I.3.2.4, the homomorphism φ gives rise to a morphism of disconnected twisted data $\{\check{I}_A, A^{(e)}, \nu_{e_1 e_2}\} \to \{J, B^{(f)}, \nu_{f_1 f_2}\}$ and, for every $f \in J$, $\mathcal{M}(B^{(f)})$ is the preimages of the connected component $X^{(f')}$ of X that corresponds to the idempotent $f' = \sup\{e \in \check{I}_A \big| \varphi(e) \leq f\}$. Applying the induction hypothesis to the intersections $U \cap X^{(f')}$, we get the required bounded homomorphism $A_U \to B$. ∎

6.3.2. Corollary. *Let $K \to K'$ be an isometric homomorphism of real valuation $\mathbf{F}_1$-fields,* *and $A \to A'$ a bounded homomorphism from a K-affinoid algebra A to a K'-affinoid algebra A' which is consistent with the previous homomorphism and induces a homeomorphism $\varphi : X' = \mathcal{M}(A') \to X = \mathcal{M}(A)$. Then the correspondence $U \mapsto \varphi^{-1}(U)$ gives rise to a bijection between the families of affinoid domains in X and in X'.* ∎

6.3.3. Example. The assumptions of Corollary 6.3.2 are satisfied in the following cases:

(1) $A' = A\widehat{\otimes}_K K'$;

(2) $K' = |K| = K/K^{**}$ and $A' = A/K^{**}$;

(3) $K' = K$ and $A/\mathrm{z}\mathbf{n}(A) \xrightarrow{\sim} A'/\mathrm{z}\mathbf{n}(A')$;

(4) if the cokernel of the homomorphism of abelian groups $K^* \to K'^*$ is finitely generated then, any K'-affinoid algebra A', A is A' viewed as a K-affinoid algebra.

6.3.4. Corollary. *Let k be a non-Archimedean field, $K \to k^{\cdot}$ an isometric homomorphism of $\mathbf{F}_1$-algebras, and $\mathcal{B}$ a k-affinoid algebra. Given a bounded homomorphism of K-algebras $A \to \mathcal{B}^{\cdot}$, the following is true:*

(i) the preimage of any affinoid subdomain U of $X = \mathcal{M}(A)$ with respect to the induced map $\varphi : Y = \mathcal{M}(\mathcal{B}) \to X$ is an affinoid subdomain of Y;

(ii) if $\varphi(Y) \subset U$, then the image of the map $\mathcal{M}(\mathcal{B}^{\cdot}) \to X$ also lies in U.

Proof. Since the Banach K-algebra $\mathcal{B}^{\cdot}$ is quasi-affinoid, both statements easily follow from Theorem 6.3.2. ∎

6.4. The relative interior and boundary of a morphism. Let $\varphi : Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be a morphism of K-affinoid spaces.

6.4.1. Definition. The *relative interior* of φ is the subset $\mathrm{Int}(Y/X) \subset Y$ consisting of the points $y \in Y$ for which the K-algebra $\widetilde{\chi}_y(\widetilde{B})$ is integral over $\widetilde{\chi}_y(\widetilde{A})$. The *relative boundary* of φ is the complement $\delta(Y/X)$ of $\mathrm{Int}(Y/X)$ in Y. If $A = K$, the set $\mathrm{Int}(Y/X)$ (resp. $\delta(Y/X)$) is denoted by $\mathrm{Int}(Y)$ (resp. $\delta(Y)$) and is called the *interior* (resp. *boundary*) of Y.

In other words, the relative interior $\mathrm{Int}(Y/X)$ consists of the points $y \in Y$ with the property that, for every non-nilpotent element $g \in B$ with $|g(y)| = \rho(g)$, one has $g(y)^n = f(y)$ for some $n \ge 1$ and $f \in A$ with $|f(y)| = \rho(f)$. Notice that the above objects do not change if we replace X and Y by $\mathcal{M}(A/\mathbf{n}(A))$ and $\mathcal{M}(B/\mathbf{n}(B))$, or by $\mathcal{M}(A/K^{**})$ and $\mathcal{M}(B/K^{**})$, respectively.

For example, suppose that $B = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$, Then, for an element $g = fT_1^{\nu_1}\cdot\ldots\cdot T_n^{\nu_n} \in B$ with $f \in A$, one has $\rho(g) = \rho(f)r_1^{\nu_1}\cdot\ldots\cdot r_n^{\nu_1}$. It follows that, if $|g(y)| = \rho(g) \ne 0$, then $|T_i(y)| = r_i$ for all i with $\nu_i \ne 0$. This implies that $y \in \mathrm{Int}(Y/X)$ if and only if $|T_i(y)| < r_i$ for all $1 \le i \le n$.

6.4.2. Proposition. *(i) If there exists an admissible epimorphism* $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B : T_i \mapsto g_i$ *such that* $|g_i(y)| < r_i$ *for all* $1 \le i \le n$*, then* $y \in \mathrm{Int}(Y/X)$*;*

(ii) if φ *is a finite morphism, then* $\mathrm{Int}(Y) = Y$*;*

(iii) given a second morphism $\psi : Z \to Y$*, one has*

$$\mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Z/X) \subset \mathrm{Int}(Z/Y)$$

and, if the kernel of the canonical homomorphism $\mathcal{H}(\psi(z))^* \to \mathcal{H}(z)^*$ *lies in the image of* K^{**} *for all points* $z \in Z$*, then the first inclusion is an equality;*

(iv) for a morphism $\psi : X' \to X$*, one has* $\psi'^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Y'/X')$ *where* ψ' *is the canonical morphism* $Y' = Y \times_X X' \to Y$*;*

(v) for an $\mathbf{F}_1$*-valuation field* K' *over* K*, one has* $\psi^{-1}(\mathrm{Int}(Y/X)) = \mathrm{Int}(Y\widehat{\otimes}K'/X\widehat{\otimes}K')$*, where* ψ *is the canonical map* $Y\widehat{\otimes}K' \to Y$ *(which is a bijection).*

The converse implication in (i) is not true in general (but see Proposition 6.4.3), and the first inclusion in (iii) is not necessarily an equality (see Remark 6.4.11). But the converse implication in (ii) is true (see Proposition 6.4.9). By Lemma 6.1.4, the assumption on the morphism ψ in (iii) holds in the case when Z is an affinoid subdomain of Y.

Proof. (i) Let $\psi : C = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B$ be the epimorphism considered. By Proposition 5.3.8, the induced homomorphism $\widetilde{C} \to \widetilde{B}$ is finite and, therefore, $\widetilde{\chi}_y(\widetilde{B})$ is integral over $\widetilde{\chi}_y(\widetilde{\psi}(\widetilde{C}))$. The latter ring coincides with $\widetilde{\chi}_y(\widetilde{A})$ because $\rho(\psi(T_i)) < r_i$, $1 \le i \le n$. It follows that $\widetilde{\chi}_y(\widetilde{B})$ is integral over $\widetilde{\chi}_y(\widetilde{A})$.

(ii) If φ is finite, then $\widetilde{B}$ is a finite $\widetilde{A}$-algebra, by Proposition 4.3.8, and it follows that $\mathrm{Int}(Y/X) = Y$.

(iv) follows from Corollary 5.3.10, and (iii) and (v) are trivial. ∎

6.4.3. Proposition. *(i) If* $y \in \mathrm{Int}(Y/X)$*, then there exists a rational neighborhood* V *of* y *and an admissible epimorphism* $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B_V : T_i \mapsto g_i$ *such that, for* $X' = \mathcal{M}(A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\})$*, one has* $y \in \mathrm{Int}(V/X) = V \cap \mathrm{Int}(X'/X)$ *and, in particular,* $|g_i(y)| < r_i$ *for all* $1 \le i \le n$*;*

(ii) if the quotient of B *by any Zariski prime ideal is integral (e.g.,* B *is quasi-integral), then (i) is true for* $V = Y$*;*

(iii) in the situation of (ii), one has $\mathrm{Int}(Y) = \pi^{-1}(\{\mathfrak{p} \in \mathrm{Zspec}(\widetilde{A}) \big| \kappa(\mathfrak{p})$ *is a finite extension of* $\widetilde{K} = K\})$*.*

Proof. (i) and (ii). Let $C_r = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B : T_i \mapsto g_i$ be an admissible epimorphism. Then $|g_i(z)| \le r_i$ for all $z \in Y$ and $1 \le i \le n$. For $z \in \mathrm{Int}(Y/X)$, let $I(z) = \{i \big| |g_i(z)| = r_i\}$. If $i \in I(z)$, then $g_i(z)^{m_i} = f_i(z)$ in $\mathcal{H}(z)$ for some $m_i \ge 1$ and $f_i \in A$ with $\rho(f_i) = |f_i(z)|$. In the situation of (ii), the quotient $B/\mathrm{Zker}(\chi_z)$ is integral. It follows that it embeds in $\mathcal{H}(z)$ and, therefore, g_i^m is equal to the image of f_i in B. Lemma 2.3.6 implies that, for any system of numbers $s_i > r_i$ with $i \in I(z)$, the epimorphism $C_{r'} \to B$ is admissible, where $r'_i = s_i$ for $i \in I(z)$ and $r'_i = r_i$ for $i \notin I(z)$. Setting now $I = \bigcup_{z \in \mathrm{Int}(Y/X)} I(z)$ and taking numbers $s_i > r_i$ for each $i \in I$, we get an admissible epimorphism $C_{r'} \to B$ with the required property, where $r'_i = s_i$ for $i \in I$ and $r'_i = r_i$ for $i \notin I$.

In the general case of (i), we can find an element $h \in B$ with $h(y) \ne 0$ and $g_i^{m_i} h = f_i h$ for all $1 \le i \le l$. If $0 < p < |h(y)|$, then $V = \{z \in Y \big| |h(z)| \ge p\}$ is an affinoid neighborhood of the point y, and the images of $g_i^{m_i}$ and f_i in B_V coincide. Furthermore, ψ can be extended to an admissible epimorphism $C\{pT_{n+1}\}\{r_1^{-1}T_1, \dots, r_l^{-1}T_l\} : T_{n+1} \mapsto h^{-1}$. By the construction, we have $|h^{-1}(y)| < p^{-1}$, and the required fact follows from Lemma 2.3.6 as above.

(ii) The set on the left hand side contains that on the right hand side because the homomorphism $\widetilde{\chi}_y : \widetilde{B} \to \widetilde{\mathcal{H}(y)} = \mathcal{H}(y)$ is the composition of the homomorphisms $\widetilde{B} \to \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{p}) \to \widetilde{\mathcal{H}(y)} = \mathcal{H}(y)$. To prove the converse inclusion, we need the following fact. *The quotient of $\widetilde{B}$ by any Zariski ideal $\mathfrak{p}$ is integral and, if $\pi(y) = \mathfrak{p}$, the homomorphism $\kappa(\mathfrak{p}) \to \mathcal{H}(y)$ is injective.* Suppose there are elements $f, g, h \in A$ with $\widetilde{f}, \widetilde{g}, \widetilde{h} \notin \mathfrak{p}$ and $\widetilde{f}\widetilde{h} = \widetilde{g}\widetilde{h}$. Then $|f(y)| = \rho(f)$, $|g(y)| = \rho(g)$ and $|h(y)| = \rho(h)$. It follows that $|(fh)(y)| = \rho(f)\rho(h) \ge \rho(fh)$ and, therefore, $|(fh)(y)| = \rho(fh)$ and $\widetilde{fh} = \widetilde{f}\widetilde{h}$. Similarly, one has $|(gh)(y)| = \rho(gh)$ and $\widetilde{gh} = \widetilde{g}\widetilde{h}$. We get $\widetilde{fh} = \widetilde{gh}$ and, in particular, $f(y) = h(y)$. By the assumption, the homomorphism $B/\mathrm{Zker}(\chi_y) \to \mathcal{H}(y)$ is injective and, therefore, $f = g$ and, in particular, $\widetilde{f} = \widetilde{g}$. Thus, the quotient $\widetilde{B}/\mathfrak{p}$ is integral. It remains to verify injectivity of the homomorphism $\widetilde{B}/\mathfrak{p} \to \mathcal{H}(y)$. Given $f, g \in A$ with $\widetilde{f}, \widetilde{g} \notin \mathfrak{p}$ and $\widetilde{f(y)} = \widetilde{g(y)}$, it follows that $f(y) = g(y)$. The assumption implies that $f = g$ and, in particular, $\widetilde{f} = \widetilde{g}$. the fact we have just verified implies that, if $y \in \mathrm{Int}(Y)$, the $\mathbf{F}_1$-field $\kappa(\mathfrak{p})$ is algebraic over $\widetilde{K} = K$. Since it is finitely generated over K, the required statement follows. ∎

6.4.4. Proposition. *The following four subsets of Y coincide:*

(1) $\mathrm{Int}(Y/X)$*;*

(2) $\{y \in Y \big| y \in \mathrm{Int}(Y^{(\mathfrak{q})}/X)$ *for every Zariski prime ideal* $\mathfrak{q} \subset B$ *with* $y \in Y_{\mathfrak{q}}\}$*;*

(3) $\{y \in Y \big| y \in \mathrm{Int}(Y'/X)$ *for every irreducible component* Y' *of* Y *with* $y \in Y'\}$*;*

(4) $\{y \in Y \big| y \in \mathrm{Int}(Y^{(\mathfrak{q})}/X)$ *for the Zariski prime ideal* $\mathfrak{q} \subset B$ *with* $y \in \check{Y}_{\mathfrak{q}}\}$*.*

Proof. (1)$\subset$(2). By Proposition 6.4.2(ii), one has $\mathrm{Int}(Y^{(\mathfrak{q})}/Y) = Y$, and the required inclusion follows from the statement (iii) applied to the composition $Y^{(\mathfrak{q})} \to Y \to X$. The inclusions (2)$\subset$(3)$\subset$(3) are trivial.

(4)$\subset$(1). One has $\mathcal{H}(y) \xrightarrow{\sim} \mathcal{H}_{Y^{(\mathfrak{q})}}(y) = \kappa(\mathfrak{q})$, and so $y \in \mathrm{Int}(Y/X)$, by Proposition 6.4.2(iii). ∎

6.4.5. Corollary. *The subsets* $\mathrm{Int}(Y/X)$ *and* $\delta(Y/X)$ *are open and closed, respectively.*

Proof. Proposition 6.4.4 reduces the situation to the case when Y is integral, and the required fact follows from Proposition 6.4.3(i). ∎

6.4.6. Proposition. *If Y is an affinoid domain in X, then* $\mathrm{Int}(Y/X)$ *coincides with the topological interior of Y in X.*

Proof. Suppose first that a point $y \in Y$ lies in the topological interior of Y in X. Then we can find a Laurent domain $V = \{x \in X \big| |f_i(x)| \le p_i, |g_j(x)| \le q_j\}$ which is contained in Y and such that $|f_i(y)| < p_i$ and $|g_j(y)| > q_j$ for all $1 \le i \le m$ and $1 \le j \le n$. By Lemma 6.1.4, one has $\mathcal{H}_Y(y) \xrightarrow{\sim} \mathcal{H}_V(y)$. We may therefore assume that $Y = V$. Then there is an admissible epimorphism $\psi : A\{p_i^{-1}T_i, q_j S_j\} \to B : T_i \mapsto \overline{f}_i, S_j \mapsto \overline{g}_j^{-1}$ such that $|\overline{f}_i(y)| < p_i$ and $|\overline{g}_j^{-1}(y)| < q_j$, where $\overline{f}_i$ and $\overline{g}_j$ are the images of f_i and g_j in B, respectively. This means that $y \in \mathrm{Int}(Y/X)$.

Suppose now that $y \in \mathrm{Int}(Y/X)$. By Proposition 6.4.4, we may assume that Y is integral and then, by Proposition 6.4.3(i), there exists an admissible epimorphism $A\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to B : T_i \mapsto g_i$ with $|g_i(y)| < r_i$ for all $1 \le i \le n$. Then the open subset $\{x \in X \big| |g_i(x) < r_i$ for all $1 \le i \le n\}$ contains the point y and is contained in Y. ∎

6.4.7. Corollary. *If Y is an affinoid domain in X and B is a finite Banach A-algebra, then Y is an idempotent domain (and, in particular, $A \to B$ is an admissible epimorphism).*

Proof. Proposition 6.4.2(ii) implies that $\mathrm{Int}(Y/X) = Y$, i.e., Y is an open and closed subset of X, and the required fact follows from Proposition 6.1.12. ∎

6.4.8. Proposition. $\mathrm{Int}(Y/X) = Y$ *if and only if the morphism $\varphi : Y \to X$ is finite.*

Proof. The converse implication follows from Proposition 6.4.2(ii). Assume therefore that $\mathrm{Int}(Y/X) = Y$. Given a non-nilpotent element $g \in B$, let y be a point from the Shilov boundary of B with $|g(y)| = \rho(g)$.

and consider first the case when Y is integral. Given a nonzero element $g \in B$, let y be a point from the Shilov boundary of B with $|g(y)| = \rho(g)$. Then $\mathrm{Zker}(\chi_y) = 0$ and, therefore, the homomorphism $B \to \mathcal{H}(y)$ is injective. Since $\widetilde{g}^m = \widetilde{f}$ in $\widetilde{\mathcal{H}(y)} = \mathcal{H}(y)$ for some $m \le 1$ and $f \in A$, it follows that $g^m = f$, i.e., g is integral over A. In the general case, we take a morphism of quasi-integral data $\{I', A_{i'}, \nu_{i'j'}, \mathbf{a}_{j'i'}\} \to \{I, B_i, \nu_{ij}, \mathbf{b}_{ji}\}$ with a map $I \to I' : i \mapsto i'$ and homomorphisms

$f_i : A_{i'} \to B_i$ that represents the homomorphism $A \to B$. Given a non-nilpotent element $b \in B$, we can replace it by its power so that, if $b = (b_i)_{i\in I} \in \mathbf{b}^{(i)}$, then b_i is not nilpotent in B_i. By the previous case, one has $b_i^m = f_i(a_{i'})$ in B_i for some $m \geq 1$ and an element $a = (a_{i'})_{i'\in I'} \in A$. It follows that $b^{m+1} = ab$ in B, i.e., b is integral over A. ∎

6.4.9. Proposition. *Let $y \in Y$. Suppose that $U_1, \dots, U_n$ are affinoid domains in X that contain the point $x = \varphi(y)$ and such that $U_1 \cup \ldots \cup U_n$ is a neighborhood of x, and suppose that V_i is an affinoid neighborhood of y in $\varphi^{-1}(U_i)$. Then $y \in \mathrm{Int}(Y/X)$ if and only if $y \in \mathrm{Int}(V_i/U_i)$ for all $1 \leq i \leq n$.*

Proof. If $y \in \mathrm{Int}(Y/X)$, then $y \in \mathrm{Int}(\varphi^{-1}(U_i)/U_i)$ for all $1 \leq i \leq n$, by Proposition 6.4.2(v). Proposition 6.4.6 implies that $y \in \mathrm{Int}(V_i/\varphi^{-1}(U_i))$ and, therefore, $y \in \mathrm{Int}(V_i/U_i)$, by Proposition 6.4.2(iii) and (iv). Conversely, suppose that $y \in \mathrm{Int}(V_i/U_i)$ for all $1 \leq i \leq n$. Consider first the case when $n = 1$. In this case Proposition 6.4.2(iii) implies that $y \in \mathrm{Int}(\varphi^{-1}(U)/U)$ Since $x \in \mathrm{Int}(U/X)$, the same fact implies that $y \in \mathrm{Int}(\varphi^{-1}(U)/X) = \mathrm{Int}(\varphi^{-1}(U)/Y) \cap \mathrm{Int}(Y/X) \subset \mathrm{Int}(Y/X)$. Thus, we can replace X by a small affinoid neighborhood of x (and Y by its preimage), and so we may assume that $X = \bigcup_{i=1}^n U_i$ and $V_i = \varphi^{-1}(U_i)$ for all $1 \leq i \leq n$. Furthermore, we may assume that both X and Y (and therefore all U_i and V_i) are reduced. By Proposition 6.4.4, we may assume that Y is irreducible and, replacing X by the irreducible component of X that contains the image of Y, we may assume that X is also irreducible. In this case, all of the affinoid domains U_i are rational, i.e., they are of the form $\{x' \in X \big| |g(x')| \geq q, \ |f_i(x')| \leq p_i|g(x')| \text{ for all } 1 \leq i \leq n\}$, where $f_1, \dots, f_n, g \in A$ and $p_1, \dots, p_n, g > 0$. Replacing X be an affinoid neighborhood, which is the intersection of Laurent domains of the form $\{x' \big| |g(x')| \leq q'\}$ for some $0 < q' < q$, we may assume that all U_i's are Weierstrass domains.

We claim that one can find $f = (f_1, \dots, f_m) \in A^m$ and $p = (p_1, \dots, p_m) \in (\mathbf{R}_+^*)^m$ such that the covering $\{U_i\}_{1\leq i\leq n}$ of X has a Laurent refinement, i.e., a refinement of the form $\{X((p^{-1}f)^\varepsilon)\}_\varepsilon$, where

$$X((p^{-1}f)^\varepsilon) = \{x' \in X \big| |f_j(x')| \leq p_j, \text{ if } \varepsilon_j = +1, \text{ and } |f_j(x')| \geq p_j, \text{ if } \varepsilon_j = -1\} \ ,$$

and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{\pm\}^m$. Indeed, if $U_i = X(p_{i1}^{-1}f_{i1}, \dots, p_{ik_i}^{-1}f_{ik_i})$, then such a Laurent refinement is defined by the tuples $f = (f_{ij})_{1\leq i\leq n, 1\leq j\leq k_i}$ and $p = (p_{ij})_{1\leq i\leq n, 1\leq j\leq k_i}$. To verify this, take $\varepsilon = \{\varepsilon_{ij}\}_{1\leq i\leq n, 1\leq j\leq k_i}$ with nonempty Laurent domain $U = X((p^{-1}f)^\varepsilon)$ and such that for every $1 \leq i \leq n$ there is $1 \leq j \leq k_i$ with $\varepsilon_{ij} = -1$. Suppose that U is not contained in any U_i. Then for every $1 \leq i \leq n$ one has $U \cap U_i = \{x' \in U \big| |f_{ij}(x')| = p_{ij} \text{ for all } j \text{ with } \varepsilon_{ij} = -1\}$. It follows that the intersection $\check{U} \cap \check{U}_i$ is a generalized subpolytope of $\check{U}$ of dimension strictly less than that of $\check{U}$. This contradicts the inclusion $\check{U} \subset \bigcup_{i=1}^{n}(\check{U} \cap \check{U}_i)$, and the claim follows.

By the claim, we may assume that $\{U_i\}_{1 \le i \le n}$ is a Laurent covering and, by induction on n, we may assume that it is a covering of the form $X = U_1 \cup U_2$ with $U_1 = \{x' \in X \big| |f(x')| \le p\}$ and $U_2 = \{x' \in X \big| |f(x')| \ge p\}$. To verify the proposition in this case, we need the following fact (cf. [Ber1, Lemma 2.5.18]).

6.4.10. Lemma. *Let A be an $\mathbf{F}_1$-subalgebra of an $\mathbf{F}_1$-algebra B, and let f be an invertible element of A. Then the intersection of the integral closures of $A[f]$ and $A[f^{-1}]$ in B coincides with the integral closure of A in B.*

Proof. Let g be an element from the intersection. Then $g^m = af^kg^n$ and $g^p = bf^{-l}g^p$ for some $a, b \in A$, $k, l \ge 0$, $m > n \ge 0$, and $p > q \ge 0$. It follows that $g^{ml+lk} = a^lb^kg^{nl+qk}$, i.e., g is integral over A. ∎

If $V_1 = \varphi^{-1}(U_1)$ and $V_2 = \varphi^{-1}(U_2)$, the assumption implies that $\widetilde{\chi}_y(\widetilde{B}_{V_1})$ and $\widetilde{\chi}_y(\widetilde{B}_{V_2})$ are integral over $\widetilde{\chi}_y(\widetilde{A}_{U_1})$ and $\widetilde{\chi}_y(\widetilde{A}_{U_2})$, respectively, and we have to deduce that $\widetilde{\chi}_y(\widetilde{B})$ is integral over $\widetilde{\chi}_y(\widetilde{A})$. By Proposition 5.3.8 applied to the admissible epimorphisms $A\{p^{-1}T\} \to A_{U_1} : T \mapsto f$ and $A\{pT\} \to A_{U_2} : T \mapsto f^{-1}$, $\widetilde{A}_{U_1}$ and $\widetilde{A}_{U_2}$ are finite algebras over $\widetilde{A}[\widetilde{f}]$ and $\widetilde{A}[\widetilde{f}^{-1}]$, respectively. It follows that $\widetilde{\chi}_x(\widetilde{A}_{U_1})$ and $\widetilde{\chi}_x(\widetilde{A}_{U_2})$ are integral over $\widetilde{\chi}_x(\widetilde{A})[f(x)]$ and $\widetilde{\chi}_x(\widetilde{A})[f(x)^{-1}]$, respectively. It follows that $\widetilde{\chi}_y(\widetilde{B})$ is integral over $\widetilde{\chi}_y(\widetilde{A})[f(y)]$ and $\widetilde{\chi}_y(\widetilde{A})[f(y)^{-1}]$, and the required fact follows from Lemma 6.4.10. ∎

§7. Further properties of K-affinoid spaces

7.1. Affinoid subdomains of $\mathcal{M}(A)$ and open subschemes of Spec(A)**.** Let X be a K-affinoid space $\mathcal{M}(A)$, and let $\mathcal{X}$ be the affine scheme Spec(A). Recall that $\mathcal{X}^{\mathrm{an}}$ denotes the space of all homomorphisms of $\mathbf{F}_1$-algebras $A \to \mathbf{R}_+$ which extend the real valuation on K. It follows that there is a canonical continuous map $X \to \mathcal{X}^{\mathrm{an}}$ that identifies X with a compact subset of $\mathcal{X}^{\mathrm{an}}$, and its composition with the canonical continuous map $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ gives rise to a continuous map $X \to \mathcal{X}$. It follows easily from Theorem I. 4.4.2.1 that, for any open affine subscheme $\mathcal{U} \subset \mathcal{X}$, the canonical homomorphism $A \to A_{\mathcal{U}}$ gives rise to a homeomorphism of $\mathcal{U}^{\mathrm{an}}$ with the open subset of $\mathcal{X}^{\mathrm{an}}$ which is the preimage of $\mathcal{U}$ with respect to the map $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$.

7.1.1. Theorem. *Let U be a nonempty affinoid domain in X, and let $\mathcal{U}$ be the minimal open subscheme of $\mathcal{X}$ which contains the image of U in $\mathcal{X}$. Then* *(i) $\mathcal{U}$ is an open affine subscheme of $\mathcal{X}$;*

(ii) there is a canonical isomorphism of partially ordered sets $\pi_0(U) \xrightarrow{\sim} \pi_0(\mathcal{U})$

(iii) the canonical homomorphism $A_{\mathcal{U}} \to A_U$ is surjective.

For an open subscheme $\mathcal{U} \subset \mathcal{X}$, let $\mathcal{U}^{\mathrm{an}}$ denote the preimage of $\mathcal{V}$ with respect to the map $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$. (By the remark before the formulation of Theorem 7.1.1, $\mathcal{U}^{\mathrm{an}}$ coincide with the union of $\mathcal{V}^{\mathrm{an}}$ taken over open affine subschemes $\mathcal{V} \subset \mathcal{U}$.) Then $\mathcal{U}$ contains the image of U if and only if $\mathcal{U}^{\mathrm{an}}$ contains U.

Proof. (i) follows from Corollary I.4.5.2.

(ii) Let V be a connected component of U, and $\mathcal{V}$ the minimal open affine subscheme of $\mathcal{X}$ which contains the image of V in $\mathcal{X}$. Since the latter is equivalent the requirement that V lies in $\mathcal{V}^{\mathrm{an}}$ and connectedness of $\mathcal{V}^{\mathrm{an}}$ is equivalent of connectedness of $\mathcal{V}$, it follows that $\mathcal{V}$ is connected and, therefore, the map $\pi_0(U) \to \pi_0(\mathcal{U})$ is surjective. That it is injective follows from Theorem 6.3.1(iii) and the fact that the minimal elements of $\mathcal{I}(V)$ and $\mathcal{I}(\mathcal{V})$ coincide.

(iii) Since both A_U and $A_{\mathcal{U}}$ are twisted product over the same partially ordered set $\pi_0(U) \xrightarrow{\sim} \pi_0(\mathcal{U})$, the situation is reduced to the case when U is connected. Replacing X by $\mathcal{U}$, we get $U \cap \mathcal{X}_{\mathbf{m}} \neq \emptyset$. Then, by Theorem 6.3.1(ii.2), U is a Weierstrass domain in X and, therefore, the homomorphism $A \to A_U$ is surjective. ∎

7.1.2. Theorem. *For any open affine subscheme $\mathcal{U} \subset \mathcal{X}$, there exists an increasing sequence of nonempty affinoid domains $U_1 \subset U_2 \subset \dots$ with the following properties:*

(1) $X \cap \mathcal{U}^{\mathrm{an}} = \bigcup_{n=1}^{\infty} U_n$;

(2) U_n is a Weierstrass subdomain of U_{n+1} and lies in the topological interior of U_{n+1} in X;

(3) $\mathcal{U}$ is the minimal open subscheme of $\mathcal{X}$ that contains the image of U_1;

(4) for any finitely generated Banach A-module M, there exists $i \geq 1$ such that the canonical homomorphism $M_{\mathcal{U}} \to M_U$ is a bijection for every affinoid domain $U_i \subset U \subset X \cap \mathcal{U}^{\mathrm{an}}$.

Proof. Case 1: $\mathcal{U}$ is a principal open subset of $\mathcal{X}$. Let $\mathcal{U} = D(f)$ for $f \in A$ and, for $r > 0$, let $U^{(r)}$ denote the Laurent domain $\{x \in X \big| |f(x)| \geq r\}$. (Notice that $U^{(r)} \neq \emptyset$ if and only if $r \leq \rho(f)$.) If $r_1 = \rho(f) > r_2 > \dots$ is a strictly decreasing sequence of positive numbers that tend to zero, we set $U_n = U^{(r_n)}$. *We claim that the sequence $U_1 \subset U_n \subset \dots$ possesses the properties (1)-(4).* Indeed, validity of (1) and (2) is trivial. Since the image of U in $\mathcal{U}$ contains a point from $\mathcal{U}_{\mathbf{m}}$, the property (3) follows from Corollary I.4.2.4. To verify the property (4), suppose first that A is finitely presented over K. Lemma 5.2.3 then implies that there is a finite chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \dots \subset N_k = M$ which are finite Banach A-modules such that each quotient N_i/N_{i-1} is isomorphic to A/Π, where Π is a closed prime ideal of A. This reduces the situation to the case when $M = A/\Pi$. For such M, one has $M_{\mathcal{U}} = 0$ if and only if $(f, 0) \in \Pi$ and, in this case, $M_U = 0$. If $M_{\mathcal{U}} \neq 0$, Corollary 6.2.3 implies that, if $M_U \neq 0$, then $M_{\mathcal{U}} \xrightarrow{\sim} M_U$. Thus, if $M_{\mathcal{U}} \neq 0$, the property (4) holds for i such that r_i is at most the spectral radius of the image of f in A/Π. If A is arbitrary, we apply the previous case to the finitely presented $|K|$-affinoid algebra $\overline{A} = A/K^{**}$ and the finitely presented Banach $\overline{A}$-module $\overline{M} = M/K^{**}$. Since $\overline{M}_{\mathcal{U}} = \overline{M_{\mathcal{U}}}$ and $\overline{M}_U = \overline{M_U}$, the required fact follows.

Case 2: $\mathcal{U}$ is connected. By Theorem I.4.2.1, $\mathcal{U}$ is the minimal connected component of a principal open subset $\mathcal{V}$ of $\mathcal{X}$. By the previous case, there exists a sequence of affinoid domains $V_1 \subset V_2 \subset \dots$ in X with the properties (1)-(4) for $\mathcal{V}$. Since $\mathcal{V}$ is the minimal open subscheme of $\mathcal{X}$ that contains the image of V_i with $i \geq 1$, Theorem 7.1.1(ii) implies that there exists a unique connected component U_i of V_i whose image in $\mathcal{X}$ lies in $\mathcal{U}$. Then the sequence $U_1 \subset U_2 \subset \dots$ possesses the properties (1)-(4) for $\mathcal{U}$.

Case 3: $\mathcal{U}$ is arbitrary. By the previous case, for every connected component $\mathcal{U}^{(e)}$ of $\mathcal{U}$, where $e \in \check{I}_{A_{\mathcal{U}}}$, there exists an increasing sequence of affinoid domains $U_1^{(e)} \subset U_2^{(e)} \subset \dots$ with the properties (1)-(4) for $\mathcal{U}^{(e)}$. Suppose we are given a system of integers $k_e \geq 1$, $e \in \check{I}_{A_{\mathcal{U}}}$. *We claim that there is a system of integers $l_e \geq k_e$, $e \in \check{I}_{A_{\mathcal{U}}}$ such that $U = \bigcup_{e \in \check{I}_{A_{\mathcal{U}}}} U_{l_e}^{(e)}$ is an affinoid domain in X.* We construct the integers l_e inductively as follows. If e is a maximal element in $\check{I}_{A_{\mathcal{U}}}$, we set $l_e = k_e$. Suppose now $e \in \check{I}_{A_{\mathcal{U}}}$ is such that l_f is defined for every $f \in \check{I}_{A_{\mathcal{U}}}$ with $e < f$. By the validity of the property (1) for $\mathcal{U}^{(e)}$, we can find a sufficiently large integer $l_e \geq k_e$ such that $\tau_{\mathfrak{p}\mathfrak{q}}((U_{l_f}^{(f)})^{(\mathfrak{p})}) \subset (U_{l_e}^{(e)})^{(\mathfrak{q})}$ for every pair of Zariski prime ideals $\mathfrak{p} \in \mathcal{I}(\mathcal{U}^{(f)})$ and $\mathfrak{q} \in \mathcal{I}(\mathcal{U}^{(e)})$ with $\mathfrak{p} \subset \mathfrak{q}$. The claim now follows from Theorem 6.3.1. Using the claim, one easily constructs a required sequence of affinoid domains that possesses the properties (1)-(4) for $\mathcal{U}$. ■

7.1.3. Corollary. *Given open subschemes $\mathcal{U}, \mathcal{V} \subset \mathcal{X}$, one has $X \cap \mathcal{U}^{\mathrm{an}} \subset X \cap \mathcal{V}^{\mathrm{an}}$ if and only if $\mathcal{U} \subset \mathcal{V}$.*

Proof. The converse implication is trivial. Suppose that $X \cap \mathcal{U}^{\mathrm{an}} \subset X \cap \mathcal{V}^{\mathrm{an}}$. Let $\mathcal{U}$ and $\mathcal{V}$ be unions of open affine subschemes $\mathcal{U}^1 \cup \dots \cup \mathcal{U}^m$ and $\mathcal{V}^1 \cup \dots \cup \mathcal{V}^n$, respectively, and let $U_1^i \subset U_2^i \subset \dots$ and $V_1^j \subset V_2^j \subset \dots$ be sequences of affinoid domains in X provided by Theorem 7.1.2 for $\mathcal{U}^i$ and $\mathcal{V}^j$, respectively. By the property (3), $\mathcal{U}^i$ is the minimal open affine subscheme of $\mathcal{X}$ that contains the image of U_1^i and, therefore, $\mathcal{U}$ is the minimal open subscheme of $\mathcal{X}$ that contains the image of $U_1^1 \cup \dots \cup U_1^m$. By the assumption, the latter union lies in $V_k^1 \cup \dots \cup V_k^n$ for some $k \geq 1$. It follows that $\mathcal{V}$ contains the image of the same union and, therefore, $\mathcal{U} \subset \mathcal{V}$. ■

7.1.4. Definition. An open subset of a K-affinoid space $X = \mathcal{M}(A)$ is said to be a *Zariski open subset* if it is of the form $X \cap \mathcal{U}^{\mathrm{an}}$, where $\mathcal{U}$ is an open subscheme of $\mathcal{X} = \mathrm{Spec}(A)$.

Notice that the intersection of two Zariski open subsets is a Zariski open subset, and the preimage of a Zariski open subset under a morphism of K-affinoid spaces is a Zariski open subset.

7.2. A continuity property. Let $X = \mathcal{M}(A)$ be a K-affinoid space, and let V be a rational domain in a X, i.e., $V = \{x \in X \big| |f_i(x)| \leq p_i |g(x)|,\ |g(x)| \geq q\}$ for some $f_1, \ldots, f_n, g \in A$ and $p_1, \ldots, p_n, q > 0$. Given positive numbers $p'_i > p_i$ and $q' < q$, let V' be the bigger rational domain $\{x \in X \big| |f_i(x)| \leq p'_i |g(x)|,\ |g(x)| \geq q'\}$. Notice that V is a Weierstrass subdomain of V', and it lies in the topological interior of V' in X.

7.2.1. Theorem. *In the above situation, if M is a finitely generated Banach A-module, then for any $p'_1, \ldots, p'_n, q'$ sufficiently close to $p_1, \ldots, p_n, q$, respectively, the canonical homomorphism of A-modules $M_{V'} \to M_V$ is a bijection, where $M_V = M \widehat{\otimes}_A A_V$.*

Proof. Replacing X by the affinoid domain $\{x \in X \big| |g(x)| \geq r\}$ for some $r < q$, we reduce the situation to the case when g is invertible, i.e., both V and V' are Weierstrass domains. In this case, the homomorphisms $A \to A_V$ and $A \to A_{V'}$ are surjective and their kernels coincide with the Zariski kernels. It follows that the same is true for the homomorphisms $A_{V'} \to A_V$ and $M_{V'} \to M_V$. Thus, it suffices to show that $\mathrm{Zker}(M \to M_V) \subset \mathrm{Zker}(M \to M_{V'})$ for all $p'_1, \ldots, p'_n$ sufficiently close to $p_1, \ldots, p_n$, respectively. For this it suffices to consider the case $n = 1$, i.e., $V = \{x \in X \big| |f(x)| \leq p\}$ and $V' = \{x \in X \big| |f(x)| \leq p'\}$.

We have $M_V = M\{p^{-1}T\}/E$ and $M_{V'} = M\{{p'}^{-1}T\}/E'$, where E and E' are the closed submodules generated by the pairs (Tn, fn) with $n \in M$. By Example 1.2.4(ii), one has $||m||_V = \inf\{||n|| p^k\}$ and $||m||_{V'} = \inf\{||b|| {p'}^n\}$, where the infimums are taken over all representations $m = f^k n$ with $n \in M$ and $k \geq 0$. Suppose that the image of m in M_V is zero. Then the first infimum is zero, and we have to show that the second infimum is also zero for all $p' > p$ sufficiently close to p. Recall that by the same Example 1.2.4(ii) one has $m \in \bigcap_{k=1}^{\infty} f^k M$. For $k \geq 0$, we set $||m||_k = \inf ||n||$, where the infimum is taken over all representations $m = f^k n$ with $n \in M$ (and so $||m||_V = \inf\limits_k \{||m||_k p^k\}$).

7.2.2. Lemma. *In the above situation, the following is true:*

(i) for every nonzero element $m \in \bigcap_{k=1}^{\infty} f^k M$ there exist positive constants $C' \leq C''$ and $r = r_M(m)$ such that $C' r^k \leq ||m||_k \leq C'' r^k$ for all $k \geq 0$;

(ii) if the element m form (i) lies (resp. does not lie) in a Zariski A-submodule $M' \subset M$, then

$m \in \bigcap_{k=1}^{\infty} f^k M'$ (resp. $m \in \bigcap_{k=1}^{\infty} f^k(M/M')$) and $r_M(m) = r_{M'}(m)$ (resp. $r_M(m) = r_{M/M'}(m)$).

Proof. Let $N = \bigcap_{k=1}^{\infty} f^k M$. By Corollary I.1.5.3, one has $N = \{m \in M \big| m = fgm$ for some $g \in A\}$ and, in particular, the map $N \to N : m \mapsto fm$ is a bijection. It follows that, for the canonical isometric homomorphism $\overline{M} = M/K^{**} : m \mapsto \overline{m}$, one has $\overline{m} \in \bigcap_{k=1}^{\infty} f^k \overline{M}$ if and only if $m \in N$. One also has $||\overline{m}||_k = ||m||_k$. We may therefore replace A by $\overline{A} = A/K^{**}$ and M by $\overline{M} = M/K^{**}$ and assume that A and M are finitely presented.

(i) Step 1. Let $P = \{p \in M \big| f^k p \in N\}$ for some $k \geq 0\}$. Since M is Zariski noetherian, there exists $l \geq 0$ such that $f^l P \subset N$. Let m be a nonzero element of N. *We claim that there exist positive constant $C' \leq C''$ such that, for every $k \geq l$, one has $C'||m||_k \leq \inf ||f^l n|| \leq C''||m||_k$, where the infimum is taken over all representations $m = f^k n$.* Indeed, since $||f^l n|| \leq ||f^l|| \cdot ||n||$, the second inequality holds for $C'' = ||f^l||$. As for the first inequality, consider the bounded homomorphism of finite Banach A-modules $P \to N : p \mapsto f^l p$. By Proposition 2.2.8, this homomorphism is admissible, and so there exists a positive constant C such that, for every element $n' \in f^k P$, there exists $p \in P$ with $f^l p = n'$ and $||p|| \leq C||n'||$. If $m = f^k n$, we apply the latter for the element $f^l n$. It follows that there exists an element $p \in P$ with $f^l p = f^l n$ and $||p|| \leq C||f^l n||$. Since $m = f^k n = f^k p$, we get $||m||_k \leq ||p|| \leq C||f^l n||$, i.e., the first inequality holds for $C' = \frac{1}{C}$.

Suppose that $m = f^k n$, where $k \geq l$. Since $f^l n \in N$ and the multiplication by f on N is a bijection, it follows that the element $p_k = f^l n$ for which $m = f^{k-l} p_k$ is uniquely defined by m and k. The above claim implies that it suffices to verify the required behavior for the function $k \mapsto ||p_k||$.

Step 2. By Lemma 5.2.3, there is a chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \ldots \subset N_j = N$ such that each quotient N_i/N_{i-1} is isomorphic to a Banach A-module of the form A/Π, where Π is a closed prime ideal of A. Suppose that $m \in N_i \backslash N_{i-1}$. Then $p_k \in N_i \backslash N_{i-1}$. We identify N_i/N_{i-1} with a quotient A/Π and assume that the Banach norm on the latter coincides with the spectral norm. Let $x_1, \ldots, x_s$ be the points of the Shilov boundary of A/Π. Then

$$||p_k|| = \max_{1 \leq i \leq s} \frac{|f(x_i)|^l \cdot |m(x_i)|}{|f(x_i)|^k} .$$

It follows that for some positive constants $C' \leq C''$ one has $C' r^k \leq ||p_k|| \leq C'' r^k$ with the number $r = (\inf_i |f(x_i)|)^{-1}$.

(ii) If $m \notin M'$, both statements are trivial. Suppose therefore that $m \in M'$. The inclusion $m \in N' = \bigcap_{k=1}^{\infty} f^k M'$ follows from Corollary I.1.5.3, and the equality $r_M(m) = r_{M'}(m)$ follows from the proof of (i). ∎

If r is the number provided, by Lemma 7.2.2, for the element m, the equality $||m||_V = 0$ is equivalent to the inequality $rp < 1$. Thus, if $||m||_V = 0$, then $rp < 1$ and, therefore, $rp' < 1$ and $||m||_{V'} = 0$ for all $p' > p$ sufficiently close to p. ∎

7.2.3. Corollary. *In the situation of Theorem 7.2.1, given finitely generated Banach A-modules M and N, for any $p'_1, \dots, p'_n, q'$ sufficiently close to $p_1, \dots, p_n, q$, respectively, the canonical homomorphism of A-modules $\mathrm{Hom}_{A_{V'}}(M_{V'}, N_{V'}) \to \mathrm{Hom}_{A_V}(M_V, N_V)$ is a bijection.*

Proof. The statement follows from Theorem 7.2.1 applied to the finitely generated Banach A-module $\mathrm{Hom}_A(M, N)$ (see Corollary 2.2.8). ∎

7.2.4. Corollary. *For any affinoid domain V in X, there exists a decreasing sequence of affinoid domains $V_1 \supset V_2 \supset \dots$ such that*

(1) V_{n+1} is a Weierstrass subdomain of V_n and lies in the topological interior of V_n in X;

(2) $\bigcap_{n=1}^{\infty} V_n = V$;

(3) for any finitely generated Banach A-module M, there exists $k \geq 1$ the canonical homomorphism $M_{V_n} \to M_V$ is a bijection for every $n \geq k$.

Proof. If V is connected, it is a rational domain, and so the statement follows from Theorem 7.2.1. In the general case, we use the reasoning from the proof of Theorem 7.1.2, Case 3. Namely, by the previous case, for every connected component $V^{(e)}$ of V, where $e \in \check{I}_{A_\mathcal{V}}$, there exists a decreasing sequence of affinoid domains $V_1^{(e)} \supset V_2^{(e)} \supset \dots$ with the properties (1)-(3) for $V^{(e)}$. Suppose we are given a system of integers $k_e \geq 1$, $e \in \check{I}_{A_V}$. *We claim that there is a system of integers $l_e \geq k_e$, $e \in \check{I}_{A_\mathcal{V}}$ such that $U = \bigcup_{e \in \check{I}_{A_V}} V_{l_e}^{(e)}$ is an affinoid domain in X.* Indeed, we construct the integers l_e inductively as follows. If $e = 1$, the minimal element in $\check{I}_{A_\mathcal{V}}$, we set $l_e = k_e$. Suppose now $e \in \check{I}_{A_\mathcal{V}} \backslash \{1\}$ is such that l_f is defined for every $f \in \check{I}_{A_\mathcal{V}}$ with $f < e$. By the validity of the property (2) for $\mathcal{V}^{(e)}$, we can find a sufficiently large integer $l_e \geq k_e$ such that $\tau_{\mathfrak{p}\mathfrak{q}}((V_{l_e}^{(e)})^{(\mathfrak{p})}) \subset (V_{l_f}^{(f)})^{(\mathfrak{q})}$ for every idempotent $f \in \check{I}_{A_\mathcal{V}}$ with $f < e$ and every pair of Zariski prime ideals $\mathfrak{p} \in \mathcal{I}(V^{(e)})$ and $\mathfrak{q} \in \mathcal{I}(V^{(f)})$ with $\mathfrak{p} \subset \mathfrak{q}$. It follows that $V_{l_f}^{(f)} \leq V_{l_e}^{(e)}$ for every $f \in \check{I}_{A_V}$ with $f < e$, and this implies the claim. Using the claim, one easily constructs a required sequence of affinoid domains that possesses the properties (1)-(3) for V. ∎

7.2.5. Definition. Let M be a finitely generated A-module. The *stalk of M at a point $x \in X$* is the inductive limit $M_x = \varinjlim M_V$ taken over all affinoid domains V that contain the point x. If $M = A$, A_x is called the *stalk of X at x*.

7.2.6. Corollary. *For every point $x \in X$, there is a sufficiently small Laurent neighborhood V of x such that $M_V \stackrel{\sim}{\to} M_x$.*

Proof. Theorem 7.2.1 implies that M_x coincides with the inductive limit taken over all affinoid *neighborhoods* of x. Consider an admissible epimorphism $K\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\} \to A : T_i \mapsto f_i$. Then $\{x\} = \{y \in X \big| |f_i(y)| = |f_i(x)| \text{ for all } 1 \le i \le n\}$. It follows that the Laurent domains $V_\varepsilon = \{y \in X \big| |f_i(x)| - \varepsilon \le |f_i(y)| \le |f_i(x)| + \varepsilon \text{ for all } 1 \le i \le n\}$, $\varepsilon > 0$, form a fundamental system of affinoid neighborhoods of x. Let ε_0 be any positive number with the property that it is strictly less than each of the nonzero numbers $|f_i(x)|$. Then, for every pair $0 < \varepsilon < \varepsilon' \le \varepsilon_0$, $V_\varepsilon = \mathcal{M}(A_\varepsilon)$ is a Weierstrass domain in $V_{\varepsilon'} = \mathcal{M}(A_{\varepsilon'})$ and, in particular, the homomorphism $A_{\varepsilon'} \to A_\varepsilon$ is surjective and its kernel coincides with the Zariski kernel. It follows that the same is true for each of the homomorphisms $M_{V_{\varepsilon'}} \to M_{V_\varepsilon}$ and, therefore, for the homomorphisms $M_{V_\varepsilon} \to M_x$. The required statement now follows from the fact that the finitely generated A_V-modules M_V are Zariski noetherian (Proposition I.1.5.2). ∎

The following statement is a consequence of the previous results, and is an analog of the non-Archimedean analytic geometry fact that the k-affinoid algebra $\mathcal{A}_V$ of an affinoid domain $V \subset X = \mathcal{M}(\mathcal{A})$ is flat over $\mathcal{A}$.

7.2.7. Theorem. *Given a finitely generated Banach A-module M, a Zariski A-submodule $N \subset M$ and an affinoid domain $V \subset X$, the canonical map $N_V \to M_V$ is an admissible monomorphism and $M_V/N_V \xrightarrow{\sim} (M/N)_V$.*

Proof. Suppose first that V is a Weierstrass domain. By induction, it suffices to consider the case $V = \{x \in X \big| |f(x)| \le p\}$. We know that the homomorphism $M \to M_V$ is surjective and that its kernel coincides with its Zariski kernel, which consists of the elements $m \in M$ with $||m||_V = 0$. By Lemma 7.2.2(i), the latter is equivalent to the property $m \in \bigcap_{k=1}^\infty f^k M$ and $r_M(m) \cdot p < 1$. The same facts are true for the finite Banach A-modules N and M/N, and so the required statement follows from Lemma 7.2.2(ii).

Suppose now that V is an arbitrary affinoid domain, and let $\mathcal{V}$ be the minimal open subscheme of $\mathcal{X} = \mathrm{Spec}(A)$ that contains the image of V in $\mathcal{X}$. The canonical homomorphism $N_\mathcal{V} \to M_\mathcal{V}$ is injective and $M_\mathcal{V}/N_\mathcal{V} \xrightarrow{\sim} (M/N)_\mathcal{V}$. By Theorem 7.1.2, we can find an affinoid domain $U \subset X \cap \mathcal{V}^{\mathrm{an}}$ such that V is a Weierstrass subdomain of U and $M_\mathcal{V} \xrightarrow{\sim} M_U$, $N_\mathcal{V} \xrightarrow{\sim} N_U$ and $(M/N)_\mathcal{V} \xrightarrow{\sim} (M/N)_U$. Thus, replacing X by U, we reduce the situation to the previous case when V is a Weierstrass domain. ∎

For a nontrivial ideal E, we set $\overline{E} = \{(f,g) \big| f = \lambda f', g = \mu g' \text{ for some } \lambda, \mu \in K^{**} \text{ and } (f',g') \in E\}$. The latter is also an ideal, and it is closed if E is closed. We say that an ideal E of A is *prime modulo* K^{**} if the ideal $\overline{E}$ is prime. If E is closed and prime modulo K^{**}, then

$\mathcal{M}(A/E) = \mathcal{M}(A/\overline{E})$ is an irreducible closed subset of X. Lemma 5.2.3 implies that every finitely generated A-module M has a chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \ldots \subset N_k = M$ such that each quotient N_i/N_{i-1} is isomorphic to a Banach A-module of the form A/E, where E is an ideal closed and prime modulo K^{**}. (Recall that, by Proposition 2.2.7, Zariski A-submodules of A are finitely generated Banach A-modules.)

7.2.8. Corollary. *In the above situation, if V is an affinoid domain in X that contains the set $\bigcup_{i=1}^{k} \mathcal{M}(A/E_i)$, then there is a canonical isomorphism of Banach A-modules $M \widetilde{\to} M_V$.*

Proof. Theorem 7.2.7 easily implies that if, for a Zariski A-submodule $N \subset M$, one has $N \widetilde{\to} N_V$ and $M/N \widetilde{\to} (M/N)_V$, then $M \widetilde{\to} M_V$. This reduces the situation to the case when $M = A/E$ for a closed ideal E. In this case the preimage of V with respect to the canonical morphism of K-affinoid spaces $\mathcal{M}(A/E) \to X$ coincides with $\mathcal{M}(A/E)$ and, therefore, $A/E \widetilde{\to} A/E\widehat{\otimes}_A A_V$. ∎

7.3. K-affinoid germs. A *K-affinoid germ* is a pair (X, U), where X is a K-affinoid space and U is an affinoid domain in X. The affinoid K-germs form a category in which morphisms from (Y, V) to (X, U) are the morphisms $\varphi : Y \to X$ with $\varphi(V) \subset U$. The category of K-germs K-$\mathcal{A}germs$ is the localization of the latter category with respect to the system of morphisms $\varphi : (Y, V) \to (X, U)$ such that φ induces an isomorphism of Y with an affinoid neighborhood of U in X. Notice that this system admits calculus of right fractions, and so the set of morphisms $\mathrm{Hom}((Y, V), (X, U))$ in K-$\mathcal{A}germs$ is the inductive limit of the sets of morphisms $\varphi : V' \to X$ with $\varphi(V) \subset U$, where V' runs through a fundamental system of affinoid neighborhoods of V in Y. It follows that a morphism $\varphi : (Y, V) \to (X, U)$ is an isomorphism in K-$\mathcal{A}germs$ if it induces an isomorphisms between some affinoid neighborhoods of V and U. Notice that the correspondence $X \mapsto (X, X)$ gives rise to a fully faithful functor K-$\mathcal{A}ff \to K$-$\mathcal{A}germs$.

7.3.1. Theorem. *Let $\varphi : Y \to X$ and $\psi : Z \to X$ be morphisms of K-affinoid spaces, and let $V \subset Y$ and $W \subset Z$ be affinoid domains, and suppose that $W \subset \mathrm{Int}(Z/X)$. Then there is a canonical bijection*

$$\mathrm{Hom}((Y, V), (Z, W)) \widetilde{\to} \mathrm{Hom}(V, W) \ .$$

Proof. Let $X = \mathcal{M}(A)$, $Y = \mathcal{M}(B)$ and $Z = \mathcal{M}(C)$. By Corollary 7.2.4, W is the intersection of a decreasing sequence of affinoid domains $W_1 \supset W_2 \supset \ldots$ with $W_{i+1} \subset \mathrm{Int}(W_i/Z)$ and for which the canonical homomorphisms $C_{W_i} \to C_W$ are bijections. This implies that the map considered is injective. For the same reason, we can shrink Y and Z and assume that the canonical homomorphisms $B \to B_V$ and $C \to C_W$ are bijections, and both V and W are Weierstrass domains. Suppose now we are given a morphism of K-affinoid spaces $\varphi : V \to W$, i.e., a bounded homomorphism of K-affinoid algebras $\varphi^* : C_W \to B_V$. Since the homomorphisms $B \to B_V$ and $C \to C_W$ are bijections, the bounded homomorphism φ^* extends to a homomorphism $\beta : C \to B$, and we have to show that one can shrink V so that the homomorphism β becomes bounded.

Fix an admissible epimorphism $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to C : T_i \mapsto h_i$, and set $g_i = \beta(f_i)$. It suffices to verify that, given $1 \le i \le n$, every point $y \in V$ has an open neighborhood $\mathcal{V}$ in Y such that $|g_i(y')| \le r_i$ for all $y' \in \mathcal{V}$. One has $|g_i(y)| = |h_i(z)|$, where $z = \varphi(y)$. If $|h_i(z)| < r_i$, we can take $\mathcal{V} = \{y' \in Y \big| |g_i(y')| < r_i\}$. Suppose therefore that $|h_i(z)| = r_i$. Then $\rho(h_i) = r_i$ and, since $z \in \mathrm{Int}(Z/X)$, it follows that $h_i^n(z) = f(z)$ for some $f \in A$ with $\rho(f) = |f(x)| = r_i^n$, where x is the image of z in X. This implies that $h_i^n h' = f h'$ for some $h' \in C$ with $h'(z) \neq 0$. If $g' = \beta(h')$, then $g_i^n g' = f g'$ and $g'(y) \neq 0$. It follows that, for every point $y' \in Y$ with $g'(y') \neq 0$, one has $g_i^n(y') = f(x')$, where x' is the image of y' in X and, therefore, $|g_i(y')| \le \rho(f)^{\frac{1}{n}} = r_i$. ∎

7.3.2. Corollary. *In the situation of Theorem 7.3.1, suppose in addition that $V \subset \mathrm{Int}(Y/X)$. Then any isomorphism of K-affinoid spaces $V \xrightarrow{\sim} W$ extends to a unique isomorphism of K-affinoid germs $(Y, V) \xrightarrow{\sim} (Z, W)$.* ∎

7.4. Acyclic K-affinoid spaces. For a finite affinoid covering $\mathcal{U} = \{U_i\}_{i \in I}$ of a K-affinoid space $X = \mathcal{M}(A)$ and a finitely generated Banach A-module M, we set

$$M_{\mathcal{U}} = \mathrm{Ker}\Big(\prod_{i \in I} M_{U_i} \to \prod_{i,j \in I} M_{U_i \cap U_j}\Big) .$$

An element of $M_{\mathcal{U}}$ is a tuple $\{m_i\}_{i \in I}$ with $m_i \in M_{U_i}$ and $m_i\big|_{U_i \cap U_j} = m_j\big|_{U_i \cap U_j}$ for all $i, j \in I$. The supremum norm $||\{m_i\}_{i \in I}|| = \sup_{i \in I} ||m_i||$ defines the structure of a Banach A-module on $M_{\mathcal{U}}$. In this subsection we investigate properties of the canonical bounded homomorphism $M \to M_{\mathcal{U}}$ and give a sufficient condition for M to possess the following property.

7.4.1. Definition. A finitely generated Banach A-module M is said to be *acyclic over* X if, for any finite affinoid covering $\mathcal{U}$ of X, the homomorphism $M \to M_{\mathcal{U}}$ is an isomorphism of Banach A-modules. If $M = A$ satisfies this condition, then X is said to be *acyclic*.

7.4.2. Theorem. *Let M be a finitely generated Banach A-module. Then*

(i) for every finite affinoid covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, the canonical map $M \to M_{\mathcal{U}}$ is an admissible monomorphism;

(ii) if M has a chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \ldots \subset N_k = M$ with $N_i/N_{i-1} \xrightarrow{\sim} A/E_i$ for ideals E_i closed and prime modulo K^{**} and such that $\bigcap_{i=1}^k Y_{i,\mathbf{m}} \neq \emptyset$, where $Y_i = \mathcal{M}(A/E_i)$, then M is acyclic over X.

Proof. Step 1. *If M is isomorphic to A/E, where E is an ideal closed and prime modulo K^{**}, then M is acyclic over X.* Indeed, we can replace A by A/E and assume that $M = A$ and A is *integral modulo* K^{**}, i.e., the quotient $\overline{A} = A/K^{**}$ is integral. Since the canonical surjective homomorphism $A \to \overline{A}$ is isometric, it follows that the Banach norm on A is equivalent to the spectral norm. Furthermore, every affinoid domain U in X is rational, and it is Weierstrass if and only if $U \cap X_{\mathbf{m}} \neq \emptyset$. Finally, if $U = X(p^{-1}\frac{f}{g}, qg^{-1})$, then the canonical homomorphism $A_g \to A_U$ is a bijection. Indeed, the latter is a surjection and its kernel coincides with the Zariski kernel (Lemma 6.1.8). Since $\overline{A}_g \to \overline{A}_U$ is a bijection, the required fact follows. It follows that A_U is integral modulo K^{**}.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a finite affinoid covering of X, and let U_i be an element of the covering $\mathcal{U}$ with $U_i \cap X_{\mathbf{m}} \neq \emptyset$. Then U_i is a Weierstrass domain, and the canonical homomorphism $A \to A_{U_i}$ is a bijection. This implies that the homomorphism $A \to A_{\mathcal{U}}$ is injective. That it is an admissible monomorphism follows from the fact that the Banach norm on each U_j is equivalent to the spectral norm. To prove that it is in fact a bijection, *we claim that for every element $j \in I$ there exists a sequence $i_0 = i, i_1, \dots, i_n = j$ of elements of I such that each intersection $U_{i_{k-1}} \cap U_{i_k}$ is a nonempty Weierstrass domain in U_{i_k}.*

We prove the above claim by induction on the Zariski-Krull dimension d of A (see §I.1.3). If $d = 0$, then A is an $\mathbf{F}_1$-field, and so all affinoid domains in X are Weierstrass. Suppose that $d \geq 1$ and that the claim is true for K-affinoid algebras which are integral modulo K^{**} and of Zariski-Krull dimension smaller than d. If there exists a nonzero Zariski prime ideal $\mathfrak{p} \subset A$ with $U_{j,\mathfrak{p}} \neq \emptyset$ then, by the induction hypothesis applied to $X_{\mathfrak{p}} = \mathcal{M}(A/\mathfrak{p})$, we can find a sequence $i_0 = i, i_1, \dots, i_n = j$ in I such that each intersection $U_{i_{k-1},\mathfrak{p}} \cap U_{i_k,\mathfrak{p}}$ is a nonempty Weierstrass domain in $U_{i_k,\mathfrak{p}}$. This implies that $U_{i_{k-1}} \cap U_{i_k}$ is a nonempty Weierstrass domain in U_{i_k}. If $U_{j,\mathfrak{p}} = \emptyset$ for all nonzero Zariski prime ideals $\mathfrak{p} \subset A$, we can find a sequence $i_l, i_{l+1}, \dots, i_n = j$ in I with $U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$ and $U_{i_k,\mathfrak{p}} = \emptyset$ for all $l < k \leq n$ and all nonzero Zariski prime ideals $\mathfrak{p} \subset A$, and $U_{i_l,\mathfrak{p}} \neq \emptyset$ for some nonzero Zariski prime ideal $\mathfrak{p} \subset A$. Notice that all affinoid domains in U_{i_k} for $l < k \leq n$ are Weierstrass. By the previous case, we can find a sequence $i_0, i_1, \dots, i_l$ with the required property, and so the sequence $i_0 = i, i_1, \dots, i_n = j$ has the same property, and the claim follows.

Let now $\{f_j\}_{j\in I}$ be an element of $A_{\mathcal{U}}$. Since $A \xrightarrow{\sim} A_{U_i}$, we can view f_i as an element of A, and *we claim that $f_j = f_i\big|_{U_j}$ for all $j \in I$.* Indeed, let $i_0, i_1, \dots, i_n$ be a sequence as above, and suppose that $f_{i_l} = f_i\big|_{U_{i_l}}$ for all $0 \leq l \leq k-1$ with some $1 \leq k \leq n$. Since $U_{i_{k-1}} \cap U_{i_k}$ is a nonempty Weierstrass domain in U_{i_k}, the canonical map $A_{U_{i_k}} \to A_{U_{i_{k-1}} \cap U_{i_k}}$ is injective and, since

$f_{i_k}\big|_{U_{i_{k-1}}\cap U_{i_k}} = f_{i_{k-1}}\big|_{U_{i_{k-1}}\cap U_{i_k}} = f_i\big|_{U_{i_{k-1}}\cap U_{i_k}}$, it follows that $f_{i_k} = f_i\big|_{U_{i_k}}$. The claim follows.

Step 2. *The statement (i) is true.* Indeed, let $N_0 = 0 \subset N_1 \subset \ldots \subset N_k = M$ be a chain of Zariski A-submodules with $N_i/N_{i-1} \xrightarrow{\sim} A/E_i$, where E_i are ideals closed and prime modulo K^{**}. By Step 1, the statements (i) and (ii) are true for each of the quotients N_i/N_{i-1}. This easily implies that the canonical homomorphism $M \to M_{\mathcal{U}}$ is admissible, and so it remains to verify that it is injective. Let m and n be two distinct elements of M, and assume $m \in N_\mu$ and $n \in N_\nu$ with $\mu \le \nu$ and μ and ν are minimal with those inclusions. If $\mu < \nu$, take $i \in I$ with $\mathcal{M}(A/E_\nu) \cap U_i \ne \emptyset$. Then $(N_\nu)_{U_i} \ne (N_{\nu-1})_{U_i}$ and, therefore, the images of m and n in $M_{\mathcal{U}}$ are not equal. If $\mu = \nu$, then the images of m and n in $M_{\mathcal{U}}$ are not equal by Step 1 applied to the quotient $N_\mu/N_{\mu-1}$.

Step 3. *The statement (ii) is true.* We have to verify that, for every element $(m_i)_{i\in I} \in M_{\mathcal{U}}$, there exists an element $m \in M$ which gives rise to $(m_i)_{i\in I}$, i.e., $m_i = m\big|_{U_i}$ for all $i \in I$. Of course, we may assume that all of the affinoid domains U_i are connected and, in particular, rational domains.

Let $Y = \bigcap_{i=1}^k Y_{i,\mathbf{m}}$. *We claim that there exists an affinoid domain V that contains the set $\bigcup_{i=1}^k Y_i$ and such that $Y \cap V_{\mathbf{m}} \ne \emptyset$.* Indeed, let E'_i denote the preimage of the maximal Zariski ideal of A/E_i. Then $Y = \mathcal{M}(A/F)$, where F is the closed ideal generated by the ideals $E'_1, \ldots, E'_k$. Furthermore, let $\mathfrak{p}$ be the Zariski preimage in A of the maximal Zariski ideal of A/F. If $\mathfrak{p} = \mathbf{m}$, then $Y \cap X_{\mathbf{m}} \ne \emptyset$, and so the required property holds for X. Assume therefore that $\mathfrak{p} \ne \mathbf{m}$. Then we can find an element $f \in A\backslash\mathfrak{p}$ which lies in all Zariski prime ideals $\mathfrak{q} \supset \mathfrak{p}$ with $\mathfrak{q} \ne \mathfrak{p}$. Since $f(x) \ne 0$ for all points $x \in Y$, it follows that the same holds for all points of the sets $Y_{i,\mathbf{m}}$ and, therefore, for all points of the sets Y_i. Thus, $f(x) \ne 0$ for all points $x \in Z$ and, therefore, we can find $r > 0$ with $|f(x)| \ge r$ for all points $x \in Z$. Then the Laurent domain $V = \{x \in X \big| |f(x)| \ge r\}$ possesses the required property. By the above claim and Corollary 7.2.8, we can replace X by an affinoid subdomain and assume that it is connected and $Y \cap X_{\mathbf{m}} \ne \emptyset$. Let U_i be an element of the covering $\mathcal{U}$ which has nonempty intersection with the latter set. Since $U_i \cap X_{\mathbf{m}} \ne \emptyset$, Theorem 6.3.1(ii.2) implies that U is a Weierstrass domain and, in particular, the kernel of the canonical homomorphism $A \to A_{U_i}$ coincides with its Zariski kernel. It follows that, if $m_i \ne 0$, there exists a unique element $m \in M$ with $m\big|_{U_i} = m_i$. If $m_i = 0$, we set $m = 0$. *We claim that $m\big|_{U_j} = m_j$ for all $j \in I$.* Indeed, if $k = 1$, this was proved in Step 1. Assume therefore that $k \ge 2$ and the claim is true for finitely generated Banach A-modules with shorter chain of Zariski A-submodules as above.

Consider first the case when $m_i \in (N_{k-1})_{U_i}$. Since $N_{k-1} \xrightarrow{\sim} (N_{k-1})_{U_i}$, one has $m \in N_{k-1}$. By Step 1, applied to M/N_{k-1}, we get $m_j \in (N_{k-1})_{U_j}$ for all $j \in J$ and, therefore, $(m_j)_{j\in I} \in (N_{k-1})_{\mathcal{U}}$.

The induction hypothesis, applied to N_{k-1}, implies that $m_j = m\big|_{U_j}$ for all $j \in I$.

Suppose now that $m_i \notin (N_{k-1})_{U_i}$, i.e., $m \notin N_{k-1}$. By the induction hypothesis applied to M/N_1, we get $m\big|_{U_j} = m_j$, if $m_j \notin (N_1)_{U_j}$, and $m\big|_{U_j} \in (N_1)_{U_j}$, if $m_j \in (N_1)_{U_j}$. It remains to show that in the latter case one has $m\big|_{U_j} = m_j$. If in this case the intersection $U_j \cap Y_1$ is empty, then both $m\big|_{U_j}$ and m_j are equal to zero. Thus, we have to verify the equality $m\big|_{U_j} = m_j$ in the case when $U_j \cap Y_1 \neq \emptyset$ (and $m\big|_{U_j}, m_j \in (N_1)_{U_j}$). We use for this the reasoning from Step 1. Namely, we notice that, if $U \subset V$ are affinoid domains such that the intersection $U \cap Y_1$ is a nonempty Weierstrass domain in $V \cap Y_1$, then the canonical homomorphism $(N_1)_V \to (N_1)_U$ is injective. Let $j \in I$ be as above. By Step 1, there exists a sequence $i_0 = i, i_2, \dots, i_n = j$ of elements of I such that each intersection $U_{i_{l-1}} \cap U_{i_l} \cap Y_1$ is a nonempty Weierstrass domain in $U_{i_l} \cap Y_1$. Let l be maximal with the property $m_{i_l} \notin (N_1)_{U_{i_l}}$. (Notice that $1 \le l < j$.) Then $m_{i_l} = m\big|_{U_{i_l}}$ and, for every $l+1 \le \mu \le n$, one has $m_{i_\mu}, m\big|_{U_{i_\mu}} \in (N_1)_{U_{i_\mu}}$. Since each of the homomorphisms $(N_1)_{U_{i_\mu}} \to (N_1)_{U_{i_{\mu-1}} \cap U_{i_\mu}}$ is injective, it follows easily that $m_{i_\mu} = m\big|_{U_{i_\mu}}$ for all $l+1 \le \mu \le n$. ∎

Let $K \to K'$ be an isometric homomorphism of real valuation $\mathbf{F}_1$-fields, A' a K'-affinoid algebra, $A \to A'$ a bounded homomorphism compatible with the homomorphism $K \to K'$, and φ is the induced map $X' = \mathcal{M}(A') \to X = \mathcal{M}(A)$.

7.4.3. Corollary. *In the above situation, assume that the fibers of the map φ are finite. Let $x \in X$ and $\varphi^{-1}(x) = \{x'_1, \dots, x'_n\}$. Given a finitely generated Banach A'-module M', there exists an affinoid neighborhood U of x in X such that every affinoid domain $x \in V \subset U$ is acyclic, the preimage $\varphi^{-1}(V)$ is a disjoint union $\coprod_{i=1}^n V'_i$ and, for every $1 \le i \le n$, the affinoid domain V'_i is acyclic and $M'_{V'_i}$ is acyclic over V'_i.*

Proof. Suppose first that $K' = K$ and $A' = A$. Let $N_0 = 0 \subset N_1 \subset \ldots \subset N_k = M = M'$ be a chain of Zariski A-submodules such that each quotient N_i/N_{i-1} is isomorphic to A/E_i, where E_i is an ideal closed and prime modulo K^{**}, and set $Y_i = \mathcal{M}(A/E_i)$. Shrinking X, we may assume that $x \in \bigcap_{i=1}^k Y_i$. The Zariski ideal $\mathfrak{p} = \{f \in A \big| f(x) = 0\}$ is prime. If $\mathfrak{p} = \mathbf{m}$, then $x \in X_{\mathbf{m}}$. If $\mathfrak{p} \neq \mathbf{m}$, let f be an element in $A \backslash \mathfrak{p}$ which lies in all Zariski prime ideals $\mathfrak{q} \supset \mathfrak{p}$ different from $\mathfrak{p}$. If $0 < r < |f(x)|$, then replacing X by the Laurent domain $\{y \in X \big| |f(y)| \ge r\}$, we get $x \in X_{\mathbf{m}}$. In this case, one also has $x \in \bigcap_{i=1}^k Y_{i,\mathbf{m}}$, and Theorem 7.4.1 implies that M is acyclic over X. Since the similar inclusion holds in any affinoid domain $x \in V \subset X$, M'_V is acyclic over V.

Consider now the general case. By the above case, we can find an affinoid neighborhood U of the point x such that every affinoid domain $x \in V \subset U$ is acyclic. Furthermore, we can find pairwise disjoint affinoid neighborhoods $U'_1, \dots, U'_n$ of the points $x'_1, \dots, x'_n$ in X' such that every affinoid domain $x'_i \in V'_i \subset U'_i$ is acyclic and $M'_{V'_i}$ is acyclic over V'_i. Shrinking U, we may assume that $\varphi^{-1}(U) \subset \coprod_{i=1}^n U'_i$. This affinoid domain U possesses the required properties. ∎

7.4.4. Corollary. *Given finite affinoid coverings $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$, the following is true:*

(i) the canonical map $M_{\mathcal{U}} \to M_{\mathcal{V}}$ is an admissible monomorphism;

(ii) there exists $\mathcal{U}$ such that for any $\mathcal{V}$ the admissible monomorphism in (i) is a bijection. ∎

We shall denote by K-$\mathcal{C}aff$ the full subcategory of K-$\mathcal{A}ff$ consisting of acyclic K-affinoid spaces.

For a finite Banach A-module M, let $\langle M \rangle$ denote the filtered inductive limit $\varinjlim M_{\mathcal{U}}$ taken over all finite affinoid coverings $\mathcal{U}$ of X. By Corollary 7.4.4, there is a finite affinoid covering $\mathcal{U}$ of X with $\mathcal{M}_{\mathcal{U}} \stackrel{\sim}{\to} \langle M \rangle$, and so $\langle M \rangle$ is a Banach A-module which is not necessarily finitely generated. If $M = A$, we get a Banach A-algebra $\langle A \rangle$ which is not necessarily K-affinoid or even finitely generated over A (see Remark 7.4.5).

Given finitely generated Banach A-modules M and N, we set $\mathrm{Hom}^p_A(M,N) = \mathrm{Hom}_A(M, \langle N \rangle)$, where the right hand side is the set of homomorphisms of A-modules $M \to \langle N \rangle$. Let $\varphi : M \to \langle N \rangle$ be such a homomorphism. There is a finite affinoid covering $\mathcal{U} = \{U_i\}$ of X such that φ is goes through a homomorphism $M \to N_{\mathcal{U}}$ and, by Lemma 1.2.3, φ is bounded. Let $\mathcal{V} = \{V_k\}$ be a finite affinoid covering of X, which is a refinement of $\mathcal{U}$. Then every V_k lies in some U_i, and so the homomorphism $M \to N_{\mathcal{U}} \to N_{U_i} \to N_{V_k}$ goes through a unique bounded homomorphism of finite Banach A_{V_k}-modules $\varphi_{V_k} : M_{V_k} \to N_{V_k}$. It is easy to see that the latter homomorphism does not depend on the choice of U_i, and all φ_{V_k}'s give rise to a bounded homomorphism of Banach A-modules $\varphi_{\mathcal{V}} : M_{\mathcal{V}} \to N_{\mathcal{V}}$. It follows that φ extends to a bounded homomorphism of Banach A-modules $\langle \varphi \rangle : \langle M \rangle \to \langle N \rangle$ and, given a second homomorphism $\psi : N \to \langle P \rangle$, one can define a composition homomorphism $\psi \circ \varphi : M \to \langle P \rangle$ by $\psi \circ \varphi = \langle \psi \rangle \circ \varphi$. Thus, one can define a category A-$\mathcal{F}mod^p$ whose objects are finitely generated Banach A-modules and morphisms are the sets $\mathrm{Hom}^p_A(M,N)$.

7.4.5. Remark. Let A be the quotient of $\mathbf{F}_1\{T_1, T_2\}$ by the ideal generated by the pair $(T_1^2 T_2, T_1 T_2)$, and let f and g be the images of T_1 and T_2 in A. Then $A = \{0, 1, f^n, g^n, fg^n\}$ with $n \geq 1$, and $X = \mathcal{M}(A)$ is a union of the three irreducible components $X_1 = \{(0,t)\}$, $X_2 = \{(t,0)\}$ and $X_3 = \{(1,t)\}$ with $0 \leq t \leq 1$. Given $0 < \alpha < \beta < 1$, let $U_1 = X_1 \cup \{(t,0) \big| 0 \leq t \leq \beta\}$ and $U_2 = X_3 \cup \{(t,0) \big| \alpha \leq t \leq 1\}$. Then U_1, U_2 and $U_1 \cap U_2 = \{(t,0) \big| \alpha \leq t \leq \beta\}$ are acyclic rational subdomains of X and, in particular, $\langle A \rangle = A_{\mathcal{U}}$ for $\mathcal{U} = \{U_1, U_2\}$. but $g\big|_{U_1 \cap U_2} = fg\big|_{U_1 \cap U_2} = 0$. It follows that the Banach $\mathbf{F}_1$-algebra $\langle A\rangle$ is generated by f, g, and the elements u for which $u\big|_{U_1}=g$ and $u\big|_{U_2}=0$, v_n with $n\ge 2$ for which $v_n\big|_{U_1}=g^n$ and $v_n\big|_{U_2}=g$, and w_n with $n\ge 2$ for which $w_n\big|_{U_1}=g$ and $w_n\big|_{U_2}=g^n$. (One has $u_nw_n=g^{n+1}$, $uv_n=u^{n+1}$ and $uw_n=u^2$.) It is easy to see that $\langle A\rangle$ is not finitely generated over A.

7.5. The category K-$\mathcal{A}ff^p$. Let $X=\mathcal{M}(A)$ and $Y=\mathcal{M}(B)$ be K-affinoid spaces. For a finite affinoid covering $\mathcal{V}=\{V_i\}_{i\in I}$ of Y, we set

$$\mathrm{Hom}_{\mathcal{V}}(Y,X)=\mathrm{Ker}(\prod_{i\in I}\mathrm{Hom}(V_i,X)\overset{\to}{\to}\prod_{i,j\in I}\mathrm{Hom}(V_i\cap V_j,X))\ .$$

One has $\mathrm{Hom}_{\mathcal{V}}(Y,X)=\mathrm{Hom}(A,B_{\mathcal{V}})$, where the latter is the set of bounded homomorphisms of Banach K-algebras. Furthermore, we set

$$\mathrm{Hom}^p(Y,X)=\lim_{\longrightarrow}\mathrm{Hom}_{\mathcal{V}}(Y,X)\ ,$$

where the inductive limit is taken over finite affinoid coverings $\mathcal{V}$ of Y. (All transition maps in this inductive limit are injective.) One has $\mathrm{Hom}^p(Y,X)=\mathrm{Hom}(A,\langle B\rangle)$, and Corollary 7.4.4 implies that there is a finite affinoid covering $\mathcal{V}$ of Y such that $\mathrm{Hom}_{\mathcal{V}}(Y,X)\overset{\sim}{\to}\mathrm{Hom}^p(Y,X)$ for all X. In particular, if Y is acyclic, then $\mathrm{Hom}(Y,X)\overset{\sim}{\to}\mathrm{Hom}^p(Y,X)$ for all X.

Elements of $\mathrm{Hom}^p(Y,X)$ are said to be *p-morphisms* from Y to X. As in the previous subsection, one shows that any bounded homomorphism $A\to\langle B\rangle$ extends in a canonical way to a bounded homomorphism $\langle A\rangle\to\langle B\rangle$. Given a second morphism $Z=\mathcal{M}(C)\to Y$, i.e., a bounded homomorphism $B\to\langle C\rangle$, one can define a bounded homomorphism $A\to\langle C\rangle$ as the composition of $A\to\langle B\rangle$ with the extended homomorphism $\langle B\rangle\to\langle C\rangle$, i.e., one can define the composition p-morphism $Z\to X$. This means that there is a well defined category K-$\mathcal{A}ff^p$ whose family of objects coincides with that of K-$\mathcal{A}ff$, and in which the set of morphisms from Y to X are the sets of p-morphisms of K-affinoid spaces $\mathrm{Hom}^p(Y,X)$. The canonical functor K-$\mathcal{A}ff\to K$-$\mathcal{A}ff^p$ is faithful but not fully faithful. The functor K-$\mathcal{C}aff\to K$-$\mathcal{A}ff$ is fully faithful.

Notice that any p-morphism from Y to X represented by a system of compatible morphisms $\varphi_i:Y_i\to X$ defines a continuous map $\varphi:Y\to X$. Furthermore, there is a well defined homomorphism of K-algebras $A_x\to B_y$, where $x=\varphi(x)$. In particular, there is a well defined isometric homomorphism of real valuation $\mathbf{F}_1$-fields $\mathcal{H}(x)\to\mathcal{H}(y)$.

7.5.1. Lemma. *Given a finite covering $\{Y_i\}_{i\in I}$ of Y by affinoid domains, the following sequence of maps of sets is exact*

$$\mathrm{Hom}^p(Y,X)\to\prod_{i\in I}\mathrm{Hom}^p(Y_i,X)\overset{\to}{\to}\prod_{i,j\in I}\mathrm{Hom}^p(Y_i\cap Y_j,X)\ .\qquad\blacksquare$$

7.5.2. Remarks. (i) Let $A = \{0, 1, e\}$ with $e^2 = e$. Then $\langle A \rangle = \mathbf{F}_1 \times \mathbf{F}_1$ is an idempotent $\mathbf{F}_1$-algebra of 4 elements. A homomorphism $A \to \langle A \rangle$ can take e to any of the elements of $\langle A \rangle$, and so the set $\mathrm{Hom}(A, \langle A \rangle)$ consists of 4 elements. On the other hand, the set $\mathrm{Hom}(\langle A \rangle, \langle A \rangle)$ consists of 9 elements.

(ii) Let $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$ be $\mathbf{F}_1$-affinoid spaces. If A and B are idempotent $\mathbf{F}_1$-algebras with the same number of elements, then X and Y are not necessarily isomorphic in $\mathbf{F}_1$-$\mathcal{A}ff$, but they are isomorphic in $\mathbf{F}_1$-$\mathcal{A}ff^p$. Here is an example of connected non-isomorphic X and Y which are isomorphic in $\mathbf{F}_1$-$\mathcal{A}ff^p$. Let $A = \mathbf{F}_1\{T_1, T_2\}/E$, where E is the ideal generated by the pair $(T_1^2 T_2, T_1 T_2)$, and $B = \mathbf{F}_1\{T_1, T_2, T_3\}/F$, where F is the ideal generated by the pairs $(T_1 T_2, T_2)$, $(T_1 T_3, 0)$ and $(T_2 T_3, 0)$. Then X and Y are non-isomorphic affinoid polytopes that can be identified with the following subsets of $\mathbf{R}_+^2$ and $\mathbf{R}_+^3$, respectively:

$$X = \{(0,t)\big|0 \le t \le 1\} \cup \{(t,0)\big|0 \le t \le 1\} \cup \{(1,t)\big|0 \le t \le 1\} \ ,$$

$$Y = \{(0,0,t)\big|0 \le t \le 1\} \cup \{(t,0,0)\big|0 \le t \le 1\} \cup \{(1,t,0)\big|0 \le t \le 1\} \ .$$

For $0 < \alpha < \beta < 1$, let U_1 and U_2 (resp. V_1 and V_2) be the Laurent domains in X (resp. Y) defined by the inequalities $|T_1(x)| \le \beta$ and $|T_2(x)| \ge \alpha$, respectively. It is easy to see that there are canonical isomorphisms $U_1 \widetilde{\to} V_1$ and $U_2 \widetilde{\to} V_2$ which induce the same isomorphism $U_1 \cap U_2 \widetilde{\to} V_1 \cap V_2$. It follows that X and Y are isomorphic in $\mathbf{F}_1$-$\mathcal{A}ff^p$.

7.6. Classes of p-morphisms.

7.6.1. Definition. A p-morphism $\varphi : Y \to X$ is said to be a *p-affinoid domain embedding* or, for brevity, a *pad-embedding* if it possesses the following property: any p-morphism $\psi : Z \to X$ with $\psi(Z) \subset \varphi(Y)$ goes through a unique p-morphism $Z \to Y$.

It follows easily from the definition that, for a *pad*-embedding $\varphi : Y \to X$, the set $\varphi(Y)$ defines the morphism φ uniquely up to a unique isomorphism in K-$\mathcal{A}ff^p$. Such a subset of X is said to be a *p-affinoid domain*. Furthermore, applying the above property to the canonical morphism $\mathcal{M}(\mathcal{H}(x)) \to X$ for a point $x = \varphi(y)$, we see that $\varphi^{-1}(x) = \{y\}$, i.e., φ is an injective map, and $\mathcal{H}(x) \widetilde{\to} \mathcal{H}(y)$.

Notice that, by Lemma 7.5.1, to show that a p-morphism $\varphi : Y \to X$ is a *pad*-embedding, it suffices to verify the condition of Definition 7.6.1 for morphisms of K-affinoid spaces $\psi : Z \to X$ (with $\psi(Z) \subset \varphi(Y)$).

7.6.2. Theorem. *The following properties of a morphism of K-affinoid spaces $\varphi : Y \to X$ are equivalent:*

(a) φ is an ad-embedding;

(b) φ is a pad-embedding.

Proof. The implication (a)$\Longrightarrow$(b) is trivial. To prove the implication (b)$\Longrightarrow$(a), assume that φ is a *pad*-embedding.

Step 1. *For any morphism of K-affinoid spaces $f : X' \to X$, the morphism $\varphi' : Y' = Y \times_X X' \to X'$ is a pad-embedding.* Indeed, suppose we are given a morphism of K-affinoid spaces $\psi : Z \to X'$ with $\psi(Z) \subset \varphi'(Y')$. Then $(f \circ \psi)(Z) \subset \varphi(Y)$, and so the morphism $f \circ \psi : Z \to X$ goes through a unique p-morphism $g : Z \to Y$. The latter is represented by a compatible system of morphisms $g_i : Z_i \to Y$, $i \in I$. Since $\varphi \circ g_i = f \circ \psi\big|_{Z_i}$, g_i goes through a unique morphism $h_i : Z_i \to Y'$. The morphisms g_i and g_j are compatible on the intersection $Z_i \cap Z_j$, and so they give rise to the required p-morphism $h : Z \to Y'$.

Step 2. *For any isometric homomorphism of valuation $\mathbf{F}_1$-fields $K \to K'$, the morphism of K'-affinoid spaces $\varphi' : Y' = Y \widehat{\otimes}_K K' \to X' = X \widehat{\otimes}_K K'$ is a pad-embedding.* Indeed, suppose we are given a morphism of K'-affinoid spaces $\psi : Z \to X'$ with $\psi(Z) \subset \varphi'(Y')$. Since all of the affinoid algebras considered are finitely presented, the morphism ψ comes from a morphism of K''-affinoid space $\psi'' : Z'' \to X''$, where K'' is a valuation $\mathbf{F}_1$-field provided with isometric homomorphisms $K \to K'' \to K'$ such that the kernel and cokernel of the homomorphism of groups $K^* \to K''^*$ are finitely generated. In this case ψ'' can be considered as a morphism of K-affinoid spaces, and so the required fact follows from Step 1.

Step 3. *The theorem is true.* It suffices to verify that the image of Y in X is an affinoid domain. Using Step 1 and Theorem 6.3.1, we easily reduce the situation to the case when X is integral. Furthermore, using Steps 1 and 2, we reduce the situation to the case when $K \xrightarrow{\sim} |K|$ and X is K-polytopal. To show that the image V of Y in X is an affinoid domain, we are going to verify the property (d) of Theorem 6.2.1 using the reasoning and notation from the proof of the implication (a)$\Longrightarrow$(d) of that theorem.

Let $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(C)$, and let V be the image of Y in X. We fix an admissible epimorphism $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A : T_i \mapsto f_i$ with $f_i \neq 0$ for all $1 \leq i \leq n$. Then X is identified with a K-affinoid polytope in $\mathbf{R}_+^n$ and V is identified with an $K_{\mathbf{Z}_+}$-subpolytope of X. It suffices to show that $\mathrm{rec}_I(\check{X}) \subset \mathrm{rec}(\check{V})$ for all $I \in \mathcal{I}(V)$. If the latter is not true for some $I \in \mathcal{I}(V)$, one constructs as in the proof of Theorem 6.2.1 a morphism of K-affinoid spaces $Z = \mathcal{M}(C) \to X$ with $C = K'\{\beta^{-1}T\}$, where K' is a bigger $\mathbf{F}_1$-subfield of $\mathbf{R}_+$ with finite quotient K'^*/K^*, and this morphism identifies Z with an interval $L = \check{L} \cup \{x\}$ in X such that $\check{L} \subset \check{X}$ and $\check{L} \cap V = \{x\} \subset \check{X}_I$.

By Step 1, this implies that the morphism $U = \{x\} \to Z = L$ is a *pad*-embedding, and the same reasoning as in the proof of Theorem 6.2.1 implies that the latter is impossible. Namely, let g be the product of the images of the elements f_i in C, and let h be the image of g in C_U. Since $h(x) = 0$ and $\mathcal{M}(C_U) = \{x\}$, it follows that $h^m = 0$ for some $m \geq 1$. But if $\mathbf{b}$ denote the Zariski ideal of C generated by g, then $\mathbf{b}^{m+1} \neq \mathbf{b}^m$. Hence, we get a morphism of K-affinoid spaces $\mathcal{M}(C/\mathbf{b}^{m+1}) \to Z$ whose image coincides with $U = \{x\}$. It follows that it goes through a weak morphism $\mathcal{M}(C/\mathbf{b}^{m+1}) \to \mathcal{M}(C_U)$. Since both spaces are points, this weak morphism is in fact a morphism, i.e., it is induced by a bounded homomorphism $C_U \to C/\mathbf{b}^{m+1}$, which is impossible since $h^m = 0$ but the m-th power of its image in C is not zero. ∎

7.6.3. Corollary. *Let $\varphi : Y \to X$ be a p-morphism of K-affinoid spaces represented by a system of compatible morphisms $\varphi_i : Y_i \to X$, $i \in I$. Then the following are equivalent:*

(a) φ is a pad-embedding;

(b) φ is an injective map, and all of the morphisms φ_i are ad-embeddings.

Proof. (a)$\Longrightarrow$(b). We already know that the first property of (b) holds, and the second property of (b) follows from Theorem 7.6.2.

(b)$\Longrightarrow$(a). We have to show that any morphism of K-affinoid spaces $\psi : Z \to X$ with $\psi(Z) \subset \varphi(Y)$ goes through a unique p-morphism $\chi : Z \to Y$. Since $\varphi_i(Y_i)$ is an affinoid domain in X, $Z_i = \psi^{-1}(\varphi_i(Y_i))$ is an affinoid domain in Z, and the morphism $\psi_i : Z_i \to X$ goes through a unique morphism $Z_i \to Y_i$. Furthermore, since φ is an injective map, it follows that $Y_i = \varphi^{-1}(\varphi_i(Y_i))$, i.e., the composition $\chi_i : Z_i \to Y_i \to Y$ is a unique morphism whose composition with φ is ψ_i. The morphisms $\chi_i : Z_i \to Y$ are compatible, and so they give rise to a unique weak morphism $\chi : Z \to Y$ with $\varphi \circ \chi = \psi$. ∎

7.6.4. Corollary. *The functor K-$\mathcal{A}ff \to K$-$\mathcal{A}ff^p$ is conservative (i.e., any morphism in the first category, which becomes an isomorphism in the second one, is an isomorphism).* ∎

We shall denote by K-$\mathcal{A}ff^{pad}$ the category whose objects are K-affinoid spaces and morphisms are *pad*-embeddings. The canonical functor K-$\mathcal{A}ff^{ad} \to K$-$\mathcal{A}ff^{pad}$ is not fully faithful but, by Theorem 7.6.2, its restriction to K-$\mathcal{C}aff^{ad}$ is fully faithful.

7.6.5. Definition. Let $\varphi : Y \to X$ be a p-morphism of K-affinoid spaces.

(i) φ is said to be a *p-finite morphism* if there exists a finite covering of X by acyclic affinoid domains $\{U_i\}_{i \in I}$ such that, for every $i \in I$, $\varphi^{-1}(U_i)$ is a finite disjoint union of acyclic affinoid domains $\coprod_{j \in J_i} V_{ij}$ for which all of the induced p-morphisms $V_{ij} \to U_i$ are finite morphisms of K-affinoid spaces.

(ii) φ is said to be a *p-closed immersion* if there exists a finite affinoid covering $\{U_i\}_{i\in I}$ of X such that, for every $i \in I$, $\varphi^{-1}(U_i)$ is an acyclic affinoid domain and the induced p-morphism $\varphi^{-1}(U_i) \to U_i$ is a closed immersion of K-affinoid spaces.

7.6.6. Theorem. *The following properties of a morphism of K-affinoid spaces $\varphi : Y \to X$ are equivalent:*

(a) φ is a finite morphism (resp. a closed immersion);

(b) φ is a p-finite morphism (resp. a p-closed immersion).

Proof. (a)$\Longrightarrow$(b). By Corollary 7.4.3, every point $x \in X$ has an acyclic affinoid neighborhood U such that the preimage $\varphi^{-1}(U)$ is a disjoint union of acyclic affinoid domains $U'_1 \coprod \ldots \coprod U'_n$ and, if φ is a closed immersion, $n = 1$. This implies the property (b).

(b)$\Longrightarrow$(a). Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\{V_{ij}\}_{j\in J_i}$ be finite affinoid coverings of X and $\varphi^{-1}(U_i)$ as in Definition 7.6.5(i). Then $\mathrm{Int}(V_{ij}/U_i) = V_{ij}$ for all $i \in I$ and $j \in J_i$. Proposition 6.4.9 implies that $\mathrm{Int}(Y/X) = Y$ and, therefore, the morphism φ is finite, by Proposition 6.4.8. Suppose now that the morphisms φ is a p-closed immersion. In this case we have to verify the following fact. Given a bounded homomorphism of finite Banach A-modules $f : M \to N$, where $X = \mathcal{M}(A)$, if all of the homomorphisms $f_i : M_{U_i} \to N_{U_i}$ are surjective, then f is surjective. Indeed, by Proposition 2.2.8, f is an admissible homomorphism, and so replacing M by $M/\mathrm{Ker}(f)$, we may assume that M is a Zariski A-submodule of N. The assumption implies that $N_{U_i}/M_{U_i} = 0$ for all $i \in I$. By Theorem 7.2.7, this implies that $(N/M)_{U_i} = 0$ for all $i \in I$, i.e., $(N/M)_{\mathcal{U}} = 0$. Theorem 7.4.1(i) then implies that $N/M = 0$, i.e., $M = N$. ∎

§8. K-analytic spaces

8.1. The category K-$\mathcal{A}n$. Let X be a locally Hausdorff topological space, and let τ be a net of compact subsets of X. As in §I.5, we consider τ as a category and denote by $\mathcal{T}$ the canonical functor $\tau \to \mathcal{T}op$ to the category of topological spaces $\mathcal{T}op$. Let $\mathcal{T}^a$ denote the forgetful functor K-$\mathcal{A}ff^{pad} \to \mathcal{T}op$ that takes a K-affinoid space to the underlying topological space. Its restrictions to the subcategory K-$\mathcal{A}ff^{ad}$ and the full subcategory K-$\mathcal{C}aff^{ad}$ are denoted by the same way.

8.1.1. Definition. (i) An *affinoid* (resp. *acyclic affinoid*, resp. a *p-affinoid*) atlas with the net τ is a pair consisting of a functor $A : \tau \to K$-$\mathcal{A}ff^{ad}$ (resp. K-$\mathcal{C}aff^{ad}$, resp. K-$\mathcal{A}ff^{pad}$) and an isomorphism of functors $\mathcal{T}^a \circ A \stackrel{\sim}{\to} \mathcal{T}$.

(ii) A *K-analytic space* is a triple (X, A, τ), where X is a locally Hausdorff topological space, τ is a net of compact subsets of X, and A is an acyclic affinoid atlas on X with the net τ.

Let (X, A, τ) be a K-analytic space. The functor A takes each $U \in \tau$ to an acyclic K-affinoid space $\mathcal{M}(A_U)$, and the isomorphism of functors provides a homeomorphism $\mathcal{M}(A_U) \xrightarrow{\sim} U$. We consider such U as a K-affinoid space.

8.1.2. Proposition. *(i) If W is an acyclic affinoid domain in some $U \in \tau$, it is an acyclic affinoid domain in any $V \in \tau$ that contains W;*

(ii) the family $\overline{\tau}$ consisting of all W with the property (i) is a net on X, and there exists an acyclic affinoid atlas $\overline{A}$ on X with the net $\overline{\tau}$ which extends A.

Notice that the acyclic affinoid atlas $\overline{A}$ in (ii) is unique up to a unique isomorphism. It is also clear that $\overline{\overline{\tau}} = \overline{\tau}$.

8.1.3. Lemma. *Let $\varphi : X' \to U \in \tau$ be a morphism of K-affinoid spaces. Then there is a unique system of compatible morphisms of acyclic K-affinoid spaces $\varphi_{Y'/V} : Y' \to V$ for all pairs consisting of $V \in \tau$ and a morphism $\psi : Y' \to X'$ with $(\varphi \circ \psi)(Y') \subset V$ such that, if there are affinoid domains $U' \subset X'$ and $W \in \tau\big|_{U \cap V}$ with $\psi(Y') \subset U'$ and $\varphi(U') \subset W$, then $\varphi_{Y'/V}$ is the composition $Y' \to U' \to W \to V$.*

Proof. Since $(\varphi \circ \psi)(Y') \subset U \cap V$, we can find $W_1, \dots, W_n \in \tau\big|_{U \cap V}$ with $(\varphi \circ \psi)(Y') \subset W_1 \cup \dots \cup W_n$. Let $X'_i = \varphi^{-1}(W_i)$ and let $Y'_i = \psi^{-1}(X'_i)$. Then X'_i and Y'_i are affinoid domains in X' and Y', respectively, and the composition morphisms of K-affinoid spaces $Y'_i \to X'_i \to W_i \to V$ are well defined. Since the latter are compatible on intersections and Y' is acyclic, they define a morphism $Y' \to V$. ∎

Proof of Proposition 8.1.2. (i) By Lemma 8.1.3, there is a well defined morphism of acyclic affinoid spaces $W \to V$. If we apply the construction from its proof for $Y' = W$ and $X' = U$, we get a finite affinoid covering $\{Y'_i\}$ of W such that each morphism $Y'_i \to V$ is an *ad*-embedding, i.e., the morphism $W \to V$ is an *pad*-embedding. Theorem 7.6.2 then implies that the latter morphism is an *ad*-embedding.

(ii) Let $U, V \in \overline{\tau}$, and $x \in U \cap V$. Take $U', V' \in \tau$ with $U \subset U'$ and $V \subset V'$. We can find $W'_1, \dots, W'_n \in \tau\big|_{U' \cap V'}$ such that $x \in W'_1 \cap \dots \cap W'_n$ and $W'_1 \cup \dots \cup W'_n$ is a neighborhood of x in $U' \cap V'$. It follows that $U_i = U \cap W'_i$ and $V_i = V \cap W'_i$ are affinoid domains in U' and V', respectively, and $U_i \cap V_i$ is an affinoid domain in W'_i. By Theorem 7.4.5, there exists an acyclic affinoid neighborhood W_i of x in $U_i \cap V_i$. We have $W_i \in \overline{\tau}\big|_{U \cap V}$ and $x \in W_1 \cap \dots \cap W_n$. Since $\bigcup_i (U_i \cap V_i) = (U \cap V) \cap (\bigcup_i W'_i)$, it follows that $W_1 \cup \dots \cup W_n$ is a neighborhood of x in $U \cap V$. Thus, $\overline{\tau}$ is a net.

Furthermore, for each $V \in \overline{\tau}$ we fix $V' \in \tau$ with $V \subset V'$ and a structure of an affinoid domain on V, i.e., an *ad*-embedding $(V, A_V) \to (V', A_{V'})$. Lemma 8.1.3 and the reasoning from (i) imply that, for each pair $U, V \in \overline{\tau}$ with $U \subset V$, there is a canonical *ad*-domain embedding $(U, A_U) \to (V, A_V)$. This defines the required acyclic K-affinoid atlas $\overline{A}$. ∎

8.1.4. Definition. A *strong morphism* of K-analytic spaces $\varphi : (X, A, \tau) \to (X', A', \tau)$ is a pair consisting of a continuous map $\varphi : X \to X'$, such that for every $U \in \tau$ there exists $U' \in \tau'$ with $\varphi(U) \subset U'$, and of a compatible system of morphisms of K-affinoid spaces $\varphi_{U/U'} : U \to U'$ with $\varphi_{U/U'} = \varphi\big|_U$ (as maps) for all pairs $U \in \tau$ and $U' \in \tau'$ with $\varphi(U) \subset U'$.

Lemma 8.1.3 easily implies that any strong morphism $\varphi : (X, A, \tau) \to (X', A', \tau)$ extends in a unique way to a strong morphism $\overline{\varphi} : (X, \overline{A}, \overline{\tau}) \to (X, \overline{A}', \overline{\tau}')$ and that one can compose strong morphisms. In this way we get a category of K-analytic spaces $K\text{-}\widetilde{\mathcal{A}n}$ with strong morphisms as morphisms.

8.1.5. Definition. A strong morphism of K-analytic spaces $\varphi : (X, A, \tau) \to (X', A', \tau')$ is said to be a *quasi-isomorphism* if it possesses the following properties:

(1) φ induces a homeomorphism of topological spaces $X \stackrel{\sim}{\to} X'$;

(2) for every pair $U \in \tau$ and $U' \in \tau'$ with $\varphi(U) \subset U'$, $\varphi_{U/U'}$ is an *ad*-embedding.

For example, the canonical strong morphism $(X, \overline{A}, \overline{\tau}) \to (X, A, \tau)$ is a quasi-isomorphism.

8.1.6. Lemma. *The system of quasi-isomorphisms in $K\text{-}\widetilde{\mathcal{A}n}$ admits calculus of right fractions.*

Proof. It suffices to verify that the system possesses the properties c) - d) which are recalled in Step 5 of the proof of Theorem I.3.4.1 (the verification of a) and b) is trivial).

c) Suppose we are given strong morphisms $(Y, B, \sigma) \xrightarrow{\varphi} (X, A, \tau) \xleftarrow{g} (X', A', \tau')$, where g is a quasi-isomorphism. We may identify X' and X. Then $\tau' \subset \overline{\tau}$. Let σ' denote the family of all $V' \in \overline{\sigma}$ for which there exists $U' \in \tau'$ with $\varphi(V') \subset U'$. *We claim that σ' is a net.* Indeed, it suffices to verify that σ' is a quasi-net. For every point $y \in Y$, we can find $V_1, \dots, V_n \in \sigma$ with $y \in V_1 \cap \ldots \cap V_n$ and such that $V_1 \cup \ldots \cup V_n$ is a neighborhood of y and, for every $1 \le i \le n$, we can find $U_i \in \tau$ with $\varphi(V_i) \subset U_i$. Furthermore, for every $1 \le i \le n$, we can find $U_{i1}, \dots, U_{im_i} \in \overline{\tau}'$ for which $\varphi(y) \in U_{i1} \cap \ldots \cap U_{im_i}$ and $U_{i1} \cup \ldots \cup U_{im_i}$ is a neighborhood of $\varphi(y)$ in U_i. Let V_{ik} be the preimage of U_{ik} in V_i. Then $y \in \bigcap_{ik} V_{ik}$ and $\bigcup_{ik} V_{ik}$ is a neighborhood of y in Y, i.e., σ' is a quasi-net. Since $\sigma' \subset \sigma$, the K-affinoid atlas B extends to a K-affinoid atlas B' with the net σ'. The canonical strong morphism $g' : (Y, B', \sigma') \to (Y, B, \sigma)$ is clearly a quasi-isomorphism, and strong morphism φ extends in a unique way to a strong morphism $\varphi' : (Y, B', \sigma') \to (X, A', \tau')$.

d) Suppose we are given two strong morphisms $\varphi, \psi : (Y, B, \sigma) \to (X, A, \tau)$ and a quasi-isomorphism $g : (X, A, \tau) \to (X', A', \tau')$ with $g\varphi = g\psi$. *Then the strong morphisms φ and ψ coincide.* Indeed, that they coincide as maps is trivial. Let $V \in \sigma$ and $U \in \tau$ be such that $\varphi(V) \subset U$ and, therefore, $\psi(V) \subset U$. Take $U' \in \tau'$ with $g(U) \subset U'$. Then we have two morphisms of K-affinoid spaces $\varphi_{V/U}, \psi_{V/U} : V \to U$ whose compositions with $g_{U/U'} : U \to U'$ coincide. Since $g_{U/U'}$ is an *ad*-embedding, it follows that $\varphi_{V/U} = \psi_{V/U}$. ∎

8.1.7. Definition. The *category of K-analytic spaces K-$\mathcal{A}n$* is the category of fractions of K-$\widetilde{\mathcal{A}n}$ with respect to the system of quasi-isomorphisms.

By Lemma 8.1.6, morphisms in the category K-$\mathcal{A}n$ can be described as follows. Let (X, A, τ) be a K-analytic space. If σ is a net on X, we write $\sigma \prec \tau$ if $\sigma \subset \overline{\tau}$. Then $\overline{A}$ defines an acyclic K-affinoid atlas A_σ with the net σ, and there is a canonical quasi-isomorphism $(X, A_\sigma, \sigma) \to (X, A, \tau)$. The system of nets σ with $\sigma \prec \tau$ is filtered and, by Lemma 8.1.6, for any K-analytic space $(\mathcal{X}', A', \tau')$ one has

$$\mathrm{Hom}((X, A, \tau), (X', A', \tau')) = \varinjlim_{\sigma \prec \tau} \mathrm{Hom}_{\widetilde{\mathcal{A}n}}((X, A_\sigma, \sigma), (X', A', \tau')) \ .$$

Notice that all of the transition maps in this inductive system are injective. Notice also that, if $\sigma \prec \tau$, then $\overline{\sigma} \prec \overline{\tau}$.

For example, Theorem 7.4.5 implies that, for K-affinoid space $X = \mathcal{M}(A)$, the family τ_c of acyclic affinoid domains in X is a net (with $\overline{\tau}_c = \tau_c$). The functor A_c that takes $U \in \tau_c$ to the K-affinoid space $\mathcal{M}(A_U)$ is an acyclic affinoid atlas with the net τ_c, and so the triple (X, A_c, τ_c) is a K-analytic space.

8.1.8. Proposition. *The correspondence $X = \mathcal{M}(A) \mapsto (X, A_c, \tau_c)$ gives rise to a fully faithful functor K-$\mathcal{A}ff^p \to K$-$\mathcal{A}n$.*

A K-analytic space will be said to be a *K-affinoid space* it lies in the essential image of above functor (i.e., it is isomorphic to (X, A_c, τ_c) for some $X = \mathcal{M}(A)$).

Proof. Let $\varphi : Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be a p-morphism of K-affinoid spaces. It induces a morphism from every acyclic affinoid subdomain V of Y to X. Theorem 7.4.5 implies that the family σ of acyclic affinoid domains $V \subset Y$ whose image is contained in an acyclic affinoid subdomain of X is a net with $\sigma \prec \sigma_c$, where σ_c is the net of all acyclic affinoid domains in Y. Thus, φ induces a morphism $(Y, B_c, \sigma_c) \to (X, A_c, \tau_c)$, i.e., we have a functor K-$\mathcal{A}ff^p \to K$-$\mathcal{A}n$. It is clear that this functor is faithful. Furthermore, an arbitrary morphism $\varphi : (Y, B_c, \sigma_c) \to (X, A_c, \tau_c)$ is induced by a strong morphism $(Y, B_c, \sigma) \to (X, A_c, \tau_c)$ for some net $\sigma \prec \sigma_c$. Since Y is compact, we can find a finite affinoid covering $\{V_i\}$ of Y with $V_i \in \sigma$ and, therefore, we have a system of morphisms $V_i \to X$ which are compatible on intersections, i.e., a p-morphism $Y \to X$. It is easy to see that the latter induces the morphism φ, i.e., the functor considered is fully faithful. ∎

8.1.9. Corollary. *For a K-affinoid space $X = \mathcal{M}(A)$ and a K-analytic space (Y, B, σ), morphisms $(Y, B, \sigma) \to (X, A_c, \tau_c)$ can be identified with families of compatible bounded homomorphisms of Banach K-algebras $A \to B_V$, $V \in \sigma$.*

Proof. Such a morphism induces, for every $V \in \sigma$, a morphism $(V, B_{V,c}, \sigma_c) \to (X, A_c, \tau_c)$. Since V is acyclic, Proposition 8.1.9 implies that the latter is induced by a morphism of K-affinoid spaces $V \to X$, i.e., by a bounded homomorphism $A \to B_V$. In this way we get the required family of compatible morphisms $A \to B_V$. Conversely, such a family induces a continuous map $\varphi : Y \to X$, and the collection σ' of acyclic affinoid subdomains V of Y with $\varpi(V)$ lying in an acyclic affinoid subdomain of X is a net with $\sigma' \prec \sigma$. If B' is the restriction of the atlas B_c to σ, we get a strong moprhism $(Y, B', \sigma') \to (X, A_c, \tau_c)$, which induces the family we started from. ∎

8.1.10. Corollary. *The functor K-$\mathcal{A}ff \to K$-$\mathcal{A}n$ commutes with fiber products.*

Proof. Let $Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ and $X' = \mathcal{M}(A') \to X$ be morphisms of K-affinoid spaces. Proposition 8.1.8 implies that the set of morphisms from a K-analytic space (Z, D, σ) to the K-affinoid space $Y \times_X X' = \mathcal{M}(B \widehat{\otimes}_A A')$ is identified with the set of families of compatible bounded homomorphisms $B \widehat{\otimes}_A A' \to D_V$, $V \in \sigma$. Each of the latter homomorphisms is a pair of bounded homomorphisms $B \to D_V$ and $A' \to D_V$ which coincide on A. This gives the required fact. ∎

8.1.11. Proposition. *A strong morphism becomes an isomorphism in the category K-$\mathcal{A}n$ if and only if it is a quasi-isomorphism.*

Proof. The converse implication is trivial. Suppose that a strong morphism $\varphi : (X, A, \tau) \to (X', A', \tau')$ becomes an isomorphism in K-$\mathcal{A}n$. It is clear that φ is a homeomorphism. The assumption implies that one can find nets $\sigma \prec \tau$ and $\sigma' \prec \tau'$ and strong morphisms $\psi : (X', A'_{\sigma'}, \sigma') \to (X, A, \tau)$ and $\varphi' : (X, A_\sigma, \sigma) \to (X', A'_{\sigma'}, \sigma')$ such that the following diagram is commutative

$$\begin{tikzcd} (X, A, \tau) \arrow[r, "\varphi"] & (X', A', \tau') \\ (X, A_\sigma, \sigma) \arrow[u] \arrow[r, "\varphi'"] & (X', A'_{\sigma'}, \sigma') \arrow[u] \arrow[ul, "\psi"'] \end{tikzcd}$$

where the vertical arrows are the canonical quasi-isomorphisms.

Let $U \in \sigma$. We can find $U' \in \sigma'$, $V \in \tau$ and $V' \in \tau'$ with $\varphi'(U) \subset U'$, $\psi(U') \subset V$ and $\varphi(V) \subset V'$. Since U is an affinoid domain in V, its preimage $U'' = \psi^{-1}_{U'/V}(U)$ is an affinoid domain in U'. The commutativity of the lower triangle implies that the composition of the morphisms $\varphi'_{U/U'} : U \to U''$ and $\psi_{U''/U} : U'' \to U$ is the identity morphism on U. The commutativity of the higher triangle implies that the composition of the morphisms $\psi_{U''/U} : U'' \to U$ and $\varphi_{U/U'} : U \to U''$ is the identity morphism on U''. Thus, $U \xrightarrow{\sim} U''$. The required fact follows. ∎

In what follows, we do not make difference between a K-analytic space (X, A, τ) and the K-analytic spaces isomorphic to it, and denote it simply by X. We call any net τ that defines the K-analytic space structure on X a *net of definition*. The underlying topological space of X will be denoted by $|X|$.

8.2. Analytic domains. Let X be a K-analytic space. We fix a triple (X, A, τ) that represents it.

8.2.1. Definition. A subset $Y \subset X$ is said to be an *analytic domain* if, for any point $y \in Y$, there exist sets $V_1, \ldots, V_n \in \overline{\tau}\big|_Y$ such that $y \in V_1 \cap \ldots \cap V_n$ and the set $V_1 \cup \ldots \cup V_n$ is a neighborhood of y in Y (i.e., $\overline{\tau}\big|_Y$ is a net on Y). (Notice that this property does not depend on the choice of τ.)

For example, any open subset of X is an analytic domain. It is easy to see that the intersection of two analytic domains is an analytic domain, the union of two *closed* analytic domains is an analytic domain, and the preimage of an analytic domain with respect to a morphism of K-analytic spaces is an analytic domain, and the restriction of the acyclic K-affinoid atlas $\overline{A}$ to the net $\overline{\tau}\big|_Y$ defines a K-analytic space $(Y, \overline{A}, \overline{\tau}\big|_Y)$. (If $\sigma \prec \tau$, then $\overline{\sigma}\big|_Y \prec \overline{\tau}\big|_Y$.) The K-analytic space $(Y, \overline{A}, \overline{\tau}\big|_Y)$, which will be denoted by Y, possesses the following property: any morphism of K-analytic spaces $\varphi : X' \to X$ with $\varphi(X') \subset Y$ goes through a unique morphism $X' \to Y$.

8.2.2. Definition. An *acyclic affinoid domain* in X is an analytic domain isomorphic to an acyclic K-affinoid space.

8.2.3. Proposition. *(i) The family $\widehat{\tau}$ of acyclic affinoid domains is a net on X, and there is a unique (up to a canonical isomorphism) acyclic K-affinoid atlas $\widehat{A}$ on X with the net $\widehat{\tau}$ that extends A;*

(ii) the strong morphism $\varphi : (X, \overline{A}, \overline{\tau}) \to (X, \widehat{A}, \widehat{\tau})$ is a quasi-isomorphism.

Proof. (i) That $\widehat{\tau}$ is a net is trivial. We fix a K-affinoid space structure on every $W \in \widehat{\tau}$, and our purpose is to construct, for every pair $W \subset W'$ in $\widehat{\tau}$ a canonical *ad*-embedding $W \to W'$. Let $\{U_i\}$ and $\{U'_k\}$ be finite coverings of W and W' by sets from $\overline{\tau}$. Each intersection $U_i \cap U'_k$ is compact and, therefore, it is a union of a finite number of sets from $\overline{\tau}$. Replacing all U_i's by them, we may assume that each U_i lies in some U'_k and, in particular, every U_i is an acyclic affinoid domain in some U'_k. It follows that there are canonical *ad*-embeddings $U_i \to W'$. It is easy to see that they are compatible on intersections and, therefore, they give rise to a *pad*-embedding $W \to W'$. Since W is acyclic, Proposition 8.1.8 implies that the latter is a morphism of K-affinoid spaces and, by Theorem 7.6.2, it is an *ad*-embedding.

The statement (ii) is trivial. ∎

8.2.4. Corollary. *Let X and X' be K-analytic spaces. Then*

(i) there is a one-to-one correspondence between $\mathrm{Hom}(X, X')$ *and the set of pairs consisting of (1) a continuous map $\varphi : X \to X'$ for which there is a net of definition τ on $\mathcal{X}$ such that the image of every set from τ lies in some acyclic affinoid subdomain of X', and (2) a compatible system of morphisms of K-affinoid spaces $\varphi_{U/U'} : U \to U'$ for all pairs of acyclic affinoid subdomains $U \subset X$ and $U' \subset X'$ with $\varphi(U) \subset U'$;*

(ii) a morphism $\varphi : X \to X'$ is an isomorphism if and only if it is a homeomorphism and, for every acyclic affinoid subdomain $U' \subset X'$, $U = \varphi^{-1}(U')$ is an acyclic affinoid subdomain of X and $\varphi_{U/U'}$ is an isomorphism. ∎

8.2.5. Definition. A *p-affinoid domain* in X is an analytic domain isomorphic to a K-affinoid space.

The following statement is verified in the same way as Proposition 8.2.3.

8.2.6. Proposition. *(i) The family τ^p of p-affinoid domains is a net on X;*

(ii) there is a unique (up to a canonical isomorphism) p-affinoid atlas A^p with the net τ^p that extends A. ∎

8.2.7. Definition. A K-analytic space X is said to be *good* if every point of X has an acyclic affinoid neighborhood.

Notice that, by Corollary 7.4.3, the latter is equivalent to the property that every point of X has a p-affinoid neighborhood.

8.3. Grothendieck topologies on a K-analytic space. The family of analytic domains in a K-analytic space X can be considered as a category, and it gives rise to a Grothendieck topology defined by the following pretopology: the set of coverings of an analytic domain $W \subset X$ is formed by families of analytic domains which are quasinets on W. For brevity, this Grothendieck topology is said to be the G-*topology* on X, and the corresponding site is denoted by X_{G}.

8.3.1. Proposition. *Any representable presheaf is a sheaf on X_{G}.*

Proof. Let $\{X_i\}_{i\in I}$ be a covering of X in X_{G}. We have to verify that, for every K-analytic space Y, the following sequence of maps is exact

$$\mathrm{Hom}(X,Y) \to \prod_i \mathrm{Hom}(X_i,Y) \rightrightarrows \prod_{i,j} \mathrm{Hom}(X_i \cap X_j, Y) \ .$$

Let $\varphi_i : X_i \to Y$ be a family of morphisms such that, for every pair $i,j \in I$, $\varphi_i\big|_{X_i\cap X_j} = \varphi_j\big|_{X_i\cap X_j}$. Let τ be the collection of acyclic affinoid domains $U \subset X$ such that there exists $i \in I$ with $U \subset X_i$ and $\varphi_i(U) \subset V$ for some acyclic affinoid domain $V \subset Y$. It is easy to see that τ is a net of definition on X. It follows that the morphisms φ_i, $i \in I$, give rise to a morphism $\varphi : X \to Y$. ∎

For example, the presheaf representable by the K-affinoid space $\mathcal{M}(K\{r^{-1}T\})$ is a sheaf on X_{G}, it is denoted by $\mathcal{O}^r_{X_{\mathrm{G}}}$. If V is a p-affinoid domain, then $\mathcal{O}^r(V) = \{f \in \langle A_V\rangle \big| \rho(f) \le r\}$. The inductive limit $\varinjlim \mathcal{O}^r_X$ is a sheaf of K-algebras on X_{G} denoted by $\mathcal{O}_{X_{\mathrm{G}}}$ and called the *structural sheaf* on X_{G}. For a p-affinoid domain V, one has $\mathcal{O}(V) = \langle A_V\rangle$. The category of $\mathcal{O}_{X_{\mathrm{G}}}$ modules is denoted by $\mathrm{Mod}(X_{\mathrm{G}})$. If $X = \mathcal{M}(A)$ is a K-affinoid space, then any finitely generated Banach A-module M defines an $\mathcal{O}_{X_{\mathrm{G}}}$-module $\mathcal{O}_{X_{\mathrm{G}}}(M)$ by $V \mapsto \langle M_V\rangle$. Notice that, by Theorem 7.4.5, the stalk $\mathcal{O}_{X_{\mathrm{G}},x}(M)$ of $\mathcal{O}_{X_{\mathrm{G}}}(M)$ at a point $x \in X$, i.e., the inductive limit $\varinjlim \langle M_V\rangle$ taken over all p-affinoid domains V that contain the point x, coincides with the stalk M_x of M at x (see §7.2). Furthermore, it follows from Theorem 7.4.2(i) that the canonical homomorphism $\langle M\rangle \to \prod_{x\in X} M_x$ is injective.

8.3.2. Lemma. *In the above situation, the correspondence $M \mapsto \mathcal{O}_{X_{\mathrm{G}}}(M)$ gives rise to a fully faithful functor A-$\mathcal{F}mod^p \to \mathrm{Mod}(X_{\mathrm{G}})$.*

Proof. Let $\varphi : M \to \langle N\rangle$ be a morphism in A-$\mathcal{F}mod^p$. It is induced by a homomorphism $M \to N_{\mathcal{U}}$ for a finite affinoid covering $\mathcal{U} = \{U_i\}$ of X. The latter induces a compatible system of homomorphisms of A_V-modules $M_V \to N_{U_i\cap V}$ for every affinoid subdomain $V \subset X$, i.e., a bounded homomorphism of A_V-modules $M_V \to N_{\mathcal{U}\cap V}$, where $\mathcal{U}\cap V$ is the covering $\{V \cap U_i\}$ of V. It follows that φ induces a bounded homomorphism of A_V-modules $\mathcal{O}_{X_{\mathrm{G}}}(M)(V) = \langle M_V\rangle \to \mathcal{O}_{X_{\mathrm{G}}}(N)(V) = \langle N_V\rangle$. All these homomorphisms are compatible on intersections, i.e., φ gives rise to a homomorphism of $\mathcal{O}_{X_{\mathrm{G}}}$-modules $\varphi_{\mathrm{G}} : \mathcal{O}_{X_{\mathrm{G}}}(M) \to \mathcal{O}_{X_{\mathrm{G}}}(N)$. In this way we get the required functor which is evidently faithful. Let ψ be a homomorphism $\mathcal{O}_{X_{\mathrm{G}}}$-modules $\mathcal{O}_{X_{\mathrm{G}}}(M) \to \mathcal{O}_{X_{\mathrm{G}}}(N)$. It induces a homomorphism of A-modules $\varphi : M \to \langle M\rangle \to \langle N\rangle$. Since both ψ and φ_{G} induce the same homomorphisms between stalks $M_x \to N_x$, it follows that $\psi = \varphi_{\mathrm{G}}$. ∎

8.3.4. Definition. An $\mathcal{O}_{X_{\mathrm{G}}}$-module F on a K-analytic space X is said to be *coherent* if there is a quasinet τ of affinoid domains in X such that, for every $V \in \tau$, $F\big|_{V_{\mathrm{G}}}$ is isomorphic to

$\mathcal{O}_{V_G}(M)$ for some finitely generated Banach A_V-module M. The category of coherent $\mathcal{O}_{X_G}$-modules is denoted by $\mathrm{Coh}(X_G)$.

Every K-analytic space is provided with a topology which is weaker than the usual one.

8.3.5. Definition. An open subset $\mathcal{U}$ of a K-analytic space X is said to be *Zariski open* if there exists a quasinet of acyclic affinoid domains $\{U_i\}_{i\in I}$ in X such that, for every $i \in I$, $\mathcal{U} \cap U_i$ is a Zariski open subset of U_i.

The topology on X formed by the Zariski open subsets is said to be the *Zariski topology* on X, and the corresponding site is denoted by X_{Zar}. Notice that there is a canonical morphism of sites $X_G \to |X| \to X_{\mathrm{Zar}}$.

We now consider a process of gluing K-analytic spaces.

8.3.6. Definition. A morphism of K-analytic spaces $\varphi : Y \to X$ is said to be an *analytic domain embedding* if it induces an isomorphism of Y with an analytic domain in X.

If $\varphi : Y \to X$ is an analytic domain embedding, then any morphism of K-analytic spaces $\psi : Z \to X$ with $\psi(Z) \subset \varphi(Y)$ goes through a unique morphism $Z \to Y$.

Let $\{X_i\}_{i\in I}$ be a family of K-analytic spaces, and suppose that, for each pair $i, j \in I$, we are given an analytic domain $X_{ij} \subset X_i$ and an isomorphism of K-analytic spaces $\nu_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$ so that $X_{ii} = X_i$, $\nu_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$, and $\nu_{ik} = \nu_{jk} \circ \nu_{ij}$ on $X_{ij} \cap X_{ik}$. We are looking for a K-analytic space X with a family of morphisms $\mu_i : X_i \to X$ such that:

(1) μ_i is an analytic domain embedding;

(2) $\{\mu_i(X_i)\}_{i\in I}$ is a covering of X in X_G;

(3) $\mu_i(X_{ij}) = \mu_i(X_i) \cap \mu_j(X_j)$;

(4) $\mu_i = \mu_j \circ \nu_{ij}$ on X_{ij}.

If such X exists we say that it is obtained by *gluing X_i's along X_{ij}'s.* Of course, a necessary condition for existence of such X is existence of a topological space X with a family of continuous maps $\mu_i : X_i \to X$ with the properties (1') μ_i induces a homeomorphism $X_i \xrightarrow{\sim} \mu_i(X_i)$, (2') $\{\mu_i(X_i)\}_{i\in I}$ is a quasinet on X, (3) and (4).

8.3.7. Proposition. *(i) If the above necessary condition is satisfied, then a K-analytic space obtained by gluing of X_i along X_{ij} exists and is unique (up to a canonical isomorphism), and its underlying topological space is X;*

(ii) the necessary condition is satisfied in each of the following cases:

(a) all X_{ij} are open in X_i;

(b) for any $i \in I$, all X_{ij} are closed in X_i and the number of $j \in I$ with $X_{ij} \neq \emptyset$ is finite.

Furthermore, in the case (a), all $\mu_i(X_i)$ are open in X. In the case (b), all $\mu_i(X_i)$ are closed in X and, if all X_i are Hausdorff (resp. paracompact), then X is Hausdorff (resp. paracompact).

Proof. (i) Let τ denote the collection of all subsets $V \subset X$ for which there exists $i \in I$ such that $V \subset \mu_i(X_i)$ and $\mu_i^{-1}(V)$ is an acyclic affinoid domain in X_i (in this case $\mu_j^{-1}(V)$ is an acyclic affinoid domain in X_j for any j with $V \subset \mu_j(X_j)$). It is easy to see that τ is a net, and there is an evident acyclic affinoid atlas A with the net τ. In this way we get a K-analytic space (X, A, τ) that possesses the properties (1)-(4). That such a K-analytic space is unique up to a canonical isomorphism is trivial.

(ii) Let $\widetilde{X}$ be the disjoint union $\coprod_i X_i$. The system $\{\nu_{ij}\}$ defines an equivalence relation R on $\widetilde{X}$. We denote by X the quotient space $\widetilde{X}/R$ and by μ_i the induced maps $X_i \to X$. In the case (a), the equivalence relation R is open (see [Bou], Ch. I, §9, n° 6), and therefore all $\mu_i(X_i)$ are open in X. In the case (b), the equivalence relation R is closed (see *loc. cit.*, n° 7), and therefore all $\mu_i(X_i)$ are closed in X and μ_i induces a homeomorphism $X_i \stackrel{\sim}{\to} \mu_i(X_i)$. Moreover, if all X_i are Hausdorff, then X is Hausdorff, by *loc. cit.*, exerc. 6. If all X_i are paracompact, then X is paracompact because it has a locally finite covering by closed paracompact subsets ([En], 5.1.34). That X satisfies the necessary conditions is trivial. ∎

8.4. Fiber products and the ground field extension functor.

8.4.1. Proposition. *The category K-$\mathcal{A}n$ admits fibre products, and the forgetful functor K-$\mathcal{A}n \to \mathcal{T}op : X \mapsto |X|$ commutes with fibre products.*

Proof. Let $\varphi : Y \to X$ and $f : X' \to X$ be morphisms of K-analytic spaces. By Corollary 8.1.9, if both morphisms come from the category K-$\mathcal{A}ff$ (e.g., if the space X is K-affinoid, and the spaces Y and X' are acyclic K-affinoid), a fiber product $Y \times_X X'$ exists and is a K-affinoid space and, by Lemma 1.3.8, one has $|Y \times_X X'| \stackrel{\sim}{\to} |Y| \times_{|X|} |X'|$.

In the general case, we are going to use Proposition 8.3.7(i) in order to provide the topological space $Y' = |Y| \times_{|X|} |X'|$ with a K-analytic space structure that makes it a required fiber product $Y \times_X X'$. For this, we may assume that φ and f are represented by strong morphisms $(Y, B, \sigma) \to (X, A, \tau)$ and $(X', A', \tau') \to (X, A, \tau)$.

Let S denote the family of all triples (V, U, U'), where $V \in \sigma$, $U \in \tau$, $U' \in \tau'$ and $\varphi(V), f(U') \subset U$. For $\alpha = (V, U, U') \in S$ we denote by W_α the k-affinoid space $V \times_U U'$. The underlying topological space of the latter coincides with $|V| \times_{|U|} |U'|$ and, therefore, can be considered as a subset of Y'. To prove the required statement, it suffices to verify the following two fact: (a) the family $\{W_\alpha\}_{\alpha \in S}$ is a quasinet on Y', and (b) for every pair $\alpha, \beta \in S$, the set $W_{\alpha\beta} = W_\alpha \cap W_\beta$ is an analytic domain in W_α, and there is a canonical isomorphism of K-analytic spaces $\nu_{\alpha\beta} : W_{\alpha\beta} \xrightarrow{\sim} W_{\beta\alpha}$. Let $y' = (y, x')$ be a point of Y', and let x be its image in X.

(a) Since τ is a quasinet on X, there exist $U_1, \ldots, U_j \in \tau$ that contain the point x and such that their union is a neighborhood of x in X. For the same reason, for every $1 \le i \le j$ there exist $V_{i1}, \ldots, V_{im_i} \in \sigma$ and $U'_{i1}, \ldots, U'_{in_i} \in \tau'$ that contain the points y and x' and such that their unions are neighborhoods of y and x' in $\varphi^{-1}(U_i)$ and $f^{-1}(U_i)$, respectively. It follows that the point y' lies in the intersection of W_γ's with $\gamma = (V_{ik}, U_i, U'_{il})$ for all $1 \le i \le j$, $1 \le k \le m_i$ and $1 \le l \le n_i$, and the union of such W_γ's is a neighborhood of y' in Y'.

(b) Suppose now that the point y' lies in the intersection $W_\alpha \cap W_\beta$ for some $\alpha = (V, U, U')$ and $\beta = (\overline{V}, \overline{U}, \overline{U}')$. Since $x \in U \cap \overline{U}$ and τ is a net, we can find $U_1, \ldots, U_j \in \tau\big|_{U \cap \overline{U}}$ which contain the point x and such that their union is a neighborhood of x in $U \cap \overline{U}$. Similarly, for every $1 \le i \le j$ there exist $V_{i1}, \ldots, V_{im_i} \in \overline{\sigma}$ and $U'_{i1}, \ldots, U'_{in_i} \in \overline{\tau}'$ that contain the points y and x' and such that their unions are neighborhoods of y and x' in $V \cap V' \cap \varphi^{-1}(U_i)$ and $V \cap V' \cap f^{-1}(U_i)$, respectively. It follows that the point y' lies in the intersection of W_γ's with $\gamma = (V_{ik}, U_i, U'_{il})$ for all $1 \le i \le j$, $1 \le k \le m_i$ and $1 \le l \le n_i$, and the union of such W_γ's is a neighborhood of y' in $W_\alpha \cap W_\beta$. It follows that $W_{\alpha\beta} = W_\alpha \cap W_\beta$ is an analytic domain in W_α, and there is a canonical isomorphism of K-analytic spaces $\nu_{\alpha\beta} : W_{\alpha\beta} \xrightarrow{\sim} W_{\beta\alpha}$. ∎

Notice that a fiber product of good K-analytic spaces is a good K-analytic space.

Similarly, given an isometric homomorphism of real valuation $\mathbf{F}_1$-fields $K \to L$, one constructs a *ground field extension functor* $K\text{-}\mathcal{A}n \to L\text{-}\mathcal{A}n : X \mapsto X\widehat{\otimes}_K L$. The L-analytic space $X\widehat{\otimes}_K L$ has the same underlying topological space and, if τ is a net of definition of X, then the functor $\tau \to L\text{-}\mathcal{A}ff : V \mapsto V\widehat{\otimes}_K L$ is an L-affinoid atlas, and it defines a structure of an L-analytic space on $X\widehat{\otimes}_K L$. It is easy to see that both spaces have the same families of analytic domains and of Zariski open subsets.

An *analytic space over* $\mathbf{F}_1$ is a pair (K, X), where K is a valuation $\mathbf{F}_1$-field and X is a K-analytic space. A morphism $(L, Y) \to (K, X)$ is a pair consisting of an isometric homomorphism $K \to L$ and a morphism of L-analytic spaces $Y \to X\widehat{\otimes}_K L$. The category of analytic spaces over $\mathbf{F}_1$ is denoted by $\mathcal{A}n_{\mathbf{F}_1}$.

For a point x of a K-analytic space X, let $\mathcal{H}(x)$ denote the filtered inductive limit of the $\mathbf{F}_1$-valuation fields $\mathcal{H}_V(x)$ taken over all acyclic affinoid (or p-affinoid) domains V that contain x. (Notice that all transition homomorphisms in this inductive limit are isomorphisms.) The valuation

$\mathbf{F}_1$-field $\mathcal{H}(x)$ is a K-affinoid algebra (see Example 2.1.2), and so the point x defines a morphism of K-analytic spaces $\mathcal{M}(\mathcal{H}(x)) \to X$. The fiber product of the latter with a morphism of K-analytic spaces $\varphi : Y \to X$ is a K-analytic space Y_x which is said to be the *fiber of* φ *at the point* x. The canonical morphism $Y_x \to Y$ induces a homeomorphism $Y_x \xrightarrow{\sim} \varphi^{-1}(x)$. Notice that Y_x can be also considered as an $\mathcal{H}(x)$-analytic space.

8.5. Finite and separated morphisms. Let $\varphi : Y \to X$ be a morphism of K-analytic spaces.

8.5.1. Definition. (i) φ is said to be a *finite morphism* if, for every point $x \in X$, there exist acyclic affinoid domains $U_1, \dots, U_n$ in X such that $x \in U_1 \cap \dots \cap U_n$, $U_1, \cup \dots \cup U_n$ is a neighborhood of x and, for every $1 \le i \le n$, $\varphi^{-1}(U_i)$ is a finite disjoint union of acyclic affinoid domains $\coprod_{j \in J_i} V_{ij}$ for which all of the induced morphisms $V_{ij} \to U_i$ are finite morphisms of K-affinoid spaces.

(ii) φ is said to be a *closed immersion* if, for every point $x \in X$, there exists acyclic affinoid domains $U_1, \dots, U_n$ in X as in (i) such that, for every $1 \le i \le n$, $\varphi^{-1}(U_i)$ is an acyclic affinoid domain and the induced morphism $\varphi^{-1}(U_i) \to U_i$ is a closed immersion of K-affinoid spaces.

Notice that is a closed immersion $\varphi : Y \to X$ it induces a homeomorphism between $|Y|$ and its image in $|X|$.

8.5.2. Proposition. *Given a finite morphism (resp. a closed immersion)* $\varphi : Y \to X$*, if* X *is good then, for every point* $x \in X$*, the property (i) (resp. (ii)) of Definition 8.5.1 holds for* $n = 1$*. In particular, the space* Y *is also good and, when both spaces are* K*-affinoid, Definition 8.5.2 is consistent with Definition 7.6.5.*

Proof. First of all, *we claim that every point* $y \in Y$ *has an acyclic affinoid neighborhood.* Indeed, since the preimage of the point $x = \varphi(y)$ in Y is a finite set of points, we can shrink X and Y so that $\varphi^{-1}(x) = \{y\}$. Since X is good, we can shrink it and assume that X is acyclic affinoid and every affinoid domain that contains the point x is also acyclic. Shrinking X we may also assume that $x \in X_{\mathbf{m}}$ and, in particular, every connected affinoid domain that contains x is Weierstrass. The assumption implies that there are Weierstrass domains $U_1, \dots, U_m$ that contain the point x and such that $U_1 \cup \dots \cup U_m$ is a neighborhood of x and all of the induced morphisms $\varphi_i : V_i = \varphi^{-1}(U_i) \to U_i$ are finite morphisms of acyclic K-affinoid spaces. If $U = \bigcap_{i=1}^m U_i$, then $U = \{y \in X \big| |f_i(y)| \le p_i\}$ for some $p_1, \dots, p_k > 0$ and $f_1, \dots, f_k \in A$. In the further constructions, we replace X by a Weierstrass domain of the form $\{y \in X \big| |f_i(y)| \le p'_i\}$ for $p'_i > p_i$ sufficiently close to p_i. Thus, we can shrink X so that all of the induced homomorphisms $A \to A_{U_i} \to A_U$

and $B_{V_i} \to B_V$ are bijections, where $V = \varphi^{-1}(U) = \bigcap_{i=1}^m U_i$. The Banach A_U-algebra B_V is isomorphic to a quotient of $A_U\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ by a closed ideal E. Since the map $A \to A_U$ is a bijection, we can consider the quotient $B = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}/E$, and since E is closed in $A_U\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$, it is also closed in $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$, i.e., B is a Banach A-algebra which is a finitely generated Banach A-module. By the construction, there is a canonical isomorphism of finitely generated Banach A_U-modules $B\widehat{\otimes}_A A_U \stackrel{\sim}{\to} B_V$, and Corollary 7.2.3 implies that we can shrink X so that the latter extends to a system of compatible isomorphisms of finite Banach A_{U_i}-algebras $B\widehat{\otimes}_A A_{U_i} \stackrel{\sim}{\to} B_{V_i}$. This defines an isomorphism of the K-analytic space Y with the K-affinoid space $\mathcal{M}(B)$. Shrinking X, we may assume that Y is acyclic, and the claim follows.

Let now $x \in X$ and $\varphi^{-1}(x) = \{y_1, \dots, y_m\}$ with $m \geq 0$ (resp. $0 \leq m \leq 1$ if X is a closed immersion). By the above claim, we can find, for every $1 \leq i \leq m$, an acyclic affinoid neighborhood V_i of the point y_i. Furthermore, we can find a p-affinoid neighborhood U of x whose preimage in Y is contained in $\cup_{i=1}^m V_i$. Shrinking U and all V_i's, we may assume that $\varphi^{-1}(U) = \coprod_{i=1}^m V_i$ and all V_i's are acyclic. Theorem 7.6.6 now implies that all of the morphisms $V_i \to U$ are finite morphisms (resp. closed immersions) of K-affinoid spaces. ∎

8.5.3. Corollary. *The classes of finite morphisms and closed immersions are preserved under composition, any base change functor, and any ground field extension functor.* ∎

8.5.4. Definition. A morphism $\varphi : Y \to X$ is said to be a G-*locally* (resp. *locally*) *closed immersion* if there exists a quasinet σ of analytic (resp. open analytic) domains in Y and, for every $V \in \sigma$, an analytic (resp. open analytic) domain U in X such that φ induces a closed immersion $V \to U$.

Of course, any closed immersion is a G-locally closed immersion. If both spaces are good, the converse implication is also true (this follows from Theorem 7.6.6). Notice that any locally closed immersion $\varphi : Y \to X$, which is injective as a map, is a composition of a closed immersion $Y \to Z$ with an *open immersion* $\psi : Z \to Y$. (The latter means that ψ induces an isomorphism of Z with an open subset of Y.) It follows that a locally closed immersion $\varphi : Y \to X$ is a closed immersion if and only if it is injective and the image of Y in X is closed. An example of an injective G-locally closed immersion is the diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ for a morphism $\varphi : Y \to X$.

8.5.5. Definition. A morphism $\varphi : Y \to X$ is said to be *separated* (resp. *locally separated*) if the diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ is a closed (resp. locally closed) immersion. If the canonical morphism $X \to \mathcal{M}(K)$ is separated (resp. locally separated), X is said to be *separated* (resp. *locally separated*).

For example, good K-analytic spaces and morphisms between them are locally separated. If a morphism $\varphi : Y \to X$ is separated, then $|Y|$ is closed in $|Y| \times_{|X|} |Y| = |Y \times_X Y|$ and, therefore, the map $|Y| \to |X|$ is Hausdorff. In particular, if X is separated, the underlying topological space $|X|$ is Hausdorff.

8.5.6. Proposition. *A locally separated morphism $\varphi : Y \to X$ is separated if and only if the induced map $|Y| \to |X|$ is Hausdorff.*

Proof. If the map $|Y| \to |X|$ is Hausdorff, the image of $|Y|$ in $|Y \times_X Y| = |Y| \times_{|X|} |Y|$ is closed and, therefore, the diagonal morphism $\Delta_{Y/X}$ is a closed immersion. ∎

8.6. Piecewise K-affinoid spaces.

8.6.1. Definition. (i) A K-analytic space X is said to be *piecewise K-affinoid* if there is a closed immersion of X in a K-affinoid space. The full subcategory of K-$\mathcal{A}n$ formed by piecewise K-affinoid spaces is denoted by K-$\mathcal{P}aff$.

(ii) An analytic domain in a K-analytic space is said to be *piecewise affinoid* if it is isomorphic to a piecewise K-affinoid space.

8.6.2. Proposition. *(i) Piecewise K-affinoid spaces are good;*

(ii) the subcategory K-$\mathcal{P}aff$ is preserved under finite disjoint unions and fiber products.

Proof. (i) follows from Proposition 8.5.3.

(ii) Let W be a disjoint union of two piecewise K-affinoid spaces. If $X \to X'$ and $Y \to Y'$ are closed immersions to K-affinoid spaces, then the induced morphism $W \to X' \coprod Y'$ is a closed immersion. Thus, to show that W is piecewise K-affinoid, it suffices to show that $X' \coprod Y'$ piecewise K-affinoid, and so we may assume that $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$. We are going to construct a closed immersion $W \to Z$ in a K-affinoid space $Z = \mathcal{M}(C)$. For this we set $C = (A\widehat{\otimes}_K B)\{T\}$. There are the following admissible epimorphisms $\alpha : C \to A$ and $\beta : C \to B$ defined by $\alpha(a \otimes bT^n) = a$ for all $n \geq 0$ (and $b \neq 0$), and $\beta(a \otimes b) = b$ and $\beta(a \otimes bT^n) = 0$ for all $n \geq 1$ (and $a \neq 0$). The homomorphisms α and β induce closed immersions of K-affinoid spaces $X \to Z$ and $Y \to Z$ with non-intersecting images and, therefore, they give rise to a closed immersion $\varphi : W \to Z$.

Let now $\varphi : Y \to X$ and $\psi : Z \to X$ be morphisms of piecewise K-affinoid spaces. If $X \to X'$ is a closed immersion of X in a K-affinoid space, then $Y \times_X Z \stackrel{\sim}{\to} Y \times_{X'} Z$. Replacing X by X', we may assume that $X = \mathcal{M}(A)$ is a K-affinoid space. Furthermore, we can find closed immersions $Y \to Y'$ and $Z \to Z'$ in K-affinoid spaces Y' and Z' over X. Then $Y' \times_X Z'$ is a K-affinoid space, and the induced morphism $Y \times_X Z \to Y' \times_X Z'$ is a closed immersion. ∎

8.6.3. Corollary. *(i) The disjoint union of two piecewise affinoid domains is a piecewise affinoid domain;*

(ii) the intersection of two piecewise affinoid domains in a separated K-analytic space is a piecewise affinoid domain;

(iii) the preimage of a piecewise affinoid domain with respect to a morphism of piecewise K-affinoid spaces is a piecewise affinoid domain. ∎

8.6.4. Proposition. *Let $Y \to X$ be a closed immersion of a piecewise K-affinoid space Y in a K-affinoid space $X = \mathrm{Spec}(A)$, $\mathfrak{p}$ a Zariski prime ideal of A with $Y \cap \check{X}_{\mathfrak{p}} \neq \emptyset$, and Y' a connected component of the latter set. Then there exists a closed immersion $Z \to Y$ from an integral K-affinoid space Z whose image coincides with $\overline{Y'}$.*

8.6.5. Lemma. *Let $\varphi : Y \to X$ be a closed immersion of a reduced piecewise K-affinoid space Y in an artinian K-affinoid space X. Then every connected component Y' of Y is an integral K-affinoid space, and the induced morphism $Y' \to X$ is a closed immersion of K-affinoid spaces.*

Proof. First of all, we may assume that both $X = \mathcal{M}(A)$ and Y are connected and reduced and, in particular, A is an $\mathbf{F}_1$-field. Let $\{U_i\}_{i \in I}$ be a finite affinoid covering of X such that, for every $i \in I$, $V_i = \varphi^{-1}(U_i) \to U_i = \mathcal{M}(A_i)$ is a closed immersion of K-affinoid spaces. We set $J = \{i \in I \big| V_i \neq \emptyset\}$ and $V_i = \mathcal{M}(B_i)$. Notice that each B_i is an $\mathbf{F}_1$-field. By Lemma 6.1.7, all of the homomorphisms $A \to A_i$ are bijections. By the same lemma, if $i, j \in J$ are such that $V_i \cap V_j = \mathcal{M}(B_{ij}) \neq \emptyset$, then the homomorphism $B_i \to B_{ij}$ is a bijection. It follows easily that the ideal $E = \mathrm{Ker}(A \to B_i)$ with $i \in J$ does not depend on i. We set $B = A/E$ and $Y' = \mathcal{M}(B)$. Then for $i \in J$ there are compatible isomorphisms of K-affinoid algebras $B \otimes_A A_i \stackrel{\sim}{\to} B_i$. They give rise to an isomorphism of K-analytic spaces $Y \stackrel{\sim}{\to} Y'$, and the required fact follows. ∎

Proof of Proposition 8.6.4. We may assume that $A = K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. Then there is a subset $I \subset \{1, \dots, n\}$ such that $\mathfrak{p}$ is generated by the variables T_i for $i \notin I$. Replacing X by $\mathcal{M}(A/\mathfrak{p})$ (and Y by the corresponding base change), we may assume that $I = \{1, \dots, n\}$ and $\mathfrak{p} = 0$. If $\overline{Y'} \subset \check{X}$, then $\overline{Y'} = Y'$, and it is a connected component of Y. Replacing Y by Y', we may assume that there are $0 < s_i \le r_i$ with $Y \subset \mathcal{M}(K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, s_1T_1^{-1}, \dots, s_nT_n^{-1}\}) \subset [s_1, r_1] \times \dots \times [s_n, r_n]$, and the required fact follows from Lemma 8.6.5. Assume therefore that $\overline{Y'} \not\subset \check{X}$. Lemma 8.6.5 implies that Y' is the intersection of $\check{X}$ with an $|K^*|$-affine subspace $\check{L}$ of $(\mathbf{R}_+^*)^n$. It follows that $\overline{Y'}$ is the intersection of X with the $|K|$-affine subspace $L = \overline{\check{L}}$ and, in particular, $\overline{Y'}$ is the underlying space of an integral $|K|$-affinoid polytope $Z = \mathcal{M}(C)$. *We claim that the canonical closed immersion $Z \to X$ goes through a closed immersion $Z \to Y$.*

Indeed, by Proposition 8.5.2, we can find a finite covering $\{U_i\}_{i\in I}$ of X by affinoid domains such that, for every $i \in I$, the preimage V_i of U_i in Y is a K-affinoid space $\mathcal{M}(B_i)$ and $V_i \to U_i$ is a closed immersion of K-affinoid spaces and, for every $i \in I$ with $V_i \cap \overline{Y'} \neq \emptyset$, V_i contains an open subset of Y'. By Corollary 6.2.4(ii), the latter implies that, for the preimage $W_i = \mathcal{M}(C_i)$ of U_i in Z, C_i is a $|K|$-polytopal algebra. It follows that the admissible epimorphism $A \to C$ induces an admissible epimorphism $A_i \to C_i$ which goes through a unique admissible epimorphism $B_i \to C_i$. The claim follows. ∎

8.6.6. Corollary. *For every piecewise K-affinoid space X, there exists a finite family of closed immersions $\varphi : Y_i \to X$ with integral K-affinoid spaces Y_i such that $X = \bigcup_i \varphi_i(Y_i)$.* ∎

8.7. The relative interior and proper morphisms. Let $\varphi : Y \to X$ be a morphism of K-analytic spaces.

8.7.1. Definition. (i) The *relative interior* of φ is the subset $\mathrm{Int}(Y/X) \subset Y$ consisting of the points $y \in Y$ with the following property: there exist acyclic affinoid domains $U_1, \ldots, U_n$ with $x = \varphi(y) \in U_1 \cap \ldots \cap U_n$ such that $U_1 \cup \ldots \cup U_n$ is a neighborhood of x and, for every $1 \le i \le n$, there exists an acyclic affinoid neighborhood V_i of y in $\varphi^{-1}(U_i)$ with $y \in \mathrm{Int}(V_i/U_i)$.

(ii) The *relative boundary* of φ is the complement $\delta(Y/X)$ of $\mathrm{Int}(Y/X)$ in Y.

(iii) If $X = \mathcal{M}(K)$, the set $\mathrm{Int}(Y/X)$ (resp. $\delta(Y/X)$) is denoted by $\mathrm{Int}(Y)$ (resp. $\delta(Y)$) and is called the *interior* (resp. *boundary*) of Y.

Proposition 6.4.9 implies that this definition is consistent with that for morphisms of K-affinoid spaces. It follows also that the sets $\mathrm{Int}(Y/X)$ and $\delta(Y/X)$ are open and closed, respectively.

8.7.2. Proposition. *(i) For a morphism $\psi : X' \to X$, one has $\psi'^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Y'/X')$ where ψ' is the canonical morphism $Y' = Y \times_X X' \to Y$;*

(ii) for a real valuation $\mathbf{F}_1$-field K' over K, one has $\psi^{-1}(\mathrm{Int}(Y/X)) = \mathrm{Int}(Y\widehat{\otimes}K'/X\widehat{\otimes}K')$, where ψ is the canonical map $Y\widehat{\otimes}K' \to Y$ (which is a bijection);

(iii) if Y is an analytic domain in X, then $\mathrm{Int}(Y/X)$ coincides with the topological interior of Y in X;

(iv) if φ is a finite morphism, then $\mathrm{Int}(Y/X) = Y$;

(v) if $\psi : Z \to Y$ is a finite morphism, then $\psi^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Z/X)$;

(vi) if both spaces X and Y are piecewise K-affinoid, the converse implication in (v) is true.

Proof. (i) Let $y' \in Y'$, $y = \psi'(y)$, $x = \varphi(y)$ and $x' = \varphi'(y')$, where φ' is the induced morphism $Y' \to X'$. Suppose that $y \in \mathrm{Int}(Y/X)$. Then there exist acyclic affinoid domains

$x \in U_1, \dots, U_m \subset X$ and, for every $1 \le i \le m$ an acyclic affinoid neighborhood V_i of y in $\varphi^{-1}(U_i)$ such that $U_1 \cup \ldots \cup U_n$ is a neighborhood of x in X and $y \in \mathrm{Int}(V_i/U_i)$. We can find acyclic affinoid domains $x' \in U'_1, \dots, U'_n \subset X'$ such that $U'_1 \cup \ldots \cup U'_n$ is a neighborhood of x' in X' and, for every $1 \le i \le n$, there is $1 \le j \le m$ with $\psi(U'_i) \subset U_j$. The fiber product $V_j \times_{U_j} U'_i$ is a p-affinoid neighborhood of y' in $\psi'^{-1}(U'_i)$.If V'_i is an acyclic affinoid neighborhood of y' in $V_j \times_{U_j} U'_i$ then, by Propositions 6.4.2(iv) and 6.4.6, one has $y' \in \mathrm{Int}(V'_i/U'_i)$ and, therefore, $y' \in \mathrm{Int}(Y'/X')$.

The statement (ii) is verified in the same way as (i), the statement (iii) easily follows from Proposition 6.4.6, and the statement (iv) is trivial.

(v) Let $z \in Z$ is such that $y = \psi(z) \in \mathrm{Int}(Y/X)$, and set $x = \varphi(y)$. By Definition 8.7.1, there exist acyclic affinoid domains $U_1, \dots, U_n$ in X with $x \in U_1 \cap \ldots \cap U_n$ such that $U_1 \cup \ldots \cup U_n$ is a neighborhood of x and, for every $1 \le i \le n$, there exists an acyclic affinoid neighborhood V_i of y in $\varphi^{-1}(U_i)$ with $y \in \mathrm{Int}(V_i/U_i)$. By Proposition 8.5.2, we can shrink V_i so that $\psi^{-1}(V_i)$ is a finite disjoint union of acyclic domains $\coprod_{j \in J_i} W_{ij}$ for which all of the induced morphisms $W_{ij} \to V_i$ are finite morphisms of K-affinoid spaces. If $z \in W_{ij}$, then W_{ij} is an affinoid neighborhood of z in $(\varphi\psi)^{-1}(U_i)$, and Proposition 6.4.2 then implies that $z \in \mathrm{Int}(W_{ij}/U_i)$. This means that $z \in \mathrm{Int}(Z/X)$.

(vi) We may assume that X is an acyclic K-affinoid space. Let $x \in X$. *We claim that the preimage $\varphi^{-1}(x)$ is finite.* Indeed, to prove the claim, we may assume that X is acyclic K-affinoid. By Corollary 8.6.6, there is a finite family of closed immersions $\psi : Z_i \to Y$ with integral K-affinoid spaces Z_i such that $Y = \bigcup_i \psi_i(Z_i)$. the statement (i) implies that $\mathrm{Int}(Z_i/X) = Z_i$. Since each Z_i is acyclic, $\varphi\psi_i : Z_i \to X$ is a morphism of K-affinoid spaces with $\mathrm{Int}(Z_i/X) = Z_i$. Proposition 6.4.8 implies that this morphism is finite, and the claim follows.

Let $\varphi^{-1}(x) = \{y_1, \dots, y_n\}$. Then we can find affinoid domains $U_1, \dots, U_m$ that contain the point x and such that their union is a neighborhood of x and, for every $1 \le i \le m$, pairwise disjoint acyclic affinoid neighborhoods V_{ij} of the point y_j in $\varphi^{-1}(U_i)$ such that $y_j \in \mathrm{Int}(V_{ij}/U_i)$. Since both spaces X and Y are compact, we can find, for every $1 \le i \le m$, an affinoid neighborhood U'_i of x in U_i such that $\varphi^{-1}(U'_i) \subset \coprod_{1 \le j \le n} V_{ij}$. If $V'_{ij} = V_{ij} \cap \varphi^{-1}(U'_i)$, then V'_{ij} is an acyclic affinoid neighborhood of the point y_j in $\varphi^{-1}(U'_i)$ and $V'_{ij} \to U'_i$ is a morphism of K-affinoid spaces with $\mathrm{Int}(V'_{ij}/U'_i) = V'_{ij}$. Proposition 6.4.8 implies that the later morphisms are finite. This means that the morphism φ is finite. ∎

8.7.3. Theorem. *Let $\varphi : Y \to X$ be a separated morphism to a K-affinoid space X. Then for every p-affinoid domain $V \subset Y$ with $V \subset \mathrm{Int}(Y/X)$ there exists a bigger p-affinoid domain $W \subset Y$*

such that $V \subset \mathrm{Int}(W/X)$ and U is a Weierstrass domain in W.

Proof. Suppose that $B_V = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}/E$ with a closed ideal E and, for $r_1' > r_1, \dots, r_n' > r_n$, we set $C = A\{r_1'^{-1}T_1, \dots, r_n'^{-1}T_n\}/E$ and $Z = \mathcal{M}(C)$. Then V is a Weierstrass domain in Z with $V \subset \mathrm{Int}(Z/X)$, and the canonical homomorphism $C \to B_V$ is a bijection. In this way we can construct a decreasing sequence of Weierstrass domains $Z_1 \supset Z_2 \supset \dots$ in $\mathrm{Int}(Z/X)$ with $C_{V_i} \overset{\sim}{\to} B_V$ and $Z_{i+1} \subset \mathrm{Int}(Z_i/Z)$ and, by Proposition 6.4.2(iii), $Z_{i+1} \subset \mathrm{Int}(Z_i/X)$. Thus, to prove the theorem, it suffices to show that the identity isomorphism of V extends to an open immersion $\mathcal{V} \hookrightarrow Z$ of an open neighborhood $\mathcal{V}$ of V in Y.

Let y be a point of V. Since $V \subset \mathrm{Int}(Y/X)$, the point y has an affinoid neighborhood W_y' in $\mathrm{Int}(Y/X)$. In particular, $y \in \mathrm{Int}(W_y'/X)$. By Corollary 7.4.3, we can shrink W so that every affinoid subdomain of W_y' that contains the point y is acyclic. Since $y \in \mathrm{Int}(W_y'/Y) \cap \mathrm{Int}(Y/X) \subset \mathrm{Int}(W_y'/X)$, there exists an affinoid neighborhood W_y of y in $\mathrm{Int}(W_y'/X)$. If $V_y = W_y \cap V$, then there is an isomorphism of K-germs $(W_y', V_y) \overset{\sim}{\to} (Y, V_y)$, and Corollary 7.3.2 implies that the identity isomorphism on V_y extends to a unique isomorphism of K-germs $(Y, V_y) \overset{\sim}{\to} (Z, V_y)$.

It follows that there is a finite family $\{\mathcal{V}_i\}_{i \in I}$ of open subsets of $\mathrm{Int}(Y/X)$ such that $V \subset \bigcup_{i \in I} \mathcal{V}_i$ and, for every $i \in I$, the identity isomorphism on $\mathcal{V}_i \cap V$ extends to an open immersion $\varphi_i : \mathcal{V}_i \hookrightarrow Z$. Let y be a point of V, and suppose it lies precisely in $\mathcal{V}_{i_1}, \dots, \mathcal{V}_{i_n}$. By the previous paragraph, we can find an affinoid domain V_y in $V \cap (\mathcal{V}_{i_1} \cap \dots \cap \mathcal{V}_{i_n})$ that contains the point y and such that the identity isomorphism on V_y extends to a unique isomorphism of K-germs $(Y, V_y) \overset{\sim}{\to} (Z, V_y)$. It follows that there exists an open neighborhood $\mathcal{V}_y$ of y in $\mathcal{V}_{i_1} \cap \dots \cap \mathcal{V}_{i_n}$ at which all of the open immersions $\varphi_{i_1}, \dots, \varphi_{i_n}$ coincide. If $\mathcal{V}$ is the union of $\mathcal{V}_y$'s taken over all point $y \in V$, then φ_i's give rise to an open immersion $\mathcal{V} \hookrightarrow Z$ we are looking for. ∎

8.7.4. Definition. A morphism of K-analytic spaces $\varphi : Y \to X$ is said to be *proper* if it is compact as a map of topological spaces and $\delta(Y/X) = \emptyset$.

8.7.5. Corollary. *(i) Every finite morphism is proper;*

(ii) the class of proper morphisms is preserved under any base change functor and any ground field extension functor;

(iii) if $\varphi : Y \to X$ is proper and $\psi : Z \to Y$ is finite, then the composition $\varphi\psi : Z \to X$ is proper. ∎

8.7.6. Remark. In the following section we will show that if $y \in \mathrm{Int}(Y/X)$ then, for every acyclic affinoid domain $U \subset X$ that contains the point $\varphi(y)$, the point y has an acyclic affinoid neighborhood V of y in $\varphi^{-1}(U)$ with $y \in \mathrm{Int}(Y/X)$. It will follow that the classes of morphisms without boundary and of proper morphisms are preserved under composition.

§9. Local structure of K-analytic spaces

9.1. A category K-*bir*. Let K be a real valuation $\mathbf{F}_1$-field. Objects of the category K-*bir* we are going to introduce are triples $\overline{X} = (L, X, \phi)$ consisting of a real valuation K-field L with finitely generated cokernel of the canonical homomorphism $K^* \to L^*$, a connected locally compact topological space X, and a compact map $\phi : X \to \mathbf{V}_{L/K}$ possessing the following properties:

(1) there exists a finite covering of X by closed subsets $\{X_i\}_{i \in I}$ such that ϕ induces a homeomorphism of each X_i with a rational convex polyhedral cone in $\mathbf{V}_{L/K}$;

(2) for every pair $i, j \in I$, the image of $X_i \cap X_j$ in $\mathbf{V}_{L/K}$ is a finite union of rational convex polyhedral cones.

Notice that there exists $n \geq 1$ such that the fibers of the map ϕ have at most n points and, since the space X is connected, the fiber of the point of $\mathbf{V}_{L/K}$ that corresponds to the trivial homomorphism $L \to \mathbf{R}_+$ consists of one point. A subset of X_i whose image in $\mathbf{V}_{L/K}$ is a rational convex polyhedral cone will be called a rational convex polyhedral cone in X_i. If such a subset lies also in X_j, the property (2) implies that it is a rational convex polyhedral cone in X_j.

9.1.1. Lemma. *In the above situation, every closed subset $Y \subset X$ with the property that ϕ induces a homeomorphism of Y with a rational convex polyhedral cone is a finite union $Y = \bigcup_k Y^k$ such that each Y^k is a rational convex polyhedral cone in some X_i.*

Proof. We prove the required statement by induction on the cardinality of the set I. If it is equal to one, the statement is trivial. Suppose that it is bigger than one and the statement is true for X with strictly smaller cardinality of I.

A closed subset $Z \subset X$ will be said to be a domain if it is a finite union $\bigcup_k Z^k$ such that each Z^k is a rational convex polyhedral cone in some X_i. (And so we have to show that Y is domain.) The properties (1) and (2) imply that the class of domains is preserved under finite intersections and that every domains itself possesses those properties.

We claim that, for any a rational convex polyhedral cone C in $\mathbf{V}_{L/K}$, its preimage $\phi^{-1}(C)$ is a domain. Indeed, if ϕ_i denotes the restriction of the map ϕ to X_i, one has $\phi^{-1}(C) = \bigcup_{i \in I} \phi_i^{-1}(C \cap \phi(X_i))$. Since each intersection $C \cap \phi(X_i)$ is a rational convex polyhedral cone in $\mathbf{V}_{L/K}$, then so is its preimage in X_i.

It suffices to show that, for every $i \in I$, the intersection $Y \cap X_i$ is a finite union of rational convex polyhedral cones in X_i. Let $\check{X}_i$ denote the open set $X_i \backslash \bigcup_{j \neq i} X_j$. If $Y \cap \check{X}_i = \emptyset$, then the required statement is true by induction applied to the closed subset $X\backslash\check{X}_i$. Assume therefore that $Y\cap\check{X}_i\neq\emptyset$. By the above claim, we can replace X by $\phi^{-1}(\phi(Y)\cap\phi(X_i))$ and Y by its intersection with the latter, and so we may assume that $\phi(Y)=\phi(X_i)$. In this situation, *we claim that* $X_i\subset Y$.

Indeed, suppose that $X_i\not\subset Y$. Since $\check{X}_i$ is dense in X_i, it follows that $\check{X}_i\not\subset Y$. Let y be a point in the topological boundary of $Y\cap\check{X}_i$ in $\check{X}_i$. The point $x=\phi(y)$ lies in the topological boundary of $\phi(Y\cap\check{X}_i)$ in $\phi(\check{X}_i)$. We can therefore find a sequence of points $x_1,x_2,\dots$ in $\phi(\check{X}_i)\backslash\phi(Y\cap\check{X}_i)$ that tend to the point x. Since $\phi(Y)=\phi(X_i)$, there are points $y_n\in\phi^{-1}(x_n)\cap Y$ for all $n\geq 1$. Replacing the latter sequence by a subsequence, we may assume that $y_n\in X_j$ for some $j\neq i$ and all $n\geq 1$. The points y_n lie in a bounded subset of X_i and, therefore, they have a limit y' in X_j. Since Y is a closed subset of X, it follows that $y'\in Y$. The restriction of the map ϕ to Y is injective, but we get $\phi(y')=\phi(y)=x$ with $y\in Y\cap\check{X}_i$ and $y'\in Y\cap X_j$, which is a contradiction. ∎

We define a morphism $\overline{X}'=(L',X',\phi')\to\overline{X}=(L,X,\phi)$ in *K-bir* as a pair (h,i), where i is an isometric homomorphism of valuation K-fields $L\to L'$ and h is a continuous map $X'\to X$ which is compatible with the induced map $\mathbf{V}_{L'/K}\to\mathbf{V}_{L/K}$.

The category *K-bir* admits fiber products. Namely, for morphisms $\overline{Y}=(M,Y,\psi)\to\overline{X}=(L,X,\phi)$ and $\overline{Z}=(N,Z,\chi)\to\overline{X}$, one has $\overline{Y}\times_{\overline{X}}\overline{Z}=(M\otimes_L N, Y\times_X Z,\mu)$, where μ is the induced map $Y\times_X Z\to\mathbf{V}_{(M\otimes_L N)/K}=\mathbf{V}_{M/K}\times_{\mathbf{V}_{L/K}}\mathbf{V}_{N/K}$. Furthermore, given an isometric homomorphism of real valuation $\mathbf{F}_1$-fields $K\to K'$, there is an associated *ground field extension functor* *K-bir* $\to$ *K′-bir* : $\overline{X}=(L,X,\phi)\mapsto\overline{X}\otimes_K K'=(L',X',\phi')$, where $L'=L\otimes_K K'$, $X'=X\times_{\mathbf{V}_{L/K}}\mathbf{V}_{L'/K'}$, and ϕ' is the induced map $X'\to\mathbf{V}_{L'/K'}$. Notice that the canonical maps $\mathbf{V}_{L'/K'}\to\mathbf{V}_{L/K}$ and $X'\to X$ are bijections.

The (non-full) subcategory of *K-bir*, formed by objects of the form $\overline{X}=(L,X,\phi)$ for a fixed L and with morphisms $(h,i):\overline{Y}=(L,Y,\psi)$ in which i is the identity map $L\stackrel{\sim}{\to}L$, will be denoted by *K-bir*(L). The triple $(L,\mathbf{V}_{L/K},\phi)$ with the identity map $\phi:\mathbf{V}_{L/K}\stackrel{\sim}{\to}\mathbf{V}_{L/K}$ will be denoted simply by $\mathbf{V}_{L/K}$, it is a final object of the category *K-bir*(L). The object $\mathbf{V}_{K/K}$ (whose underlying space is one point) is a final object of the category *K-bir*.

9.1.2. Definition. Let $\overline{X}=(L,X,\phi)$ be an object of *K-bir*.

(i) A subset $X'\subset X$ is said to be an *affine domain* if ϕ induces a homeomorphism of X' with a rational convex polyhedral cone in $\mathbf{V}_{L/K}$. If $X'=X$, $\overline{X}$ is said to be an *affine* object of *K-bir*.

(ii) A subset $X'\subset X$ is said to be a *domain* if it is a finite union of affine subsets of X.

(iii) A morphism $(h,i):\overline{X}'=(L,X',\phi')\to\overline{X}$ in *K-bir*(L) is said to a *domain* (resp. an *affinoid domain*) *embedding* if the map h induces a homeomorphism of X' with a domain (resp. an affinoid domain) in X.

Lemma 9.1.1 implies that the class of domains is preserved by finite intersections. If X' is a domain (resp. an affine domain), it gives rise to an object $\overline{X}' = (L, X', \phi|_{X'})$ of K-$bir(L)$, and the induced morphism $\overline{X}' \to \overline{X}$ is a domain (resp. an affinoid domain) embedding. Notice that, given a morphism $\overline{Y} \to \overline{X}$, the preimage of a subdomain of $\overline{X}$ is a domain in $\overline{Y}$. The similar fact holds for the preimages of domains with respect to the ground field extension functors.

Let $\{\overline{X}_i = (X_i, L, \phi_i)\}_{i\in I}$ be a finite family of objects in K-$bir(L)$, and suppose that, for each pair $i, j \in I$, we are given a domain $\overline{X}_{ij} \subset \overline{X}_i$ and an isomorphism of objects of K-$bir(L)$, $\nu_{ij} : \overline{X}_{ij} \overset{\sim}{\to} \overline{X}_{ji}$ which satisfy the usual conditions for gluing. Then there is an object $\overline{X} = (X, L, \phi)$ of K-$bir(L)$ which is obtained by gluing of all $\overline{X}_i$'s along $\overline{X}_{ij}$'s, i.e., it possesses the usual properties of such an object (see §8.3). Indeed, the topological space X is obtained by the usual gluing of the spaces X_i's, and the maps ϕ_i's give rise to a continuous map $\phi : X \to \mathbf{V}_{L/K}$.

9.1.3. Definition. A morphism $(h, i) : \overline{X}' \to \overline{X}$ is said to be *separated* (resp. *proper*) if the induced map $X' \to X \times_{\mathbf{V}_{L/K}} \mathbf{V}_{L'/K}$ is injective (resp. bijective).

The following properties of separated and proper morphisms easily follow from their definition.

9.1.4. Proposition. *(i) The class of separated (resp. proper) morphisms in K-bir is preserved under composition, any base change functor, and any ground field extension functor;*

(ii) if for a morphism $(h, i) : \overline{X}' \to \overline{X}$ there exists a finite covering of X by domains $\{\overline{X}_k\}$ such that all of the induced morphisms $h^{-1}(\overline{X}_k) \to \overline{X}_k$ are separated (resp. proper), then so is the morphism (h, i);

(iii) a morphism $(h, i) : \overline{X}' \to \overline{X}$ is separated if and only if the induced morphism $\overline{X}' \to \overline{X}' \times_{\overline{X}} \overline{X}'$ is proper;

(iv) given morphisms $(h, i) : \overline{X}' \to \overline{X}$ and $(h', i') : \overline{X}'' \to \overline{X}'$, if the composition $\overline{X}'' \to \overline{X}$ is separated, then so is the morphism (h', i');

(v) given morphisms as in (iv), if the composition $\overline{X}'' \to \overline{X}$ is proper and the morphism (h, i) is separated, then the morphism (h', i') is proper;

(vi) in the situation of (v), if in addition the kernel of the homomorphism $L'^* \to L''^*$ lies in the image of K^{**}, then the morphism (h, i) is also proper. ■

9.2. Germs of K-analytic spaces and the reduction functor K-$\mathcal{G}erms_{pt} \to K$-$bir$. A *germ* of a K-analytic space (or simply a K-*germ*) is a pair (X, S), where X is a K-analytic space and S is a subset of the underlying topological space $|X|$ of X. If S is a point x, then (X, S) will be denoted by (X, x) or X_x, and such a K-germ is said to be *good* if x has an affinoid neighborhood in X. The K-germs form a category in which morphisms from (Y, T) to (X, S) are the morphisms $\varphi : Y \to X$ with $\varphi(T) \subset S$. The category of K-germs K-$\mathcal{G}erms$ is the localization of the latter category with respect to the system of morphisms $\varphi : (Y, T) \to (X, S)$ such that φ induces an isomorphism of Y with an open neighborhood of S in X. Notice that this system admits calculus of right fractions, and so the set of morphisms $\mathrm{Hom}((Y, T), (X, S))$ in K-$\mathcal{G}erms$ is the inductive limit of the sets of morphisms $\varphi : \mathcal{V} \to X$ with $\varphi(T) \subset S$, where $\mathcal{V}$ runs through a fundamental system of open neighborhoods of T in Y. It follows that a morphism $\varphi : (Y, T) \to (X, S)$ is an isomorphism in K-$\mathcal{G}erms$ if it induces an isomorphisms between some open neighborhoods of T and S. Notice that the correspondence $X \mapsto (X, |X|)$ gives rise to a fully faithful functor K-$\mathcal{A}n \to K$-$\mathcal{G}erms$.

The category K-$\mathcal{G}erms$ admits fiber products. Namely, given morphisms $(Y, T) \to (X, S)$ and $(X', S') \to (X, S)$, if $\varphi : (\mathcal{V}, T) \to (X, S)$ and $(\mathcal{U}', S') \to (X, S)$ are their representatives, then a required fiber product is $(\mathcal{V} \times_X \mathcal{U}', T \times_S S')$. Given an isometric homomorphism of valuation $\mathbf{F}_1$-fields $K \to K'$, the ground field extension functor K-$\mathcal{A}n \to K'$-$\mathcal{A}n : X \mapsto X\widehat{\otimes}_K K'$ extends in an evident way to a *ground field extension functor* of K-germs K-$\mathcal{G}erms \to K'$-$\mathcal{G}erms$. A morphism $(Y, T) \to (X, S)$ is said to be an *analytic domain embedding* if it has a representative $\mathcal{V} \to X$ which is an analytic domain embedding of K-analytic spaces.

We are interested here in the full subcategory K-$\mathcal{G}erms_{pt}$ of K-$\mathcal{G}erms$ consisting of *pointed K-germs*, i.e., K-germs of the form $X_x = (X, x)$. Notice that an analytic subdomain of a K-analytic space is closed at any of its points; it follows that the union of two analytic subdomains of a K-germ X_x makes sense, and is an analytic subdomain of X_x. Furthermore, since the functor $X \mapsto |X|$ commutes with fiber products, it follows that the category K-$\mathcal{G}erms_{pt}$ is preserved by fiber products in K-$\mathcal{G}erms$. For the similar reason, given an isometric homomorphism of real valuation $\mathbf{F}_1$-fields $K \to K'$, the ground field extension functor K-$\mathcal{G}erms \to K'$-$\mathcal{G}erms$ gives rise to a functor K-$\mathcal{G}erms_{pt} \to K'$-$\mathcal{G}erms_{pt}$. Finally, let K-$\mathcal{A}ff_{pt}$ denote the category of *pointed K-affinoid spaces* which is defined in the same way as above. Then Corollary 7.4.3 implies that the canonical functor from K-$\mathcal{A}ff_{pt}$ to the category of pointed good germs is an equivalence of categories.

We are going to construct a functor K-$\mathcal{G}erms_{pt} \to K$-$bir : X_x \mapsto \widetilde{X}_x$.

First of all, assume that $X = \mathcal{M}(A)$ is a K-affinoid space. Each point $x \in X$ corresponds to a character $\chi_x : A \to \mathcal{H}(x)$ which, in its turn, gives rise to a character $\widetilde{\chi}_x : \widetilde{A} \to \widetilde{\mathcal{H}(x)} = \mathcal{H}(x)$ (that takes the element $\widetilde{f} \in \widetilde{A}$ of an element $f \in A$ to $f(x) = \chi_x(f)$, if $|f(x)| = \rho(f)$, and to zero, otherwise). We set $\widetilde{X}_x = (\mathcal{H}(x), \mathbf{V}_{\mathcal{H}(x)/K}\{\widetilde{\chi}_x(\widetilde{A})\}, \phi)$, where ϕ is the canonical embedding $\mathbf{V}_{\mathcal{H}(x)/K}\{\widetilde{\chi}_x(\widetilde{A})\} \to \mathbf{V}_{\mathcal{H}(x)/K}$. If V is an affinoid domain in X that contains the point x, then there is an associated affine domain embedding $\widetilde{V}_x \to \widetilde{X}_x$. If, in addition, V is a neighborhood of x in X then, by Proposition 6.4.6, $\widetilde{\chi}_x(\widetilde{A}_V)$ is integral over $\widetilde{\chi}_x(\widetilde{A})$ and, therefore, $\widetilde{V}_x = \widetilde{X}_x$. In this way we get a functor from the category of pointed good K-germs to K-*bir*.

9.2.1. Proposition. *Let X_x be a good K-germ, and let $\{Y_x^i\}_{i\in I}$ be a finite family of good analytic subdomains of X_x. Then*

(i) $\widetilde{\bigcap_{i\in I} Y_x^i} = \bigcap_{i\in I} \widetilde{Y}_x^i$;

(ii) if the analytic domain $Y_x = \bigcup_{i\in I} Y_x^i$ is good, then $\widetilde{Y}_x = \bigcup_{i\in I} \widetilde{Y}_x^i$;

(iii) for any morphism of good K-germs $\varphi : Z_z \to X_x$ and any good analytic subdomain Y_x of X_x, one has $\widetilde{\varphi^{-1}(Y_x)} = \widetilde{\varphi}^{-1}(\widetilde{Y}_x)$.

If $f_1, \dots, f_n$ are functions analytic and invertible in a neighborhood of x, i.e., $f_i \in \mathcal{O}_{X,x}^*$, they come from an affinoid neighborhood $Y = \mathcal{M}(A)$ of x, i.e., $f_i \in A$, and so there is an affinoid domain $Y\{r_1^{-1}f_1, \dots, r_n^{-1}T_n\}$, where $r_i = |f_i(x)|$. The latter defines an affinoid subdomain of X_x denoted by $X_x\{r_i^{-1}f_i\}_{1\le i\le n} = X_x\{r_1^{-1}f_1, \dots, r_n^{-1}f_n\}$.

9.2.2. Lemma. *Every affinoid subdomain of X_x has the above form $X_x\{r_i^{-1}f_i\}_{1\le i\le n}$, and one has $\widetilde{X_x\{r_i^{-1}f_i\}}_{1\le i\le n} = \widetilde{X}_x\{f_i(x)\}_{1\le i\le n}$.*

Proof. We may assume that $X = \mathcal{M}(A)$ is K-affinoid and the affinoid subdomain considered comes from a connected affinoid domain Y in X. By Theorem 6.3.1(ii), Y is rational, i.e., $Y = \{y \in X \big| |g(y)| \ge q, |f_i(y)| \le p_i \text{ for } 1 \le i \le n\}$, where $f_1, \dots, f_n, g \in A$ and $p_1, \dots, p_n, q > 0$. Since $g(x) \ne 0$, we can shrink X so that g is invertible in A, i.e., we can replace each f_i by $f_i g_i^{-1}$ and assume that Y is a Weierstrass domain $X\{r_1^{-1}f_1, \dots, r_n^{-1}f_n\}$. Suppose that $|f_i(x)| = r_i$ for $1 \le i \le m$, and $|f_i(x)| < r_i$ for $m+1 \le i \le n$. Then the Weierstrass domain $X\{r_{m+1}^{-1}f_{m+1}, \dots, r_n^{-1}f_n\}$ is a neighborhood of the point x in X and so, shrinking X, we achieve the required form of Y and, therefore, of Y_x.

Thus, let $Y = X\{r_1^{-1}f_1, \dots, r_n^{-1}f_n\}$ with $r_i = |f_i(x)|$ for all $1 \le i \le n$. Then the map $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A_Y : T_i \mapsto f_i$ is an admissible epimorphism. Proposition 5.3.8 implies that the induced homomorphism $\widetilde{A}[r_1^{-1}T_1, \dots, r_n^{-1}T_n] \to \widetilde{A}_Y : T_i \mapsto \widetilde{f_i\big|_Y}$ is finite, and the required statement follows from Corollary I.7.1.2. ∎

If $f_1, \dots, f_n \in \mathcal{O}_{X,x}^*$, the family $\{X_x\{(\frac{r_i}{r_j})^{-1}\frac{f_i}{f_j}\}_{1\le i\le n}\}_{1\le j\le n}$ with $r_i = |f_i(x)|$ is a finite affinoid covering of X_x. Such a covering of X_x is said to be *rational*.

9.2.3. Lemma. *Any finite affinoid covering of X_x has a rational refinement.*

Proof. We may assume that $X = \mathcal{M}(A)$ is K-affinoid, and let $\{Y^i\}_{1\le i\le m}$ be a finite affinoid covering of X. By Lemma 9.2.2, we can shrink X and assume that $Y^i = X\{r_{ij}^{-1}f_{ij}\}_{1\le j\le n_i}$ for $f_{ij}\in A^*$ with $|f_{ij}(x)| = r_{ij}$. Adding ones to the system of functions $\{f_{ij}\}$, we may assume that $n_i = n$, $f_{in} = 1$ and $r_{in} = 1$ for all $1\le i\le m$. Let J be the set of all sequences $\mathbf{j} = (j_1,\dots,j_m)$ with $1\le j_i\le n$ and $\max\limits_i\{j_i\} = n$. *We claim that the rational covering of X defined by the functions $g_{\mathbf{j}} = f_{1j_1}\cdot\ldots\cdot f_{mj_m}$ for $\mathbf{j}\in J$ refines the covering we started from.* Indeed, setting $r_{\mathbf{j}} = r_{1j_1}\cdot\ldots\cdot r_{mj_m}$, it suffices to show that each domain $V_{\mathbf{j}} = X\{(\frac{r_{\mathbf{i}}}{r_{\mathbf{j}}})^{-1}\frac{g_{\mathbf{i}}}{g_{\mathbf{j}}}\}_{\mathbf{i}\in J}$ is contained in Y^p for $1\le p\le m$ such that $j_p = n$. To prove the latter, we have to verify that, for every point $y\in V_{\mathbf{j}}$, one has $|f_{pk}(y)|\le r_{pk}$ for all $1\le k\le n$. The point y lies in some Y^l (we may assume that $l\ne p$) and, in particular, $|f_{lj_l}(y)|\le r_{lj_l}$. On the other hand, if $\mathbf{i}\in J$ is such that $i_l = n$ and $i_k = j_k$ for $k\ne l$, then $\frac{g_{\mathbf{i}}}{g_{\mathbf{j}}} = \frac{1}{f_{lj_l}}$ and $\frac{r_{\mathbf{i}}}{r_{\mathbf{j}}} = \frac{1}{r_{lj_l}}$. It follows that $|f_{lj_l}(y)| = r_{lj_l}$. Finally, given $1\le k\le n$, let $\mathbf{i}\in J$ be such that $i_p = k$, $i_l = n$ and $i_q = j_q$ for all $q\ne p,l$. Then $\frac{g_{\mathbf{i}}}{g_{\mathbf{j}}} = \frac{f_{pk}}{f_{lj_l}}$ and $\frac{r_{\mathbf{i}}}{r_{\mathbf{j}}} = \frac{r_{pk}}{r_{lj_l}}$ and, therefore, $|f_{pk}(y)|\le r_{pk}$. ∎

Proof of Proposition 9.2.1. We may assume that X_x and all of the analytic domains considered are K-affinoid. In this case (i) and (iii) follow from Lemma 9.2.2, and (ii) follows from Lemma 9.2.3 applied to the finite affinoid covering $\{Y_x^i\}_{i\in I}$ of Y_x. ∎

Let now X_x be a separated K-germ (i.e., the point x has a separated open neighborhood). If we fix a finite covering $\{Y_x^i\}_{i\in I}$ of X_x by good (e.g., affinoid) analytic domains then, for every pair $i,j\in I$, the intersection $Y_x^{ij} = Y_x^i\cap Y_x^j$ is a good analytic domain. Proposition 9.2.1 implies that the family $\{\widetilde{Y}_x^i\}_{i\in I}$ of objects of K-$bir(\mathcal{H}(x))$ together with the family of affine domains $\{\widetilde{Y}_x^{ij}\}_{i,j\in I}$ satisfy the gluing conditions, and so we can glue all $\widetilde{Y}_x^i$'s along $\widetilde{Y}_x^{ij}$, and we get an object $\widetilde{X}_x$ of K-$bir(\mathcal{H}(x))$. This object does not depend up to a canonical isomorphism from the choice of the covering, and the correspondence $X_x\mapsto\widetilde{X}_x$ gives rise to a functor from the category of separated pointed K-germs to K-bir, which possesses the naturally extended properties (i) and (ii) of Proposition 9.2.1.

Finally, if X_x is an arbitrary K-germ, we fix a finite covering $\{Y_x^i\}_{i\in I}$ of X_x by separated (e.g., affinoid) analytic domains. Then, for every pair $i,j\in I$, the intersection $Y_x^{ij} = Y_x^i\cap Y_x^j$ is a separated analytic domain. As above, the family of domains $\{\widetilde{Y}_x^{ij}\}_{i,j\in I}$ satisfy the gluing conditions, and so we can glue all $\widetilde{Y}_x^i$'s along $\widetilde{Y}_x^{ij}$, and we get an object $\widetilde{X}_x$ of K-$bir(\mathcal{H}(x))$, which does not depend up to a canonical isomorphism from the choice of the covering. In this way we get the required functor $K\text{-}\mathcal{G}erms_{pt}\to K\text{-}bir : X_x\mapsto\widetilde{X}_x$. This functor takes analytic (resp. affine) domain embedding to domain (resp. affine domain) embeddings and also possesses the naturally extended properties (i) and (ii) of Proposition 9.2.1.

9.2.4. Proposition. *Let $\varphi : Y_y \to X_x$ be a morphism of pointed germs. Then*

(i) if φ has no boundary (i.e., $y \in \mathrm{Int}(Y/X)$), then the morphism $\widetilde{\varphi} : \widetilde{Y}_y \to \widetilde{X}_x$ is proper;

(ii) for any morphism $X'_{x'} \to X_x$, the canonical morphism $\widetilde{Y_y \times_{X_x} X'_{x'}} \to \widetilde{Y}_y \times_{\widetilde{X}_x} \widetilde{X}'_{x'}$ is proper.

We will show in §9.5 that the converse implication in (i) is also true.

Proof. (i) By Definition 8.7.1, it suffices to consider the case when $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$ are K-affinoid. In this case, one has $\widetilde{X}_x = \mathbf{V}_{\mathcal{H}(x)/K}\{\widetilde{\chi}_x(\widetilde{A})\}$ and $\widetilde{Y}_y = \mathbf{V}_{\mathcal{H}(y)/K}\{\widetilde{\chi}_y(\widetilde{B})\}$. Since $\widetilde{\chi}_y(\widetilde{B})$ is integral over $\widetilde{\chi}_y(\widetilde{A})$, the required fact follows.

(ii) As in (i), the situation is easily reduced to the case when $X = \mathcal{M}(A)$, $Y = \mathcal{M}(B)$ and $X' = \mathcal{M}(A')$ are K-affinoid. In this case the required fact follows from Corollary 5.3.10. ∎

9.3. Bijection between domains in X_x and $\widetilde{X}_x$.

9.3.1. Theorem. *Given a K-germ X_x, the reduction functor gives rise to a bijection between the set of analytic domains in X_x and the set of domains in $\widetilde{X}_x$.*

Proof. In Steps 1-4, the K-germ X_x is assumed to be good and, in Step 5, we consider the general case.

Step 1. *If Y_x and Z_x are good analytic subdomains of X_x with $\widetilde{Y}_x \subset \widetilde{Z}_x$, then $Y_x \subset Z_x$.* Indeed, by Lemma 9.2.2, we may assume that $X = \mathcal{M}(A)$ is K-affinoid, $Y = X\{r_i^{-1}f_i\}_{1\le i\le m}$ and $Z = X\{s_j^{-1}g_j\}_{1\le j\le n}$ with $f_i, g_j \in A^*$, $r_i = |f_i(x)|$ and $s_j = |g_j(x)|$. Then $\widetilde{X}_x = \mathbf{V}_{\mathcal{H}(x)/K}\{\widetilde{\chi}_x(\widetilde{A})\}$ and, by Lemma 9.2.2, $\widetilde{Y}_x = \mathbf{V}_{\mathcal{H}(x)/K}\{\widetilde{\chi}_x(\widetilde{A})[f_i(x)]_{1\le i\le m}\}$ and $\widetilde{Z}_x = \mathbf{V}_{\mathcal{H}(x)/K}\{\widetilde{\chi}_x(\widetilde{A})[g_j(x)]_{1\le j\le n}\}$. Since $\widetilde{Y_x \cap Z_x} = \widetilde{Y}_x \cap \widetilde{Z}_x = \widetilde{Y}_x$, Corollary I.7.1.2 implies that each element $g_j(x)$ is integral over $\widetilde{\chi}_x(\widetilde{A})[f_i(x)]_{1\le i\le m}$, i.e., $g_j(x)^k = (af_1^{l_1}\cdot\ldots\cdot f_m^{l_m})(x)$ for some $a \in A$ with $|a(x)| = \rho(a)$ and $l_1, \ldots, l_m \ge 0$. It follows that $(g_j^k, af_1^{l_1}\cdot\ldots\cdot f_m^{l_m}) \in \Pi_{\mathfrak{p}_x}$, i.e., $g_j^k h = af_1^{l_1}\cdot\ldots\cdot f_m^{l_m} h$ for some element $h \in A\backslash\mathfrak{p}_x$. Shrinking X, we may assume that $h \in A^*$ and, therefore, $g_j^k = af_1^{l_1}\cdot\ldots\cdot f_m^{l_m}$. The absolute value of the restriction of the function on the right hand side to Y achieves its maximum $s_j^k = |g_j(x)|^k$ at the point x and, therefore, the same is true for the function $g_j\big|_Y$. It follows that $Y \subset Z$.

Step 2. *If Y_x is an arbitrary subdomain of X_x with $\widetilde{Y}_x = \widetilde{X}_x$, then $Y_x = X_x$.* (Of course, if Y_x is good, the claim follows from Step 1.) To prove the claim in general, we need the following analog of Lemma 9.2.3. Let L be an $\mathbf{F}_1$-field, and let X be a subset of $\mathbf{V}_L$. Then for any set $f_1, \ldots, f_n \in L^*$ the system $\{X\{\frac{f_1}{f_j}, \ldots, \frac{f_n}{f_j}\}\}_{1\le j\le n}$ is a covering of X. Such a covering is said to be *rational*.

9.3.2. Lemma. *Any finite covering of a set $X \subset \mathbf{V}_L$ by subsets of the form $X\{f_1, \dots, f_n\}$ has a rational refinement.*

Proof. Let $\{Y^i\}_{1\le i\le m}$ be a finite covering of X with $Y^i = X\{f_{ij}\}_{1\le j\le n_i}$. Adding ones to the system of functions f_{ij}, we may assume that $n_i = n$ and $f_{in} = 1$ for all $1 \le i \le m$. Let J be the set of all sequences $\mathbf{j} = (j_1, \dots, j_m)$ with $1 \le j_i \le n$ and $\max\limits_i \{j_i\} = n$. *We claim that the rational covering defined by the elements $g_{\mathbf{j}} = f_{ij_1} \cdot \ldots \cdot f_{nj_m}$ for $\mathbf{j} \in J$ refines the covering we started from.* Indeed, it suffices to show that each domain $V_{\mathbf{j}} = X\{\frac{g_{\mathbf{i}}}{g_{\mathbf{j}}}\}_{\mathbf{i}\in J}$ is contained in Y^p for $1 \le p \le m$ with $j_p = n$. To prove the latter, we have to verify that, for every point $x \in V_{\mathbf{j}}$, one has $f_{pk}(x) \le 1$ for all $1 \le k \le n$. The point x lies in some Y^l (we may assume that $l \ne p$) and, in particular, $f_{lj_l}(x) \le 1$. On the other hand, if $\mathbf{i} \in J$ is such that $i_l = n$ and $i_k = j_k$ for $k \ne l$, then $\frac{g_{\mathbf{i}}}{g_{\mathbf{j}}} = \frac{1}{f_{lj_l}}$. It follows that $f_{lj_l}(x) = 1$. Finally, given $1 \le k \le n$, let $\mathbf{i} \in J$ be such that $i_p = k$, $i_l = n$ and $i_q = j_q$ for all $q \ne p, l$. Then $\frac{g_{\mathbf{i}}}{g_{\mathbf{j}}} = \frac{f_{pk}}{f_{lj_l}}$ and , therefore, $f_{pk}(x) \le 1$. ∎

By Lemma 9.2.2, Y_x has a finite affinoid covering $\{V_i\}_{1\le i\le m}$, where each V_i has the form $X_x\{r_j^{-1} f_j\}_{1\le j\le n}$ with $r_j = |f_j(x)|$, and $\widetilde{V}_i$ has the form $\widetilde{X}_x\{f_j(x)\}_{1\le j\le n}$. Then $\{\widetilde{V}_i\}_{1\le i\le m}$ is a covering of $\widetilde{X}_x = \widetilde{Y}_x$, and Lemma 9.3.2 implies that this covering has a rational refinement $\{W_j\}_{1\le j\le n}$, i.e., there are elements $g_1, \dots, g_n \in \mathcal{O}^*_{X,x}$ such that $W_j = \widetilde{X}_x\{\frac{g_k(x)}{g_j(x)}\}_{1\le k\le n}$ for all $1 \le j \le n$. If $r_j = |f_j(x)|$ and $U_j = X_x\{(\frac{r_k}{r_j})^{-1}\frac{f_k}{f_j}\}_{1\le k\le n}$, then $\widetilde{U}_j = W_j$. Since $\widetilde{U}_j \subset \widetilde{V}_i$ for some $1 \le i \le m$, Step 1 implies that $U_j \subset V_i$ and, therefore, the rational covering $\{U_j\}_{1\le j\le n}$ of X_x is in fact a covering of Y_x, i.e., $Y_x = X_x$.

Step 3. *If Y_x and Z_x are arbitrary analytic subdomains of X_x with $\widetilde{Y}_x \subset \widetilde{Z}_x$, then $Y_x \subset X_x$.* Indeed, let $\{Y^i_x\}_{i\in I}$ and $\{Z^i_x\}_{j\in J}$ be finite coverings of Y_x and Z_x by good K-germs. Then, for every $i \in I$, $\{\widetilde{Y}^i_x \cap \widetilde{Z}^i_j\}_{j\in J}$ is a covering of $\widetilde{Y}^i_x$. Since Y^i_x is good, Step 2 implies that $\{Y^i_i \cap Z^j_x\}_{j\in J}$ is a covering of Y^i_x and, therefore, $Y_x \subset Z_x$.

Step 4. *Every subdomain V of $\widetilde{X}_x$ is the reduction of an analytic subdomain of X_x.* It suffices to consider the case when V is affine and, by Lemma 9.2.2, it suffices to show that V is of the form $\widetilde{X}_x\{\alpha_1, \dots, \alpha_n\}$ for some $\alpha_1, \dots, \alpha_n \in \mathcal{H}(x)^*$. This is clear since $\widetilde{X}_x$ is identified with a rational convex polyhedral cone in $\mathbf{V}_{\mathcal{H}(x)/K}$ and any rational convex polyhedral subcone in it has the required form.

Thus, the theorem is true for good X_x.

Step 5. *The theorem is true for arbitrary X_x.* It suffices to verify the properties of Steps 3 and 4. Let $\{X^i_x\}_{i\in I}$ be a finite covering of X_x by good analytic domains. If Y_x and Z_x are analytic subdomains of X_x with $\widetilde{Y}_x \subset \widetilde{Z}_x$ then, for every $i \in I$, $Y_x \cap X^i_x$ and $Z_x \cap X^i_x$ are analytic subdomains of the good analytic domain X_x^i with $\widetilde{Y_x \cap X_x^i} \subset \widetilde{Z_x \cap X_x^i}$. Step 3 implies that $Y_x \cap X_x^i \subset Z_x \cap X_x^i$ and, therefore, $Y_x \subset Z_x$. Furthermore, if V is a subdomain of $\widetilde{X}_x$ then for every $i \in I$, $V \cap \widetilde{X}_x^i$ is a subdomain of $\widetilde{X}_x^i$. Step 4 implies that $V \cap \widetilde{X}_x^i = \widetilde{Y}_x^i$ for an analytic subdomain Y_x^i of X_x^i. Then $Y_x = \bigcup_{i \in I} Y_x^i$ is an analytic subdomain of X_x with $\widetilde{Y}_x = V$. ∎

9.3.3. Corollary. *Let $\varphi : Y_y \to X_x$ is a G-locally closed immersion of pointed K-germs. Then it is a locally closed immersion if and only if the morphism $\widetilde{\varphi} : \widetilde{Y}_y \to \widetilde{X}_x$ is proper.*

Proof. The direct implication follows from Proposition 9.2.4(i). Conversely, suppose that the morphism $\widetilde{\varphi} : \widetilde{Y}_y \to \widetilde{X}_x$ is proper, i.e., $\widetilde{Y}_y \stackrel{\sim}{\to} \widetilde{X}_x \times_{\mathbf{V}_{\mathcal{H}(x)/K}} \mathbf{V}_{\mathcal{H}(y)/K}$. Since φ is a G-locally closed immersion, the morphism $Y_y \to X_x$ goes through a closed immersion $Y_x \to X'_x$ to an analytic domain X'_x and, in particular, the induced morphism $\widetilde{Y}_y \to \widetilde{X}'_x$ is proper, i.e., $\widetilde{Y}_y \stackrel{\sim}{\to} \widetilde{X}'_x \times_{\mathbf{V}_{\mathcal{H}(x)/K}} \mathbf{V}_{\mathcal{H}(y)/K}$. It follows that $\widetilde{X}'_x \stackrel{\sim}{\to} \widetilde{X}_x$, and Theorem 9.3.1 implies that $X'_x \stackrel{\sim}{\to} X_x$, i.e., φ is a closed immersion. ∎

9.3.4. Corollary. *Let $\varphi : Y_y \to X_x$ be a morphism of pointed K-germs. then*

(i) φ is separated if and only if the induced morphism $\widetilde{\varphi} : \widetilde{Y}_y \to \widetilde{X}_x$ is separated;

(ii) given a second morphism $\psi : Z_z \to Y_y$, if the composition $\varphi\psi : Z_z \to X_x$ is separated, then so is the morphism ψ.

Proof. (i) A morphism $\varphi : Y_y \to X_x$ is separated if and only if the diagonal morphism $Y_y \to Y_x \times_{X_x} Y_y$ is a closed immersion. Since the diagonal morphism is always a G-closed immersion, Corollary 9.3.3 implies that it is a closed immersion if and only if the induced morphism $\widetilde{Y}_y \to \widetilde{Y_x \times_{X_x} Y_y}$ is proper. By Proposition 9.2.4, the latter is equivalent to the property that the diagonal morphism $\widetilde{Y}_y \to \widetilde{Y}_y \times_{\widetilde{X}_x} \widetilde{Y}_y$ is proper and, by Proposition 9.1.4(iii), it is equivalent to the property that the induced morphism $\widetilde{\varphi} : \widetilde{Y}_y \to \widetilde{X}_x$ is separated.

(ii) follows from (i) and Proposition 9.1.4(iv). ∎

9.3.5. Corollary. *Let X be a K-affinoid space. A locally closed subset $Y \subset X$ is an analytic domain if and only if its intersection $Y \cap X'$ with each irreducible component X' of X is an analytic domain in X'.*

Proof. We have to verify that, for every point $y \in Y$, there is an analytic domain U in X that contains the point y and such that $U \cap \mathcal{U} = Y \cap \mathcal{U}$ for some open neighborhood $\mathcal{U}$ of y in X. Let $\mathfrak{p} = \mathfrak{p}_y$ and $Z = \mathcal{M}(A/\Pi_\mathfrak{p})$. Then the canonical morphism of pointed germs $Z_y \to X_y$ induces an isomorphism $\widetilde{Z}_y \stackrel{\sim}{\to} \widetilde{X}_y$. The assumption implies that the point y has a neighborhood V in Z which is an analytic domain in Z. By Theorem 9.3.1, there is an analytic domain V in X which contains the point y and such that $V_y \stackrel{\sim}{\to} U_y \times_{X_y} Z_y$. *We claim that U possesses the required property.*

Indeed, if X' is an irreducible component of X, the assumption implies that there is an analytic domain V' in X' that contains the point y and such that $V' \cap \mathcal{U}' = (Y \cap X') \cap \mathcal{U}'$ for an open neighborhood $\mathcal{U}'$ of y in X', and that there is an isomorphism of pointed germs $V_y \stackrel{\sim}{\to} V'_y \times_{X_y} Z_y$. It follows that we can find an open neighborhood $\mathcal{U}$ of y in X such that $(U \cap X') \cap \mathcal{U} = (Y \cap X') \cap \mathcal{U}$ for each irreducible component X' of X and, therefore, $U \cap \mathcal{U} = Y \cap \mathcal{U}$. ∎

9.4. A characterization of good germs.

9.4.1. Theorem. *A K-germs X_x is good if and only if its reduction $\widetilde{X}_x$ is affine.*

Let L be an $\mathbf{F}_1$-field, and let X be a subset of $\mathbf{V}_L$. Then for any set $f_1, \dots, f_n \in L^*$ the system $\{X\{f_1^{\varepsilon_1}, \dots, f_n^{\varepsilon_n}\}\}_{\varepsilon \in \{\pm\}^n}$ is a covering of X. Such a covering is said to be *Laurent.*

9.4.2. Lemma. *Any finite covering of a set $X \subset \mathbf{V}_L$ by subsets of the form $X\{f_1, \dots, f_n\}$ has a Laurent refinement.*

Proof. By Lemma 9.3.2, we may assume that we are given a rational covering of X, i.e., $\{X\{\frac{f_1}{f_j}, \dots, \frac{f_n}{f_j}\}\}_{1 \le j \le n}$ with $f_1, \dots, f_n \in L^*$. *We claim that the Laurent covering of X defined by the elements $g_{ij} = \frac{f_i}{f_j}$ with $1 \le i < j \le n$ refines the above covering.* Indeed, let V be a set of that Laurent covering which is defined by a choice of ε_{ij} with $1 \le i < j \le n$. Given $1 \le i \ne j \le n$, we write $i \prec j$ if either $i < j$ and $\varepsilon_{ij} = 1$, or $i > j$ and $\varepsilon_{ij} = -1$. This defines a transitive relation on the set $\{1, \dots, n\}$. Let i be a maximal element with respect to this relation, i.e., for every $j \ne i$ one has either $j < i$ and $\varepsilon_{ji} = 1$, or $j > i$ and $\varepsilon_{ji} = -1$. Then $V \subset X\{\frac{f_1}{f_i}, \dots, \frac{f_n}{f_i}\}$. ∎

Proof of Theorem 9.4.1. The direct implication is trivial. To prove the converse implication, we may assume, Corollary 9.3.4, that X is a compact K-analytic space. A finite affinoid covering of X gives rise to a finite affine covering of of $\widetilde{X}_x$ and, by Lemma 9.4.2, the latter has a Laurent refinement $\{V_\varepsilon = \widetilde{X}_x\{\alpha_1^{\varepsilon_1}, \dots, \alpha_n^{\varepsilon_n}\}\}_{\varepsilon \in \{\pm\}^n}$, where $\alpha_1, \dots, \alpha_n$ nonzero elements of $\widetilde{\mathcal{H}(x)} = \mathcal{H}(x)$. By Theorem 9.3.1, each V_ε is the reduction $\widetilde{Y}_x^{(\varepsilon)}$ of an affinoid domain $Y^{(\varepsilon)} \subset X$. Shrinking X, we may assume that $X = \bigcup_\varepsilon Y^{(\varepsilon)}$. Induction on n reduces the theorem to verification of the following fact.

Given a separated compact K-analytic space X and a point $x \in X$, assume that $\widetilde{X}_x$ is affine and that X is a union of two affinoid domains Y and Z such that $x \in Y \cap Z$, $\widetilde{Y}_x = \widetilde{X}_x\{\alpha\}$ and $\widetilde{Z}_x = \widetilde{X}_x\{\alpha^{-1}\}$ for a nonzero element $\alpha \in \widetilde{\mathcal{H}(x)} = \mathcal{H}(x)$. Then the point x has an affinoid neighborhood in X.

In the construction which follows, we replace X by an analytic subdomain of the form $Y' \cup Z'$, where Y' and Z' are marked Laurent neighborhoods of the point x in Y and Z, respectively.

Our purpose is to shrink X and construct a closed immersion $X \to X'$ in a K-affinoid space X'. (Proposition 8.6.2(i) will then imply that the point x has an affinoid neighborhood in X.)

Let $Y = \mathcal{M}(B)$, $Z = \mathcal{M}(C)$ and $Y \cap Z = \mathcal{M}(A)$. Shrinking X, we may assume that all affinoid subdomains of Y and of Z that contain the point x are acyclic. We may also assume that $\alpha = f(x) = g(x)$ for some $f \in B^*$ and $g \in C^*$ with $t = |\alpha| = \rho(f)$ and $\rho(g^{-1}) = t^{-1}$.

Step 1. *Given affinoid neighborhoods Y' and Z' of the point x in Y and Z, respectively, one can find smaller marked Laurent neighborhoods $x \in Y'' \subset Y'$ and $x \in Z'' \subset Z'$ with $Y'' \cap Z'' = Y''\{tf^{-1}\} = Z''\{t^{-1}g\}$.* Indeed, we can shrink X and assume that $Y' = Y$ and $Z' = Z$. Since the reductions of the K-germs of $Y \cap Z$, $Y\{tf^{-1}\}$ and $Z\{t^{-1}g\}$ at x coincide, Theorem 9.3.1 implies that there are marked Laurent neighborhoods $Y' = \mathcal{M}(B')$ and $Z' = \mathcal{M}(C')$ of x in Y and Z, respectively, such that $Y' \cap Z = Y'\{tf^{-1}\}$ and $Y \cap Z' = Z'\{t^{-1}g\}$. Furthermore, since $Y' \cap Z'$ is a neighborhood of x in $Y \cap Z$, there is a marked Laurent neighborhood W of x in $Y \cap Z$ which is contained in $Y' \cap Z'$. Let $W = (Y \cap Z)\{p_i^{-1}u_i, q_j v_j\}$ with $u_i, v_j \in A$ such that $|u_i(x)| > p_i$ and $|v_j(x)| < q_j$. It is also a marked Laurent neighborhood of x in $Y' \cap Z = Y'\{tf^{-1}\}$ and $Y \cap Z' = Z'\{t^{-1}g\}$, and the latter are Weierstrass domains in Y' and Z', respectively. It follows that the elements $u_i\big|_{Y'\cap Z}$ and $v_j\big|_{Y\cap Z'}$ can be extended to elements $u_i', v_j' \in B'$ and $u_i'', v_j'' \in C'$, respectively. Then $Y'' = Y'\{p_i^{-1}u_i', q_j v_j'\}$ and $Z'' = Z'\{p_i^{-1}u_i'', q_j v_j''\}$ are marked Laurent neighborhoods of x in Y' and Z', respectively. Finally, since $W = Y'' \cap (Y' \cap Z) = Z'' \cap (Y \cap Z')$, it follows that $W = Y''\{tf^{-1}\} = Z''\{t^{-1}g\} = Y'' \cap Z''$.

Step 2. *Given elements $b \in B$ and $c \in C$ with $b(x) = c(x) \neq 0$, we can shrink X so that the images of b and c in A are equal.* Indeed, let b' and c' be the images of b and c in A, respectively, and set $\mathfrak{p} = \{a \in A \big| a(x) = 0\}$. Then there exists an element $a \in A \backslash \mathfrak{p}$ such that $ab' = ac'$. Since that homomorphisms $B \to A$ and $C \to A$ are surjective, we can find elements $u \in B$ and $v \in C$ whose images in A coincide with a. If $0 < r < |a(x)|$, then $Y\{ru^{-1}\}$ and $Z\{rv^{-1}\}$ are marked Laurent neighborhoods of x in Y and Z, respectively. We can therefore shrink X so that the element a becomes invertible in A, and we get $b' = c'$.

The above claim implies that we can shrink X and assume that the images of the elements f and g in A are equal and, by Step 1, we may assume that $Y \cap Z = Y\{tf^{-1}\} = Z\{t^{-1}g\}$. Let $0 < p < t < q$ be numbers with $|f(y)| > p$ for all $y \in Y$ and $|g(z)| < q$ for all $z \in Z$.

Step 3. Since $\widetilde{X}_x$ is affine, it coincides with $\mathbf{V}_{\mathcal{H}(x)/K}\{\beta_1, \dots, \beta_m\}$ for nonzero elements $\beta_i \in \widetilde{\mathcal{H}(x)} = \mathcal{H}(x)$. We can shrink X so that one has $\beta_i = f_i(x) = g_i(x)$ for some elements $f_i \in B$ and $g_i \in C$ with $\rho(f_i) = \rho(g_i) = |\beta_i| = r_i$. We can shrink X and assume that $f_i \in B^*$ and $g_i \in C^*$ and, by Steps 1 and 2, we can shrink X and assume that the images of f_i and g_i in A coincide and

$Y \cap Z = Y\{tf^{-1}\} = Z\{t^{-1}g\}$.

Step 4. Consider the bounded homomorphisms

$$B' = K\{r_1^{-1}T_1, \ldots, r_m^{-1}T_m, t^{-1}S_1, pS_2\} \to B : T_i \mapsto f_i, S_1 \mapsto f, S_2 \mapsto f^{-1},$$

$$C' = K\{r_1^{-1}T_1, \ldots, r_m^{-1}T_m, q^{-1}S_1, tS_2\} \to C : T_i \mapsto g_i, S_1 \mapsto g, S_2 \mapsto g^{-1}.$$

By the construction, the point x lies in the interior of the morphisms $Y \to \mathcal{M}(B')$ and $Z \to \mathcal{M}(C')$. By Proposition 6.4.3(i) and Step 1, one can shrink X so that the above homomorphisms can be extended to admissible epimorphisms

$$B'\{r_{m+1}^{-1}T_1, \ldots, r_{m+\mu}^{-1}T_\mu\} \to B : T_i \mapsto f_{m+i}$$

$$C'\{r_{m+\mu+1}^{-1}T_1, \ldots, r_{m+\mu+\nu}^{-1}T_\nu\} \to C : T_j \mapsto g_{m+\mu+j}$$

such that $|f_{\mu+i}(x)| < r_{\mu+i}$ and $|g_{m+\mu+j}(x)| < r_{m+\mu+j}$ for all $1 \le i \le \mu$ and $1 \le j \le \nu$, and $Y \cap Z = Y\{tf^{-1}\} = Z\{t^{-1}g\}$. Since $Y \cap Z$ is a Weierstrass domain in both Y and Z, we can find elements $g_{m+1}, \ldots, g_{m+\mu} \in C$ and $f_{m+\mu+1}, \ldots, f_{m+\mu+\nu} \in B$ with $g_{m+i}\big|_{Y\cap Z} = f_{m+i}\big|_{Y\cap Z}$ and $f_{m+\mu+j}\big|_{Y\cap Z} = g_{m+\mu+j}\big|_{Y\cap Z}$. By the construction, the Weierstrass domains $Y' = Y\{r_{m+\mu+1}^{-1}f_{m+\mu+1}, \ldots, r_{m+\mu+\nu}^{-1}f_{m+\mu+\nu}\}$ and $Z' = Z\{r_{m+1}^{-1}g_{m+1}, \ldots, r_{m+\mu}^{-1}g_{m+\mu}\}$ are marked Laurent neighborhoods of x in Y and Z, respectively, that contain $Y \cap Z$. One also has $Y'\{tf^{-1}\} = Y \cap Z = Z'\{t^{-1}f\}$ and, in particular, this set coincides with $Y' \cap Z'$. Thus, we can shrink X by replacing Y by Y' and Z by Z' and, setting $n = m + \mu + \nu$, we get admissible epimorphisms

$$K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, t^{-1}S_1, pS_2\} \to B : T_i \mapsto f_i, S_1 \mapsto f, S_2 \mapsto f^{-1},$$

$$K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, q^{-1}S_1, tS_2\} \to C : T_i \mapsto f_i, S_1 \mapsto g, S_2 \mapsto g^{-1},$$

with $p < t < q$, $f_i\big|_{Y\cap Z} = g_i\big|_{Y\cap Z}$, $f\big|_{Y\cap Z} = g\big|_{Y\cap Z}$, and $Y \cap Z = Y\{tf^{-1}\} = Z\{t^{-1}g\}$.

Step 5. *The K-analytic space X is piecewise K-affinoid.* Indeed, let X' be the K-affinoid space $\mathcal{M}(K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, q^{-1}S_1, pS_2\})$, and let $Y' = X'\{t^{-1}S_1\}$ and $Z' = X'\{tS_2\}$. The above admissible epimorphisms give rise to closed immersions $Y \to Y'$ and $Z \to Z'$ which are compatible on the intersection $Y \cap Z$. This means that we have a closed immersion of the K-analytic space X in the K-affinoid space X' and, therefore, X is a piecewise K-affinoid space. In particular, it is good at x. ∎

9.4.3. Corollary. *A morphism of pointed K-germs $\varphi : Y_y \to X_x$ has no boundary if and only if the induced morphism $\widetilde{\varphi} : \widetilde{Y}_y \to \widetilde{X}_x$ is proper.*

Proper. The direct implication follows from Proposition 9.2.4(i). Conversely, suppose that the morphism $\widetilde{\varphi}$ is proper. This means that the map $\widetilde{Y}_y \to \widetilde{X}_x \times_{\mathbf{V}_{\mathcal{H}(x)/K}} \mathbf{V}_{\mathcal{H}(y)/K}$ is a bijection.

To verify that φ has no boundary at x, we may assume that $X = \mathcal{M}(A)$ is K-affinoid and, in particular, the canonical map $\widetilde{X}_x \to \mathbf{V}_{\mathcal{H}(x)/K}$ identifies $\widetilde{X}_x$ with the rational convex polyhedral cone $\mathbf{V}_{\mathcal{H}(x)/K}\{\widetilde{\chi}_x(\widetilde{A})\}$. It follows that the canonical map $\widetilde{Y}_y \to \mathbf{V}_{\mathcal{H}(y)/K}$ identifies $\widetilde{Y}_y$ with a rational convex polyhedral cone in $\mathbf{V}_{\mathcal{H}(y)/K}$. Theorem 9.4.1 implies that the pointed germ Y_x is also K-affinoid, and so we may assume that $Y = \mathcal{M}(B)$ is also K-affinoid. Then $\widetilde{Y}_y = \mathbf{V}_{\mathcal{H}(y)/K}\{\widetilde{\chi}_y(\widetilde{B})\}$. Since the later coincides with $\mathbf{V}_{\mathcal{H}(y)/K}\{\widetilde{\chi}_y(\widetilde{A})\}$, it follows that $\widetilde{\chi}_y(\widetilde{B})$ is integral over $\widetilde{\chi}_y(\widetilde{A})$ and, therefore, $y \in \mathrm{Int}(Y/X)$. ∎

9.4.4. Corollary. *Let $\varphi : Y \to X$ be a morphism of K-analytic spaces. Then*

(i) if $y \in \mathrm{Int}(Y/X)$ then, for every acyclic affinoid domain $U \subset X$ that contains the point $x = \varphi(y)$, the point y has an acyclic affinoid neighborhood V in $\varphi^{-1}(U)$ with $y \in \mathrm{Int}(V/U)$;

(ii) for a second morphisms $\psi : Z \to Y$, one has $\mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Z/X)$ and, if the morphism φ is locally separated, then $\mathrm{Int}(Z/X) \subset \mathrm{Int}(Z/Y)$;

(iii) if in addition to (ii) the kernel of the canonical homomorphism $\mathcal{H}(\psi(z))^* \to \mathcal{H}(z)^*$ lies in the image of K^{**} for all points $z \in Z$, then $\mathrm{Int}(Z/X) = \mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X))$.

Proof. (i) We may assume that $U = X$. The reasoning from the proof of Corollary 9.4.3 shows that $\widetilde{Y}_y$ is affine, the pointed germ Y_y is affine and, therefore, there exists an acyclic affinoid neighborhood V of the point y with $y \in \mathrm{Int}(V/X)$.

(ii) Suppose a point $z \in Z$ lies in the set on the left hand side, and let $y = \psi(z)$ and $x = \varphi(y)$. Then the morphisms $\widetilde{Z}_z \to \widetilde{Y}_y$ and $\widetilde{Y}_y \to \widetilde{X}_x$ are proper, and Proposition 9.1.4(i) implies that the morphism $\widetilde{Z}_y \to \widetilde{X}_x$ is proper, i.e., $z \in \mathrm{Int}(Z/X)$. ∎

9.4.5. Corollary. *The classes of morphisms without boundary and of proper morphisms are preserved under composition.* ∎

§10. Examples of K-analytic spaces

10.1. GAGA. Let K-$\mathcal{S}ch$ denote the category of schemes of *locally finite type over* K, i.e., schemes $\mathcal{X}$ with the property that every point of $\mathcal{X}$ has an open affine neighborhood which is finitely generated over K. We are going to associate to such a scheme $\mathcal{X}$ a K-analytic space $\mathcal{X}^{\mathrm{an}}$. Before doing this, we introduce as follows the notion of a morphism from a K-analytic space to $\mathcal{X}$.

Thus, let $(\mathcal{X}, A, \tau)$ and (Y, B, σ) be a scheme of locally finite type over K and a K-analytic space, respectively. A *strong morphism* $\varphi : (Y, B, \sigma) \to (\mathcal{X}, A, \tau)$ is a pair consisting of a continuous map $\varphi : Y \to \mathcal{X}$, such that for every $V \in \sigma$ there exists $\mathcal{U} \in \tau$ with $\varphi(V) \subset \mathcal{U}$, and of a system of compatible morphisms of affine schemes $\varphi_{\mathcal{V}/\mathcal{U}} : \mathcal{V} = \mathrm{Spec}(B_V) \to \mathcal{U}$ with $\varphi^{\mathrm{an}}_{\mathcal{V}/\mathcal{U}}\big|_V = \varphi\big|_V$ (as maps) for all pairs $V \in \sigma$ and $\mathcal{U} \in \tau$ with $\varphi(V) \subset \mathcal{U}$. (The map on the left hand side is the composition $V \to \mathcal{V}^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}} \to \mathcal{U}$, where the middle map is induced by the morphism $\varphi_{\mathcal{V}/\mathcal{U}}$.)

10.1.1. Lemma. *Any strong morphism $\varphi : (Y, B, \sigma) \to (\mathcal{X}, A, \tau)$ extends in a unique way to a strong morphism $\overline{\varphi} : (Y, \overline{B}, \overline{\sigma}) \to (\mathcal{X}, \widehat{A}, \widehat{\tau})$.*

Proof. Given a pair $V \in \overline{\sigma}$ and $\mathcal{U} \in \widehat{\tau}$ with $\varphi(V) \subset \mathcal{U}^{\mathrm{an}}$, we can find $V' \in \sigma$ and $\mathcal{U}' \in \tau$ with $V \subset V'$ and $\varphi(V') \subset \mathcal{U}$. The composition of the morphism $\varphi_{\mathcal{V}'/\mathcal{U}'} : \mathcal{V}' = \mathrm{Spec}(B_{V'}) \to \mathcal{U}'$ with the canonical morphism $\mathcal{V} = \mathrm{Spec}(B_V) \to \mathcal{V}'$ gives rise to a morphism $\varphi_{\mathcal{V}/\mathcal{U}'} : \mathcal{V} \to \mathcal{U}'$ with $\varphi_{\mathcal{V}/\mathcal{U}'}(V) \subset \mathcal{U} \cap \mathcal{U}'$, i.e., $\varphi_{\mathcal{V}/\mathcal{U}'}$ gives rise to a morphism of schemes $\mathcal{V} \to \mathcal{U} \cap \mathcal{U}'$. The composition of the latter with the canonical morphism $\mathcal{U} \cap \mathcal{U}' \to \mathcal{U}$ gives rise to the required morphism of affine schemes $\varphi_{\mathcal{V}/\mathcal{U}} : \mathcal{V} \to \mathcal{U}$. ∎

Lemma 10.1.1 easily implies that if, in addition to the strong morphism considered, we are given strong morphisms $(Y', B', \sigma') \to (Y, B, \sigma)$ and $(\mathcal{X}, A, \tau) \to (\mathcal{X}', A', \tau')$, there is a well defined strong composition morphism $(Y', B', \sigma') \to (\mathcal{X}', A', \tau')$. We define the set of morphisms $\mathrm{Hom}((Y, B, \sigma), (\mathcal{X}, A, \tau))$ as the filtered inductive limit of the sets of strong morphisms $(Y, B', \sigma') \to (\mathcal{X}, A, \tau)$ taken over all nets $\sigma' \prec \sigma$. We now return to our brief notation for schemes and K-analytic spaces.

10.1.2. Corollary. *There is a one-to-one correspondence between* $\mathrm{Hom}(Y, \mathcal{X})$ *and the set of pairs consisting of*

(1) a continuous map $\varphi : Y \to \mathcal{X}$ with the property that there is a net of definition σ of Y such that, for every $V \in \sigma$, there exists an open connected affine subscheme $\mathcal{U} \subset \mathcal{X}$ with $\varphi(V) \subset \mathcal{U}$;

(2) a system of compatible morphisms of affine schemes $\varphi_{\mathcal{V}/\mathcal{U}} : \mathcal{V} = \mathrm{Spec}(A_V) \to \mathcal{U}$ for all pairs V and $\mathcal{U}$, an acyclic affinoid subdomain of Y and an open connected affine subscheme of $\mathcal{X}$, respectively, with $\varphi_{V/\mathcal{U}}(V) \subset \mathcal{U}$ and $\varphi^{\mathrm{an}}_{V/\mathcal{U}}\big|_V = \varphi\big|_V$ (as maps). ∎

10.1.3. Corollary. *If $\mathcal{X} = \mathrm{Spec}(A)$ is affine, then* $\mathrm{Hom}(Y, \mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}(\langle A \rangle, \mathcal{O}(Y))$. ∎

The right hand side is the set of homomorphisms of K-algebras.

Let $\Phi_{\mathcal{X}}$ be the functor from the category of K-analytic spaces K-$\mathcal{A}n$ to the category of sets that takes a K-analytic space Y to the set of morphisms $\mathrm{Hom}(Y, \mathcal{X})$.

10.1.4. Theorem. *(i) The functor $\Phi_{\mathcal{X}}$ is representable by a K-analytic space without boundary $\mathcal{X}^{\mathrm{an}}$ and a morphism $\pi : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$;*

(ii) the canonical functor $\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathcal{X}^{\mathrm{an}})$ *is fully faithful;*

(iii) there is an isomorphism of sites $\mathcal{X}^{\rm an}_{\rm Zar} \widetilde{\to} \mathcal{X}_{\rm Zar}$*;*

(iv) the correspondence $\mathcal{X} \mapsto \mathcal{X}^{\rm an}$ *gives rise to a fully faithful functor* K-$\mathcal{S}ch \to K$-$\mathcal{A}n$ *that commutes with fiber products and extensions of the ground field.*

The morphism of sites in (iii) is induced by the correspondence $\mathcal{U} \mapsto \pi^{-1}(\mathcal{U})$, and the latter also induces a morphism of sites $\pi_{\rm G} : \mathcal{X}^{\rm an}_{\rm pG} \to \mathcal{X}_{\rm G}$. The functor from (ii) is defined by $\pi_{\rm G}$, i.e., it takes a coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ to $\pi^*_{\rm G}\mathcal{F} = \pi^{-1}_{\rm G}\mathcal{F} \otimes_{\pi^{-1}_{\rm G}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}^{\rm an}}$.

Proof. Step 1. Let $\{Y_i\}_{i\in I}$ be a covering of a K-analytic space Y in $Y_{\rm pG}$. *Then the following sequence of maps is exact:*

$$\mathrm{Hom}(Y, \mathcal{X}) \to \prod_i \mathrm{Hom}(Y_i, \mathcal{X}) \rightrightarrows \prod_{i,j} \mathrm{Hom}(Y_i \cap Y_j, \mathcal{X})$$

(i.e., the functor $\Phi_{\mathcal{X}}$ is a sheaf on $X_{\rm G}$). Indeed, let $\varphi_i : Y_i \to \mathcal{X}$ be a family of morphisms such that, for any pair $i, j \in I$, $\varphi_i\big|_{Y_i\cap Y_j} = \varphi_j\big|_{Y_i\cap Y_j}$, and let τ be the collection of affinoid subdomains $V \subset Y$ such that there exists $i \in I$ with $V \subset Y_i$ and $\varphi_i(V) \subset \mathcal{U}$ for some open affine subscheme $\mathcal{U} \subset \mathcal{X}$. It is easy to see that τ is a net of definition and, therefore, the morphisms φ_i, $i \in I$, give rise to a morphism $\varphi : Y \to \mathcal{X}$. That φ is unique is trivial.

Step 2. Let $\mathcal{X}$ be the scheme affine space $\mathrm{Spec}(K[T_1, \dots, T_n])$. In this case one has

$$\mathcal{X}^{\rm an} = \bigcup_{r>0} E(0; r) \ ,$$

where $E(0; r)$ is the closed polydisc of radius $r = (r_1, \dots, r_n) \in (\mathbf{R}^*_+)^n$ with center at zero defined as the set $\{x \in \mathcal{X}^{\rm an} \big| |T_i(x)| \le r_i \text{ for all } 1 \le i \le n\}$. The latter set is canonically identified with the spectrum of the K-affinoid algebra $K\{r^{-1}T\} = K\{r^{-1}T_1, \dots, r_n^{-1}T_n\}$. This defines a K-affinoid atlas on $\mathcal{X}^{\rm an}$ with the net $\{E(0; r)\}_{r>0}$, and the corresponding triple is a K-analytic space, which is called the *n-dimensional affine space* and denoted by $\mathbf{A}^n$. It follows from Corollary 10.1.3 that $\mathbf{A}^n$ and the canonical morphism $\mathbf{A}^n \to \mathcal{X}$ represent the functor $\Phi_{\mathcal{X}}$.

Step 3. Let M be a finite A-module for $A = K[T_1, \dots, T_n]$ and, for $r = (r_1, \dots, r_n) \in (\mathbf{R}^*_+)^n$, we set $A(r) = K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. By Lemma 1.2.2, there is a unique Zariski A-submodule $N(r) \subset M$ such that the quotient $M(r) = M/N(r)$ has the structure of a finite Banach $A(r)$-module. *We claim that there exist* $r'_1, \dots, r'_n > 0$ *such that for every* $r_i \ge r'_i$, $1 \le i \le n$, *one has* $N(r) = 0$ *or, equivalently,* $M \widetilde{\to} M(r)$. Indeed, by Corollary 2.4.3, there is a finite chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \dots \subset N_k = M$ such that each quotient N_i/N_{i-1} is isomorphic to an A-module of the form A/Π, where Π is a prime ideal of A. This reduces the situation to the case when M is the quotient K-algebra $B = A/\Pi$ for a prime ideal Π. If $\mathcal{Y} = \mathrm{Spec}(B)$, the set $\mathcal{Y}^{\mathrm{an}}$ is identified with an irreducible affine subspace of $\mathbf{R}_+^n$, and the spectrum $\mathcal{Y}^{\mathrm{an}}(r) = \mathcal{M}(B(r))$ is identified with the $|K|$-affinoid polytope $\{y \in \mathcal{Y}^{\mathrm{an}} \big| |T_i(y)| \le r_i \text{ for all } 1 \le i \le n\}$. If all r_i's are large enough, the $|K|$-affinoid polytope $\mathcal{Y}^{\mathrm{an}}(r)$ generates the affine subspace $\mathcal{Y}^{\mathrm{an}}$. This implies that a nonzero element of B cannot be equal to zero on $\mathcal{Y}^{\mathrm{an}}$ and, therefore, $B \xrightarrow{\sim} B(r)$.

Step 4. Let $\mathcal{X} = \mathrm{Spec}(A)$ be an arbitrary finitely presented affine scheme over K. We fix an epimorphism of K-algebras $A' = K[T_1, \dots, T_n] \to A : T_i \mapsto f_i$ and denote by F its kernel and, for $r = (r_1, \dots, r_n) \in (\mathbf{R}_+^*)^n$, we set $A'(r) = K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. Let $\mathbf{a}(r)$ be the unique minimal Zariski $A(r)$-submodule of A from Lemma 1.2.2(ii), i.e., the quotient $A(r) = A/\mathbf{a}(r)$ is a K-affinoid algebra. Furthermore, the above epimorphism gives rise to a homeomorphism of $\mathcal{X}^{\mathrm{an}}$ with a closed subset of $\mathbf{A}^n$. The set $\mathcal{X}^{\mathrm{an}}(r) = \mathcal{X}^{\mathrm{an}} \cap E(0; r) = \{x \in \mathcal{X}^{\mathrm{an}} \big| |f_i(x)| \le r_i \text{ for all } 1 \le i \le n\}$ is identified with the spectrum $\mathcal{M}(A(r))$. This defines a K-affinoid atlas A on $\mathcal{X}^{\mathrm{an}}$ with the net $\tau = \{\mathcal{X}^{\mathrm{an}}(r)\}_{r>0}$, and the corresponding triple $(\mathcal{X}^{\mathrm{an}}, A, \tau)$ is a K-analytic space which will be denoted simply by $\mathcal{X}^{\mathrm{an}}$. *We claim that the functor $\Phi_{\mathcal{X}}$ is representable by $\mathcal{X}^{\mathrm{an}}$ and the canonical morphism $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$.* (In particular, the K-analytic space $\mathcal{X}^{\mathrm{an}}$ does not depend on the choice of the above epimorphism.) Indeed, we have to verify that, for any K-analytic space Y, the morphism $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ gives rise to a bijection $\mathrm{Hom}(Y, \mathcal{X}^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}(Y, \mathcal{X}) = \mathrm{Hom}(A, \mathcal{O}(Y))$. Step 1 and Proposition 7.4.5 reduce the situation to the case when $Y = \mathcal{M}(B)$ is K-affinoid, and so we have to verify that $\mathrm{Hom}(\mathcal{M}(B), \mathcal{X}^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}(A, B)$. The set on the left hand side is the inductive limit $\varinjlim \mathrm{Hom}(A(r), B)$. By Step 3, if r_i's are large enough, then $A \xrightarrow{\sim} A(r)$, and so we have to verify that for any homomorphism of K-algebras $A \to B : f_i \mapsto g_i$ one can find larger r_i-s such that the induced homomorphism of K-affinoid algebras $A'(r) \to B$ is bounded. But Corollary 2.2.2 guarantees this property if $r_i \ge \rho(g_i)$ for all $1 \le i \le n$.

Step 5. Let $\varphi : \mathcal{Y} = \mathrm{Spec}(B) \to \mathcal{X} = \mathrm{Spec}(A)$ be a closed (resp. open) immersion of finitely presented affine schemes over K. *Then the canonical morphism $\varphi^{\mathrm{an}} : \mathcal{Y}^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ is a closed (resp. open) immersion of K-analytic spaces.* Indeed, if φ is a closed immersion, the statement follows from Step 4. Assume that φ is an open immersion. It this case, it suffices to consider the following two cases (1) $\mathcal{Y}$ is the minimal connected component of $\mathcal{X}$, and (2) $\mathcal{Y}$ is a principal open subset of $\mathcal{X}$. In the case (1), $\mathcal{Y}^{\mathrm{an}}$ is the minimal connected component of $\mathcal{X}^{\mathrm{an}}$, the morphism φ is also a closed immersion, and the required fact follows. Suppose that $\mathcal{Y}$ is a principal open subset of $\mathcal{X}$, i.e., $\mathcal{Y} = D(f)$ for some $f \in A$. If we fix an epimorphism $A' = K[T_1, \dots, T_n] \to A$ as in Step 4, it extends to an epimorphism $B' = A'[T_{n+1}] \to B : T_{n+1} \mapsto \frac{1}{f}$. For $r' = (r, r_{n+1}) \in (\mathbf{R}_+^*)^n \times \mathbf{R}_+^*$, one has $\mathcal{Y}^{\rm an}(r') = \{x \in \mathcal{X}^{\rm an}(r) \big| |f(x)| \ge r_{n+1}\}$, i.e., $\mathcal{Y}^{\rm an}(r')$ is a Laurent subdomain of $\mathcal{X}^{\rm an}$. This implies that $\varphi^{\rm an}$ identifies $\mathcal{Y}^{\rm an}$ with an open analytic subdomain of $\mathcal{X}^{\rm an}$.

Step 6. Let $\mathcal{U}$ be an open subscheme of an affine scheme $\mathcal{X} = \mathrm{Spec}(A)$ as above. Then $\mathcal{U}^{\rm an}$ is an open analytic subdomain of $\mathcal{X}^{\rm an}$. *We claim that the functor $\Phi_{\mathcal{U}}$ is representable by $\mathcal{U}^{\rm an}$ and the canonical morphism $\mathcal{U}^{\rm an} \to \mathcal{U}$.* Indeed, by Step 1 and Proposition 7.4.5, we have to verify that for any K-affinoid space $Y = \mathcal{M}(B)$ one has $\mathrm{Hom}(Y, \mathcal{U}^{\rm an}) \stackrel{\sim}{\to} \mathrm{Hom}(Y, \mathcal{U})$, i.e., for any morphism of affine schemes $\varphi : \mathcal{Y} = \mathrm{Spec}(B) \to \mathcal{X}$ with $\varphi(\mathcal{Y}) \subset \mathcal{U}$, the morphism $\varphi^{\rm an} : Y \to \mathcal{X}^{\rm an}$ is induced by a unique morphism of K-analytic spaces $Y \to \mathcal{U}^{\rm an}$. Since Y is compact, there is $r \in (\mathbf{R}_+^*)^n$ with $\varphi^{\rm an}(Y) \subset \mathcal{X}^{\rm an}(r)$, i.e., the morphism $\varphi^{\rm an} : Y \to \mathcal{X}^{\rm an}$ is induced by a unique morphism of K-affinoid spaces $\varphi^{\rm an}(r) : Y \to \mathcal{X}^{\rm an}(r)$. Since the image of Y is contained in $\mathcal{U}^{\rm an} \cap \mathcal{X}^{\rm an}(r)$ and the latter set is an open analytic subdomain of $\mathcal{X}^{\rm an}(r)$, the morphism $\varphi^{\rm an}(r)$ is induced by a unique morphism of K-analytic spaces $Y \to \mathcal{U}^{\rm an} \cap \mathcal{X}^{\rm an}(r)$, and the required fact follows.

Step 7. Let now $\mathcal{X}$ be an arbitrary scheme locally finitely presented over K. We define $\mathcal{X}^{\rm an}$ as the K-analytic space obtained by gluing of the K-analytic spaces $\mathcal{U}^{\rm an}$ along the open analytic domains $(\mathcal{U} \cap \mathcal{V})^{\rm an}$ for all open affine subschemes $\mathcal{U}, \mathcal{V} \subset \mathcal{X}$. *We claim that the functor $\Phi_{\mathcal{X}}$ is representable by $\mathcal{X}^{\rm an}$ and the canonical morphism $\mathcal{X}^{\rm an} \to \mathcal{X}$.* Indeed, by Step 1 and Proposition 7.4.5, we have to verify that $\mathrm{Hom}(Y, \mathcal{X}^{\rm an}) \stackrel{\sim}{\to} \mathrm{Hom}(Y, \mathcal{X})$, where $Y = \mathcal{M}(B)$ is a K-affinoid space with either at most two connected components, or with three connected components U, V_1 and V_2 such that V_1 and V_2 are not comparable in $\pi_0(Y)$ and $U = \inf(V_1, V_2)$. In the first case, Y belongs to any net of definition on Y and, therefore, for any morphism $\varphi : Y \to \mathcal{X}$ one has $\varphi^{\rm an}(Y) \subset \mathcal{U}^{\rm an}$, where $\mathcal{U}$ is an open affine subscheme of $\mathcal{X}$, i.e., φ is induced by a morphism $Y \to \mathcal{U}$. By Step 4, the latter is induced by a unique morphism $Y \to \mathcal{U}^{\rm an}$, and the required fact follows. In the second case, there exist open affine subschemes $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W} \subset \mathcal{X}$ and an affinoid subdomain $V_1 \cup V_2 \subset W \subset Y$ such that φ induces unique morphisms $U \cup V_1 \to \mathcal{V}_1$, $U \cup V_2 \to \mathcal{V}_2$ and $W \to \mathcal{W}$. By Step 4, they give rise to unique morphisms of K-analytic spaces $U \cup V_1 \to \mathcal{V}_1^{\rm an}$, $U \cup V_2 \to \mathcal{V}_2^{\rm an}$ and $W \to \mathcal{U}^{\rm an}$, and the latter are compatible on intersections. It follows that φ is induced by a unique morphism of K-analytic spaces $Y \to \mathcal{X}^{\rm an}$. Theorem 7.1.4 implies that the K-analytic space $\mathcal{X}^{\rm an}$ has no boundary, i.e., (i) is true.

Step 8. It suffices to prove (ii) in the case when $\mathcal{X} = \mathrm{Spec}(A)$ is an affine scheme. In this case, every coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ is of the form $\mathcal{O}_{\mathcal{X}}(M)$ for some finite A-module M. Furthermore, the K-analytic space $\mathcal{X}^{\rm an}$ is a union of an increasing sequence of affinoid domains $V_1 \subset V_2 \subset \dots$, and Step 3 implies that, for any finite A-module M, there exists $k \ge 1$ such that $M \stackrel{\sim}{\to} M_{V_n}$

for all $n \geq k$, i.e., $\mathcal{F}(\mathcal{X}) \xrightarrow{\sim} \pi_{\mathrm{G}}^* \mathcal{F}(V_n)$ for all $n \geq k$. This immediately implies that the functor $\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathcal{X}^{\mathrm{an}}) : \mathcal{F} \mapsto \pi_{\mathrm{G}}^* \mathcal{F}$ is fully faithful, i.e., (ii) is true.

Step 9. We have to show that every Zariski open subset $\mathcal{W} \subset \mathcal{X}^{\mathrm{an}}$ is of the form $\mathcal{U}^{\mathrm{an}}$, where $\mathcal{U}$ is an open subscheme $\mathcal{X}$. Suppose first that $\mathcal{X} = \mathrm{Spec}(A)$ is affine. Let U be an affinoid domain in $\mathcal{X}^{\mathrm{an}}$ with $A \xrightarrow{\sim} A_U$. Then $U \cap \mathcal{W} = U \cap \mathcal{U}^{\mathrm{an}}$, where $\mathcal{U}$ is an open affine subscheme of $\mathcal{X}$. *We claim that* $\mathcal{W} = \mathcal{U}^{\mathrm{an}}$. Indeed, if V is a bigger affinoid domain with $A \xrightarrow{\sim} A_V$, then $V \cap \mathcal{W} = V \cap \mathcal{V}^{\mathrm{an}}$, where $\mathcal{V}$ is also an open affine subscheme of $\mathcal{X}$. It follows that $U \cap \mathcal{U}^{\mathrm{an}} = U \cap \mathcal{V}^{\mathrm{an}}$ and, by Corollary 7.1.5, one has $\mathcal{U} = \mathcal{V}$. Since $\mathcal{X}^{\mathrm{an}}$ is a union such affinoid domains V, it follows that $\mathcal{W} = \mathcal{U}^{\mathrm{an}}$. In the general case, the intersection of $\mathcal{W}$ with $\mathcal{Y}^{\mathrm{an}}$ for every open affine subscheme $\mathcal{Y} \subset \mathcal{X}$ coincides with $\mathcal{V}^{\mathrm{an}}$ for some open affine subscheme $\mathcal{V} \subset \mathcal{Y}$. If $\mathcal{U}$ is the union of all such $\mathcal{V}$'s, then $\mathcal{W} = \mathcal{U}^{\mathrm{an}}$. Finally, we have to show that, given an open subscheme $\mathcal{U} \subset \mathcal{X}$, a family $\{\mathcal{U}_i\}_{i \in I}$ of open subschemes of $\mathcal{U}$ is a covering of $\mathcal{U}$ in $\mathcal{X}_{\mathrm{G}}$ if and only if the family $\{\mathcal{U}_i^{\mathrm{an}}\}_{i \in I}$ is a covering of $\mathcal{U}^{\mathrm{an}}$ in $\mathcal{X}^{\mathrm{an}}_{\mathrm{Zar}}$. The direct implication is trivial. To verify the converse implication, we have to show that every open affine subscheme $\mathcal{V} = \mathrm{Spec}(B) \subset \mathcal{U}$, which is of the form $\mathcal{V}' \cup \mathcal{V}'' \cup \inf(\mathcal{V}', \mathcal{V}'')$ for some connected components $\mathcal{V}'$ and $\mathcal{V}''$ of $\mathcal{V}$, is contained in some $\mathcal{U}_i$. We know that the K-analytic space $\mathcal{V}^{\mathrm{an}}$ is a union of an increasing sequence of affinoid domains $V_1 \subset V_2 \subset \dots$ with $B \xrightarrow{\sim} B_{V_n}$ for all $n \geq 1$. Since $\{\mathcal{U}_i^{\mathrm{an}}\}_{i \in I}$ is a covering of $\mathcal{U}^{\mathrm{an}}$ in $\mathcal{X}^{\mathrm{an}}_{\mathrm{Zar}}$, it follows that, for every $n \geq 1$, there exists $i_n \in I$ with $V_n \subset \mathcal{U}_{i_n}^{\mathrm{an}}$ and, therefore, $V_n \subset (\mathcal{V} \cap \mathcal{U}_{i_n})^{\mathrm{an}}$. Since the number of open subschemes of $\mathcal{V}$ is finite, it follows that $\mathcal{V} \subset \mathcal{U}_{i_n}$ for some $n \geq 1$, i.e., (iii) is true.

Step 10. Let $\mathcal{X}$ and $\mathcal{Y}$ be schemes locally finitely presented over K. We have to show that the canonical map $\mathrm{Hom}(\mathcal{Y}, \mathcal{X}) \to \mathrm{Hom}(\mathcal{Y}^{\mathrm{an}}, \mathcal{X}^{\mathrm{an}})$ is a bijection, and in fact it suffices to verify this only for affine $\mathcal{Y}$'s. Indeed, if the map considered is a bijection for affine $\mathcal{Y}$'s, it is also a bijection for separated $\mathcal{Y}$'s, by Proposition I.5.5.1 and 8.4.4, and then for the same reason it is a bijection for arbitrary $\mathcal{Y}$'s. Thus, assume that $\mathcal{Y} = \mathrm{Spec}(B)$ is affine. Notice that if $\mathcal{X}$ is also affine, the required bijectivity easily follows from Steps 3 and 4.

Step 11. Suppose that $\mathcal{X}$ is arbitrary, and we are given two morphisms $\varphi, \psi : \mathcal{Y} \to \mathcal{X}$ with $\varphi^{\mathrm{an}} = \psi^{\mathrm{an}}$. To show that they coincide it suffices to verify that their restrictions to every connected component of $\mathcal{Y}$ coincide, and so we may assume that $\mathcal{Y}$ is connected. In this case there exist open affine subschemes $\mathcal{U}, \mathcal{W} \subset \mathcal{X}$ with $\varphi(\mathcal{Y}) \subset \mathcal{U}$ and $\psi(\mathcal{Y}) \subset \mathcal{W}$. Since $\varphi^{\mathrm{an}}(\mathcal{Y}^{\mathrm{an}}) = \psi^{\mathrm{an}}(\mathcal{Y}^{\mathrm{an}}) \subset \mathcal{U}^{\mathrm{an}} \cap \mathcal{W}^{\mathrm{an}} = (\mathcal{U} \cap \mathcal{W})^{\mathrm{an}}$, it follows that $\varphi(\mathcal{Y}), \psi(\mathcal{Y}) \subset \mathcal{U} \cap \mathcal{W}$. This means that φ and ψ go through morphisms to the open subscheme $\mathcal{U} \cap \mathcal{W}$. By the same argument, there exist open affine subscheme $\mathcal{U}', \mathcal{W}' \subset \mathcal{U} \cap \mathcal{W}$ with $\varphi(\mathcal{Y}), \psi(\mathcal{Y}) \subset \mathcal{U}' \cap \mathcal{W}'$. Since the number of open subschemes in $\mathcal{U}$ (and $\mathcal{W}$)

is finite, we can find an open affine subscheme in $\mathcal{U} \cap \mathcal{W}$ which contains $\varphi(\mathcal{Y})$ and $\psi(\mathcal{Y})$, i.e., we reduce the situation to the case when $\mathcal{X}$ is also affine and, therefore, $\varphi = \psi$.

Step 12. Suppose now we are given a morphism of K-analytic spaces $\varphi : \mathcal{Y}^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$. To verify that φ is induced by a morphism of schemes $\mathcal{Y} \to \mathcal{X}$, we use the result and a reasoning from Step 9. It suffices to show that, if $\mathcal{Y}$ is a union $\mathcal{Y}' \cup \mathcal{Y}'' \cup \inf(\mathcal{Y}', \mathcal{Y}'')$ for connected components $\mathcal{Y}'$ and $\mathcal{Y}''$ of $\mathcal{Y}$, the image $\varphi(\mathcal{Y}^{\mathrm{an}})$ lies in $\mathcal{U}^{\mathrm{an}}$, where $\mathcal{U}$ is an open affine subscheme of $\mathcal{X}$. We know that the K-analytic space $\mathcal{Y}^{\mathrm{an}}$ is a union of an increasing sequence of affinoid domains $V_1 \subset V_n \subset \dots$ with $B \stackrel{\sim}{\to} B_{V_n}$ for all $n \geq 1$ and, for every $n \geq 1$, one has $\varphi(V_n) \subset \mathcal{U}_n^{\mathrm{an}}$, where $\mathcal{U}_n$ is an open affine subscheme of $\mathcal{X}$. The preimage of $\mathcal{U}_n^{\mathrm{an}}$ in $\mathcal{Y}^{\mathrm{an}}$ is a Zariski open subset and, therefore, Step 9 implies that $\varphi^{-1}(\mathcal{U}_n^{\mathrm{an}}) = \mathcal{V}_n^{\mathrm{an}}$, where $\mathcal{V}_n$ is an open affinoid subscheme of $\mathcal{Y}$. Since $\mathcal{Y}^{\mathrm{an}}$ is a union of V_n's and the number of open affine subschemes of $\mathcal{Y}$ is finite, it follows that $\mathcal{V}_n = \mathcal{Y}$ for a sufficiently large n and, therefore, $\varphi(\mathcal{Y}^{\mathrm{an}}) \subset \mathcal{U}_n^{\mathrm{an}}$.

Thus, the functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ is fully faithful. Since it commutes with fiber products and the ground field extension functor on affine schemes, the same properties hold for arbitrary schemes. ∎

§11. Non-Archimedean analytic spaces associated to analytic spaces over $\mathbf{F}_1$

11.1. Construction of a functor $X \mapsto X^{(\phi)}$. Let K be a real valuation $\mathbf{F}_1$-field, and let k be a non-Archimedean field, and suppose we are given an isometric homomorphism of $\mathbf{F}_1$-fields $\phi : K \to k^{\cdot}$. The latter allows one to view any Banach k-algebra $\mathcal{B}$ as a Banach K-algebra $\mathcal{B}^{\cdot}$. If A is a Banach K-algebra, the set of bounded ϕ-homomorphisms $A \to \mathcal{B}^{\cdot}$ will be denoted by $\mathrm{Hom}_{\phi}(A, \mathcal{B}^{\cdot})$.

11.1.1. Definition. A *ϕ-morphism* from a k-analytic space Y to a K-analytic space X is a pair consisting of the following:

(1) a continuous map $\varphi : Y \to X$ such that, for every point $y \in Y$, there exist affinoid domains $V_1, \dots, V_n \subset Y$ such that $y \in V_1 \cap \dots \cap V_n$, $V_1 \cup \dots \cup V_n$ is a neighborhoods of y and, for every $1 \leq i \leq n$, $\varphi(V_i)$ lies in an acyclic affinoid subdomain of X;

(2) a system of compatible bounded ϕ-homomorphisms $A_U \to \mathcal{B}_V^{\cdot}$ for all pairs consisting of an affinoid domain $V = \mathcal{M}(\mathcal{B}_V) \subset Y$ and an acyclic affinoid domain $U = \mathcal{M}(A_U) \subset X$ with $\varphi(V) \subset U$ such that the induced map $V \to U$ coincides with $\varphi\big|_V$.

It is easy to see that, given a ϕ-morphism $\varphi : Y \to X$, a morphism of k-analytic spaces $\psi : Y' \to Y$, and a morphism of K-analytic spaces $\chi : X \to X'$, there is a well defined composition morphism $\chi\varphi\psi : Y' \to X'$. Let Φ_X be the functor from the category of k-analytic spaces to the category of sets that takes a k-analytic space Y to the set of ϕ-morphisms $\mathrm{Hom}_\phi(Y,X)$.

11.1.2. Theorem. *(i) The functor Φ_X is representable by a k-analytic space $X^{(\phi)}$ and a ϕ-morphism $\pi = \pi_X : X^{(\phi)} \to X$ which is compact as a map;*

(ii) the preimage $\pi^{-1}(Y)$ of an analytic (resp. piecewise affinoid) domain $Y \subset X$ is an analytic (resp. affinoid) domain in $X^{(\phi)}$;

(iii) given a morphism of K-analytic spaces $Y \to X$, one has $\pi_Y^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Y^{(\phi)}/X^{(\phi)})$;

(iv) the functor $X \mapsto X^{(\phi)}$ commutes with fiber products and takes open and closed immersions, and finite and proper morphisms to morphisms of the same type;

(v) the functor $X \mapsto X^{(\phi)}$ gives rise to a functor K-$\mathcal{P}aff \to k$-$\mathcal{A}ff$;

(vi) for any scheme $\mathcal{X}$ of locally finite type over K, there is a canonical isomorphism of k-analytic spaces $(\mathcal{X}^{(\phi)})^{\mathrm{an}} \xrightarrow{\sim} (\mathcal{X}^{\mathrm{an}})^{(\phi)}$.

If Y is a k-analytic space, a ϕ-homomorphism $\alpha : A \to \mathcal{O}(Y)^{\cdot}$ from a Banach K-algebra A is said to be *bounded* if, for any affinoid domain $V \subset Y$, the induced ϕ-homomorphism $A \to \mathcal{B}_V^{\cdot}$ is bounded.

11.1.3. Lemma. *If $X = \mathcal{M}(A)$ is K-affinoid, then $\mathrm{Hom}_\phi(Y,X)$ coincides with the set of bounded ϕ-homomorphisms $A \to \mathcal{O}(Y)^{\cdot}$.*

Proof. The statement is easily reduced to the case when $Y = \mathcal{M}(B)$ is a k-affinoid space. Given a ϕ-morphism $\varphi : Y \to X$, let $\{V_i\}_{i\in I}$ be a finite affinoid covering of Y such that, for every $i \in I$, $\varphi(V_i)$ lies in an acyclic affinoid subdomain U_i of X. This gives rise to a family of bounded homomorphisms $A_{U_i} \to \mathcal{B}_{V_i}^{\cdot}$ which are compatible on intersections. Since $\mathcal{B} \xrightarrow{\sim} \mathrm{Ker}(\prod_{i\in I} \mathcal{B}_{V_i} \rightrightarrows \prod_{i,j\in I} \mathcal{B}_{V_i\cap V_j})$, we get a bounded ϕ-homomorphism $A \to \mathcal{B}^{\cdot}$ Conversely, given such a bounded ϕ-homomorphism $A \to \mathcal{B}^{\cdot}$, let φ be the induced map $Y \to X$, and let $U \subset X$ and $V \subset Y$ be affinoid subdomains with $\varphi(V) \subset U$. Corollary 6.3.4 implies that the bounded ϕ-homomorphism $A \to \mathcal{B}^{\cdot} \to \mathcal{B}_V^{\cdot}$ goes through a unique bounded ϕ-homomorphism $A_U \to \mathcal{B}_V^{\cdot}$ such that the corresponding map $V \to U$ coincides with the restriction of φ to V. This means that the ϕ-homomorphism φ gives rise to a unique ϕ-morphism $Y \to X$. ∎

11.1.4. Lemma. *For any K-affinoid algebra A, the functor $\mathcal{B} \to \mathrm{Hom}_\phi(A, \mathcal{B}^{\cdot})$ that takes a k-affinoid algebra $\mathcal{B}$ to the set of bounded ϕ-homomorphisms $A \to \mathcal{B}^{\cdot}$ is representable by a k-affinoid algebra $k_\phi\{A\}$.*

Proof. First of all, if A is the K-affinoid algebra $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$, then the functor considered is representable by the k-affinoid algebra $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. In the general case, we represent A as a quotient B/E of a K-affinoid algebra B of the above form. Then the functor considered is representable by the K-affinoid algebra $k_\phi\{B\}/\mathbf{b}_E$, where $\mathbf{b}_E$ is the ideal of $k_\phi\{B\}$ generated by the elements $f-g$ for $(f,g)\in E$. (Recall that all ideals of a k-affinoid algebra are closed.) ∎

The k-affinoid algebra $k_\phi\{A\}$ can be also constructed as follows. Let $\{f_i\}_{i\in I}$ be a system of elements of A which represent nonzero elements of the quotient A/K^*. Then there is an isomorphism of K-Banach spaces $\oplus_{i\in I}Kf_i \widetilde{\to} A$. Let I' be the set of all $i\in I$ such that the stabilizer of f_i in K^* lies in the kernel of the homomorphism of groups $K^*\to k^*$. Then the above isomorphism gives rise to an isomorphism of k-Banach spaces $\widehat{\oplus}_{i\in I'}kf_i \widetilde{\to} k_\phi\{A\}$, where the space on the left hand side consists of sums $F=\sum_{i\in I'}\lambda_i f_i$ such that $|\lambda_i|\cdot||f_i||\to 0$ with respect to the filter of complements of finite sets in I' and $||F||=\max\limits_i|\lambda_i|\cdot||f_i||$.

Notice that the correspondence $A\mapsto k_\phi\{A\}$ is a covariant functor left adjoint to the functor from the category of k-affinoid algebras to that of quasi-affinoid K-algebras (induced by ϕ).

11.1.5. Lemma. *(i) For an element $f\in A$ and a number $r>0$, there are canonical isomorphisms $k_\phi\{A\{r^{-1}f\}\}\widetilde{\to} k_\phi\{A\}\{r^{-1}f\}$ and $k_\phi\{A\{rf^{-1}\}\}\widetilde{\to} k_\phi\{A\}\{rf^{-1}\}$;*

(ii) for A-affinoid algebras B and C, there is a canonical isomorphism

$$k_\phi\{B\widehat{\otimes}_A C\}\widetilde{\to} k_\phi\{B\}\widehat{\otimes}_{k_\phi\{A\}}k_\phi\{C\}\ ;$$

(iii) if a bounded homomorphism of K-affinoid algebras $A\to B$ is surjective admissible (resp. finite), then so is the homomorphism of k-affinoid algebras $k_\phi\{A\}\to k_\phi\{B\}$;

(iv) the kernel of the admissible epimorphism $k_\phi\{A\}\to k_\phi\{A/\mathbf{r}(A)\}$ is a nilpotent ideal.

Proof. The statements (i) and (ii) follow from the fact that both k-affinoid algebras represent the same functor, and (iii) follows from the construction of $k_\phi\{A\}$. The kernel of the homomorphism in (iv) is generated by elements of the form $a-b$ with $(a,b)\in\mathbf{r}(A)$. By the proof of Lemma I.8.1.4(iv), one has $(a-b)^{2n+1}=0$, and the claim follows. ∎

Let I be an $\mathbf{F}_1$-subalgebra of I_A, the finite idempotent $\mathbf{F}_1$-subalgebra of A. For an idempotent $e\in\check{I}$, we set $A^{(e)}=A/F_e$, where F_e is the ideal of A generated by the prime ideal Π_e of I. (Recall that, by Example 1.1.4, the ideal F_e is closed.)

11.1.6. Lemma. *In the above situation, there is a canonical isomorphism of k-affinoid algebras*

$$k_\phi\{A\}\widetilde{\to}\prod_{e\in\check{I}}k_\phi\{A^{(e)}\}\ .$$

Proof. As in the proof of Lemma I.8.1.5, it suffices to consider the case when I has only one nontrivial idempotent e. In this case, $F_e = \{(a,b)\big| ae = be\}$ and F_1 is the ideal associated with the Zariski ideal Ae, i.e., $A^{(1)} = A/Ae$. It follows that there are canonical isomorphisms of k-algebras $k_\phi\{A^{(e)}\} \stackrel{\sim}{\to} k_\phi\{A\}/k_\phi\{A\}(1-e)$ and $k_\phi\{A^{(1)}\} \stackrel{\sim}{\to} k_\phi\{A\}/k_\phi\{A\}e$. The required fact follows. ∎

For a K-affinoid space $X = \mathcal{M}(A)$, let $X^{(\phi)}$ denote the k-affinoid space $\mathcal{M}(k_\phi\{A\})$. There is a canonical continuous map $\pi = \pi_X : X^{(\phi)} \to X$ that takes a bounded multiplicative seminorm $k_\phi\{A\} \to \mathbf{R}_+$ to its composition with the canonical bounded homomorphism $A \to k_\phi\{A\}$.

11.1.7. Lemma. *(i) The correspondence $X \mapsto X^{(\phi)}$ gives rise to a functor*

$$K\text{-}\mathcal{A}ff^p \to k\text{-}\mathcal{A}ff \ ;$$

(ii) this functor takes p-affinoid domain embeddings, p-closed immersions and p-finite morphisms to p-morphisms of the same type;

(iii) given a p-morphism $Y \to X$, one has $\pi_Y^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Y^{(\phi)}/X^{(\phi)})$.

Proof. (i) Let $\varphi : Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be a piecewise affinoid morphism of k-affinoid spaces represented by a compatible system of morphisms $V_i = \mathcal{M}(B_i) \to X$, i.e., by a compatible system of bounded homomorphisms of K-affinoid algebras $\alpha_i : A \to B_i$, where $\{V_i\}_{i\in I}$ is a finite covering of Y by affinoid domains. Lemmas 11.1.5(i) and 11.1.6 imply that the preimage W_i of each V_i in $\mathcal{Y}^{(\phi)} = \mathcal{M}(\mathcal{B})$, where $\mathcal{B} = k_\phi\{B\}$, is an affinoid domain, and one has $W_i = \mathcal{M}(\mathcal{B}_i)$, where $\mathcal{B}_i = k_\phi\{B_i\}$. Tate's Acyclicity Theorem ([Ber1, 2.2.5]) implies that, for the finite affinoid covering $\{W_i\}_{i\in I}$ of $Y^{(\phi)}$, there is an exact sequence of admissible homomorphisms of k-affinoid algebras $\mathcal{B} \to \prod_{i\in I} \mathcal{B}_i \rightrightarrows \prod_{i,j\in I} \mathcal{B}_{ij}$, where $\mathcal{B}_{ij} = k_\phi\{B_{ij}\}$ and $B_{ij} = B_{V_i\cap V_j}$. The compatible system of bounded homomorphisms $\{\alpha_i\}_{i\in I}$ induces a compatible system of bounded homomorphisms of k-affinoid algebras $k_\phi\{A\} \to \mathcal{B}_i$, and the above exact sequence implies that the system $\{\alpha_i\}_{i\in I}$ is induced by a bounded homomorphism $k_\phi\{A\} \to \mathcal{B}$, which gives rise to the required morphism of k-affinoid spaces $Y^{(\phi)} \to X^{(\phi)}$.

The statement (ii) follows from Lemma 11.1.5 and the properties of k-affinoid algebras.

(iii) Let $y' \in \pi_Y^{-1}(\mathrm{Int}(Y/X))$, and let y, x' and x be its images in Y, $X^{(\phi)}$ and X, respectively. First of all, replacing X and Y by acyclic affinoid neighborhoods of the points x and y, we may assume that $Y \to X$ is a morphism of K-affinoid spaces. By Proposition 6.4.3(i), we can replace Y by an affinoid neighborhood of y such that there is an admissible epimorphism $C = A\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to B : T_i \mapsto g_i$ with $|g(y)| < r_i$ for all $1 \le i \le n$. It gives rise to an admissible epimorphism $k_\phi\{C\} = k_\phi\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to k_\phi\{B\} : T_i \mapsto g_i$ with $|g(y')| < r_i$ for all $1 \le i \le n$ and, therefore, $y' \in \mathrm{Int}(Y^{(\phi)}/X^{(\phi)})$. ∎

Proof of Theorem 11.1.2. (i) First of all, if $X = \mathcal{M}(A)$ is K-affinoid then, by Lemma 11.1.3, for any k-analytic space $Y = \mathcal{M}(\mathcal{B})$, $\mathrm{Hom}_\phi(Y, X)$ coincides with the set of bounded ϕ-homomorphisms $A \to \mathcal{B}^{\cdot}$ and, by Lemma 11.1.4, the latter coincides with $\mathrm{Hom}(k\{A\}, \mathcal{B}) = \mathrm{Hom}(Y, X^{(\phi)})$. This means that the k-affinoid space $X^{(\phi)}$ and the morphism $\pi : X^{(\phi)} \to X$ represent the functor Φ_X. This easily implies that, for any analytic domain $U \subset X$, its preimage $\pi^{-1}(U)$ is an analytic subdomain of $X^{(\phi)}$ which represents the functor Φ_U, i.e., $U^{(\phi)} = \pi^{-1}(U)$.

Furthermore, suppose that X is a paracompact K-analytic space. Then we can find a locally finite covering of X by affinoid subdomains $\{U_i\}_{i\in I}$. This allows us to view X as the k-analytic space obtained by gluing of all of the affinoid domains U_i along their joint intersections $U_i \cap U_j$. Let $X^{(\phi)}$ be the Hausdorff k-analytic space obtained by gluing the k-affinoid spaces $U_i^{(\phi)}$ along the analytic subdomains $(U_i \cap U_j)^{(\phi)}$ (see [Ber2, 1.3.3]). The morphisms $U_i^{(\phi)} \to U_i$ give rise to a morphism $\pi : X^{(\phi)} \to X$, and $X^{(\phi)}$ and π represent the functor Φ_X. as above, this implies that, for any open analytic $\mathcal{U} \subset X$, its preimage $\pi^{-1}(U)$ is an open analytic domain of $X^{(\phi)}$ which represents the functor Φ_U, i.e., $U^{(\phi)} = \pi^{-1}(U)$.

Finally, suppose that X is an arbitrary K-analytic space. We take an open covering of X by paracompact open subsets $\{\mathcal{U}_i\}_{i\in I}$. By the previous case, each functor $\Phi_{\mathcal{U}_i}$ is representable by a Hausdorff k-analytic space $\mathcal{U}_i^{(\phi)}$ and a morphism $\pi_i : \mathcal{U}_i^{(\phi)} \to \mathcal{U}_i$. If $X^{(\phi)}$ is the k-analytic space obtained by gluing of the $\mathcal{U}_i^{(\phi)}$ along the open analytic subdomains $(\mathcal{U}_i \cap \mathcal{U}_j)^{(\phi)}$ and π is the morphism $X^{(\phi)} \to X$ induced by the morphisms π_i, then the pair $(X^{(\phi)}, p)$ represents the functor Φ_X.

The statement (ii) follows from the construction, (iii) follows from Lemma 11.1.7(iii), (iv) follows from Lemma 11.1.5 and (ii).

(v) It suffices to consider the case when $\mathcal{X} = \mathrm{Spec}(A)$ is an affine scheme. Then A is a quotient of the K-algebra $K[T_1, \dots, T_n]$, and (iv) reduces the situation when $A = K[T_1, \dots, T_n]$. In this case both $(\mathcal{X}^{(\phi)})^{\mathrm{an}}$ and $(\mathcal{X}^{\mathrm{an}})^{(\phi)}$ coincide with the affine space $\mathbf{A}^n$ of dimension n over k. ∎

11.2. k-analytic spaces with a topologized prelogarithmic K-structure. The following definition is given in a slightly more general setting than that considered in this subsection in order to be used in §11.5.

11.2.1. Definition. Given an isometric homomorphism of real valuation fields $\phi : K \to L$, a K-analytic space X is said to be *ϕ-nontrivial* if, for every acyclic affinoid domain $U \subset X$, the stabilizer of each non-nilpotent element of A_U in K^* is contained in $\mathrm{Ker}(K^* \stackrel{\phi}{\to} L^*)$.

Notice that the property of X to be ϕ-nontrivial is equivalent to the following one: for every point $x \in X$, one has $\mathrm{Ker}(K^* \to \mathcal{H}(x)^*) \subset \mathrm{Ker}(K^* \stackrel{\phi}{\to} L^*)$.

Let us turn to our situation when $L = k^{\cdot}$ for a non-Archimedean field k.

11.2.2. Lemma. *The following properties of a K-analytic space X are equivalent:*

(a) X is ϕ-nontrivial;

(b) the map $\pi : X^{(\phi)} \to X$ is surjective.

Proof. We may assume that $X = \mathcal{M}(A)$ is K-affinoid.

(b)$\Longrightarrow$(a). Suppose that there exist elements $f \in A \backslash \mathrm{z}\mathbf{n}(A)$ and $\alpha \in K^* \backslash \mathrm{Ker}(K^* \stackrel{\phi}{\to} k^*)$ with $\alpha f = f$. By Theorem 6.4.4, we can find a sufficiently large $r > 0$ such that for the rational domain $V = \{x \in X \big| |f(x) \geq r\}$ the canonical homomorphism $A_f \to A_V$ is a bijection. Since $\alpha \in \mathrm{Ker}(K^* \to A_f^*)$, it follows that $\alpha \in \mathrm{Ker}(K^* \to A_V^*)$ and, therefore, $k_\phi\{A_V\} = 0$, which contradicts surjectivity of the map $\mathcal{M}(k_\phi\{A\}) \to \mathcal{M}(A)$.

(a)$\Longrightarrow$(b). It suffices to show that, for any point $x \in X = \mathcal{M}(A)$, the k-affinoid algebra $k_\phi\{\mathcal{H}(x)\}$ is nontrivial. Replacing K by its quotient by $\mathrm{Ker}(K^* \to \mathcal{H}(x)^*)$, we may assume that $\mathrm{Ker}(K^* \to \mathcal{H}(x)^*) = 1$. The group $\mathcal{H}(x)^*/K^*$ is isomorphic to a direct sum $\mathbf{Z}^m \oplus (\oplus_{i=m+1}^n \mathbf{Z}/d_i\mathbf{Z})$. Take representatives $f_1, \dots, f_n$ of the canonical generators of the direct summands, and set $r_i = |f_i|$ for $1 \leq i \leq n$ and $\alpha_i = f_i^{d_i} \in K^*$ for $m+1 \leq i \leq n$. Then the k-affinoid algebra $k_\phi\{\mathcal{H}(x)\}$ is isomorphic to the quotient of $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, r_1T_1^{-1}, \dots, r_nT_n^{-1}\}$ by the ideal generated by the polynomials $T_i^{d_i} - \phi(\alpha_i)$ for $m+1 \leq i \leq n$ and, therefore, it is nontrivial. ∎

The category of ϕ-nontrivial K-analytic spaces is denoted by $K\text{-}\mathcal{A}n^{[\phi]}$.

11.2.3. Definition. (i) A k-analytic space with a *topologized prelogarithmic K-structure* is a quadruple $(Y, \sigma, \mathcal{A}, \alpha)$ consisting of a k-analytic space Y, a Grothendieck topology σ on Y with a base formed by affinoid domains, a σ-sheaf of Banach K-algebras $\mathcal{A}$, and a bounded ϕ-homomorphism of σ-sheaves $\mathcal{A} \to \mathcal{O}_{Y_\mathrm{G}}\big|_\sigma$.

(ii) A morphism $(Y, \sigma, \mathcal{A}, \alpha) \to (Y', \sigma', \mathcal{A}', \alpha')$ is a pair consisting of a morphism of k-analytic spaces $\varphi : Y \to Y'$, which induces a morphism of sites $Y_\sigma \to Y'_{\sigma'}$, and a bounded homomorphism of σ'-sheaves of Banach K-algebras $\mathcal{A}' \to \varphi_*\mathcal{A}$, which is compatible with the homomorphism $\mathcal{O}_{Y'_\mathrm{G}} \to \varphi_*\mathcal{O}_{Y_\mathrm{G}}$.

(iii) The category of k-analytic spaces with a topologized prelogarithmic K-structure is denoted by $k\text{-}\mathcal{A}n^{[\phi]}$.

11.2.4. Theorem. *The correspondence $X \mapsto X^{(\phi)}$ gives rise to a fully faithful functor*

$$K\text{-}\mathcal{A}n^{[\phi]} \to k\text{-}\mathcal{A}n^{[\phi]} \ .$$

Proof. Let X be a ϕ-nontrivial K-analytic space. Then the k-analytic space $X^{(\phi)}$ is provided with the Grothendieck topology σ generated by the following pretopology: it consists of the preimages $\pi^{-1}(U)$ of analytic domains $U \subset X$, and the set of coverings of $\pi^{-1}(U)$ consists of the families $\{\pi^{-1}(U_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is a covering of U in $X_{\rm G}$. If π_σ denotes the morphism of sites $X^{(\phi)}_\sigma \to X_{\rm G}$, the homomorphism $\mathcal{O}_{X_{\rm G}} \to \pi_{\sigma*}\mathcal{O}_{\mathcal{X}^{(\phi)}_{\rm G}}$ induces a homomorphism $\alpha : \pi^*_\sigma\mathcal{O}_{X_{\rm G}} \to \mathcal{O}_{\mathcal{X}^{(\phi)}_{\rm G}}\big|_\sigma$. The tuple $(X^{(\phi)}, \sigma, \pi^*_\sigma\mathcal{O}_{X_{\rm G}}, \alpha)$ is an object of the category k-$\mathcal{A}n^{[\phi]}$. That the correspondence $X \mapsto (X^{(\phi)}, \sigma, \pi^*_\sigma\mathcal{O}_{X_{\rm G}}, \alpha)$ is a faithful functor is easy. To show that it is fully faithful, we notice that there is a canonical isomorphism $\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} \pi_{\sigma*}\pi^*_\sigma\mathcal{O}_{\mathcal{X}}$, which follows from the definition of the Grothendieck topology σ and the fact that the map $\pi : X^{(\phi)} \to X$ is surjective.

Let $\varphi : (X^{(\phi)}, \sigma, \pi^*_\sigma\mathcal{O}_{X_{\rm G}}, \alpha) \to (X'^{(\phi)}, \sigma', \pi'^*_{\sigma'}\mathcal{O}_{X'_{\rm G}}, \alpha')$ be a morphism in k-$\mathcal{A}n^{[\phi]}$. A construction of a morphism of K-analytic spaces $\psi : X \to X'$ which induces φ is done in several steps.

Step 1. *There exists a unique continuous map $\psi : X \to X'$ which is compatible with the map $\varphi : X^{(\phi)} \to X'^{(\phi)}$.* Indeed, since both maps $\pi : X^{(\phi)} \to X$ and $\pi' : X'^{(\phi)} \to X'$ are compact, they are factor maps (see [En, §2.4]), and so it suffices to verify that the map φ takes fibers of π to fibers of π'. Let x be a point of X, and let y be a point of $X^{(\phi)}$ with $x = \pi(y)$. Let also $y' = \varphi(y)$ and $x' = \pi'(y')$. We have to show that $\varphi(\pi^{-1}(x)) \subset \pi'^{-1}(x')$. For this it suffices to verify that $\varphi(\pi^{-1}(x)) \subset \pi'^{-1}(U')$ for every affinoid domain $U' \subset X'$ that contains the point x'. By the definition of the morphism φ, the preimage of the affinoid domain $U'^{(\phi)} = \pi'^{-1}(U')$ coincides with $U^{(\phi)} = \pi^{-1}(U)$ for an analytic domain $U \subset X$. Since $x \in U$, it follows that $\pi^{-1}(x) \subset \varphi^{-1}(U'^{(\phi)})$, and the required fact follows.

Step 2. *The map $\psi : X \to X'$ possesses the property (1) from Corollary 8.2.4(i).* Indeed, in notation of Step 1, one has $\psi^{-1}(U') = U$ and, therefore, there exist acyclic affinoid domains $U_1, \dots, U_n \subset U$ such that $x \in U_1 \cap \ldots \cap U_n$ and $U_1 \cup \ldots \cup U_n$ is a neighborhood of x in U. This implies the required fact.

Step 3. Let $U \subset X$ and $U' \subset X'$ be acyclic affinoid domains with $\psi(U) \subset U'$. The morphism φ defines a homomorphism of σ'-sheaves of K-algebras $\beta : \pi'^*_{\sigma'}\mathcal{O}_{X'_{\rm G}} \to \varphi_*\pi^*_\sigma\mathcal{O}_{X_{\rm G}}$ which, in its turn, defines a bounded homomorphism $\gamma : A_{U'} = (\pi'^*_{\sigma'}\mathcal{O}_{X'_{\rm G}})(U'^{(\phi)}) \to A_U = (\pi^*_\sigma\mathcal{O}_{X_{\rm G}})(U^{(\phi)})$. Since the homomorphism γ is compatible with the bounded homomorphism $k_\phi\{A_{U'}\} \to k_\phi\{A\}$, which induces the restriction of the map φ to $U^{(\phi)}$, it follows that the map $U = \mathcal{M}(A_{U'}) \to U' = \mathcal{M}(A_U)$, induced by γ, coincides with the restriction of ψ to U. It follows also that the system of homomorphisms γ is compatible and, by Corollary 8.2.4(i), we get a morphism of K-analytic spaces $\psi : X \to X'$ which induces the morphism φ we started from. ∎

11.3. ϕ-special K-analytic spaces.

11.3.1. Definition. Given an isometric homomorphism of real valuation fields $\phi : K \to L$, a K-analytic space X is said to be *ϕ-special* if, for every acyclic affinoid domain $U \subset X$, the following is true:

(1) A_U is ϕ-nontrivial;

(2) for every Zariski prime ideal $\mathfrak{p} \subset A_U$, the group $\mathrm{Coker}(K^* \to \kappa(\mathfrak{p})^*)$ has no torsion;

(3) A_U is special in the sense of Definition I.1.2.9.

Again we turn to our situation when $L = k^{\cdot}$ for a non-Archimedean field k.

11.3.2. Theorem. *Let X be a ϕ-special K-analytic space. Then*

(i) for every point $x \in X$ the k-affinoid algebra $k_\phi\{\mathcal{H}(x)\}$ is integral, and its spectral norm is multiplicative;

(ii) the map $\sigma : X \to X^{(\phi)}$ that takes a point $x \in X$ to the point from $\pi^{-1}(x) = \mathcal{M}(k_\phi\{\mathcal{H}(x)\})$, which corresponds to the norm on $k_\phi\{\mathcal{H}(x)\}$, is continuous;

(iii) the map σ is continuous with respect to the G-topologies on both spaces and, in particular, it induces a morphism of sites $X_{\mathrm{G}} \to X^{(\phi)}_{\mathrm{G}}$;

(iv) there is a strong deformation retraction $\Phi : X^{(\phi)} \times [0,1] \to X^{(\phi)}$ of $X^{(\phi)}$ to $\sigma(X)$.

Proof. (i) It suffices to show that, for any ϕ-nontrivial valuation K-field L with finitely generated and torsion free quotient group L^*/K^*, the canonical Banach norm on the k-affinoid algebra $k_\phi\{L\}$ is multiplicative. If $f_1, \ldots, f_n$ are elements of L^* whose images generate the quotient group L^*/K^* and $r_i = |f_i|$, then $k_\phi\{L\} \stackrel{\sim}{\to} k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, r_1T_1^{-1}, \ldots, r_nT_n^{-1}\}$, and the required fact follows.

(ii) It suffices to consider the case when $X = \mathcal{M}(A)$ is K-affinoid, and we have to verify that, for every function $F \in k_\phi\{A\}$, the map $X \to \mathbf{R}_+ : x \mapsto |F(\sigma(x))|$ is continuous. For this we can replace A by $A/\mathbf{n}(A)$ and assume that X is reduced. Since X is ϕ-special, the real valuation $\mathbf{F}_1$-field $\mathcal{H}(x)$ of a point $x \in X$ does not change if we replace X by an irreducible component that contains x (see Corollary I. 1.2.11). It suffices therefore to verify continuity of the restriction of the above function to every irreducible component of X, i.e., we may assume that X is integral. In this case, for every point $x \in X$, the kernel of the character $A \to \mathcal{H}(x) : f \mapsto f(x)$ coincides with its Zariski kernel $\mathfrak{p}_x$ and, therefore, if $f(x) = \lambda g(x) \neq 0$ for some $\lambda \in K^*$ and $f, g \in A$, then $f = \lambda g$. It follows that, if $\{f_i\}_{i \in I}$ is a system of representatives of nonzero elements of A/K^*, then for every element $F = \sum_{i \in I} \lambda_i f_i \in k_\phi\{A\}$ and every point $x \in X$, one has $|F(\sigma(x))| = \max_{i \in I} |\lambda_i| \cdot |f_i(x)|$. We have to verify that, for every $r > 0$, the sets $\mathcal{U}_{>r} = \{x \in X \big| |F(\sigma(x))| > r\}$ and $\mathcal{U}_{<r} = \{x \in$

$X\big||F(\sigma(x))| < r\}$ are open. By the above remark, we have $\mathcal{U}_{>r} = \bigcup_{i\in I}\{x \in X\big||\lambda_i| \cdot |f_i(x)| > r\}$ and, therefore, this set is open. If now J is the finite subset of $i \in I$ with $|\lambda_i| \cdot ||f_i|| \geq r$, the above remark implies that $\mathcal{U}_{<r} = \bigcap_{i\in J}\{x \in X\big||\lambda_i| \cdot |f_i(x)| < r\}$ and, therefore, this set is also open.

(iii) It suffices to consider the case when $X = \mathcal{M}(A)$ is reduced K-affinoid and, by Corollary 9.3.5, we may even assume that X is integral. Let V be a rational subdomain of $X^{(\phi)}$ with $V \cap \sigma(X) \neq \emptyset$, i.e., $V = \{y \in X^{(\phi)}\big||G(y)| \geq q$ and $|F_k(y)| \leq p_i|G(y)|$ for all $1 \leq k \leq n\}$, where $F_1, \dots, F_n, G \in k_\phi\{A\}$ and $p_1, \dots, p_n, q > 0$. In notations from the proof of (ii), let $F_k = \sum_{i\in I} \lambda_{ki} f_i$ and $G = \sum_{i\in I} \mu_i f_i$. Furthermore, let J be the (nonempty) finite set consisting of all $i \in I$ with $|\mu_i| \cdot ||g_i|| \geq q$ and, for $i \in J$, define a rational subdomain of X

$$U_i = \{x \in X\big||f_i(x)| \geq \frac{q}{|\mu_i|} \text{ and } |f_j(x)| \leq \left|\frac{\mu_i}{\mu_j}\right| \cdot |f_i(x)| \text{ for all } j \in J\}\ .$$

Notice that the restriction of f_i to U_i is invertible. Finally, for $1 \leq k \leq n$, let J_k be the finite set of all $i \in I$ with $|\lambda_{ki}| \cdot ||f_i|| \geq p_k q$ and, for $i \in J$, consider the following Weierstrass subdomain of U_i

$$U_{ki} = \{x \in U_i\big| \left|\frac{f_j}{f_i}(x)\right| \leq p_k \left|\frac{\mu_i}{\lambda_{kj}}\right| \text{ for all } j \in J_k\}\ .$$

Then $\sigma^{-1}(V) = \bigcup_{1\leq k\leq n, i\in J} U_{ki}$.

(iv) The proof is done in several steps.

Step 1. For a finitely generated abelian group Γ, we set $\mathcal{D}_\Gamma = \mathrm{Fspec}(K[\Gamma])$. The homomorphism $K[\Gamma] \to K[\Gamma] \otimes_K K[\Gamma]$ defines the structure of a group object on $\mathcal{D}_\Gamma$ in the category of schemes over K and, therefore, $\mathcal{D}_\Gamma^{\mathrm{an}}$ is a group object in K-$\mathcal{A}n$. The latter has a K-affinoid subgroup G_Γ defined by $G_\Gamma = \{x \in \mathcal{D}_\Gamma^{\mathrm{an}}\big||\gamma(x)| = 1$ for all $\gamma \in \Gamma\}$ and whose underlying topological space consists of one point. One has $G_\Gamma = \mathcal{M}(K\{\Gamma\})$, where $K\{\Gamma\}$ is the K-affinoid algebra provided with the norm $||\gamma|| = 1$ for all $\gamma \in \Gamma$. It follows that $G_\Gamma^{(\phi)} = \mathcal{M}(k\{\Gamma\})$ is a k-affinoid group.

Step 2. In the situation of Step 1, assume that the group Γ has no torsion. Then $G_\Gamma^{(\phi)}$ is isomorphic (as a k-analytic space) to a closed poly-annulus with center at zero of radius one. For $0 \leq t \leq 1$, we set $G_{\Gamma,t}^{(\phi)} = \{y \in G_\Gamma^{(\phi)}\big||(\gamma-1)(y)| \leq t$ for all $\gamma \in \Gamma\}$. One has $G_{\Gamma,0}^{(\phi)} = \{1\}$, $G_{\Gamma,1}^{(\phi)} = G_\Gamma^{(\phi)}$ and, for $0 < t < 1$, $G_{\Gamma,t}^{(\phi)}$ is a K-affinoid subgroup isomorphic (as a k-analytic space) to a closed polydisc with center at zero of radius one. In all cases, the k-affinoid space $G_{\Gamma,t}^{(\phi)}$ has a maximal point denoted by g_t. The points g_t are peaked and, therefore, if there is an action of $G_\Gamma^{(\phi)}$ on a k-analytic space Z, each point g_t defines a continuous map $Z \to Z : z \mapsto g_t * z$ (see [Ber1, §5.2]). For example, $g_t * g_{t'} = g_{\max(t,t')}$ and $g_1 * y = g_1$ for all $t, t' \in [0,1]$ and $y \in G_\Gamma^{(\phi)}$. Recall also that the map $[0,1] \to G_\Gamma^{(\phi)} : t \mapsto g_t$ is continuous and, by [Ber1, 6.1.1], the map $G_\Gamma^{(\phi)} \times Z \to Z : (z,t) \mapsto g_t * z$ is continuous. Notice that any injective homomorphism $\Gamma \hookrightarrow \Gamma'$ to a similar group Γ' gives rise to a surjective homomorphism of k-affinoid groups $G_{\Gamma'}^{(\phi)} \to G_\Gamma^{(\phi)}$ that takes the point $g_{\Gamma',t}$ to the point $g_{\Gamma,t}$ for all $0 \le t \le 1$.

We now turn to the theorem.

Step 3. Suppose first that $X = \mathcal{M}(A)$ is an integral K-affinoid space. Let F be the fraction $\mathbf{F}_1$-field of A. We set $\mathcal{G} = \mathcal{G}_F = G^{(\phi)}_{F^*/K^*}$ and, for $0 \le t \le 1$, denote by $\mathcal{G}_t = \mathcal{G}_{F,t}$ (resp. $g_t = g_{F,t}$) the corresponding subgroup (resp. point) of $\mathcal{G}$ introduced in Step 2. Then the homomorphism $A \to A \otimes_K K\{F^*/K^*\}$ that takes a nonzero element $f \in A$ to $f \otimes \overline{f}$, where $\overline{f}$ is the image f in F^*/K^*, is isometric, and it defines an action of G_{F^*/K^*} on X, and the latter defines an action of $\mathcal{G}$ on $X^{(\phi)}$. We can therefore construct a continuous homotopy $\Phi = \Phi_X : X^{(\phi)} \times [0,1] \to X^{(\phi)}$ that takes a pair (y,t) to the point $y_t = g_t * y$ (see [Ber1, §6.1]). Since $g_t * g_{t'} = g_{\max(t,t')}$, Φ is a strong deformation retraction of $X^{(\phi)}$ to the subset $g_1 * X^{(\phi)}$. We are going to show (among other things) that the latter set coincides with $\sigma(X^{(\phi)})$.

Step 4. Let $\mathfrak{p}$ be a Zariski prime ideal of A. Then the canonical injective homomorphism $A^{(\mathfrak{p})} = A/\mathfrak{p} \to A$ gives rise to an injective homomorphism $\kappa(\mathfrak{p}) \to F$, and the homomorphisms $A \to A \otimes_K K\{F/K^*\}$ and $A^{(\mathfrak{p})} \to A^{(\mathfrak{p})} \otimes_K K\{\kappa(\mathfrak{p})/K^*\}$ are compatible. It follows that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{G}_F \times X^{(\phi)} & \longrightarrow & X^{(\phi)} \\ \downarrow & & \downarrow \tau_{\mathfrak{p}} \\ \mathcal{G}_{\kappa(\mathfrak{p})} \times (X^{(\mathfrak{p})})^{(\phi)} & \longrightarrow & (X^{(\mathfrak{p})})^{(\phi)} \end{array}$$

The remark at the end of Step 2 implies that, for every point $y \in X^{(\phi)}$, one has $\tau_{\mathfrak{p}}(y_t) = (\tau_{\mathfrak{p}}(y))_t$. Thus, to show that $y_1 = \sigma(y)$, it suffices to consider the case when $\mathfrak{p}_x = 0$, where $x = \pi(y)$. In this case the canonical homomorphism $A \to \mathcal{H}(x)$ induces an isomorphism of K-fields $F \xrightarrow{\sim} \mathcal{H}(x)$ and a morphism of K-affinoid spaces $\varphi : Z = \mathcal{M}(\mathcal{H}(x)) \to X$ which give rise to the commutative diagram

$$\begin{array}{ccc} \mathcal{G}_F \times X^{(\phi)} & \longrightarrow & X^{(\phi)} \\ \uparrow & & \uparrow \\ \mathcal{G}_{\mathcal{H}(x)} \times Z^{(\phi)} & \longrightarrow & Z^{(\phi)} \end{array}$$

It follows that, for every point $z \in Z^{(\phi)}$, one has $\varphi(z_t) = (\varphi(z))_t$. This reduces the situation to the case $A = \mathcal{H}(x)$. In this case $X^{(\phi)}$ is isomorphic to a closed poly-annulus and, by [Ber1, 6.1.3(ii), one gets the equality $y_1 = \sigma(y)$ for all points $y \in X^{(\phi)}$.

Step 5. In the general case, it suffices to verify that the homotopy maps Φ constructed on the irreducible components of X are compatible on intersections. But this follows from Step 5 and the fact that the real valuation field $\mathcal{H}(x)$ of a point $x \in X$ is not changed if we replace X by an irreducible component that contains x. ∎

The full subcategory of K-$\mathcal{A}n$ consisting of ϕ-special K-analytic spaces is denoted by K-$\mathcal{A}n^{(\phi)}$.

11.4. k-analytic spaces with a prelogarithmic K-structure.

11.4.1. Definition. (i) A k-analytic space with a *prelogarithmic K-structure* is a triple $(Y, \mathcal{A}, \alpha)$ consisting of a k-analytic space Y, a sheaf of Banach K-algebras $\mathcal{A}$ on Y_{G}, and a bounded ϕ-homomorphism of sheaves of Banach K-algebras $\mathcal{A} \to \mathcal{O}_{Y_{\mathrm{G}}}$.

(ii) A morphism $(Y, \mathcal{A}, \alpha) \to (Y', \mathcal{A}', \alpha')$ is a pair consisting of a morphism of k-analytic spaces $\varphi : Y \to Y'$ and a bounded homomorphism of sheaves of Banach K-algebras $\mathcal{A}' \to \varphi_* \mathcal{A}$, which is compatible with the homomorphism $\mathcal{O}_{Y'_{\mathrm{G}}} \to \varphi_* \mathcal{O}_{Y_{\mathrm{G}}}$.

(iii) The category of k-analytic spaces with a prelogarithmic K-structure is denoted by k-$\mathcal{A}n^{(\phi)}$.

11.4.2. Theorem. *The correspondence $X \mapsto X^{(\phi)}$ gives rise to a fully faithful functor*

$$K\text{-}\mathcal{A}n^{(\phi)} \mapsto k\text{-}\mathcal{A}n^{(\phi)} .$$

Proof. The functor considered takes a ϕ-special K-analytic space to $(X^{(\phi)}, \pi^*_{\mathrm{G}} \mathcal{O}_{X_{\mathrm{G}}}, \alpha)$, where π_G is the morphism of sites $X^{(\phi)}_{\mathrm{G}} \to X_{\mathrm{G}}$ and α is the induced homomorphism $\pi^*_{\mathrm{G}} \mathcal{O}_{X_{\mathrm{G}}} \to \mathcal{O}_{X^{(\phi)}_{\mathrm{G}}}$.

11.4.3. Lemma. *There is a canonical isomorphism $\mathcal{O}_{X_{\mathrm{G}}} \stackrel{\sim}{\to} \pi_{\mathrm{G}*} \pi^*_{\mathrm{G}} \mathcal{O}_{X_{\mathrm{G}}}$.*

Proof. We may assume that $X = \mathcal{M}(A)$ is acyclic K-affinoid, and consider the commutative diagram of morphisms of sites

$$\begin{array}{ccc} X^{(\phi)} & \stackrel{\pi}{\longrightarrow} & X \\ \uparrow \tau^{(\phi)} & & \uparrow \tau \\ X^{(\phi)}_{\mathrm{G}} & \stackrel{\pi_{\mathrm{G}}}{\longrightarrow} & X_{\mathrm{G}} \end{array}$$

Let F be the restriction of the sheaf $\mathcal{O}_{X_{\mathrm{G}}}$ to the usual topology of X, i.e., $F = \tau_* \mathcal{O}_{X_{\mathrm{G}}}$. By Theorem 7.2.1, there is a canonical isomorphism $\tau^* F \stackrel{\sim}{\to} \mathcal{O}_{X_{\mathrm{G}}}$. Since the map $\pi : X^{(\phi)} \to X$ is compact and, by Proposition 11.3.2, its fibers are connected, there is a canonical isomorphism $F \stackrel{\sim}{\to} \pi_* \pi^* F$. Since $\pi^* F \stackrel{\sim}{\to} \tau^{(\phi)}_* \tau^{(\phi)*} \pi^* F$ (see [Ber2, §1.3]), we have $F \stackrel{\sim}{\to} \pi_* \tau^{(\phi)}_* \tau^{(\phi)*} \pi^* F \stackrel{\sim}{\to} \tau_* \pi_{\mathrm{G}*} \pi^*_{\mathrm{G}} \tau^* F$. It follows that $\mathcal{O}_{X_{\mathrm{G}}} = \tau^* F \stackrel{\sim}{\to} \pi_{\mathrm{G}*} \pi^*_{\mathrm{G}} \tau^* F = \pi_{\mathrm{G}*} \pi^*_{\mathrm{G}} \mathcal{O}_{X_{\mathrm{G}}}$. ∎

Let $\varphi : (X^{(\phi)}, \pi^*_X \mathcal{O}_{X_{\mathrm{G}}}, \alpha) \to (X'^{(\phi)}, \pi'^*_{X'} \mathcal{O}_{X'_{\mathrm{G}}}, \alpha')$ be a morphism in K-$\mathcal{A}n^{(\phi)}$. As in the proof of Theorem 11.2.4, we construct a morphism of K-analytic spaces $\psi : X \to X'$, which induces φ, in several steps.

Step 1. *There exists a unique continuous map $\psi : X \to X'$ which is compatible with the map $\varphi : X^{(\phi)} \to X'^{(\phi)}$.* Indeed, since both maps $\pi : X^{(\phi)} \to X$ and $\pi' : X'^{(\phi)} \to X'$ are compact, they are factor maps (see [En, §2.4]), and so it suffices to verify that the map φ takes fibers of π to fibers of π'. Let x be a point of X, and let y be a point of $X^{(\phi)}$ with $x = \pi(y)$. Let also $y' = \varphi(y)$ and $x' = \pi'(y')$. We have to show that $\varphi(\pi^{-1}(x)) \subset \pi'^{-1}(x')$. For this it suffices to verify that $\varphi(\pi^{-1}(x)) \subset \pi'^{-1}(U')$ for every affinoid domain $U' \subset X'$ that contains the point x'.

Let $U' = \mathcal{M}(A')$. Since the fiber $\pi^{-1}(x)$ is connected, it suffices to verify that, for every point $z \in \pi^{-1}(x)$ from a neighborhood of the point y and every $f' \in A'$, one has $|F(y)| = |F(z)|$, where F is the image of f' under the composition homomorphism $A' \to \mathcal{O}(U'^{(\phi)}) \to \mathcal{O}(U^{(\phi)})$. Since there exist affinoid domains $V_1, \ldots, V_n \subset \varphi^{-1}(U'^{(\phi)})$ that contain the point y and such that $V_1 \cup \ldots \cup V_n$ is a neighborhood of y in $\varphi^{-1}(U'^{(\phi)})$, it suffices to verify that, give an affinoid domain $y \in V \subset \varphi^{-1}(U'^{(\phi)})$, on has $|F(y)| = |F(z)|$ for all points $z \in V \cap \pi^{-1}(x)$. For this we recall that $\mathcal{O}_{X,x} = \varinjlim \mathcal{O}(W)$ and set $\mathcal{O}_{V,x} = \varinjlim \mathcal{O}(V \cap \pi^{-1}(W))$, where both inductive limits are taken over all affinoid domains W of X that contain the point x. The morphism in K-$\mathcal{A}n^{(\phi)}$ we started from defines a homomorphism $\mathcal{O}_{X',x'} \to \mathcal{O}_{X,x}$ for which the following diagram is commutative

$$\begin{array}{ccccc} \mathcal{O}_{X',x'} & \longrightarrow & \mathcal{O}_{X,x} & & \\ & & & \searrow & \\ \uparrow & & & & \mathcal{O}_{V,x} \\ & & & \nearrow & \\ A' & \stackrel{f' \mapsto F}{\longrightarrow} & \mathcal{O}(U^{(\phi)}) & & \end{array}$$

This immediately implies the required equality.

Step 2. *The map $\psi : X \to X'$ possesses the property (1) from Corollary 8.2.4(i).* First of all, if U' is an analytic domain in X', then $\psi^{-1}(U')$ is an analytic domain in X. Indeed, by Step 1, the latter set coincides with $\sigma^{-1}(\varphi^{-1}(U'^{(\phi)}))$, and Proposition 11.3.2(iii) implies that it is an analytic domain in X. Let now $x \in X$. We can find acyclic affinoid domains $U'_1, \ldots, U'_n$ in X' that contain the point $x' = \psi(x)$ and such that $U'_1 \cup \ldots, \cup U'_n$ is a neighborhood of x' in X'. By the above claim, each preimage $U_i = \psi^{-1}(U'_i)$ is an analytic domain in X, and $U_1 \cup \ldots \cup U_n$ is a neighborhood of x in X. For every $1 \leq i \leq n$, we can find acyclic affinoid domains $U_{i1}, \ldots, U_{im_i} \subset X$ that contain the point x and such that their union is a neighborhood of x in U_i. Then $\bigcup_{1 \leq i \leq n, 1 \leq j \leq m_i} U_{ij}$ is a neighborhood of x in X, and the required fact follows.

Step 3. Let $U \subset X$ and $U' \subset X'$ be acyclic affinoid domains with $\psi(U) \subset U'$. Then $\varphi(U^{(\phi)}) \subset U'^{(\phi)}$ and, therefore, φ defines a bounded homomorphism $\gamma : A_{U'} = (\pi^*_{X'} \mathcal{O}_{X'_{\mathrm{G}}})(U'^{(\phi)}) \to A_U = (\pi^*_X \mathcal{O}_{X_{\mathrm{G}}})(U^{(\phi)})$. The same reasoning as in Step 3 from the proof of Theorem 11.2.4 shows that the system of homomorphisms γ gives rise to a morphism of K-analytic spaces $\psi : X \to X'$ which induces the morphism φ we started from. ∎