Number Theory/*Théorie des nombres*

The automorphism group of the Drinfeld half-plane

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Abstract – Let k be a local non-Archimedean field. We prove that the group of analytic automorphisms of the Drinfeld half-plane Ω^d of dimension $d-1$ over k coincides with $\mathrm{PGL}_d(k)$. This is applied to prove the Rigidity Conjecture of Mustafin which states that, if Γ_1 and Γ_2 are torsion free discrete subgroups of $\mathrm{PGL}_d(k)$, then the quotient spaces $\Gamma_1\backslash\Omega^d$ and $\Gamma_2\backslash\Omega^d$ are isomorphic if and only if Γ_1 and Γ_2 are conjugate.

Le groupe des automorphismes du demi-plan de Drinfeld

Résumé – *Soit k un corps local non archimédien. Nous prouvons que le groupe des automorphismes analytiques du demi-plan de Drinfeld de dimension $d-1$ sur k coïncide avec $\mathrm{PGL}_d(k)$. Cela est appliqué pour prouver la conjecture de rigidité de Mustafin qui affirme que, si Γ_1 and Γ_2 sont des sous-groupes discrets sans torsion de $\mathrm{PGL}_d(k)$, alors les espaces quotients $\Gamma_1\backslash\Omega^d$ et $\Gamma_2\backslash\Omega^d$ sont isomorphes si et seulement si les sous-groupes Γ_1 and Γ_2 sont conjugués.*

Version française abrégée – Soient k un corps local non archimédien et $d \geq 2$. Le demi-plan de Drinfeld de dimension $d-1$ sur k est un espace k-analytique Ω^d qui est le sous-ensemble ouvert des points de l'espace projectif $\mathbf{P}^{d-1}$ non contenus dans les hyperplans definis sur k. L'espace Ω^d est un analogue non archimédien du demi-plan de Poincaré. Il est lié étroitement à l'immeuble de Bruhat-Tits $\mathcal{B}^d$ de $\mathrm{SL}_d(k)$, et il est utilisé dans l'étude de l'uniformisation par des sous-groupes discrets de $\mathrm{PGL}_d(k)$ et des représentations de $\mathrm{GL}_d(k)$ (voir [4], [5], [6]). Le résultat principal de cette note est le suivant.

Théorème 1. – *Pour tout corps non archimédien K sur k, on a*

$$\mathrm{PGL}_d(k) \xrightarrow{\sim} \mathrm{Aut}_K(\Omega^d \widehat{\otimes} K) \ .$$

La démonstration est donnée dans le cadre de géométrie analytique de [1] et [2]. Nous construisons une immersion $\mathrm{PGL}_d(k)$-équivariante $\mathcal{B}^d \to \Omega^d \widehat{\otimes} K$ qui est inverse à l'application $\tau : \Omega^d \widehat{\otimes} K \to \Omega^d \to \mathcal{B}^d$ de [4] et, donc, nous permet à identifier $\mathcal{B}^d$ avec son image dans $\Omega^d \widehat{\otimes} K$. (Une telle immersion est construite dans [1], §5, pour touts les groupes reductifs déployés.) Nous prouvons que $\mathcal{B}^d$ est l'ensemble des points maximaux de $\Omega^d \widehat{\otimes} K$ relativement à l'ordre partiel suivant : $x \leq y$ si $|f(x)| \leq |f(y)|$ pour toute $f \in \mathcal{O}(\Omega^d \widehat{\otimes} K)$. On en deduit que tout automorphisme analytique φ de $\Omega^d \widehat{\otimes} K$ induit un automorphisme simplicial de $\mathcal{B}^d$ et que $\varphi \circ \tau = \tau \circ \varphi$. Après, pour tout appartement Λ de $\mathcal{B}^d$ nous construisons une retraction $\Omega^d \widehat{\otimes} K \to \Lambda$. Nous prouvons que cette retraction induite une retraction simpliciale $\tau_\Lambda : \mathcal{B}^d \to \Lambda$ et que $\varphi \circ \tau_\Lambda = \tau_{\varphi(\Lambda)} \circ \varphi$ pour tout automorphisme simplicial φ de $\mathcal{B}^d$. Cela reduit le théorème 1 à la verification du fait que toute fonction analytique bornée sur $\Omega^d \widehat{\otimes} K$ est constante.

Pour un sous-groupe discret $\Gamma \subset \mathrm{PGL}_d(k)$, on désigne par X_Γ l'éspace quotient $\Gamma\backslash\Omega^d$ et, pour des sous-groupes $\Gamma_1, \Gamma_2 \subset \mathrm{PGL}_d(k)$, on désigne par $C(\Gamma_1, \Gamma_2)$ l'ensemble des $g \in \mathrm{PGL}_d(k)$ avec $g\Gamma_1 g^{-1} = \Gamma_2$.

Théorème 2. – *Soient Γ_1 and Γ_2 des sous-groupes discrets sans torsion de* $\mathrm{PGL}_d(k)$. *Alors, pour tout corps non archimédien K sur k, il y a une bijection canonique*

$$C(\Gamma_1, \Gamma_2)/\Gamma_1 \xrightarrow{\sim} \mathrm{Isom}_K(X_{\Gamma_1} \widehat{\otimes} K, X_{\Gamma_2} \widehat{\otimes} K) \ .$$

Si $\Gamma_1 = \Gamma_2 = \{1\}$, c'est le théorème 1. Si Γ_1 and Γ_2 sont co-compacts dans $\mathrm{PGL}_d(k)$ et $K = k$, c'est la conjecture de rigidité de Mustafin qui affirme que X_{Γ_1} et X_{Γ_2} sont isomorphes si et seulement si les sous-groupes Γ_1 and Γ_2 sont conjugués. Rappelons ([6], §4) que dans ce cas X_{Γ_1} et X_{Γ_2} sont les espaces analytiques associés à des variétés projectives sur k et que l'ensemble $C(\Gamma_1, \Gamma_2)$ est fini. Dans le cadre de géométrie analytique de [1] et [2], le théorème 2 est une conséquence immédiate du théorème 1. En fait, des espaces analytiques sont localement compacts et localement connexes par arc, l'espace $\Omega^d \widehat{\otimes} K$ est simplement connexe (et même contractible) et, donc, $\Omega^d \widehat{\otimes} K$ est un revêtement universel des espaces $X_{\Gamma_1} \widehat{\otimes} K$ et $X_{\Gamma_2} \widehat{\otimes} K$. La demonstration du théorème 2 est, simplement, un rappel de quelques faits fondamentals sur la factorisation des espaces analytiques par une action des groupes discretes, mis dans le cadre de [1] et [2]. Ces faits ont une forme particulièrement simple et naturelle pour la classe des espaces analytique bons ([2], §1), c'est-à-dire, tels que tout leur point a un voisinage affinoide. (Par exemple, tout sous-ensemble ouvert de l'espace analytique associé à une variété algébrique est bon.)

Introduction. – Let k be a local non-Archimedean field and $d \geq 2$. The Drinfeld half-plane over k of dimension $d-1$ is a k-analytic space Ω^d which is the open subset of the projective space $\mathbf{P}^{d-1}$ that consists of all points not lying in any hyperplane defined over k. The space Ω^d is a non-Archimedean analog of the Poincaré half-plane. It is closely related to the Bruhat-Tits building $\mathcal{B}^d$ of $\mathrm{SL}_d(k)$, and is used in the study of uniformization by discrete subgroups of $\mathrm{PGL}_d(K)$ and representations of $\mathrm{GL}_d(k)$ (see [4], [5], [6]). The main result of this note is the following

Theorem 1. – *For any non-Archimidean field K over k, one has*

$$\mathrm{PGL}_d(k) \widetilde{\rightarrow} \mathrm{Aut}_K(\Omega^d \widehat{\otimes} K) \ .$$

The proof is given in the framework of the analytic geometry from [1] and [2]. We construct a $\mathrm{PGL}_d(k)$-equivariant embedding $\mathcal{B}^d \to \Omega^d \widehat{\otimes} K$ which is inverse to the map $\tau : \Omega^d \widehat{\otimes} K \to \Omega^d \to \mathcal{B}^d$ from [4] and, therefore, allows us to identify $\mathcal{B}^d$ with its image in $\Omega^d \widehat{\otimes} K$. (Such an embedding is constructed in [1], §5, for an arbitrary split reductive group.) We show that $\mathcal{B}^d$ is the set of all maximal points of $\Omega^d \widehat{\otimes} K$ with respect to the following partial ordering: $x \leq y$ if $|f(x)| \leq |f(y)|$ for all $f \in \mathcal{O}(\Omega^d \widehat{\otimes} K)$. This is used to show that any analytic automorphism φ of $\Omega^d \widehat{\otimes} K$ induces a simplicial automorphism of $\mathcal{B}^d$ and that $\varphi \circ \tau = \tau \circ \varphi$. Furthermore, for each apartment Λ of $\mathcal{B}^d$ we construct a retraction map $\Omega^d \widehat{\otimes} K \to \Lambda$. We show that this retraction induces a simplicial retraction $\tau_\Lambda : \mathcal{B}^d \to \Lambda$ and that $\varphi \circ \tau_\Lambda = \tau_{\varphi(\lambda)} \circ \varphi$ for any simplicial automorphism φ of $\mathcal{B}^d$. This reduces Theorem 1 to the verification of the fact that any bounded analytic function on $\Omega^d \widehat{\otimes} K$ is constant.

For a discrete subgroup $\Gamma \subset \mathrm{PGL}_d(k)$ let X_Γ denote the quotient space $\Gamma \backslash \Omega^d$, and for subgroups $\Gamma_1, \Gamma_2 \subset \mathrm{PGL}_d(k)$ let $C(\Gamma_1, \Gamma_2)$ denote the set $\{g \in \mathrm{PGL}_d(k) \big| g\Gamma_1 g^{-1} = \Gamma_2\}$.

Theorem 2. – *Let Γ_1 and Γ_2 be torsion free discrete subgroups of $\mathrm{PGL}_d(k)$. Then for any non-Archimedean field K over k there is a canonical bijection*

$$C(\Gamma_1, \Gamma_2)/\Gamma_1 \widetilde{\rightarrow} \mathrm{Isom}_K(X_{\Gamma_1} \widehat{\otimes} K, X_{\Gamma_2} \widehat{\otimes} K) \ .$$

If $\Gamma_1 = \Gamma_2 = \{1\}$, this is Theorem 1. If Γ_1 and Γ_2 are cocompact in $\mathrm{PGL}_d(k)$ and $K = k$, this is the Rigidity Conjecture of Mustafin that states that X_{Γ_1} and X_{Γ_2} are isomorphic if and only if the subgroups Γ_1 and Γ_2 are conjugate in $\mathrm{PGL}_d(k)$. Recall ([6], §4) that in this case X_{Γ_1} and X_{Γ_2} are the analytifications of projective varieties over k and the set $C(\Gamma_1, \Gamma_2)/\Gamma_1$ is finite. In the framework of [1] and [2], Theorem 2 is an immediate consequence of Theorem 1.

In fact, analytic spaces are locally compact and locally arcwise connected, the space $\Omega^d \widehat{\otimes} K$ is simply connected (and even contractible) and, therefore, $\Omega^d \widehat{\otimes} K$ is a universal covering of the spaces $X_{\Gamma_1} \widehat{\otimes} K$ and $X_{\Gamma_2} \widehat{\otimes} K$. The proof of Theorem 2 is simply a recall of basic facts on the factorisation of analytic spaces by an action of discrete groups. These facts have an especially simple and natural form for the class of good analytic spaces ([2], §1), i.e., those ones in which every point has an affinoid neighborhood. (For example, any open subset of the analytification of an algebraic variety is good.)

Proof of Theorem 1. – For a k-analytic space X we set $X_K = X \widehat{\otimes} K$. We introduce a partial ordering on the space X_K as follows: $x \le y$ if $|f(x)| \le |f(y)|$ for all analytic functions $f \in \mathcal{O}(X_K)$. Notice that if $x \le y$ then $|f(x)| = |f(y)|$ for all $f \in \mathcal{O}(X_K)^*$.

1. Recall (see [1], §1.5) that the affine space $\mathbf{A}^d_K$ is the space of multiplicative seminorms on the ring of polynomials $R_K := K[T_0, \dots, T_{d-1}]$ that extend the valuation on K. The following simple fact describes fibres of the canonical morphism $\mathbf{A}^d_K \backslash \{0\} \to \mathbf{P}^{d-1}_K$.

The images of two points $x, y \in \mathbf{A}^d_K \backslash \{0\}$ coincide in $\mathbf{P}^{d-1}_K$ if and only if there exists $\lambda > 0$ such that for all $n \ge 0$ and all $f \in R_{n,K}$, where $R_{n,K}$ is the space of homogeneous polynomials in R_K of degree n, one has $|f(y)| = \lambda^n |f(x)|$.

2. Let $\mathcal{N}$ be the space of norms on the k-vector space $R_1 = R_{1,k}$ endowed with the weakest topology with respect to which all real valued functions on $\mathcal{N}$ of the form $N \mapsto N(f)$, where $f \in R_1$, are continuous. The group $\mathbf{R}^*_+$ acts on $\mathcal{N}$, and it is known that the quotient space $\mathcal{B}^d := \mathcal{N}/\mathbf{R}^*_+$ is the Bruhat-Tits building of the group $\mathrm{SL}_d(k)$ ([3], §10, p. 238). Let $\widetilde{\Omega}^d_K$ denote the preimage of Ω^d_K in $\mathbf{A}^d_K \backslash \{0\}$. Then the continuous $\mathrm{GL}_d(k)$-equivariant map $\widetilde{\tau} : \widetilde{\Omega}^d_K \to \mathcal{N}$ that takes a point of $\widetilde{\Omega}^d_K$, which is a multiplicative seminorm on R_K, to its restriction on R_1 (canonically embedded in R_K) induces, by Step 1, a continuous $\mathrm{GL}_d(k)$-equivariant map $\tau : \Omega^d_K \to \mathcal{B}^d$.

Let now $N \in \mathcal{N}$. Since the field k is locally compact, there is an orthogonal basis $e_0, \dots, e_{d-1}$ of R_1, i.e., such that $N(\sum_{i=0}^{d-1} a_i e_i) = \max |a_i| N(e_i)$. Let $\widetilde{j}(N)$ be the point of $\widetilde{\Omega}^d_K$ defined by $|(\sum_\nu a_\nu e^\nu)(\widetilde{j}(N))| = \max |a_\nu| N(e)^\nu$. By Step 1, $\widetilde{j}$ induces a $\mathrm{GL}_d(k)$-equivariant continuous map $j : \mathcal{B}^d \to \Omega^d_K$ with $\tau \circ j = 1_{\mathcal{B}^d}$. It follows $\mathcal{B}^d$ is homeomorphic to its image $j(\mathcal{B}^d)$, and the set $j(\mathcal{B}^d)$ is closed in Ω^d_K. In what follows we identify $\mathcal{B}^d$ with $j(\mathcal{B}^d)$ and consider $\tau : \Omega^d_K \to \mathcal{B}^d$ as a retraction map.

3. *(1) $x \le \tau(x)$ for all $x \in \Omega^d_K$;*

(2) if $x, y \in \mathcal{B}^d$ and $x \ne y$, then none of the inequalities $x \le y$ and $y \le x$ is true.

We fix an apartment Λ of $\mathcal{B}^d$ as follows. Let $T_0, \dots, T_{d-1}$ be a fixed basis of R_1, and let $\widetilde{\Lambda}$ be the set of norms on R_1 of the form $N(\sum_{i=0}^{d-1} a_i T_i) = \max |a_i| r_i$ for $(r_0, \dots, r_{d-1}) \in (\mathbf{R}^*_+)^d$. The apartment Λ is the image of $\widetilde{\Lambda}$ in $\mathcal{B}^d$.

(2) Since each pair of points of $\mathcal{B}^d$ is contained in an apartment and the group $\mathrm{SL}_d(k)$ acts transitively on the set of apartments, we may assume that $x, y \in \Lambda$. In this case we can find among the functions $t_i^{\pm 1}$, where $t_i = \frac{T_i}{T_0}$ are the coordinate functions on $\mathbf{A}^{d-1}$, such f and g that $|f(x)| < |f(y)|$ and $|g(x)| > |g(y|$.

(1) It suffices to verify that for $x \in \mathcal{B}^d$ the set $\tau^{-1}(x)$ is an affinoid domain in Ω^d_K and x is a unique maximal point of $\tau^{-1}(x)$. Since $\mathrm{SL}_d(k)$ acts transitively on the set of chambers of $\mathcal{B}^d$, we may assume that x is contained in the chamber $\Delta := \{y \in \Lambda \big| 1 \ge |t_1(y)| \ge \dots \ge |t_{d-1}(y)| \ge q^{-1}\}$, where $q = |\pi^{-1}|$ and π is a uniformizing element of k. Assume first that x is contained in the interior $\overset{\circ}{\Delta}$ of Δ, i.e., $1 > |t_1(x)| > \dots > |t_{d-1}(x)| > q^{-1}$. Then $\tau^{-1}(x)$ is the closed annulus $A(x) := \{y \in \mathbf{A}^{d-1}_K \big| |t_i(y)| = |t_i(x)|,\ 1 \le i \le d-1\}$, and x is a unique maximal point of $A(x)$. Indeed, if $i \ne j$, then the equality $|\alpha||t_i(x)| = |\beta||t_j(x)|$ for $\alpha, \beta \in k^*$ is impossible, and

therefore for $y \in A(x)$ one has $|(\sum_{i=1}^{d-1} a_i t_i)(y)| = \max |a_i||t_i(x)|$, i.e., $\tau^{-1}(x) = A(x)$. Each analytic function f on $A(x)$ has a unique representation in the form $\sum_{\nu \in \mathbf{Z}^{d-1}} a_\nu t^\nu$, where $a_\nu \in K$ and $|a_\nu||t(x)|^\nu \to 0$ for $|\nu| \to \infty$. The norm $||f|| = \max |a_\nu||t(x)|^\nu$ is multiplicative on $\mathcal{O}(A(x))$, and the point x corresponds to this norm. In the general case, we take linear polynomials $L_1, \dots, L_m$ of the form $\sum_{i=1}^n a_i t_i + \sum_{i=n+1}^{d-1} \pi^{-1} a_i t_i$, where $1 \le n \le d-1$ and, for each $1 \le i \le d-1$, a_i runs through representatives of the residue field of k in the ring of integers. Then $\tau^{-1}(x) = \{y \in A(x) \big| |L_j(y)| = |L_j(x)|, 1 \le j \le m\}$. The latter is an affinoid domain in $A(x)$, and each analytic function on $\tau^{-1}(x)$ can be approximated by a function of the form f/L, where $f \in \mathcal{O}(A(x))$ and $L = L_1^{n_1} \dots L_m^{n_m}$ (see [1], 2.2.2). It follows that x is a unique maximal point of $\tau^{-1}(x)$.

4. Let φ be a K-analytic automorphism of Ω_K^d. Our purpose is to show φ is induced by an element of $\mathrm{GL}_d(k)$. First of all, we claim that *φ induces a simplicial automorphism of the building $\mathcal{B}^d$ and $\varphi \circ \tau = \tau \circ \varphi$.* Indeed, from Step 3 it follows that φ induces a homeomorphism of $\mathcal{B}^d$ in itself and $\varphi \circ \tau = \tau \circ \varphi$. It follows that φ takes chamber interiors to chamber interiors and, therefore, chambers to chambers. To verify the claim, it suffices to show that the maps between the chamber interiors are affine. For this it suffices to show that for any $f \in \mathcal{O}(\Omega_K^d)^*$ the function $\mathcal{B}^d \to \mathbf{R} : x \mapsto \log_q |f(x)|$ is affine on each chamber interior. Since the group $\mathrm{SL}_d(k)$ acts transitively on the set of chambers, the latter fact should be verified only for the chamber Δ from Step 3. But any invertible analytic function on $\tau^{-1}(\overset{\circ}{\Delta})$ is of the form $a t_1^{n_1} \dots t_{d-1}^{n_{d-1}}(1+h)$ with $a \in K^*$, $n_i \in \mathbf{Z}$ and an analytic function h such that $|h(y)| < 1$ for all $y \in \tau^{-1}(\overset{\circ}{\Delta})$.

5. Since the group $\mathrm{SL}_d(k)$ acts transitively on the set of apartments, we may assume that $\varphi(\Lambda) = \Lambda$ for the apartment Λ from Step 3. Furthermore, since any simplicial automorphism of Λ is induced by an element of the normalizer of the torus corresponding to Λ, we may assume that $\varphi\big|_\Lambda = 1_\Lambda$. We have to show that in this case φ is induced by a diagonal matrix whose non-zero entries are units of k. For this we introduce a retraction map $\tau_\Lambda : \Omega_K^d \to \Lambda$ which is the restriction of the retraction map $(\mathbf{A}_K^1 \backslash \{0\})^{d-1} \to \Lambda$ for which $|(\sum a_\nu t^\nu)(\tau_\Lambda(x))| = \max |a_\nu||t(x)|^\nu$. *We claim that* $\tau_\Lambda \circ \varphi = \tau_\Lambda$. Since $\varphi \circ \tau = \tau \circ \varphi$ and $\tau_\Lambda \circ \tau = \tau_\Lambda$, it suffices to verify the above equality only for the restrictions of the both maps to $\mathcal{B}^d$.

5.1. *The retraction map $\tau_\Lambda : \mathcal{B}^d \to \Lambda$ is simplicial.*

Let Δ' be a chamber of $\mathcal{B}^d$, and let $g = (a_{i,j})_{0 \le i,j \le d-1} \in \mathrm{SL}_d(k)$ be such that $g(\Delta) = \Delta'$. Then for $x \in \Delta$ one has $|t_i(g(x))| = \frac{|a_{i,j_i}| \cdot |t_{j_i}(x)|}{|a_{0,j_0}| \cdot |t_{j_0}(x)|}$, where $t_0 = 1$ and j_i is the minimal j with $|a_{i,j}| = \max_l |a_{i,l}|$. The claim follows.

5.2. By Step 5.1, it suffices to verify that $\tau_\Lambda(\varphi(x)) = \tau_\Lambda(x)$ only for zero-dimensional simplices (vertices) of $\mathcal{B}^d$. We claim that *the above equality holds for any simplicial automorphism φ of $\mathcal{B}^d$ with $\varphi\big|_\Lambda = 1_\Lambda$.* To show this, it is convenient to use the interpretation of the vertices of $\mathcal{B}^d$ as the similarity classes $\{M\}$ of lattices $M \subset R_1$. Namely, if a vertex x corresponds to the class of a norm $N \in \mathcal{N}$, then $x = \{M\}$, where $M = \{f \in R_1 \big| N(f) \le 1\}$. Let $M \subset R_1$ be a lattice. From the definition of τ_Λ it follows that $\tau_\Lambda(\{M\}) = \{L\}$, where the lattice L is generated by $\pi^{n_i} T_i$, $0 \le i \le d-1$, and n_i are such that $\pi^{n_i} T_i \in M \backslash \pi M$. Let L' be a sublattice of M with $\{L'\} \in \Lambda$. Then L' is generated by $\pi^{n'_i} T_i$, $0 \le i \le d-1$, and since $L' \subset M$ one has $n'_i \ge n_i$, i.e., $L' \subset L$. In particular, if $L' \ne L$, then $[M : L] < [M : L']$. It follows that $\tau_\Lambda(\{M\})$ is the class of the lattice $L \subset M$ with $\{L\} \in \Lambda$ for which $[M : L]$ is minimal. Thus, to prove our claim, it suffices to show that the function $(\{M\}, \{L\}) \mapsto \min\{[M : L] \big| M \in \{M\}, L \in \{L\}$ and $L \subset M\}$ is invariant under all automorphisms of $\mathcal{B}^d$. But this is clear because this function is invariant under $\mathrm{SL}_d(k)$ and any pair of points in $\mathcal{B}^d$ is contained in one apartment.

6. Since $|t_i(x)| = |t_i(\tau_\Lambda(x))|$ for all $x \in (\mathbf{A}_K^1 \backslash \{0\})^{d-1}$, from Step 5 it follows that $|\varphi^* t_i(x)| =$

$|t_i(x)|$ for all $x \in \Omega_K^d$, and therefore the invertible analytic functions $(\varphi^* t_i)/t_i$ on Ω_K^d are bounded. Theorem 1 now follows from the following lemma.

Lemma 3. – *Any bounded analytic function on Ω_K^d is constant.*

Proof. The lemma follows by induction from the following statement applied to the projection (to the first $d-2$ coordinates) $\Omega_K^d \to \Omega_K^{d-1}$. *Let X be a reduced K-analytic space, and Y an open subset of $X \times \mathbf{P}_K^1$ such that its complement is contained in a union of Zariski closed subsets and the projection $\varphi : Y \to X$ is surjective. Then any analytic function $f \in \mathcal{O}(Y)$ bounded on the fibres of φ is of the form $\varphi^* g$ for some $g \in \mathcal{O}(X)$.* Since the statement is local with respect to the G-topology of X, we may assume that $X = \mathcal{M}(\mathcal{A})$ is K-affinoid. If $\mathcal{A} = K$, the statement is well known. We also remark that in this case the set Y contains the closed annulus $A(r, r') := \{y \in \mathbf{A}_K^1 \big| r \leq |t(y)| \leq r'\}$ for some $0 < r < r'$. It follows that, in the general case, for each point $x \in X$ we can replace X by an affinoid neighborhood of x and assume that Y contains the affinoid domain $V = X \times A(r, r)$ for some $r > 0$. The function $f\big|_V$ has a unique representation in the form $\sum_{i=-\infty}^{\infty} g_i t^i$, where $g_i \in \mathcal{A}$ and $||g_i|| r^i \to 0$ for $i \to \pm\infty$. Using the assumptions, we get $f = g_0$. ∎

Remarks. – (i) The reasoning from Step 3 shows that the retraction map $\tau : \Omega_K^d \to \mathcal{B}^d$ is proper.

(ii) The simplicial retraction map $\tau_\Lambda : \mathcal{B}^d \to \Lambda$ sends the chambers that are not contained in Λ to simplices of smaller dimension. Indeed, if $\dim(\tau_\Lambda(\Delta')) = d-1$ (in the situation of Step 5.1), then for $i \neq l$ one should have $j_i \neq j_l$. This easily implies that $\Delta' = g(\Lambda) \subset \Lambda$.

(iii) It follows from the proof that the retraction map $\tau_\Lambda : \mathcal{B}^d \to \Lambda$ is defined for every apartment Λ of $\mathcal{B}^d$ and one has $\varphi \circ \tau_\Lambda = \tau_{\varphi(\Lambda)} \circ \varphi$ for any automorphism φ of $\mathcal{B}^d$.

Proof of Theorem 2. – All the analytic spaces considered are assumed to be Hausdorff. Recall that an action of a group Γ on a locally compact space X is said to be discrete if for any compact subset $V \subset X$ the set $\Gamma_V := \{\gamma \in \Gamma \big| \gamma(V) \cap V \neq \emptyset\}$ is finite. For such an action the quotient space $\Gamma \backslash X$ is locally compact. Furthermore, the above action is said to be free if $\Gamma_x = \{1\}$ for all $x \in X$. For such an action the canonical map $X \to \Gamma \backslash X$ is a topological covering map.

Let a group Γ act discretely on a K-analytic space X. We say that *the quotient space* $\Gamma \backslash X$ *exists* if one can endow the topological space $\Gamma \backslash X$ with a K-analytic space structure and construct a morphism $p : X \to \Gamma \backslash X$ such that, for any morphism φ from X to a K-analytic space Y with $\varphi \circ \gamma = \varphi$ for all $\gamma \in \Gamma$, there exists a unique morphism $\psi : \Gamma \backslash X \to Y$ with $\varphi = \psi \circ p$. We say that an affinoid domain $V \subset X$ is Γ-*marked* if the set Γ_V is a group and $\gamma(V) = V$ for all $\gamma \in \Gamma_V$. A morphism $p : T \to S$ is said to be an *analytic covering* if each point of S has an open neighborhood $\mathcal{U}$ such that $p^{-1}(\mathcal{U}) = \coprod_{i \in I} \mathcal{V}_i$ and all of the induced maps $\mathcal{V}_i \to \mathcal{U}$ are isomorphisms.

Lemma 4. – *Assume that either (1) the action of Γ on X is free, or (2) the space X is separated and each point of X has a neighborhood of the form $V_1 \cup \ldots \cup V_n$, where V_i are Γ-marked affinoid domains. Then the quotient space $\Gamma \backslash X$ exists. In the case (1), $p : X \to \Gamma \backslash X$ is an analytic covering.*

Proof. – Let τ be (1) the family of all affinoid domains $V \subset X$ with $\gamma(V) \cap V = \emptyset$ for $\gamma \neq 1$, and (2) the family of all Γ-marked affinoid domains. It follows from the assumptions that τ is a net stable under the action of Γ, and therefore $\sigma := \{p(V) \big| V \in \tau\}$ is a net on $\Gamma \backslash X$. Using the fact that the subalgebra of invariants of a K-affinoid algebra under a finite automorphism group is K-affinoid ([4], 6.3; [1], 2.1.14), one constructs in the evident way a K-affinoid atlas with the net σ that gives rise to the required K-analytic space structure on $\Gamma \backslash X$. ∎

Corollary 5. *If Γ acts discretely on a good separated K-analytic space X, then the quotient space $\Gamma \backslash X$ exists, and $\Gamma \backslash X$ is a good separated K-analytic space.*

Proof. It suffices to show that each point $x \in X$ has a Γ-marked affinoid neighborhood. Let Σ be a compact neighborhood of x with $\Gamma_\Sigma = \Gamma_x$. If U is an affinoid neighborhood of x with $U \subset \Sigma$, then $V = \cap_{\gamma \in \Gamma_x} \gamma(U)$ is a Γ-marked affinoid neighborhood of x with $\Gamma_V = \Gamma_x$. ∎

Let us return to our situation.

Lemma 6. – *(i) The following properties of a subgroup $\Gamma \subset \mathrm{PGL}_d(k)$ are equivalent:*

(a) the action of Γ on Ω^d_K is discrete (resp. discrete and free);

(b) the action of Γ on $\mathcal{B}^d$ is discrete (resp. discrete and free);

(c) Γ is discrete (resp. torsion free and discrete) in $\mathrm{PGL}_d(k)$.

(ii) The following properties of a discrete subgroup $\Gamma \subset \mathrm{PGL}_d(k)$ are equivalent:

(a) the K-analytic space $\Gamma \backslash \Omega^d_K$ is proper;

(b) the topological space $\Gamma \backslash \mathcal{B}^d$ is compact;

(c) Γ is cocompact in $\mathrm{PGL}_d(k)$.

Proof. – Everything easily follows from the facts that the retraction map $\tau : \Omega^d_K \to \mathcal{B}^d$ is $\mathrm{PGL}_d(k)$-equivariant and proper, the fixed point set of any compact subgroup of $\mathrm{PGL}_d(k)$ on $\mathcal{B}^d$ is nonempty, and the vertices of $\mathcal{B}^d$ correspond bijectively to the right cosets of the compact subgroup $\mathrm{PGL}_d(k^\circ) \subset \mathrm{PGL}_d(k)$, where k° is the ring of integers of k. ∎

Theorem 2 now follows in the evident way from Theorem 1, the fact ([1], §6.1) that the space Ω^d_K is simply connected (and even contractible) and the following

Lemma 7. – *Let $p : Y \to X$ be an analytic covering. Then for any morphism $\varphi : Y' \to X$ with simply connected Y' and for any pair of points $y \in Y$, $y' \in Y'$ with $p(y) = \varphi(y')$ there exists a unique morphism $\psi : Y' \to Y$ with $\varphi = p \circ \psi$ and $y = \psi(y')$.*

Proof. – Since analytic spaces are locally compact and locally arcwise connected, then such ψ exists and is unique, at least, as a map of topological spaces. We may assume that φ is represented by a strong morphism (see [2], §1) $(Y', \mathcal{B}', \sigma') \to (X, \mathcal{A}, \tau)$, where the net τ is such that for any $V \in \tau$ one has $p^{-1}(V) = \coprod_{i \in I} W_i$ and $W_i \tilde{\to} V$. The family σ of all affinoid domains $W \subset Y$ with $p(W) \subset V$ for some $V \in \tau$ is a net on Y, and for any $W' \in \sigma'$ there exists $W \in \sigma$ with $\varphi(W') \subset W$. Then p is defined by a strong morphism $(Y, \mathcal{B}, \sigma) \to (X, \mathcal{A}, \tau)$, and there is an evident strong morphism $(Y', \mathcal{B}', \sigma') \to (Y, \mathcal{B}, \sigma)$ that defines the required morphism ψ. ∎

Remark. – One can construct a $\mathrm{PGL}_d(k)$-equivariant homotopy between the identity map on Ω^d_K and the retraction map $\tau : \Omega^d_K \to \mathcal{B}^d \subset \Omega^d_K$. It follows that, for any discrete subgroup $\Gamma \subset \mathrm{PGL}_d(k)$, $\Gamma \backslash \mathcal{B}^d$ is a strong deformation retract of $\Gamma \backslash \Omega^d_K$.

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