Smooth p-adic analytic spaces are locally contractible. II

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0 Introduction

Let k be a field complete with respect to a non-Archimedean valuation, k° its ring of integers, and $\widetilde{k}$ its residue field. Every formal scheme $\mathfrak{X}$ locally finitely presented over k° has a closed fiber $\mathfrak{X}_s$, which is a scheme of locally finite type over $\widetilde{k}$, and a generic fiber $\mathfrak{X}_\eta$, which is a strictly k-analytic space (in the sense of [Ber2]) whose underlying

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topological space is a paracompact locally compact space of dimension $\dim(\mathfrak{X}_\eta)$, and there is a reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$.

Given a formal scheme $\mathfrak{X}$ for which there is a sequence of morphisms from a certain class $\underline{\mathfrak{X}} = (\mathfrak{X} = \mathfrak{X}_l \stackrel{f_{l-1}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1 \stackrel{f_0}{\to} \mathfrak{X}_0 = \mathrm{Spf}(k^\circ))$, in [Ber7] we constructed a strong deformation retraction of the generic fiber $\mathfrak{X}_\eta$ to a closed subset $S(\underline{\mathfrak{X}})$ called the skeleton of $\underline{\mathfrak{X}}$. (The morphisms from that class are called poly-stable, such a sequence $\underline{\mathfrak{X}}$ is called a poly-stable fibration, and such a formal scheme $\mathfrak{X}$ is called pluri-stable.) We also constructed a canonical homeomorphism between the skeleton $S(\underline{\mathfrak{X}})$ and the geometric realization of a simplicial set associated with the closed fiber of $\underline{\mathfrak{X}}$. This homotopy description of the spaces $\mathfrak{X}_\eta$ together with the results of J. de Jong from [deJ] were used in [Ber7] to prove that in the case, when the valuation on k is nontrivial, any strictly analytic subdomain of a smooth k-analytic space is locally contractible.

In our work in progress on integration on p-adic analytic spaces, the following stronger property turns out to play an important role. Assume that the valuation on k is nontrivial, and let X be a strictly analytic domain in a smooth k-analytic space. Then each point $x \in X$ has a fundamental system of open neighborhoods V such that: (a) there is a contraction Φ of V to a point $x_0 \in V$; (b) there is an increasing sequence of compact strictly analytic domains $X_1 \subset X_2 \subset \cdots \subset V$ which exhaust V and are preserved under Φ; (b) for any bigger non-Archimedean field K, $V\widehat{\otimes}K$ has a finite number of connected components and Φ lifts to a contraction of each of them to a point over x_0; and (d) there is a finite separable extension L of k such that, if K from (c) contains L, then the map $V\widehat{\otimes}K \to V\widehat{\otimes}L$ induces a bijection between the sets of connected components.

One of the main purposes of this paper is to prove the above property. The proof is based on a further study of the skeleton $S(\underline{\mathfrak{X}})$ for those poly-stable fibrations $\underline{\mathfrak{X}}$ in which all of the poly-stable morphisms $f_i : \mathfrak{X}_{i+1} \to \mathfrak{X}_i$ are so called nondegenerate. This study has an independent interest. It turns out that $S(\underline{\mathfrak{X}})$ depends only on $\mathfrak{X} = \mathfrak{X}_l$ (it is therefore denoted by $S(\mathfrak{X})$), and that it is provided with a canonical piecewise linear structure of a special type. This piecewise linear structure on the skeleton $S(\mathfrak{X})$ is closely related to the analytic structure on the generic fiber $\mathfrak{X}_\eta$, and is in fact reflected in many familiar properties and objects related to analytic functions (such as the growth and Newton polygon of an analytic function). We now give a summary of the material which follows.

In §1, we introduce and study a subcategory of the category of piecewise linear spaces. The exposition is slightly non-traditional in the sense that the model vector space for us is the multiplicative group $(\mathbb{R}_+^*)^n$ provided with the following action of $\mathbb{R}$: $(s, (t_1, \dots, t_n)) \mapsto (t_1^s, \dots, t_n^s)$. Similarly, linear functions considered are maps to $\mathbb{R}_+^*$ of the form $(t_1, \dots, t_n) \mapsto rt_1^{s_1} \dots t_n^{s_n}$. The subcategory introduced consists of the piecewise linear spaces which are built from the polytopes defined by linear inequalities with certain restrictions on their coefficients. Namely, the coefficients at the linear terms are required to belong to a sub-semiring $S \subset \mathbb{R}$, and the constant terms are required to belong to a submonoid $R \subset \mathbb{R}_+^*$ such that for any $r \in R$ and $s \in S$

one has $r^s \in R$. The polytopes defined in such a way are called R_S-polytopes, and the spaces obtained are called piecewise R_S-linear. If $S = \mathbb{R}$ and $R = \mathbb{R}_+^*$, one gets the whole category of piecewise linear spaces. The skeleton $S(\mathfrak{X})$ of a nondegenerate pluri-stable formal scheme over k° is provided (in §5) with a piecewise $R_{\mathbb{Z}_+}$-linear structure for $R = |k^*| \cap [0, 1]$.

There is at least a formal similarity between piecewise linear and k-analytic spaces. Namely, both are provided with a Grothendieck topology formed by piecewise linear subspaces in the former and by analytic subdomains in the latter. Coverings are defined in the same way: a family $\{Y_i\}_{i\in I}$ of subspaces of Y is a covering if every point $y \in Y$ has a neighborhood of the form $Y_{i_1} \cup \cdots \cup Y_{i_n}$ with $y \in Y_{i_1} \cap \cdots \cap Y_{i_n}$. In §6, a direct relation between the Grothendieck topologies on $S(\mathfrak{X})$ and $\mathfrak{X}_\eta$ is established, and it is very important for applications in §7 and §8.

To describe the constructions of §2 and §3, recall that in [Ber7] we associated with the closed fiber of a poly-stable fibration $\underline{\mathfrak{X}}$ over k° of length l a polysimplicial set, i.e., an object of the category $\Lambda^\circ\mathcal{E}ns$ of contravariant functors from a certain category Λ to the category of sets $\mathcal{E}ns$. (The simplicial set mentioned at the beginning of the introduction was in fact derived from the latter.) If $l = 1$, we associated with the formal scheme $\mathfrak{X} = \mathfrak{X}_1$ itself a more refined object, an R-colored polysimplicial set, i.e., an object of the category $\Lambda_R^\circ\mathcal{E}ns$, where the category Λ_R was associated with a submonoid $R \subset [0, 1]$. (In the case considered, $R = |k| \cap [0, 1]$.) The geometric realization of an R-colored polysimplicial set was provided with an extra structure, a monoid of continuous functions to $[0, 1]$ (which were eventually related to the absolute values of the functions from the monoid $\mathcal{O}(\mathfrak{X}) \cap \mathcal{O}(\mathfrak{X}_\eta)^*$).

Let $\mathcal{R}$ be a category provided with a geometric realization functor that takes an object A to a pair $(|A|, M_A)$, where $|A|$ is a topological space and M_A is a semiring of continuous functions on $|A|$ with values in $[0, 1]$. (The semirings are considered with the usual multiplication and the following addition: $f \dot{+} g = \max(f, g)$.) In §2, we construct a category $\Lambda_{\mathcal{R}}$ provided with a similar geometric realization functor. It gives rise to a category of $\mathcal{R}$-colored polysimplicial sets $\Lambda_{\mathcal{R}}^\circ\mathcal{E}ns$ and a similar geometric realization functor on it. If $\mathcal{R}$ is a one point category with the geometric realization functor that takes the only object of $\mathcal{R}$ to a one point space with a submonoid $R \subset [0, 1]$, one gets the category Λ_R introduced in [Ber7, §4]. The only difference is that the monoids, considered in *loc. cit.*, are submonoids of the semirings considered here, but the former can be characterized inside the latter.

In §3, we study the category obtained by iteration of the latter construction. Namely, given a submonoid $R \subset [0, 1]$, we set $\Lambda_{R,1} = \Lambda_R$ and $\Lambda_{R,l} = \Lambda_{\Lambda_{R,l-1}}$ for $l \geq 2$. In this way we get the category $\Lambda_{R,l}^\circ\mathcal{E}ns$ of R-colored polysimplicial sets of length l. The main facts established here are as follows. The geometric realization of an R-colored polysimplicial set of length l is always Hausdorff and, if the set is locally finite and $0 \notin R$, the geometric realization is provided with a canonical piecewise $R_{\mathbb{Z}_+}$-linear structure so that the semiring associated with it consists of certain piecewise $R_{\mathbb{Z}_+}$-linear functions.

In §4, we recall the notion of a poly-stable morphism and introduce an additional property of nondegenerateness. (A pluri-stable formal scheme over k° is nondegenerate if and only if its generic fiber is a normal strictly k-analytic space.) We introduce a partial ordering on the generic fiber $\mathfrak{X}_\eta$ of a formal scheme $\mathfrak{X}$ locally finitely presented over k°, and prove that the skeleton $S(\underline{\mathfrak{X}})$ of a nondegenerate poly-stable fibration $\underline{\mathfrak{X}}$ of length l coincides with the set of maximal points with respect to the ordering on $\mathfrak{X}_{l,\eta}$. This implies that $S(\underline{\mathfrak{X}})$ depends only on $\mathfrak{X}_l$, and so the skeleton $S(\mathfrak{X})$ of a nondegenerate pluri-stable formal scheme $\mathfrak{X}$ is well defined. We also recall the construction of the retraction map $\tau : \mathfrak{X}_\eta \to S(\mathfrak{X})$, which in general depends on the choice of $\underline{\mathfrak{X}}$ with $\mathfrak{X}_l = \mathfrak{X}$, and introduce a class of so called strongly nondegenerate pluri-stable formal schemes for which τ does not depend on the choice of $\underline{\mathfrak{X}}$.

In §5, we associate with every nondegenerate poly-stable fibration $\underline{\mathfrak{X}}$ over k° of length l a locally finite R-colored polysimplicial set $\mathbb{D}(\underline{\mathfrak{X}})$ of length l, where $R = |k^*| \cap [0, 1]$, and construct a canonical homeomorphism $|\mathbb{D}(\underline{\mathfrak{X}})| \stackrel{\sim}{\to} S(\underline{\mathfrak{X}})$ such that, for any $f \in \mathcal{O}'(\mathfrak{X}_l)$, the function $x \mapsto |f(x)|$ on $S(\underline{\mathfrak{X}})$ is contained in the semiring $M_{\underline{\mathfrak{X}}}$ associated with the geometric realization of $\mathbb{D}(\underline{\mathfrak{X}})$. (Here $\mathcal{O}'(\mathfrak{X})$ is the set of all $f \in \mathcal{O}(\mathfrak{X})$ whose restriction to every connected component of $\mathfrak{X}$ is not zero.) This provides the skeleton $S(\underline{\mathfrak{X}})$ with a piecewise $R_{\mathbb{Z}_+}$-linear structure and a semiring of piecewise $R_{\mathbb{Z}_+}$-linear functions $M_{\underline{\mathfrak{X}}}$.

In §6.1, we prove that the latter depend only on $\mathfrak{X}_l$, i.e., given a nondegenerate pluri-stable formal scheme $\mathfrak{X}$ over k°, a piecewise $R_{\mathbb{Z}_+}$-linear structure on $S(\mathfrak{X})$ and a semiring of piecewise $R_{\mathbb{Z}_+}$-linear functions $M_{\mathfrak{X}}$ on it are well defined and, for any $f \in \mathcal{O}'(\mathfrak{X})$, the function $x \mapsto |f(x)|$ on $S(\mathfrak{X})$ is contained in $M_{\mathfrak{X}}$. We also prove that any pluri-stable morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ from a similar formal scheme $\mathfrak{X}'$ gives rise to a piecewise $R_{\mathbb{Z}_+}$-linear map $S(\mathfrak{X}') \to S(\mathfrak{X})$ and it takes functions from $M_{\mathfrak{X}}$ to functions from $M_{\mathfrak{X}'}$. In §6.2, we get a first application of the above results whose elementary particular case tells the following. Given a compact strictly analytic domain X in the analytification of a separated scheme of finite type over k and invertible analytic functions $f_1, \dots, f_n$ on X, the image of the mapping $X \to (\mathbb{R}_+^*)^n : x \mapsto (|f_1(x)|, \dots, |f_n(x)|)$ is a finite union of $R_{\mathbb{Z}_+}$-polytopes of dimension at most $\dim(X)$. (This result was recently extended by A. Ducros to arbitrary compact strictly k-analytic spaces.) Moreover, if such X is connected, the quotient group $\mathcal{O}(X)^*/(k^*\mathcal{O}(X)^1)$ is finitely generated, where $\mathcal{O}(X)^1 = \{f \in \mathcal{O}(X)^* \mid |f(x)| = 1$ for all $x \in X\}$.

Let $\mathfrak{X}$ be a nondegenerate pluri-stable formal scheme over k°. In §6.3, we prove that, for any strictly analytic subdomain $V \subset \mathfrak{X}_\eta$, the intersection $V \cap S(\mathfrak{X})$ is a piecewise $R_{\mathbb{Z}_+}$-linear subspace of $S(\mathfrak{X})$ and, for any analytic function $f \in \mathcal{O}'(V)$, the function $x \mapsto |f(x)|$ on $V \cap S(\mathfrak{X})$ is piecewise $|k^*|_{\mathbb{Z}_+}$-linear. In particular, the canonical embedding $S(\mathfrak{X}) \hookrightarrow \mathfrak{X}_\eta$ is continuous with respect to the Grothendieck topologies of $S(\mathfrak{X})$ and $\mathfrak{X}_\eta$ formed by piecewise $R_{\mathbb{Z}_+}$-linear subspaces and strictly analytic subdomains, respectively. In §6.4, we prove that the retraction map $\tau : \mathfrak{X}_\eta \to S(\mathfrak{X})$ is continuous with respect to the same Grothendieck topologies on $S(\mathfrak{X})$ and $\mathfrak{X}_\eta$. (This result is used in §7 and §8.) We also prove that, given an arbitrary morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ from a similar formal scheme $\mathfrak{X}'$ over k°, the composition map $S(\mathfrak{X}') \stackrel{\varphi}{\to} \mathfrak{X}_\eta \stackrel{\tau}{\to} S(\mathfrak{X})$ is piecewise $(\sqrt{|k^*|})_{\mathbb{Q}_+}$-linear, where $\sqrt{|k^*|} = \{\alpha \in \mathbb{R}_+^* \mid \alpha^n \in |k^*| \text{ for some } n \geq 1\}$.

In §7, we prove the property mentioned at the beginning of the introduction.

In §8, we prove results which have a direct relation to p-adic integration. Assume that the characteristic of k is zero. The sheaf of constant functions $\mathfrak{c}_X$ on a reduced strictly k-analytic space X is the étale sheaf of k-vector spaces $\mathrm{Ker}(\mathcal{O}_X \stackrel{d}{\to} \Omega^1_X)$. If k is algebraically closed, it is the constant sheaf k_X associated with k, but in general it is much bigger. Assume X is smooth. It is well known that the de Rham complex $\mathcal{O}_X \stackrel{d}{\to} \Omega^1_X \stackrel{d}{\to} \Omega^2_X \stackrel{d}{\to} \cdots$ is not exact. On the other hand, the similar de Rham complex for the sheaf of naive analytic functions (i.e., the functions analytic in an open neighborhood of each point from the dense subset $X_0 = \{x \in X \mid [\mathcal{H}(x) : k] < \infty\}$) is exact, but the kernel of the first differential is too large. One of the purposes of a p-adic integration theory is to find an intermediate class of functions between the analytic and naive analytic ones such that the corresponding de Rham complex is an exact resolution of the sheaf of constant functions $\mathfrak{c}_X$. It is what was essentially done by R. Coleman in [Col] and [CoSh] for smooth k-analytic curves. In our generalization of his work, the following two facts are of crucial importance. The first one (Theorem 8.2.1) tells that each point of X has a fundamental system of open neighborhoods V such that $H^n(V, \mathfrak{c}_X) = 0$ for all $n \geq 1$. The second one (Corollary 8.3.3) tells that, given a nondegenerate strictly pluri-stable formal scheme $\mathfrak{X}$ over k°, an irreducible component $\mathcal{Y} \subset \mathfrak{X}_s$, and a Zariski closed subset $Z \subset \mathfrak{X}_\eta$, then for $X = \pi^{-1}(\mathcal{Y})\backslash Z$ one has $H^n(X, \mathfrak{c}_X) = 0$ for all $n \geq 1$.

To give some idea on how these two facts are used (in our work in progress), notice that, if the above integration theory exists and X is a smooth k-analytic space with $H^1(X, \mathfrak{c}_X) = 0$, then every closed analytic one-form on X has a primitive (of course, in a bigger class of functions) which is defined uniquely up to an element of $\mathfrak{c}(X)$. The second of the above facts provides a class of spaces (of the form $X = \pi^{-1}(\mathcal{Y})$) where one constructs such a primitive. The construction depends on $\mathfrak{X}$ and $\mathcal{Y}$ (and not only on X), and the first fact is used to show that the primitive constructed actually depends only on X.

In another work in progress, we generalize many of the results of this paper to the whole class of pluri-stable formal schemes. In particular, we show that the skeleton $S(\underline{\mathfrak{X}})$ always depends only on $\mathfrak{X} = \mathfrak{X}_l$, but in the general case $S(\mathfrak{X})$ is provided with a so called piecewise monomial structure which is more general than the piecewise linear structure considered here (see Remark 1.3.2(ii)). It is for that reason certain constructions in §2, §3 and §5 are considered in a more general setting.

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1 Piecewise R_S-linear spaces

1.1 R_S-polytopes

Recall that a (compact) polytope in a vector space is the convex hull of a finite set of points. This object is a building block of the classical notion of a piecewise linear space. A basic fact is that a compact subset of a vector space is a polytope if and only if it can be defined by a finite number of linear inequalities (see [Zie, Theorem 1.1]).

We say that a set is a *semiring* if it is a commutative monoid by multiplication and addition related by the identity $a(b + c) = ab + ac$ and which contains 1. An example of a semiring is the set of all continuous non-negative real valued functions on a topological space provided with the usual multiplication and the following addition: $f \dot{+} g = \max(f, g)$. In this section we consider only sub-semirings of the field of real numbers $\mathbb{R}$.

Let S be a sub-semiring of $\mathbb{R}$ that contains 0, and let R be a nonempty S-submonoid of $\mathbb{R}_+^*$, i.e., it is a nonempty submonoid of $\mathbb{R}_+^*$ such that for any $r \in R$ and $s \in S$ one has $r^s \in R$. The simplest example is $S = \mathbb{R}$ and $R = \mathbb{R}_+^*$, and the main examples considered in the paper are provided by a non-Archimedean field k and are as follows: $S = \mathbb{Z}_+$ and $R = |k^*| \cap [0, 1]$ or $|k^*|$, and $S = \mathbb{Q}_+$ and $R = \sqrt{|k^*|} = \{\alpha \in \mathbb{R}_+^* \mid \alpha^n \in |k^*| \text{ for some } n \geq 1\}$. If $R = \{1\}$ (e.g., if the valuation on k is trivial), everything we are going to consider is trivial, but has a meaning.

We denote by $\widetilde{S}$ (resp. $\overline{S}$) the subring (resp. subfield) of $\mathbb{R}$ generated by S and by $\widetilde{R}$ (resp. $\overline{R}$) the $\widetilde{S}$-submodule (resp. $\overline{S}$-vector subspace) of $\mathbb{R}_+^*$ generated by R, and we denote by $\langle R\rangle$ the convex hull of R in $\mathbb{R}_+^*$, which is also an S-submonoid of $\mathbb{R}_+^*$. (Here are all possible values of $\langle R\rangle$: $\{1\}$, $[1, \infty[$, $]0, 1]$ and $\mathbb{R}_+^*$.) For $n \geq 0$, we denote by $A^n(R_S)$ the S-monoid of functions on $(\mathbb{R}_+^*)^n$ of the form $(t_1, \dots, t_n) \mapsto r t_1^{s_1} \dots t_n^{s_n}$, where $r \in R$ and $s_1, \dots, s_n \in S$, and, for a subset $V \subset (\mathbb{R}_+^*)^n$, we denote by $A_V(R_S)$ the set of the restrictions to V of the functions from $A^n(R_S)$.

An R_S*-polytope in* $(\mathbb{R}_+^*)^n$ is a compact subset of $\langle R\rangle^n$ which is defined by a finite system of inequalities of the form $f(t) \leq g(t)$ with $f, g \in A^n(R_S)$. Of course, any R_S-polytope is also an $\overline{R}_{\overline{S}}$-polytope. An easy criterion for the latter is as follows.

A point of $(\mathbb{R}_+^*)^n$ is said to be an $\overline{R}$*-point* if all of its coordinates are contained in $\overline{R}$, and a line in $(\mathbb{R}_+^*)^n$ is said to be $\overline{S}$*-rational* if there exist $s_1, \dots, s_n \in \overline{S}$ such that, for some (and therefore every) pair of distinct points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of the line, one has $\frac{x_i}{y_i} = t^{s_i}$ with $t \in \mathbb{R}_+^*$, $1 \leq i \leq n$. Notice that, if the above x and y are $\overline{R}$-points, then $t \in \overline{R}$ and, in fact, $\frac{f(y)}{f(x)} \in \overline{R}$ for all $f \in A^n(R_S)$.

1.1.1 Lemma. *The following properties of a polytope $V \subset (\mathbb{R}_+^*)^n$ are equivalent:*

(a) *V is an $\overline{R}_{\overline{S}}$-polytope;*

(b) *V is defined by a finite system of inequalities of the form $f(t) \leq g(t)$ with $f, g \in A^n(R_S)$;*

(c) *all vertices of V are $\overline{R}$-points and all edges of V are $\overline{S}$-rational.*

Notice that if $\dim_{\overline{S}}(\overline{R}) = 1$ then the second property in (c) follows from the first one.

Proof. The equivalence of (a) and (b) is trivial, and the equivalence of (b) and (c) is a simple linear algebra. ◻

1.1.2 Corollary. *Let V be an R_S-polytope in $(\mathbb{R}_+^*)^n$. Then any subset of V, which is defined by a finite system of inequalities of the form $t_1^{s_1} \dots t_n^{s_n} \le r$ with $s_1, \dots, s_n \in \overline{S}$ and $r \in \overline{R}$, is an R_S-polytope. In particular, all faces of V and the intersection of two R_S-polytopes are R_S-polytopes.* ◻

An (abstract) *R_S-polytope* is a topological space X provided with a set of continuous functions A_X for which there exists a homeomorphism $\varphi : X \xrightarrow{\sim} V$, where V is an R_S-polytope in $(\mathbb{R}_+^*)^n$, such that φ^* induces a bijection $A_V(R_S) \xrightarrow{\sim} A_X$. For example, a subset $V \subset (\mathbb{R}_+^*)^n$ provided with the set of functions $A_V(R_S)$ is an (abstract) R_S-polytope if and only if V is an R_S-polytope in $(\mathbb{R}_+^*)^n$. A *morphism of R_S-polytopes* $\psi : X' \to X$ is a continuous map that takes functions from A_X to functions from $A_{X'}$. In this way we get a category of R_S-polytopes.

For example, there is an evident anti-equivalence between the category of zero dimensional R_S-polytopes and the category of S-monoids $R \subset R' \subset \langle R \rangle \cap \overline{R}$, which are generated over S by R and a finite number of elements, and with inclusions as morphisms. In particular, if the S-monoid R is not divisible, the minimal dimension of an affine space which contains a zero dimensional R_S-polytope isomorphic to a given one may be sufficiently large.

A subset Y of an R_S-polytope X is said to be an *R_S-polytope in X* if one of the above maps φ takes it to an R_S-polytope in V. Such a subset is provided with the evident R_S-polytope structure.

1.1.3 Corollary. *Let $\varphi : X' \to X$ be a morphism of R_S-polytopes. Then*

(i) *the image $\varphi(X')$ is an R_S-polytope in X;*

(ii) *φ induces an isomorphism $X' \xrightarrow{\sim} \varphi(X')$ if and only if the map $A_X \to A_{X'}$ is surjective;*

(iii) *for any R_S-polytope Y in X, the preimage $\varphi^{-1}(Y)$ is an R_S-polytope in X', and the induced map $\varphi^{-1}(Y) \to Y$ is a morphism of R_S-polytopes.* ◻

A morphism of R_S-polytopes $\varphi : X' \to X$ is said to be an *immersion* if it satisfies the equivalent properties of Corollary 1.1.3(ii).

1.2 R_S-polyhedra

An *R_S-polyhedron in* $(\mathbb{R}_+^*)^n$ is a finite union of R_S-polytopes. Let V be an R_S-polyhedron. A continuous function $f : V \to \mathbb{R}_+^*$ is said to be *piecewise R_S-linear* if V can be represented as a union of R_S-polytopes $V = V_1 \cup \cdots \cup V_k$ such that $f|_{V_i} \in A_{V_i}(R_S)$ for all $1 \le i \le k$. Let $P_V(R_S)$ denote the set of all piecewise R_S-linear functions on V. From Corollary 1.1.2 it follows that, given $f_1, \ldots, f_m \in P_V(R_S)$, one can find R_S-polytopes $V_1, \ldots, V_k \subset V$ such that $V = V_1 \cup \cdots \cup V_k$ and $f_i|_{V_j} \in A_{V_j}(R_S)$ for all $1 \le i \le m$ and $1 \le j \le k$. In particular, $P_V(R_S)$ is an S-monoid, and it contains the functions $\max\{f_1, \ldots, f_n\}$ and $\min\{f_1, \ldots, f_n\}$.

1.2.1 Lemma. *Let $V \subset (\mathbb{R}_+^*)^n$ and $U \subset (\mathbb{R}_+^*)^m$ be R_S-polyhedra. Then the following properties of a continuous map $\varphi : V \to U$ are equivalent:*

(a) *there exist R_S-polytopes $V_1, \ldots, V_k \subset V$ and $U_1, \ldots, U_k \subset U$ such that $V = V_1 \cup \cdots \cup V_k$ and φ induces morphisms of R_S-polytopes $V_i \to U_i$, $1 \le i \le k$;*

(b) *φ^* takes functions from $P_U(R_S)$ to $P_V(R_S)$.*

Proof. The implication (a)$\Longrightarrow$(b) easily follows from Corollary 1.1.3(iii). Assume that φ^* takes functions from $P_U(R_S)$ to $P_V(R_S)$, and let $f_1, \ldots, f_m$ be the preimages of the coordinate functions on $(\mathbb{R}_+^*)^m$ in $P_V(R_S)$. We can find R_S-polytopes $V_1, \ldots, V_k \subset V$ such that $V = V_1 \cup \cdots \cup V_k$ and $f_i|_{V_j} \in A_{V_j}(R_S)$ for all $1 \le i \le m$ and $1 \le j \le k$. Then the image U_i of each V_i under φ is an R_S-polytope in $(\mathbb{R}_+^*)^m$, which is contained in U, and the induced maps $V_i \to U_i$ are morphisms of R_S-polytopes. □

A continuous map between R_S-polyhedra $\varphi : V' \to V$ is said to be *piecewise R_S-linear* if it possesses the equivalent properties of Lemma 1.2.1.

An (abstract) *R_S-polyhedron* is a topological space X provided with a set of continuous functions P_X for which there exists a homeomorphism $\varphi : X \xrightarrow{\sim} V$, where V is an R_S-polyhedron in $(\mathbb{R}_+^*)^n$, such that φ^* induces a bijection $P_V(R_S) \xrightarrow{\sim} P_X$. A *morphism of R_S-polyhedra* $\varphi : X' \to X$ is a continuous map that takes functions from P_X to functions from $P_{X'}$. A subset Y of an R_S-polyhedron X is said to be an *R_S-polyhedron in X* if the above map φ takes it to an R_S-polyhedron in V. This property of Y does not depend on the choice of φ, and in this case Y is provided with the evident R_S-polyhedron structure.

1.2.2 Lemma. *Let $\varphi : X' \to X$ be a morphism of R_S-polyhedra. Then*

(i) *the image $\varphi(X')$ is an R_S-polyhedron in X;*

(ii) *for any R_S-polyhedron Y in X, $\varphi^{-1}(Y)$ is an R_S-polyhedron in X', and the induced map $\varphi^{-1}(Y) \to Y$ is a morphism of R_S-polyhedra.* □

We say that a morphism of R_S-polyhedra $\varphi : X' \to X$ is an *immersion* if it induces an isomorphism $X' \xrightarrow{\sim} \varphi(X')$.

1.2.3 Lemma. *The following properties of a morphism of R_S-polyhedra $\varphi : X' \to X$ are equivalent:*

(a) *φ is an isomorphism (resp. an immersion);*

(b) *for every R_S-polyhedron Y in X, the induced morphism $\varphi^{-1}(Y) \to Y$ is an isomorphism (resp. an immersion);*

(c) *there exists a finite covering of X by R_S-polyhedra $\{Y_i\}$ such that the induced morphisms $\varphi^{-1}(Y_i) \to Y_i$ are isomorphisms (resp. immersions).* □

Notice that, if a morphism of R_S-polytopes is an isomorphism (resp. immersion) as a morphism of R_S-polyhedra, then it is an isomorphism (resp. immersion) as a morphism of R_S-polytopes.

1.3 Piecewise R_S-linear spaces

Let X be a locally compact space. (All locally compact spaces are assumed to be Hausdorff.) An *R_S-polyhedron chart* on X is a compact subset $V \subset X$ provided with an R_S-polyhedron structure. Two charts U and V are said to be *compatible* if $U \cap V$ is an R_S-polyhedron in U as well as in V, and the R_S-polyhedron structures on it induced from U and V are the same. An *piecewise R_S-linear atlas* on X is a family τ of compatible R_S-polyhedron charts with the property that every point $x \in X$ has a neighborhood of the form $V_1 \cup \cdots \cup V_n$ with $V_1, \dots, V_n \in \tau$.

Given a piecewise R_S-linear atlas τ on X, we say that an R_S-polyhedron chart on X is *compatible with* τ if it is compatible with every chart from τ. Two piecewise R_S-linear atlases on X are said to be *compatible* if every chart of one atlas is compatible with the other atlas. From Lemma 1.2.3 it follows that, if two R_S-polyhedron charts are compatible with a piecewise R_S-linear atlas, then they are compatible. It follows that compatibility is an equivalence relation on the set of piecewise R_S-linear atlases on X.

A *piecewise R_S-linear space* is a locally compact space X provided with an equivalence class of piecewise R_S-linear atlases. Notice that each equivalence class has a unique maximal atlas. It consists of all R_S-polyhedron charts which are compatible with some (and, therefore, with any) piecewise R_S-linear atlas from the equivalence class. The charts from the maximal atlas will be called *R_S-polyhedra in X*. A function $f : X \to \mathbb{R}_+^*$ is said to be *piecewise R_S-linear* if its restriction to every R_S-polyhedron Y in X is contained in P_Y. The set of such functions on X will be denoted by P_X.

A *morphism of piecewise R_S-linear spaces* is a continuous map $\varphi : X' \to X$ with the following property. There exist piecewise R_S-linear atlases τ on X and τ' on X'

that define the piecewise R_S-linear structures on X and X' and such that for every $V' \in \tau'$ there exists $V \in \tau$ for which $\varphi(V') \subset V$ and the induced map $V' \to V$ is a morphism of R_S-polyhedra. Notice that in this case, for every pair of R_S-polyhedra $V \subset X$ and $V' \subset X$ with $\varphi(V') \subset V$, the induced map $V' \to V$ is a morphism of R_S-polyhedra. It follows that one can compose piecewise R_S-linear morphisms, and so we get a category of piecewise R_S-linear spaces PL_S^R. This category admits finite direct products.

A subset Y of a piecewise R_S-linear space X is said to be a *piecewise R_S-linear subspace* if every point $y \in Y$ has a neighborhood in Y of the form $V_1 \cup \cdots \cup V_n$, where $V_1, \dots, V_n$ are R_S-polyhedra in X. Such a subset Y is locally closed in X, and has a canonical structure of a piecewise R_S-linear space. Given a morphism of piecewise R_S-linear spaces $\varphi : X' \to X$, the preimage of any piecewise R_S-linear subspace of X is a piecewise R_S-linear subspace of X'. If φ is proper, then the image $\varphi(X')$ is a piecewise R_S-linear subspace of X. The morphism φ is said to be an *immersion* if it induces an isomorphism between X' and a piecewise R_S-linear subspace of X.

Let X be a piecewise R_S-linear space. The family of its piecewise R_S-linear subspaces can be considered as a category, and it gives rise to a Grothendieck topology X_{G} generated by the pretopology in which the set of coverings of a piecewise R_S-linear subspace Y consists of families $\{Y_i\}_{i \in I}$ of piecewise R_S-linear subspaces of Y such that every point $y \in Y$ has a neighborhood of the form $Y_{i_1} \cup \cdots \cup Y_{i_n}$ with $y \in Y_{i_1} \cap \cdots \cap Y_{i_n}$. Since all open subsets of X are piecewise R_S-linear subspaces, there is a morphism of sites $X_{\mathrm{G}} \to X$. Moreover, every morphism of piecewise R_S-linear spaces $\varphi : X' \to X$ gives rise to a morphisms of sites $X'_{\mathrm{G}} \to X_{\mathrm{G}}$. The correspondence $Y \mapsto P_Y$ is a sheaf in the Grothendieck topology X_{G}, denoted by $\mathcal{P}_{X_{\mathrm{G}}}$. Its restriction to the usual topology of X will be denoted by $\mathcal{P}_X$. More generally, for any piecewise R_S-linear space X', the correspondence $Y \mapsto \mathrm{Hom}(Y, X')$ is a sheaf of sets on X_{G}.

A morphism of piecewise R_S-linear spaces $\varphi : Y \to X$ is said to be a *G-local immersion* (G stands for Grothendieck topology) if for every point $y \in Y$ there exist R_S-polyhedra $V_1, \dots, V_n \subset Y$ such that $V_1 \cup \cdots \cup V_n$ is a neighborhood of y in Y and all of the induced morphisms $V_i \to X$ are immersions. Notice that a G-local immersion $\varphi : Y \to X$, which induces a homeomorphism of Y with its image in X, is an immersion.

If S' is a sub-semiring of $\mathbb{R}$ that contains S and R' is an S'-submonoid of $\mathbb{R}_+^*$ that contains R, then there is the evident functor $\mathrm{PL}_S^R \to \mathrm{PL}_{S'}^{R'}$. Of course, this functor does not change the underlying topological spaces, but it can change their Grothendieck topology. From Corollary 1.1.2 it follows that the Grothendieck topology is not changed if $S' \subset \overline{S}$ and $R' \subset \overline{R}$.

Let $\{X_i\}_{i \in I}$ be a family of piecewise R_S-linear spaces, and suppose that, for each pair $i, j \in I$, we are given a piecewise R_S-linear subspace $X_{ij} \subset X_i$ and an isomorphism $\nu_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$ so that $X_{ii} = X_i$, $\nu_{ij}(X_{ij} \cap X_{il}) = X_{ji} \cap X_{jl}$ and $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $X_{ij} \cap X_{il}$. In this case one can construct a topological space X obtained by gluing of X_i along X_{ij}. (It is the quotient space $\widetilde{X}/E$, where $\widetilde{X}$ is the disjoint union $\coprod_i X_i$

and E is the equivalence relation on $\widetilde{X}$ defined by the system $\{\nu_{ij}\}$.) Let μ_i denote the induced map $X_i \to X$.

1.3.1 Lemma. *In each of the following cases, there exists a unique piecewise R_S-linear structure on X such that all μ_i are immersions:*

(a) *all X_{ij} are open in X_i and X is Hausdorff;*

(b) *for any $i \in I$, all X_{ij} are closed in X_i and the number of $j \in I$ with $X_{ij} \neq \emptyset$ is finite.*

Furthermore, in the case (a), *all $\mu_i(X_i)$ are open in X and, in the case* (b), *all $\mu_i(X_i)$ are closed in X.*

In the situation of the lemma, X is said to be obtained by *gluing of X_i along X_{ij}*.

Proof. In the case (a), the equivalence relation E is open (see [Bou, Ch. I, §9, n° 6]) and, therefore, all $\mu_i(X_i)$ are open in X. In the case (b), the equivalence relation E is closed (see *loc. cit.*, n° 7) and, therefore, all $\mu_i(X_i)$ are closed in X, μ_i induce homeomorphisms $X_i \xrightarrow{\sim} \mu_i(X_i)$, and X is Hausdorff.

Let τ denote the family of all subsets $V \subset X$ for which there exists $i \in I$ such that $V \subset \mu_i(X_i)$ and $\mu_i^{-1}(V)$ is an R_S-polyhedron in X_i (in this case $\mu_i^{-1}(V)$ is an R_S-polyhedron in X_j for every j with $V \subset \mu_j(X_j)$). The family τ is a piecewise R_S-linear atlas on X and, for the piecewise R_S-linear space structure on X it defines, all μ_i are immersions. That the piecewise R_S-linear structure on X with the latter property is unique is trivial. ◻

1.3.2 Remarks. (i) The definition of a piecewise linear space given in this subsection is an easy version of the definition of a non-Archimedean analytic space in [Ber2]. Both are examples of a global object defined by gluing local objects (affinoid spaces in the former and polyhedra in the latter) which are *closed* subsets. The main difference between our definition and that in [Hud] is in the freeing of the requirement that every point has a neighborhood isomorphic to a polyhedron. The latter property (appropriately adjusted) is established in the following subsection and used in §7 (see also Remark 1.4.5).

(ii) If $R = \{1\}$, then any R_S-polyhedron is a point and any piecewise R_S-linear space is a discrete topological space with the only one piecewise R_S-linear function which takes value 1.

(iii) The piecewise monomial spaces introduced in our work in progress and mentioned in the introduction are glued from certain compact subsets of $\mathbb{R}_+^n$ which are defined by a finite number of inequalities $f(t) \leq g(t)$ with f and g of the form $rt_1^{s_1} \dots t_n^{s_n}$, where s_i are elements of a sub-semiring $S \subset \mathbb{R}$ and r are elements of an S-submonoid $R \subset \mathbb{R}_+$ such that if $0 \in R$ then $S \subset \mathbb{R}_+$.

1.4 An embedding property

1.4.1 Proposition. *Every point of a piecewise R_S-linear space has a compact piecewise R_S-linear neighborhood which admits a piecewise $\overline{R}_{\overline{S}}$-linear isomorphism with an $\overline{R}_{\overline{S}}$-polyhedron.*

The statement is trivial if $R = \{1\}$, and so we assume that $R \neq \{1\}$.

Let X be an R_S-polyhedron in $(\mathbb{R}_+^*)^n$. An *R_S-polytopal subdivision of* X is a finite family τ of R_S-polytopes that cover X and are such that (1) if $V \in \tau$, then all faces of V are contained in τ, and (2) if $U, V \in \tau$, then $U \cap V$ is a face in U and in V. The subdivision τ is a *refinement* of a similar subdivision τ' if each $V \in \tau$ is contained in some $V' \in \tau'$. If τ is a family of subsets of a set and U is a subset of the same set, then $\tau|_U$ denotes the family $\{V \in \tau \mid V \subset U\}$.

1.4.2 Lemma. *Let X be an R_S-polyhedron in $(\mathbb{R}_+^*)^n$, and let σ be a finite family of R_S-polyhedra in X. Then there exists an R_S-polytopal subdivision τ of X such that for every $U \in \sigma$ the following is true:*

(a) *$\tau|_U$ is an R_S-polytopal subdivision of U;*

(b) *if $V \in \tau$, then $U \cap V$ is a face in V.*

Proof. Step 1. *There exists τ that satisfies* (a). Indeed, replacing each polyhedron $U \in \sigma$ by a finite set of R_S-polytopes whose union is U, we may assume that σ consists of R_S-polytopes. We may also assume that σ contains a finite set of R_S-polytopes whose union is X. For each $U \in \sigma$, we fix a finite set $F(U)$ of pairs (f, g) of functions from $A^n(R_S)$ such that $U = \{x \in (\mathbb{R}_+^*)^n \mid f(x) \le g(x)$ for all $(f, g) \in F(U)\}$. Let F be the union of $F(U)$ for all $U \in \sigma$. Then the required R_S-polytopal subdivision τ consists of the polytopes W for which there exist subsets $T \subset \sigma$ and $F_\le, F_\ge \subset F$ with $F_\le \cap F_\ge = \emptyset$ such that W is the set of all points $x \in \bigcap_{U \in T} U$ satisfying the inequalities $f(x) \le g(x)$ for $(f, g) \in F_\le$ and $f(x) \ge g(x)$ for $(f, g) \in F_\ge$ and the equalities $f(x) = g(x)$ for $(f, g) \in F \backslash (F_\le \cup F_\ge)$.

Indeed, let $W(T, F_\le, F_\ge)$ denote the above polytope. Since $W(T', F'_\le, F'_\ge) \cap W(T'', F''_\le, F''_\ge) = W(T' \cup T'', F'_\le \cap F''_\le, F'_\ge \cap F''_\ge)$, it suffices to check that, if $W' = W(T', F'_\le, F'_\ge)$ is contained in $W = W(T, F_\le, F_\ge)$, then W' is a face of W. For this we can replace T' by $T' \cup T$, $F'_\le$ by $F'_\le \cap F_\le$ and $F'_\ge$ by $F'_\ge \cap F_\ge$ and, therefore, we may assume that $T' \supset T$, $F'_\le \subset F_\le$ and $F'_\ge \subset F_\ge$. Since $W(T, F'_\le, F'_\ge)$ is evidently a face of W, we may assume that $F'_\le = F_\le$ and $F'_\ge = F_\ge$. It remains, therefore, to consider the case when $T' = T \cup \{U\}$ for some $U \in \sigma$. In this case, one has $W' = W(T, F_\le, F_\ge \backslash F(U))$, and the latter is evidently a face of W.

Step 2. *If τ satisfies* (a)*, there exists a refinement of τ that satisfies* (b). If $U \in \sigma$ and $V \in \tau$, $U \cap V$ is a union of faces of V. Let $M(V, U)$ denote the set of the faces of V in $V \cap U$ which are maximal by inclusion. For each pair of distinct faces $W_1, W_2 \in M(V, U)$ of V, we fix a hyperplane $L \subset (\mathbb{R}_+^*)^n$ defined by an equation $f(x) = g(x)$ with $f, g \in A^n(R_S)$ and such that $L \cap W_1 = L \cap W_2 = W_1 \cap W_2$

and, for every pair of points $x_1 \in W_1 \backslash W_2$ and $x_2 \in W_2 \backslash W_1$, the interval connecting them intersects L. Let σ' be the union of σ, τ and of $\{L \cap X\}$ for all quadruples (U, V, W_1, W_2) as above. By Step 1, there exists an R_S-polytopal subdivision τ' of X with the property (a) for σ'. *We claim that τ' satisfies the property (b) for σ.* Indeed, suppose there exist $U \in \sigma$ and $V' \in \tau'$ for which there exist two distinct faces $W'_1, W'_2 \in M(V', U)$, and let $V \in \tau$ contain V'. Then W'_1 and W'_2 cannot lie in one face of V in $V \cap U$ because they are maximal among the faces of V' in $V' \cap U$. Thus, there exist two distinct faces $W_1, W_2 \in M(V, U)$ such that $W'_1 \subset W_1$, $W'_2 \subset W_2$, $W'_1 \not\subset W_1 \cap W_2$ and $W'_2 \not\subset W_1 \cap W_2$. Let x_1 and x_2 be points from the interiors of W'_1 and W'_2, respectively, which do not lie in $W_1 \cap W_2$, and let L be the hyperplane associated with (U, V, W_1, W_2). Then L contains a point from the interval connecting x_1 and x_2. Such a point lies in the interior of a face of V' that contains W'_1 and W'_2. Since $\tau'|_{L \cap X}$ is a subdivision of $L \cap X$, it follows that $W'_1, W'_2 \subset L$. This contradicts the equalities $L \cap W_1 = L \cap W_2 = W_1 \cap W_2$. ◻

An R_S-polytopal subdivision τ is said to be *simplicial* if all polytopes from τ are simplices.

1.4.3 Lemma. *If* $\dim_{\overline{S}}(\overline{R}) = 1$, *then any R_S-polytopal subdivision of an R_S-polyhedron $X \subset (\mathbb{R}^*_+)^n$ has an R_S-simplicial refinement with the same set of vertices.*

Proof. The assumption implies that the convex hull of any subset of the set of vertices of an R_S-polytope is an R_S-polytope and, therefore, the proof of the corresponding classical fact (see [RoSa, Proposition 2.9]) is applicable. (The same reasoning will be used in the proof of Lemma 1.4.4 below.) ◻

Proof of Proposition 1.4.1. First of all, we may assume that S is a field and, therefore, R is a vector space over S. It suffices to show that every point x of a piecewise R_S-linear space X, which is a union of two R_S-polyhedra X' and X'', has an R_S-polyhedron neighborhood. Of course, we may assume that $x \in X' \cap X''$. Let R' be a fixed one-dimensional S-vector subspace of R. *We claim that there exists a compact piecewise R_S-linear neighborhood of x, which is isomorphic to a piecewise R'_S-linear space.*

(1) By Lemma 1.4.2, there exists an R_S-polytopal subdivision τ' of X' with the properties (a) and (b) for $\sigma = \{X' \cap X''\}$. Furthermore, we can find an R_S-polytopal subdivision τ'' of X'' with the properties (a) and (b) for $\sigma = \tau'|_{X' \cap X''}$.

(2) Let W be the minimal polytope from τ'' that contains the point x, and let τ be the family of all polytopes from $\tau' \cup \tau''$ that contain W. (Notice that τ is preserved under intersections.) Then $\bigcup_{V \in \tau} V$ is a neighborhood of x in X. The point x lies in the interior $\mathring{W}$ of W. Let x_0 be a fixed R-point in $\mathring{W}$. We say that a point y from the above union is *marked* if for some (and therefore any) $V \in \tau$ with $y \in V$ one has $\frac{f(y)}{f(x_0)} \in R'$ for all $f \in A_V(R_S)$. A polytope in $V \in \tau$ is said to be *special* if it contains the point x_0 and all its vertices are marked points, and a polyhedron in V is said to be *special* if it is a finite union of special polytopes. Notice that a line in

V passing through two different marked points is S-rational and, by Lemma 1.1.1, special polytopes are R_S-polytopes, and special polyhedra are R_S-polyhedra. Notice also that a polytope, which is special as a polyhedron, is special as a polytope.

(3) We are going to construct for every $V \in \tau$ a special polyhedron $\widetilde{V} \subset V$, which is a neighborhood of the point x in V and such that if $U \in \tau$ and $U \subset V$ then $\widetilde{U} = \widetilde{V} \cap U$. The construction is made inductively and, at the beginning, for polytopes from $\tau \cap \tau''$. First of all, since the set of marked points is dense in $\mathring{W}$, we can find a special polytope $\widetilde{W} \subset W$, which is a neighborhood of the point x in W and is contained in $\mathring{W}$. Let V be a bigger polytope from $\tau \cap \tau''$, and assume that $\widetilde{U}$ are already constructed for all $U \in \tau \cap \tau''$ with $U \subset \dot{V}$, where $\dot{V} = V \backslash \mathring{V}$ is the boundary of V. Then the polyhedron $V_1 = \cup \widetilde{U}$, where the union is taken over all $U \in \tau \cap \tau''$ with $U \subset \dot{V}$, is a neighborhood of the point x in $\dot{V}$. We take an arbitrary marked point $y \in \mathring{V}$ and define $\widetilde{V}$ as the join of y and V_1 in V (i.e., the set $\{\lambda y + \mu z\}$, where $z \in V_1$, $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$). After the special polyhedra $\widetilde{V}$ are constructed for all $V \in \tau \cap \tau''$, we continue the same construction for polytopes $V \in \tau \cap \tau'$. Namely, assume first that $V \subset X' \cap X''$. Then V is a union of some $U \in \tau''$, and we define $\widetilde{V}$ as the union $\cup \widetilde{U}$, taken over all $U \in \tau \cap \tau''$ with $U \subset V$. Assume now that V is minimal among those polytopes from $\tau \cap \tau'$ that contain a point from $X \backslash X''$. Then the intersection $V'' = V \cap X''$ is a face of V of smaller dimension. It follows that the special polyhedron $\widetilde{V}''$ is a neighborhood of the point x in the boundary $\dot{V}$ of V. We take an arbitrary special point $y \in \mathring{V}$ and define $\widetilde{V}$ as the join of y and $\widetilde{V}''$ in V. If a polytope $V \in \tau \cap \tau'$ is not minimal among those, that contain a point from $X \backslash X''$, and the special polyhedra $\widetilde{U}$ are constructed for all $U \in \tau \cap \tau'$ with $U \subset \dot{V}$, we denote by V_1 the union of the corresponding $\widetilde{U}$'s and define $\widetilde{V}$ as the join of some special point $y \in \mathring{V}$ and V_1.

(4) The union $Y = \bigcup_{V \in \tau} \widetilde{V}$ is a compact piecewise R_S-linear neighborhood of the point x in X. *We claim that Y is isomorphic to a piecewise R'_S-linear space.* Indeed, assume that $V \in \tau$ is an R_S-polytope in $(\mathbb{R}^*_+)^n$, and let the coordinates of the point x_0 be $(\alpha_1, \ldots, \alpha_n)$. Then the automorphism φ of $(\mathbb{R}^*_+)^n : (y_1, \ldots, y_n) \mapsto (\frac{y_1}{\alpha_1}, \ldots, \frac{y_n}{\alpha_n})$ takes marked points to R'-points and, therefore, it takes every special polytope U in V to an R'_S-polytope $\varphi(U)$ in $\varphi(V)$. Moreover, φ induces a bijection between $A_{\varphi(U)}(R'_S)$ and the subspace of $A_U(R_S)$ consisting of functions of the form $y \mapsto r' \frac{f(y)}{f(x_0)}$ with $r' \in R'$ and $f \in A_V(R_S)$. It follows that this R'_S-polytope structure on U does not depend on the embedding of V in a vector space, and it gives rise to R'_S-polyhedron structures on special polyhedra in V. Moreover, if $V', V'' \in \tau$, then the R'_S-polyhedron structures on special polyhedra in $V' \cap V''$, induced from V' and V'', are compatible. In this way we get a piecewise R'_S-linear structure on Y.

The proposition now follows from the following lemma, which is a straightforward generalization of the classical result for $S = \mathbb{R}$ and $R = \mathbb{R}^*_+$.

1.4.4 Lemma. *If S is a field and $\dim_S(R) = 1$, then any compact piecewise R_S-linear space is isomorphic to an R_S-polyhedron.*

Proof. It suffices to show that a compact piecewise R_S-linear space X, which is a union of two R_S-polyhedra X' and X'', is isomorphic to an R_S-polyhedron.

(A) An R_S*-polytope chart* on X is a compact subset $V \subset X$ provided with an R_S-polytope structure which gives rise to an R_S-polyhedron in X. Two R_S-polytope charts U and V are said to be *compatible* if $U \cap V$ is an R_S-polytope in U as well as in V, and the R_S-polytope structures on it induced from U and V are the same. *We claim that X can be covered by a finite family τ of R_S-simplex charts such that* (1) *if $V \in \tau$, then all faces of V are contained in τ, and* (2) *if $U, V \in \tau$, then $U \cap V$ is a face in U and in V.* Indeed, by Lemma 1.4.2, there exists an R_S-polytopal subdivision τ' of X' with the properties (a) and (b) for $\sigma = \{X' \cap X''\}$, and we can find an R_S-polytopal subdivision τ'' of X'' with the properties (a) and (b) for $\sigma = \tau'|_{X' \cap X''}$. Since $\dim_S(R) = 1$, we may apply Lemma 1.4.3 and assume that τ'' is simplicial. Let $V_1, \ldots, V_m$ be all of the polytopes from τ', which are not contained in $X' \cap X''$ and such that if V_i is a face of V_j then $i \leq j$. We set $Y_1 = X''$ and $Y_{i+1} = Y_i \cup V_i$, and provide as follows each Y_i with a family of R_S-simplex charts τ_i possessing the properties (1) and (2) and such that $\tau_1 = \tau''$ and $\tau_{i+1}|_{Y_i} = \tau_i$ for all $1 \leq i \leq m$. For this we fix an ordering of the set of the vertices in τ' outside $X' \cap X''$, and assume that, for some $1 \leq i \leq m$, τ_i is already constructed. If x is the first vertex of V_i outside $X' \cap X''$, we define τ_{i+1} as consisting of all simplices from τ_i and the joins of x and $U \in \tau_i$ with $U \subset \dot{V}_i$. (The latter are R_S-simplices since $\dim_S(R) = 1$.) The family $\tau = \tau_{m+1}$ on $Y_{m+1} = X$ is the required one.

(B) Let $\{x_1, \ldots, x_{n+1}\}$ be the set of all vertices in τ, and let $\{y_1, \ldots, y_{n+1}\}$ be a set of independent R-points in $(\mathbb{R}_+^*)^n$. For a simplex $V \in \tau$, let $\varphi(V)$ be the R_S-simplex in $(\mathbb{R}_+^*)^n$, which is the convex hull of those points from $\{y_1, \ldots, y_{n+1}\}$ which corresponds to the vertices of V. Then the correspondence $x_i \mapsto y_i$ gives rise to an isomorphism between X and the R_S-polyhedron which is the union of all $\varphi(V)$ with $V \in \tau$. ◻

1.4.5 Remarks. (i) It is not true in general that every point of a piecewise R_S-linear space has a compact piecewise R_S-linear neighborhood isomorphic to an R_S-polyhedron. For example, assume that $S = \mathbb{Z}_+$ and R is an arbitrary submonoid of $\mathbb{R}_+^*$ that contains a number $0 < r < 1$, and let W be the triangle in $(\mathbb{R}_+^*)^2$ defined by the inequalities $t_1 \leq 1$ and $r \leq t_2 \leq t_1$. If U_1 and U_2 are the edges of W defined by the equalities $t_2 = r$ and $t_1 = t_2$, respectively, there is an isomorphism $U_1 \stackrel{\sim}{\to} U_2$ that takes a point (t_1, r) to the point (t_1, t_1), and it defines an involutive automorphism φ of $V = U_1 \cup U_2$. Let X be the piecewise $R_{\mathbb{Z}_+}$-linear space obtained by gluing of two copies of W along the isomorphism φ of V (see Lemma 1.3.1). Then the point $x = (r, r)$ has no an $R_{\mathbb{Z}_+}$-polyhedron neighborhood in X. Indeed, let f be a piecewise $R_{\mathbb{Z}_+}$-linear function in a neighborhood of x in X. The preimage of the neighborhood in W contains a triangle W' defined in W by the inequality $t_1 \leq r'$ for some $r' \in \overline{R}$ with $r < r' < 1$, and one has $f(y) = f(\varphi(y))$ for all $y \in W' \cap U_1$. But the restriction of f to a vertical interval in W' (defined by the equality $t_1 = \alpha$ for $r < \alpha \leq r'$) is nondecreasing as a function on t_2. It follows that the restriction of

f to each vertical interval is constant. Since piecewise $R_{\mathbb{Z}_+}$-linear functions separate points of an $R_{\mathbb{Z}_+}$-polyhedron, the point x has no an $R_{\mathbb{Z}_+}$-polyhedron neighborhood.

(ii) Although Proposition 1.4.1 is enough for an application in §7, it would be interesting to know if its statement is true for $\widetilde{S}$ and $\widetilde{R}$ (instead of $\overline{S}$ and $\overline{R}$), and if Lemma 1.4.4 is true without the assumption $\dim_S(R) = 1$.

2 $\mathcal{R}$-colored polysimplicial sets

2.1 Categories with a geometric realization functor

Given a topological space X, the set of all non-negative real valued functions on X forms a semiring with respect to the usual multiplication and the following addition: $f \dot{+} g = \max(f, g)$. We denote by $\mathcal{T}op^{\mathrm{sr}}$ the category of the pairs (X, M) consisting of a topological space X and a semiring M of continuous functions on X with values in $[0, 1]$ such that $1 \in M$. The set of morphisms $\mathrm{Hom}((X', M'), (X, M))$ consists of the continuous maps $X' \to X$ that take functions from M to M'. The category $\mathcal{T}op^{\mathrm{sr}}$ admits direct limits.

Let $\mathcal{R}$ be a small category provided with a functor $\mathcal{R} \to \mathcal{T}op^{\mathrm{sr}} : A \mapsto (|A|, M_A)$ (which will be called a *geometric realization functor*). In this section we introduce certain categories which are related to $\mathcal{R}$ and also provided with a geometric realization functor. The first example is the category $\mathcal{R}^\circ\mathcal{E}ns$ of contravariant functors from $\mathcal{R}$ to the category of sets $\mathcal{E}ns$. The category $\mathcal{R}$ can be considered as its full subcategory under the fully faithful functor $\mathcal{R} \to \mathcal{R}^\circ\mathcal{E}ns : A \mapsto \mathcal{R}_A$ that takes an object to the contravariant functor represented by it. The geometric realization functor $\mathcal{R}^\circ\mathcal{E}ns \to \mathcal{T}op^{\mathrm{sr}} : C \mapsto (|C|, M_C)$ is the one that extends $\mathcal{R} \to \mathcal{T}op^{\mathrm{sr}}$ to the functor which commutes with direct limits. For an object $A \in \mathrm{Ob}(\mathcal{R})$ and an element $c \in C_A$, where C_A is the value of C at A, we denote by σ_c the corresponding map $|A| \to |C|$.

2.2 The category $\Lambda_{\mathcal{R}}$

Recall the definition of the category Λ from [Ber7, §3]. First of all, for a tuple $\boldsymbol{n} = (n_0, \dots, n_p)$ with either $p = n_0 = 0$ or $p \ge 0$ and $n_i \ge 1$ for all $0 \le i \le p$, let $[\boldsymbol{n}]$ denote the set $[n_0] \times \cdots \times [n_p]$, where $[n] = \{0, 1, \dots, n\}$. The set $[\boldsymbol{n}] \in \mathrm{Ob}(\Lambda)$ is endowed with a metric as follows. The distance between two elements $\boldsymbol{i}$ and $\boldsymbol{j}$ of $[\boldsymbol{n}]$ is the number of distinct coordinates of $\boldsymbol{i}$ and $\boldsymbol{j}$. Objects of the category Λ are the sets $[\boldsymbol{n}]$ for the tuples $\boldsymbol{n}$ as above, and morphisms are isometric maps. By [Ber7, Lemma 3.1], each isometric map $\gamma : [\boldsymbol{n}'] \to [\boldsymbol{n}]$ can be described as follows. First of all, we set $\omega(\boldsymbol{n}) = [p]$, if $[\boldsymbol{n}] \ne [0]$, and $\omega(\boldsymbol{n}) = \emptyset$, otherwise. Then there is a pair (f, α) consisting of an injective map $f : \omega(\boldsymbol{n}') \to [p]$ and $\alpha = \{\alpha_i\}_{0 \le i \le p}$, where α_i is an injective map $[n'_{f^{-1}(i)}] \to [n_i]$ for $i \in \mathrm{Im}(f)$, and is a map $[0] \to [n_i]$

for $i \notin \mathrm{Im}(f)$. The map γ takes an element $\boldsymbol{i}' = (i'_0, \dots, i'_{p'}) \in [\boldsymbol{n}']$ to the element $\boldsymbol{i} = (i_0, \dots, i_p) \in [\boldsymbol{n}]$ with $i_j = \alpha_j(i'_{f^{-1}(j)})$ for $j \in \mathrm{Im}(f)$, and $i_j = \alpha_j(0)$ for $j \notin \mathrm{Im}(f)$. It follows that, for every subset $J \subset \omega(\boldsymbol{n})$, the morphism $\gamma : [\boldsymbol{n}'] \to [\boldsymbol{n}]$ gives rise to a morphism $[\boldsymbol{n}'_{f^{-1}(J)}] \to [\boldsymbol{n}_J]$, where $\boldsymbol{n}_J$ denotes the tuple $(n_{j_0}, \dots, n_{j_t})$ if $J = \{j_0, \dots, j_t\}$ is non-empty and $j_0 < \dots < j_t$, and the zero tuple 0, otherwise.

Assume we are given a category $\mathcal{R}$ and a functor $\mathcal{R} \to \mathcal{T}op^{\mathrm{sr}} : A \mapsto (|A|, M_A)$ (as in §2.1). We introduce as follows a category $\Lambda_{\mathcal{R}}$, whose objects are denoted by $[\boldsymbol{n}]_{A,\boldsymbol{r}}$, and a functor $\Lambda_{\mathcal{R}} \to \mathcal{T}op^{\mathrm{sr}} : [\boldsymbol{n}]_{A,\boldsymbol{r}} \mapsto (\Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}}, M^{\boldsymbol{n}}_{A,\boldsymbol{r}})$. First of all, the objects $[\boldsymbol{n}]_{A,\boldsymbol{r}}$ correspond to the following data: $[\boldsymbol{n}] = [n_0] \times \dots \times [n_p] \in \mathrm{Ob}(\Lambda)$, $A \in \mathrm{Ob}(\mathcal{R})$ and $\boldsymbol{r} = (r_0, \dots, r_p) \in M_A^{p+1}$, which satisfy the condition that $r_0 = 1$, if $[\boldsymbol{n}] = [0]$, and $r_i \neq 1$ for all $0 \le i \le p$, if $[\boldsymbol{n}] \neq [0]$. Given an object $[\boldsymbol{n}]_{A,\boldsymbol{r}}$ and a morphism $\psi : A' \to A$, let $J(\psi, \boldsymbol{r})$ denote the set of all $j \in \omega(\boldsymbol{n})$ with $r_j(x) < 1$ for some $x \in \mathrm{Im}(|\psi|)$, where $|\psi|$ is the map $|A'| \to |A|$. A morphism $[\boldsymbol{n}']_{A',\boldsymbol{r}'} \to [\boldsymbol{n}]_{A,\boldsymbol{r}}$ is a pair consisting of a morphism $\psi : A' \to A$ in $\mathcal{R}$ and a morphism $\gamma : [\boldsymbol{n}'] \to [\boldsymbol{n}_J]$ in Λ, where $J = J(\psi, \boldsymbol{r})$, which satisfy the following condition: if γ is associated with a pair (f, α) as above, then $r'_j = |\psi|^*(r_{f(j)})$ for all $j \in \omega(\boldsymbol{n}')$.

Furthermore, we set

$$\Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}} = \{(x, \boldsymbol{t}) \in |A| \times [0,1]^{[\boldsymbol{n}]} \mid t_{i0} \dots t_{in_i} = r_i(x),\ 0 \le i \le p\}$$

and denote by $M^{\boldsymbol{n}}_{A,\boldsymbol{r}}$ the semiring of continuous functions on $\Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}}$ generated by all functions from M_A and the coordinate functions $\boldsymbol{t} \mapsto t_{ij}$. Given a morphism $(\gamma, \psi) : [\boldsymbol{n}']_{A',\boldsymbol{r}'} \to [\boldsymbol{n}]_{A,\boldsymbol{r}}$ as above, the corresponding map $\Sigma^{\boldsymbol{n}'}_{A',\boldsymbol{r}'} \to \Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}}$ takes a point $(x', \boldsymbol{t}')$ to the point $(x, \boldsymbol{t})$, where $x = |\psi|(x')$ and (a) if $i \notin J(\psi, \boldsymbol{r})$, then $t_{ij} = 1$ for all $0 \le j \le n_i$, (b) if $i \in J(\psi, \boldsymbol{r}) \backslash \mathrm{Im}(f)$, then $t_{ij} = r_i(x)$ for $j = \alpha_i(0)$ and $t_{ij} = 1$ for $j \neq \alpha_i(0)$, and (c) if $i \in \mathrm{Im}(f)$, then $t_{ij} = t'_{f^{-1}(i),\alpha_i^{-1}(j)}$ for $j \in \mathrm{Im}(\alpha_i)$ and $t_{ij} = 1$ for $j \notin \mathrm{Im}(\alpha_i)$. In this way we get a geometric realization functor $\Lambda_{\mathcal{R}} \to \mathcal{T}op^{\mathrm{sr}}$.

2.3 Connections between the categories $\Lambda_{\mathcal{R}}$ and $\mathcal{R}$

First of all, there is a fully faithful functor $\mathcal{R} \to \Lambda_{\mathcal{R}} : A \mapsto [0]_{A,1}$ and a functor $\Lambda_{\mathcal{R}} \to \mathcal{R} : [\boldsymbol{n}]_{A,\boldsymbol{r}} \mapsto A$. The latter makes $\Lambda_{\mathcal{R}}$ a fibered category over the category $\mathcal{R}$ in the sense of [SGA1, Exp. VI] and can be seen using the following general construction.

Let $\mathcal{R}'$ be another small category provided with a functor $\mathcal{R}' \to \mathcal{T}op^{\mathrm{sr}} : A' \mapsto (|A'|, M_{A'})$, and assume we are given a functor $\mathcal{R}' \to \mathcal{R} : A' \mapsto A$ and a morphism of functors from $\mathcal{R}'$ to $\mathcal{T}op^{\mathrm{sr}}$: $(|A'|, M_{A'}) \stackrel{h_A}{\to} (|A|, M_A)$. Then one can define a functor

$$\Lambda_{\mathcal{R}} \times_{\mathcal{R}} \mathcal{R}' \to \Lambda_{\mathcal{R}'} : ([\boldsymbol{n}]_{A,\boldsymbol{r}}, A') \mapsto [\boldsymbol{n}']_{A',\boldsymbol{r}'} ,$$

where $\boldsymbol{n}' = \boldsymbol{n}_J$, $\boldsymbol{r}' = h_A^*(\boldsymbol{r}_J)$ and $J = \{j \in \omega(\boldsymbol{n}) \mid r_j(x) < 1 \text{ for some } x \in \mathrm{Im}(h_A)\}$. (The truncation $\boldsymbol{r}_J$ has the same meaning as $\boldsymbol{n}_J$.) Notice that there is an isomorphism of functors from $\Lambda_{\mathcal{R}} \times_{\mathcal{R}} \mathcal{R}'$ to $\mathcal{T}op^{\mathrm{sr}}$: $\Sigma^{\boldsymbol{n}'}_{A',\boldsymbol{r}'} \stackrel{\sim}{\to} \Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}} \times_{|A|} |A'|$.

2.3.1 Examples. (i) Given an object $A \in \mathrm{Ob}(\mathcal{R})$, let $\{A\}$ denote the category consisting of one object A (with only identity morphism) provided with the following functor to $\mathcal{T}op^{\mathrm{sr}}$: $A \mapsto (|A|, M_A)$. The above construction, applied to the canonical functor $\{A\} \to \mathcal{R}$, gives an equivalence of categories $\Lambda_{\mathcal{R}} \times_{\mathcal{R}} \{A\} \stackrel{\sim}{\to} \Lambda_{\{A\}}$.

(ii) Given a morphism $\psi : A' \to A$ in $\mathcal{R}$, the above construction, applied to the functor $\{A'\} \to \mathcal{R} : A' \mapsto A$ and the morphism $(|A'|, M_{A'}) \stackrel{|\psi|}{\to} (|A|, M_A)$, gives the inverse image functor $\psi^* : \Lambda_{\mathcal{R}} \times_{\mathcal{R}} \{A\} \to \Lambda_{\mathcal{R}} \times_{\mathcal{R}} \{A'\}$ that makes $\Lambda_{\mathcal{R}}$ a fibered category over $\mathcal{R}$.

(iii) Given an object $A \in \mathrm{Ob}(\mathcal{R})$ and a point $x \in |A|$, let $\{x\}$ denote the category consisting of one object x (with only identity morphism) provided with the functor to $\mathcal{T}op^{\mathrm{sr}}$: $x \mapsto (x, M_x)$, where $M_x = \{f(x) \mid f \in M_A\}$. The above construction, applies to the functor $\{x\} \mapsto \mathcal{R} : x \mapsto A$ and the canonical morphism $(x, M_x) \to (|A|, M_A)$, gives a functor $\Lambda_{\mathcal{R}} \times_{\mathcal{R}} \{A\} \to \Lambda_{\{x\}}$.

Recall that one can associate with each small category $\mathcal{L}$ a partially ordered set $O(\mathcal{L})$ (see [GaZi, Ch. II, §5.1]). Namely, it is the partially ordered set associated with the set $\mathrm{Ob}(\mathcal{L})$ provided with the following partial preorder structure: $C \le D$ if there is a morphism $C \to D$. As a set, $O(\mathcal{L})$ is the set of equivalence classes in $\mathrm{Ob}(\mathcal{L})$ with respect to the following equivalence relation: $C \sim D$ if there are morphisms $C \to D$ and $D \to C$. The partially ordered set $O(\mathcal{L})$ can be considered as a category so that the map $\mathrm{Ob}(\mathcal{L}) \to O(\mathcal{L})$ is the underlying map of the evident functor $\mathcal{L} \to O(\mathcal{L})$. A functor $O(\mathcal{L}) \to \mathcal{L}$, whose composition with the latter is the identity functor on $O(\mathcal{L})$, will be said to be a *section* of $\mathcal{L} \to O(\mathcal{L})$.

The following simple lemma describes the partially ordered set $O([\boldsymbol{n}]_{A,\boldsymbol{r}})$, associated with the category $\Lambda_{\mathcal{R}}/[\boldsymbol{n}]_{A,\boldsymbol{r}}$, in terms of the partially ordered set $O(A)$, associated with the category $\mathcal{R}/A$. First of all, we notice that, given $[\boldsymbol{n}]_{A,\boldsymbol{r}}$ and two morphisms $A'' \stackrel{\varphi}{\to} A' \stackrel{\psi}{\to} A$, one has $J(\psi \circ \varphi, \boldsymbol{r}) \subset J(\psi, \boldsymbol{r})$ and, in particular, the subset $J(\psi, \boldsymbol{r})$ depends only on the equivalence class of ψ in $\mathrm{Ob}(\mathcal{R}/A)$. We also say that a non-empty subset $C \subset [\boldsymbol{n}] = [n_0] \times \cdots \times [n_p]$ is of the direct product type if $C = C_0 \times \cdots \times C_p$, where C_i is the image of C under the canonical projection $[\boldsymbol{n}] \to [n_i]$.

2.3.2 Lemma.

(i) *There is a one-to-one correspondence between* $O([\boldsymbol{n}]_{A,\boldsymbol{r}})$ *and the set of pairs* (ψ, C) *consisting of an element* $\psi \in O(A)$ *and a subset* $C \subset [\boldsymbol{n}_J]$ *of the direct product type, where* $J = J(\psi, \boldsymbol{r})$*;*

(ii) $(\psi', C') \le (\psi'', C'')$ *if and only if* $\psi' \le \psi''$ *and* C' *is contained in the image of* C'' *under the canonical projection* $[\boldsymbol{n}_{J''}] \to [\boldsymbol{n}_{J'}]$*, where* $J' = J(\psi', \boldsymbol{r})$ *and* $J'' = J(\psi'', \boldsymbol{r})$*;*

(iii) *any section* $O(A) \to \mathcal{R}/A$ *of the functor* $\mathcal{R}/A \to O(A)$ *can be lifted to a section* $O([\boldsymbol{n}]_{A,\boldsymbol{r}}) \to \Lambda_{\mathcal{R}}/[\boldsymbol{n}]_{A,\boldsymbol{r}}$ *of the functor* $\Lambda_{\mathcal{R}}/[\boldsymbol{n}]_{A,\boldsymbol{r}} \to O([\boldsymbol{n}]_{A,\boldsymbol{r}})$.

Proof. Given a morphism $\psi : A' \to A$ in $\mathcal{R}$, let $J = J(\psi, \boldsymbol{r}) = \{0 \le j_0 < \cdots < j_q \le p\}$. For a non-empty subset $C = C_0 \times \cdots \times C_q \subset [\boldsymbol{n}_J]$, let $\boldsymbol{n}^C$ be the tuple consisting of the numbers $\#C_i - 1$ that are greater than zero, and let $\boldsymbol{r}^C$ be the corresponding subtuple of $\boldsymbol{r}$. Then the canonical injective maps $C_i \to [n_{j_i}]$ define a morphism $\mu_{\psi,C} : [\boldsymbol{n}^C]_{A',\boldsymbol{r}^C} \to [\boldsymbol{n}]_{A,\boldsymbol{r}}$ in $\Lambda_{\mathcal{R}}$. It is easy to see that, when ψ runs through a system of representatives of $O(A)$ in $\mathrm{Ob}(\mathcal{R}/A)$, the morphisms $\mu_{\psi,C}$ run through a system of representatives of $O([\boldsymbol{n}]_{A,\boldsymbol{r}})$ in $\mathrm{Ob}(\Lambda/[\boldsymbol{n}]_{A,\boldsymbol{r}})$, i.e., (i) is true. The statements (ii) and (iii) also easily follow from the construction. □

Assume that $\mathcal{R}$ has a structure of a symmetric strict monoidal category, i.e., there is a multiplication bifunctor $\mathcal{R} \times \mathcal{R} \stackrel{\square}{\to} \mathcal{R} : (A', A'') \mapsto A'\square A''$ which satisfies certain conditions (see [Mac, Ch. VII]). Assume also that the canonical morphisms of partially ordered sets $O(A') \times O(A'') \to O(A'\square A'')$ are isomorphisms, and that there is an isomorphism of functors from $\mathcal{R} \times \mathcal{R}$ to $\mathcal{T}op^{\mathrm{sr}}$: $(|A'|, M_{A'}) \times (|A''|, M_{A''}) \stackrel{\sim}{\to} (|A'\square A''|, M_{A'\square A''})$. Then this structure is naturally extended to the category $\Lambda_{\mathcal{R}}$ and the same properties also hold. Namely, the multiplication bifunctor $\Lambda_{\mathcal{R}} \times \Lambda_{\mathcal{R}} \stackrel{\square}{\to} \Lambda_{\mathcal{R}} : ([\boldsymbol{n}']_{A',\boldsymbol{r}'}, [\boldsymbol{n}'']_{A'',\boldsymbol{r}''}) \mapsto [\boldsymbol{n}]_{A,\boldsymbol{r}} = [\boldsymbol{n}']_{A',\boldsymbol{r}'}\square[\boldsymbol{n}'']_{A'',\boldsymbol{r}''}$ is defined as follows: $A = A'\square A''$ and (a) $\boldsymbol{n} = \boldsymbol{n}'$ and $\boldsymbol{r} = \boldsymbol{r}'$, if $[\boldsymbol{n}''] = [0]$, (b) $\boldsymbol{n} = \boldsymbol{n}''$ and $\boldsymbol{r} = \boldsymbol{r}''$, if $[\boldsymbol{n}'] = [0]$, and (c) $\boldsymbol{n} = (n'_0, \ldots, n'_{p'}, n''_0, \ldots, n''_{p''})$ and $\boldsymbol{r} = (r'_0, \ldots, r'_{p'}, r''_0, \ldots, r''_{p''})$, otherwise. The first property follows from Lemma 2.3.2, and the second one follows from the definition (of $(\Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}}, M^{\boldsymbol{n}}_{A,\boldsymbol{r}})$).

2.4 $\mathcal{R}$-colored polysimplicial sets

The category of *$\mathcal{R}$-colored polysimplicial sets* is the category $\Lambda^\circ_{\mathcal{R}}\mathcal{E}ns$. By §2.1, there is a geometric realization functor $\Lambda^\circ_{\mathcal{R}}\mathcal{E}ns \to \mathcal{T}op^{\mathrm{sr}} : D \mapsto (|D|, M_D)$ which commutes with direct limits and extends the functor $[\boldsymbol{n}]_{A,\boldsymbol{r}} \mapsto (\Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}}, M^{\boldsymbol{n}}_{A,\boldsymbol{r}})$.

The functor representable by an object $[\boldsymbol{n}]_{A,\boldsymbol{r}} \in \mathrm{Ob}(\Lambda_{\mathcal{R}})$ is denoted by $\Lambda[\boldsymbol{n}]_{A,r}$ and, for $D \in \mathrm{Ob}(\Lambda^\circ_{\mathcal{R}}\mathcal{E}ns)$, the image of $[\boldsymbol{n}]_{A,\boldsymbol{r}}$ under D is denoted by $D^{\boldsymbol{r}}_{A,\boldsymbol{n}}$. One evidently has $\mathrm{Hom}(\Lambda[\boldsymbol{n}]_{A,r}, D) \stackrel{\sim}{\to} D^{\boldsymbol{r}}_{A,\boldsymbol{n}}$ and, therefore, there is a canonical bijection between the set $\coprod D^{\boldsymbol{r}}_{A,\boldsymbol{n}}$ of polysimplices of D and the set of objects of the category $\Lambda_{\mathcal{R}}/D$. In particular, there is an equivalence relation on the set of polysimplices of D, and the set of equivalence classes is provided with a partial ordering. It is denoted by $O(D)$. Notice that $O(\Lambda[\boldsymbol{n}]_{A,\boldsymbol{r}})$ coincides with the partially ordered set $O([\boldsymbol{n}]_{A,\boldsymbol{r}})$ considered in §2.3. The correspondence $D \mapsto O(D)$ is a functor from $\Lambda^\circ_{\mathcal{R}}\mathcal{E}ns$ to the category of partially ordered sets $\mathcal{O}r$, and this functor commutes with direct limits (cf. [Ber7, 3.3]).

There is a fully faithful functor $\Lambda_{\mathcal{R}^\circ\mathcal{E}ns} \to \Lambda^\circ_{\mathcal{R}}\mathcal{E}ns : [\boldsymbol{n}]_{C,\boldsymbol{r}} \mapsto \Lambda[\boldsymbol{n}]_{C,\boldsymbol{r}}$ which commutes with direct limits and extends the functor $\Lambda_{\mathcal{R}} \to \Lambda^\circ_{\mathcal{R}}\mathcal{E}ns : [\boldsymbol{n}]_{A,\boldsymbol{r}} \mapsto \Lambda[\boldsymbol{n}]_{A,\boldsymbol{r}}$. Namely, $\Lambda[\boldsymbol{n}]_{C,\boldsymbol{r}}$ is the polysimplicial set D with the property that, for $[\boldsymbol{m}]_{A,\boldsymbol{s}} \in \mathrm{Ob}(\Lambda_{\mathcal{R}})$, $D^{\boldsymbol{s}}_{A,\boldsymbol{m}}$ is the set of pairs consisting of an element $c \in C_A$ and a morphism $\gamma = (f, \alpha) : [\boldsymbol{m}] \to [\boldsymbol{n}_I]$ in Λ with $I = I(c, \boldsymbol{r}) = \{i \in \omega(\boldsymbol{n}) \mid r_i(x) < 1$ for some $x \in \mathrm{Im}(\sigma_c)\}$ such that $s_j = \sigma_c^*(r_{f(j)})$ for all $j \in \omega(\boldsymbol{m})$.

2.4.1 Lemma. *There is a canonical isomorphism of functors from $\Lambda_{\mathcal{R}^\circ\mathcal{E}ns}$ to $\mathcal{T}op^{\mathrm{sr}}$:*

$$(|\Lambda[\boldsymbol{n}]_{C,\boldsymbol{r}}|, M_{\Lambda[\boldsymbol{n}]_{C,\boldsymbol{r}}}) \stackrel{\sim}{\to} (\Sigma^{\boldsymbol{n}}_{C,\boldsymbol{r}}, M^{\boldsymbol{n}}_{C,\boldsymbol{r}}) \ .$$

Proof. If $\boldsymbol{n} = (n_0, \dots, n_p)$ and $\boldsymbol{r} = (r_0, \dots, r_p)$, then

$$\Sigma^{\boldsymbol{n}}_{C,\boldsymbol{r}} = \{(x, \boldsymbol{t}) \in |C| \times [0,1]^{[\boldsymbol{n}]} \mid t_{i0} \dots t_{in_i} = r_i(x),\ 0 \le i \le p\}$$

and $M^{\boldsymbol{n}}_{C,\boldsymbol{r}}$ is the semiring generated by M_C and the coordinate functions t_{ij}. On the other hand, there are canonical isomorphisms $\varinjlim \mathcal{R}_A \stackrel{\sim}{\to} C$ and $\varinjlim \Lambda[\boldsymbol{n}_I]_{A,\boldsymbol{r}_I} \stackrel{\sim}{\to} D$, where both limits are taken over the category $\mathcal{R}/C$ (whose objects are morphisms $\mathcal{R}_A \stackrel{c}{\to} C$) and $I = I(c, \boldsymbol{r})$. The required isomorphism is defined by the canonical maps $\Sigma^{\boldsymbol{n}_I}_{A,\boldsymbol{r}_I} \to \Sigma^{\boldsymbol{n}}_{C,\boldsymbol{r}}$ that take a point $(x, \boldsymbol{t}')$ to the point $(\sigma_c(x), \boldsymbol{t})$ with $t_{ij} = t'_{ij}$ and $t_{ij} = 1$ for all $0 \le j \le n_i$, if $i \in I$ and $i \notin I$, respectively. □

The canonical functor $\Lambda_{\mathcal{R}} \to \mathcal{R}^\circ\mathcal{E}ns : [\boldsymbol{n}]_{A,\boldsymbol{r}} \mapsto \mathcal{R}_A$ can be extended to a functor

$$\Lambda^\circ_{\mathcal{R}}\mathcal{E}ns \to \mathcal{R}^\circ\mathcal{E}ns : D \mapsto \overline{D}$$

which commutes with direct limits. (It is left adjoint to the functor $\mathcal{R}^\circ\mathcal{E}ns \to \Lambda^\circ_{\mathcal{R}}\mathcal{E}ns$ induced by the functor $[\boldsymbol{n}]_{A,\boldsymbol{r}} \mapsto A$.) One can describe $\overline{D}$ as follows. Given $A \in \mathrm{Ob}(\mathcal{R})$, let $\widetilde{D}_A$ denote the set of the polysimplices of D over A, i.e., the union $\cup D^{\boldsymbol{r}}_{A,\boldsymbol{n}}$ taken over all $[\boldsymbol{n}]_{A,\boldsymbol{r}} \in \mathrm{Ob}(\Lambda_{\mathcal{R}})$. Since $\Lambda_{\mathcal{R}}$ is a fibered category over $\mathcal{R}$, the correspondence $A \mapsto \widetilde{D}_A$ is an object of $\mathcal{R}^\circ\mathcal{E}ns$. We provide the set $\widetilde{D}_A$ with the minimal equivalence relation with respect to which any two elements $d, d' \in \widetilde{D}_A$ with the following property are equivalent: $d \in D^{\boldsymbol{r}}_{A,\boldsymbol{n}}$, $d' \in D^{\boldsymbol{r}'}_{A,\boldsymbol{n}'}$ and there exists a morphism $\gamma : [\boldsymbol{n}']_{A,\boldsymbol{r}'} \to [\boldsymbol{n}]_{A,\boldsymbol{r}}$ over the identity morphism of A with $d' = D(\gamma)(d)$. Then $\overline{D}_A$ is the quotient of $\widetilde{D}_A$ with respect to the above equivalence relation (i.e., $\overline{D}_A$ is the set of connected components of $\widetilde{D}_A$). The following properties of the functor $D \mapsto \overline{D}$ easily follow from the construction.

2.4.2 Lemma.

(i) *For every $C \in \mathrm{Ob}(\mathcal{R}^\circ\mathcal{E}ns)$, there is a canonical isomorphism $\overline{\Lambda[\boldsymbol{n}]_{C,\boldsymbol{r}}} \stackrel{\sim}{\to} C$;*

(ii) *the functor $D \mapsto \overline{D}$ makes $\Lambda^\circ_{\mathcal{R}}\mathcal{E}ns$ a fibered category over $\mathcal{R}^\circ\mathcal{E}ns$, namely, given a morphism $\psi : C \to \overline{D}$ in $\mathcal{R}^\circ\mathcal{E}ns$, the inverse image $\psi^* D$ is as follows: $(\psi^* D)^{\boldsymbol{r}}_{A,\boldsymbol{n}} = D^{\boldsymbol{r}}_{A,\boldsymbol{n}} \times_{\overline{D}_A} C_A$;*

(iii) *the structure of a fibered category $\Lambda^\circ_{\mathcal{R}}\mathcal{E}ns$ over $\mathcal{R}^\circ\mathcal{E}ns$ extends that on $\Lambda_{\mathcal{R}}$ over $\mathcal{R}$.* □

Notice that the canonical surjective projections $\Sigma^{\boldsymbol{n}}_{A,\boldsymbol{r}} \to |A|$ give rise to functorial surjective projections $|D| \to |\overline{D}|$.

3 R-colored polysimplicial sets of length l

3.1 The category $\Lambda_{R,l}$

Let R be a nontrivial submonoid of $[0, 1]$ that contains 1. (In §3.5, it will be assumed that $0 \notin R$.) We can consider R as a semiring of continuous functions on a one point space. If $\mathcal{R}$ is a one point category and Σ is the functor that associates with the only object of $\mathcal{R}$ the above space, then $\Lambda_{\mathcal{R}}$ is the category Λ_R introduced in [Ber7, §4]. We iterate this construction by setting $\Lambda_{R,1} = \Lambda_R$ and $\Lambda_{R,l} = \Lambda_{\Lambda_{R,l-1}}$ for $l \geq 2$. We also denote by $\Sigma_{R,l}$ the corresponding functor $\Lambda_{R,l} \to \mathcal{T}op^{\mathrm{sr}}$.

We represent objects of the category $\Lambda_{R,l}$ as pairs $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ of the following form, and we denote the image of $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ under the functor $\Sigma_{R,l}$ by $(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}, M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$. First of all, $\underline{\boldsymbol{n}}$ is a tuple $(\boldsymbol{n}^{(1)}, \dots, \boldsymbol{n}^{(l)})$ with $[\boldsymbol{n}^{(i)}] = [n^{(i)}_0] \times \cdots \times [n^{(i)}_{p_i}] \in \mathrm{Ob}(\Lambda)$. Furthermore, $\underline{\boldsymbol{r}}$ is a tuple $(\boldsymbol{r}^{(1)}, \dots, \boldsymbol{r}^{(l)})$ with $\boldsymbol{r}^{(i)} = (r^{(i)}_0, \dots, r^{(i)}_{p_i})$ of the following type: $r^{(1)}_0, \dots, r^{(1)}_{p_1} \in R$ and, for $i \geq 2$, $r^{(i)}_0, \dots, r^{(i)}_{p_i} \in M^{\underline{\boldsymbol{n}}^{\leq i-1}}_{\underline{\boldsymbol{r}}^{\leq i-1}}$, where $\underline{\boldsymbol{n}}^{\leq i} = (\boldsymbol{n}^{(1)}, \dots, \boldsymbol{n}^{(i)})$ and $\underline{\boldsymbol{r}}^{\leq i} = (\boldsymbol{r}^{(1)}, \dots, \boldsymbol{r}^{(i)})$ for $1 \leq i \leq l$. Finally, the tuples $\boldsymbol{r}^{(i)}$ satisfy the condition that $r^{(i)}_0 = 1$, if $[\boldsymbol{n}^{(i)}] = [0]$, and $r^{(i)}_j \neq 1$ for all $0 \leq j \leq p_i$, otherwise. The object with $[\boldsymbol{n}^{(i)}] = [0]$ for all $1 \leq i \leq l$ will be denoted by $[\underline{0}]_{\underline{1},l}$. One has

$$\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} = \{\underline{\boldsymbol{t}} = (\boldsymbol{t}^{(1)}, \dots, \boldsymbol{t}^{(l)}) \in [0,1]^{[\boldsymbol{n}^{(1)}]} \times \cdots \times [0,1]^{[\boldsymbol{n}^{(l)}]} \mid t^{(i)}_{j0} \dots t^{(i)}_{jn^{(i)}_j} = r^{(i)}_j(\underline{\boldsymbol{t}}^{\leq i-1})\},$$

where $\underline{\boldsymbol{t}}^{\leq i-1} = (\boldsymbol{t}^{(1)}, \dots, \boldsymbol{t}^{(i-1)})$, and $M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is the semiring of continuous functions generated by R and the coordinate functions $\underline{\boldsymbol{t}} \mapsto t^{(i)}_{jk}$. Notice that for any morphism $[\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ the corresponding map $\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'} \to \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is injective. By Lemma 2.3.2(iii), the canonical functor $\Lambda_{R,l}/[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \to O([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$ has a section $O([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}) \to \Lambda_{R,l}/[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$.

We set $|\underline{\boldsymbol{n}}| = \sum_{i=1}^{l} |\boldsymbol{n}^{(i)}|$, where $|\boldsymbol{n}^{(i)}| = \sum_{j=0}^{p_i} n^{(i)}_j$. Furthermore, let $\mathring{\Sigma}^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ denote the open subset of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ that consists of the points as above with the additional conditions $t^{(i)}_{j0} < 1, \dots, t^{(i)}_{jn^{(i)}_j} < 1$ for all $1 \leq i \leq l$ and $0 \leq j \leq p_i$ with $[\boldsymbol{n}^{(i)}] \neq [0]$. It is called the *interior of* $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$. The *boundary* $\dot{\Sigma}^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ *of* $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is the complement of $\mathring{\Sigma}^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$. The proof of the following lemma is trivial.

3.1.1 Lemma. *If $r \in M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ and $r \neq 1$, then $r(x) < 1$ for all $x \in \mathring{\Sigma}^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$. Furthermore, $\mathring{\Sigma}^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is dense in $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$, and it coincides with the set of points that have an open neighborhood homeomorphic to an open ball (of dimension $|\underline{\boldsymbol{n}}|$).* □

A subset of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$, which is the image of the interior $\mathring{\Sigma}^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}$ with respect to the injective map $\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'} \to \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ that corresponds to a morphism $[\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$, is called a *cell* of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$. The closure of a cell will be called a *cell closure*. (It coincides with the image of $\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}$ under the above map.) Notice that a cell depends only on the equivalence class of the morphism $[\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ in the partially ordered set $O([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$. Let $O(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$ denote the set of cells of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ provided with the following partial ordering: $A \le B$ if $A \subset \overline{B}$.

3.1.2 Lemma. (i) *A cell closure is a disjoint union of cells;*

(ii) *two distinct cells are disjoint* (*and, therefore,* $O(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$ *can be also viewed as the set of all cell closures partially ordered by inclusion*)*;*

(iii) *there is an isomorphism of partially ordered sets* $O([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}) \xrightarrow{\sim} O(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$.

Proof. Assume that the statements are true for $l-1$. By Lemma 3.1.1(ii), to prove (i), it suffices to verify that $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is a disjoint union of cells. First of all, if $[\boldsymbol{n}^{(l)}] = [0]$, then $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \xrightarrow{\sim} \Sigma^{\underline{\boldsymbol{n}}^{\le l-1}}_{\underline{\boldsymbol{r}}^{\le l-1}}$, and the required fact for $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ easily follows from that for $[\underline{\boldsymbol{n}}^{\le l-1}]_{\underline{\boldsymbol{r}}^{\le l-1}}$. Assume therefore that $[\boldsymbol{n}^{(l)}] \ne [0]$, and let $\underline{\boldsymbol{t}} \in \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$. To show that the point $\underline{\boldsymbol{t}}$ is contained in a cell, we may assume, by the induction hypothesis, that $\underline{\boldsymbol{t}}^{\le l-1} \in \mathring{\Sigma}^{\underline{\boldsymbol{n}}^{\le l-1}}_{\underline{\boldsymbol{r}}^{\le l-1}}$. For $0 \le i \le p_l$, let C_i denote the subset of all $j \in [n_i^{(l)}]$ with $t_{ij}^{(l)} < 1$. (The subset C_i is non-empty since $r_i^{(l)}(\underline{\boldsymbol{t}}^{\le l-1}) < 1$.) Furthermore, let J be the subset of all $j \in \omega(\boldsymbol{n}^{(l)})$ with $\#C_j > 1$, and let $\boldsymbol{m}$ be the tuple of the numbers $\#C_j - 1$ for $j \in J$, if $J \ne \emptyset$, and $\boldsymbol{m} = (0)$, if $J = \emptyset$. Then the sets C_j define a morphism $[\boldsymbol{m}] \to [\boldsymbol{n}^{(l)}]$ in Λ. Let $\boldsymbol{s}$ be the tuple of the functions $r_j^{(l)}$ for $j \in J$, if $J \ne \emptyset$, and $\boldsymbol{s} = (1)$, if $J = \emptyset$. Then there is a well defined morphism $[\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ in $\Lambda_{R,l}$, where $[\underline{\boldsymbol{n}}'^{\le l-1}]_{\underline{\boldsymbol{r}}'^{\le l-1}} = [\underline{\boldsymbol{n}}^{\le l-1}]_{\underline{\boldsymbol{r}}^{\le l-1}}$, $\underline{\boldsymbol{n}}'^{(l)} = \boldsymbol{m}$ and $\underline{\boldsymbol{r}}'^{(l)} = \boldsymbol{s}$, and the point $\underline{\boldsymbol{t}}$ is contained in the cell that corresponds to this morphism. Notice that in this way we described all cells of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ over $\mathring{\Sigma}^{\underline{\boldsymbol{n}}^{\le l-1}}_{\underline{\boldsymbol{r}}^{\le l-1}}$, and all of them are pair-wise disjoint, i.e., (i) and (ii) are true. The statement (iii) now easily follows from the induction hypothesis and Lemma 2.3.2. □

Notice that the symmetric strict monoidal category structure on the category Λ_R in the sense of [Mac, Ch. VII], defined in [Ber7, §3], extends naturally to the category $\Lambda_{R,l}$. Namely, the multiplication bifunctor $\Lambda_{R,l} \times \Lambda_{R,l} \xrightarrow{\square} \Lambda_{R,l} : ([\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}, [\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''}) \mapsto [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} = [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \square [\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''}$ is defined as follows: (a) $[\boldsymbol{n}^{(i)}] = [\boldsymbol{n}''^{(i)}]$ and $\boldsymbol{r}^{(i)} = \boldsymbol{r}''^{(i)}$, if $[\boldsymbol{n}'^{(i)}] = [0]$, (b) $[\boldsymbol{n}^{(i)}] = [\boldsymbol{n}'^{(i)}]$ and $\boldsymbol{r}^{(i)} = \boldsymbol{r}'^{(i)}$, if $[\boldsymbol{n}''^{(i)}] = [0]$, and (c) $\boldsymbol{n}^{(i)} = (n_0'^{(i)}, \dots, n_{p_i'}'^{(i)}, n_0''^{(i)}, \dots, n_{p_i''}''^{(i)})$ and $\boldsymbol{r}^{(i)} = (r_0'^{(i)}, \dots, r_{p_i'}'^{(i)}, r_0''^{(i)}, \dots, r_{p_i''}'^{(i)})$, otherwise. Notice also that there is a canonical isomorphism of partially ordered sets $O([\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}) \times O([\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''}) \xrightarrow{\sim} O([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$ and of objects of $\mathcal{T}op^{\mathrm{sr}}$: $(\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}, M^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}) \times (\Sigma^{\underline{\boldsymbol{n}}''}_{\underline{\boldsymbol{r}}''}, M^{\underline{\boldsymbol{n}}''}_{\underline{\boldsymbol{r}}''}) \xrightarrow{\sim} (\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}, M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$.

3.2 R-colored polysimplicial sets of length l

The category of *R-colored polysimplicial sets of length l* is the category $\Lambda^{\circ}_{R,l}\mathcal{E}ns$ of contravariant functors from $\Lambda_{R,l}$ to the category of sets $\mathcal{E}ns$. If R' is a bigger submonoid of $[0, 1]$, there are fully faithful functors $\Lambda_{R,l} \to \Lambda_{R',l}$ and $\Lambda^{\circ}_{R,l}\mathcal{E}ns \to \Lambda^{\circ}_{R',l}\mathcal{E}ns$. The standard *$\underline{\boldsymbol{r}}$-colored $\underline{\boldsymbol{n}}$-polysimplex* $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ is the functor representable by $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$. If $D \in \mathrm{Ob}(\Lambda^{\circ}_{R,l}\mathcal{E}ns)$, the image of $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ under D is denoted by $D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ (the set of $\underline{\boldsymbol{r}}$-colored $\underline{\boldsymbol{n}}$-polysimplices of D). One evidently has $\mathrm{Hom}(\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}, D) \xrightarrow{\sim} D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ and, therefore, there is a canonical bijection between the set $\coprod D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ of all polysimplices of D and the set of objects of the category $\Lambda_{R,l}/D$. In particular, there is an equivalence relation on the set of polysimplices of D, and the set of equivalence classes is provided with a partial ordering. It is denoted by $\mathcal{O}(D)$. Notice that $\mathcal{O}(\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$ coincides with the partially ordered set $\mathcal{O}([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$. The correspondence $D \mapsto \mathcal{O}(D)$ is a functor from $\Lambda^{\circ}_{R,l}\mathcal{E}ns$ to the category of partially ordered sets $\mathcal{O}r$, and this functor commutes with direct limits. A polysimplicial set is said to be *finite* if it has a finite number of polysimplices. It is said to be *locally finite* if each polysimplex is contained in a finite number of other polysimplices (i.e., the corresponding element of $\mathcal{O}(D)$ is smaller than at most a finite number of other elements of $\mathcal{O}(D)$).

The *dimension* of a polysimplex $d \in D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ is $|\underline{\boldsymbol{n}}|$. Notice that it is equal to the topological dimension of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$. Let $m \geq 0$. The *m-skeleton* $\mathrm{Sk}^m(D)$ of a polysimplicial set D is the polysimplicial subset of D which is formed by the polysimplices of dimension at most m. We also set $\mathrm{Sk}^{-1}(C) = \emptyset$. For example, $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} = \mathrm{Sk}^m(\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$, where $m = |\underline{\boldsymbol{n}}|$, and we set $\dot{\Lambda}[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} = \mathrm{Sk}^{m-1}(\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$ (the *boundary* of $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$). For $d \in D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$, let G_d denote the stabilizer of d in the automorphism group $\mathrm{Aut}([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$.

3.2.1 Lemma. *Let P^m be a set of representatives of the equivalence classes of polysimplices of D of dimension m. Then the following diagram is cocartesian:*

$$\begin{array}{ccc} \coprod_{d \in P^m} G_d \backslash \dot{\Lambda}[\underline{\boldsymbol{n}}_d]_{\underline{\boldsymbol{r}}_d} & \longrightarrow & \mathrm{Sk}^{m-1}(D) \\ \big\downarrow & & \big\downarrow \\ \coprod_{d \in P^m} G_d \backslash \Lambda[\underline{\boldsymbol{n}}_d]_{\underline{\boldsymbol{r}}_d} & \longrightarrow & \mathrm{Sk}^{m}(D). \end{array}$$

Proof. Let E be the cocartesian product, and let N and S denote the polysimplicial sets at the north-west and the south-west of the diagram, respectively. Given $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$, if $|\underline{\boldsymbol{n}}| < m$, one evidently has $N^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}} \xrightarrow{\sim} S^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ and $\mathrm{Sk}^{m-1}(D)^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}} \xrightarrow{\sim} \mathrm{Sk}^m(D)^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ and, therefore, $E^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}} \xrightarrow{\sim} \mathrm{Sk}^m(D)^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$. On the other hand, if $|\underline{\boldsymbol{n}}| = m$, then $N^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}} = \mathrm{Sk}^{m-1}(D)^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}} = \emptyset$ and $S^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}} \xrightarrow{\sim} \mathrm{Sk}^m(D)^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ and, therefore, $E \xrightarrow{\sim} \mathrm{Sk}^m(D)$. $\square$

The canonical functor $\Lambda_{R,l} \to \Lambda^{\circ}_{R,l-1}\mathcal{E}ns : [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \mapsto \Lambda[\underline{\boldsymbol{n}}^{\leq l-1}]_{\underline{\boldsymbol{r}}^{\leq l-1}}$ can be extended to a functor

$$\Lambda^{\circ}_{R,l}\mathcal{E}ns \to \Lambda^{\circ}_{R,l-1}\mathcal{E}ns : D \mapsto D^{\leq l-1}$$

which commutes with direct limits. (It is the functor $D \mapsto \overline{D}$ from §2.4.) By Lemma 2.4.2, the latter functor makes $\Lambda^{\circ}_{R,l}\mathcal{E}ns$ a fibered category over $\Lambda^{\circ}_{R,l-1}\mathcal{E}ns$ which is compatible with the fibered category structure of $\Lambda_{R,l}$ over $\Lambda_{R,l-1}$.

The symmetric strict monoidal structure on the category $\Lambda_{R,l}$ is naturally extended to the category $\Lambda^{\circ}_{R,l}\mathcal{E}ns$, i.e., there is a bifunctor $\Lambda^{\circ}_{R,l}\mathcal{E}ns \times \Lambda^{\circ}_{R,l}\mathcal{E}ns \overset{\square}{\to} \Lambda^{\circ}_{R,l}\mathcal{E}ns : (D', D'') \mapsto D'\square D''$ that commutes with direct limits and extends the functor $([\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}, [\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''}) \mapsto [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}\square[\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''}$. One easily sees that the canonical morphism $D'\square D'' \to D' \times D''$ is injective and that there is an isomorphism of partially ordered sets $O(D') \times O(D'') \overset{\sim}{\to} O(D'\square D'')$.

The functor $\Lambda_{R,l} \to \mathcal{T}op^{\mathrm{sr}} : [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \mapsto (\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}, M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$ can be extended to a *geometric realization functor* $\Lambda^{\circ}_{R,l}\mathcal{E}ns \to \mathcal{T}op^{\mathrm{sr}} : D \mapsto (|D|, M_D)$ which commutes with direct limits. Notice that there are functorial projections $(|D|, M_D) \to (|D^{\leq l-1}|, M_{D^{\leq l-1}})$, which are surjective on the underlying topological spaces, and that there are functorial bijective continuous maps $|D'\square D''| \to |D'| \times |D''|$.

3.3 Elementary functions

Given a semiring M of continuous non-negative real valued functions on a topological space X, we say that a nonzero function $f \in M$ is *elementary* if it possesses the following property: if $f = \max(g, h)$ $(= g\dot{+}h)$ for some nonzero $g, h \in M$, then either $f = g$ or $f = h$. The subset of elementary functions in M will be denoted by $e(M)$.

3.3.1 Proposition.

(i) *Given $f, g \in e(M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$, if $f|_U = g|_U$ for a non-empty open subset $U \subset \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$, then $f = g$;*

(ii) *given a nonzero $f \in M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$, there exists a unique finite subset $\{f_i\}_{i\in I} \subset e(M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$ such that $f = \max_{i\in I}\{f_i\}$, but $f \neq \max_{i\in J}\{f_i\}$ for strictly smaller subsets $J \subset I$.*

3.3.2 Lemma. *For every $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \in \mathrm{Ob}(\Lambda_{R,l})$ different from $[\underline{0}]_{\underline{1},l}$, the object $(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}, M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$ of $\mathcal{T}op^{\mathrm{sr}}$ is isomorphic to an object $(\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}, M^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'})$ with the tuple $\underline{\boldsymbol{n}}'$ of the form $((1), \dots, (1))$ (of length $|\underline{\boldsymbol{n}}|$).*

Proof. We may assume that $[\boldsymbol{n}^{(i)}] \neq [0]$ for all $1 \leq i \leq l$, and we notice that there is an evident isomorphism $(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}, M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}) \overset{\sim}{\to} (\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}, M^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'})$, where $\boldsymbol{n}'$ and $\boldsymbol{r}'$ are the tuples $((n_0^{(1)}), \dots, (n_{p_1}^{(1)}), (n_0^{(2)}), \dots, (n_{p_l}^{(l)}))$ and $((r_0^{(1)}), \dots, (r_{p_1}^{(1)}), (r_0^{(2)}), \dots, (r_{p_l}^{(l)}))$ of length $\sum_{i=1}^{l}(p_i + 1)$. Thus, we may assume that all p_i's are zero, i.e., $\underline{\boldsymbol{n}} = ((n^{(1)}), \dots, (n^{(l)}))$ and $\underline{\boldsymbol{r}} = ((r^{(1)}), \dots, (r^{(l)}))$. We now notice that the equation $t_0 \dots t_n = r$ is equivalent to the system of two equations $t_0 \dots t_{n-2} \cdot t'_{n-1} = r$ and $t_{n-1} \cdot t_n = t'_{n-1}$. Thus, if $n^{(i)} > 1$ for some $1 \le i \le l$, then $(\Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}, M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}) \widetilde{\to} (\Sigma_{\underline{\boldsymbol{r}}'}^{\underline{\boldsymbol{n}}'}, M_{\underline{\boldsymbol{r}}'}^{\underline{\boldsymbol{n}}'})$, where $\boldsymbol{n}'$ and $\boldsymbol{r}'$ are the tuples $(\dots, (n^{(i-1)}), (n^{(i)} - 1), (1), (n^{(i+1)}), \dots)$ and $(\dots, (r^{(i-1)}), (r^{(i)}), (t'_{n^{(i)}-1}), (r^{(i+1)}), \dots)$. Repeating this procedure, we construct the required isomorphism. ◻

Proof of Proposition 3.3.1. Lemmas 3.1.1 and 3.3.2 reduce the proposition to the verification of the following fact.

Assume we are given an object (X, M) of $\mathcal{T}op^{\mathrm{sr}}$, which possesses the properties (i) and (ii). Given a function $r \in M$ such that the open set $V = \{x \in X \mid r(x) < 1\}$ is dense in X, we set $X' = \{(x, t_0, t_1) \in X \times [0, 1]^2 \mid t_0 \cdot t_1 = r(x)\}$. Let M' denote the monoid of continuous functions on X' generated by M and the coordinate functions t_0 and t_1, and let $\overline{M}'$ denote the semiring of continuous functions generated by M'. (Notice that $e(\overline{M}') \subset M'$.) Then

(1) every nonzero function from M' has a unique representation in either the form $f t_0^m$ or the form $f t_1^n$ with $f \in M \backslash \{0\}$, $m \ge 0$ and $n \ge 1$;

(2) the elementary functions among them are precisely those with $f \in e(M)$;

(3) the semiring $\overline{M}'$ possesses the properties (i) and (ii).

Notice that the statements (1)–(3) hold when X is a one point space and that the canonical projection $\pi : X' \to X$ is an open map. That any nonzero function $F \in M'$ is of the form considered is trivial. The form of the restriction of F to the fiber $\pi^{-1}(x)$ of a point $x \in V$ is unique and, since V is dense in X, we see that the form of F is unique, i.e., (1) is true. If F is of the form from (1), let us call the function $f \in M$ the base of F.

Assume the restrictions of two nonzero functions $F, G \in M'$ to a non-empty open subset $U' \subset X'$ coincide. Then for every point x from the non-empty open set $U = \pi(U') \cap V$ the restrictions of F and G to $\pi^{-1}(x)$ coincide. It follows that F and G have similar forms and for their bases f and g one has $f|_U = g|_U$. It follows that $f = g$, i.e., (i) is true for the nonzero functions from M' with an elementary base. Let E denote the latter class of functions. It is clear that any nonzero function from $\overline{M}'$ is the maximum of a finite set of functions from E and, in particular, $e(\overline{M}') \subset E$.

Assume that for $F \in E$ one has $F = \max\{F_1, \dots, F_n\}$ with $F_1, \dots, F_n \in E$ and that the family $F_1, \dots, F_n$ is minimal. Then there exists a non-empty set $U' \subset X'$ such that $F_1(x') > F_i(x')$ for all $x' \in U'$ and $2 \le i \le n$. It follows that $F|_{U'} = F_1|_{U'}$, and the validity of the property (i) for functions from E implies that $F = F_1$, i.e., $E = e(\overline{M}')$ and (2) is true.

Assume now that $\max\{F_1, \dots, F_m\} = \max\{G_1, \dots, G_n\}$ for $F_i, G_j \in E$ and that the families of functions on both sides are minimal. Given $1 \le i \le m$, there exists a non-empty open subset $U' \subset M'$ such that $F_i(x') > F_k(x')$ for all $x' \in U'$ and $k \ne i$. Furthermore, we can find $1 \le j \le n$ and a non-empty open subset $U'' \subset U'$ such that $G_j(x') > G_l(x')$ for all $x' \in U''$ and $l \ne j$. It follows that $F_i|_{U''} = G_j|_{U''}$ and therefore $F_i = G_j$. Hence, $\{F_1, \dots, F_m\} \subset \{G_1, \dots, G_n\}$. By symmetry, the converse inclusion also holds, i.e., (3) is true. □

3.3.3 Corollary. *The set $e(M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$ consists of the functions which can be uniquely represented in the form of a product $\lambda \prod (t^{(i)}_{jk})^{a^{(i)}_{jk}}$ taken over all $1 \le i \le l$ with $[\boldsymbol{n}^{(i)}] \ne [0]$, $0 \le j \le p_i$ and $0 \le k \le n^{(i)}_j$, where $\lambda \in R \backslash \{0\}$ and $a^{(i)}_{jk} \in \mathbb{Z}_+$ are such that for every i and j there is k with $a^{(i)}_{jk} = 0$.* □

3.3.4 Corollary. *The family of cell closures in $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ coincides with the family of all non-empty subsets of the form $\{x \in \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \mid f(x) = 1\}$ with $f \in e(M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}})$. In particular, any isomorphism $(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}, M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}) \xrightarrow{\sim} (\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}, M^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'})$ in $\mathcal{T}op^{\mathrm{sr}}$ gives rise to an isomorphism of partially ordered sets $O([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}) = O(\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}) \xrightarrow{\sim} O(\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}) = O([\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'})$.*

Proof. Assume that the statement is true for $l - 1$. To prove the direct implication it suffices to consider the cells of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ over the interior of $\Sigma^{\underline{\boldsymbol{n}}^{\le l-1}}_{\underline{\boldsymbol{r}}^{\le l-1}}$. Such a cell corresponds to a subset $C \subset [\boldsymbol{n}^{(l)}] = [n^{(l)}_0] \times \cdots \times [n^{(l)}_{p_l}]$ of the form $C_0 \times \cdots \times C_{p_l}$ with $C_i \subset [n^{(l)}_i]$, and its closure coincides with the set $\{x \in \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \mid f(x) = 1\}$ for the elementary function $f = \prod_{i=0}^{p_l} \prod_{j \notin C_i} t^{(l)}_{ij}$. To prove the converse implication, it suffices to consider an elementary function f represented in the form of Corollary 3.3.3 with $\lambda = 1$ and $a^{(i)}_{jk} = 0$ for all $1 \le i \le l - 1$. For $0 \le i \le p_l$, we set $C_i = \{j \in [n^{(l)}_i] \mid a^{(l)}_{ij} = 0\}$. Then the set $\{x \in \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \mid f(x) = 1\}$ coincides with the closure of the cell that corresponds to the subset $C = C_0 \times \cdots \times C_{p_l} \subset [\boldsymbol{n}^{(l)}]$. □

3.4 Hausdorffness of the geometric realization

3.4.1 Proposition. *For every $D \in \mathrm{Ob}(\Lambda^{\circ}_{R,l}\mathcal{E}ns)$, the topological space $|D|$ is Hausdorff.*

3.4.2 Lemma. *The morphism $\dot{\Lambda}[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \to \Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ induces a homeomorphism*

$$|\dot{\Lambda}[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}| \xrightarrow{\sim} \dot{\Sigma}^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} .$$

Proof. Step 1. For $i \geq 0$, we define as follows a subset P_i of the set of cells of $\Sigma_{\underline{r}}^{\underline{n}}$: $P_0 = \{\mathring{\Sigma}_{\underline{r}}^{\underline{n}}\}$ and, for $i \geq 1$, P_i is the set of maximal cells in the complement of the union of all cells from $\bigcup_{j=0}^{i-1} P_j$. Let $\overline{P}_i$ denote the set of the closures of cells from P_i. (Recall that the map $P_i \to \overline{P}_i : A \mapsto \overline{A}$ is a bijection.) *We claim that*

(a) *every cell from P_2 is contained in exactly two cell closures from $\overline{P}_1$;*

(b) *if a cell A is contained in $B \cap C$ for $B, C \in \overline{P}_1$ with $B \neq C$, then there exist $B_1 = B, B_2, \dots, B_k = C \in \overline{P}_1$ and $D_1, \dots, D_{k-1} \in \overline{P}_2$ such that $A \subset D_1 \cap \cdots \cap D_{k-1}$ and $D_i \subset B_i \cap B_{i+1}$ with $B_i \neq B_{i+1}$ for all $1 \leq i \leq k-1$.*

Indeed, assume the claim is true for $l-1$. By Lemma 3.3.2 and Corollary 3.3.4, we may assume that $\boldsymbol{n}^{(l)} = (1)$. Let $\underline{\boldsymbol{m}} = \underline{\boldsymbol{n}}^{\leq l-1}$, $\underline{\boldsymbol{s}} = \underline{\boldsymbol{r}}^{\leq l-1}$, $r = r_0^{(l)}$ and $S = \Sigma_{\underline{s}}^{\underline{m}}$. One has $\Sigma_{\underline{r}}^{\underline{n}} = \{(x, t_0, t_1) \in S \times]0,1]^2 \mid t_0 \cdot t_1 = r(x)\}$. Let π denote the canonical projection $\Sigma_{\underline{r}}^{\underline{n}} \to \Sigma_{\underline{s}}^{\underline{m}}$, and let Q_i and $\overline{Q}_i$ denote the sets of cells and cell closures in $\Sigma_{\underline{s}}^{\underline{m}}$ similar to P_i and $\overline{P}_i$. For $X \in Q_i$, the preimage $\pi^{-1}(X)$ is a disjoint union of three cells $X' \in P_i$, $X^0 \in P_{i+1}$ (defined by $t_0 = 1$) and $X^1 \in P_{i+1}$ (defined by $t_1 = 1$), if $r|_X \neq 1$, and is a cell $\widetilde{X} \in P_{i+1}$, if $r|_X = 1$. For $Y = \overline{X}$, we denote by Y', Y^0, Y^1 and $\widetilde{Y}$ the closures of X', X^0, X^1 and $\widetilde{X}$, respectively. For example, $S' = \Sigma_{\underline{r}}^{\underline{n}}$. We now verify (a) and (b) case by case.

(a) Let $A \in P_2$. If $A = \widetilde{X}$ for $X \in Q_1$ with $r|_X = 1$, then A is contained only in S^0 and S^1. If $A = X^i$ for $i = 0, 1$ and $X \in Q_1$ with $r|_X \neq 1$, then A is contained only in X' and S^i. If $A = X'$ for $X \in Q_2$ with $r|_X \neq 1$, then A is contained only in Y' and Z', where Y and Z are the cell closures from $\overline{Q}_1$ that contain X.

(b) Assume first that $B = S^0$ and $C = S^1$. Then $A = \widetilde{X}$ for a cell X in S with $r|_X = 1$. Let Y be a cell closure from $\overline{Q}_1$ that contains X. If $r|_Y = 1$, then $\widetilde{Y} \in \overline{P}_2$ and $A \subset \widetilde{Y} \subset S^0 \cap S^1$. If $r|_Y \neq 1$, then $Y^0, Y^1 \in \overline{P}_2$, $Y' \in \overline{P}_1$, and one has $A \subset Y^0 \cap Y^1$, $Y^0 \subset S^0 \cap Y'$ and $Y^1 \subset Y' \cap S^1$. Assume now that $B = S^0$ and $C = Y'$, where $Y \in \overline{Q}_1$. Then $Y^0 \in \overline{P}_2$, and one has $A \subset Y^0 \subset S^0 \cap Y'$. Assume finally that $B = Y'$ and $C = Z'$ with $Y, Z \in \overline{Q}_1$, and let X be the image of A in S. If $r|_X = 1$ (and, therefore, $A = \widetilde{X}$), then $Y^0, Z^0 \in \overline{P}_2$, and one has $A \subset Y^0 \cap Z^0$, $Y^0 \subset Y' \cap S^0$ and $Z^0 \subset S^0 \cap Z'$. If $r|_X \neq 1$, we apply induction and find $Y_1 = Y, Y_2, \dots, Y_k = Z \in \overline{Q}_1$ and $V_1, \dots, V_{k-1} \in \overline{Q}_2$ such that $X \subset V_1 \cap \cdots \cap V_{k-1}$ and $V_i \subset Y_i \cap Y_{i+1}$ with $Y_i \neq Y_{i+1}$ for all $1 \leq i \leq k-1$. Since $r|_{V_i} \neq 1$, then $V_i' \in \overline{P}_2$ and $Y_i' \in \overline{P}_1$, and one has $A \subset V_1' \cap \cdots \cap V_k'$ and $V_i' \subset Y_i' \cap Y_{i+1}'$ for all $1 \leq i \leq k-1$.

Step 2. Let us fix a section $O(\Sigma_{\underline{r}}^{\underline{n}}) = O([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}) \to \Lambda/[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} : A \mapsto ([\underline{\boldsymbol{n}}_A]_{\underline{\boldsymbol{r}}_A} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$ of the canonical functor $\Lambda_{R,l}/[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ (see Lemma 2.3.2(iii)). By Step 1, for every cell $B \in P_2$ there are exactly two cells $B_1, B_2 \in P_1$ with $B \leq B_1$ and $B \leq B_2$. *We claim that there is a canonical isomorphism of polysimplicial sets*

$$\mathrm{Coker}\Big(\coprod_{B \in P_2} \Lambda[\underline{\boldsymbol{n}}_B]_{\underline{\boldsymbol{r}}_B} \rightrightarrows \coprod_{A \in P_1} \Lambda[\underline{\boldsymbol{n}}_A]_{\underline{\boldsymbol{r}}_A}\Big) \xrightarrow{\sim} \dot{\Lambda}[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}\,,$$

where the upper and lower morphisms are induced by the canonical morphisms $[\underline{\boldsymbol{n}}_B]_{\underline{\boldsymbol{r}}_B} \mapsto [\underline{\boldsymbol{n}}_{B_1}]_{\underline{\boldsymbol{r}}_{B_1}}$ and $[\underline{\boldsymbol{n}}_B]_{\underline{\boldsymbol{r}}_B} \mapsto [\underline{\boldsymbol{n}}_{B_2}]_{\underline{\boldsymbol{r}}_{B_2}}$, respectively. Indeed, let C denote the cokernel. From Step 1 it follows that the morphism $C \to \dot{\Lambda}[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ induces an isomorphism of partially ordered sets $O(C) \overset{\sim}{\to} O(\dot{\Lambda}[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$. The claim now follows from the following simple observation. Given a morphism of polysimplicial sets $C \to D$ which induces an isomorphism of partially ordered sets $O(C) \overset{\sim}{\to} O(D)$, assume that the stabilizer of every polysimplex $d \in D^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ in $\mathrm{Aut}([\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}})$ is trivial. Then $C \overset{\sim}{\to} D$.

The statement of the lemma now follows from the fact that the geometric realization functor commutes with cokernels. □

3.4.3 Corollary. *In the situation of Lemma* 3.2.1, *the following diagram of topological spaces is cocartesian:*

$$\begin{array}{ccc} \coprod_{d \in P^m} G_d \backslash \dot{\Sigma}^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d} & \longrightarrow & |\mathrm{Sk}^{m-1}(D)| \\ \downarrow & & \downarrow \\ \coprod_{d \in P^m} G_d \backslash \Sigma^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d} & \longrightarrow & |\mathrm{Sk}^{m}(D)|. \end{array}$$

Proof. The statement follows from Lemmas 3.2.1 and 3.4.2 and the fact that the geometric realization functor commutes with direct limits. □

Proof of Proposition 3.4.1. By Corollary 3.4.3, the canonical map $|\mathrm{Sk}^{m-1}(D)| \to |\mathrm{Sk}^m(D)|$ identifies the first space with a closed subspace of the second one. It follows also that a subset $\mathcal{U} \subset |\mathrm{Sk}^m(D)|$ is open in $|\mathrm{Sk}^m(D)|$ if and only if the intersection $\mathcal{U} \cap |\mathrm{Sk}^{m-1}(D)|$ is open in $|\mathrm{Sk}^{m-1}(D)|$ and the preimages of $\mathcal{U}$ under all maps $\Sigma^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d} \to |\mathrm{Sk}^m(D)|$ that correspond to the polysimplices $d \in P^m$ are open in $\Sigma^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d}$. Given a polysimplicial set C, let us say that two subsets $\mathcal{U}, \mathcal{V} \subset |C|$ are strongly disjoint if the closures of their preimages in $\Sigma^{\underline{\boldsymbol{n}}_c}_{\underline{\boldsymbol{r}}_c}$ are disjoint for every $c \in C$. *We claim that*

(a) *given strongly disjoint open subsets $\mathcal{U}, \mathcal{V} \subset |\mathrm{Sk}^{m-1}(D)|$, there exist strongly disjoint open subsets $\mathcal{U}', \mathcal{V}' \subset |\mathrm{Sk}^m(D)|$ with $\mathcal{U}' \cap |\mathrm{Sk}^{m-1}(D)| = \mathcal{U}$ and $\mathcal{V}' \cap |\mathrm{Sk}^{m-1}(D)| = \mathcal{V}$;*

(b) *given an open subset $\mathcal{U} \subset |\mathrm{Sk}^{m-1}(D)|$, a polysimplex $d \in P^m$, and a set X in the image of the interior $\mathring{\Sigma}^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d}$ under the corresponding map $\Sigma^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d} \to |\mathrm{Sk}^m(D)|$ such that the preimage of X in $\mathring{\Sigma}^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d}$ is relatively compact, there exists an open subset $\mathcal{U}' \subset |\mathrm{Sk}^m(D)|$ with $\mathcal{U}' \cap |\mathrm{Sk}^{m-1}(D)| = \mathcal{U}$ which is strongly disjoint from X.*

(a) For a polysimplex $d \in P^m$, let $\mathcal{U}_{(d)}$ denote the preimage of $\mathcal{U}$ in $G_d \backslash \dot{\Sigma}^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d}$. Since the closures of $\mathcal{U}_{(d)}$ and $\mathcal{V}_{(d)}$ are disjoint and $G_d \backslash \Sigma^{\underline{\boldsymbol{n}}_d}_{\underline{\boldsymbol{r}}_d}$ is a compact space, it contains open subsets $\mathcal{U}^{(d)}$ and $\mathcal{V}^{(d)}$ whose closures are disjoint and such that $\mathcal{U}^{(d)} \cap (G_d\backslash\dot{\Sigma}^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}) = \mathcal{U}_{(d)}$ and $\mathcal{V}^{(d)} \cap (G_d\backslash\dot{\Sigma}^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}) = \mathcal{V}_{(d)}$. The required sets $\mathcal{U}'$ and $\mathcal{V}'$ are constructed as the unions of the images of $\mathcal{U}^{(d)}$ and $\mathcal{V}^{(d)}$ in $|\mathrm{Sk}^m(D)|$, respectively, taken over all $d \in P^m$.

(b) For the given polysimplex d, we can find an open subset $\mathcal{U}^{(d)} \subset G_d\backslash\Sigma^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}$ with $\mathcal{U}^{(d)} \cap (G_d\backslash\dot{\Sigma}^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}) = \mathcal{U}_{(d)}$ and such that its closure does not intersect with the closure of the preimage of X in $G_d\backslash\mathring{\Sigma}^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}$. If e is a polysimplex from P^m different from d, we take for $\mathcal{U}^{(e)}$ an arbitrary open subset of $G_d\backslash\Sigma^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}$ with $\mathcal{U}^{(e)} \cap (G_d\backslash\dot{\Sigma}^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}) = \mathcal{U}_{(e)}$. The required set $\mathcal{U}'$ is the union of the images of $\mathcal{U}^{(d)}$ and $\mathcal{U}^{(e)}$ in $|\mathrm{Sk}^m(D)|$ taken over $e \in P^m$ different from d.

Step 2. $|D|$ *is a Hausdorff space.* Let x and y be two distinct points of $|D|$. They are contained in the images of $\mathring{\Sigma}^{\boldsymbol{m}}_{\underline{\boldsymbol{s}}}$ and $\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ under the maps $\Sigma^{\boldsymbol{m}}_{\underline{\boldsymbol{s}}} \to |D|$ and $\Sigma^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}} \to |D|$ that corresponds to (unique) polysimplices $d \in P^m$ and $e \in P^n$. Assume that $m \le n$. First of all, to construct disjoint open neighborhoods $\mathcal{U}'$ of x and $\mathcal{V}'$ of y in $|D|$, it suffices to construct strongly disjoint open neighborhoods $\mathcal{U}$ of x and $\mathcal{V}$ of y in $|\mathrm{Sk}^n(D)|$. Indeed, if $\mathcal{U}$ and $\mathcal{V}$ are already constructed then, by Step 1(a), there exist increasing sequences of subsets $\mathcal{U}_n = \mathcal{U} \subset \mathcal{U}_{n+1} \subset \cdots$ and $\mathcal{V}_n = \mathcal{V} \subset \mathcal{V}_{n+1} \subset \cdots$ such that $\mathcal{U}_i$ and $\mathcal{V}_i$ are open and strongly disjoint in $|\mathrm{Sk}^i(D)|$, $\mathcal{U}_{i+1} \cap |\mathrm{Sk}^i(D)| = \mathcal{U}_i$ and $\mathcal{V}_{i+1} \cap |\mathrm{Sk}^i(D)| = \mathcal{V}_i$. Since $|D|$ is a direct limit of the spaces $|\mathrm{Sk}^i(D)|$, it follows that the unions $\mathcal{U}'$ and $\mathcal{V}'$ of all $\mathcal{U}_i$ and $\mathcal{V}_i$, respectively, are open and disjoint in $|D|$.

Assume first that $m = n$. By Corollary 3.4.3, $|\mathrm{Sk}^n(D)|\backslash|\mathrm{Sk}^{n-1}(D)|$ is a disjoint union of open subsets of $|\mathrm{Sk}^n(D)|$, which are evidently Hausdorff and locally compact, and therefore any two open neighborhoods of x and y with disjoint closures are also open and strongly disjoint in $|\mathrm{Sk}^n(D)|$. Assume now that $m < n$. Let $\mathcal{U}$ be an arbitrary open neighborhood of the point x in $|\mathrm{Sk}^{n-1}(D)|$, and let $\mathcal{V}$ be an open neighborhood of the point y in the image of $\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ in $|\mathrm{Sk}^n(D)|$ such that the preimage of $\mathcal{V}$ in $\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ is relatively compact. By Step 1(b), there exists an open subset $\mathcal{U}' \subset |\mathrm{Sk}^n(D)|$ with $\mathcal{U}' \cap |\mathrm{Sk}^{n-1}(D)| = \mathcal{U}$ which is strongly disjoint from $\mathcal{V}$, and we are done. □

A subset of $|D|$, which is the image of the interior $\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ under the map $\Sigma^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}} \to |D|$ that corresponds to a polysimplex $d \in D^{\underline{\boldsymbol{r}}}_{\boldsymbol{n}}$, is called a *cell* of $|D|$. Corollary 3.4.3 implies that such a cell is homeomorphic to $G_d\backslash|\mathring{\Sigma}^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}|$ and that $|\mathrm{Sk}^m(D)|\backslash|\mathrm{Sk}^{m-1}(D)|$ is a disjoint union of the cells that correspond to polysimplices from P^m. Proposition 3.4.1 implies that the closure of the above cell in $|D|$ coincides with the image of $\Sigma^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ in $|D|$. Such a compact subset of $|D|$ is called a *cell closure*. Let $O(|D|)$ denote the set of cells of $|D|$ provided with the following partial ordering: $A \le B$ if $A \subset \overline{B}$.

3.4.4 Corollary. (i) *A cell closure is a disjoint union of cells;*

(ii) *two distinct cells are disjoint* (*and, therefore,* $O(|D|)$ *can be also viewed as the set of all cell closures partially ordered by inclusion*)*;*

(iii) *there is an isomorphism of partially ordered sets* $O(D) \xrightarrow{\sim} O(|D|)$. □

3.4.5 Corollary.

(i) *Given an injective morphism of polysimplicial sets $D' \to D$, the corresponding map $|D'| \to |D|$ identifies $|D'|$ with a closed subset of $|D|$;*

(ii) *for any polysimplicial set D, there is a one-to-one correspondence between polysimplicial subsets of D and the closed subsets of $|D|$, which are unions of cells.* ◻

A polysimplicial set D is said to be *free* if for every polysimplex $d \in D^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ the corresponding morphism $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \to D$ is injective. Notice that every polysimplicial set that admits a morphism to free polysimplicial set is also free.

3.4.6 Lemma. *If D is a free polysimplicial set, the following properties of a morphism $D' \to D$ are equivalent:*

(a) *the morphism $D' \to D$ is injective;*

(b) *the map $|D'| \to |D|$ identifies $|D'|$ with a closed subset of $|D|$;*

(c) *the map of partially ordered sets $O(D') \to O(D)$ is injective.*

Proof. The implications (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) follow from Corollaries 3.4.5(i) and 3.4.4(iii) (and do not require the assumption on D). Assume (c) is true, and let two polysimplices $d_1, d_2 \in D'^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ have the same image d in $D^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$. Then there is an automorphism γ of $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ with $D'(\gamma)(d_1) = d_2$ and, therefore, $D(\gamma)(d) = d$. Since D is free, γ is the identity automorphism and, therefore, $d_1 = d_2$. ◻

Recall that a Kelley space is a Hausdorff topological space X possessing the property that a subset of X is closed whenever its intersection with each compact subset of X is closed. For example, every locally compact space is Kelley. Proposition 3.4.1 implies that the geometric realization $|D|$ of any polysimplicial set D is a Kelley space. It is locally compact if and only if D is locally finite. Given polysimplicial sets D' and D'', there is a homeomorphism $|D' \square D''| \widetilde{\to} |D'| \times |D''|$, where the latter direct product is taken in the category of Kelley spaces.

3.5 A piecewise $R_{\mathbb{Z}_+}$-linear structure on the geometric realization

In this subsection we assume that the monoid R does not contain zero. In this case, $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is evidently an $R_{\mathbb{Z}_+}$-polyhedron in $(\mathbb{R}^*_+)^{[\boldsymbol{n}^{(1)}]} \times \cdots \times (\mathbb{R}^*_+)^{[\boldsymbol{n}^{(l)}]}$. The semiring $M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is generated by R and the coordinate functions and, in particular, all functions from $M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ are piecewise $R_{\mathbb{Z}_+}$-linear. We remark that one can easily see, by induction on l, that the inverse of any coordinate function on $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ is piecewise $\widetilde{R}_{\mathbb{Z}_+}$-linear.

Given a function $f \in M_{\underline{r}}^{\boldsymbol{n}}$, let $\{f_i\}_{i\in I}$ be the finite set of elementary function from Proposition 3.3.1(ii) that are associated with f. For $i \in I$, we set $V_i(f) = \{x \in \Sigma_{\underline{r}}^{\boldsymbol{n}} \mid f_i(x) \geq f_j(x)$ for all $j \in I\}$. (Notice that each $V_i(f)$ contains a point x with $f_i(x) > f_j(x)$ for all $j \neq i$.) We set $\sigma(f) = \{V_i(f)\}_{i\in I}$ and, for a subset $F = \{f_1, \dots, f_m\} \subset M_{\underline{r}}^{\boldsymbol{n}}$, we denote by $\sigma(F)$ the family of all sets of the form $V_1 \cap \dots \cap V_m$ with $V_i \in \sigma(f_i)$. Notice that the union of all $V \in \sigma(F)$ coincides with $\Sigma_{\underline{r}}^{\boldsymbol{n}}$. Finally, we set $F_{\underline{r}}^{\boldsymbol{n}} = \{r_j^{(i)}\}_{1\leq i\leq l, 0\leq j\leq p_i}$ and $\sigma_{\underline{r}}^{\boldsymbol{n}} = \sigma(F_{\underline{r}}^{\boldsymbol{n}})$.

3.5.1 Lemma. *Let F be a finite subset of $M_{\underline{r}}^{\boldsymbol{n}}$ that contains $F_{\underline{r}}^{\boldsymbol{n}}$. Then*

(i) *every $V \in \sigma(F)$ is an $R_{\mathbb{Z}_+}$-polytope, and the restriction to V of each function from the monoid generated by F and the coordinate functions is $R_{\mathbb{Z}_+}$-linear on V;*

(ii) *if $U, V \in \sigma(F)$, then $U \cap V$ is a face in U and in V;*

(iii) *if Σ is a cell closure in $\Sigma_{\underline{r}}^{\boldsymbol{n}}$ and $V \in \sigma(F)$, then $\Sigma \cap V$ is a face of V.*

Proof. (i) The set V is defined in $[0,1]^{[\boldsymbol{n}^{(1)}]} \times \dots \times [0,1]^{[\boldsymbol{n}^{(l)}]}$ by the following equalities and inequalities for all $1 \leq i \leq l$, $0 \leq j \leq p_i$ and $f \in F$: (1) $t_{j0}^{(i)}(x) \dots t_{jn_j^{(i)}}^{(i)}(x) = r_j^{(i)}(x)$, and (2) $f_k(x) \geq f_{k'}(x)$ for some k and all k', where $\{f_{k'}\}$ is the finite set of elementary functions associated with f. Since $r_j^{(i)} \in F$ and is the maximum of the corresponding $f_{k'}$'s, (2) implies that (1) is equivalent to the equality $t_{j0}^{(i)}(x) \dots t_{jn_j^{(i)}}^{(i)}(x) = f_k(x)$, and the statement follows.

(ii) The polytopes U and V are defined by the same equalities (1) and similar inequalities (2) with different k's, and their intersection $U \cap V$ is defined by the additional equalities of the corresponding elementary functions f_k's. It follows that $U \cap V$ is a face in U and in V.

(iii) Since Σ is defined in $\Sigma_{\underline{r}}^{\boldsymbol{n}}$ by the equalities $t_{jk}^{(i)} = 1$ for some i, j and k (see Corollary 3.3.4), it follows that $\Sigma \cap V$ is a face of V. □

From Lemma 3.5.1 it follows that the family $\tau(F)$ of all of the faces of the polytopes from $\sigma(F)$ is an $R_{\mathbb{Z}_+}$-polytopal subdivision of $\Sigma_{\underline{r}}^{\boldsymbol{n}}$. It follows also that every cell closure Σ in $\Sigma_{\underline{r}}^{\boldsymbol{n}}$ is an $R_{\mathbb{Z}_+}$-polyhedron and $\tau(F)|_\Sigma$ is an $R_{\mathbb{Z}_+}$-polytopal subdivision of Σ. The subdivision $\tau(F_{\underline{r}}^{\boldsymbol{n}})$ will be denoted by $\tau_{\underline{r}}^{\boldsymbol{n}}$.

3.5.2 Corollary. *Every morphism $\gamma : [\underline{\boldsymbol{n}}']_{\underline{r}'} \to [\underline{\boldsymbol{n}}]_{\underline{r}}$ in $\Lambda_{R,l}$ gives rise to an immersion of $R_{\mathbb{Z}_+}$-polyhedra $\Sigma_{\underline{r}'}^{\boldsymbol{n}'} \to \Sigma_{\underline{r}}^{\boldsymbol{n}}$, and the restriction of $\tau_{\underline{r}}^{\boldsymbol{n}}$ to the image of the latter gives rise to an $R_{\mathbb{Z}_+}$-polytopal subdivision of $\Sigma_{\underline{r}'}^{\boldsymbol{n}'}$, which is a refinement of $\tau_{\underline{r}'}^{\boldsymbol{n}'}$. If γ is an isomorphism, both subdivisions coincide.* □

Thus, the correspondence $[\underline{\boldsymbol{n}}]_{\underline{r}} \mapsto \Sigma_{\underline{r}}^{\boldsymbol{n}}$ gives rise to a functor from $\Lambda_{R,l}$ to the category of $R_{\mathbb{Z}_+}$-polyhedra in which morphisms are immersions.

3.5.3 Proposition. *One can provide the geometric realization $|D|$ of every* locally finite *R-colored polysimplicial set D of length l with a unique piecewise $R_{\mathbb{Z}_+}$-linear structure so that*

(a) *if $D = \Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$, it is the canonical $R_{\mathbb{Z}_+}$-polyhedron structure on $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$;*

(b) *for any morphism $D' \to D$ between locally finite R-colored polysimplicial sets of length l, the induced map $|D'| \to |D|$ is a G-local immersion of piecewise $R_{\mathbb{Z}_+}$-linear spaces.*

3.5.4 Lemma. *Assume we are given a piecewise $R_{\mathbb{Z}_+}$-linear space X and an equivalence relation E on X which is a piecewise $R_{\mathbb{Z}_+}$-linear subspace of $X \times X$ and satisfies the following two properties:*

(1) *both projections $p_1, p_2 : E \to X$ are proper G-local immersions of piecewise $R_{\mathbb{Z}_+}$-linear spaces;*

(2) *for every point $x \in X$, there exist $R_{\mathbb{Z}_+}$-polyhedra $X_1, \dots, X_n$ in X with the property that any two equivalent points of X_i are equal and such that $X_1 \cup \dots \cup X_n$ is a neighborhood of x in X.*

Then the quotient space $Y = X/E$ can be provided with a unique piecewise $R_{\mathbb{Z}_+}$-linear structure such that the canonical map $X \to Y$ is a G-local immersion.

Proof. First of all, the space Y is locally compact since both projections $p_1, p_2 : E \to X$ are proper. Let σ be the family of $R_{\mathbb{Z}_+}$-polyhedrons U in X such that any two equivalent points of U are equal. By (2), σ is a piecewise $R_{\mathbb{Z}_+}$-linear atlas on X. Furthermore, let τ be the family of the compact subsets V of Y for which there exists $U \in \sigma$ with $U \xrightarrow{\sim} V$. Since the fibers of both projections $p_1, p_2 : E \to X$ are finite, it follows that for every point $y \in Y$ there exist $V_1, \dots, V_n \in \tau$ such that $V_1 \cup \dots \cup V_n$ is a neighborhood of y in Y. Finally, let $V', V'' \in \tau$, and let $U', U'' \in \sigma$ be such that $U' \xrightarrow{\sim} V'$ and $U'' \xrightarrow{\sim} V''$. The set $W = (U' \times U'') \cap E$ is an $R_{\mathbb{Z}_+}$-polyhedron and, by the assumptions, the projections $p_1 : W \to U'$ and $p_2 : W \to U''$ are injective G-local immersions, i.e., they are immersions. It follows that the $R_{\mathbb{Z}_+}$-polyhedron structures on V' and V'', provided by the homeomorphisms with U' and U'', respectively, are compatible on the intersection $V' \cap V''$. Thus, τ is a piecewise $R_{\mathbb{Z}_+}$-linear atlas on Y, and the canonical map $X \to Y$ is a G-local immersion. That the piecewise $R_{\mathbb{Z}_+}$-linear structure on Y with the latter property is unique is already clear. ◻

Proposition 3.5.3 is established using the construction of Lemma 3.5.4 and the following two simple facts which are proved without the assumption $0 \notin R$.

Given a polysimplex $d \in D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$, let E_d denote the equivalence relation on $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ induced by the canonical map $\lambda_d : \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \to |D|$. We consider E_d as a subset of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \times \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$. Furthermore, for a morphism $\gamma : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ in $\Lambda_{R,l}$, let $\overline{\gamma}$ denote the induced map $\lambda_{D(\gamma)(d)} = \lambda_d \circ \Sigma(\gamma) : \Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'} \to \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \to |D|$.

3.5.5 Lemma. *E_d coincides with the union $\cup(\overline{\gamma}_1, \overline{\gamma}_2)(\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'})$ taken over all pairs of morphisms $\gamma_1, \gamma_2 : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ with $D(\gamma_1)(d) = D(\gamma_2)(d)$.*

Proof. It is clear that the union is contained in E_d. On the other hand, assume that the images of two points $x_1, x_2 \in \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ coincide in $|D|$. We have to verify that there exist two morphisms $\gamma_1, \gamma_2 : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ with $D(\gamma_1)(d) = D(\gamma_2)(d)$ and a point $x' \in \Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}$ such that $x_1 = \Sigma(\gamma_1)(x')$ and $x_2 = \Sigma(\gamma_2)(x')$. First of all, let x_1 and x_2 lie in the cells of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ associated with morphisms $\gamma_1 : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ and $\gamma'_2 : [\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$, respectively. Since the images of $\mathring{\Sigma}^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}$ and $\mathring{\Sigma}^{\underline{\boldsymbol{n}}''}_{\underline{\boldsymbol{r}}''}$ in $|D|$ coincide, from Corollary 3.4.4(iii) it follows that there exists an isomorphism $\alpha : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \xrightarrow{\sim} [\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''}$ with $D(\gamma_1)(d) = D(\gamma'_2 \circ \alpha)(d)$. Thus, replacing γ'_2 by $\gamma'_2 \circ \alpha$, we may assume that $[\underline{\boldsymbol{n}}'']_{\underline{\boldsymbol{r}}''} = [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}$. Furthermore, let $x_1 = \Sigma(\gamma_1)(x')$ and $x_2 = \Sigma(\gamma'_2)(x'')$ for some $x', x'' \in \mathring{\Sigma}^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}$. Since the images of the points x' and x'' in $|D|$ coincide, from Corollary 3.4.3 it follows that there is an automorphism α of $[\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}$ with $D(\alpha)(d') = d'$, where $d' = D(\gamma_1)(d) = D(\gamma'_2)(d)$, such that x'' is the image of x' under the corresponding automorphism of $\Sigma^{\underline{\boldsymbol{n}}'}_{\underline{\boldsymbol{r}}'}$. Hence, we get the required fact for the morphisms γ_1 and $\gamma_2 = \gamma'_2 \circ \alpha$ and the point x'. ◻

Assume that for every $1 \le i \le l$ we are given an ordering on the set $[\boldsymbol{n}^{(i)}] = [n^{(i)}_0] \times \cdots \times [n^{(i)}_{p_i}]$. Let us represent elements of $[\boldsymbol{n}^{(i)}]$ as pairs (j, μ), where $0 \le j \le p_i$ and $0 \le \mu \le n^{(i)}_j$, and consider the following subset of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$

$$X = \{x = (x^{(i)}_{jk}) \in \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \mid x^{(i)}_{j\mu} \le x^{(i)}_{k\nu} \text{ for } (j, \mu) \le (k, \nu) \text{ in } [\boldsymbol{n}^{(i)}],\ 1 \le i \le l\}\,.$$

Notice that the sets of this form cover $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$, but some of them may be empty.

3.5.6 Lemma. *If the images of two points $x, y \in X$ in $|D|$ coincide, then $x = y$.*

Proof. (A) *Given $[\underline{\boldsymbol{n}}'] \in \mathrm{Ob}(\Lambda)$, the set X has a non-empty intersection with at most one cell which corresponds to an equivalence class of $[\underline{\boldsymbol{n}}']$-polysimplices of $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$.* (An $[\underline{\boldsymbol{n}}']$-polysimplex is an $\underline{\boldsymbol{r}}'$-colored $[\underline{\boldsymbol{n}}']$-polysimplex for some $[\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \in \mathrm{Ob}(\Lambda_{R,l})$.) Assume that the statement is true for $l-1$. We set $[\underline{\boldsymbol{m}}]_{\underline{\boldsymbol{s}}} = [\underline{\boldsymbol{n}}^{\le l-1}]_{\underline{\boldsymbol{r}}^{\le l-1}}$ and $[\underline{\boldsymbol{m}}'] = [\underline{\boldsymbol{n}}'^{\le l-1}]$. The image of X under the canonical projection $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} \to \Sigma^{\underline{\boldsymbol{m}}}_{\underline{\boldsymbol{s}}}$ is contained in a set of the same type and, therefore, it has a non-empty intersection with at most one cell which corresponds to an equivalence class of $[\underline{\boldsymbol{m}}']$-polysimplices of $\Lambda[\underline{\boldsymbol{m}}]_{\underline{\boldsymbol{s}}}$. If the latter cell exists, we may assume that $[\underline{\boldsymbol{m}}'] = [\underline{\boldsymbol{m}}]$. By Lemma 2.3.2, the equivalences classes of $[\underline{\boldsymbol{n}}']$-polysimplices of $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ correspond bijectively to non-empty subsets of $[\boldsymbol{n}^{(l)}] = [n^{(l)}_0] \times \cdots \times [n^{(l)}_{p_l}]$ of the form $C = C_0 \times \cdots \times C_{p_i}$ with $|C_{j_k}| = n'^{(l)}_k + 1$ for $0 \le k \le p'_l$ and $|C_j| = 1$ for $j \notin \{j_0, \dots, j_{p'_l}\}$. Given such a subset C, the corresponding cell of $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ consists of the points x over $\mathring{\Sigma}^{\underline{\boldsymbol{m}}}_{\underline{\boldsymbol{s}}}$ with $x^{(l)}_{j\mu} < 1$ for $\mu \in C_j$

and $x^{(l)}_{j\mu} = 1$ for $\mu \notin C_j$. After a permutation of the coordinate functions $\{t^{(l)}_{j\mu}\}_\mu$, we may assume that the ordering on the set $[\boldsymbol{n}^{(l)}]$ satisfies the property $(j,\mu) < (j,\nu)$ for $\mu < \nu$. Hence, if the above cell has a non-empty intersection with X, then $C_j = \{0,\dots,c_j\}$, where $c_j = {n'}^{(l)}_k$, if $j = j_k$, and $c_j = 0$, if $j \notin \{j_0,\dots,j_{p'_l}\}$. Moreover, in this case one has $(j,c_j) < (k,c_k+1)$ for all $0 \le j,k \le p_l$ with $c_k < n^{(l)}_k$. These inequalities uniquely determine the sequence $c_0, c_1, \dots, c_{p_l}$ among those obtained from it by a permutation, and this implies the required fact.

(B) By (A), the points x and y are contained in one cell of $\Sigma^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ and, therefore, the claim follows from Corollary 3.4.3 and the following elementary fact. If, for a non-decreasing sequence of numbers $x_1 \le \cdots \le x_n$ and a permutation $\sigma \in S_n$, one has $x_{\sigma(1)} \le \cdots \le x_{\sigma(n)}$, then $x_{\sigma(i)} = x_i$ for all $1 \le i \le n$. ◻

Proof of Proposition 3.5.3. We apply the construction of Lemma 3.5.4 to the disjoint union $X = \coprod \Sigma^{\boldsymbol{n}_d}_{\underline{\boldsymbol{r}}_d}$, taken over all polysimplices d of D, and the equivalence relation $E \subset X \times X$ induced on X by the canonical surjective map $X \to |D|$. Since the validity of the properties (1) and (2) follows from Lemmas 3.5.5 and 3.5.6, respectively, we are done. ◻

3.5.7 Corollary. *Let D be a locally finite R-colored polysimplicial set of length l. Then*

(i) *all cells and cell closures are piecewise $R_{\mathbb{Z}_+}$-linear subspaces of $|D|$;*

(ii) *all functions from M_D are piecewise $R_{\mathbb{Z}_+}$-linear.* ◻

Thus, if D is a locally finite R-colored polysimplicial set of length l, its geometric realization $|D|$ is a piecewise $R_{\mathbb{Z}_+}$-linear space provided with a semiring M_D of piecewise $R_{\mathbb{Z}_+}$-linear functions and a locally finite stratification by relatively compact piecewise $R_{\mathbb{Z}_+}$-linear subspaces, cells, with the property that the closure of a cell, a cell closure, is also a piecewise $R_{\mathbb{Z}_+}$-linear subspace and a (finite) union of cells. Furthermore, given a morphism $D' \to D$ between R-colored polysimplicial sets of length l, the corresponding map $|D'| \to |D|$ is a G-local immersion of piecewise $R_{\mathbb{Z}_+}$-linear spaces that takes functions from M_D to functions from $M_{D'}$ and induces a surjective open map from every cell of $|D'|$ to a cell of $|D|$.

3.5.8 Remarks. (i) It is very likely that the property (2) in Lemma 3.5.4 always follows from (1).

(ii) It follows from the remark at the beginning of this subsection that, given a piecewise $R_{\mathbb{Z}_+}$-linear subspace X of the geometric realization $|D|$ of a locally finite R-colored polysimplicial set D, every piecewise $\widetilde{R}_{\mathbb{Z}}$-linear (resp. $\overline{R}_{\mathbb{Q}}$-linear) function on X is in fact piecewise $\widetilde{R}_{\mathbb{Z}_+}$-linear (resp. $\overline{R}_{\mathbb{Q}_+}$-linear).

4 The skeleton of a nondegenerate pluri-stable formal scheme

4.1 Poly-stable fibrations and pluri-stable morphisms

Let k be a non-Archimedean field whose valuation is not assumed to be nontrivial. For a strictly k-analytic space X, we denote by $\mathcal{O}'(X)$ the multiplicative monoid of all analytic functions $f \in \mathcal{O}(X)$ for which the Zariski closed set $\{x \in X \mid f(x) = 0\}$ is nowhere dense in X. If X is normal (i.e., all strictly affinoid subdomains of X are normal), then $\mathcal{O}'(X)$ coincides with the set of all $f \in \mathcal{O}(X)$ whose restriction to every connected component of X is not zero. For a formal scheme $\mathfrak{X}$ locally finitely presented over k°, we denote by $\mathcal{O}'(\mathfrak{X})$ the multiplicative monoid of all $f \in \mathcal{O}(\mathfrak{X})$ whose image in $\mathcal{O}(\mathfrak{X}_\eta)$ is contained in $\mathcal{O}'(\mathfrak{X}_\eta)$.

For an affine formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$ finitely presented over k°, an element $a \in A$ and an integer $n \geq 0$, we set

$$\mathfrak{X}(n, a) = \mathrm{Spf}(A\{T_0, \dots, T_n\}/(T_0 \dots T_n - a)) ,$$

and for $m \geq 0$ we set $\mathfrak{X}(m) = \mathfrak{X}(m, 1)$. (If $n = 0$, we assume that $a = 1$ and set $\mathfrak{X}(0, 1) = \mathfrak{X}$.) Furthermore, given tuples $\boldsymbol{n} = (n_0, \dots, n_p) \in \mathbb{Z}^{p+1}$ and $\boldsymbol{a} = (a_0, \dots, a_p) \in A^{p+1}$ such that $n_i \geq 1$ and each a_i is not invertible in A, or $p = n_0 = 0$ and $a_0 = 1$, we set

$$\mathfrak{X}(\boldsymbol{n}, \boldsymbol{a}) = \mathfrak{X}(n_0, a_0) \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \mathfrak{X}(n_p, a_p) .$$

(If $\mathfrak{X} = \mathrm{Spf}(k^\circ)$, the latter is the formal scheme which was denoted in [Ber7] by $\mathfrak{T}(\boldsymbol{n}, \boldsymbol{a})$.) If, in addition, a non-negative integer m is given, we set $\mathfrak{X}(\boldsymbol{n}, \boldsymbol{a}, m) = \mathfrak{X}(\boldsymbol{n}, \boldsymbol{a}) \times_{\mathfrak{X}} \mathfrak{X}(m)$.

Recall ([Ber7, §1]) that a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ of formal schemes locally finitely presented over k° is said to be *strictly poly-stable* if, for every point $\boldsymbol{y} \in \mathfrak{Y}$, there exist an open affine neighborhood $\mathfrak{X}' = \mathrm{Spf}(A)$ of $\varphi(\boldsymbol{y})$ and an open neighborhood $\mathfrak{Y}' \subset \varphi^{-1}(\mathfrak{X}')$ of $\boldsymbol{y}$ such that the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ goes through an étale morphism $\mathfrak{Y}' \to \mathfrak{X}'(\boldsymbol{n}, \boldsymbol{a}, m)$ for some triple $(\boldsymbol{n}, \boldsymbol{a}, m)$ as above. If the latter morphisms can be found in such a way that $a_i \in \mathcal{O}'(\mathfrak{X}') \subset A$ for all $0 \leq i \leq p$, then φ will be said to be *nondegenerate*. Furthermore, φ is said to be *(nondegenerate) poly-stable* if there exists a surjective étale morphism $\mathfrak{Y}' \to \mathfrak{Y}$ for which the induced morphism $\mathfrak{Y}' \to \mathfrak{X}$ is (nondegenerate) strictly poly-stable.

A (*nondegenerate, strictly*) *poly-stable fibration over* k° *of length* $l \geq 0$ is a sequence of (nondegenerate, strictly) poly-stable morphisms

$$\underline{\mathfrak{X}} = (\mathfrak{X}_l \overset{f_{l-1}}{\to} \cdots \overset{f_1}{\to} \mathfrak{X}_1 \overset{f_0}{\to} \mathfrak{X}_0 = \mathrm{Spf}(k^\circ)) .$$

For the above $\underline{\mathfrak{X}}$, we denote by $\underline{\mathfrak{X}}^{\leq l-1}$ the poly-stable fibration $(\mathfrak{X}_{l-1} \overset{f_{l-2}}{\to} \cdots \overset{f_1}{\to} \mathfrak{X}_1)$ of length $l-1$. (We omit f_0 and $\mathfrak{X}_0 = \mathrm{Spf}(k^\circ)$ if their presence is evident.) Recall that in [Ber7] we denoted by $k^\circ\text{-}\mathcal{P}st_l$ the category of poly-stable fibrations over k°

of length l, and we considered the category $k^\circ\text{-}\mathcal{P}st_l^{\text{ét}}$ with the same family of objects as $k^\circ\text{-}\mathcal{P}st_l$ but with étale morphisms between them. We denote by $k^\circ\text{-}\mathcal{P}st_{\text{nd},l}$ and $k^\circ\text{-}\mathcal{P}st_{\text{nd},l}^{\text{ét}}$ the full subcategories of the latter consisting of the objects for which all of the morphisms f_i, $0 \le i \le l-1$ are nondegenerate.

A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ of formal schemes locally finitely presented over k° is said to be (*nondegenerate, strictly*) *pluri-stable* if it is a composition of (nondegenerate, strictly) poly-stable morphisms. For example, a formal scheme $\mathfrak{X}$ over k° is (nondegenerate, strictly) pluri-stable (i.e., the morphism $\mathfrak{X} \to \mathrm{Spf}(k^\circ)$ is a such one) if there exists a (nondegenerate, strictly) poly-stable fibration $\underline{\mathfrak{X}}$ over k° of some length l with $\mathfrak{X}_l = \mathfrak{X}$. The category of pluri-stable formal schemes over k° will be denoted by $k^\circ\text{-}\mathcal{P}\ell st$, and $k^\circ\text{-}\mathcal{P}\ell st^{\text{ét}}$ and $k^\circ\text{-}\mathcal{P}\ell st^{\text{pl}}$ will denote the categories with the same family of objects but with étale and pluri-stable morphisms between them, respectively. The full subcategories of the latter consisting of the nondegenerate pluri-stable formal schemes will be denoted by $k^\circ\text{-}\mathcal{P}\ell st_{\text{nd}}$, $k^\circ\text{-}\mathcal{P}\ell st_{\text{nd}}^{\text{ét}}$ and $k^\circ\text{-}\mathcal{P}\ell st_{\text{nd}}^{\text{pl}}$.

Pluri-stable morphisms and formal schemes are examples of pluri-nodal morphisms and formal schemes introduced in [Ber7, §1] (see Remark 4.1.5). Recall that a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ between formal schemes locally finitely presented over k° is called *strictly pluri-nodal* if locally in the Zariski topology it is a composition of étale morphisms and morphisms of the form $\mathrm{Spf}(A\{u, v\}/(uv-a)) \to \mathrm{Spf}(A)$, $a \in A$, and it is called *pluri-nodal* if there exists a surjective étale morphism $\mathfrak{Y}' \to \mathfrak{Y}$ such that the induced morphism $\mathfrak{Y}' \to \mathfrak{X}$ is strictly pluri-nodal. We also say that such a morphism is *nondegenerate* if the above morphisms $\mathrm{Spf}(A\{u, v\}/(uv - a)) \to \mathrm{Spf}(A)$ can be found in such a way that $a \in \mathcal{O}'(\mathrm{Spf}(A)) \subset A$. (Notice that this is consistent with the notion of a nondegenerate pluri-stable morphism.) Recall that for any pluri-nodal formal scheme $\mathfrak{X}$ over k° the reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ is surjective (see [Ber7, Corollary 1.7]).

4.1.1 Lemma. *Every pluri-nodal morphism is flat.*

Proof. Since étale morphisms are evidently flat, it suffices to consider morphisms of the form $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ with $B = A\{u, v\}/(uv - a)$, $a \in A$. Let α be an element of the maximal ideal $k^{\circ\circ}$ which is not equal to zero if the valuation on k is nontrivial. Each element of B has a unique representation in the form $\sum_{i=-\infty}^{\infty} f_i w_i$, where $f_i \to 0$ in the α-adic topology of A, and $w_i = u^{-i}$ for $i < 0$ and $w_i = v^i$ for $i \ge 0$. It follows that, if the valuation on k is trivial, B is a free A-module. If the valuation on k is nontrivial, it follows that, for every $n \ge 1$, $B/(\alpha^n B)$ is a free module over $A/(\alpha^n A)$ and, by [BoLü1, Lemma 1.6], B is flat over A. ◻

4.1.2 Corollary. *Given a pluri-nodal morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, one has $\varphi_\eta(\mathfrak{Y}_\eta) = \pi^{-1}(\varphi_s(\mathfrak{Y}_s))$.*

Proof. First of all, increasing the field k, we may assume that its valuation is nontrivial. It suffices to show that, given a faithfully flat morphism $\mathfrak{Y} = \mathrm{Spf}(B) \to \mathfrak{X} = \mathrm{Spf}(A)$, the induces map $Y = \mathfrak{Y}_\eta \to X = \mathfrak{X}_\eta$ is surjective. For this we notice that the set $X_0 = \{x \in X \mid [\mathcal{H}(x) : k] < \infty\}$ coincides with the set of prime ideals $\wp \subset A$ with $\dim(A/\wp) = 1$ and for which the canonical homomorphism $k^\circ \to A/\wp$ is injective. It follows that X_0 is contained in the image of Y. Since X_0 is dense in X and both spaces X and Y are compact, the map $Y \to X$ is surjective. □

4.1.3 Lemma. *A pluri-nodal formal scheme $\mathfrak{X}$ over k° is nondegenerate if and only if its generic fiber $\mathfrak{X}_\eta$ is a normal strictly k-analytic space.*

Proof. The direct implication follows straightforwardly from [Ber7, Lemma 1.5]. To prove the direct implication (and the corollary that follows), it suffices to verify the following fact. *Let $\varphi : \mathfrak{Z} = \mathrm{Spf}(C) \to \mathfrak{X} = \mathrm{Spf}(A)$ be a morphism of pluri-nodal formal schemes that goes through an étale morphism $\psi : \mathfrak{Z} \to \mathfrak{Y} = \mathrm{Spf}(B)$ with $B = A\{u, v\}/(uv)$, and assume that $\mathfrak{Z}_\eta$ is normal. Then*

(a) *$\varphi_s(\mathfrak{Z}_s)$ is an open subscheme of $\mathfrak{X}_s$;*

(b) *the strictly k-analytic space $\pi^{-1}(\varphi_s(\mathfrak{Z}_s))$ is normal;*

(c) *$\psi(\mathfrak{Z}) \subset \mathfrak{U} \cup \mathfrak{V}$, where $\mathfrak{U} = \mathrm{Spf}(B_{\{u\}}) \stackrel{\sim}{\to} \mathrm{Spf}(A\{u, \frac{1}{u}\})$ and $\mathfrak{V} = \mathrm{Spf}(B_{\{v\}}) \stackrel{\sim}{\to} \mathrm{Spf}(A\{v, \frac{1}{v}\})$.*

Indeed, (a) is true since the morphism of schemes $\varphi_s : \mathfrak{Z}_s \to \mathfrak{X}_s$ is flat and of finite type. Furthermore, (b) is true since $\mathfrak{Z}_\eta$ is normal, C is flat over A and $\pi^{-1}(\varphi_s(\mathfrak{Z}_s)) = \varphi_\eta(\mathfrak{Z}_\eta)$, by Corollary 4.1.2. Finally, since the reduction map $\pi : \mathfrak{Z}_\eta \to \mathfrak{Z}_s$ is surjective, to prove (c) it suffices to show that for every point $y \in \psi_\eta(\mathfrak{Z}_\eta)$ either $|u(y)| = 1$ or $|v(y)| = 1$. Assume this is not true, i.e., there exists a point $y \in \psi_\eta(\mathfrak{Z}_\eta)$ with $|u(y)| < 1$ and $|v(y)| < 1$ (since $uv = 0$ then in fact either $u(y) = 0$ or $v(y) = 0$). Then for the point $\boldsymbol{y} = \pi(y) \in \mathfrak{Y}_s$ one has $u(\boldsymbol{y}) = v(\boldsymbol{y}) = 0$. Consider the closed immersion $\chi : \mathfrak{X} \to \mathfrak{Y}$ defined by the surjection $B \to A$ that takes u and v to zero. Since the reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ is surjective, it follows that there exists a point $y' \in \pi^{-1}(\boldsymbol{y})$ with $u(y') = v(y') = 0$. Since $\pi^{-1}(\boldsymbol{y}) \subset \pi^{-1}(\psi_s(\mathfrak{Z}_s)) = \psi_\eta(\mathfrak{Z}_\eta)$, the latter contradicts [Ber7, Lemma 1.5]. □

4.1.4 Corollary. *Any pluri-nodal morphism from a* nondegenerate *pluri-nodal to a pluri-nodal formal scheme over k° is always nondegenerate.* □

The closed fiber $\mathfrak{X}_s$ of a pluri-stable formal scheme $\mathfrak{X}$ over k° is provided with a stratification, i.e., a partition of $\mathfrak{X}_s$ by locally closed irreducible normal subschemes with the property that the closure of a stratum is a union of strata (see [Ber7, §2]). The set of the generic points of the strata is denoted by $\mathrm{str}(\mathfrak{X}_s)$. It is a partially ordered set with respect to the following ordering: $\boldsymbol{x} \le \boldsymbol{y}$ if $\boldsymbol{y}$ is contained in the closure of $\boldsymbol{x}$. A pluri-stable (and, in particular, an étale) morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ induces a map of partially ordered sets $\mathrm{str}(\mathfrak{Y}_s) \to \mathrm{str}(\mathfrak{X}_s)$ and, in fact, $\mathrm{str}(\mathfrak{Y}_s) = \cup \mathrm{str}(\mathfrak{Y}_{s,\boldsymbol{x}})$, where the union is taken over all $\boldsymbol{x} \in \mathrm{str}(\mathfrak{X}_s)$. If k' is a bigger non-Archimedean field, then the morphism $\mathfrak{X}' = \mathfrak{X}\widehat{\otimes}_{\mathrm{Spf}(k^\circ)}\mathrm{Spf}(k'^\circ) \to \mathfrak{X}$ induces a surjective map of partially ordered sets $\mathrm{str}(\mathfrak{X}'_s) \to \mathrm{str}(\mathfrak{X}_s)$. If all of the strata of $\mathfrak{X}_s$ are geometrically irreducible, the latter map is an isomorphism. If the valuation on k is trivial then, by [Ber7, 1.5], the closed fiber $\mathfrak{X}_s$ of a nondegenerate pluri-nodal formal scheme $\mathfrak{X}$ is normal and, therefore, $\mathrm{str}(\mathfrak{X}_s)$ coincides with the set $\mathrm{gen}(\mathfrak{X}_s)$ of generic points of the irreducible components of $\mathfrak{X}_s$.

As in [Ber7], we introduce categories $\mathcal{P}st_{\mathrm{nd},l}$ and $\mathcal{P}\ell st_{\mathrm{nd}}$ whose objects are pairs $(k, \underline{\mathfrak{X}})$ and $(k, \mathfrak{X})$, where k is a non-Archimedean field and $\underline{\mathfrak{X}}$ is from k°-$\mathcal{P}st_{\mathrm{nd},l}$ and $\mathfrak{X}$ is from k°-$\mathcal{P}\ell st_{\mathrm{nd}}$, and morphisms $(K, \underline{\mathfrak{Y}}) \to (k, \underline{\mathfrak{X}})$ and $(K, \mathfrak{Y}) \to (k, \mathfrak{X})$ are pairs consisting of an isometric embedding $k \hookrightarrow K$ and morphisms $\underline{\mathfrak{Y}} \to \underline{\mathfrak{X}}\widehat{\otimes}_{k^\circ}K^\circ$ in K°-$\mathcal{P}st_{\mathrm{nd},l}$ and $\mathfrak{Y} \to \mathfrak{X}\widehat{\otimes}_{k^\circ}K^\circ$ in K°-$\mathcal{P}\ell st_{\mathrm{nd}}$, respectively. Similarly, $\mathcal{P}st^{\text{ét}}_{\mathrm{nd},l}$, $\mathcal{P}\ell st^{\text{ét}}_{\mathrm{nd}}$ and $\mathcal{P}\ell st^{\mathrm{pl}}_{\mathrm{nd}}$ denote the categories with the same families of objects but with the above morphisms for which the morphisms $\underline{\mathfrak{Y}} \to \underline{\mathfrak{X}}\widehat{\otimes}_{k^\circ}K^\circ$ and $\mathfrak{Y} \to \mathfrak{X}\widehat{\otimes}_{k^\circ}K^\circ$ are étale and pluri-stable, respectively.

Notice that $\mathcal{P}st_{\mathrm{nd},l}$ and $\mathcal{P}\ell st^{\text{ét}}_{\mathrm{nd}}$ are full subcategories of the categories $\mathcal{P}st_l$ and $\mathcal{P}st^{\text{ét}}_l$ from [Ber7], respectively, and all of these categories are fibered ones over the category dual to the category of non-Archimedean fields. Notice also that the correspondence $\underline{\mathfrak{X}} \mapsto \underline{\mathfrak{X}}^{\le l-1}$ gives rise to a functor $\mathcal{P}st_l \to \mathcal{P}st_{l-1}$. For brevity, the pairs $(k, \underline{\mathfrak{X}})$ and $(k, \mathfrak{X})$ will be denoted by $\underline{\mathfrak{X}}$ and $\mathfrak{X}$, respectively.

4.1.5 Remarks. Assume that the valuation on k is nontrivial, and let $a \in k^{\circ\circ}\backslash\{0\}$, $A = k^\circ\{u, v\}/(uv - a)$ and $B = A\{x, y\}/(xy - (u + v))$. The localization $B_{\{u\}}$ is canonically isomorphic to $k^\circ\{u, x, \frac{1}{u}, \frac{1}{x}\}$. Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be two copies of $\mathrm{Spf}(B)$, X_{12} and X_{21} two copies of $\mathrm{Spf}(B_{\{u\}})$ considered as open subschemes of X_1 and X_2, respectively, and $\mathfrak{X}$ the separated formal scheme constructed by gluing X_1 and X_2 along the isomorphism $X_{12} \stackrel{\sim}{\to} X_{21}$ that takes u to $\frac{1}{x}$ and x to $\frac{1}{u}$. We believe that the strictly pluri-nodal formal scheme $\mathfrak{X}$ is not pluri-stable.

4.2 The skeleton of a poly-stable fibration

Recall that in [Ber7] we constructed for every poly-stable fibration $\underline{\mathfrak{X}}$ over k° of length l a closed subset $S(\underline{\mathfrak{X}}) \subset \mathfrak{X}_{l,\eta}$, the skeleton of $\underline{\mathfrak{X}}$, and a proper strong deformation retraction $\Phi : \mathfrak{X}_{l,\eta} \times [0, l] \to \mathfrak{X}_{l,\eta}$ of $\mathfrak{X}_{l,\eta}$ to $S(\underline{\mathfrak{X}})$. The retraction map $\mathfrak{X}_{l,\eta} \to S(\underline{\mathfrak{X}}) : x \mapsto x_\tau = \Phi(x, l)$ is denoted by τ. In this subsection we briefly recall a part of the construction and some basic facts from [Ber7]. (The construction of the retraction map τ will be recalled in §4.4.)

First of all, if $\mathfrak{X} = \mathfrak{T}(\boldsymbol{n}, \boldsymbol{a}, m)$ with $\mathfrak{T} = \mathrm{Spf}(k^\circ)$, then $\mathfrak{X}_\eta = \mathcal{M}(\mathcal{B})$, where $\mathcal{B} = \mathcal{C}/\boldsymbol{b}$, $\mathcal{C} = \mathcal{A}\{T_{00}, \dots, T_{pn_p}\}$, $\mathcal{A} = k\{T_1, \dots, T_m, \frac{1}{T_1}, \dots, \frac{1}{T_m}\}$, and $\boldsymbol{b}$ is the ideal of $\mathcal{C}$ generated by the elements $T_{i0} \dots T_{in_i} - a_i$, $0 \le i \le p$. If we provide $\mathcal{A}$ and $\mathcal{C}$ with the canonical norms and $\mathcal{B}$ with the quotient norm, then the set D, consisting of the elements $\sum_\mu a_\mu T^\mu$ such that $a_\mu = 0$ for all $\mu = \{\mu_{ij}\}_{0\le i\le p, 0\le j\le n_i}$

with $\min_{0\le j\le n_i}\{\mu_{ij}\} \ge 1$ for some $0 \le i \le p$, is a Banach $\mathcal{A}$-submodule of $\mathcal{C}$, and the canonical surjection $\mathcal{C} \to \mathcal{B}$ induces an isometric isomorphism $D \stackrel{\sim}{\to} \mathcal{B}$. The skeleton $S(\mathfrak{X})$ is the image of the set $S = \{\boldsymbol{t} \in [0,1]^{[\boldsymbol{n}]} \mid t_{i0} \dots t_{in_i} = |a_i|, 0 \le i \le p\}$ under the following injective mapping $S \to \mathfrak{X}_\eta$. It takes a point $\boldsymbol{t} \in S$ to the bounded multiplicative seminorm on $\mathcal{B}$ which is induced by the function $D \to \mathbb{R}_+ : f = \sum_\mu a_\mu T^\mu \mapsto \max_\mu\{||a_\mu||\boldsymbol{t}^\mu\}$.

If $\mathfrak{X}$ is a formal scheme over k° that admits an étale morphism to some $\mathfrak{Y} = \mathfrak{T}(\boldsymbol{n}, \boldsymbol{a}, m)$, then the skeleton $S(\mathfrak{X})$ is the preimage of $S(\mathfrak{Y})$ under the induced map $\mathfrak{X}_\eta \to \mathfrak{Y}_\eta$. One show that this subset of $\mathfrak{X}_\eta$ is well defined. If the closed fiber $\mathfrak{X}_s$ has a unique maximal stratum, then the map $S(\mathfrak{X}) \to S(\mathfrak{Y})$ is injective, and if, in addition, this maximal stratum goes to the unique maximal stratum of $\mathfrak{Y}_s$, then $S(\mathfrak{X}) \stackrel{\sim}{\to} S(\mathfrak{Y})$. If $\mathfrak{X}$ is an arbitrary strictly poly-stable formal scheme over k°, one defines the skeleton $S(\mathfrak{X})$ as the union $\bigcup_{i\in I} S(\mathfrak{X}_i)$, where $\{\mathfrak{X}_i\}_{i\in I}$ is a covering of $\mathfrak{X}$ by open subschemes that admit an étale morphism to a formal scheme of the form $\mathfrak{T}(\boldsymbol{n}, \boldsymbol{a}, m)$. If $\mathfrak{X}$ is an arbitrary poly-stable formal scheme over k°, one takes a surjective étale morphism $\mathfrak{X}' \to \mathfrak{X}$ from a strictly poly-stable formal scheme $\mathfrak{X}'$ and defines the skeleton $S(\mathfrak{X})$ as the image of $S(\mathfrak{X}')$ under the induced map $\mathfrak{X}'_\eta \to \mathfrak{X}_\eta$.

Furthermore, one defines the skeleton $S(\mathfrak{Y}/\mathfrak{X})$ of a poly-stable morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ as follows. Given a point $x \in \mathfrak{X}_\eta$, $\mathfrak{Y}_x = \mathfrak{Y} \times_{\mathfrak{X}} \mathrm{Spf}(\mathcal{H}(x)^\circ)$ is a poly-stable formal scheme over $\mathcal{H}(x)^\circ$, and there are canonical isomorphisms $\mathfrak{Y}_{x,\eta} \stackrel{\sim}{\to} \mathfrak{Y}_{\eta,x}$ and $\mathfrak{Y}_{x,s} \stackrel{\sim}{\to} \mathfrak{Y}_{s,\boldsymbol{x}} \otimes_{\widetilde{k}(\boldsymbol{x})} \widetilde{\mathcal{H}(x)}$, where $\boldsymbol{x}$ is the image of x under the reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$. The skeleton of φ is the closed set

$$S(\mathfrak{Y}/\mathfrak{X}) = \bigcup_{x\in\mathfrak{X}_\eta} S(\mathfrak{Y}_x)\ .$$

One also constructs a strong deformation retraction $\Psi_\varphi : \mathfrak{Y}_\eta \times [0,1] \to \mathfrak{Y}_\eta$ of $\mathfrak{Y}_\eta$ to $S(\mathfrak{Y}/\mathfrak{X})$.

Finally, let $\underline{\mathfrak{X}} = (\mathfrak{X}_l \stackrel{f_{l-1}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ be a poly-stable fibration over k° of length $l \ge 0$. If $l = 0$, then $S(\underline{\mathfrak{X}}) = \mathfrak{X}_{0,\eta} = \mathcal{M}(k)$. If $l = 1$, then $S(\underline{\mathfrak{X}}) = S(\mathfrak{X}_1)$ and, if $l \ge 2$, then the skeleton $S(\underline{\mathfrak{X}})$ is the closed set

$$S(\underline{\mathfrak{X}}) = S(\mathfrak{X}_l/\mathfrak{X}_{l-1}) \cap f_{l-1}^{-1}(S(\underline{\mathfrak{X}}^{\le l-1}))\ .$$

The correspondence $\underline{\mathfrak{X}} \mapsto S(\underline{\mathfrak{X}})$ is a subfunctor of the following functor from $\mathcal{P}st_l^{\text{ét}}$ to the category of paracompact locally compact spaces: $\underline{\mathfrak{X}} \mapsto \mathfrak{X}_{l,\eta}$. This functor is denoted by S^l. Notice that there is a canonical morphism of functors $S(\underline{\mathfrak{X}}) \to S(\underline{\mathfrak{X}}^{\le l-1})$. One constructs the strong deformation retraction $\Phi : \mathfrak{X}_{l,\eta} \times [0,l] \to \mathfrak{X}_{l,\eta}$ of $\mathfrak{X}_{l,\eta}$ to $S(\underline{\mathfrak{X}})$ inductively as a composition of the strong deformation retraction $\Psi_{f_{l-1}}$ of $\mathfrak{X}_{l,\eta}$ to $S(\mathfrak{X}_l/\mathfrak{X}_{l-1})$ with a strong deformation retraction of $S(\mathfrak{X}_l/\mathfrak{X}_{l-1})$ to $S(\underline{\mathfrak{X}})$, which is a certain lifting of the strong deformation retraction $\Phi : \mathfrak{X}_{l-1,\eta} \times [0, l-1] \to \mathfrak{X}_{l-1,\eta}$. One has $(x_t)_{t'} = x_{\max(t,t')}$ for all points $x \in \mathfrak{X}_{l,\eta}$ and all $t, t' \in [0,l]$, where $x_t = \Phi(x,t)$.

Recall that the image of every point from $S(\underline{\mathfrak{X}})$ under the reduction map $\pi : \mathfrak{X}_{l,\eta} \to \mathfrak{X}_{l,s}$ is contained in $\mathrm{str}(\mathfrak{X}_{l,s})$. The preimage of a point from $\mathrm{str}(\mathfrak{X}_{l,s})$ in $S(\underline{\mathfrak{X}})$ is a locally closed subset called a *cell of* $S(\underline{\mathfrak{X}})$. The cells form a partially ordered set $O(S(\underline{\mathfrak{X}}))$ with respect to the following ordering: $A \le B$ if $A \subset \overline{B}$. The reduction map induces an isomorphism of partially ordered sets $O(S(\underline{\mathfrak{X}})) \stackrel{\sim}{\to} \mathrm{str}(\mathfrak{X}_{l,s})$.

For example, if the valuation on k is trivial and $\underline{\mathfrak{X}}$ is nondegenerate, then $\mathrm{str}(\mathfrak{X}_{l,s})$ coincides with the set $\mathrm{gen}(\mathfrak{X}_{l,s})$ of generic points of the irreducible components of $\mathfrak{X}_{l,s}$. By [Ber7, Corollary 1.7], for any point $\boldsymbol{x} \in \mathrm{gen}(\mathfrak{X}_{l,s})$, there exists a unique point $x \in \mathfrak{X}_{l,\eta}$ with $\pi(x) = \boldsymbol{x}$. It follows that $S(\underline{\mathfrak{X}})$ is a discrete set which is the preimage of $\mathrm{gen}(\mathfrak{X}_{l,s})$ in $\mathfrak{X}_{l,\eta}$. In particular, if $\mathfrak{X}_l$ is connected, $\mathfrak{X}_{l,\eta}$ is contractible.

Given a formal scheme $\mathfrak{X}$ locally finitely presented over k°, one provides its generic fiber $\mathfrak{X}_\eta$ with a partial ordering as follows (see [Ber7, §5]). If $\mathfrak{X} = \mathrm{Spf}(A)$ is affine, then $x \le y$ if $|f(x)| \le |f(y)|$ for all $f \in A$. If $\mathfrak{X}$ is arbitrary, the partial orderings on the generic fibers of open affine subschemes of $\mathfrak{X}$ are compatible, and they define a partial ordering on $\mathfrak{X}_\eta$. Given a poly-stable fibration $\underline{\mathfrak{X}}$ over k° of length l, one has $x \le x_t$ for all $x \in \mathfrak{X}_{l,\eta}$ and all $t \in [0, t]$ and, in particular, $x \le x_\tau$, where $x_\tau = \tau(x)$.

One of the key ingredients of the above constructions is the following fact, which will be also used here. Recall (see [Ber7, §7) that a strictly poly-stable morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is said to be geometrically elementary if, for every point $\boldsymbol{x} \in \mathfrak{X}_s$, the partially ordered set $\mathrm{str}(\mathfrak{Y}_{s,\boldsymbol{x}})$ has a unique maximal element and all of the strata of $\mathfrak{Y}_{s,\boldsymbol{x}}$ are geometrically irreducible. Notice that if $\varphi' : \mathfrak{Y}' \to \mathfrak{X}$ is another strictly poly-stable morphism, which is also geometrically elementary, and we are given an étale morphism $\mathfrak{Y}' \to \mathfrak{Y}$ over $\mathfrak{X}$, then the induced map $S(\mathfrak{Y}'/\mathfrak{X}) \to S(\mathfrak{Y}/\mathfrak{X})$ is injective. The fact is as follows (see [Ber7, Corollary 7.4]). Given a strictly poly-stable morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, for every point $\boldsymbol{y} \in \mathfrak{Y}_s$ there exists an étale morphism $\mathfrak{X}' \to \mathfrak{X}$ and an open subscheme $\mathfrak{Y}' \subset \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ such that the image of $\mathfrak{Y}'_s$ in $\mathfrak{Y}_s$ contains the point $\boldsymbol{y}$ and the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ is geometrically elementary.

4.3 The dependence of $S(\underline{\mathfrak{X}})$ on $\mathfrak{X}_l$

Given a formal scheme $\mathfrak{X}$ locally finitely presented over k°, we introduce as follows a partial ordering $\preceq$ on the generic fiber $\mathfrak{X}_\eta$, which is stronger than the partial ordering $\le$ considered in [Ber7]: $x \preceq y$ if, for every étale morphism $\mathfrak{X}' \to \mathfrak{X}$ and every point $x' \in \mathfrak{X}'_\eta$ over x, there exists a point $y' \in \mathfrak{X}'_\eta$ over y such that $|f(x')| \le |f(y')|$ for all $f \in \mathcal{O}(\mathfrak{X}'_\eta)$. Notice that, given a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, for any pair of points $x, y \in \mathfrak{Y}_\eta$ with $x \preceq y$ one has $\varphi_\eta(x) \preceq \varphi_\eta(y)$.

4.3.1 Theorem. *Let* $\underline{\mathfrak{X}} = (\mathfrak{X}_l \stackrel{f_{l-1}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ *be a poly-stable fibration over* k°. *Then*

(i) *for all points* $x \in \mathfrak{X}_{l,\eta}$ *and all* $t \in [0, l]$, *one has* $x \preceq x_t$;

(ii) *if $\underline{\mathfrak{X}}$ is nondegenerate, the skeleton $S(\underline{\mathfrak{X}})$ coincides with the set of the points of $\mathfrak{X}_{l,\eta}$ which are maximal with respect to the partial ordering $\preceq$.*

Proof. (i) Given an étale morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}_l$ and a point $x' \in \mathfrak{X}'_\eta$ over x, let $\underline{\mathfrak{X}}'$ be the poly-stable fibration $(\mathfrak{X}' \xrightarrow{f_{l-1}\circ\varphi} \mathfrak{X}_{l-1} \xrightarrow{f_{l-2}} \cdots \xrightarrow{f_1} \mathfrak{X}_1)$. By [Ber7, Theorem 8.1(viii)], one has $\varphi_\eta(x'_t) = x_t$. Since $x' \preceq x'_t$, it follows that $x \preceq x_t$.

(ii) By (i), the skeleton $S(\underline{\mathfrak{X}})$ contains the set of maximal points and, therefore, it remains to show that for any pair of distinct points $x, y \in S(\underline{\mathfrak{X}})$ none of the inequalities $x \preceq y$ or $y \preceq x$ is true. Since this property is local in the étale topology, we may assume that all formal schemes $\mathfrak{X}_i$ are affine, i.e., $\mathfrak{X}_i = \mathrm{Spf}(A_i)$, and every morphism $f_i : \mathfrak{X}_{i+1} \to \mathfrak{X}_i$ goes through an étale $\mathfrak{X}_{i+1} \to \mathfrak{X}_i(\boldsymbol{n}_i, \boldsymbol{a}_i, m_i)$ and is geometrically elementary. It suffices to show that there exist two functions $f, g \in A_l$ with $|f(x)| < |f(y)|$ and $|g(x)| > |g(y)|$. This is trivially true for $l = 0$, and assume that $l \geq 1$ and that this is true for $l - 1$. We may assume that the images of x and y in $S(\underline{\mathfrak{X}}^{\leq l-1})$ coincide. Let z be this image. If $\boldsymbol{n}_l = (n_0, \dots, n_p)$ and $\boldsymbol{a}_l = (a_0, \dots, a_p)$, then for every $0 \leq i \leq p$ one has $|(T_{i0} \dots T_{in_i})(x)| = |(T_{i0} \dots T_{in_i})(y)| = |a_i(z)|$. Notice that $|a_i(z)| \neq 0$ because $\underline{\mathfrak{X}}$ is nondegenerate. Since the morphisms $\mathfrak{X}_l \to \mathfrak{X}_{l-1}$ and $\mathfrak{Y} = \mathfrak{X}_{l-1}(\boldsymbol{n}_l, \boldsymbol{a}_l, m_l) \to \mathfrak{X}_{l-1}$ are geometrically elementary, it follows that the canonical map $S(\mathfrak{X}_l/\mathfrak{X}_{l-1}) \to S(\mathfrak{Y}/\mathfrak{X}_{l-1})$ is injective and, therefore, there exist $0 \leq i \leq p$ and $0 \leq j \leq n_i$ with $|T_{ij}(x)| \neq |T_{ij}(y)|$. Assume that $|T_{ij}(x)| < |T_{ij}(y)|$. Then for $g = T_{i0} \dots T_{i,j-1}T_{i,j+1} \dots T_{in_i}$ one has $|g(x)| > |g(y)|$. □

Thus, the skeleton $S(\mathfrak{X})$ is well defined for any nondegenerate pluri-stable formal scheme $\mathfrak{X}$.

4.3.2 Corollary. *Let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be a pluri-stable morphism between nondegenerate pluri-stable formal schemes over k°. Then*

(i) $\varphi_\eta(S(\mathfrak{Y})) \subset S(\mathfrak{X})$*;*

(ii) *if φ is étale, then $S(\mathfrak{Y}) = \varphi_\eta^{-1}(S(\mathfrak{X}))$.*

Proof. (i) By Corollary 4.1.4, it suffices to consider the case when the morphism φ is nondegenerate poly-stable. Assume that $\mathfrak{X} = \mathfrak{X}_{l-1}$ for a nondegenerate poly-stable fibration $(\mathfrak{X}_{l-1} \xrightarrow{f_{l-2}} \cdots \xrightarrow{f_1} \mathfrak{X}_1)$ of length $l - 1$, and we set $\underline{\mathfrak{X}} = (\mathfrak{Y} \xrightarrow{\varphi} \mathfrak{X}_{l-1} \xrightarrow{f_{l-2}} \cdots \xrightarrow{f_1} \mathfrak{X}_1)$. Then the morphism φ takes $S(\underline{\mathfrak{X}})$ to $S(\underline{\mathfrak{X}}^{\leq l-1})$. Since $S(\mathfrak{Y}) = S(\underline{\mathfrak{X}})$ and $S(\mathfrak{X}) = S(\underline{\mathfrak{X}}^{\leq l-1})$, the required fact follows.

(ii) By (i), one has $S(\mathfrak{Y}) \subset \varphi_\eta^{-1}(S(\mathfrak{X}))$. Let $x \in S(\mathfrak{X})$ and $y \in \varphi_\eta^{-1}(x)$. By Theorem 4.3.1(i), one has $y \preceq y_t$ and, therefore, $y_t \in \varphi_\eta^{-1}(x)$ for all $t \in [0, l]$. Since $\varphi_\eta^{-1}(x)$ is a discrete topological space, it follows that $y = y_l \in S(\mathfrak{Y})$. □

Corollary 4.3.2 implies that the correspondence $\mathfrak{X} \mapsto S(\mathfrak{X})$ is a subfunctor of the functor $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ on the category $\mathcal{P}\ell st^{\mathrm{pl}}_{\mathrm{nd}}$.

4.3.3 Remarks. (i) For any nondegenerate pluri-nodal formal scheme $\mathfrak{X}$, there exists a surjective étale morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ from a nondegenerate strictly pluri-stable formal scheme $\mathfrak{Y}$. From Corollary 4.3.2 it follows that the image of $S(\mathfrak{Y})$ in $\mathfrak{X}_\eta$ does not depend on the choice of φ and coincides with the set of maximal points with respect to the partial ordering $\preceq$ on $\mathfrak{X}_\eta$. It can be called the skeleton $S(\mathfrak{X})$ of $\mathfrak{X}$, and both statements of Corollary 4.3.2 hold for any pluri-nodal morphism φ.

(ii) In our work in progress, we give a similar description of the skeleton $S(\underline{\mathfrak{X}})$ of an arbitrary poly-stable fibration $\underline{\mathfrak{X}}$ of length l as the set of maximal points with respect to a partial ordering on $\mathfrak{X}_{l,\eta}$ which is stronger than the above one (but coincides with it if $\underline{\mathfrak{X}}$ is nondegenerate).

4.4 The retraction map $\tau : \mathfrak{X}_{l,\eta} \to S(\mathfrak{X}_l)$

Let $\underline{\mathfrak{X}} = (\mathfrak{X}_l \stackrel{f_{l-1}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ be a poly-stable fibration over k° of length l. In this subsection we recall the construction of the retraction map $\tau = \tau_{\underline{\mathfrak{X}}} : \mathfrak{X}_{l,\eta} \to S(\underline{\mathfrak{X}})$, and we introduce a class of nondegenerate poly-stable fibrations $\underline{\mathfrak{X}}$ for which the retraction map τ depends only on $\mathfrak{X}_l$.

Assume that $l \geq 1$ and that the retraction map is already constructed for poly-stable fibrations of length $l-1$. Consider first the case when $\mathfrak{X}_{l-1}$ is affine and $\mathfrak{X}_l = \mathfrak{X}_{l-1}(\boldsymbol{n}, \boldsymbol{a}, m)$ with $\boldsymbol{n} = (n_0, \dots, n_p)$ and $\boldsymbol{a} = (a_0, \dots, a_p)$. The continuous mapping

$$\mathfrak{X}_{l,\eta} \to \mathfrak{X}_{l-1,\eta} \times [0,1]^{[\boldsymbol{n}]} : y \mapsto (f_{l-1}(y); |T_{i0}(y)|, \dots, |T_{i,n_i}(y)|)_{0 \leq i \leq p}$$

induces a homeomorphism between $S(\mathfrak{X}_l/\mathfrak{X}_{l-1})$ and the closed set

$$S = \{(x; \boldsymbol{t}) \in \mathfrak{X}_{l-1,\eta} \times [0,1]^{[\boldsymbol{n}]} \mid t_{0i} \dots t_{in_i} = |a_i(x)|,\ 0 \leq i \leq p\},$$

and it gives rise to a retraction map $\rho : \mathfrak{X}_{l,\eta} \to S$. If $l = 1$, then $S(\underline{\mathfrak{X}}) = S$ and $\tau = \rho$. Assume therefore that $l \geq 2$. In this case the retraction map τ is a composition of the above map ρ with a retraction map $\gamma : S \to S(\underline{\mathfrak{X}})$ constructed as follows (see [Ber7, p. 62]).

First of all, one defines for each $n \geq 0$ a strong deformation retraction $\psi_n : [0,1]^{[n]} \times [0,1] \to [0,1]^{[n]}$ to the point $(1, \dots, 1)$. The map ψ_n is required to possess the property that $\psi_n(\sigma(\boldsymbol{t}), s) = \sigma(\psi_n(\boldsymbol{t}, s))$ for all permutations σ of degree $n+1$, and so it suffices to define $\psi_n(\boldsymbol{t}, s)$ only for the points $\boldsymbol{t} \in [0,1]^{[n]}$ with $t_0 \leq t_1 \leq \cdots \leq t_n$. First, if $s \leq t_0 \dots t_n$, then $\psi_n(\boldsymbol{t}, s) = \boldsymbol{t}$. Furthermore, if $t_i^{i+1} t_{i+1} \dots t_n \leq s < t_{i+1}^{i+2} t_{i+2} \dots t_n$ for some $0 \leq i \leq n-1$, then

$$\psi_n(\boldsymbol{t}, s) = \left(\left(\frac{s}{t_{i+1} \dots t_n}\right)^{\frac{1}{i+1}}, \dots, \left(\frac{s}{t_{i+1} \dots t_n}\right)^{\frac{1}{i+1}}, t_{i+1}, \dots, t_n\right).$$

Finally, if $s \geq t_n^{n+1}$, then $\psi_n(\boldsymbol{t}, s) = \left(s^{\frac{1}{n+1}}, \ldots, s^{\frac{1}{n+1}}\right)$. The retraction map $\gamma : S \to S(\underline{\mathfrak{X}})$ is as follows

$$\gamma(x, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_p) = \left(x_\tau; \psi_{n_0}(\boldsymbol{t}_0, |a_0(x_\tau)|), \ldots, \psi_{n_p}(\boldsymbol{t}_p, |a_p(x_\tau)|)\right) .$$

If $\underline{\mathfrak{X}}$ is such that $\mathfrak{X}_{l-1}$ is affine, the morphism $f_{l-1} : \mathfrak{X}_l \to \mathfrak{X}_{l-1}$ is geometrically elementary and goes through an étale morphism $\varphi : \mathfrak{X}_l \to \mathfrak{Y} = \mathfrak{X}_{l-1}(\boldsymbol{n}, \boldsymbol{a}, m)$, then the morphism $\underline{\mathfrak{X}} \to \underline{\mathfrak{Y}} = (\mathfrak{Y} \to \mathfrak{X}_{l-1} \stackrel{f_{l-2}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ gives rise to embeddings $S(\mathfrak{X}_l/\mathfrak{X}_{l-1}) \hookrightarrow S(\mathfrak{Y}/\mathfrak{X}_{l-1})$ and $S(\underline{\mathfrak{X}}) \hookrightarrow S(\underline{\mathfrak{Y}})$, and the above retraction maps $\rho : \mathfrak{Y}_\eta \to S(\mathfrak{Y}/\mathfrak{X}_{l-1})$ and $\gamma : S(\mathfrak{Y}/\mathfrak{X}_{l-1}) \to S(\underline{\mathfrak{Y}})$ give rise to retractions maps $\rho : \mathfrak{X}_{l,\eta} \to S(\mathfrak{X}_l/\mathfrak{X}_{l-1})$ and $\gamma : S(\mathfrak{X}_l/\mathfrak{X}_{l-1}) \to S(\underline{\mathfrak{X}})$. The latter do not depend on the choice of φ, and one has $\tau = \gamma \circ \rho$.

If $\underline{\mathfrak{X}}$ is arbitrary, one can find surjective étale morphisms $\underline{\mathfrak{X}}' \to \underline{\mathfrak{X}}$ and $\underline{\mathfrak{X}}'' \to \underline{\mathfrak{X}}' \times_{\underline{\mathfrak{X}}} \underline{\mathfrak{X}}'$ such that the morphisms $f'_{l-1} : \mathfrak{X}'_l \to \mathfrak{X}'_{l-1}$ and $f''_{l-1} : \mathfrak{X}''_l \to \mathfrak{X}''_{l-1}$ are disjoint unions of morphisms satisfying the assumptions of the previous paragraph. Since the retraction maps $\tau' : \mathfrak{X}'_{l,\eta} \to S(\underline{\mathfrak{X}}')$ and $\tau'' : \mathfrak{X}''_{l,\eta} \to S(\underline{\mathfrak{X}}'')$ are compatible, they give rise to a retraction map $\tau : \mathfrak{X}_{l,\eta} \to S(\underline{\mathfrak{X}})$.

We say that a strictly poly-stable morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is *strongly nondegenerate* if, for every point $\boldsymbol{y} \in \mathfrak{Y}$, there exist an open affine neighborhood $\mathfrak{X}' = \mathrm{Spf}(A)$ of $\varphi(\boldsymbol{y})$ and an open neighborhood $\mathfrak{Y}' \subset \varphi^{-1}(\mathfrak{X}')$ of $\boldsymbol{y}$ such that the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ goes through an étale morphism $\mathfrak{Y}' \to \mathfrak{X}'(\boldsymbol{n}, \boldsymbol{a}, m)$, where all a_i are invertible in $\mathcal{A} = A \otimes_{k^\circ} k$ (i.e., $a_i(x) \neq 0$ for all $x \in \mathfrak{X}'_\eta$). A poly-stable morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is said to be *strongly nondegenerate* if there exists a surjective étale morphism $\mathfrak{Y}' \to \mathfrak{Y}$ for which the induced morphism $\mathfrak{Y}' \to \mathfrak{X}$ is strongly nondegenerate strictly poly-stable. For example, a poly-stable formal scheme $\mathfrak{X}$ over k° is strongly nondegenerate if and only if it is nondegenerate.

One can easily see that a poly-stable morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is strongly nondegenerate if and only if the induced morphism of strictly k-analytic spaces $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is smooth in the sense of rigid geometry (or rig-smooth). (A morphism of strictly k-analytic spaces $\varphi : Y \to X$ is said to be *rig-smooth* if every point $y \in Y$ has a neighborhood of the form $V_1 \cup \cdots \cup V_n$, where each V_i is a strictly affinoid subdomain of Y such that the induced morphism $V_i \to X$ goes through a quasi-étale morphism (see [Ber5, §3]) to an affine space $\boldsymbol{A}^m_X$ over X. A morphism between good strictly k-analytic spaces is smooth in the sense of [Ber2] if and only if it is rig-smooth and has no boundary.)

A pluri-stable formal scheme $\mathfrak{X}$ over k° is said to be *strongly nondegenerate* if the canonical morphism $\mathfrak{X} \to \mathrm{Spf}(k^\circ)$ is a sequence of strongly nondegenerate poly-stable morphisms. Similarly, a poly-stable fibration $\underline{\mathfrak{X}}$ of length l is said to be *strongly nondegenerate* if all the morphisms $f_i : \mathfrak{X}_{i+1} \to \mathfrak{X}_i$ are strongly nondegenerate.

4.4.1 Theorem. *Let $\underline{\mathfrak{X}}$ be a strongly nondegenerate poly-stable fibration of formal schemes. Then, for every point $x \in \mathfrak{X}_{l,\eta}$, x_τ is a unique point of $S(\underline{\mathfrak{X}}) = S(\mathfrak{X}_l)$ which is greater than or equal to x (with respect to the partial ordering $\preceq$ on $\mathfrak{X}_{l,\eta}$).*

Proof. As in the proof of Theorem 4.3.1(ii), the property considered is local in the étale topology and, therefore, we may assume that all formal schemes $\mathfrak{X}_i$ are affine, and every morphism $f_i : \mathfrak{X}_{i+1} \to \mathfrak{X}_i$ goes through an étale $\mathfrak{X}_{i+1} \to \mathfrak{X}_i(\boldsymbol{n}_i, \boldsymbol{a}_i, m_i)$ and is geometrically elementary. For every $0 \le i \le p$, one has $T_{i0} \dots T_{in_i} = a_i$. Since a_i are invertible on $\mathfrak{X}_{l-1,\eta}$, it follows that all of the coordinate functions T_{ij} are invertible on $\mathfrak{X}_{l,\eta}$. Since $x \le x_\tau$, the latter implies that $|T_{ij}(x)| = |T_{ij}(x_\tau)|$. But, by the proof of Theorem 4.3.1(ii), we know that for any pair of distinct points $y, z \in S(\underline{\mathfrak{X}})$ there exist functions $f, g \in A_l$ which are representable in the form of products of coordinate functions and such that $|f(y)| < |g(z)|$ and $|f(y)| > |g(z)|$. The required fact follows. ◻

From Theorem 4.4.1 it follows that, for any strongly nondegenerate poly-stable formal scheme $\mathfrak{X}$, there is a well defined retraction map $\tau : \mathfrak{X}_\eta \to S(\mathfrak{X})$.

4.4.2 Corollary. *Let $\mathfrak{X}$ be a strongly nondegenerate pluri-stable formal scheme. Given a poly-stable fibration $\underline{\mathfrak{X}}'$ of length l and a morphism of formal schemes $\varphi : \mathfrak{X}'_l \to \mathfrak{X}$, the following diagram is commutative:*

$$\begin{array}{ccc} S(\underline{\mathfrak{X}}') & \xrightarrow{\tau\circ\varphi_\eta} & S(\mathfrak{X}) \\ {\scriptstyle\tau'}\uparrow & & \uparrow{\scriptstyle\tau} \\ \mathfrak{X}'_{l,\eta} & \xrightarrow[\varphi_\eta]{} & \mathfrak{X}_\eta . \end{array}$$

where τ' is the retraction map associated with $\underline{\mathfrak{X}}'$.

Proof. For every point $x' \in \mathfrak{X}_{l,\eta}$, one has $x' \preceq x'_\tau$ and, therefore, $\varphi_\eta(x') \preceq \varphi_\eta(x'_\tau)$. Theorem 4.4.1 implies that $\varphi_\eta(x')_\tau = \varphi_\eta(x'_\tau)_\tau$. ◻

5 A colored polysimplicial set associated with a nondegenerate poly-stable fibration

5.1 Formulation of the result

In this section we construct for every non-Archimedean field k and every $l \ge 0$ a functor $k^\circ\text{-}\mathcal{P}st^{\text{ét}}_{\text{nd},l} \to \Lambda^{\circ,\text{lf}}_{R^k,l}\mathcal{E}ns$, where $R^k = |k^*| \cap [0,1]$. This family of functors for different k's forms a functor between fibered categories over the category dual to that of non-Archimedean fields. The first one is the category $\mathcal{P}st^{\text{ét}}_{\text{nd},l}$, and the second one is the category $\widetilde{\Lambda}^{\circ,\text{lc}}_l\mathcal{E}ns$ introduced as follows. Its objects are pairs (k, D) consisting of a non-Archimedean field k and a locally finite polysimplicial set $D \in \text{Ob}(\Lambda^{\circ,\text{lf}}_{R^k,l}\mathcal{E}ns)$, and morphisms $(k', D') \to (k, D)$ are pairs consisting of an isometric embedding

$k \hookrightarrow k'$ and a morphism $D' \to D$ in $\Lambda^{\circ,\mathrm{lf}}_{R^{k'},l}\mathcal{E}ns$. For brevity, a pair (k, D) is denoted by D.

5.1.1 Theorem-Construction. *One can construct for every $l \geq 0$:*

(a) *a functor of fibered categories $\mathbb{D}^l : \mathcal{P}st^{\text{ét}}_{\mathrm{nd},l} \to \widetilde{\Lambda}^{\circ,\mathrm{lc}}_l\mathcal{E}ns$ (it takes $\underline{\mathfrak{X}}$ to $\mathbb{D}(\underline{\mathfrak{X}})$),*

(b) *an isomorphism of functors $\theta_l : |\mathbb{D}(\underline{\mathfrak{X}})| \stackrel{\sim}{\to} S(\underline{\mathfrak{X}})$, and*

(c) *a morphism of functors $\mathbb{D}(\underline{\mathfrak{X}})^{\leq l-1} \to \mathbb{D}(\underline{\mathfrak{X}}^{\leq l-1})$ compatible with θ_l and θ_{l-1},*

which possess the following properties:

(1) *if $\underline{\mathfrak{X}}$ is strictly poly-stable, the polysimplicial set $\mathbb{D}(\underline{\mathfrak{X}})$ is free;*

(2) *given a surjective étale morphism $\underline{\mathfrak{X}}' \to \underline{\mathfrak{X}}$, there is an isomorphism of polysimplicial sets $\mathrm{Coker}(\mathbb{D}(\underline{\mathfrak{X}}' \times_{\underline{\mathfrak{X}}} \underline{\mathfrak{X}}') \stackrel{\to}{\to} \mathbb{D}(\underline{\mathfrak{X}}')) \stackrel{\sim}{\to} \mathbb{D}(\underline{\mathfrak{X}})$;*

(3) *the homeomorphism $\theta_l : |\mathbb{D}(\underline{\mathfrak{X}})| \stackrel{\sim}{\to} S(\underline{\mathfrak{X}})$ induces an isomorphism of partially ordered sets $O(|\mathbb{D}(\underline{\mathfrak{X}})|) \stackrel{\sim}{\to} O(S(\underline{\mathfrak{X}}))$;*

(4) *for every $g \in \mathcal{O}'(\mathfrak{X}_l)$, one has $\theta^*_l(|g|) \in M_{\mathbb{D}(\underline{\mathfrak{X}})}$, where $|g|$ is the function $x \mapsto |g(x)|$;*

(5) *if $\underline{\mathfrak{X}}$ is strictly poly-stable, then each point of $\mathfrak{X}_l$ has an open affine neighborhood $\mathfrak{X}' = \mathrm{Spf}(A)$ such that, for $\underline{\mathfrak{X}}' = (\mathfrak{X}' \stackrel{f_{l-1}}{\to} \mathfrak{X}_{l-2} \stackrel{f_{l-2}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$, $\mathbb{D}(\underline{\mathfrak{X}}')$ is a standard polysimplex $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ and the map $A\backslash\{0\} \to M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}} : g \mapsto \theta^*_l(|g|)$ is surjective.*

The construction is done by induction in §§5.2–5.5. If $l = 0$, then $\underline{\mathfrak{X}} = (\mathfrak{X}_0 = \mathrm{Spf}(k^\circ))$, $S(\underline{\mathfrak{X}}) = \mathfrak{X}_{0,\eta} = \mathcal{M}(k)$ and $\mathbb{D}(\underline{\mathfrak{X}}) = \Lambda[0]_1$. Assume that $l \geq 1$ and that the above objects are already constructed for $l - 1$. For a polysimplex $d \in \mathbb{D}(\underline{\mathfrak{X}})^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$, we shall denote by $\overline{\sigma}_d$ the map $\Sigma^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}} \stackrel{\sigma_d}{\to} |\mathbb{D}(\underline{\mathfrak{X}})| \stackrel{\theta_l}{\to} S(\underline{\mathfrak{X}})$.

5.1.2 Remark. In our work in progress, we extend the above construction to the whole class of poly-stable fibrations. Namely, we construct a functor $\underline{\mathfrak{X}} \mapsto \mathbb{D}(\underline{\mathfrak{X}})$ from the category of all poly-stable fibrations of length l over k° to the category of $|k^\circ|$-colored polysimplicial sets of length l, an isomorphism of functors $\theta_l : |\mathbb{D}(\underline{\mathfrak{X}})| \stackrel{\sim}{\to} S(\underline{\mathfrak{X}})$, and a morphism of functors $\mathbb{D}(\underline{\mathfrak{X}})^{\leq l-1} \to \mathbb{D}(\underline{\mathfrak{X}}^{\leq l-1})$. They possess the same properties (1)–(5) with the only difference that, in (4), $\theta^*_l(|g|) \in M_{\mathbb{D}(\underline{\mathfrak{X}})}$ for all $g \in \mathcal{O}(\mathfrak{X}_l)$ and, in (5), the map $A \to M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}} : g \mapsto \theta^*_l(|g|)$ is surjective. The combinatorial part of the proof of Theorem 5.1.1 in §§5.2–5.4 works also in the general case. The assumption on nondegenerateness of $\underline{\mathfrak{X}}$ is used here only for the verification of the property (4) in §5.5 since, in the general case, its verification is more involved.

5.2 Construction of $\mathbb{D}(\underline{\mathfrak{X}})$ for strictly poly-stable $\underline{\mathfrak{X}}$

Before starting the construction, we recall some facts from [Ber7, §3]. Let $\mathcal{X}$ be a strictly poly-stable scheme over a field K. For a point $x \in \mathrm{str}(\mathcal{X})$, the set $\mathrm{irr}(\mathcal{X}, x)$ of the irreducible components of $\mathcal{X}$ passing through x is provided with a metric as follows: the distance between two components $X, X' \in \mathrm{irr}(\mathcal{X}, x)$ is the codimension of the intersection $X \cap X'$ at the point x. Given an étale morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ and a point $y \in \mathrm{str}(\mathcal{Y})$, for any point $x \in \mathrm{str}(\mathcal{X})$ with $\varphi(y) \le x$ the canonical map $\mathrm{irr}(\mathcal{Y}, y) \to \mathrm{irr}(\mathcal{X}, x)$ is isometric. For example, if $\mathcal{T} = \mathcal{T}_0 \times \cdots \times \mathcal{T}_p \times \mathcal{S}$, where $\mathcal{T}_i = \mathrm{Spec}(K[T_{i0}, \dots, T_{in_i}]/(T_{i0} \dots T_{in_i}))$ with $n_i \ge 1$ and $\mathcal{S} = \mathrm{Spec}(K[S_1, \dots, S_m, S_1^{-1}, \dots, S_m^{-1}])$, then there is an isometric bijection $[\boldsymbol{n}] \overset{\sim}{\to} \mathrm{irr}(\mathcal{T}, t)$ that takes $\boldsymbol{j} = (j_0, \dots, j_p) \in [\boldsymbol{n}]$ to the irreducible component defined by the equations $T_{0j_0} = \cdots = T_{pj_p} = 0$, where t is the maximal point in $\mathrm{str}(\mathcal{T})$. Thus, any étale morphism $\varphi : \mathcal{X}' \to \mathcal{T}$ from an open neighborhood $\mathcal{X}'$ of the point x to the above scheme $\mathcal{T}$, that takes x to the above point t, gives rise to an isometric bijection $\mu_\varphi : [\boldsymbol{n}] \overset{\sim}{\to} \mathrm{irr}(\mathcal{X}, x)$. The latter property of φ is equivalent to the fact that all of the coordinate functions T_{ij} vanish at the point x.

Let $\underline{\mathfrak{X}} = (\mathfrak{X}_l \overset{f_{l-1}}{\to} \cdots \overset{f_1}{\to} \mathfrak{X}_1)$ be a nondegenerate strictly poly-stable fibration over k°. We set $\mathfrak{X} = \mathfrak{X}_{l-1}$, $\mathfrak{Y} = \mathfrak{X}_l$ and $\varphi = f_{l-1}$. By induction, there is a free locally finite polysimplicial set $C = \mathbb{D}(\underline{\mathfrak{X}}^{\le l-1})$ and a continuous map $|C| \to \mathfrak{X}_\eta$ that identifies $|C|$ with $S(\underline{\mathfrak{X}}^{\le l-1})$. Since $O(C) \overset{\sim}{\to} O(|C|)$ and $O(S(\underline{\mathfrak{X}}^{\le l-1})) \overset{\sim}{\to} \mathrm{str}(\mathfrak{X}_s)$, the latter map induces an isomorphism of partially ordered sets $O(C) \overset{\sim}{\to} \mathrm{str}(\mathfrak{X}_s) : (\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \overset{c}{\to} C) \mapsto \overline{c}$. We construct as follows an R^k-colored polysimplicial set D of length l.

Given $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \in \mathrm{Ob}(\Lambda_{R^k,l})$, let $D_{\underline{\boldsymbol{n}}}^{\underline{\boldsymbol{r}}}$ be the set of all triples $d = (\boldsymbol{y}, c, \mu)$ consisting of a point $\boldsymbol{y} \in \mathrm{str}(\mathfrak{Y}_s)$, a polysimplex $c \in C_{\underline{\boldsymbol{n}}^{\le l-1}}^{\underline{\boldsymbol{r}}^{\le l-1}}$ with $\overline{c} = \boldsymbol{x}$, where $\boldsymbol{x} = \varphi_s(\boldsymbol{y}) \in \mathrm{str}(\mathfrak{X}_s)$, and an isometric bijection $\mu : [\boldsymbol{n}^{(l)}] \overset{\sim}{\to} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}}, \boldsymbol{y})$ such that there exists an open affine neighborhood $\mathfrak{X}' \subset \mathfrak{X}$ of $\boldsymbol{x}$ and an open neighborhood $\mathfrak{Y}' \subset \varphi^{-1}(\mathfrak{X}')$ of $\boldsymbol{y}$ for which the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ goes through an étale morphism $\psi : \mathfrak{Y}' \to \mathfrak{X}'(\boldsymbol{n}^{(l)}, \boldsymbol{a}, m)$ such that all of the coordinate functions T_{ij} of $\mathfrak{X}'(\boldsymbol{n}^{(l)}, \boldsymbol{a}, m)$ vanish at $\boldsymbol{y}$, $\mu_\psi = \mu$ and $\overline{\sigma}_c^*(|\boldsymbol{a}|) = \boldsymbol{r}^{(l)}$. From [Ber7, Proposition 4.3] it follows that the object $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ is uniquely defined by the triple $d = (\boldsymbol{y}, c, \mu)$.

Furthermore, let $\gamma : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ be a morphism in $\Lambda_{R^k,l}$. It gives rise to a morphism $\gamma^{\le l-1} : [\underline{\boldsymbol{n}}'^{\le l-1}]_{\underline{\boldsymbol{r}}'^{\le l-1}} \to [\underline{\boldsymbol{n}}^{\le l-1}]_{\underline{\boldsymbol{r}}^{\le l-1}}$, and we set $c' = C(\gamma^{\le l-1})(c) \in C_{\underline{\boldsymbol{n}}'^{\le l-1}}^{\underline{\boldsymbol{r}}'^{\le l-1}}$ and $\boldsymbol{x}' = \overline{c}' \in \mathrm{str}(\mathfrak{X}_s)$. One has $\boldsymbol{x}' \le \boldsymbol{x}$ and, by [Ber7, Proposition 2.9], the set of points $\boldsymbol{y}' \in \mathrm{str}(\mathfrak{Y}_s)$ with $\varphi_s(\boldsymbol{y}') = \boldsymbol{x}'$ and $\boldsymbol{y}' \le \boldsymbol{y}$ is non-empty and has a unique maximal point. Let $\boldsymbol{y}''$ be this point. By [Ber7, Lemma 6.1], there exists a unique pair (J, μ'') consisting of a subset $J \subset \omega(\boldsymbol{n}^{(l)})$ and an isometric bijection

$\mu'' : [\boldsymbol{n}_J^{(l)}] \xrightarrow{\sim} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}'')$ for which the following diagram is commutative

$$\begin{array}{ccc} [\boldsymbol{n}^{(l)}] & \xrightarrow[\mu]{\sim} & \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}}, \boldsymbol{y}) \\ \downarrow & & \downarrow \\ [\boldsymbol{n}_J^{(l)}] & \xrightarrow[\mu'']{\sim} & \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}''). \end{array} \tag{5.1}$$

(Here the left vertical arrow is the canonical projection, and the right one is from [Ber7, Proposition2.9].) By the proof of *loc. cit.*, one has $J = \{j \in \omega(\boldsymbol{n}^{(l)}) \mid a_j(\boldsymbol{x}') = 0$ in $\widetilde{k}(\boldsymbol{x}')\}$, i.e., $J = \{j \in \omega(\boldsymbol{n}^{(l)}) \mid |a_j(x)| < 1$ for some (and therefore all) $x \in \pi^{-1}(\boldsymbol{x}')\}$. It follows that J is precisely the set of all $j \in \omega(\boldsymbol{n}^{(l)})$ with $r_j(x) < 1$ for some $x \in \mathrm{Im}(\Sigma(\gamma^{\le l-1}))$ and, therefore, the morphism γ gives rise to a morphism $\gamma^{(l)} : [\boldsymbol{n}'^{(l)}] \to [\boldsymbol{n}_J^{(l)}]$ in Λ such that $r'_j = r_{f(j)} \circ \Sigma(\gamma^{\le l-1})$ for all $j \in \omega(\boldsymbol{n}'^{(l)})$, where f is the map $\omega(\boldsymbol{n}'^{(l)}) \to J$ defined by $\gamma^{\le l-1}$ (see §2.1). By [Ber7, Lemma 3.13], there exists a unique pair $(\boldsymbol{y}', \mu')$ consisting of a point $\boldsymbol{y}' \in \mathrm{str}(\mathfrak{Y}_{s,\boldsymbol{x}'})$ with $\boldsymbol{y}' \le \boldsymbol{y}''$ and an isometric bijection $\mu' : [\underline{\boldsymbol{n}}'^{(l)}] \xrightarrow{\sim} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}')$ for which the following diagram is commutative

$$\begin{array}{ccc} [\boldsymbol{n}_J^{(l)}] & \xrightarrow[\mu'']{\sim} & \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}'') \\ {\scriptstyle \gamma^{(l)}}\uparrow & & \uparrow \\ [\boldsymbol{n}'^{(l)}] & \xrightarrow[\mu']{\sim} & \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}'). \end{array} \tag{5.2}$$

Let now $\mathfrak{Y}''$ be the open subscheme of $\mathfrak{Y}'$ where all of the coordinate functions of $\mathfrak{X}'(\boldsymbol{n}^{(l)}, \boldsymbol{a}, m)$, which do not vanish at the point $\boldsymbol{y}'$, are invertible. We also set $a'_j = a_{f(j)}$ for $j \in \omega(\boldsymbol{n}'^{(l)})$. Then the morphism $\mathfrak{Y}'' \to \mathfrak{X}'$ goes through an étale morphism $\psi' : \mathfrak{Z}'' \to \mathfrak{X}'(\boldsymbol{n}'^{(l)}, \boldsymbol{a}', m')$ (for some $m' \ge 0$). Thus, the triple $d' = (\boldsymbol{y}', c', \mu')$ is an element of $D_{\underline{\boldsymbol{n}}'}^{\boldsymbol{r}'}$, and we get an R^k-colored polysimplicial set D of length l.

We claim that the following is true:

(i) *the polysimplicial set D is free and locally finite;*

(ii) *the correspondence $d = (\boldsymbol{y}, c, \mu) \mapsto \boldsymbol{y}$ defines an isomorphism of partially ordered sets $O(D) \xrightarrow{\sim} \mathrm{str}(\mathfrak{Y}_s)$ over the isomorphism $O(C) \xrightarrow{\sim} \mathrm{str}(\mathfrak{X}_s)$;*

(iii) *the morphism $D^{\le l-1} \to C : d = (\boldsymbol{y}, c, \mu) \mapsto c$ (see § 2.4) is surjective (resp. injective) if and only if the map $\mathrm{str}(\mathfrak{Y}_s) \to \mathrm{str}(\mathfrak{X}_s)$ is surjective (resp. for every $\boldsymbol{x} \in \mathrm{str}(\mathfrak{X}_s)$, $\mathfrak{Y}_{s,\boldsymbol{x}}$ is connected).*

(i) That D is locally finite is trivial. To show that it is free, we have to verify that, given $d = (\boldsymbol{y}, c, \mu) \in D_{\underline{\boldsymbol{n}}}^{\boldsymbol{r}}$ and two morphisms $\gamma_1, \gamma_2 : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ with $D(\gamma_1)(d) = D(\gamma_2)(d)$, then $\gamma_1 = \gamma_2$. Let $d' = (\boldsymbol{y}', c', \mu') = D(\gamma_1)(d)$. Since

$c' = C(\gamma_1^{\leq l-1})(c) = C(\gamma_2^{\leq l-1})(c)$ and C is free, it follows that $\gamma_1^{\leq l-1} = \gamma_2^{\leq l-1}$. The equality $\gamma_1^{(l)} = \gamma_2^{(l)}$ now follows from the fact that both morphisms appear as left vertical arrows in the corresponding diagrams (5.2) with the same sets and three other arrows.

(ii) Given a point $\boldsymbol{y} \in \mathrm{str}(\mathfrak{Y}_s)$, let $c \in C^{\underline{s}}_{\underline{\boldsymbol{m}}}$ be a polysimplex with $\overline{c} = \boldsymbol{x} = \varphi_s(\boldsymbol{y})$. One can find an open affine neighborhood $\mathfrak{X}'$ of $\boldsymbol{x}$ and an open neighborhood $\mathfrak{Y}' \subset \varphi^{-1}(\mathfrak{X}')$ of $\boldsymbol{y}$ for which the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ goes through an étale morphism $\psi : \mathfrak{Y}' \to \mathfrak{X}'(\boldsymbol{n}, \boldsymbol{a}, m)$ such that all of the coordinate functions T_{ij} on $\mathfrak{X}'(\boldsymbol{n}, \boldsymbol{a}, m)$ vanish at $\boldsymbol{y}$. Then the étale morphism gives rise to an isometric bijection $\mu : [\boldsymbol{n}] \stackrel{\sim}{\to} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}}, \boldsymbol{y})$. Let $\boldsymbol{a} = (a_0, \ldots, a_p)$. By the property (4), $r_i = \overline{\sigma}_c^*(|a_i|) \in M^{\underline{\boldsymbol{m}}}_{\underline{s}}$ for all $0 \leq i \leq p$. Thus, if $\underline{\boldsymbol{n}} = (\underline{\boldsymbol{m}}, \boldsymbol{n})$ and $\underline{\boldsymbol{r}} = (\underline{s}, \boldsymbol{r})$, where $\boldsymbol{r} = (r_0, \ldots, r_p)$, then the triple $d = (\boldsymbol{y}, c, \mu)$ gives rise to an element of $D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$, i.e., the canonical map $O(D) \to \mathrm{str}(\mathfrak{Y}_s) : d = (\boldsymbol{y}, c, \mu) \mapsto \boldsymbol{y}$ is surjective.

Assume now that there are two polysimplices $d = (\boldsymbol{y}, c, \mu) \in D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ and $d' = (\boldsymbol{y}', c', \mu') \in D^{\underline{\boldsymbol{r}}'}_{\underline{\boldsymbol{n}}'}$ with $\boldsymbol{y}' \leq \boldsymbol{y}$. Then for $\boldsymbol{x} = \varphi_s(\boldsymbol{y})$ and $\boldsymbol{x}' = \varphi_s(\boldsymbol{y}')$ one has $\boldsymbol{x}' \leq \boldsymbol{x}$. Since $\overline{c} = \boldsymbol{x}$, $\overline{c}' = \boldsymbol{x}'$ and $O(C) \stackrel{\sim}{\to} \mathrm{str}(\mathfrak{X}_s)$, there is a morphism $\alpha : [\underline{\boldsymbol{n}}'^{\leq l-1}]_{\underline{\boldsymbol{r}}'^{\leq l-1}} \to [\underline{\boldsymbol{n}}^{\leq l-1}]_{\underline{\boldsymbol{r}}^{\leq l-1}}$ with $c' = C(\alpha)(c)$. Let $\boldsymbol{y}'' \in \mathrm{str}(\mathfrak{Y}_{s,x'})$ be the unique maximal point with the property $\boldsymbol{y}'' \leq \boldsymbol{y}$. As above, there exists a unique pair (J, μ'') consisting of a subset $J \subset \omega(\boldsymbol{n}^{(l)})$ and an isometric bijection $\mu'' : [\boldsymbol{n}_J^{(l)}] \stackrel{\sim}{\to} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}'')$ for which the diagram (5.1) is commutative, and we know that $J = \{j \in \omega(\boldsymbol{n}^{(l)}) \mid |a_j(x)| < 1$ for some $x \in \mathrm{Im}(\Sigma(\alpha))\}$. Let β denote the isometric map

$$[\boldsymbol{n}'^{(l)}] \overset{\mu'}{\stackrel{\sim}{\to}} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}') \to \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}'}, \boldsymbol{y}'') \overset{\mu''^{-1}}{\stackrel{\sim}{\to}} [\boldsymbol{n}_J^{(l)}]\,.$$

It induces an injective map $f : \omega(\boldsymbol{n}'^{(l)}) \to J$. From [Ber7, Proposition 4.3] it follows that $r'^{(l)}_j = \Sigma(\alpha)^*(r^{(l)}_{f(j)})$ for all $j \in \omega(\boldsymbol{n}'^{(l)})$ and, therefore, the pair (α, β) induces a morphism $\gamma : [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'} \to [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ for which $d' = D(\gamma)(d)$. It follows that the map $O(D) \to \mathrm{str}(\mathfrak{Y}_s)$ is an isomorphism of partially ordered sets.

(iii) The direct implications follows straightforwardly from the description of $D^{\leq l-1}$ in terms of D. Assume first that the map $\mathrm{str}(\mathfrak{Y}_s) \to \mathrm{str}(\mathfrak{X}_s)$ is surjective. We have to show that for every $c \in C^{\underline{s}}_{\underline{\boldsymbol{m}}}$ there exists $d = (\boldsymbol{y}, c, \mu) \in D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ with $[\underline{\boldsymbol{n}}^{\leq l-1}]_{\underline{\boldsymbol{r}}^{\leq l-1}} = [\underline{\boldsymbol{m}}]_{\underline{s}}$. By (ii), there exists $d' = (\boldsymbol{y}, c', \mu') \in D^{\underline{\boldsymbol{r}}'}_{\underline{\boldsymbol{n}}'}$ with $\overline{c}' = \overline{c}$. Since $O(C) \stackrel{\sim}{\to} \mathrm{str}(\mathfrak{X}_s)$, there exists an isomorphism $\gamma : [\underline{\boldsymbol{m}}]_{\underline{s}} \stackrel{\sim}{\to} [\underline{\boldsymbol{n}}'^{\leq l-1}]_{\underline{\boldsymbol{r}}'^{\leq l-1}}$ with $c = C(\gamma)(c')$. If $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$ is the inverse image of $[\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}$ under γ (in the sense of Example 2.3.1(ii)) and μ is the composition of the isometric bijection $[\boldsymbol{n}^{(l)}] \stackrel{\sim}{\to} [\boldsymbol{n}'^{(l)}]$ with μ', then the triple $d = (\boldsymbol{y}, c, \mu)$ is an element of $D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$. Assume now that, for every $\boldsymbol{x} \in \mathfrak{X}_s$, $\mathfrak{Y}_{s,\boldsymbol{x}}$ is connected. We have to show that any two polysimplices $d = (\boldsymbol{y}, c, \mu) \in D^{\underline{\boldsymbol{r}}}_{\underline{\boldsymbol{n}}}$ and $d' = (\boldsymbol{y}', c, \mu') \in D^{\underline{\boldsymbol{r}}'}_{\underline{\boldsymbol{n}}'}$ (over the same c) are equivalent. By the assumption, it suffices to consider the case when $\boldsymbol{y}' \le \boldsymbol{y}$, but in this case the required fact is obtained from the construction in the proof of (ii) (with the identity morphism α).

We set $\mathbb{D}(\underline{\mathfrak{X}}) = D$. It is easy to see that the correspondence $\underline{\mathfrak{X}} \mapsto \mathbb{D}(\underline{\mathfrak{X}})$ is functorial on the full subcategory of $\mathcal{P}st^{\text{ét}}_{\text{nd},l}$ that consists of strictly poly-stable fibrations.

5.3 Construction of $\mathbb{D}(\underline{\mathfrak{X}})$ for arbitrary $\underline{\mathfrak{X}}$

5.3.1 Lemma. *Assume we are given a surjective étale morphisms $\underline{\mathfrak{X}}' \to \underline{\mathfrak{X}}$ between nondegenerate strictly poly-stable fibrations of length l. Then there is an isomorphism of polysimplicial sets* $\mathrm{Coker}(\mathbb{D}(\underline{\mathfrak{X}}' \times_{\underline{\mathfrak{X}}} \underline{\mathfrak{X}}') \overset{\to}{\to} \mathbb{D}(\underline{\mathfrak{X}}')) \overset{\sim}{\to} \mathbb{D}(\underline{\mathfrak{X}})$.

Proof. We set $\underline{\mathfrak{X}}'' = \underline{\mathfrak{X}}' \times_{\underline{\mathfrak{X}}} \underline{\mathfrak{X}}'$, $\mathfrak{X} = \mathfrak{X}_{l-1}$, $C = \mathbb{D}(\underline{\mathfrak{X}}^{\le l-1})$, $\mathfrak{Y} = \mathfrak{X}_l$, $D = \mathbb{D}(\underline{\mathfrak{X}})$ and so on. By the induction hypothesis, there is an isomorphism of polysimplicial sets $\mathrm{Coker}(C'' \overset{\to}{\to} C') \overset{\sim}{\to} C$.

The morphism of polysimplicial sets $D' \to D$ is surjective. Indeed, let $d = (\boldsymbol{y}, c, \mu) \in D^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$. By [Ber7, Corollary 2.8], there exists a point $\boldsymbol{y}' \in \mathrm{str}(\mathfrak{Y}'_s)$ over the point $\boldsymbol{y} \in \mathfrak{Y}_s$. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be the images of $\boldsymbol{y}$ and $\boldsymbol{y}'$ in $\mathfrak{X}_s$ and $\mathfrak{X}'_s$, respectively. One has $\overline{c} = \boldsymbol{x}$. To prove the claim, it suffices to show that there exists a polysimplex $c' \in C'^{\boldsymbol{r}^{\le l-1}}_{\underline{\boldsymbol{n}}^{\le l-1}}$ over c with $\overline{c}' = \boldsymbol{x}'$ (since the triple $d' = (\boldsymbol{y}', c', \mu')$ will then represent an element of $D'^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ over d, where μ' is the composition of μ with the inverse of the canonical isometric bijection $\mathrm{irr}(\mathfrak{Y}'_{s,\boldsymbol{x}'}, \boldsymbol{y}') \overset{\sim}{\to} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}}, \boldsymbol{y})$). Since $O(C) \overset{\sim}{\to} \mathrm{str}(\mathfrak{X}_s)$ and $O(C') \overset{\sim}{\to} \mathrm{str}(\mathfrak{X}'_s)$, the necessary fact is a consequence of the following simple observation. *Given a morphism $E' \to E$ in $\Lambda^{\circ}_{R,l}\mathcal{E}ns$, the canonical map $E'^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}} \to E^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}} \times_{O(E)} O(E')$ is surjective for every $[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \in \mathrm{Ob}(\Lambda_{R,l})$.* To see the latter, let us consider a pair of polysimplices $e \in E^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ and $e' \in E'^{\boldsymbol{r}'}_{\underline{\boldsymbol{n}}'}$ such that the class of d, the image of e' in $E^{\boldsymbol{r}'}_{\underline{\boldsymbol{n}}'}$, coincides with that of e in $O(E)$. It follows that there is an isomorphism $\gamma : [\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}} \overset{\sim}{\to} [\underline{\boldsymbol{n}}']_{\underline{\boldsymbol{r}}'}$ with $e = E(\gamma)(d)$. Then the image of the polysimplex $E'(\gamma)(e')$ in $E^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ is e and its class in $O(E')$ coincides with that of e'.

The morphism $\mathrm{Coker}(D'' \overset{\to}{\to} D') \to D$ *is an isomorphism.* Assume there are two polysimplices $d_1 = (\boldsymbol{y}_1, c_1, \mu_1)$ and $d_2 = (\boldsymbol{y}_2, c_2, \mu_2)$ in $D'^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ whose images in $D^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ coincide. Then $\overline{c}_1 = \boldsymbol{x}_1$ and $\overline{c}_2 = \boldsymbol{x}_2$ are the images of the points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ in $\mathfrak{X}'_s$, respectively. By [Ber7, Corollary 2.8], we can find a point $\boldsymbol{y}'' \in \mathrm{str}(\mathfrak{Y}''_s)$ over the pair of points $(\boldsymbol{y}_1, \boldsymbol{y}_2)$. Let $\boldsymbol{x}''$ be the image of $\boldsymbol{y}''$ in $\mathfrak{X}''_s$. It suffices to show that there exists a polysimplex $c'' \in C''^{\boldsymbol{r}}_{\underline{\boldsymbol{n}}}$ over the pair of polysimplices (c_1, c_2) with $\overline{c}'' = \boldsymbol{x}''$. Since $O(C') \overset{\sim}{\to} \mathrm{str}(\mathfrak{X}'_s)$ and $O(C'') \overset{\sim}{\to} \mathrm{str}(\mathfrak{X}''_s)$, this follows from the above observation applied to the morphism $C'' \to C' \times_C C'$. □

We fix for each nondegenerated poly-stable fibrations $\underline{\mathfrak{X}}$ of length l a surjective étale morphism $\underline{\mathfrak{X}}' \to \underline{\mathfrak{X}}$ so that, if $\underline{\mathfrak{X}}$ is strictly poly-stable, then $\underline{\mathfrak{X}}' = \underline{\mathfrak{X}}'' = \underline{\mathfrak{X}}$, and define $\mathbb{D}(\underline{\mathfrak{X}})$ as the cokernel $\mathrm{Coker}(\mathbb{D}(\underline{\mathfrak{X}}' \times_{\underline{\mathfrak{X}}} \underline{\mathfrak{X}}') \stackrel{\rightarrow}{\rightarrow} \mathbb{D}(\underline{\mathfrak{X}}'))$. We get a functor $\mathbb{D}^l : \mathcal{P}st^{\text{ét}}_{\text{nd},l} \to \widetilde{\Lambda}^{\circ,\text{lf}}_l \mathcal{E}ns$ that possesses the properties (1) and (2). We also get a morphism of functors $\mathbb{D}(\underline{\mathfrak{X}})^{\le l-1} \to \mathbb{D}(\underline{\mathfrak{X}}^{\le l-1})$ and functorial isomorphisms of partially ordered sets $O(\mathbb{D}(\underline{\mathfrak{X}})) \stackrel{\sim}{\to} \mathrm{str}(\mathfrak{X}_{l,s})$.

5.4 Construction of an isomorphism of functors $\theta_l : |\mathbb{D}^l| \stackrel{\sim}{\to} S^l$

5.4.1 Lemma. *Given an open immersion $\underline{\mathfrak{Y}} \hookrightarrow \underline{\mathfrak{X}}$, the induced morphism $\mathbb{D}(\underline{\mathfrak{Y}}) \to \mathbb{D}(\underline{\mathfrak{X}})$ is injective (and, therefore, it identifies $\mathbb{D}(\underline{\mathfrak{Y}})$ with the polysimplicial subset of $\mathbb{D}(\underline{\mathfrak{X}})$ which corresponds in $O(\mathbb{D}(\underline{\mathfrak{X}})) = \mathrm{str}(\mathfrak{X}_{l,s})$ to the subset $\mathrm{str}(\mathfrak{Y}_{l,s})$).*

Proof. If $\underline{\mathfrak{X}}$ is strictly poly-stable, the statement follows for $l-1$ (resp. l) from the induction hypothesis and Lemma 3.4.6 (resp. the explicit construction of $\mathbb{D}(\underline{\mathfrak{X}})$). In the general case, assume that two polysimplices d_1 and d_2 of $\mathbb{D}(\underline{\mathfrak{Y}})$ go to the same polysimplex of $\mathbb{D}(\underline{\mathfrak{X}})$. Let $\underline{\mathfrak{X}}' \to \underline{\mathfrak{X}}$ be a surjective étale morphism with strictly poly-stable $\underline{\mathfrak{X}}'$, and let $\underline{\mathfrak{Y}}'$ be the preimage of $\underline{\mathfrak{Y}}$ in $\underline{\mathfrak{X}}'$. We can find polysimplices $d'_1 = (\boldsymbol{y}'_1, c'_1, \mu'_1)$ and $d'_2 = (\boldsymbol{y}'_2, c'_2, \mu'_2)$ in $\mathbb{D}(\underline{\mathfrak{Y}}')$ over d_1 and d_2, respectively. The assumption implies that there exist polysimplices $d''_i = (\boldsymbol{x}''_i, c''_i, \mu''_i)$ of $\mathbb{D}(\underline{\mathfrak{X}}' \times_{\underline{\mathfrak{X}}} \underline{\mathfrak{X}}')$, $1 \le i \le n$, with $p_1(d''_1) = d'_1$, $p_2(d''_i) = p_1(d''_{i+1})$ for $1 \le i \le n-1$ and $p_2(d''_n) = d'_2$. It follows that $p_1(\boldsymbol{x}''_1) = \boldsymbol{y}'_1$ and, therefore, $p_2(\boldsymbol{x}''_1) \in \mathfrak{Y}'_{l,s}$, i.e., $\boldsymbol{x}''_1 \in \mathrm{str}(\mathfrak{Y}'_{l,s} \times_{\mathfrak{Y}_{l,s}} \mathfrak{Y}'_{l,s})$. For the same reason, the same is true for all points $\boldsymbol{x}''_i$ and, therefore, all of the polysimplices d''_i come from $\mathbb{D}(\underline{\mathfrak{Y}}' \times_{\underline{\mathfrak{Y}}} \underline{\mathfrak{Y}}')$, i.e., $d_1 = d_2$. □

Notice that it suffices to construct an isomorphism of functors $|\mathbb{D}^l| \stackrel{\sim}{\to} S^l$ on a full subcategory of $\mathcal{P}st^{\text{ét}}_{\text{nd},l}$ with the property that any object of the whole category is the image of an object of the subcategory under a surjective étale morphism. It suffices therefore to construct functorial homeomorphisms $|\mathbb{D}(\underline{\mathfrak{X}})| \stackrel{\sim}{\to} S(\underline{\mathfrak{X}})$ for $\underline{\mathfrak{X}}$ which are strictly poly-stable and such that $\mathfrak{X}_{l-1}$ is affine, and the morphism $f_{l-1} : \mathfrak{X}_l \to \mathfrak{X}_{l-1}$ is geometrically elementary and goes through an étale morphism $\mathfrak{X}_l \to \mathfrak{X}_{l-1}(\boldsymbol{n}, \boldsymbol{a}, m)$. (Notice that in this case the formal scheme $\mathfrak{X}_l$ is quasi-compact.) We set $\mathfrak{X} = \mathfrak{X}_{l-1} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathfrak{X}_l$, $\varphi = f_{l-1}$, $C = \mathbb{D}(\underline{\mathfrak{X}}^{\le l-1})$ and $D = \mathbb{D}(\underline{\mathfrak{X}})$. The first example of a geometrically elementary morphism is a morphism of the form $\mathfrak{X}(\boldsymbol{n}, \boldsymbol{a}, m) \to \mathfrak{X}$.

5.4.2 Lemma. *If $\mathfrak{Y} = \mathfrak{X}(\boldsymbol{n}, \boldsymbol{a}, m)$, then $D \stackrel{\sim}{\to} \Lambda[\boldsymbol{n}]_{C,|\boldsymbol{a}|}$ (see §2.4).*

Proof. Given a polysimplex $d = (\boldsymbol{y}, c, \mu) \in D^{\boldsymbol{s}}_{\underline{\boldsymbol{m}}}$, the set $I = \{i \in \omega(\boldsymbol{n}) \mid |a_i(x)| < 1$ for some $x \in \mathrm{Im}(\overline{\sigma}_c)\}$ coincides with the set $I(c, |\boldsymbol{a}|)$ defined in §2.3. If $\boldsymbol{y}'$ is the maximal point in $\mathfrak{Y}_{s,\boldsymbol{x}}$, where $\boldsymbol{x} = \varphi_s(\boldsymbol{y}) = \overline{c}$, there is a canonical isometric bijection $[\boldsymbol{n}_I] \stackrel{\sim}{\to} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}})$ and, therefore, the isometric bijection $\mu : [\boldsymbol{m}^{(l)}] \stackrel{\sim}{\to} \mathrm{irr}(\mathfrak{Y}_{s,\boldsymbol{x}}, \boldsymbol{y})$ defines a morphism $\gamma = (f, \alpha) : [\boldsymbol{m}^{(l)}] \to [\boldsymbol{n}_I]$ in Λ such that $s_j = \overline{\sigma}^*_c(|a_{f(j)}|)$ for all

$j \in \omega(\boldsymbol{m}^{(l)})$. The pair, consisting of $c \in C^{\underline{\boldsymbol{s}}^{\leq l-1}}_{\underline{\boldsymbol{m}}^{\leq l-1}}$ and the morphism γ, represents an $\underline{\boldsymbol{s}}$-colored $\underline{\boldsymbol{m}}$-polysimplex of $\Lambda[\boldsymbol{n}]_{C,|\boldsymbol{a}|}$, and the correspondence $d = (\boldsymbol{y}, c, \mu) \mapsto (c, \gamma)$ gives rise to the required isomorphism. ◻

Assume that $\mathfrak{Y} = \mathfrak{X}(\boldsymbol{n}, \boldsymbol{a}, m)$, where $\boldsymbol{n} = (n_1, \dots, n_p)$ and $\boldsymbol{a} = (a_0, \dots, a_p)$. Recall (see Step 1 from [Ber7, §5]) that the continuous mapping

$$\mathfrak{Y}_\eta \to \mathfrak{X}_\eta \times [0,1]^{[\boldsymbol{n}]} : y \mapsto (\varphi(y); |T_{00}(y)|, \dots, |T_{pp}(y)|)$$

induces a homeomorphism between $S(\mathfrak{Y}/\mathfrak{X})$ and the closed set

$$S = \{(x; \boldsymbol{t}) \in \mathfrak{X}_\eta \times [0,1]^{[\boldsymbol{n}]} \mid t_{i0} \dots t_{in_i} = |a_i(x)|, 1 \leq i \leq p\} .$$

Since $S(\underline{\mathfrak{X}}) = S(\mathfrak{Y}/\mathfrak{X}) \cap \varphi^{-1}(S(\underline{\mathfrak{X}}^{\leq l-1}))$, the isomorphism of Lemma 2.4.1 defines a homeomorphism $|D| \xrightarrow{\sim} S(\underline{\mathfrak{X}})$ which possesses the property (3). Indeed, it suffices to verify that, given a function $g \in \mathcal{O}'(\mathfrak{Y})$ and a polysimplex $d \in D^{\underline{\boldsymbol{s}}}_{\underline{\boldsymbol{m}}}$, one has $\overline{\sigma}^*_d(|g|) \in M^{\underline{\boldsymbol{m}}}_{\underline{\boldsymbol{s}}}$. This easily follows from [Ber7, Lemma 5.6].

Consider now a geometrically elementary morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ that goes through an étale morphism $\mathfrak{Y} \to \mathfrak{Z} = \mathfrak{X}(\boldsymbol{n}, \boldsymbol{a}, m)$. We set $\underline{\mathfrak{Z}} = (\mathfrak{Z} \to \mathfrak{X}_{l-1} \stackrel{f_{l-2}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ and $E = \mathbb{D}(\underline{\mathfrak{Z}})$. By the claim (iii) from §5.2, the morphisms of polysimplicial sets $D^{\leq l-1} \to C$ and $E^{\leq l-1} \to C$ are injective and bijective, respectively, and, by the above construction, there is a homeomorphism $|E| \xrightarrow{\sim} S(\underline{\mathfrak{Z}})$ that possesses the property (3). Since for every point $\boldsymbol{x} \in \mathrm{str}(\mathfrak{X}_s)$ the induced map of partially ordered sets $\mathrm{str}(\mathfrak{Y}_{s,\boldsymbol{x}}) \to \mathrm{str}(\mathfrak{Z}_{s,x})$ is injective, from Lemma 3.4.6 it follows that the morphism of polysimplicial sets $D \to E$ is injective. On the other hand, let x be a point of $|C| = S(\underline{\mathfrak{X}}^{\leq l-1})$ and $\boldsymbol{x}$ its image in $\mathfrak{X}_s$. Notice that $\boldsymbol{x} \in \mathrm{str}(\mathfrak{X}_s)$ (see [Ber7, Theorem 8.1(v)]). Since $\mathfrak{Y}_{s,\boldsymbol{x}}$ is geometrically irreducible, the maps $\mathrm{str}(\mathfrak{Y}_{x,s}) \to \mathrm{str}(\mathfrak{Z}_{x,s})$ and $\mathbb{D}(\mathfrak{Y}_x) \to \mathbb{D}(\mathfrak{Z}_x)$ are injective and, by [Ber7, Theorem 5.4], the map $S(\mathfrak{Y}_x) \to S(\mathfrak{Z}_x)$ is injective, and its image is the union of the cells of $S(\mathfrak{Z}_x)$ that are the preimages of the points coming from $\mathrm{str}(\mathfrak{Y}_{x,s})$. It follows that the map $S(\underline{\mathfrak{X}}) \to S(\underline{\mathfrak{Z}})$ is injective, and its image is the union of the cells of $S(\underline{\mathfrak{Z}})$ that are the preimages of the points coming from $\mathrm{str}(\mathfrak{Y}_s)$. Since $O(D) \xrightarrow{\sim} \mathrm{str}(\mathfrak{Y}_s)$, we get a homeomorphism $|D| \xrightarrow{\sim} S(\underline{\mathfrak{X}})$. The restriction of the latter to the fibers at the point x gives rise to a homeomorphism $\mathbb{D}(\mathfrak{Y}_x) \xrightarrow{\sim} S(\mathfrak{Y}_x)$ which coincides with that of [Ber7, Theorem 5.4]. It follows that the homeomorphism $|D| \xrightarrow{\sim} S(\underline{\mathfrak{X}})$ is well defined and, in fact, functorial.

Thus, an isomorphism of functors $\theta_l : |\mathbb{D}^l| \xrightarrow{\sim} S^l$ that possesses the property (3) is constructed. It follows from the construction that the morphism $\mathbb{D}(\underline{\mathfrak{X}})^{\leq l-1} \to \mathbb{D}(\underline{\mathfrak{X}}^{\leq l-1})$ is compatible with θ_l and θ_{l-1}.

5.5 Verification of the properties (4) and (5)

In this subsection we use the assumption that the poly-stable fibrations considered are nondegenerate. If the valuation on k is trivial, both properties are evidently true, and so we assume that the valuation on k is nontrivial.

It is clear that it suffices to verify the property (4) only for strictly poly-stable $\underline{\mathfrak{X}}$. Let $\boldsymbol{y}_0 \in \mathfrak{X}_{l,s}$, $\boldsymbol{y}$ the generic point of the stratum of $\mathfrak{X}_{l,s}$ that contains the point $\boldsymbol{y}_0$, and $\boldsymbol{x}$ the image of $\boldsymbol{y}$ in $\mathfrak{X}_{l-1,s}$. First of all, we can shrink $\underline{\mathfrak{X}}$ so that $\mathfrak{X}_{l-1} = \mathrm{Spf}(A)$ is affine, the point $\boldsymbol{x}$ is a unique maximal one in the partially ordered set $\mathrm{str}(\mathfrak{X}_{l-1,s})$, $\mathbb{D}(\underline{\mathfrak{X}}^{\le l-1}) = \Lambda[\underline{\boldsymbol{m}}]_{\underline{\boldsymbol{s}}}$, and the map $A\backslash\{0\} \to M_{\underline{\boldsymbol{s}}}^{\underline{\boldsymbol{m}}} : f \mapsto \theta_{l-1}^*(|f|)$ is surjective. Furthermore, we can shrink $\mathfrak{X}_l$ so that $\mathfrak{X}_l = \mathrm{Spf}(B)$ is affine, the point $\boldsymbol{y}$ is a unique maximal one in $\mathrm{str}(\mathfrak{X}_{l,s})$, and the canonical morphism $\mathfrak{X}_l \to \mathfrak{X}_{l-1}$ goes through an étale morphism $\varphi : \mathfrak{X}_l \to \mathfrak{Z} = \mathfrak{X}_{l-1}(\boldsymbol{n}, \boldsymbol{a}, m)$ such that the image $\boldsymbol{z}$ of $\boldsymbol{y}$ in $\mathfrak{Z}_s$ is a unique maximal point in $\mathrm{str}(\mathfrak{Z}_s)$. It follows that $\mathbb{D}(\underline{\mathfrak{X}}) \stackrel{\sim}{\to} \Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$, where $\underline{\boldsymbol{n}} = (\underline{\boldsymbol{m}}, \boldsymbol{n})$ and $\underline{\boldsymbol{r}} = (\underline{\boldsymbol{s}}, |\boldsymbol{a}|)$, and that $S(\underline{\mathfrak{X}}) \stackrel{\sim}{\to} S(\underline{\mathfrak{Z}})$, where $\underline{\mathfrak{Z}} = (\mathfrak{Z} \to \mathfrak{X}_{l-1} \stackrel{f_{l-1}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$. Since the retraction maps $\mathfrak{Y}_\eta \to S(\underline{\mathfrak{X}})$ and $\mathfrak{Z}_\eta \to S(\underline{\mathfrak{Z}})$ commute with φ, it follows that $S(\underline{\mathfrak{X}}) = \varphi^{-1}(S(\underline{\mathfrak{Z}}))$. From [Ber7, Lemma 5.6] it follows that $\theta_l^*(|h|) \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ for all $h \in C\backslash\{0\}$, where $\mathfrak{Z} = \mathrm{Spf}(C)$, and that the map $C\backslash\{0\} \to M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}} : h \mapsto \theta_l^*(|h|)$ is surjective. Thus, to prove the claim, *it suffices to show that $\theta_l^*(|g|) \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ for all $g \in B\backslash\{0\}$*. For this we need, first of all, the following criterion for a real valued continuous function on $\Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ to be contained in $M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$.

Let $\widetilde{M}_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ denote the set of all continuous functions $\alpha : \Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}} \to \mathbb{R}_+^*$ with the property that, for every relatively compact open subset $\mathcal{U} \subset \mathring{\Sigma}_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$, there exists a function $f \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ with $\alpha|_{\mathcal{U}} = f|_{\mathcal{U}}$. One evidently has $M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}} \subset \widetilde{M}_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$.

5.5.1 Lemma. *Assume that for $\alpha \in \widetilde{M}_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ there exists $\beta \in \widetilde{M}_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ with $\alpha \cdot \beta \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$. Then $\alpha \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$.*

Proof. Given a function $f \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$, let $\{f_i\}_{i\in I}$ be the finite set of elementary functions from Proposition 3.3.1(ii) that are associated with f. For $i \in I$, $U_i(f) = \{x \in \Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}} \mid f_i(x) > f_j(x) \text{ for all } j \in I,\ j \neq i\}$ is a nonempty open subset of $\Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$, and the union $\bigcup_{i\in I} U_i(f)$ is dense in $\Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$. Furthermore, we set $A(f) = \{U_i(f)\}_{i\in I}$ and, for a subset $F = \{f_1, \dots, f_m\} \subset M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$, we denote by $A(F)$ the family of all sets of the form $U_1 \cap \cdots \cap U_m$ with $U_i \in A(f_i)$. (Notice that the union of all $U \in A(F)$ is dense in $\Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$.) Finally, for $f \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ we set $B(f) = A(\{f\} \cup F_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}})$, where $F_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}} = \{r_j^{(i)}\}_{1\le i\le l, 0\le j\le p_i}$. Each set $U \in B(f)$ is contained in an $R_{\mathbb{Z}_+}^k$-subpolytope of $\Sigma_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$ and is convex in it, and the restriction $f|_U$ is a linear function on U (see Lemma 3.5.1(i)).

Let α and β be from the formulation, and set $h = \alpha \cdot \beta \in M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}}$. *We claim for every $U \in B(h)$ there exists a unique $f^{(U)} \in e(M_{\underline{\boldsymbol{r}}}^{\underline{\boldsymbol{n}}})$ with $\alpha|_U = f^{(U)}|_U$.* Indeed, the uniqueness of $f^{(U)}$ follows from Proposition 3.3.1(i). Let $\mathcal{U}$ be a relatively compact convex open subset of $U\cap\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$, and let f and g be functions from $M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ with $\alpha|_{\mathcal{U}} = f|_{\mathcal{U}}$ and $\beta|_{\mathcal{U}} = g|_{\mathcal{U}}$. Then $h|_{\mathcal{U}} = f|_{\mathcal{U}}\cdot g|_{\mathcal{U}}$. The function on the left hand side is linear. On the other hand, both functions $f|_{\mathcal{U}}$ and $g|_{\mathcal{U}}$ are maxima of a finite number of linear functions. It follows that they are in fact linear. This easily implies that $\mathcal{U}$ is a subset of some set from $A(f)$, and if $f^{(U)}$ is the corresponding elementary component of f then $\alpha|_{\mathcal{U}} = f^{(U)}|_{\mathcal{U}}$. From Proposition 3.3.1(i) it follows that $f^{(U)}$ does not depend on the choice of the set $\mathcal{U}$ and the function f and, by the continuity of α, $\alpha|_U = f^{(U)}|_U$.

Thus, $\alpha = \max_{U\in B(h)}\{f^{(U)}\}$ since this equality is true for the restrictions of both sides to every relatively compact open subset of $\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$. It follows that $\alpha\in M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$. ◻

Let $g\in B\backslash\{0\}$. By [Ber7, Theorem 8.1(vi)], the local ring of every point from $S(\underline{\mathfrak{X}})$ is a field. It follows that $\varepsilon = \min\{|g(y)| \mid y\in S(\underline{\mathfrak{X}})\} > 0$.

A. *The function $\theta_l^*(|g|)$ is contained in $\widetilde{M}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$.* First of all, we recall that the interior $\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ is the preimage of $S(\underline{\mathfrak{X}})\cap\pi^{-1}(\boldsymbol{y})$ under θ_l, and that the morphism $\mathfrak{X}_l\to\mathfrak{X}_{l-1}$ goes through an étale morphism $\varphi:\mathfrak{X}_l\to\mathfrak{Z}=\mathfrak{X}_{l-1}(\boldsymbol{n},\boldsymbol{a},m)$.

1. *We may assume that $m=0$.* Indeed, consider first the case $l=1$. If t is the maximal point of $\mathfrak{X}_0(m)_\eta$ (it corresponds to the supremum norm of the algebra $k\{T_1,\dots,T_m,T_1^{-1},\dots,T_m^{-1}\}$), then $\mathbb{D}(\mathfrak{X}'_1)\stackrel{\sim}{\to}\mathbb{D}(\mathfrak{X}_1)$, where $\mathfrak{X}'_1 = (\mathfrak{X}_1)_t = \mathfrak{X}_1\times_{\mathfrak{X}_0(m)}\mathrm{Spf}(\mathcal{H}(t)^\circ)$, and $S(\mathfrak{X}'_1)\stackrel{\sim}{\to}S(\mathfrak{X}_1)$. Since $|\mathcal{H}(t)| = |k|$, the situation is reduced to $\mathfrak{X}'_1$ (for which $m=0$). In the case $l\ge 2$, one has $\mathbb{D}(\underline{\mathfrak{X}}')\stackrel{\sim}{\to}\mathbb{D}(\underline{\mathfrak{X}})$ and $S(\underline{\mathfrak{X}}')\stackrel{\sim}{\to}S(\underline{\mathfrak{X}})$, where $\underline{\mathfrak{X}}' = (\mathfrak{X}_l\stackrel{f'_{l-1}}{\to}\mathfrak{X}_{l-1}(m)\stackrel{f'_{l-2}}{\to}\mathfrak{X}_{l-2}\stackrel{f_{l-3}}{\to}\cdots\stackrel{f_1}{\to}\mathfrak{X}_1)$, f'_{l-1} is the composition of φ with the canonical projection $\mathfrak{X}_{l-1}(\boldsymbol{n},\boldsymbol{a},m)\to\mathfrak{X}_{l-1}(m)$, and f'_{l-2} is the composition of the canonical projection $\mathfrak{X}_{l-1}(m)\to\mathfrak{X}_{l-1}$ with f_{l-2}.

2. *We may assume that $[\boldsymbol{n}]\ne[0]$.* Indeed, if $[\boldsymbol{n}]=[0]$, then the morphism f_{l-1} is étale. If $l=1$, the whole statement of this subsection is trivial. If $l\ge 2$, there is an isomorphism $\mathbb{D}(\underline{\mathfrak{X}})\stackrel{\sim}{\to}\Lambda[0]_{\mathbb{D}(\underline{\mathfrak{X}}'),1}$ (see §2.4), where $\underline{\mathfrak{X}}' = (\mathfrak{X}_l\xrightarrow{f_{l-2}\circ f_{l-1}}\mathfrak{X}_{l-2}\stackrel{f_{l-3}}{\to}\cdots\stackrel{f_1}{\to}\mathfrak{X}_1)$ is of length $l-1$.

3. *We may assume that the étale morphism from the maximal stratum Y of $\mathfrak{X}_{l,s}$ to the maximal stratum Z of $\mathfrak{Z}_s$, induced by φ, is an open immersion.* Indeed, let $\boldsymbol{n}=(n_0,\dots,n_p)$ and $\boldsymbol{a}=(a_0,\dots,a_p)$. The reductions of the functions $a_0,\dots,a_p$ in $\widetilde{A}$ vanish at the maximal stratum X of $\mathfrak{X}_{l-1,s}$. (Notice that X is closed in $\mathfrak{X}_{l-1,s}$.) The maximal stratum Z of $\mathfrak{Z}_s$, which is defined in the preimage of X by vanishing of all coordinate functions T_{ij} for $0\le i\le p$ and $0\le j\le n_i$, maps isomorphically onto X, and the maximal stratum Y of $\mathfrak{X}_{l,s}$ is the preimage of Z in $\mathfrak{X}_{l,s}$. The induced morphism $Y\to X$ is étale, and we can find an étale morphism $\mathfrak{X}'_{l-1}=\mathrm{Spf}(A')\to\mathfrak{X}_{l-1}$ such that $\mathfrak{X}'_{l-1,s}$ contains a closed subset X' provided with an open immersion $X'\hookrightarrow Y$ compatible with the étale morphisms $Y\to X$ and $X'\to X$. Shrinking $\mathfrak{X}_l$, we may assume that $X'\stackrel{\sim}{\to}Y$. Let $\mathfrak{X}'_l$ be the connected component $\mathfrak{X}_l\times_{\mathfrak{X}_{l-1}}\mathfrak{X}'_{l-1}$ that contains the image of Y under the evident morphism to the closed fiber of the latter.

Then the required property is true for $\underline{\mathfrak{X}}' = (\underline{\mathfrak{X}}'_l \to \underline{\mathfrak{X}}'_{l-1} \to \underline{\mathfrak{X}}_{l-2} \to \dots \mathfrak{X}_1)$ and $\mathbb{D}(\underline{\mathfrak{X}}') \widetilde{\to} \mathbb{D}(\underline{\mathfrak{X}})$.

4. Shrinking $\mathfrak{X}_l$, we may assume that φ identifies Y with a closed subset Z' of $\mathfrak{Z}'_s$, where $\mathfrak{Z}'$ is an open subset of $\mathfrak{Z}$ of the form $\mathrm{Spf}(C')$ with $C' = C_{\{c\}}$, and we may also assume that the image of $\mathfrak{X}_l$ is contained in $\mathfrak{Z}'$. By [Ber7, Lemma 4.4], there is an isomorphism of analytic spaces $\pi^{-1}(Y) \widetilde{\to} \pi^{-1}(Z')$ and of completions $\widehat{C}' \widetilde{\to} \widehat{B}$ with respect to the ideals JC' and JB, respectively, where $J = (\alpha, h_1, \dots, h_m) \subset C$, $h_1, \dots, h_m$ are elements of C, whose reductions in $\widetilde{C}$ generate the ideal of Z, and α is a fixed non-zero element of $k^{\circ\circ}$. Any relatively compact subset of $\mathring{\Sigma}^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ is contained in $\theta_l^{-1}(V_\delta)$ for some $\delta > 0$, where $V_\delta = \{y \in S(\underline{\mathfrak{X}}) \mid |h_i(y)| < 1 - \delta,\ 1 \le i \le m\}$. Let n be a sufficiently large integer with $(1-\delta)^j |\alpha|^{n-j} < \varepsilon$ for all $0 \le j \le n$. Then $|h(y)| < \varepsilon$ for all $y \in V_\delta$ and all $h \in J^n B$. Finally, we can find an element $h \in C$ and an integer $\nu \ge 0$ such that $g - \frac{h}{c^\nu} \in J^n B$. Since $|c(y)| = 1$ for all $y \in \mathfrak{X}_{l,\eta}$, it follows that $|g(y)| = |h(y)|$ for all $y \in V_\delta$.

B. *The function $\theta_l^*(|g|)$ is contained in $M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$.* We can shrink $\mathfrak{X}_l$ so that $B = B'_{\{f\}}$ with $B' = C[T]/(P)$ and $f \in B'$, where $P(T)$ is a monic polynomial in $C[T]$ such that the image of its derivative in B is invertible. Furthermore, we can find $g' \in B'$ and $m \ge 0$ such that $|(g - \frac{g'}{f^m})(y)| < \varepsilon$ for all $y \in \mathfrak{X}_{l,\eta}$. Since $|f(y)| = 1$ for all $y \in \mathfrak{X}_{l,\eta}$, it follows that $|g(y)| = |g'(y)|$ for all $y \in S(\underline{\mathfrak{X}})$. Thus, we may assume that $g \in B'$. Since the strictly k-affinoid algebra $\mathcal{C} = C \otimes_{k^\circ} k$ is normal, the coefficients of the minimal polynomial $T^n + h_1 T^{n-1} + \dots + h_n$ of g over its fraction field are in fact elements of $\mathcal{C}$. From [BGR, Proposition 3.8.1/7(a)] it follows that $h_i \in \mathcal{C}^\circ$, and since $\mathcal{C}^\circ = C$, by [Ber7, Proposition 1.4], it follows that $h_i \in C$ for all $1 \le i \le n$. One has $h_n \ne 0$ and $h_n = -g(g^{n-1} + h_1 g^{n-2} + \dots + h_{n-1})$, and the required fact follows from Lemma 5.5.1 □

6 *p*-Adic analytic and piecewise linear spaces

6.1 A piecewise linear structure on the skeleton of a pluri-stable formal scheme

Let $\underline{\mathfrak{X}} = (\mathfrak{X}_l \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_1} \mathfrak{X}_1)$ be a nondegenerate poly-stable fibration over k° of length l. By Theorem 5.1.1, there is a canonical homeomorphism between the geometric realization of the R^k-colored polysimplicial set $\mathbb{D}(\underline{\mathfrak{X}})$ of length l and the skeleton $S(\underline{\mathfrak{X}})$. This homeomorphism provides $S(\underline{\mathfrak{X}})$ with a piecewise $R^k_{\mathbb{Z}_+}$-linear structure and a semiring $M_{\underline{\mathfrak{X}}}$ of piecewise $R^k_{\mathbb{Z}_+}$-linear functions on $S(\underline{\mathfrak{X}})$. Recall that the skeleton $S(\underline{\mathfrak{X}})$, as a subset of $\mathfrak{X}_{l,\eta}$, depends only on $\mathfrak{X}_l$ (see §4.3). Let $\underline{\mathfrak{X}}' = (\mathfrak{X}'_{l'} \xrightarrow{f'_{l'-1}} \cdots \xrightarrow{f'_1} \mathfrak{X}'_1)$ be another nondegenerate poly-stable fibration of length l' over k'°.

6.1.1 Theorem. *For any morphism $\varphi : \mathfrak{X}'_{l'} \to \mathfrak{X}_l$ in $\mathcal{P}\ell st^{\text{ét}}_{\text{nd}}$, the induced map $S(\underline{\mathfrak{X}}') \to S(\underline{\mathfrak{X}})$ is a G-local immersion of piecewise $R^{k'}_{\mathbb{Z}_+}$-linear spaces, and it takes functions from $M_{\underline{\mathfrak{X}}}$ to functions from $M_{\underline{\mathfrak{X}}'}$.*

Proof. Since the statement is true for the morphism $\mathfrak{X}_l\widehat{\otimes}_{k^\circ}k'^\circ \to \mathfrak{X}_l$, we can replace $\underline{\mathfrak{X}}$ by $\underline{\mathfrak{X}}\widehat{\otimes}_{k^\circ}k'^\circ$ so that we may assume that $k' = k$ and φ is an étale k°-morphism. Furthermore, if $\underline{\mathfrak{Y}} = (\mathfrak{X}_{l'} \stackrel{f_{l-1}\circ\varphi}{\to} \mathfrak{X}_{l-1} \stackrel{f_{l-2}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$, then $S(\underline{\mathfrak{Y}}) = S(\mathfrak{X}'_{l'}) = S(\underline{\mathfrak{X}}')$ and, by Theorem 5.1.1, applied to the canonical morphism $\underline{\mathfrak{Y}} \to \underline{\mathfrak{X}}$, we can replace $\underline{\mathfrak{X}}$ by $\underline{\mathfrak{Y}}$ so that we may assume that φ is an isomorphism. Finally, given a surjective étale morphism $\psi : \mathfrak{Y} \to \mathfrak{X}_l$, we denote by ψ' the surjective étale morphism $\mathfrak{Y}' = \mathfrak{X}'_{l'} \times_{\mathfrak{X}_l} \mathfrak{Y} \to \mathfrak{X}'_{l'}$, and we set $\underline{\mathfrak{Y}} = (\mathfrak{Y} \stackrel{f_{l-1}\circ\psi}{\to} \mathfrak{X}_{l-1} \stackrel{f_{l-2}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ and $\underline{\mathfrak{Y}}' = (\mathfrak{Y}' \stackrel{f_{l'-1}\circ\psi'}{\to} \mathfrak{X}'_{l'-1} \stackrel{f'_{l'-2}}{\to} \cdots \stackrel{f'_1}{\to} \mathfrak{X}'_1)$. Since the canonical maps $S(\underline{\mathfrak{Y}}) \to S(\underline{\mathfrak{X}})$ and $S(\underline{\mathfrak{Y}}') \to S(\underline{\mathfrak{X}}')$ are surjective G-local immersions of piecewise $R^{k'}_{\mathbb{Z}_+}$-linear spaces, we may always replace $\underline{\mathfrak{X}}$ by $\underline{\mathfrak{Y}}$ and $\underline{\mathfrak{X}}'$ by $\underline{\mathfrak{Y}}'$. This reduces the situation to the case when $\underline{\mathfrak{X}}$ is strictly poly-stable, $\mathfrak{X}_l = \text{Spf}(A)$ is affine, $\mathbb{D}(\underline{\mathfrak{X}})$ is a standard polysimplex $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$, and the map $A\backslash\{0\} \to M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}} : g \mapsto \theta^*_l(|g|)$ is surjective. It follows that the homeomorphism $S(\underline{\mathfrak{X}}') \to S(\underline{\mathfrak{X}})$ takes functions from $M_{\underline{\mathfrak{X}}} = M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ to functions from $M_{\underline{\mathfrak{X}}'}$. Since $S(\underline{\mathfrak{X}})$ is isomorphic to the $R^k_{\mathbb{Z}_+}$-polyhedron $\Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$, the map $S(\underline{\mathfrak{X}}') \to S(\underline{\mathfrak{X}})$ is piecewise $R^k_{\mathbb{Z}_+}$-linear. Applying the latter to the inverse morphism $\varphi^{-1} : \mathfrak{X}_l \to \mathfrak{X}'_{l'}$, we deduce that the map $S(\underline{\mathfrak{X}}') \to S(\underline{\mathfrak{X}})$ is in fact a piecewise $R^k_{\mathbb{Z}_+}$-linear isomorphism. □

Thus, for any nondegenerate pluri-stable formal scheme $\mathfrak{X}$ over k°, the skeleton $S(\mathfrak{X})$ is provided with a well defined piecewise $R^k_{\mathbb{Z}_+}$-linear structure and a semiring $M_{\mathfrak{X}}$ of piecewise $R^k_{\mathbb{Z}_+}$-linear functions.

6.1.2 Corollary. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a pluri-stable morphism between nondegenerate pluri-stable formal schemes over k°. Then the induced map $S(\mathfrak{X}') \to S(\mathfrak{X})$ is piecewise $R^k_{\mathbb{Z}_+}$-linear and it takes functions from $M_{\mathfrak{X}}$ to functions from $M_{\mathfrak{X}'}$.*

Proof. The statement is deduced from Theorem 6.1.1 in the same way as Corollary 4.3.2(i) is deduced from Theorem 4.3.1. □

6.1.3 Corollary. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism between nondegenerate pluri-stable formal schemes, and assume that $\mathfrak{X}$ is strongly nondegenerate. Then the induced map $\tau \circ \varphi_\eta : S(\mathfrak{X}') \to S(\mathfrak{X})$ is piecewise $R^k_{\mathbb{Z}_+}$-linear and it takes functions from $M_{\mathfrak{X}}$ functions from $M_{\mathfrak{X}'}$.*

Proof. Let $\underline{\mathfrak{X}}$ be a strongly nondegenerate poly-stable fibration of length l with $\mathfrak{X}_l = \mathfrak{X}$. As in the proof of Theorem 6.1.1, the situation is reduced to the case when all formal schemes $\mathfrak{X}_i = \mathrm{Spf}(A_i)$ are affine and every morphism $f_i : \mathfrak{X}_{i+1} \to \mathfrak{X}_i$ goes through an étale $\mathfrak{X}_{i+1} \to \mathfrak{X}_i(\boldsymbol{n}_i, \boldsymbol{a}_i, m_i)$ and is geometrically elementary. In this case, $|f(x)| = |f(x_\tau)|$ for all coordinate function f from A_l and all points $x \in \mathfrak{X}_\eta$. It follows that for every point $x' \in S(\mathfrak{X}')$ one has $|f(\tau(\varphi_\eta(x')))| = |\varphi_\eta^*(f)(x')|$. Since the restriction of the function $|\varphi_\eta^*(f)|$ to $S(\mathfrak{X}')$ is contained in $M_{\mathfrak{X}'}$, it follows that the map $\tau \circ \varphi_\eta$ takes functions from $M_{\mathfrak{X}}$ to functions from $M_{\mathfrak{X}'}$ and, in particular, it is piecewise $R^k_{\mathbb{Z}_+}$-linear. $\square$

Given a nondegenerate pluri-stable formal scheme $\mathfrak{X}$ over k°, let $\widetilde{M}_{\mathfrak{X}}$ denote the semiring of real valued functions f on $S(\mathfrak{X})$ with the following property: for every quasi-compact open subscheme $\mathfrak{Y} \subset \mathfrak{X}$, there exists $\alpha \in |k^*|$ such that $(\alpha f)|_{S(\mathfrak{Y})} \in M_{\mathfrak{Y}}$. Notice that $\widetilde{M}_{\mathfrak{X}}$ consists of piecewise $|k^*|_{\mathbb{Z}_+}$-linear functions. Let $\widetilde{M}^*_{\mathfrak{X}}$ denote the subset of the functions f invertible in $\widetilde{M}_{\mathfrak{X}}$ (i.e., such that there exists $g \in \widetilde{M}_{\mathfrak{X}}$ with $fg = 1$). It is a group by multiplication that contains $|k^*|$.

6.1.4 Corollary.

(i) *If $f \in \mathcal{O}'(\mathfrak{X}_\eta)$, the restriction of $|f|$ to $S(\mathfrak{X})$ is contained in $\widetilde{M}_{\mathfrak{X}}$;*

(ii) *if $f \in \mathcal{O}(\mathfrak{X}_\eta)^*$, the restriction of $|f|$ to $S(\mathfrak{X})$ is contained in $\widetilde{M}^*_{\mathfrak{X}}$, and it gives rise to an embedding $\mathcal{O}(\mathfrak{X}_\eta)^*/\mathcal{O}(\mathfrak{X})^* \hookrightarrow \widetilde{M}^*_{\mathfrak{X}}$.*

Proof. (i) If $f \in \mathcal{O}'(\mathfrak{X}_\eta)$, one can find for every quasi-compact open subscheme $\mathfrak{Y} \subset \mathfrak{X}$ an element $\alpha \in k^*$ with $(\alpha f)|_{\mathfrak{Y}_\eta} \in \mathcal{O}'(\mathfrak{Y})$. It follows that $|\alpha f||_{S(\mathfrak{Y})} \in M_{\mathfrak{Y}}$, i.e., $|f||_{S(\mathfrak{X})} \in \widetilde{M}_{\mathfrak{X}}$.

(ii) If $f \in \mathcal{O}(\mathfrak{X}_\eta)^*$, there exists $g \in \mathcal{O}(\mathfrak{X}_\eta)^*$ with $fg = 1$, and the inclusion $|f||_{S(\mathfrak{X})} \in \widetilde{M}^*_{\mathfrak{X}}$ follows from (i). Furthermore, since $x \le x_\tau$ for all points $x \in \mathfrak{X}_\eta$, it follows that $|f(x)| = |f(x_\tau)|$ for all $f \in \mathcal{O}(\mathfrak{X}_\eta)^*$ and, therefore, the kernel of the homomorphism $\mathcal{O}(\mathfrak{X}_\eta) \to \widetilde{M}^*_{\mathfrak{X}} : f \mapsto |f||_{S(\mathfrak{X})}$ coincides with the set of the functions $f \in \mathcal{O}(\mathfrak{X}_\eta)$ with $|f(x)| = 1$ for all $x \in \mathfrak{X}_\eta$. But from [Ber4, Proposition 1.4] it follows that the latter set coincides with $\mathcal{O}(\mathfrak{X})^*$. $\square$

6.1.5 Corollary. *If $\mathfrak{X}$ is quasi-compact and connected, $\mathcal{O}(\mathfrak{X}_\eta)^*/(k^*\mathcal{O}(\mathfrak{X})^*)$ is a finitely generated torsion free group.*

Proof. By Corollary 6.1.4, the group considered is embedded in $\widetilde{M}^*_{\mathfrak{X}}/|k^*|$. If $\{\mathfrak{Y}_j\}_{j\in J}$ is a finite étale covering of $\mathfrak{X}$ with connected $\mathfrak{Y}_j$'s, then $\widetilde{M}^*_{\mathfrak{X}}/|k^*|$ is embedded in the direct product of $\widetilde{M}^*_{\mathfrak{Y}_j}/|k^*|$. We may therefore assume that $\mathfrak{X} = \mathfrak{X}_l$ for a strictly pluri-stable fibration $\underline{\mathfrak{X}}$ over k° of length l with affine $\mathfrak{X}_i$'s and for which $\mathbb{D}(\underline{\mathfrak{X}})$ is a standard polysimplex $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$. In this case one can easily show that $\widetilde{M}^*_{\mathfrak{X}}$ is generated by $\widetilde{M}^*_{\mathfrak{X}_{l-1}}$ and the coordinate functions $t^{(l)}_{j\nu}$ with $r^{(l)}_j \in \widetilde{M}^*_{\mathfrak{X}_{l-1}}$. Since $\widetilde{M}^*_{\mathfrak{X}_0} = |k^*|$, the required statement easily follows. $\square$

6.1.6 Remark. To represent the above results in a functorial form, let us introduce as follows a fibered category $\widetilde{\mathrm{PL}}^{\mathrm{sr}}$ over the category dual to the category of non-Archimedean fields. Its objects are triples (k, X, M_X) consisting of a non-Archimedean field k, a piecewise $R^k_{\mathbb{Z}_+}$-linear space X, and a semiring M_X of piecewise $R^k_{\mathbb{Z}_+}$-linear functions on X. Morphisms $(k', X', M_{X'}) \to (k, X, M_X)$ are pairs consisting of an isometric embedding $k \hookrightarrow k'$ and a piecewise $R^{k'}_{\mathbb{Z}_+}$-linear map $X' \to X$ that takes functions from M_X to functions from $M_{X'}$. Let also $\widetilde{\mathrm{PL}}^{\mathrm{sr}}_{\mathrm{G}}$ be the category with the same family of objects but with those of the above morphisms for which the map $X' \to X$ is a G-local immersion of piecewise $R^{k'}_{\mathbb{Z}_+}$-linear spaces. Then the correspondence $\mathfrak{X} \mapsto (S(\mathfrak{X}), M_{\mathfrak{X}})$ gives rise to functors between fibered categories $\mathcal{P}\ell st^{\text{ét}}_{\mathrm{nd}} \to \widetilde{\mathrm{PL}}^{\mathrm{sr}}_{\mathrm{G}}$, $\mathcal{P}\ell st^{\mathrm{pl}}_{\mathrm{nd}} \to \widetilde{\mathrm{PL}}^{\mathrm{sr}}$ and $\mathcal{P}\ell st_{\mathrm{snd}} \to \widetilde{\mathrm{PL}}^{\mathrm{sr}}$.

6.2 The image of an analytic space in the skeleton

Recall that a strictly k-analytic space X is said to be quasi-algebraic if every point of X has a neighborhood of the form $V_1 \cup \cdots \cup V_n$, where each V_i is a strictly affinoid subdomain of X isomorphic to an affinoid domain in the analytification of a scheme of finite type over k. Recall also that a morphism of k-analytic spaces is said to be compact if it induces a proper map between the underlying topological spaces.

6.2.1 Theorem. *Let $\mathfrak{X}$ be a strongly nondegenerate pluri-stable formal scheme over k°, τ the retraction map $\mathfrak{X}_\eta \to S(\mathfrak{X})$, and Y a quasi-algebraic strictly k-analytic space. Then for any compact morphism $\varphi : Y \to \mathfrak{X}_\eta$ the image $\tau(\varphi(Y))$ is a piecewise $R^k_{\mathbb{Z}_+}$-linear closed subspace of $S(\mathfrak{X})$ of dimension at most* $\dim(Y)$.

Proof. It suffices to consider the case when the formal scheme $\mathfrak{X}$ is affine and Y is a strictly affinoid domain in $\mathcal{Z}^{\mathrm{an}}$, where $\mathcal{Z}$ is an integral affine scheme of finite type over k. Replacing k by the separable closure of k in $\mathcal{O}(\mathcal{Z})$, we may assume that $\mathcal{Z}$ is geometrically irreducible. By [Ber7, Lemma 9.4], there is an open embedding of $\mathcal{Z}$ in $\mathcal{Y}_\eta$, where $\mathcal{Y}$ is an integral scheme proper finitely presented and flat over k°, and an open subscheme $\mathcal{W}$ of $\mathcal{Y}_s$ such that $Y = \pi^{-1}(\mathcal{W}) = (\widehat{\mathcal{Y}}_{/\mathcal{W}})_\eta$, where π is the reduction map $\widehat{\mathcal{Y}}_\eta = \mathcal{Y}^{\mathrm{an}}_\eta \to \mathcal{Y}_s$. Since $\mathcal{Z}$ is geometrically irreducible, then so is $\mathcal{Y}_\eta$. By de Jong's results [deJ] (in the form of [Ber7, Lemma 9.2]), there exist a finite normal extension k' of k and a poly-stable fibration $\underline{\mathcal{Y}}' = (\mathcal{Y}'_l \stackrel{f'_{l-1}}{\to} \cdots \stackrel{f'_1}{\to} \mathcal{Y}'_1)$ over k'°, where all morphisms f'_i are projective of dimension one and have smooth geometrically irreducible generic fibers, and a dominant morphism $\mathcal{Y}'_l \to \mathcal{Y}$ that induces a proper generically finite morphism $\mathcal{Y}'_{l,\eta} \to \mathcal{Y}_\eta$. Notice that, since the morphisms f'_i have smooth geometrically irreducible generic fibers, the poly-stable fibration $\widehat{\underline{\mathcal{Y}}}'$ is nondegenerate.

Let $\mathcal{W}'$ be the preimage of $\mathcal{W}$ in $\mathcal{Y}'_{l,s}$, $\mathfrak{Y}'$ the formal completion of $\mathcal{Y}'_l$ along $\mathcal{W}'$, and $Y' = \mathfrak{Y}'_\eta$. The morphism φ gives rise to a surjective generically finite morphism of strictly k-analytic spaces $Y' \to Y$. *We claim that the induced morphism $Y' \to \mathfrak{X}_\eta$ comes from a unique morphism of formal schemes $\varphi' : \mathfrak{Y}' \to \mathfrak{X}$.* Indeed, let $\mathfrak{X} = \mathrm{Spf}(A)$, and let $\mathfrak{Y}'' = \mathrm{Spf}(B)$ be an open affine subscheme of $\mathfrak{Y}'$. The morphism of strictly k-affinoid spaces $\mathfrak{Y}''_\eta \to \mathfrak{X}_\eta$ is defined by a homomorphism of strictly k-affinoid algebras $\mathcal{A} = A \otimes_{k^\circ} k \to \mathcal{B} = B \otimes_{k'^\circ} k'$. By [Ber7, Proposition 1.4], one has $A \stackrel{\sim}{\to} \mathcal{A}^\circ$ and $B \stackrel{\sim}{\to} \mathcal{B}^\circ$. It follows that the homomorphism $\mathcal{A} \to \mathcal{B}$ defines a unique homomorphism $A \to B$ which, in its turn, defines a morphism of affine formal schemes $\mathfrak{Y}'' \to \mathfrak{X}$ that induces the morphism $\mathfrak{Y}''_\eta \to \mathfrak{X}_\eta$ we started from.

Thus, we have $\varphi(Y) = \varphi'_\eta(\mathfrak{Y}'_\eta)$. By Corollary 4.4.2, the image of $\varphi'_\eta(\mathfrak{Y}'_\eta)$ under the retraction map $\tau : \mathfrak{X}_\eta \to S(\mathfrak{X})$ coincides with the image of the skeleton $S(\mathfrak{Y}')$ under the map $S_{\varphi'} : S(\mathfrak{Y}') \to S(\mathfrak{X})$. But, by Corollary 6.1.3, the latter map is piecewise $R^{k'}_{\mathbb{Z}_+}$-linear. Hence, the image of $S(\mathfrak{Y}')$ under $S_{\varphi'}$ is a piecewise $R^k_{\mathbb{Z}_+}$-linear closed subspace of $S(\mathfrak{X})$ of dimension at most $\dim(Y') = \dim(Y)$. □

6.2.2 Corollary. *Let Y be a compact quasi-algebraic strictly k-analytic space, and $f_1, \ldots, f_n$ invertible analytic functions on Y. Then the image of Y under the map*

$$Y \mapsto (\mathbb{R}^*_+)^n : y \mapsto (|f_1(y)|, \ldots, |f_n(y)|)$$

is a $|k^*|_{\mathbb{Z}_+}$-polyhedron in $(\mathbb{R}^*_+)^n$ of dimension at most $\dim(Y)$.

Proof. Since Y is compact, we can multiply all of the functions by an element of k^* so that the image is contained in the set $S = \{\boldsymbol{t} \in (\mathbb{R}^*_+)^n \mid |a| \le |t_i| \le 1$ for all $1 \le i \le n\}$ with $a \in k^*$. Let $\mathfrak{X}$ be the direct product of n copies of the affine formal scheme $\mathrm{Spf}(k^\circ\{u, v\}/(uv - a))$. It is a strongly nondegenerate poly-stable formal scheme. The projection of $\mathfrak{X}_\eta$ to the coordinate v of each of the affine formal schemes identifies $\mathfrak{X}_\eta$ with the poly-annulus $\{x \in \boldsymbol{A}^n \mid |a| \le |T_i(x)| \le 1$ for all $1 \le i \le n\}$, and the functions $f_1, \ldots, f_n$ give rise to a morphism of strictly k-analytic spaces $\varphi : Y \to \mathfrak{X}_\eta$. Furthermore, the continuous map $(\boldsymbol{A}^1 \backslash \{0\})^n \to (\mathbb{R}^*_+)^n : x \mapsto (|T_1(x)|, \ldots, |T_n(x)|)$ identifies the skeleton $S(\mathfrak{X})$ with the set S, and gives rise to the retraction map $\tau : \mathfrak{X}_\eta \to S(\mathfrak{X}) = S$. Thus, the map from the statement of the corollary coincides the composition $\tau \circ \varphi : Y \to S(\mathfrak{X}) = S$ and, by Theorem 6.2.1, its image is a $R^k_{\mathbb{Z}_+}$-polyhedron in S. □

The following is a consequence of Corollary 6.1.5 and the proof of Theorem 6.2.1. For an analytic space Y, we set $\mathcal{O}(Y)^1 = \{f \in \mathcal{O}(Y) \mid |f(y)| = 1$ for all $y \in Y\}$.

6.2.3 Corollary. *If a quasi-algebraic strictly k-analytic space Y is compact and connected, then the group $\mathcal{O}(Y)^*/(k^*\mathcal{O}(Y)^1)$ is finitely generated.*

Proof. As in the proof of Theorem 6.2.1, one can apply de Jong's results to show that there is a finite surjective family of morphisms $Y_i \to Y$, where each Y_i is the generic fiber $\mathfrak{X}^i_\eta$ of a connected pluri-stable formal scheme $\mathfrak{X}^i$ over k_i°, where k_i is a finite extension of k. Then the group considered is embedded in the direct product of the groups $\mathcal{O}(Y_i)^*/(k^*\mathcal{O}(Y_i)^1)$. Since the groups $k_i^*/(k^*k_i^1)$ are finite, the required statement follows from Corollary 6.1.5. $\square$

6.3 Continuity of the embedding $S(\mathfrak{X}) \hookrightarrow \mathfrak{X}_\eta$ in the Grothendieck topology

Let $\mathfrak{X}$ be a nondegenerate pluri-stable formal scheme over k°. The piecewise $R^k_{\mathbb{Z}_+}$-linear structure on the skeleton $S(\mathfrak{X})$ provides it with a Grothendieck topology formed by piecewise $R^k_{\mathbb{Z}_+}$-linear subspaces. Recall (see [Ber2, §1.3]) that $\mathfrak{X}_\eta$ is also provided with a Grothendieck topology formed by strictly analytic subdomains.

6.3.1 Theorem. *For any strictly analytic subdomain $V \subset \mathfrak{X}_\eta$, the intersection $V \cap S(\mathfrak{X})$ is a piecewise $R^k_{\mathbb{Z}_+}$-linear subspace of $S(\mathfrak{X})$ and, for any $f \in \mathcal{O}'(V)$, the restriction of the function $|f|$ to $V \cap S(\mathfrak{X})$ is piecewise $|k^*|_{\mathbb{Z}_+}$-linear. In particular, the canonical embedding $S(\mathfrak{X}) \hookrightarrow \mathfrak{X}_\eta$ is continuous with respect to the Grothendieck topologies of $S(\mathfrak{X})$ and $\mathfrak{X}_\eta$.*

Proof. It suffices to consider the case when $\mathfrak{X} = \mathrm{Spf}(A)$ is affine and connected. By Gerritzen–Grauert Theorem ([BGR, 7.3.5/2]), a basis of the Grothendieck topology on a strictly k-affinoid space is formed by rational strictly affinoid domains, and so we may assume that V is such a domain. This means that there are functions $f_1, \dots, f_n, g \in \mathcal{A} = A \otimes_{k^\circ} k$ without common zeros on $\mathfrak{X}_\eta$ such that $V = \{x \in \mathfrak{X}_{l,\eta} \mid |f_i(x)| \le |g(x)|$ for all $1 \le i \le n\}$. Multiplying all of the above functions by an element of k^*, we may assume that $f_1, \dots, f_n, g \in A$. Since any function on $S(\mathfrak{X})$ of the form $x \mapsto |f(x)|$ with $f \in A \backslash \{0\}$ is piecewise $R_{\mathbb{Z}_+}$-linear, it follows that $V \cap S(\mathfrak{X})$ is a piecewise $R^k_{\mathbb{Z}_+}$-linear subspace of $S(\mathfrak{X})$.

Furthermore, let $f \in \mathcal{O}'(V)$. Then $\varepsilon = \min\{|f(x)| \mid x \in V \cap S(\mathfrak{X})\} > 0$, and one can find an element $h \in \mathcal{A}$ and an integer $n \ge 0$ such that $\left|(f - \frac{h}{g^n})(x)\right| < \varepsilon$ for all $x \in V$ and, therefore, the restrictions of the functions $|f|$ and $\frac{|h|}{|g|^n}$ to $V \cap S(\mathfrak{X})$ coincide. The latter function is evidently piecewise $|k^*|_{\mathbb{Z}}$-linear. That it is in fact $|k^*|_{\mathbb{Z}_+}$-linear follows from Remark 3.5.8(ii). $\square$

6.4 Continuity of the retraction map $\tau : \underline{\mathfrak{X}}_\eta \to S(\underline{\mathfrak{X}})$ in the Grothendieck topology

Let $\mathfrak{X}$ be a nondegenerate pluri-stable formal scheme over k°. We choose a nondegenerate poly-stable fibration $\underline{\mathfrak{X}} = (\mathfrak{X}_l \stackrel{f_{l-1}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ over k° of length l with $\mathfrak{X}_l = \mathfrak{X}$ and denote by $\tau = \tau_{\underline{\mathfrak{X}}}$ the corresponding retraction map $\mathfrak{X}_\eta \to S(\mathfrak{X})$.

6.4.1 Theorem. *For any piecewise $R^k_{\mathbb{Z}_+}$-linear subspace $E \subset S(\mathfrak{X})$, $\tau^{-1}(E)$ is a strictly analytic subdomain of $\mathfrak{X}_\eta$. In particular, the retraction map τ is continuous with respect to the Grothendieck topologies of $S(\mathfrak{X})$ and $\mathfrak{X}_\eta$.*

Assume that the above $\underline{\mathfrak{X}}$ possesses the following properties:

(1) for every $1 \le i \le l$, $\mathfrak{X}_i = \mathrm{Spf}(A_i)$ is affine;

(2) $\mathbb{D}(\underline{\mathfrak{X}})$ is a standard polysimplex $\Lambda[\underline{\boldsymbol{n}}]_{\underline{\boldsymbol{r}}}$, and $\mathbb{D}(\underline{\mathfrak{X}}^{\le i})$ are the standard polysimplices $\Lambda[\underline{\boldsymbol{n}}^{\le i}]_{\underline{\boldsymbol{r}}^{\le i}}$ for all $1 \le i \le l$;

(3) the maps $A_i \backslash \{0\} \to M^{\underline{\boldsymbol{n}}^{\le i}}_{\underline{\boldsymbol{r}}^{\le i}} : f \mapsto \theta_i^*(|f|)$, are surjective for all $1 \le i \le l$;

(4) for every $1 \le i \le l$, the morphism $f_{i-1} : \mathfrak{X}_i \to \mathfrak{X}_{i-1}$ goes through an étale morphism $\mathfrak{X}_i \to \mathfrak{X}_{i-1}(\boldsymbol{n}^{(i)}, \boldsymbol{a}^{(i)}, m_i)$.

In what follows we identify $S(\underline{\mathfrak{X}}^{\le i}) = S(\mathfrak{X}_i)$ with $\Sigma^{\underline{\boldsymbol{n}}^{\le i}}_{\underline{\boldsymbol{r}}^{\le i}}$. Furthermore, we introduce as follows a positive integer $\nu(\underline{\boldsymbol{n}})$. If $l = 1$, then $\nu(\underline{\boldsymbol{n}}) = 1$. If $l \ge 2$, then $\nu(\underline{\boldsymbol{n}}) = \nu(\underline{\boldsymbol{n}}^{\le l-1}) \cdot \mu(\boldsymbol{n}^{(l)})$ where, for $\boldsymbol{n} = (n_0, \dots, n_p)$, $\mu(\boldsymbol{n})$ is the least common multiple of the integers $1, 2, \dots, \max_{0 \le i \le p}\{n_i\} + 1$.

6.4.2 Lemma. *In the above situation, for every element $\alpha \in M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ there exist the following data:*

(a) *a finite covering of $S(\mathfrak{X}) = \Sigma^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ by $R^k_{\mathbb{Z}_+}$-polyhedra $\{E_i\}_{i \in I}$;*

(b) *for every $i \in I$, a finite covering of the preimage $\tau^{-1}(E_i)$ by strictly analytic domains $\{V_{ij}\}_{j \in J_i}$ with $\tau(V_{ij}) = V_{ij} \cap S(\mathfrak{X})$;*

(c) *for every $i \in I$ and $j \in J_i$, functions $f_{ij}, g_{ij} \in A_l$ such that for all $x \in V_{ij}$ one has $|f_{ij}(x_\tau)| = |f_{ij}(x)|$, $|g_{ij}(x_\tau)| = |g_{ij}(x)|$ and*

$$\alpha(x_\tau) = \left| \frac{f_{ij}(x)}{g_{ij}(x)} \right|^{\frac{1}{\nu(\underline{\boldsymbol{n}})}} .$$

Proof. First of all, we notice that if an element $\alpha \in M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$ possesses the properties of the lemma then, for any function $f \in A_l$, the sets $\{x \in \mathfrak{X}_\eta \mid |f(x)| \le \alpha(x_\tau)\}$ and $\{x \in \mathfrak{X}_\eta \mid |f(x)| \ge \alpha(x_\tau)\}$ are strictly analytic subdomains of $\mathfrak{X}_\eta$.

We prove the lemma by induction on l. Since it is evidently true for $l = 0$, we assume that $l \geq 1$ and that the statement is true for $\underline{\mathfrak{X}}^{\leq l-1}$. The morphism $f_{l-1} : \mathfrak{X} \to \mathfrak{X}_{l-1}$ goes through an étale morphism $\mathfrak{X} \to \mathfrak{X}' = \mathfrak{X}_{l-1}(\boldsymbol{n}, \boldsymbol{a}, m)$ with $\boldsymbol{n} = \boldsymbol{n}^{(l)}$, $\boldsymbol{a} = \boldsymbol{a}^{(l)}$ and $m = m_l$. Since $\mathbb{D}(\underline{\mathfrak{X}}) \stackrel{\sim}{\to} \mathbb{D}(\underline{\mathfrak{X}}')$ and the map $B\backslash\{0\} \to M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}} : g \mapsto \theta^*_l(|g|)$ is surjective, where $\underline{\mathfrak{X}}' = (\mathfrak{X}' \to \mathfrak{X}_{l-1} \stackrel{f_{l-2}}{\to} \cdots \stackrel{f_1}{\to} \mathfrak{X}_1)$ and $\mathfrak{X}' = \mathrm{Spf}(B)$, we may assume that $\mathfrak{X} = \mathfrak{X}'$. Of course, we assume that $[\boldsymbol{n}] \neq [0]$.

Step 1. *We may assume that the element* α *is a coordinate function on* $\Sigma^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$. Indeed, it suffices to show that if the lemma is true for two elements $\alpha, \alpha' \in M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$, then it is also true for their product $\alpha \cdot \alpha'$ and their maximum $\max\{\alpha, \alpha'\}$. Let us take the data provided by the assumption for the functions α and α', and mark the data for α' with the prime sign. Then the data for the product $\alpha \cdot \alpha'$ consist of the $R^k_{\mathbb{Z}_+}$-polyhedra $E_i \cap E'_{i'}$, the strictly analytic domains $V_{ij} \cap V_{i'j'}$, and the functions $f_{ij} \cdot f'_{i'j'}$ and $g_{ij} \cdot g'_{i'j'}$. The data for the maximum $\max\{\alpha, \alpha'\}$ consist of the same $R^k_{\mathbb{Z}_+}$-polyhedra $E_i \cap E'_{i'}$, the strictly analytic subdomains of $V_{ij} \cap V_{i'j'}$, defined in it by the inequalities $\left|\frac{f_{ij}(x)}{g_{ij}(x)}\right| \geq \left|\frac{f'_{i'j'}(x)}{g'_{i'j'}(x)}\right|$ and $\left|\frac{f_{ij}(x)}{g_{ij}(x)}\right| \leq \left|\frac{f'_{i'j'}(x)}{g'_{i'j'}(x)}\right|$, respectively, and the functions $f_{ij} \cdot g'_{i'j'}$, $f'_{i'j'} \cdot g_{ij}$ and $\{g_{ij} \cdot g'_{i'j'}\}$.

Step 2. By Step 1, we may assume that the element $\alpha \in M^{\boldsymbol{n}}_{\underline{\boldsymbol{r}}}$ is one of the coordinate functions $t_{0j} = \theta^*_l(|T_{0j}|)$. We denote n_0, a_0 and T_{0j} by n, a and T_j, respectively. For a point $y \in \mathfrak{X}_\eta$, we denote by x its image in $\mathfrak{X}_{l-1,\eta}$, and we denote by y_τ and x_τ the images of y and x in $S(\underline{\mathfrak{X}}) = S(\mathfrak{X})$ and $S(\underline{\mathfrak{X}}^{\leq l-1})$, respectively. First of all, we define the following covering of $S(\mathfrak{X})$ by $R^k_{\mathbb{Z}_+}$-polyhedra which correspond to permutations $\sigma \in S_{n+1}$:

$$E_\sigma = \{y \in S(\mathfrak{X}) \mid |T_{\sigma(0)}(y)| \leq |T_{\sigma(1)}(y)| \leq \cdots \leq |T_{\sigma(n)}(y)|\} .$$

It suffices to consider the restrictions of the coordinate functions to E, which correspond to the trivial permutation. From the description of τ, recalled in §4.4, it follows that $\tau^{-1}(E) = \bigcup_{i=0}^n V_i$, where V_i consists of all points $y \in \mathfrak{X}_\eta$ that satisfy the following three inequalities:

$$|a(x_\tau)| \leq |(T^{i+2}_{i+1} T_{i+2} \dots T_n)(y)|\} ,$$

$$\max_{0 \leq j \leq i} \{|(T^{i+1}_j T_{i+1} \dots T_n)(y)|\} \leq |a(x_\tau)| ,$$

$$\max_{0 \leq j \leq i} \{|T_j(y)|\} \leq |T_{i+1}(y)| \leq \cdots \leq |T_n(y)| .$$

Applying the induction hypothesis to the function $\theta^*_{l-1}(|a|)$, the first two inequalities define a finite union of rational strictly affinoid subdomains of $\mathfrak{X}_\eta$, and the functions $T_{i+1}, \dots, T_n$ are invertible on each of them. It follows that the third inequality also defines a rational strictly affinoid subdomain in each of them and, therefore, V_i is a finite union of rational strictly affinoid domains. Furthermore, the description of τ implies that for $y \in V_i$ one has

$$|T_j(y_\tau)| = \left|\frac{a(x_\tau)}{(T_{i+1}\dots T_n)(y)}\right|^{\frac{1}{i+1}}$$

for $0 \le j \le i$, and $|T_j(y_\tau)| = |T_j(y)|$ for $i+1 \le j \le n$. It follows that

$$\tau(V_i) = V_i \cap S(\mathfrak{X}) = \{y \in S(\underline{\mathfrak{X}}) \mid |T_0(y)| = \dots = |T_i(y)| \le |T_{i+1}(y)| \le \dots \le |T_n(y)|\}.$$

Applying again the induction hypothesis to the function $\theta^*_{l-1}(|a|)$, we get the required fact. □

Proof of Theorem 6.4.1. First of all, since the retraction map τ is proper, the statement is local in the Zariski topology. Furthermore, by Raynaud's theorem (see [BoLü2, Corollary 5.11), given a flat morphism of strictly k-affinoid spaces $\varphi : Y \to X$, for any strictly affinoid domain $V \subset Y$ the image $\varphi(V)$ is a finite union of strictly affinoid subdomains of X, i.e., is a compact strictly analytic subdomain of X. It follows that the statement of the theorem is local in the étale topology and, in particular, we may assume that $\underline{\mathfrak{X}}$ is strictly poly-stable. Of course, we may assume that all $\mathfrak{X}_i = \mathrm{Spf}(A_i)$ are affine. After that we can shrink $\underline{\mathfrak{X}}$ so that it satisfies the assumptions of Lemma 6.4.2. It suffices to show that, given two elements $\alpha, \alpha' \in M^{\boldsymbol{n}}_{\boldsymbol{r}}$, the preimage $\tau^{-1}(D)$ of $D = \{x \in S(\mathfrak{X}) \mid \alpha(x) \le \alpha'(x)\}$ is a strictly analytic subdomain of $\mathfrak{X}_\eta$. Let us take the data provided by Lemma 6.4.2 for the functions α and α', and mark the data for α' with the prime sign. It suffices to show that, for every quadruple $i \in I$, $j \in J_i$, $i' \in I'$ and $j' \in J'_{i'}$, the intersection $\tau^{-1}(D) \cap V_{ij} \cap V'_{i'j'}$ is a strictly analytic subdomains of $V_{ij} \cap V'_{i'j'}$. We have

$$\tau^{-1}(D) \cap V_{ij} \cap V'_{i'j'} = \left\{x \in V_{ij} \cap V'_{i'j'} \mid \left|\frac{f_{ij}(x)}{g_{ij}(x)}\right| \le \left|\frac{f'_{i'j'}(x)}{g'_{i'j'}(x)}\right|\right\}$$

Since all of the functions in the inequality are invertible on $V_{ij} \cap V'_{i'j'}$, the set considered is a strictly analytic subdomain of $V_{ij} \cap V'_{i'j'}$. □

6.4.3 Corollary. *The following properties of a subset $E \subset S(\mathfrak{X})$ are equivalent:*

(a) *E is a piecewise $R^k_{\mathbb{Z}_+}$-linear subspace of $S(\mathfrak{X})$;*

(b) *$\tau^{-1}(E)$ is a strictly analytic subdomain of $\mathfrak{X}_\eta$.* □

The following result is a consequence of Lemma 6.4.2. Let $\mathfrak{X}$ and $\mathfrak{X}'$ be nondegenerate pluri-stable formal schemes over k° and k'°, respectively, and let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism in $\mathcal{P}\ell st_{\mathrm{nd}}$. We fix a nondegenerate poly-stable fibration of length l over k° with $\mathfrak{X}_l = \mathfrak{X}$ which gives rise to a retraction map $\tau : \mathfrak{X}_\eta \to S(\mathfrak{X})$.

6.4.4 Theorem. *The map $\tau \circ \varphi_\eta : S(\mathfrak{X}') \to S(\mathfrak{X})$ is piecewise $\left(\sqrt{|k'^*|}\right)_{\mathbb{Q}_+}$-linear. If $l = 1$, this map is in fact $|k'^*|_{\mathbb{Z}_+}$-linear.*

Proof. Replacing $\underline{\mathfrak{X}}$ by $\underline{\mathfrak{X}}\widehat{\otimes}_{k^\circ} k'^\circ$, we may assume that $k' = k$ and that φ is a k-morphism. Furthermore, since the statement is local in the étale topology of $\underline{\mathfrak{X}}$ and $\mathfrak{X}'$, we may assume that $\underline{\mathfrak{X}}$ satisfies the assumptions of Lemma 6.4.2 and that $\mathfrak{X}'$ is strictly pluri-stable and small enough so that $S(\mathfrak{X}')$ is an $R^k_{\mathbb{Z}_+}$-polyhedron. To prove the statement, it suffices to show that if $l \geq 2$ (resp. $l = 1$) then, for every $\alpha \in M_{\mathfrak{X}} = M^{\underline{\boldsymbol{n}}}_{\underline{\boldsymbol{r}}}$, $\varphi^*_\eta(\tau^*(\alpha))$ is a piecewise $(\sqrt{|k^*|})_{\mathbb{Q}_+}$-linear (resp. $|k^*|_{\mathbb{Z}_+}$-linear) function on $S(\mathfrak{X}')$.

By Lemma 6.4.2, there exists a finite covering of $\mathfrak{X}_\eta$ by strictly analytic domains $\{V_{ij}\}_{i \in I, j \in J_i}$ and, for each $i \in I$ and $j \in J_i$, functions $f_{ij}, g_{ij} \in A_l$ such that for all $x \in V_{ij}$ one has $|f_{ij}(x_\tau)| = |f_{ij}(x)|$, $|g_{ij}(x_\tau)| = |g_{ij}(x)|$ and

$$\alpha(x_\tau) = \left| \frac{f_{ij}(x)}{g_{ij}(x)} \right|^{\frac{1}{\nu(\underline{\boldsymbol{n}})}} .$$

By Theorem 6.3.1, each $E'_{ij} = S(\mathfrak{X}') \cap \varphi^{-1}_\eta(V_{ij})$ is a piecewise $R^k_{\mathbb{Z}_+}$-linear subspace of the $R^k_{\mathbb{Z}_+}$-polyhedron $S(\mathfrak{X}')$ and, by the above formula, the restriction of $\varphi^*_\eta(\tau^*(\alpha))$ to E'_{ij} coincides with the restriction of the piecewise $(\sqrt{|k^*|})_{\mathbb{Q}}$-linear function

$$x' \mapsto \left| \frac{(\varphi^* f_{ij})(x')}{(\varphi^* g_{ij})(x')} \right|^{\frac{1}{\nu(\underline{\boldsymbol{n}})}} .$$

The latter function is piecewise $(\sqrt{|k^*|})_{\mathbb{Q}_+}$-linear, by Remark 3.5.8(ii). If $l = 1$, then $\nu(\underline{\boldsymbol{n}}) = 1$ and, therefore, it is even piecewise $|k^*|_{\mathbb{Z}_+}$-linear. Since $S(\mathfrak{X}')$ is a union of all E'_{ij}, the required fact follows. □

7 Strong local contractibility of smooth analytic spaces

7.1 Formulation of the result

Let k be a non-Archimedean field with a non-trivial valuation. Recall (see [Ber7, §9]) that a k-analytic space is said to be locally embeddable to a smooth space if each point $x \in X$ has an open neighborhood isomorphic to a strictly analytic domain in a smooth k-analytic space. This class includes the class of spaces smooth in the sense of [Ber2], their strictly analytic subdomains, and is contained in the class of spaces smooth in the sense of rigid geometry (i.e., rig-smooth spaces). Notice also that any rig-smooth affinoid space is locally embeddable in a smooth space.

Recall also that a strong deformation retraction of a topological space X to a subset $S \subset X$ is a continuous mapping $\Phi : X \times [0, 1] \to X$ such that $\Phi(x, 0) = x$ and $\Phi(x, 1) \in S$ for all $x \in X$, and $\Phi(x, t) = x$ for all $x \in S$ and $t \in [0, 1]$. We say that a subspace $Y \subset X$ is preserved under Φ if $\Phi(Y \times [0,1]) \subset Y$. If S is a point, Φ is said to be a contraction of X to the point.

7.1.1 Theorem. *Let X be a k-analytic space locally embeddable in a smooth space. Each point $x \in X$ has a fundamental system of open neighborhoods V which possess the following properties:*

(a) *there is a contraction Φ of V to a point $x_0 \in V$;*

(b) *there is an increasing sequence of compact strictly analytic domains $X_1 \subset X_2 \subset \cdots$ which are preserved under Φ and such that $V = \bigcup_{n=1}^{\infty} X_n$;*

(c) *given a non-Archimedean field K over k, $V \widehat{\otimes} K$ has a finite number of connected components, and Φ lifts to a contraction of each of the connected components to a point over x_0;*

(d) *there is a finite separable extension L of k such that, if K from* (c) *contains L, then the map $V \widehat{\otimes} K \to V \widehat{\otimes} L$ induces a bijection between the sets of connected components.*

Recall that [Ber7, Theorem 9.1] states that each point $x \in X$ has a fundamental system of contractible open neighborhoods V. In §7.2, we recall the main construction from the proof of *loc. cit.*. After that, instead of using [Ber7, Theorem 8.2], we use results from §1 and §6. But before doing this, we establish a simple fact which will be used in the last step of the proof and is true without the assumption that the valuation on k is nontrivial.

Let k' be a finite extension of k. Then every strictly k'-affinoid algebra $\mathcal{A}$ is evidently a strictly k-affinoid algebra, and so the strictly k'-affinoid space $X = \mathcal{M}(\mathcal{A})$ can be considered as a strictly k-affinoid space, i.e., there is a canonical functor from the category of strictly k'-affinoid spaces to that of strictly k-affinoid ones. From the following proposition it follows that the latter can be extended to a functor st-k'-$\mathcal{A}n \to st$-k-$\mathcal{A}n$ from the category of strictly k'-analytic spaces to that of strictly k-analytic ones, and it takes strictly k'-analytic domains to strictly k-analytic ones. Notice that the above functor is left adjoint to the ground field extension functor st-k-$\mathcal{A}n \to st$-k'-$\mathcal{A}n : X \mapsto X \widehat{\otimes} k'$.

7.1.2 Proposition. *Let X be a strictly k'-affinoid space. Then any strictly k'-affinoid subdomain $V \subset X$ is a strictly k-affinoid subdomain of X, considered as a strictly k-affinoid space.*

7.1.3 Lemma. *Assume that the valuation on k is trivial, and let $\varphi : Y = \mathcal{M}(\mathcal{B}) \to X = \mathcal{M}(\mathcal{A})$ be a morphism of strictly k-affinoid spaces. Then the following are equivalent:*

(a) *φ identifies Y with a strictly affinoid subdomain of X;*

(b) *the induced morphism of affine schemes* $\mathcal{Y} = \mathrm{Spec}(\mathcal{B}) \to \mathcal{X} = \mathrm{Spec}(\mathcal{A})$ *is an open immersion.*

Proof. (a)$\Longrightarrow$(b) For any point $y \in Y$ with $[\mathcal{H}(y) : k] < \infty$, one has $\mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{Y,y}$, where x is the image of y in X. But the images $\boldsymbol{x}$ of x in $\mathcal{X}$ corresponds to a maximal ideal of $\mathcal{A}$, and $\mathcal{O}_{X,x}$ coincides with the completion $\widehat{\mathcal{O}}_{\mathcal{X},\boldsymbol{x}}$ of $\mathcal{O}_{\mathcal{X},\boldsymbol{x}}$ by the maximal ideal (see [Ber1, Theorem 3.5.1]), and the same is true for the image $\boldsymbol{y}$ of y in $\mathcal{Y}$. It follows that the morphism of schemes induces an isomorphism $\widehat{\mathcal{O}}_{\mathcal{X},\boldsymbol{x}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{Y},\boldsymbol{y}}$ and, therefore, it is an étale morphism. On the other hand, since for any bigger field K (also provided with the trivial valuation) the map $\mathcal{Y}(K) = Y(K) \to \mathcal{X}(K) = X(K)$ is injective, the morphism of schemes is radicial. It remains to use the fact that any étale and radicial morphism between affine schemes of finite type over a field is an open immersion.

(b)$\Longrightarrow$(a) If $\mathcal{Y}$ is identified with a principal open subset $\{\boldsymbol{x} \in \mathcal{X} \mid f(\boldsymbol{x}) \neq 0\}$, then Y is identified with the rational subdomain $\{x \in X \mid |f(x)| = 1\}$. In the general case, $\mathcal{Y}$ is a finite union of principal open subsets, and so $Y = \bigcup_{i=1}^{n} Y_i$, where each Y_i is identified with a rational subdomain of X of the above forms. From [Ber2, Remark 1.2.1] it follows that φ identifies Y with a strictly affinoid subdomain of X. □

7.1.4 Corollary. *If the valuation on k is trivial, then any strictly k-analytic space is Hausdorff.*

Proof. By [Ber2, Lemma 1.1.1(ii)], it suffices to show that any strictly analytic subdomain Y of a strictly k-analytic space $X = \mathcal{M}(\mathcal{A})$ is compact. From Lemma 7.1.3 it follows that Y corresponds to an open subscheme of $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$. Since the ring $\mathcal{A}$ is Noetherian, any open subscheme of $\mathcal{X}$ is quasicompact and, therefore, Y is compact. □

Proof of Proposition 7.1.2. If the valuation on k is trivial, the statement follows from Lemma 7.1.3. Thus, assume that the valuation on k is nontrivial, and let $X = \mathcal{M}(\mathcal{A})$ and $V = \mathcal{M}(\mathcal{A}_V)$. The statement is trivial if V is a rational domain since it is defined by the inequalities $|f_i(x)| \leq |g(x)|$, where $f_1, \dots, f_n, g$ are elements of $\mathcal{A}$ that generate the unit ideal. Assume V is arbitrary. By Gerritzen–Grauert Theorem ([BGR, 6.3.5/2]), it is a finite union $\bigcup_{i=1}^{n} V_i$ of rational strictly affinoid subdomains of X. By Tate's Acyclicity Theorem, there an isomorphism of commutative Banach k-algebras $\mathcal{A}_V \xrightarrow{\sim} \mathrm{Ker}(\prod_i \mathcal{A}_{V_i} \rightrightarrows \prod_{i,j} \mathcal{A}_{V_i \cap V_j})$. Since $\mathcal{A}_V$ is strictly k-affinoid and the canonical map $V \to \mathcal{M}(\mathcal{A}_V)$ is a bijection, V is a strictly k-affinoid subdomain of X (see [Ber2, Remark 1.2.1]). □

7.2 Proof: Step 1

We follow the proof of [Ber7, Theorem 9.1]. It is done by induction on the dimension of X at x. First of all, we may assume that X is a strictly analytic domain in $\mathcal{X}^{\mathrm{an}}$, where $\mathcal{X} = \mathrm{Spec}(A)$ is a smooth irreducible affine scheme over k. Let $\boldsymbol{x}$ be the image of the point x in $\mathcal{X}$. There are the following two cases:

(α) $\boldsymbol{x}$ is not the generic point of $\mathcal{X}$;

(β) $\boldsymbol{x}$ is the generic point of $\mathcal{X}$.

Case (α). As in *loc. cit.*, Steps 1 and 2 of Case (a), one reduces the situation to the case when the field $k(\boldsymbol{x})$ is separable over k and, after that, one shows that there is a sufficiently small open neighborhood of x isomorphic to $Y \times D(0; r)$ with $x = (y, 0)$, where Y is a strictly analytic domain in the analytification of a smooth scheme over k and $D(0; r)$ is the open disc with center at zero and of radius $r > 0$. Thus, we may assume that $X = Y \times D(0; r)$, and it suffices to show that the point $x = (y, 0)$ has an open neighborhood with the properties (a)–(d). In *loc. cit.*, Step 3, one constructs a continuous mapping $X \times [0, 1] \to X : (x', t) \mapsto x'_t$, which is a retraction of X to a closed subset homeomorphic to $Y \times [0, r[$ and such that $|T(x'_t)| = |T(x')|$ for all $x' \in X$ and $t \in [0, 1]$. Thus, if $\mathcal{V}$ is an open neighborhood of the point y and $Y_1 \subset Y_2 \subset \cdots$ is an increasing sequence of compact strictly analytic domains in $\mathcal{V}$ possessing the properties (a)–(d), then the open neighborhood $\mathcal{V} \times D(0, r)$ of the point x and the sequence of compact strictly analytic domains $Y_1 \times E(0; r_1) \subset Y_2 \times E(0, r_2) \subset \cdots$ possess the same properties, where $r_1 < r_2 < \cdots$ is an increasing sequence of numbers from $\sqrt{|k^*|}$ with $r_i \to r$ as $i \to \infty$, and $E(0; r)$ is the closed disc of radius r.

Case (β). As in *loc. cit.*, Case (b), we may assume that X is compact and $\mathcal{X}$ is geometrically irreducible, and it suffices to show that, given a rational strictly affinoid neighborhood W of x in $\mathcal{X}^{\mathrm{an}}$, there exists an open neighborhood of x in X which possesses the properties (a)–(d) and is contained in $W \cap X$. By *loc. cit.*, Lemma 9.4, there is an open embedding of $\mathcal{X}$ in $\mathcal{Y}_\eta$, where $\mathcal{Y}$ is an integral scheme proper finitely presented and flat over k°, open subschemes $\mathcal{Z}$ and $\mathcal{W}$ of $\mathcal{Y}_s$, and a closed subscheme $\mathcal{V}$ of $\mathcal{Y}_s$ such that

(1) $X = \pi^{-1}(\mathcal{Z})$, $W = \pi^{-1}(\mathcal{W})$ and $\pi(x) \in \mathcal{V}$;

(2) $\mathcal{V} \subset \mathcal{W}$;

(3) $\mathcal{V}$ and $\mathcal{Y}_s \backslash \mathcal{Z}$ are unions of irreducible components of $\mathcal{Y}_s$.

By J. de Jong results [deJ] (in the form of [Ber7, Lemma 9.2]), there exist a finite normal extension k' of k, a poly-stable fibration $\underline{\mathcal{Y}}'$ of length l over k'° such that all morphisms $f'_i : \mathcal{Y}'_{i+1} \to \mathcal{Y}'_i$ are projective of dimension one and have smooth geometrically irreducible generic fibers, an action of a finite group G on $\underline{\mathcal{Y}}'$ over k°, and a dominant G-equivariant morphism $\varphi : \mathcal{Y}'_l \to \mathcal{Y}$ that induces a proper generically finite morphism $\mathcal{Y}'_{l,\eta} \to \mathcal{Y}_\eta$ and such that the field $R(\mathcal{Y}'_{l,\eta})^G$ is purely inseparable over $R(\mathcal{Y})$. Notice that the poly-stable fibration $\widehat{\underline{\mathcal{Y}}}'$ over k'° is nondegenerate.

Let $\mathcal{Z}'$, $\mathcal{W}'$ and $\mathcal{V}'$ be the preimages of $\mathcal{Z}$, $\mathcal{W}$ and $\mathcal{V}$ in $\mathcal{Y}'_{l,s}$, respectively. Then $\mathcal{V}'$ and $\mathcal{Y}'_{l,s} \backslash \mathcal{Z}'$ are unions of irreducible components of $\mathcal{Y}'_{l,s}$ and $\mathcal{V}' \subset \mathcal{W}'$. For

$X' = \pi^{-1}(\mathcal{Z}')$ and $W' = \pi^{-1}(\mathcal{W}')$, one has $X' = \varphi_\eta^{-1}(X)$ and $W' = \varphi_\eta^{-1}(W)$. Moreover, $\pi^{-1}(\mathcal{V}') \cap X'$ is an open subset of X' contained in $W' \cap X'$. By the construction, we can find a nonempty open affine subscheme $\mathcal{U} \subset \mathcal{X}$ such that the morphism $\mathcal{U}' := \varphi_\eta^{-1}(\mathcal{U}) \to \mathcal{U}$ is finite and the finite morphism $G\backslash\mathcal{U}' \to \mathcal{U}$ is radicial. By the assumption (β), the point x is contained in $\mathcal{U}^{\rm an}$. It follows that the set $U := \pi^{-1}(\mathcal{V}) \cap X \cap \mathcal{U}^{\rm an}$ is an open neighborhood of x in X contained in $W \cap X$. The set $U' := \pi^{-1}(\mathcal{V}') \cap X' \cap \mathcal{U}'^{\rm an}$ is open in X' and dense Zariski open in $\pi^{-1}(\mathcal{V}') \cap X' = \pi^{-1}(\mathcal{V}' \cap \mathcal{Z}')$, and the radicial morphism $G\backslash\mathcal{U}' \to \mathcal{U}$ induces a homeomorphism $G\backslash U' \overset{\sim}{\to} U$. Since $\mathcal{V}'$ and $\mathcal{Y}'_{l,s}\backslash\mathcal{Z}'$ are unions of irreducible components of $\mathcal{Y}'_{l,s}$, it follows that $\mathcal{V}' \cap \mathcal{Z}'$ is a strata subset of $\mathcal{Y}'_{l,s}$.

By [Ber7, Theorems 8.1], there is a G-equivariant strong deformation retraction $\Phi' : \mathcal{Y}'^{\rm an}_{l,\eta} \times [0,1] \to \mathcal{Y}'^{\rm an}_{l,\eta} : (y', t) \mapsto y'_t$ to the skeleton $S' = S(\widehat{\underline{\mathcal{Y}}}')$ of the formal completion of $\underline{\mathcal{Y}}'$ along the closed fiber. (Notice that $\mathcal{Y}'^{\rm an}_{l,\eta} = \widehat{\mathcal{Y}}'_{l,\eta}$.) Furthermore, Φ' induces a G-equivariant strong deformation retraction of the set $\pi^{-1}(\mathcal{V}' \cap \mathcal{Z}')$ to its intersection $\widetilde{S}'$ with the skeleton S' of $\underline{\mathcal{Y}}'$. This intersection $\widetilde{S}'$ is contained in the Zariski open subset U' of $\pi^{-1}(\mathcal{V}' \cap \mathcal{Z}')$, and U' is preserved under Φ'. Thus, Φ' induces a strong deformation retraction $\Phi : \mathcal{U}^{\rm an} \times [0,1] \to \mathcal{U}^{\rm an}$ to the closed subset $S = G\backslash S'$, as well as a strong deformation retraction of U to $\widetilde{S} = G\backslash\widetilde{S}'$.

7.3 Proof: Step 2

We can shrink $\mathcal{U}$ so that the finite morphisms $\mathcal{U}' \to G\backslash\mathcal{U}'$ and $G\backslash\mathcal{U}' \to \mathcal{U}$ are flat. In this case, the induced morphisms between the analytifications are also flat (see [Ber2, Proposition 3.2.10), and M. Raynaud's theorem (see [BoLü2, Corollary 5.11]) implies that the image of any strictly analytic subdomain of $\mathcal{U}'^{\rm an}$ and $G\backslash\mathcal{U}'^{\rm an}$ is a strictly analytic domain in $G\backslash\mathcal{U}'^{\rm an}$ and $\mathcal{U}^{\rm an}$, respectively. In particular, we can replace $\mathcal{Y}$ by the quotient $G\backslash\mathcal{Y}'_{l,\eta}$, and so we may assume that there are isomorphisms of schemes $G\backslash\mathcal{U}' \overset{\sim}{\to} \mathcal{U}$ and of analytic spaces $G\backslash U' \overset{\sim}{\to} U$, and we may assume that $k'^G = k$ and, in particular, that k' is a Galois extension of k.

We now claim that *there exists a sequence of compact strictly analytic domains* $Y_1 \subset Y_2 \subset \cdots$ *in* $\mathcal{Y}^{\rm an}$ *which are preserved under* Φ *and such that* $\mathcal{U}^{\rm an} = \bigcup_{n=1}^\infty Y_n$. Indeed, $\mathcal{Y}'^{\rm an}_{l,\eta}$ is the generic fiber of the formal completion $\widehat{\mathcal{Y}}'_l$ of $\mathcal{Y}'_l$ along its closed fiber. The latter formal scheme is a finite union of G-invariant open affine subschemes $\mathfrak{Y}^i$. If we can find an exhausting sequence of G-invariant compact strictly analytic domains $Y'^i_1 \subset Y'^i_2 \subset \cdots$ in $\mathfrak{Y}^i_\eta \cap \mathcal{U}'^{\rm an}$ which are preserved under Φ' and for which the quotients $Y^i_n = G\backslash Y'^i_n$ exist, then the sequence of the compact analytic domains $Y_n = \bigcup_i Y^i_n$ possesses the required properties. It suffices therefore to consider an open affine formal subscheme $\mathfrak{Y}'$ of $\widehat{\mathcal{Y}}'_l$.

Let $\mathfrak{Y}' = {\rm Spf}(A')$. The generic fiber $\mathfrak{Y}'_\eta$ is the strictly k-affinoid space $\mathcal{M}(\mathcal{A}')$, where $\mathcal{A}' = A' \otimes_{k'^\circ} k'$. The complement of $\mathcal{U}'^{\rm an}$ in $\mathfrak{Y}'_\eta$ is defined by a finite number of equations $f_i(x') = 0$, $1 \le i \le m$, with $f_i \in A'^G$. We take a decreasing sequence of positive numbers $r_1 > r_2 > \cdots$ in R^k with $\lim_{n\to\infty} r_n = 0$, and consider the G-invariant strictly affinoid domains $Y_n^i = \{x \in \mathfrak{Y}'_\eta \mid |f_i(y')| \ge r_i\}$. Since $|f(y'_t)| \ge |f'(y')|$ for all elements $f' \in \mathcal{A}'$ and all $t \in [0,1]$ (see [Ber7, Theorem 8.1(iii)]), Y_n^i are preserved under Φ'. It follows that the compact strictly analytic domains $Y'_n = \bigcup_{i=1}^m Y_n^i$ are preserved under Φ' and G-invariant, and the quotients $G\backslash Y'_n$ exist. One also has $Y'_1 \subset Y'_2 \subset \cdots$ and $\bigcup_{n=1}^\infty Y'_n = \mathfrak{Y}'_\eta \cap \mathcal{U}'^{\mathrm{an}}$.

7.4 Proof: Step 3

Consider the $R^{k'}$-colored polysimplicial sets $D' = \mathbb{D}(\underline{\widehat{\mathcal{Y}}'})$ and $\widetilde{D}' = \mathbb{D}(\underline{\mathfrak{Z}}')$, where $\underline{\mathfrak{Z}}'$ is the poly-stable fibration $(\mathfrak{Z}' \to \widehat{\mathcal{Y}}'_{l-1} \to \cdots \to \mathcal{Y}'_1)$ over k'° and $\mathfrak{Z}'$ is the formal completion of $\widehat{\mathcal{Y}}'_l$ along the open subset $\mathcal{Z}'$ of $\mathcal{Y}'_{l,s}$. By §4.3, there are canonical homeomorphisms $|D'| \stackrel{\sim}{\to} S' = S(\underline{\widehat{\mathcal{Y}}'})$ and $|\widetilde{D}'| \stackrel{\sim}{\to} S(\underline{\mathfrak{Z}}')$. Setting $D = G\backslash D'$ and $\widetilde{D} = G\backslash\widetilde{D}'$, we can identify S with $|D|$ and $\widetilde{S}$ with an open subset of $|\widetilde{D}|$. (Notice that $|\widetilde{D}|$ is a closed subset of $|D|$.)

Let x_0 be the image of the point x under the retraction map $\tau : \mathcal{U}^{\mathrm{an}} \to S$ induced by Φ. By Proposition 1.4.1, one can find a compact $R^{k'}_{\mathbb{Z}_+}$-piecewise linear neighborhood E of the point x_0 in $|\widetilde{D}|$, which is isomorphic to an $(\sqrt{|k^*|})_{\mathbb{Q}}$-polyhedron in an affine space $(\mathbb{R}_+^*)^d$. For $0 < r < 1$, let $B(x_0, r)$ denote the open box $\{y \in (\mathbb{R}_+^*)^d \mid r < |\frac{t_i(y)}{t_i(x_0)}| < r^{-1}, 1 \le i \le d\}$. One can find $0 < r_0 < 1$ such that, for every $r_0 \le r < 1$, the open set $\widetilde{E}(r) = E \cap B(x_0, r)$ is contained in $\widetilde{S}$ and possesses the property that, for each point $y \in \widetilde{E}(r)$, the interval $\{x_0^t \cdot y^{1-t}\}_{t\in[0,1]}$, connecting the points x_0 and y, is contained in $\widetilde{E}(r)$. Let us fix such r, and let Ψ be the contraction $\widetilde{E}(r) \times [0,1] \to \widetilde{E}(r) : (y,t) \mapsto x_0^t \cdot y^{1-t}$ of $\widetilde{E}(r)$ to the point x_0. Furthermore, let $1 \ge r_1 > r_2 > \cdots > r$ be a sequence of numbers from $\sqrt{|k^*|}$ with $\lim_{n\to\infty} r_n = r$. Then the $R^{k'}_{\mathbb{Z}_+}$-polyhedrons $E_n = \{y \in \widetilde{E}(r) \mid r_n \le |\frac{t_i(y)}{t_i(x_0)}| \le r_n^{-1}, 1 \le i \le d\}$ are preserved under Ψ and $\widetilde{E}(r) = \bigcup_{n=1}^\infty E_n$. Since $\widetilde{E}(r) \subset \widetilde{S}$, the set $V(r) = \tau^{-1}(\widetilde{E}(r)) \cap \mathcal{U}^{\mathrm{an}}$ is an open neighborhood of the point x in U.

We claim that, for every $r_0 \le r < 1$, $V(r)$ possesses the properties (a) *and* (b)*, and that one can find $r_0 \le r'_0 \le 1$ such that, for every $r'_0 \le r < 1$, $V(r)$ also possesses the properties* (c) *and* (d).

(a) The composition of the strong deformation retraction of $\tau^{-1}(\widetilde{E}(r))$ to $\widetilde{E}(r)$, induced by Φ, and of the contraction Ψ of $\widetilde{E}(r)$ to x_0, gives rise to a contraction of $V(r)$ to the point x_0.

(b) We claim that $Z_n = \tau^{-1}(E_n)$ *is a strictly analytic subdomain of* $\mathcal{U}^{\mathrm{an}}$. Indeed, let E'_n be the preimage of E_n in S'. By Theorem 6.4.1, $Z'_n = \tau'^{-1}(E'_n)$ is a strictly k'-analytic subdomain of $\mathcal{U}'^{\mathrm{an}}$, where τ' is the retraction map $\mathcal{U}'^{\mathrm{an}} \to S'$ induced by Φ'. Proposition 7.1.2 implies that Z'_n is a strictly k-analytic subdomain of $\mathcal{U}'^{\mathrm{an}}$ considered as a strictly k-analytic space. Since Z_n is the image of Z'_n under the flat morphism $\mathcal{U}'^{\rm an} \to \mathcal{U}^{\rm an}$, the claim follows from M. Raynaud's theorem. Thus, the intersection $X_n = Y_n \cap Z_n$, where Y_n is constructed in §7.2, is a compact strictly analytic subdomain of $V(r)$, and it is evidently preserved under the contraction from (a). One also has $V(r) = \bigcup_{n=1}^{\infty} X_n$ since $\mathcal{U}^{\rm an} = \bigcup_{n=1}^{\infty} Y_n$ and $\widetilde{E}(r) = \bigcup_{n=1}^{\infty} E_n$.

To establish the properties (c) and (d), we need the following additional fact.

Given a G-local immersion of compact piecewise $R^{k'}$-linear spaces $g : T \to S$, one can find $r_0 \le r'_0 < 1$ such that, for every $r'_0 \le r < 1$, the contraction Ψ of the set $\widetilde{E}(r)$ to x_0 lifts to a contraction of each of the connected component of $g^{-1}(\widetilde{E}(r))$ to a point above x_0. Indeed, let $y_1, \dots, y_n$ be the preimages of the point x_0 in T. We can find pairwise disjoint neighborhoods $D_1, \dots, D_n$ of the points $y_1, \dots, y_n$, respectively, with the following property: for every $1 \le i \le n$, $D_i = \bigcup_{j=1}^{m_i} D_{ij}$, where each D_{ij} is a compact piecewise $R^{k'}$-linear subspace of T that contains the point y_i and such that g induces an isomorphism of D_{ij} with an $R^{k'}$-polyhedron E_{ij} in E. One can find $r_0 \le r'_0 < 1$ such that, for every $1 \le i \le n$, $1 \le j \le m_i$ and every point $y \in E_{ij} \cap \widetilde{E}(r'_0)$, the interval, connecting the points x_0 and y, is contained in $E_{ij} \cap \widetilde{E}(r'_0)$. This construction guarantees the required property of $\widetilde{E}(r)$ and Ψ for all $r'_0 \le r < 1$.

(c) and (d). For a non-Archimedean field K over k, $V(r)\widehat{\otimes}K$ is a strictly analytic domain in $\mathcal{Y}^{\rm an}_\eta\widehat{\otimes}K$. The latter is a quotient of $\widehat{\mathcal{Y}}'_{l,\eta}\widehat{\otimes}K$ under the action of the group G. Since k' is a finite Galois extension of k, the tensor product $k' \otimes_k K$ is isomorphic to a direct product of m copies of a finite Galois extension K' of K with $m\cdot[K' : K] = [k' : k]$. This isomorphism gives rise to an action of G on the direct product and, therefore, to an action of G on the corresponding disjoint union $\mathfrak{Y}^K_i$ of m copies of each of the formal schemes $\widehat{\mathcal{Y}}'_i\widehat{\otimes}_{k'^\circ}K'^\circ$, $1 \le i \le l$. Thus, we have a nondegenerate poly-stable fibration $\underline{\mathfrak{Y}}^K = (\mathfrak{Y}^K_l \to \cdots \to \mathfrak{Y}^K_1)$ over K'° provided with an action of the group G over K°, and an isomorphism of strictly K-analytic spaces $G\backslash\mathfrak{Y}^K_{l,\eta} \stackrel{\sim}{\to} \mathcal{Y}^{\rm an}_\eta\widehat{\otimes}K$.

Let D'_K be the $R^{K'}$-colored polysimplicial set associated with $\underline{\mathfrak{Y}}^K$, and S'_K the skeleton of $\underline{\mathfrak{Y}}^K$. There is a G-equivariant homeomorphism $|D'_K| \stackrel{\sim}{\to} S'_K$. It gives rise to a homeomorphism $|D_K| \stackrel{\sim}{\to} S_K = G\backslash S'_K$, where $D_K = G\backslash D'_K$. Furthermore, the G-equivariant strong deformation retraction Φ'_K of $\mathfrak{Y}^K_{l,\eta}$ to S'_K gives rise to a strong deformation retraction Φ_K of $\mathcal{Y}^{\rm an}_\eta\widehat{\otimes}K$ to S_K compatible with the strong deformation retraction Φ of $\mathcal{Y}^{\rm an}_\eta$ to S. If g_K denotes the canonical G-local immersion of compact piecewise $R^{K'}$-linear spaces $S_K \to S$, then Φ_K induces a strong deformation retraction of $V\widehat{\otimes}K$ to $g_K^{-1}(\widetilde{E}(r))$. It follows that the number of connected components of $V\widehat{\otimes}K$ is finite.

Furthermore, we can find finite unramified extensions $L_1, \dots, L_n$ of k' such that for any K, as above, there is an embedding of some L_i into K' which induces an isomorphism of partially ordered sets $\mathrm{str}(\mathcal{Y}'_{l,s} \otimes_{\widetilde{k}'} \widetilde{K}') \stackrel{\sim}{\to} \mathrm{str}(\mathcal{Y}'_{l,s} \otimes_{\widetilde{k}'} \widetilde{L}_i)$ and, therefore, it induces isomorphisms of $R^{K'}$-colored polysimplicial sets $D'_{K'} \stackrel{\sim}{\to} D'_{L_i}$ and

$D_{K'} \xrightarrow{\sim} D_{L_i}$. The composition of the morphism inverse to the latter with the canonical surjection $D_{K'} \to D_K$ gives rise to a surjective morphism of $R^{K'}$-colored polysimplicial sets $D_{L_i} \to D_K$. Since the polysimplicial sets D_{L_i} are finite, it follows that there are only finite many possible polysimplicial sets D_K and all of them are $R^{k'}$-colored (because $|L_i^*| = |k'^*|$). Let $D_1, \dots, D_\mu$ be these $R^{k'}$-colored polysimplicial sets. We apply the above additional fact to the G-local immersion of compact piecewise $R^{k'}$-linear spaces $\coprod_{i=1}^{\mu} |D_i| \to S$. It follows that there is a number $r_0 \le r_0' < 1$ such that for any K and any $r_0' \le r < 1$ the contraction Ψ of $\widetilde{E}(r)$ to the point x_0 lifts to a contraction of each of the connected component of $g_K^{-1}(\widetilde{E}(r))$ to a point above x_0. The composition of Φ_K with such a lifting gives rise to a contraction of each connected component of $V\widehat{\otimes}K$ to a point above x_0, i.e., (c) is true.

Finally, let L be a finite unramified extension of k' such that all of the strata of the scheme $\mathcal{Y}'_{l,s} \otimes_{\widetilde{k}'} \widetilde{L}$ are geometrically irreducible over $\widetilde{L}$. Then for any K as above with $L \subset K$ there are isomorphisms of R^K-colored polysimplicial sets $D'_K \xrightarrow{\sim} D'_L$ and $D_K \xrightarrow{\sim} D_L$. (Notice that in this case $K' = K$ since $k' \subset K$.) It follows that the canonical map $S_K \to S_L$ is a homeomorphism and, therefore, it induces a homeomorphism $g_K^{-1}(\widetilde{E}(r)) \xrightarrow{\sim} g_L^{-1}(\widetilde{E}(r))$. This implies (d). □

8 Cohomology with coefficients in the sheaf of constant functions

8.1 The sheaf of constant functions

Let k be a non-Archimedean field with a non-trivial valuation. Recall that in every strictly k-analytic space X the subset $X_0 = \{x \in X \mid [\mathcal{H}(X) : k] < \infty\}$ is dense.

For a reduced strictly k-analytic space X, we denote by $\mathfrak{c}(X)$ the set of all analytic functions $f \in \mathcal{O}(X)$ such that the image of each connected component of X under the morphism $f : X \to \mathbf{A}^1$ is a point. (Since such a point should lie in $(\mathbf{A}^1)_0$, a function $f \in \mathcal{O}(X)$ is contained in $\mathfrak{c}(X)$ if and only if the restriction of f to each connected component of X is algebraic over k.) The correspondence $U \mapsto \mathfrak{c}(U)$ is a sheaf of k-algebras in the étale topology of X (as well as in the G-topology of X), denoted by $\mathfrak{c}_X$. Of course, if k is algebraically closed, it is the constant sheaf k_X associated with k.

8.1.1 Lemma. *Assume that X is connected. Then*

(i) $\mathfrak{c}(X)$ *is a field finite over k;*

(ii) *assume that the algebra of any connected strictly affinoid subdomain of X has no zero divisors (e.x., X is normal); if the restriction of a function $f \in \mathcal{O}(X)$ to a non-empty open subset $\mathcal{U}$ is in $\mathfrak{c}(\mathcal{U})$, then $f \in \mathfrak{c}(X)$.*

Proof. (i) Let f be a nonzero element of $\mathfrak{c}(X)$. Then the image of X under the morphism $f : X \to \boldsymbol{A}^1$ is a nonzero point from $(\boldsymbol{A}^1)_0$ and, therefore, $P(f) = 0$ for a monic polynomial $P(T) = T^n + a_1 T^{n-1} + \cdots + a_n \in k[T]$ with $a_n \neq 0$. It follows that f is invertible in $\mathfrak{c}(X)$, i.e., $\mathfrak{c}(X)$ is a field. It is embedded in the field $\mathcal{H}(x)$ of every point $x \in X$. Since there is a point x with $[\mathcal{H}(x) : k] < \infty$, $\mathfrak{c}(X)$ is finite over k.

(ii) We may assume that $X = \mathcal{M}(\mathcal{A})$ is strictly k-affinoid, and we can find a nonzero polynomial $P(T)$ over k with $P(f|_{\mathcal{U}}) = 0$, i.e., for the element $g = P(f) \in \mathcal{A}$ one has $g|_{\mathcal{U}} = 0$. It follows that the image of g in the local ring $\mathcal{O}_{X,x}$ of any point $x \in \mathcal{U}$ is zero. This local ring is faithfully flat over the local ring $\mathcal{O}_{\mathcal{X},\boldsymbol{x}}$ of the affine scheme $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$ at the image $\boldsymbol{x}$ of x in $\mathcal{X}$ (see [Ber2, 2.1.4]). It follows that the image of g in the localization of $\mathcal{A}$ with respect to the prime ideal of the point $\boldsymbol{x}$ is zero and, therefore, g is a zero divisor in $\mathcal{A}$. The assumption implies that $g = 0$. □

A strictly k-analytic space X is said to be *geometrically reduced* (resp. *geometrically normal*) if the strictly $\widehat{k^{\mathrm{a}}}$-analytic space $\overline{X} = X \widehat{\otimes} \widehat{k^{\mathrm{a}}}$ is reduced (resp. normal). For example, the generic fiber of $\mathfrak{X}_\eta$ of a nondegenerate pluri-stable formal scheme $\mathfrak{X}$ over k° is geometrically normal.

8.1.2 Lemma. *Let X be a geometrically reduced strictly k-analytic space. Then*

(i) *the set of points $x \in X_0$ such that X is smooth at x and the field $\mathcal{H}(x)$ is separable over k is dense in X;*

(ii) *if x is a point from X_0 with the properties* (i)*, then there is an open neighborhood of x isomorphic to an open polydisc in an affine space over $\mathcal{H}(x)$.*

Proof. (i) We can replace X by an open neighborhood of any point from X_0 in the interior of X so that it may be assumed to be closed. Since the field $\widehat{k^{\mathrm{a}}}$ is algebraically closed and the regular locus of $\overline{X}$ is non-empty, from [Ber5, Theorem 5.2] it follows that the smooth locus of X is dense in X. Replacing X by the smooth locus, we may assume that X is smooth. We then can shrink it and assume that there is an étale morphism $\varphi : X \to \boldsymbol{A}^n$. For each point $x \in X$, $\mathcal{H}(x)$ is a finite separable extension of $\mathcal{H}(\varphi(x))$. We may therefore assume that $X = \boldsymbol{A}^n$. In this case the statement follows from the well known fact that the set of all elements of an algebraic closure k^{a} of k, which are separable over k, is dense in k^{a} (see [BGR, 3.4.1/6]).

(ii) As in (i), we can shrink X and assume that there is an étale morphism $X \to \boldsymbol{A}^n : x \mapsto y$. It induces an étale morphism $X' = X \widehat{\otimes} \mathcal{H}(x) \to \boldsymbol{A}^n_{\mathcal{H}(x)}$. The point x has an $\mathcal{H}(x)$-rational preimage x' in X' and, therefore, the étale morphism $X' \to X$ is a local isomorphism at the point x'. Thus, shrinking X, we get an étale morphism $X \to \boldsymbol{A}^n_{\mathcal{H}(x)} : x \mapsto y'$ with $\mathcal{H}(y') \stackrel{\sim}{\to} \mathcal{H}(x)$. It follows that the latter morphism is a local isomorphism at the point x. □

8.1.3 Corollary. *Let X be a geometrically reduced strictly k-analytic space. Then*

(i) *if X is connected, $\mathfrak{c}(X)$ is a finite separable extension of k;*

(ii) *the stalk* $\mathfrak{c}_{X,x}$ *of* $\mathfrak{c}_X$ *at a point* $x \in X$ *coincides with the algebraic separable closure of* k *in* $\mathcal{H}(x)$*;*

(iii) *the pullback of the étale sheaf* $\mathfrak{c}_X$ *to* $\overline{X}$ *is the constant sheaf* $k^{\mathrm{s}}_{\overline{X}}$ *associated with the separable closure* k^{s} *of* k *in* k^{a}.

Proof. (i) trivially follows from Lemma 8.1.2, and it implies that the image of $\mathfrak{c}_{X,x}$ in $\mathcal{H}(x)$ is contained in the algebraic separable closure. Let k' be a finite separable subextension of k in $\mathcal{H}(x)$, and consider the canonical étale morphism $X' = X\widehat{\otimes}k' \to X$. The canonical character $\mathcal{H}(x) \otimes k' \to \mathcal{H}(x)$ defines a point x' over x with $\mathcal{H}(x) \overset{\sim}{\to} \mathcal{H}(x')$. From [Ber2, Proposition 3.4.2] it follows that the above étale morphism is a local isomorphism at the point x' and, therefore, k' is contained in the image of $\mathfrak{c}_{X,x}$ in $\mathcal{H}(x)$. The statement (iii) is already trivial. □

8.1.4 Lemma. *The following properties of a geometrically reduced strictly* k*-analytic space* X *are equivalent:*

(a) $\mathfrak{c}(X) = k$*;*

(b) $X\widehat{\otimes}k'$ *is connected for every finite extension* k' *of* k*;*

(c) $\overline{X}$ *is connected.*

Proof. (a)$\Longrightarrow$(b) Assume that there is k' such that $X\widehat{\otimes}k'$ is not connected. If k'' is the maximal subextension of k' separable over k, then the canonical map $X\widehat{\otimes}k' \to X\widehat{\otimes}k''$ is a homeomorphism and, therefore, we may assume that $k' = k''$. We may also assume that k' is a Galois extension of k. Let X' be a connected component of $X\widehat{\otimes}k'$. The morphism $X' \to X$ is a finite étale Galois covering of X of degree less than $[k' : k]$. If G is the Galois group of this covering, then $\mathfrak{c}(X) = \mathfrak{c}(X')^G \supset k'^G$. The latter field is bigger than k, and this contradicts the assumption (a).

(b)$\Longrightarrow$(c) Assume that $\overline{X}$ is a disjoint union of non-empty open subsets $\mathcal{U}_1$ and $\mathcal{U}_2$. Since for every compact analytic subdomain $Y \subset X$ the canonical map $\overline{Y} \to \varprojlim Y\widehat{\otimes}k'$ is a homeomorphism, where the inverse limit is taken over finite separable extensions k' of k in k^{a}, it follows that the images of $\mathcal{U}_1$ and $\mathcal{U}_2$ in every $X\widehat{\otimes}k'$ are open and closed and, therefore, the maps $\mathcal{U}_i \to X\widehat{\otimes}k'$ are surjective. But we can find k' such that $X\widehat{\otimes}k'$ has a k'-rational point. Since the preimage of the latter in $\overline{X}$ is a one point subset, we get a contradiction.

(c)$\Longrightarrow$(a) From (c) it follows that X is connected and, in particular, $\mathfrak{c}(X)$ is a finite separable extension of k. One has $X\widehat{\otimes}\widehat{k^{\mathrm{a}}} \overset{\sim}{\to} X\widehat{\otimes}_{\mathfrak{c}(X)}(\mathfrak{c}(X) \otimes_k \widehat{k^{\mathrm{a}}})$. The latter tensor product is a direct product of $[\mathfrak{c}(X) : k]$ copies of $\widehat{k^{\mathrm{a}}}$, and so the connectedness of $X\widehat{\otimes}\widehat{k^{\mathrm{a}}}$ implies that $\mathfrak{c}(X) = k$. □

8.1.5 Corollary. *Let* Y *be a strictly analytic domain in a geometrically reduced strictly* k*-analytic space* X*. Then the sheaf* $\mathfrak{c}_Y$ *is canonically isomorphic to the pullback of the sheaf* $\mathfrak{c}_X$ *on* Y*.*

Proof. It suffices to show that, given a compact strictly analytic domain Y in X, there exists a compact neighborhood U of Y with $\mathfrak{c}(U) \xrightarrow{\sim} \mathfrak{c}(Y)$. For this we may assume that Y and X are connected. Furthermore, we may shrink X so that $\mathfrak{c}(X) \xrightarrow{\sim} \mathfrak{c}(U)$ for any connected compact neighborhood U of Y in X. Finally, we may assume that $\mathfrak{c}(X) = k$ (see Remark 8.1.7). We claim that in this case $\mathfrak{c}(Y) = k$. Indeed, if this is not true, we can find a finite separable extension k' of k such that $Y\widehat{\otimes}k'$ is not connected. Let $\{Y_i\}_{1\le i\le n}$ be the connected components of $Y\widehat{\otimes}k'$, and let $\{U_i\}_{1\le i\le n}$ be their pairwise disjoint compact neighborhoods. Then there exists a connected compact neighborhood U of Y whose preimage in $X\widehat{\otimes}k'$ is contained in $\coprod_{i=1}^n \mathcal{U}_i$. It follows that $U\widehat{\otimes}k'$ is not connected. Since $\mathfrak{c}(U) = k$, this contradicts Lemma 8.1.4. □

8.1.6 Lemma. *Assume that the characteristic of k is zero, and let X be a reduced strictly k-analytic space that satisfies the assumption of Lemma* 8.1.1 (ii). *Then* $\mathfrak{c}_X = \mathrm{Ker}(\mathcal{O}_X \xrightarrow{d} \Omega^1_X)$.

Proof. We may assume that X is connected. Let f be a function from $\mathcal{O}(X)$ with $df = 0$. Any strictly affinoid subdomain $V \subset X$ is regular at a dense open subset $\mathcal{V} \subset V$ and, therefore, V is smooth at each point from $\mathcal{V} \cap X_0$ (see [Ber5, 5.2]). By Lemma 8.1.2, there exists a non-empty open subset $\mathcal{W} \subset \mathcal{V}$ isomorphic to an open polydisc in an affine space over k', a finite extension of k. It follows that $f|_{\mathcal{W}} \in \mathfrak{c}(\mathcal{W})$, and Lemma 8.1.1 (ii) implies that $f \in \mathfrak{c}(X)$. □

8.1.7 Remark. Let $X = \mathcal{M}(\mathcal{A})$ be a strictly k-affinoid space, and V a strictly k-affinoid subdomain of X. Assume that $\mathcal{A}$ contains a finite extension k' of k. Then X and V can be considered as strictly k'-affinoid spaces, and it is easy to see (in comparison to Proposition 7.1.2) that V is a strictly k'-affinoid subdomain of X.

8.2 Local cohomological triviality of the sheaf $\mathfrak{c}_X$

8.2.1 Theorem. *Assume that the characteristic of k is zero, and let X be a k-analytic space locally embeddable in a smooth space. Then each point of X has a fundamental system of open neighborhoods V with $H^n(V, \mathfrak{c}_X) = 0$ for all $n \ge 1$.*

Since the characteristic of k is zero, the stalks of $\mathfrak{c}_X$ are uniquely divisible abelian groups, and since the Galois cohomology of such a group is trivial, [Ber2, Proposition 4.2.4] implies that, for any reduced strictly k-analytic space X, the étale cohomology groups $H^n(X, \mathfrak{c}_X)$ of X coincide with the cohomology groups $H^n(|X|, \mathfrak{c}_X)$ of the underlying topological space $|X|$.

Proof. By Theorem 7.1.1, each point of X has a fundamental system of open neighborhoods V with the properties (a)–(d). *We claim that, for such V, one has $H^n(V, \mathfrak{c}_X) = 0$, $n \ge 1$.*

Let $X_1 \subset X_2 \subset \cdots$ be the increasing sequence of compact strictly analytic subdomains of V from the property (b). By Corollary 8.1.5, the pullback of the étale sheaf $\mathfrak{c}_X$ to X_m coincides with $\mathfrak{c}_{X_m}$. From [Ber2, Lemma 6.3.12] it follows that to prove the claim it suffices to show that $H^n(X_m, \mathfrak{c}_{X_m}) = 0$ for all $n \geq 1$.

By Corollary 8.1.3(ii), the pullback of the étale sheaf $\mathfrak{c}_{X_m}$ to $\overline{X}_m$ is the constant sheaf $k^{\mathrm{s}}_{\overline{X}_m}$. Since X_m is compact, there is a Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(\overline{X}_m, k^{\mathrm{s}})) \Longrightarrow H^{p+q}(X_m, \mathfrak{c}_{X_m}) ,$$

where G is the Galois group of k^{s} over k. The étale cohomology groups $H^q(\overline{X}_m, k^{\mathrm{s}})$ coincide with $H^q(|\overline{X}_m|, k^{\mathrm{s}})$. Since all of the connected components of $\overline{X}_m$ are contractible, it follows that $H^q(\overline{X}_m, k^{\mathrm{s}}) = 0$ and, therefore, $E_2^{p,q} = 0$ for all $q \geq 1$. Furthermore, since $H^p(G, k^{\mathrm{s}}) = 0$ for all $p \geq 1$, the spectral sequence implies that $H^n(X_m, \mathfrak{c}_{X_m}) = 0$ for all $n \geq 1$. □

8.3 Cohomology of certain analytic spaces

In this subsection, k is assumed to be of characteristic zero. Let $\mathfrak{X}$ be a nondegenerate pluri-stable formal scheme over k°, and let $\mathcal{Y}$ be a quasi-compact locally closed strata subset of the closed fiber $\mathfrak{X}_s$ (i.e., $\mathcal{Y}$ is a locally closed subset which is a finite union of strata of $\mathfrak{X}_s$). The set $S(\mathfrak{X}_{/\mathcal{Y}}) = S(\mathfrak{X}) \cap \pi^{-1}(\mathcal{Y})$ is a piecewise $R^k_{\mathbb{Z}_+}$-linear subspace of $S(\mathfrak{X})$. It is a union of strata and contained in each dense Zariski open subset of $\pi^{-1}(\mathcal{Y})$. We also set $\overline{\mathfrak{X}} = \mathfrak{X}\widehat{\otimes}_{k^\circ}(\widehat{k}^{\mathrm{a}})^\circ$. It is a nondegenerate pluri-stable formal scheme over $(\widehat{k}^{\mathrm{a}})^\circ$ with the closed fiber $\overline{\mathfrak{X}}_s = \mathfrak{X}_s \otimes_{\widetilde{k}} \widetilde{k}^{\mathrm{a}}$, $\overline{\mathcal{Y}} = \mathcal{Y} \otimes_{\widetilde{k}} \widetilde{k}^{\mathrm{a}}$ is a subscheme of the latter, and so a piecewise $R^{k^{\mathrm{a}}}_{\mathbb{Z}_+}$-linear subspace $S(\overline{\mathfrak{X}}_{/\overline{\mathcal{Y}}})$ of $S(\overline{\mathfrak{X}})$ is defined. Let G be the Galois group of k^{s} over k.

8.3.1 Theorem. *Let $X = \pi^{-1}(\mathcal{Y})\backslash Z$, where Z is a nowhere dense Zariski closed subset of $\mathfrak{X}_\eta$. Then the canonical maps $S(\overline{\mathfrak{X}}_{/\overline{\mathcal{Y}}}) \hookrightarrow \overline{X} \to X$ induce isomorphisms of finitely dimensional vector spaces over k*

$$H^n(X, \mathfrak{c}_X) \overset{\sim}{\to} H^n(\overline{X}, k^{\mathrm{s}})^G \overset{\sim}{\to} H^n(S(\overline{\mathfrak{X}}_{/\overline{\mathcal{Y}}}), k^{\mathrm{s}})^G, \quad n \geq 0 .$$

Since the characteristic of k is zero, the first two groups can be considered in the étale as well as in the usual topology. The third group $H^n(S(\overline{\mathfrak{X}}_{/\overline{\mathcal{Y}}}), k^{\mathrm{s}})$ is of course considered in the usual topology, it coincides with the singular cohomology group and is evidently finitely dimensional over k^{s}. From [Ber7, Theorem 8.1] it follows that $S(\overline{\mathfrak{X}}_{/\overline{\mathcal{Y}}})$ is a strong deformation retraction of $\overline{X}$, and this implies the second isomorphism. Furthermore, if $\mathcal{Y}$ is open in $\mathfrak{X}_s$ and X coincides with $\pi^{-1}(\mathcal{Y})$, then X is compact and, therefore, the first isomorphism follows from the Hochschild–Serre spectral sequence. The non-triviality of the first isomorphism is in the fact that such a spectral sequence does not hold if X is not compact.

Proof. By [Ber7, Theorem 8.1], there is a proper strong deformation retraction Φ of $\mathfrak{X}_\eta$ to the skeleton $S(\mathfrak{X})$, and it lifts to a strong deformation retraction $\overline{\Phi}$ of $\overline{\mathfrak{X}}$ to $S(\overline{\mathfrak{X}})$. Let τ and $\overline{\tau}$ denote the corresponding retraction maps $\mathfrak{X}_\eta \to S(\mathfrak{X})$ and $\overline{\mathfrak{X}}_\eta \to S(\overline{\mathfrak{X}})$. We set $S = S(\mathfrak{X}_{/\mathcal{Y}})$ and $\overline{S} = S(\overline{\mathfrak{X}}_{\overline{\mathcal{Y}}})$. From *loc. cit.* it follows that $\pi^{-1}(\mathcal{Y}) = \tau^{-1}(S)$, $\pi^{-1}(\overline{\mathcal{Y}}) = \overline{\tau}^{-1}(\overline{S})$, and that X and $\overline{X}$ contain S and $\overline{S}$ and are preserved under Φ and $\overline{\Phi}$, respectively.

8.3.2 Lemma. *There is an increasing sequence $X_1 \subset X_2 \subset \cdots$ of compact strictly analytic subdomains of $\pi^{-1}(\mathcal{Y})$ with the following properties:*

(a) $X = \bigcup_{n=1}^\infty X_n$;

(b) *all X_n are preserved under Φ;*

(c) *all $\tau(X_n)$ are compact piecewise $R^k_{\mathbb{Z}_+}$-linear subspaces of S.*

Proof. First of all, shrinking $\mathfrak{X}$ we may assume that it is quasi-compact and $\mathcal{Y}$ is closed in $\mathfrak{X}_s$. *We claim that it suffices to consider the case when $\mathfrak{X}$ is affine.* Indeed, assume the lemma is true in this case, and let $\{\mathfrak{X}_i\}_{i\in I}$ be a finite covering of $\mathfrak{X}$ by open affine subschemes. By the assumption, we can find, for every $i \in I$, an increasing sequence $X_1^i \subset X_2^i \subset \cdots$ of compact strictly analytic domains of $\pi^{-1}(\mathcal{Y}_i)$ with the properties (a)–(c) for $X \cap \pi^{-1}(\mathcal{Y}_i)$, where $\mathcal{Y}_i = \mathcal{Y} \cap \mathfrak{X}_{i,s}$. Then the properties (a)–(b) hold for the compact strictly analytic domains $X_n = \bigcup_{i\in I} X_n^i$. Thus, let $\mathfrak{X} = \mathrm{Spf}(A)$.

Let $f_1, \dots, f_m$ be nonzero elements of A with $Z = \{x \in \mathfrak{X}_\eta \mid f_i(x) = 0$ for all $1 \le i \le m\}$. Let ε be a positive integer which is smaller than all of the minima of the functions $x \mapsto |f_i(x)|$ on the skeleton $S(\mathfrak{X})$, and let $\varepsilon \ge r_1 > r_2 > \cdots$ be a decreasing sequence of numbers from $|k^*|$ tending to zero. By [Ber7, Theorem 8.1(iii)], for every $1 \le i \le m$ and $n \ge 1$, the strictly affinoid domain $Y_n^i = \{x \in \mathfrak{X}_\eta \mid |f_i(x)| \ge r_n\}$ is preserved under Φ. Then the same is true for the compact strictly analytic domain $Y_n = \bigcup_{i=1}^m Y_n^i$. Thus, we have an increasing sequence $Y_1 \subset Y_2 \subset \cdots$ of compact strictly analytic domains in $\mathfrak{X}_\eta$ which contain $S(\mathfrak{X})$, are preserved under Φ and such that $\pi^{-1}(\mathcal{Y})\backslash Z = \bigcup_{n=1}^\infty Y_n$.

Let $E_1 \subset E_2 \subset \cdots$ be an increasing sequence of compact piecewise $R^k_{\mathbb{Z}_+}$-linear subspaces of S with $S = \bigcup_{n=1}^\infty E_n$. By Theorem 6.4.1, each $\tau^{-1}(E_n)$ is a compact strictly analytic domain in $\pi^{-1}(\mathcal{Y})$. Then $X_n = \tau^{-1}(E_n) \cap Y_n$ is a compact strictly analytic domain in $X = \pi^{-1}(\mathcal{Y})\backslash Z$, it is preserved under Φ and its image under τ is E_n, i.e., the sequence $X_1 \subset X_2 \subset \cdots$ possesses the properties (a)–(c). □

Lemma 8.3.2 implies that the compact strictly analytic domains $\overline{X}_n$ of $\overline{X}$ are preserved under $\overline{\Phi}$ and $\overline{\tau}(\overline{X}_n)$ are piecewise $R^{k^{\mathrm{a}}}_{\mathbb{Z}_+}$-linear subspaces of $\overline{S}$. In particular, $H^q(\overline{X}_n, k^{\mathrm{s}})$ are of finite dimension over k^s. Since $\overline{X} = \bigcup_{n=1}^\infty \overline{X}_n$, there is an isomorphism of finitely dimensional vector spaces over k^{s}

$$H^q(\overline{X}, k^{\mathrm{s}}) \xrightarrow{\sim} \varprojlim_n H^q(\overline{X}_n, k^{\mathrm{s}}),\ q \ge 0 .$$

Let $\overline{h}_n^q$ denote the dimension over $k^{\rm s}$ of the image of $H^q(X_m, k^{\rm s})$ in $H^q(\overline{X}_n, k^{\rm s})$ for sufficiently large m. One has $\overline{h}_1^q \le \overline{h}_2^q \le \dots$ and $\overline{h}_n^q = \overline{h}^q$ for sufficiently large n, where $\overline{h}^q$ is the dimension of $H^q(\overline{X}, k^{\rm s})$ over $k^{\rm s}$. Recall that, by the Hochschield–Serre spectral sequence, one has $H^q(X_n, \mathfrak{c}_{X_n}) \stackrel{\sim}{\to} H^q(\overline{X}_n, k^{\rm s})^G$.

Let K be a finite unramified Galois extension of k such that all of the strata of the closed fiber of $\mathfrak{X}\widehat{\otimes}_{k^\circ} K^\circ$ are geometrically irreducible. Then the action of G on the skeleton $S(\overline{\mathfrak{X}})$ goes through an action of its finite quotient $\mathrm{Gal}(K/k)$. It follows that $H^q(\overline{X}_n, k^{\rm s})^{\mathrm{Gal}(k^{\rm s}/K)} = H^q(\overline{X}_n, K)$, and we get

$$H^q(X_n, \mathfrak{c}_{X_n}) \stackrel{\sim}{\to} H^q(\overline{X}_n, k^{\rm s})^G = H^q(\overline{X}_n, K)^{\mathrm{Gal}(K/k)} .$$

The latter space has finite dimension over k and, in particular, there is an isomorphism

$$H^q(X, \mathfrak{c}_X) \stackrel{\sim}{\to} \lim_{\substack{\longleftarrow \\ n}} H^q(X_n, \mathfrak{c}_{X_n}),\ q \ge 0 .$$

It follows also that the image of $H^q(X_m, \mathfrak{c}_{X_m})$ in $H^q(X_n, \mathfrak{c}_{X_n})$ for sufficiently large m is of dimension at most $[K : k]\cdot\overline{h}_n^q$ over k. Hence, the dimension of $H^q(X, \mathfrak{c}_X)$ over k is at most $[K : k]\cdot\overline{h}^q$, and there is a canonical isomorphism $H^q(X, \mathfrak{c}_X) \stackrel{\sim}{\to} H^q(\overline{X}, k^{\rm s})^G$. □

8.3.3 Corollary. *Let $\mathfrak{X}$ be a nondegenerate strictly pluri-stable formal scheme over k°, $\mathcal{Y}$ an irreducible component of $\mathfrak{X}_s$, and $X = \pi^{-1}(\mathcal{Y})\backslash Z$, where Z is a Zariski closed subset of $\mathfrak{X}_\eta$. Then $H^n(X, \mathfrak{c}_X) = 0$ for all $n \ge 1$.*

Proof. By Theorem 8.3.1, we may assume that k is algebraically closed, and it suffices to show that $S(\mathfrak{X}_{/\mathcal{Y}})$ is contractible. (Of course, at this point the assumption on the characteristic of k is already not important.) To prove the contractibility, it is more convenient to use [Ber7, Theorem 8.2] instead of Theorem 5.1.1 of this paper.

Let $\underline{\mathfrak{X}}$ be a strictly poly-stable fibration over k° with $\mathfrak{X}_l = \mathfrak{X}$. Recall that [Ber7, Theorem 8.2] identifies the skeleton $S(\mathfrak{X}) = S(\underline{\mathfrak{X}})$ with the geometric realization $|C|$ of a polysimplicial set $C = \boldsymbol{C}(\underline{\mathfrak{X}})$ associated with $\underline{\mathfrak{X}}$. The polysimplicial set C here is an object of the category $\Lambda^\circ\mathcal{E}ns$, where Λ is a category with the same family of objects as Λ but with larger sets of morphisms, and the geometric realization functor extends the functor that takes $[\boldsymbol{n}] \in \mathrm{Ob}(\Lambda)$ with $\boldsymbol{n} = (n_0, \dots, n_p)$ to

$$\Sigma^{\boldsymbol{n}} = \{(u_{ij})_{0\le i\le p, 0\le j\le n_i} \in [0,1]^{[\boldsymbol{n}]} \mid u_{i0} + \dots + u_{in_i} = 1,\ 0 \le i \le p\} .$$

Since $\underline{\mathfrak{X}}$ is strictly poly-stable, the polysimplicial set C is interiorly free, i.e., the stabilizer of any nondegenerate $\boldsymbol{n}$-polysimplex of C in $\mathrm{Aut}([\boldsymbol{n}])$ is trivial. It follows that the corresponding map $\Sigma^{\boldsymbol{n}} \to |C|$ is injective on the interior $\mathring{\Sigma}^{\boldsymbol{n}}$ of $\Sigma^{\boldsymbol{n}}$. Let y be the vertex of $|C|$ that corresponds to the generic point of $\mathcal{Y}$. Then $S(\mathfrak{X}_{/\mathcal{Y}})$ is identified with the union S of all cells of $|C|$ whose closure contains the vertex y. We define a map $\Phi : S \times [0,1] \to S$ as follows $\Phi(x,t) = ty + (1-t)x$. (Notice that the latter makes sense in S.) The map Φ is evidently continuous and defines a contraction of S to the point y. □

8.3.4 Corollary. *Let X be a reduced strictly k-analytic space isomorphic to $W \backslash \mathcal{V}^{\mathrm{an}}$, where W is a compact strictly analytic domain in the analytification $\mathcal{X}^{\mathrm{an}}$ of a separated scheme $\mathcal{X}$ of finite type over k and $\mathcal{V}$ is a Zariski closed subset of $\mathcal{X}$. Then there are canonical isomorphisms of finitely dimensional vector spaces over k*

$$H^n(X, \mathfrak{c}_X) \stackrel{\sim}{\to} H^n(\overline{X}, k^{\mathrm{s}})^G, \ n \geq 0 .$$

Proof. By [Ber7, Theorem 10.1], the abelian group $H^n(\overline{X}, \mathbb{Z})$ is of finite rank and G acts on it through a finite quotient. Since $H^n(\overline{X}, k^{\mathrm{s}}) = H^n(\overline{X}, \mathbb{Z}) \otimes_{\mathbb{Z}} k^{\mathrm{s}}$, it follows that the action of G on $H^n(\overline{X}, k^{\mathrm{s}})$ is discrete. It follows that, if there exists a proper hypercovering $X_\bullet \to X$ such that the statement is true for all X_n's, then it is also true for X. Using this remark and de Jong's results [deJ] (as in the proof of *loc. cit.*), the situation is reduced to the case when X is of the form considered in Theorem 8.3.1. $\square$

8.3.5 Remark. Assume that k is a finite extension of $\mathbb{Q}_p$, and let $\mathcal{X}$ be a separated reduced scheme of finite type over k. By [Ber8, Theorem 1.1(a$''$)], there are canonical isomorphism $H^n(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbb{Q}_p) \stackrel{\sim}{\to} H^n(\overline{\mathcal{X}}, \mathbb{Q}_p)^{\mathrm{sm}}$, where $H^n(|\overline{\mathcal{X}}^{\mathrm{an}}|, \mathbb{Q}_p)$ are the cohomology groups of the underlying topological space of $\overline{\mathcal{X}}^{\mathrm{an}} = (\mathcal{X} \otimes \widehat{k}^{\mathrm{a}})^{\mathrm{an}}$, $H^n(\overline{\mathcal{X}}, \mathbb{Q}_p)$ are the p-adic étale cohomology groups of $\overline{\mathcal{X}} = \mathcal{X} \otimes k^{\mathrm{a}}$ and, for a p-adic representation V, V^{sm} denotes the subspace of V consisting of the elements with open stabilizer in G. Together with Corollary 8.3.4, this implies that there are canonical isomorphisms

$$H^n(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) \stackrel{\sim}{\to} (H^n(\overline{\mathcal{X}}, \mathbb{Q}_p)^{\mathrm{sm}} \otimes_{\mathbb{Q}_p} k^{\mathrm{s}})^G = (H^n(\overline{\mathcal{X}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} k^{\mathrm{s}})^G .$$

It follows that $\dim_k H^n(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) = \dim_{\mathbb{Q}_p} H^n(\overline{\mathcal{X}}, \mathbb{Q}_p)^{\mathrm{sm}}$.

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