COMPLEX ANALYTIC VANISHING CYCLES FOR FORMAL SCHEMES

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Contents

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0. Introduction

0.1. **Previous work on vanishing cycles for formal schemes.** Let k be a non-Archimedean field with nontrivial discrete valuation, k° its ring of integers, $k^{\circ\circ}$ the maximal ideal of k°, and $\widetilde{k} = k^\circ/k^{\circ\circ}$ the residue field of k. A formal scheme $\mathfrak{X}$ over k° is said to be special if it is a locally finite union of open affine subschemes of the form $\mathrm{Spf}(A)$ with A isomorphic to a quotient of $k^\circ\{T_1, \dots, T_m\}[[S_1, \dots, S_n]]$. If all of these open affine subschemes can be found with $n = 0$, such $\mathfrak{X}$ is said to be of locally finite type (or of finite type if in addition $\mathfrak{X}$ is quasicompact). Each special formal scheme $\mathfrak{X}$ over k° has a generic fiber $\mathfrak{X}_\eta$, which is a paracompact strictly k-analytic space, and a closed fiber $\mathfrak{X}_s$, which is a scheme of locally finite type over $\widetilde{k}$. The class of formal schemes of locally finite type is preserved under formal completion $\mathfrak{X}_{/\mathcal{Y}}$ of $\mathfrak{X}$ along an open subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, and the class of special formal schemes is preserved under formal completion of $\mathfrak{X}$ along an arbitrary subscheme of $\mathfrak{X}_s$. For example, if $\mathcal{Y}$ is a scheme of finite type over k°, then the formal completion $\widehat{\mathcal{Y}}$ (resp. $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$) of $\mathcal{Y}$ along its closed fiber $\mathcal{Y}_s = \mathcal{Y} \otimes_{k^\circ} \widetilde{k}$ (resp. along an arbitrary subscheme $\mathcal{Z} \subset \mathcal{Y}_s$) is a formal scheme of finite type (resp. a quasicompact special formal scheme) over k°. Till the end of the introduction, we assume for simplicity that the residue field $\widetilde{k}$ is algebraically closed and all of the special formal schemes considered are quasicompact.

In [Ber96b] and [Ber15, §3.1], we constructed, for every special formal scheme $\mathfrak{X}$ over k°, a vanishing cycles functor $\Psi_\eta : \mathfrak{X}_\eta^\sim \to \mathfrak{X}_s(G)^\sim$ from the category of étale sheaves on $\mathfrak{X}_\eta$ to the category of étale sheaves on $\mathfrak{X}_s$ provided with a continuous discrete action of $G = \mathrm{Gal}(k^{\mathrm{a}}/k)$, where k^{a} is a fixed algebraic closure of k. In particular, for any continuous discrete G-module Λ there is an associated complex $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ of sheaves on $\mathfrak{X}_s$, where $\Lambda_{\mathfrak{X}_\eta}$ is the locally constant sheaf on $\mathfrak{X}_\eta$ induced by Λ. The construction is functorial and, therefore, any morphism of special formal schemes $\varphi : \mathfrak{Y} \to \mathfrak{X}$ gives rise to a morphism

$$\theta_\eta(\varphi, \Lambda) : \varphi_s^*(R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})) \to R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta}) \ .$$

The corresponding homomorphism between q-th cohomology sheaves is denoted by $\theta_\eta^q(\varphi, \Lambda)$. Among other things, we proved the following results. Suppose Λ is finite of order not divisible by $\mathrm{char}(\widetilde{k})$. Then

(i) the sheaves $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ are constructible;

(ii) one has $H^q(\mathfrak{X}_{\overline{\eta}}, \Lambda) = R^q\Gamma(\mathfrak{X}_s, R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta}))$, where $\mathfrak{X}_{\overline{\eta}} = \mathfrak{X}_\eta \widehat{\otimes}_k \widehat{k^{\mathrm{a}}}$;

(iii) given $\mathfrak{X}$, $\mathfrak{Y}$ and Λ, as above, there exists an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ such that, for any pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ congruent modulo $\mathcal{J}$ and any q, one has $\theta_\eta^q(\varphi, \Lambda) = \theta_\eta^q(\psi, \Lambda)$;

(iv) given a scheme $\mathcal{Y}$ of finite type over k° and a subscheme $\mathcal{Z} \subset \mathcal{Y}_s$, there is a canonical isomorphism $R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta})\big|_{\mathcal{Z}} \widetilde{\to} R\Psi_\eta(\Lambda_{(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta})$, where $R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta})$ is the vanishing cycles complex of the scheme $\mathcal{Y}$.

0.2. **The purpose of the paper.** Although the above functor Ψ_η gives rise to vanishing cycles complexes for arbitrary discrete G-modules Λ, e.g., $\mathbf{Z}$, those complexes do not possess good properties, and the reason is that such properties are not satisfied by the integral étale cohomology groups of algebraic varieties and non-Archimedean analytic spaces.

On the other hand, if $\mathcal{Y}$ is a scheme of finite type over the ring $\mathcal{O}_{\mathbf{C},0}$ of functions analytic in a neighborhood of zero in the complex plane, one can define vanishing cycles complexes $R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta^h})$ on the analytification $\mathcal{Y}_s^h$ of $\mathcal{Y}_s = \mathcal{Y} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathbf{C}$ for arbitrary $\pi_1(\mathbf{C}^*)$-modules Λ (i.e., abelian groups provided with an action of the fundamental group of the punctured complex plane). By [SGA7, Exp. XIV], if Λ is finite, there is a canonical isomorphism $R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta})^h \widetilde{\to} R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta^h})$, and the above property (iv) implies that, for any subscheme $\mathcal{Z} \subset \mathcal{Y}_s$, there is a canonical isomorphism

$$R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta^h})\big|_{\mathcal{Z}^h} \widetilde{\to} R\Psi_\eta(\Lambda_{(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta})^h \ .$$

A natural question (mentioned, for example, by Kontsevich and Soibelman in [KS11, 7.1, 7.4]) is as follows. Can one extend the construction of the vanishing cycles complexes for special formal schemes over the completion $\widehat{\mathcal{O}}_{\mathbf{C},0}$ of $\mathcal{O}_{\mathbf{C},0}$ and for arbitrary $\pi_1(\mathbf{C}^*)$-modules Λ so that, in the case of the formal scheme $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$, one gets the complex $R\Psi_\eta(\Lambda_{\mathcal{Y}_\eta^h})\big|_{\mathcal{Z}^h}$?

The purpose of this paper is to give a positive answer to this question and to derive a construction of integral "étale" cohomology groups for a class of non-Archimedean analytic spaces over the fraction field of $\widehat{\mathcal{O}}_{\mathbf{C},0}$, which includes the analytifications of proper schemes over that field.

0.3. **Complex analytic vanishing cycles for formal schemes.** Let now K be a non-Archimedean field of the same type as k and, in addition, assume its ring of integers K° contains the field of complex numbers $\mathbf{C}$ and $\mathbf{C}\widetilde{\to}\widetilde{K}$. There is a canonical isomorphism $G\widetilde{\to}\varprojlim \mu_n$ of the Galois group $G = \mathrm{Gal}(K^{\mathrm{a}}/K)$ of any algebraic closure K^{a} of K. The element $\sigma = (e^{\frac{2\pi i}{n}})_{n\geq 1}$ of the projective limit generates a subgroup Π isomorphic to $\mathbf{Z}$ and defines an isomorphism $G\widetilde{\to}\widehat{\mathbf{Z}}$.

Our purpose is to construct, for every special formal scheme over K°, a functor similar to that from §0.1 but from the category of arbitrary Π-modules. It is clear that such an object should depend on a certain choice related to the field K. For example, the construction from §0.1 depends on the choice of an algebraic closure of K, and the object obtained is in fact a functor from the étale fundamental groupoid of the field K. The role of the latter in our construction is played by the following groupoid Π_K which is naturally equivalent to a (non-full) subgroupoid of the étale fundamental groupoid of K.

The family of objects of Π_K is the set of generators of the maximal ideal $K^{\circ\circ}$ of K°. If ϖ and ϖ' are two generators, then $\mathrm{Hom}_{\Pi_K}(\varpi,\varpi')$ is the set of elements $\beta \in K^\circ$ with $\exp(\beta) = \frac{\varpi}{\varpi'}$. Composition of morphisms corresponds to addition in K°. For example, $\mathrm{Hom}_{\Pi_K}(\varpi,\varpi)$ is the subgroup $\mathbf{Z}(1) = 2\pi i\mathbf{Z} \subset i\mathbf{R}$, whose generator $2\pi i$ will be denoted by $\sigma^{(\varpi)}$, and it is canonically isomorphic to the group Π under the homomorphism that takes $\sigma^{(\varpi)}$ to the element σ. We are now going to construct a faithful functor from Π_K to the étale fundamental groupoid of K.

First of all, consider the following algebraic closure $\mathcal{K}^{\mathrm{a}}$ of the fraction field $\mathcal{K}$ of $\mathcal{O}_{\mathbf{C},0}$. (The complex plane $\mathbf{C}$ is provided with a fixed coordinate function z.) Let $\exp : \mathbf{C} \to \mathbf{C}^*$ be the exponential map $b \mapsto e^b$. Then $\mathcal{K}^{\mathrm{a}}$ is the field of functions meromorphic in some half plane $\{b \in \mathbf{C}\big|\mathrm{Re}(b) < r\}$ and algebraic over $\mathcal{K}$. (It is algebraically closed.) We denote by $K^{(\varpi)}$ the field $\mathcal{K}^{\mathrm{a}} \otimes_{\mathcal{K}} K$, where the tensor product is taken with respect to the embedding of fields $\mathcal{K} \hookrightarrow K : z \mapsto \varpi$. (By the way, the latter embedding provides $\mathcal{K}$ with a valuation which does not depend on the choice of ϖ.) Let G_K be the groupoid whose objects are the fields $K^{(\varpi)}$ for $\varpi \in \Pi_K$ and in which the set of morphisms $\mathrm{Hom}_{G_K}(K^{(\varpi)}, K^{(\varpi')})$ is the profinite set of isomorphisms of fields $K^{(\varpi)}\widetilde{\to}K^{(\varpi')}$ over K. For example, $\mathrm{Hom}_{G_K}(K^{(\varpi)}, K^{(\varpi)})$ is canonically isomorphic to the Galois group G. The canonical functor from G_K to the étale fundamental groupoid of K is an equivalence of categories, and here is a construction of a faithful functor $\Pi_K \to G_K$.

This functor takes each ϖ to the field $K^{(\varpi)}$. In order to define an isomorphism $\nu_\varphi : K^{(\varpi)}\widetilde{\to}K^{(\varpi)}$ that corresponds to a morphism $\varphi : \varpi \to \varpi'$ in Π_K as above, we notice that the field $\mathcal{K}^{\mathrm{a}}$ is generated over $\mathcal{K}$ by the functions $b \mapsto e^{\frac{b}{n}}$ for $n \geq 1$. If ϖ_n and ϖ'_n are the images of those functions in $K^{(\varpi)}$ and $K^{(\varpi')}$, respectively, then the isomorphism ν_φ is defined by $\nu_\varphi(\varpi_n) = \exp(\frac{\beta}{n})\varpi'_n$. For example, $\nu_{\sigma^{(\varpi)}}(\varpi_n) = e^{\frac{2\pi i}{n}}\varpi_n$.

Applying the above construction to the field $\widehat{\mathcal{K}}$, we get a groupoid $\Pi_{\widehat{\mathcal{K}}}$ and a faithful functor $\Pi_{\widehat{\mathcal{K}}} \to G_{\widehat{\mathcal{K}}}$. Each $\varpi \in \Pi_K$, gives rise to isomorphisms of groupoids $\Pi_{\widehat{\mathcal{K}}}\widetilde{\to}\Pi_K$ and $G_{\widehat{\mathcal{K}}}\widetilde{\to}G_K$ (that take $z \in \Pi_{\widehat{\mathcal{K}}}$ to ϖ). We now notice that, for each element $\beta \in \mathcal{K}^\circ$, one has $\exp(\beta) \in \mathcal{K}^\circ$. This means that one can define a full subcategory $\Pi_{\mathcal{K}} \subset \Pi_{\widehat{\mathcal{K}}}$ whose objects are generators of the maximal ideal $\mathcal{K}^{\circ\circ}$ of $\mathcal{K}^\circ$. The category $\Pi_{\mathcal{K}}$ is a subgroupoid of $G_{\mathcal{K}}$.

If $\mathcal{P}$ is a groupoid, a $\mathcal{P}$-space is a contravariant functor $P \mapsto X^{(P)}$ from $\mathcal{P}$ to the category of topological (or analytic) spaces. A $\mathcal{P}$-sheaf F on a $\mathcal{P}$-space X is a family of sheaves $F^{(P)}$ on $X^{(P)}$ satisfying natural properties of compatibility with respect to morphisms in $\mathcal{P}$ (see §3.3). In §3.4 we show that the category of $\mathcal{P}$-sheaves on X is a topos. The derived category of abelian $\mathcal{P}$-sheaves on X is denoted by $D(X(\mathcal{P}))$. If X is a trivial $\mathcal{P}$-space, i.e., the corresponding functor takes all objects to the same space X and all morphisms to the identity map, a $\mathcal{P}$-sheaf is just a covariant functor from $\mathcal{P}$ to the category of sheaves on X. If it is a one point space, the abelian $\mathcal{P}$-sheaves on it are called $\mathcal{P}$-modules and their category is denoted by $\mathcal{P}$-Mod. The map from X to a one point space defines a functor $\Lambda \mapsto \underline{\Lambda}_X$ from the category of $\mathcal{P}$-modules to that of abelian $\mathcal{P}$-sheaves on X.

For example, for a K-analytic space X the functor $\varpi \mapsto X^{(\varpi)} = X\widehat{\otimes}_K\widehat{K^{(\varpi)}}$ is an analytic Π_K-space (as well as a G_K-space), denoted by $\overline{X}$, and various cohomology groups of $\overline{X}$ with coefficients in a Π_K-sheaf are Π_K-modules.

The purpose of this paper is to construct, for every special formal scheme $\mathfrak{X}$ over K°, an exact functor

$$D^b(\Pi_K\text{-Mod}) \to D^b(\mathfrak{X}^h_s(\Pi_K)) : \Lambda^\cdot \mapsto R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \ .$$

(The notation $R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$ for the resulting complex is suggestive.) We prove that the complexes $R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$ possess the following properties:

(i) they are functorial in $\mathfrak{X}$, i.e., every morphism of special formal schemes $\varphi : \mathfrak{Y} \to \mathfrak{X}$ gives rise to a morphism of complexes

$$\theta^h_\eta(\varphi, \Lambda^\cdot) : \varphi^{h*}_s(R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})) \to R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{Y}_\eta})$$

which, in its turn, induces homomorphisms $\theta^{h,q}_\eta(\varphi, \Lambda^\cdot)$ between q-th cohomology sheaves;

(ii) there is a canonical isomorphism

$$R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) = R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}) \otimes^{\mathbf{L}}_{\mathbf{Z}_{\mathfrak{X}^h_s}} \underline{\Lambda}^\cdot_{\mathfrak{X}^h_s} \ ;$$

(iii) the sheaves $R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ are (algebraically) constructible in the sense of [Ver76, §2], and the action of Π_K on them is quasi-unipotent;

(iv) if a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is formally smooth, then $\theta^h_\eta(\varphi, \Lambda^\cdot)$ is an isomorphism (φ is formally smooth if locally in the étale topology of $\mathfrak{Y}$ it is a composition of morphisms of the form $\mathfrak{Z}_{/\mathcal{Y}} \to \mathfrak{Z}$ for a subscheme $\mathcal{Y} \subset \mathfrak{Z}_s$ and $\mathfrak{Z} \times \mathfrak{A}^1 \to \mathfrak{Z}$, where $\mathfrak{A}^1 = \mathrm{Spf}(K^\circ\{T\})$);

(v) given $\mathfrak{X}$ with rig-smooth generic fiber, there exists $n \geq 1$ such that, for every $\mathfrak{Y}$ *of finite type over* K°, every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ congruent modulo $(K^{\circ\circ})^n$, every Π_K-module Λ which is either finite or has no $\mathbf{Z}$-torsion, and every q, one has $\theta^{h,q}_\eta(\varphi, \Lambda) = \theta^{h,q}_\eta(\psi, \Lambda)$;

(vi) given $\mathfrak{X}$ and $\mathfrak{Y}$ both with rig-smooth generic fibers, there exists an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ such that, for every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ congruent modulo $\mathcal{J}$, every Π_K-module Λ as in (v), and every q, one has $\theta^{h,q}_\eta(\varphi, \Lambda) = \theta^{h,q}_\eta(\psi, \Lambda)$;

(vii) given a complex of discrete $\mathbf{Z}/n\mathbf{Z}[G_K]$-modules $\Lambda^\cdot$ with finite cohomology modules, there is a canonical isomorphism

$$(R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}))^h \widetilde{\to} R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \ ,$$

where $R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})$ is the vanishing cycles complex on $\mathfrak{X}_s$ from §0.1;

(viii) given a morphism of germs of complex analytic spaces $(B,b) \to (\mathbf{C},0)$, a scheme $\mathcal{Y}$ of finite type over $\mathcal{O}_{B,b}$, a subscheme $\mathcal{Z} \subset \mathcal{Y}_s = \mathcal{Y} \otimes_{\mathcal{O}_{B,b}} \mathbf{C}$, and $\Lambda^\cdot \in D(\Pi_{\widehat{\mathcal{K}}}\text{-Mod})$, there is a canonical isomorphism

$$R\Psi_\eta(\Lambda^\cdot_{\mathcal{Y}^h_\eta})\big|_{\mathcal{Z}^h} \widetilde{\to} R\Psi^h_\eta(\Lambda^\cdot_{(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta}) \ .$$

Here is an explanation of the objects on both sides of the isomorphism in (viii).

First of all, the formal completion $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ of $\mathcal{Y}$ along the subscheme $\mathcal{Z}$ is a special formal scheme over $\widehat{\mathcal{K}}^\circ$, and the right hand side in (viii) is the value at $\Lambda^\cdot$ of the above exact functor $R\Psi^h_\eta$ associated to it.

Furthermore, the scheme $\mathcal{Y}$ defines a complex analytic space $\mathcal{Y}^h$ over an open neighborhood of b in B. If the neighborhood is small enough, there is an induced morphism $\mathcal{Y}^h \to \mathbf{C}$. The same construction applied to the schemes $\mathcal{Y}_s$ and $\mathcal{Y}_\eta = \mathcal{Y} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathcal{K}$ gives the usual complex analytification $\mathcal{Y}^h_s$ of $\mathcal{Y}_s$ and a space $\mathcal{Y}^h_\eta$, which can be identified with the preimage of $\mathbf{C}^*$ under the above morphism. The complex of $\Pi_{\widehat{\mathcal{K}}}$-modules $\Lambda^\cdot$ defines a complex of $\Pi_{\mathcal{K}}$-modules which, in its turn, defines a complex of locally constant sheaves on $\mathbf{C}^*$ whose pullback on $\mathcal{Y}^h_\eta$ is denoted by $\Lambda^\cdot_{\mathcal{Y}^h_\eta}$ (see §3.3). The complex $R\Psi_\eta(\Lambda^\cdot_{\mathcal{Y}^h_\eta})$ on the left hand side in (viii) is the value at $\Lambda^\cdot_{\mathcal{Y}^h_\eta}$ of the derived functor of the complex analytic vanishing cycles functor Ψ_η from [SGA7, Exp. XIV] (its definition is recalled in §1.3).

The continuity properties (v) and (vi) are stronger than corresponding results from [Ber96b] and [Ber15] (mentioned in §0.1(iii)), but the assumptions on rig-smoothness are probably superfluous. In any case, if $\mathfrak{X} = \widehat{\mathcal{Y}}_{/\mathcal{Z}}$ as in (viii), then $\mathfrak{X}_\eta$ is rig-smooth if and only if there exists an open neighborhood V of $\mathcal{Z}^h$ in $\mathcal{Y}^h$ such that the induced morphism $V \to \mathbf{C}$ is smooth outside the preimage of zero.

Remark 0.3.1. Let f be a smooth complex valued function on an open neighborhood of 0 in $\mathbf{R}^n$ and equal to zero at 0. Its Taylor series expansion $\mathrm{T}(f)$ is an element of the maximal ideal of $\mathbf{C}[[T_1,\dots,T_n]]$. (Recall that, by Borel's Lemma ([GG73, Ch. IV, §2]), each element of the latter ring is the Taylor series expansion of some smooth complex valued function on an open neighborhood of 0 in $\mathbf{R}^n$.) Thus, if $\mathfrak{X} = \mathrm{Spf}(\mathbf{C}[[T_1,\dots,T_n]])$, $\mathrm{T}(f)$ defines a morphism of formal schemes $\mathfrak{X} \to \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathbf{C},0})$. Since $\mathfrak{X}_s$ is a one point space, $\psi^q_f = R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ are just finitely generated abelian groups provided with a quasi-unipotent action of the fundamental group $\pi_1(\mathbf{C}^*)$. The groups ψ^q_f are functorial in f, i.e., each morphism (resp. isomorphism) of smooth germs $(\mathbf{R}^m,0) \to (\mathbf{R}^n,0)$ defines homomorphisms (resp. isomorphisms) $\psi^q_f \to \psi^q_g$, where g is the lift of f to $(\mathbf{R}^m,0)$. The continuity property (vi) implies that, given f on $(\mathbf{R}^n,0)$ and g on $(\mathbf{R}^m,0)$, there exists $k \geq 1$ such that, for any pair of morphisms $(\mathbf{R}^m,0) \to (\mathbf{R}^n,0)$ that have the same k-jets and take f to g, the corresponding homomorphisms $\psi^q_f \to \psi^q_g$ coincide. Notice that if, after an automorphism of $(\mathbf{R}^n,0)$, the Taylor series $\mathrm{T}(f)$ coincides with that of a function analytic in an open neighborhood of 0 in $\mathbf{C}^n$, then ψ^q_f are isomorphic to the usual vanishing cycles cohomology groups of that analytic function. But there exist f's without this property (see [Sh76]). It would be interesting to know the geometric meaning of the groups ψ^q_f for arbitrary smooth complex (or real) valued functions f.

0.4. **Ingredients of the construction.** The main ingredients used in the construction of the vanishing cycles complexes and establishing their properties are Michael Temkin's work on functorial desingularization of quasi-excellent schemes in characteristic zero ([Tem08], [Tem18]), the work of Kazuya Kato and his collaborators on log geometry ([Kato89], [KN99], [Nak98]), and author's work on vanishing cycles for formal schemes ([Ber93], [Ber96b], [Ber15]) and on the structure of polystable formal schemes ([Ber99]).

Namely, a scheme $\mathcal{Y}$ of locally finite type over a discrete valuation Henselian ring R (such as K° or $\mathcal{K}^\circ = \mathcal{O}_{\mathbf{C},0}$) is said to be distinguished if locally in the étale topology it is isomorphic to an affine scheme of the form $\mathrm{Spec}(A)$ for $A = R[T_1, \dots, T_n]/(T_1^{e_1} \cdot \ldots \cdot T_m^{e_m} - \varpi)$, where $1 \le m \le n$, $e_i \ge 1$ for all $1 \le i \le m$, and ϖ is a generator of the maximal ideal of R. We always consider such $\mathcal{Y}$ as a log scheme provided with the canonical log structure (which is, for the above affine scheme, is generated by the coordinate functions $T_1, \dots, T_m$).

A special formal scheme $\mathfrak{X}$ over K° is said to be distinguished if locally in the étale topology it is isomorphic to an affine formal scheme of the form $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$, where $\mathcal{Y}$ is a distinguished scheme over K° and $\mathcal{Z}$ is the union of some of the irreducible components of $\mathcal{Y}_s = \mathcal{Y} \otimes_{K^\circ} \widetilde{K}$. The log structure on the scheme $\mathcal{Y}$ induces a log structure on the formal completion $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$. Using results from [Ber99], we show that the latter log structure coincides with the canonical one, i.e., the value of the monoid sheaf on $\mathfrak{U}$ étale over $\mathfrak{X}$ is the multiplicative submonoid of $\mathcal{O}(\mathfrak{U})$ consisting of the functions invertible on the generic fiber $\mathfrak{U}_\eta$. In particular, this log structure on $\mathfrak{X}$ as well as that induced on the complex analytification $\mathfrak{X}_s^h$ of the closed fiber $\mathfrak{X}_s$ is functorial in $\mathfrak{X}$.

Furthermore, Temkin's results from [Tem08] and [Tem18] imply that each special formal scheme $\mathfrak{X}$ over K° admits a proper hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ by distinguished formal schemes $\mathfrak{Y}_n$, $n \ge 0$. Each complex analytic space $Y_n = \mathfrak{Y}_{n,s}^h$ provided with the log structure induced from $\mathfrak{Y}_n$ defines, by the construction of Kato and Nakayama from [KN99], a topological space $Y_n^{\log}$. By the above, the latter form an augmented simplicial topological space $a^{\log} : Y_\bullet^{\log} = (Y_n^{\log})_{n \ge 0} \to \mathfrak{X}_s^h$. We define the vanishing cycles complexes $R\Psi_\eta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ on $\mathfrak{X}_s^h$ in terms of this augmented simplicial topological space, and show that their cohomology sheaves $R^q\Psi_\eta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ are (algebraically) constructible in the sense of [Ver76].

Finally, in order to establish properties of those complexes and, in particular, to verify that they do not depend on the choice of the proper hypercovering, we use results from [KN99] and [Nak99] to show that the same construction for the groups $\mathbf{Z}/n\mathbf{Z}$ gives the analytification of the vanishing cycles complexes $R\Psi_\eta((\mathbf{Z}/n\mathbf{Z})_{\mathfrak{X}_\eta})$ introduced in [Ber96b] and [Ber15].

0.5. **Integral "étale" cohomology of restricted analytic spaces.** For a quasicompact special formal scheme flat over K° and a Π_K-module Λ, we set

$$H^q(\mathfrak{X}_{\overline{\eta}}, \Lambda) = R^q\Gamma(\mathfrak{X}_s^h, R\Psi_\eta^h(\Lambda_{\mathfrak{X}_\eta})) \ .$$

This definition imitates the property (ii) from §0.1 and, if Λ comes from a finite discrete G_K-module, gives the usual étale cohomology groups of the analytic space $\mathfrak{X}_{\overline{\eta}}$ with coefficients in Λ. We believe that the groups on the left hand side depend only on the K-analytic space $\mathfrak{X}_\eta$ for arbitrary Λ's, but can deduce from results of the previous subsection only the following fact. For any admissible proper morphism

$\mathfrak{X}' \to \mathfrak{X}$ (i.e., a proper morphism with $\mathfrak{X}'_\eta \widetilde{\to} \mathfrak{X}_\eta$), the induced maps $H^q(\mathfrak{X}_{\overline{\eta}}, \Lambda) \to H^q(\mathfrak{X}'_{\overline{\eta}}, \Lambda)$ are isomorphisms. This leads us to introduction of the category $K\text{-}\widehat{\mathcal{A}n}$ of *restricted K-analytic spaces*, which is the localization of the category quasicompact special formal schemes flat over K° with respect to admissible proper morphisms. Its objects are denoted by $\widehat{X}$, $\widehat{Y}$ and so on. There is an evident faithful (but not fully faithful) functor $K\text{-}\widehat{\mathcal{A}n} \to K\text{-}\mathcal{A}n : \widehat{X} \mapsto X$ so that the generic fiber functor $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ goes through it. Raynaud theory implies that this functor gives rise to an equivalence between the full subcategory of $K\text{-}\widehat{\mathcal{A}n}$ formed by formal schemes flat and of finite type over K° and the category of compact strictly K-analytic spaces.

We fix for every restricted K-analytic space $\widehat{X}$ a formal model $\mathfrak{X}$ and, for a Π_K-module Λ, we set $H^q(\overline{\widehat{X}}, \Lambda) = H^q(\mathfrak{X}_{\overline{\eta}}, \Lambda)$. For $\varpi \in \Pi_K$, the ϖ-component of the latter is denoted by $H^q(\widehat{X}^{(\varpi)}, \Lambda)$. If Λ has no $\mathbf{Z}$-torsion, one has $H^q(\overline{\widehat{X}}, \Lambda) = H^q(\overline{\widehat{X}}, \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$. We prove that

(i) the Π_K-modules $H^q(\overline{\widehat{X}}, \Lambda)$ are well defined, and the correspondence $\widehat{X} \mapsto H^q(\overline{\widehat{X}}, \Lambda)$ is functorial in $\widehat{X}$;

(ii) $H^q(\overline{\widehat{X}}, \mathbf{Z})$ are quasi-unipotent Π_K-modules and finitely generated over $\mathbf{Z}$;

(iii) for every prime l, there are canonical Π_K-equivariant isomorphisms

$$H^q(\overline{\widehat{X}}, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_l \widetilde{\to} H^q(\overline{X}_{\text{ét}}, \mathbf{Z}_l) = \varprojlim H^q(\overline{X}_{\text{ét}}, \mathbf{Z}/l^n\mathbf{Z}) \ ,$$

where $H^q(\overline{X}_{\text{ét}}, \mathbf{Z}/l^n\mathbf{Z})$ are the Π_K-modules $\varpi \mapsto H^q(X^{(\varpi)}_{\text{ét}}, \mathbf{Z}/l^n\mathbf{Z})$ and the latter are étale cohomology groups of $X^{(\varpi)} = X \widehat{\otimes}_K \widehat{K^{(\varpi)}}$ from [Ber93];

(iv) there are canonical Π_K-equivariant homomorphisms

$$H^q(|\overline{X}|, \mathbf{Z}) \to H^q(\overline{\widehat{X}}, \mathbf{Z})$$

compatible with the canonical homomorphisms

$$H^q(|\overline{X}|, \mathbf{Z}/n\mathbf{Z}) \to H^q(\overline{\widehat{X}}_{\text{ét}}, \mathbf{Z}/n\mathbf{Z}) \ ,$$

where the groups on the left hand side are the cohomology groups of the underlying topological Π_K-space $|\overline{X}|$ of $\overline{X}$;

(v) in the situation of (viii) from §0.3, if $\mathcal{Y}$ is separated, then for $\widehat{X}$ represented by $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ there are canonical $\Pi_{\mathcal{K}}$-equivariant isomorphisms

$$H^q(\mathcal{Y}^h(\mathcal{Z}^h)_{\overline{\eta}}, \mathbf{Z}) \widetilde{\to} H^q(\overline{\widehat{X}}, \mathbf{Z}) \ ,$$

where $H^q(\mathcal{Y}^h(\mathcal{Z}^h)_{\overline{\eta}}, \mathbf{Z}) = \varinjlim H^q(V_{\overline{\eta}}, \mathbf{Z})$ with the inductive limit taken over open neighborhoods V of $\mathcal{Z}^h$ in $\mathcal{Y}^h$ and $V_{\overline{\eta}}$ is the preimage of $\mathbf{C}^*$ in $\overline{V}$;

(vi) in the situation of (viii) from §0.3, if $\mathcal{Y}$ is separated and $\mathcal{Y} = \mathcal{Y}_\eta$, then every morphism $X \to \mathcal{Y}^{\text{an}}$ from a compact strictly K-analytic space X gives rise to canonical Π_K-equivariant homomorphisms $H^q(\overline{\mathcal{Y}^h}, \mathbf{Z}) \to H^q(\overline{X}, \mathbf{Z})$, which are also functorial in X and $\mathcal{Y}$.

The property (iii), applied to $X = \mathcal{Y}^{\text{an}}$ for a proper scheme $\mathcal{Y}$ over K, gives rise to a Π_K-equivariant isomorphism

$$H^q(\overline{\mathcal{Y}^{\text{an}}}, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_l \widetilde{\to} H^q(\overline{\mathcal{Y}}, \mathbf{Z}_l) \ ,$$

where the right hand side is the Π_K-module $\varpi \mapsto H^q(\mathcal{Y}^{(\varpi)}, \mathbf{Z}_l)$ and the latter is the l-adic étale cohomology group of the scheme $\mathcal{Y}^{(\varpi)} = \mathcal{Y} \otimes_K K^{(\varpi)}$.

In (v), if $\mathcal{Y}$ comes from a separated scheme $\mathcal{Y}'$ of finite type over $\mathbf{C}$, i.e., $\mathcal{Y} = \mathcal{Y}' \otimes_{\mathbf{C}} \mathcal{K}^\circ$ and $\mathcal{Z} \subset \mathcal{Y}_s = \mathcal{Y}'$, then $H^q(\overline{\widehat{X}}, \mathbf{Z})$ is just the cohomology group $H^q(\mathcal{Z}^h, \mathbf{Z})$ provided with the trivial action of $\Pi_{\mathcal{K}}$.

In (vi), $\mathcal{Y}^{\mathrm{an}}$ is the K-analytic space associated (in [Ber15, §3.2]) to the scheme $\mathcal{Y} \otimes_{\mathcal{O}_{B,b}} (\widehat{\mathcal{O}}_{B,b} \otimes_{K^\circ} K)$, and $\overline{\mathcal{Y}^h} = \mathcal{Y}^h \times_{\mathbf{C}^*} \mathbf{C}$. The group $H^q(\overline{\mathcal{Y}^h}, \mathbf{Z})$ is in fact an inductive limit of the corresponding cohomology groups taken over open neighborhoods of the point b in B (see §1). If the above $\mathcal{Y}$ is proper over $\mathcal{K}$, the property (v) implies that there is a canonical isomorphism $H^q(\overline{\mathcal{Y}^h}, \mathbf{Z}) \widetilde{\rightarrow} H^q(\overline{\mathcal{Y}^{\mathrm{an}}}, \mathbf{Z})$.

We conjecture that the above Π_K-modules $H^q(\overline{\widehat{X}}, \mathbf{Z})$ are provided with a mixed Hodge structure which is functorial in $\widehat{X}$ and such that, if $X = \mathcal{Y}^{\mathrm{an}}$ for a proper scheme $\mathcal{Y}$ over $\mathcal{K}$ as in the previous paragraph, it coincides with the limit mixed Hodge structure on the groups $H^q(\overline{\mathcal{Y}^h}, \mathbf{Z})$.

0.6. **Comparison with de Rham cohomology.** A restricted K-analytic space $\widehat{X}$ is said to be rig-smooth, if the K-analytic space X is rig-smooth. For such $\widehat{X}$, its distinguished formal models form a cofinal family in that of all formal models, and the de Rham cohomology groups $H^q_{\mathrm{dR}}(\widehat{X}/K^\circ)$ are defined as the hypercohomology of the complex $\omega^{\cdot}_{\mathfrak{X}/K^\circ}$ of logarithmic differential forms over K° of a fixed distinguished formal model $\mathfrak{X}$ of $\widehat{X}$. Notice that, if X is compact and, in particular, $\mathfrak{X}$ is of finite type over K°, then there are canonical isomorphisms $H^q_{\mathrm{dR}}(\widehat{X}/K^\circ) \otimes_{K^\circ} K \widetilde{\rightarrow} H^q_{\mathrm{dR}}(X/K)$, where the latter are the usual de Rham cohomology groups of X, i.e., the hypercohomology groups of the de Rham complex of differential forms $\Omega^{\cdot}_{X/K}$ considered in the G-topology of X. We show that the groups $H^q_{\mathrm{dR}}(\widehat{X}/K^\circ)$ do not depend on the choice of a distinguished formal model up to a canonical isomorphism, and there are canonical isomorphisms

$$H^q(\overline{\widehat{X}}, \mathbf{C}) \otimes_{\mathbf{C}} K^\circ \widetilde{\rightarrow} H^q_{\mathrm{dR}}(\widehat{X}/K^\circ) \ .$$

Notice that $H^q(\overline{\widehat{X}}, \mathbf{C})$ are Π_K-modules, and $H^q_{\mathrm{dR}}(\widehat{X}/K^\circ)$ are provided with the Gauss-Manin connection $\nabla : H^q_{\mathrm{dR}}(\widehat{X}/K^\circ) \to H^q_{\mathrm{dR}}(\widehat{X}/K^\circ) \otimes_{K^\circ} \omega^1_{K^\circ}$. Notice also that $\omega^1_{K^\circ}$ is a free module of rank one over K° generated by the one form $d\log(\varpi) = \frac{d\varpi}{\varpi}$ for each generator ϖ of $K^{\circ\circ}$. The results we are going to describe relate these structures on both sides of the above isomorphism in a form which reminds Fontaine's p-adic Hodge theory.

First of all, the field K can be considered as a Π_K-field, which will be denoted by $\underline{K}$. Namely, one associated to each $\varpi \in \Pi_K$ the field K and to each morphism $\varpi \to \varpi'$ in Π_K the automorphism of K that takes $f(\varpi)$ to $f(\varpi')$ for $f \in \mathbf{C}((T))$. In the same way, the ring of integers K° can be considered as a Π_K-ring, which will be denoted by $\underline{K}^\circ$.

Furthermore, let W_K be the algebra of $\mathbf{C}$-linear endomorphisms K generated by multiplications by elements of K and derivations $\frac{\partial}{\partial\varpi}$ for generators ϖ of the maximal ideal $K^{\circ\circ}$. If ϖ is fixed, each element of W_K has a unique representation in the form $g_n \frac{\partial^n}{\partial\varpi^n} + g_{n-1}\frac{\partial^{n-1}}{\partial\varpi^{n-1}} + \ldots + g_0$ with $n \geq 0$ and $g_i \in K$. The algebra W_K can be considered as a Π_K-ring, which will be denoted by $\underline{W}_K$. Namely, one associated to each $\varpi \in \Pi_K$ the algebra W_K and to each morphism $\varpi \to \varpi'$ in

Π_K the automorphism of W_K that takes $f(\varpi)$ to $f(\varpi')$ as above and $\frac{\partial}{\partial\varpi}$ to $\frac{\partial}{\partial\varpi'}$. Notice that $\underline{K}$ is a left $\underline{W}_K$-module.

Finally, for a generator ϖ of $K^{\circ\circ}$ let δ_ϖ denote the derivation $\varpi\frac{\partial}{\partial\varpi}$ on K which preserves K° and all of its ideals. Let W_{K° be the K°-subalgebra of W_K generated by the derivations δ_ϖ. By the way, the Gauss-Manin connection on the groups $H^q_{\rm dR}(\widehat{X}/K^\circ)$ gives rise to the structure of a W_{K°-module on them. (The action of δ_ϖ is the composition of ∇ with the isomorphism $\omega^1_{K^\circ}\widetilde{\to}K^\circ : d\log(\varpi)\mapsto 1$.) The Π_K-ring structure on W_K induces a Π_K-structure on W_{K°, and the latter Π_K-ring will be denoted by $\underline{W}_{K^\circ}$. Notice that $\underline{K}^\circ$ is a left $\underline{W}_{K^\circ}$-module.

For a left $\underline{W}_{K^\circ}$-module D, a complex number λ and an element $\varpi\in\Pi_K$, we set $D^{(\varpi)}_\lambda = \{x\in D^{(\varpi)}\big|(\delta_\varpi-\lambda)^n(x)=0 \text{ for some } n\geq 1\}$. If λ is fixed, the correspondence $\varpi\mapsto D^{(\varpi)}$ is a Π_K-submodule of D denoted by D_λ. For a subset $I\subset\mathbf{C}$, we set $D_I=\oplus_{\lambda\in I}D_\lambda$. We also denote by $\widetilde{D}$ the Π_K-module $D/(K^{\circ\circ}\cdot D)$. A left $\underline{W}_{K^\circ}$-module D is said to be *distinguished* if it possesses the following properties:

1. (1) D is free of finite rank over K°;
2. (2) there exists a finite subset $I=I(D)\subset\mathbf{Q}\cap[0,1)$ such that the canonical map $D\to\widetilde{D}$ induces an isomorphism of Π_K-modules $D_I\widetilde{\to}\widetilde{D}$;
3. (3) for $\varpi\in\Pi_K$, the actions of $\sigma^{(\varpi)}$ and δ_ϖ on $D^{(\varpi)}$ are related by the equality $\sigma^{(\varpi)}=\exp(-2\pi i\delta_\varpi)$.

If D is a distinguished $\underline{W}_{K^\circ}$-module, the action of the automorphism $\sigma^{(\varpi)}$ on the vector space $\widetilde{D}$ of finite dimension over $\mathbf{C}$ is quasi-unipotent. A Π_K-module with the latter property will be said to be quasi-unipotent.

It is easy to show (see Proposition 3.5.3) that the functor $D\mapsto\widetilde{D}$ from the category of distinguished $\underline{W}_{K^\circ}$-modules to that of quasi-unipotent Π_K-modules of finite dimension over $\mathbf{C}$ is an equivalence of categories. Conversely, if V is a quasi-unipotent Π_K-module of finite dimension over $\mathbf{C}$, one can provide the tensor product $V\otimes_{\mathbf{C}}\underline{K}^\circ$ with the structure of a distinguished $\underline{W}_{K^\circ}$-module (see §3.5) so that the correspondence $V\mapsto V\otimes_{\mathbf{C}}\underline{K}^\circ$ defines a functor inverse to the above one.

The comparison result mentioned at the beginning of this subsection states that, for a rig-smooth restricted K-analytic space $\widehat{X}$, the de Rham cohomology group $H^q_{\rm dR}(\widehat{X}/K^\circ)$ is provided with the structure of a distinguished $\underline{W}_{K^\circ}$-module which extends that of a left W_{K°-module induced by the Gauss-Manin connection, and there is an isomorphism of distinguished $\underline{W}_{K^\circ}$-modules

$$H^q(\overline{\widehat{X}},\mathbf{C})\otimes_{\mathbf{C}}\underline{K}^\circ\widetilde{\to}H^q_{\rm dR}(\widehat{X}/K^\circ)\ .$$

This result implies that, for each $\varpi\in\Pi_K$, one has

$$H^q(\widehat{X}^{(\varpi)},\mathbf{C})=\{x\in H^q_{\rm dR}(\widehat{X}/K^\circ)\big|(\delta_\varpi-\lambda)^n(x)=0 \text{ for } \lambda\in\mathbf{Q}\cap[0,1) \text{ and } n\geq 1\}\ ,$$

and the action of $\sigma^{(\varpi)}$ on the left hand side coincides with that of $\exp(-2\pi i\delta_\varpi)$.

We are now going to describe the de Rham cohomology group considered and the above isomorphism in the case when $\widehat{X}$ comes from a geometric object as in the situation of (viii) from §0.3.

First of all, we notice that in the same way one can consider $\Pi_{\mathcal{K}}$-rings $\underline{\mathcal{K}}^\circ$ and $\underline{W}_{\mathcal{K}^\circ}$, introduce distinguished $\underline{W}_{\mathcal{K}^\circ}$-modules, and show the similar equivalence between the category of distinguished $\underline{W}_{\mathcal{K}^\circ}$-modules and that of quasi-unipotent $\Pi_{\mathcal{K}}$-modules of finite dimension over $\mathbf{C}$.

Suppose we are given a quasicompact distinguished scheme $\mathcal{Y}$ over $\mathcal{K}^\circ$ and a closed subscheme $\mathcal{Z} \subset \mathcal{Y}_s$ which is the union of some of the irreducible components of $\mathcal{Y}_s$. One defines the de Rham cohomology groups

$$H^q_{\mathrm{dR}}(\mathcal{Y}^h(\mathcal{Z}^h)/\mathcal{K}^\circ) = \varinjlim H^q_{\mathrm{dR}}(V/\mathbf{C})$$

with the inductive limit taken over open neighborhoods V of $\mathcal{Z}^h$ in $\mathcal{Y}^h$, where $H^q_{\mathrm{dR}}(V/\mathbf{C})$ is the relative de Rham cohomology group of the log analytic space V (with the log structure induced from that of $\mathcal{Y}$) over the log space $\mathbf{C}$ (with the log structure generated by the coordinate function z).

The last result states that the groups $H^q_{\mathrm{dR}}(\mathcal{Y}^h(\mathcal{Z}^h)/\mathcal{K}^\circ)$ are distinguished $\underline{W}_{\mathcal{K}^\circ}$-modules, there is a canonical isomorphism

$$H^q_{\mathrm{dR}}(\mathcal{Y}^h(\mathcal{Z}^h)/\mathcal{K}^\circ) \otimes_{\mathcal{K}^\circ} \widehat{\mathcal{K}}^\circ \widetilde{\to} H^q_{\mathrm{dR}}(\widehat{\mathcal{Y}}_{/\mathcal{Z}}/\widehat{\mathcal{K}}^\circ) \ ,$$

and there is an isomorphism of distinguished $\underline{W}_{\mathcal{K}^\circ}$-modules

$$H^q(\mathcal{Y}^h(\mathcal{Z}^h)_{\overline{\eta}}, \mathbf{C}) \otimes_{\mathbf{C}} \underline{\mathcal{K}}^\circ \widetilde{\to} H^q_{\mathrm{dR}}(\mathcal{Y}^h(\mathcal{Z}^h)/\mathcal{K}^\circ) \ ,$$

which induces the above isomorphism for the formal completion $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$.

The latter is a refinement of results of Steenbrink from [Ste76, §2], which are considered in the case when $\mathcal{Y}$ is proper over $\mathcal{K}^\circ$ and $\mathcal{Z} = \mathcal{Y}_s$.

0.7. **Plan of the paper.** In §1, we recall the framework of pro-analytic spaces and their cohomology which is convenient for dealing with the analytifications $\mathcal{X}^h$ of schemes $\mathcal{X}$ finitely presented over a Stein germ, i.e., a germ (X, Σ) of a complex analytic space in which Σ is a compact subset of X that has a fundamental system of open Stein neighborhoods. We then consider the main example of such a situation and give a characterization of rig-smoothness of the generic fiber of the formal scheme $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ from the property §0.3(viii) in complex analytic terms (Theorem 1.2.1). In §1.3, we definitions of nearby and vanishing cycles functors from [SGA7, Exp. XIV] and, in §1.4, we prove a comparison theorem 1.4.1 for the class of schemes from the same property §0.3(viii), which is more general than that in *loc. cit.*. In §1.5, we recall some notions of log geometry and especially a beautiful construction of Kato and Nakayama from [KN99] that associates to every fine log complex analytic space (X, M_X) a topological space $X^{\log}$ and a proper surjective map $\tau : X^{\log} \to X$. Their results easily imply a description of the vanishing cycles complex $R\Psi_\eta(\Lambda_{X_\eta}^{\cdot})$ of a vertical log smooth analytic space X over the log open disc (D, M_D) with $M_D = \mathcal{O}_D \cap \mathcal{O}^*_{D^*}$ in terms of the space $X_s^{\log}$ associated to the log structure on X_s induced from X (Theorem 1.5.2).

In §2, k is an arbitrary non-Archimedean field with non-trivial discrete valuation. We introduce distinguished schemes and special formal schemes over k°, and deduce from Temkin's result [Tem18] that, if $\mathrm{char}(\widetilde{k}) = 0$, every reduced special formal scheme $\mathfrak{X}$ flat over k° admits a local blow-up $\mathfrak{Y} \to \mathfrak{X}$ which induces an isomorphism over the rig-smooth locus of $\mathfrak{X}_\eta$ and such that $\mathfrak{Y}$ is distinguished. This implies that every special formal scheme $\mathfrak{X}$ admits a distinguished proper hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ (i.e., such that each $\mathfrak{Y}_n$ is distinguished and the morphism $\mathfrak{Y}_n \to \mathfrak{X}$ is proper). Furthermore, let $\mathfrak{X}$ be the formal scheme $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ with $\mathcal{Y}$ a distinguished scheme over k° and $\mathcal{Z}$ the union of some of the irreducible components of $\mathcal{Y}_s$. Using results from [Ber99], we prove that the log structure on $\mathfrak{X}$ generated by the canonical log structure on $\mathcal{Y}$ coincides with the canonical log structure on $\mathfrak{X}$ whose value on $\mathfrak{U}$ étale over $\mathfrak{X}$ is $\mathcal{O}(\mathfrak{U}) \cap \mathcal{O}(\mathfrak{U}_\eta)^*$. This implies that distinguished special formal schemes over k° are formally log smooth over k°, i.e., as log special formal schemes provided with the canonical structure, they are étale locally isomorphic to the formal completion $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ of a vertical log smooth scheme over k° along a subscheme $\mathcal{Z} \subset \mathcal{Y}_s$, and whose log structure is induced by that of $\mathcal{Y}$.

In §3.1, we introduce various groupoids related to the field K. They include the groupoids Π_K and $\Pi_{\mathcal{K}}$, already mentioned in §0.3, as well as a groupoid $\Pi_{K_r^\circ}$ related to the log scheme $\mathrm{pt}_{K_r^\circ} = \mathrm{Spec}(K_r^\circ)$, where $K_r^\circ = K^\circ/(K^{\circ\circ})^r$, $r \geq 1$, with the log structure induced by the canonical one on $\mathrm{Spec}(K^\circ)$. In §3.2, we consider examples of $\mathcal{P}$-spaces for those groupoids and, in §3.3, we introduce the notion of a $\mathcal{P}$-sheaf and a $\mathcal{P}$-cosheaf on a $\mathcal{P}$-space and consider important examples of those objects. In addition to the $\Pi_K$ring $\underline{W}_{K^\circ}$ and the $\Pi_{\mathcal{K}}$-ring $\underline{W}_{\mathcal{K}^\circ}$, mentioned in §0.6, we introduce a related $\Pi_{K_r^\circ}$-ring $\underline{W}_{K_r^\circ}$. In §3.4, we show that the category of $\mathcal{P}$-sheaves on a $\mathcal{P}$-space X is equivalent to the category of sheaves on an explicitely constructed site $X(\mathcal{P})_{\text{ét}}$. Finally, in §3.5, we introduce distinguished modules over $\underline{W}_{K^\circ}$, $\underline{W}_{\mathcal{K}^\circ}$ and $\underline{W}_{K_r^\circ}$, and construct an equivalence of each of their categories with a corresponding category of quasi-unipotent modules of finite dimension over $\mathbf{C}$ similar to that mentioned in §0.6.

In §4.1, we introduce distinguished log complex analytic spaces over the complex analytification $\mathbf{pt}_{K_r^\circ} = \mathrm{pt}^h_{K_r^\circ}$ of the log scheme $\mathrm{pt}_{K_r^\circ}$ mentioned in the previous paragraph. They include log spaces obtained from distinguished special formal schemes over K° and from distinguished log complex analytic spaces over (D, M_D) from §1.5. In §4.2, we describe a certain $\Pi_{K_r^\circ}$-cosheaf on a distinguished log analytic space X over $\mathbf{pt}_{K_r^\circ}$ in terms of its log structure and, in §4.3, we describe in terms of the same log structure the $\Pi_{K_r^\circ}$-sheaves on X that appear in Theorem 1.5.2, and use it for a description of vanishing cycles sheaves in the situation of §0.3(viii) for a class of schemes $\mathcal{Y}$.

Our purpose in §5 is to prove that, for a formally log smooth formal scheme $\mathfrak{X}$ over K°, the analytification $(R\Psi_\eta(\mathbf{Z}/n\mathbf{Z})_{\mathfrak{X}_\eta})^h$ of the vanishing cycles complex, introduced in [Ber96b], has the same description in terms of the topological space $(\mathfrak{X}_s^h)^{\log}$ as in Theorem 1.5.2 (Theorem 5.1.1). For this we use, among other things, the log étale cohomology developed by Kazuya Kato and his collaborators.

In §6, we introduce the complex $R\Psi_\eta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ for an arbitrary special formal scheme $\mathfrak{X}$ over K° in terms of the simplicial topological space $(\mathfrak{Y}^h_{\bullet,s})^{\log}$ associated to a distinguished proper hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$. We prove the property §0.3(iii) and use it together with the main result of §5 to show that the construction does not depend on the choice of the hypercovering and is functorial in $\mathfrak{X}$. We then extend the construction to an exact functor $R\Psi_\eta^h$ on arbitrary complexes $\Lambda^\cdot$ taking the property §0.3(ii) as a definition, and prove the comparison property §0.3(vii). In §6.2 we prove the property §0.3(iv) and, in §6.3, we prove the comparison property §0.3(viii).

In §7, we prove the continuity properties §0.3(v) and (vi).

In §8, we introduce the category of restricted K-analytic spaces K-$\widehat{\mathcal{A}n}$, define the groups $H^q(\widehat{\widehat{X}}, \mathbf{Z})$ for such a space $\widehat{X}$, and prove all of their properties listed in §0.5.

In §9, we study a purely complex analytic object, the complex $\omega^\cdot_{X/K_r^\circ}$ of log differential forms on a distinguished log analytic space X over the log space $\mathbf{pt}_{K_r^\circ}$. We construct a complex of $\underline{W}_{K_r^\circ}$-sheaves $L^\cdot_X$ and a quasi-isomorphism $L^\cdot_X \widetilde{\to} \omega^\cdot_{X/K_r^\circ}$.

This implies, for example, that the de Rham cohomology groups $H^q_{\mathrm{dR}}(X/K^\circ_r)$ have the structure of a $\underline{W}_{K^\circ_r}$-module. We also construct a quasi-isomorphism of $L^\cdot_X$ with a complex closely related to that from the construction of the vanishing cycles complex in §6.1. Our construction is a refinement of that from Steenbrink's paper [Ste76, §2], but it is done in the framework of log geometry of Kato-Nakayama [KN99].

In §10, we prove the comparison results formulated in §0.6.

We remark that the terms "nearby" and "vanishing cycles", introduced in [Ber94] and used in this paper (as well as in [Ber96b] and [Ber15]) for the functors Θ and Ψ_η, are not standard ones used in literature. Nevertheless, all of these functors have the same meaning as the corresponding functors with the same notations from [SGA7], and we recall their definition.

1. Vanishing cycles in complex analytic geometry

1.1. **The analytification of a scheme over a Stein germ.** Recall that a Stein compact is a compact subset Σ of a complex analytic space X which has a fundamental system of open neighborhoods which are Stein spaces. For example, if $\Sigma = \{x\}$ is just a point, it is a Stein compact and $\mathcal{O}_X(\Sigma) = \mathcal{O}_{X,x}$ is the stalk of the structural sheaf of X at x. A natural framework for dealing with the analytification of schemes finitely presented over the ring $\mathcal{O}_X(\Sigma)$ is that of pro-analytic spaces. This framework is developed in [SGA4, Exp. I] (see also [Ber96a, §2]). We recall briefly some notations and facts.

The category $\mathrm{Pro}(C)$ of pro-objects of a category C is defined as follows. Its objects are covariant functors $I \to C : i \mapsto X_i$, where I is a small cofiltered category, and they are denoted by $"\varprojlim_I" X_i$. Morphisms between such objects are defined as follows: $\mathrm{Hom}("\varprojlim_J" Y_j, "\varprojlim_I" X_i) = \varprojlim_I \varinjlim_{J^\circ} \mathrm{Hom}(Y_j, X_i)$. The category $\mathrm{Pro}(C)$ admits cofiltered projective limits, and if C admits fiber products, then so is $\mathrm{Pro}(C)$. If C is the category of complex analytic spaces $\mathbf{C}$-$\mathcal{A}n$, we get the category of pro-analytic spaces $\mathrm{Pro}(\mathbf{C}\text{-}\mathcal{A}n)$. A pro-analytic space $"\varprojlim_I" X_i$ gives rise to the underlying locally ringed space $|\mathbf{X}|$ of $\mathbf{X}$. Namely, the underlying topological space $|\mathbf{X}|$ of $\mathbf{X}$ is the projective limit of the underlying topological spaces $|X_i|$ of X_i and $\mathcal{O}_{\mathbf{X},x} = \varinjlim_{I^\circ} \mathcal{O}_{X_i,x_i}$, where x_i is the image of x in X_i. We remark that the space $|\mathbf{X}|$ may be empty even when $\mathbf{X}$ is nontrivial.

An example of pro-analytic spaces is provided by $\mathbf{C}$-germs of analytic spaces. Recall (see [Ber93, §3.4]) that the latter are pairs (X, Σ), where X is a complex analytic space and Σ is a subset of X, and the set of morphisms $\mathrm{Hom}((X', \Sigma'), (X, \Sigma))$ is the inductive limit of the sets of morphisms $\varphi : \mathcal{U}' \to X$ with $\varphi(\Sigma') \subset \Sigma$, where $\mathcal{U}'$ runs through open neighborhoods of Σ' in X'. If Σ is a Stein compact, the germ (X, Σ) is said to be *Stein.*

There is a fully faithful functor $\mathbf{C}$-*Germs* $\to \mathrm{Pro}(\mathbf{C}\text{-}\mathcal{A}n)$ from the category of $\mathbf{C}$-germs $\mathbf{C}$-*Germs* that takes (X, Σ) to $X(\Sigma) = "\varprojlim" \mathcal{U}$, where $\mathcal{U}$ runs through open neighborhoods of Σ in X. This functor commutes with direct products, but does not commute in general with fiber products. For example, let $\varphi : Y \to X$ be a morphism of complex analytic spaces and $x \in X$. Then the fiber product $Y \times_X (X, x)$ in the category $\mathbf{C}$-*Germs* is the $\mathbf{C}$-germ $(Y, \varphi^{-1}(x))$, i.e., it gives rise to $Y(\varphi^{-1}(x)) = "\lim\limits_{\longleftarrow}"\mathcal{V}$, where $\mathcal{V}$ runs through *all* open neighborhoods of the fiber $\varphi^{-1}(x)$. The corresponding fiber product $Y(x) := Y \times_X X(x)$ in the category $\mathrm{Pro}(\mathbf{C}\text{-}\mathcal{A}n)$ is $"\lim\limits_{\longleftarrow}"\varphi^{-1}(\mathcal{U})$, where $\mathcal{U}$ runs through open neighborhoods of x. We remark that the canonical morphism $Y(\varphi^{-1}(x)) \to Y(x)$ induces an isomorphism between the underlying locally ringed spaces, and there is a morphism $Y_x \to Y(\varphi^{-1}(x))$ which induces a homeomorphism between the underlying topological spaces. (Here Y_x is the analytic space which is the fiber of Y at x in the usual sense.)

For a complex analytic space X, the category of morphisms of complex analytic spaces $Y \to X$ is denoted by $X\text{-}\mathcal{A}n$. Such an Y is said to be an X*-analytic space*. If $\mathbf{X} = "\lim\limits_{\substack{\longleftarrow \\ I}}"X_i$ is a pro-analytic space, then an $\mathbf{X}$*-analytic space* is an object of the category $\mathbf{X}\text{-}\mathcal{A}n := \lim\limits_{\substack{\longrightarrow \\ I^\circ}} X_i\text{-}\mathcal{A}n$. If P is a class of morphisms between k-analytic spaces which is preserved under any base change, then one can extend in the evident way the class P to morphisms between $\mathbf{X}$-analytic spaces.

Construction 1.1.1. Let (X, Σ) be a Stein germ. We are going to construct an *analytification* functor $\mathcal{O}_X(\Sigma)\text{-}\mathcal{S}ch \to X(\Sigma)\text{-}\mathcal{A}n : \mathcal{Y} \mapsto \mathcal{Y}^h$ where, for a commutative ring A, $A\text{-}\mathcal{S}ch$ denotes the category of schemes finitely presented over A. This is done in two steps.

(1) For a Stein space U, there is an analytification functor

$$\mathcal{O}(U)\text{-}\mathcal{S}ch \to U\text{-}\mathcal{A}n : \mathcal{Y} \mapsto \mathcal{Y}^h \ .$$

Namely, for a scheme $\mathcal{Y}$ finitely presented over $\mathcal{O}(U)$, $\mathcal{Y}^h$ represents the functor on $U\text{-}\mathcal{A}n$ that takes a morphism $Z \to U$ to the set of morphisms of locally ringed spaces $Z \to \mathcal{Y}$ over $\mathcal{O}(U)$. For example, if $\mathcal{Y} = \mathrm{Spec}(A)$, where $A = \mathcal{O}(X)[T_1, \ldots, T_m]/\mathbf{a}$ with finitely generated ideal $\mathbf{a}$, then $\mathcal{Y}^h$ is the closed analytic subspace of $U \times \mathbf{C}^m$ defined by the coherent subsheaf of ideals $\mathcal{J}$ generated by $\mathbf{a}$.

(2) An $X(\Sigma)$*-scheme* is an object of the category

$$X(\Sigma)\text{-}\mathcal{S}ch = \lim_{\substack{\longrightarrow \\ U \supset \Sigma}} \mathcal{O}(U)\text{-}\mathcal{S}ch \ ,$$

where the inductive limit is taken over the open Stein neighborhoods of Σ in S. There is a natural fully faithful functor $\mathcal{O}_X(\Sigma)\text{-}\mathcal{S}ch \to X(\Sigma)\text{-}\mathcal{S}ch : \mathcal{Y} \mapsto \underline{\mathcal{Y}}$. Namely, if $\mathcal{Y}$ is finitely presented over $\mathcal{O}_X(\Sigma)$, it follows from [EGA4, Théorème (8.8.2)] that there exists a scheme $\mathcal{Y}_U$ finitely presented over $\mathcal{O}(U)$ for an open Stein neighborhood U of Σ, and $\underline{\mathcal{Y}}$ is defined by this $\mathcal{Y}_{\mathcal{U}}$. The analytification functor from (1) defines a functor $X(\Sigma)\text{-}\mathcal{S}ch \to X(\Sigma)\text{-}\mathcal{A}n : \mathcal{Z} \mapsto \mathcal{Z}^h$, and the required analytification functor $\mathcal{O}_X(\Sigma)\text{-}\mathcal{S}ch \to X(\Sigma)\text{-}\mathcal{A}n$ is the composition of the latter with the functor $\mathcal{O}_X(\Sigma)\text{-}\mathcal{S}ch \to X(\Sigma)\text{-}\mathcal{S}ch$, i.e., $\mathcal{Y}^h = (\underline{\mathcal{Y}})^h$ for $\mathcal{Y}$ as above is defined by $\mathcal{Y}_U^h$. We notice that there is a canonical morphism of pro-objects in the category of locally ringed spaces $\mathcal{Y}^h \to \underline{\mathcal{Y}}$. We also notice that, given morphisms of Stein germs $(X', \Sigma') \to (X, \Sigma)$, there is a canonical isomorphism of $X'(\Sigma')$-analytic spaces

$$(\mathcal{Y} \otimes_{\mathcal{O}_X(\Sigma)} \mathcal{O}_{X'}(\Sigma'))^h \widetilde{\to} \mathcal{Y}^h \times_{X(\Sigma)} X'(\Sigma') \ .$$

Lemma 1.1.2. *If a morphism $\varphi : \mathcal{Z} \to \mathcal{Y}$ of schemes finitely presented over $\mathcal{O}_X(\Sigma)$ is separated (resp. proper, resp. finite, resp. closed immersion, resp. open immersion, resp. étale, resp. smooth), then so is the induced morphism of $X(\Sigma)$-analytic spaces $\varphi^h : \mathcal{Z}^h \to \mathcal{Y}^h$.* $\square$

For a complex pro-analytic space $\mathbf{X} = "\varprojlim_I" X_i$, the category of sheaves of sets $\mathbf{T}(\mathbf{X})$ is defined as the inductive limit of the categories of sheaves of sets $\mathbf{T}(X_i)$ on X_i. There are also the abelian categories of abelian sheaves $\mathbf{S}(\mathbf{X})$AAG0@$\mathbf{T}(\mathbf{X})$, $\mathbf{S}(\mathbf{X})$: the categories of sheaves of sets and of abelian groups on $\mathbf{X}$—) and of sheaves of R-module $\mathbf{S}(\mathbf{X}, R)$, where R is a commutative ring. Their derived categories are denoted by $D(\mathbf{X})$ and $D(\mathbf{X}, R)$. If all of the transition morphisms $X_i \to X_j$ are local isomorphisms (e.g., open immersions), then the category $\mathbf{S}(\mathbf{X})$ has injectives, and so the values of the left exact functor $\mathbf{S}(\mathbf{X}) \to \mathcal{A}b : F \mapsto F(\mathbf{X}) = \varinjlim_{I^\circ} F(X_i)$ are $H^q(\mathbf{X}, F) = \varinjlim_{I^\circ} H^q(X_i, F)$.

Given a morphism of pro-analytic spaces $\varphi : \mathbf{Y} = "\varprojlim_J" Y_j \to \mathbf{X} = "\varprojlim_I" X_i$, there is a well defined inverse image functor $\varphi^* : \mathbf{T}(\mathbf{X}) \to \mathbf{T}(\mathbf{Y})$ and, in the situations we really need, there is a direct image functor $\varphi_* : \mathbf{T}(\mathbf{Y}) \to \mathbf{T}(\mathbf{X})$ which is right adjoint to φ^* (see [Ber96a, §2]). Namely, the functor φ_* is defined if the morphism φ makes $\mathbf{Y}$ an $\mathbf{X}$-analytic space. In this case we may assume that $I = J$ and φ is defined by a morphism of analytic spaces $Y_i \to X_i$ for some $i \in I$. If F is a sheaf on $\mathbf{Y}$, we can increase i and assume that it is defined by a sheaf F_i on Y_i. Then φ_* is defined by the sheaf $\varphi_{i*}(F)$ on X_i. The restriction of φ_* to the category of abelian sheaves is a left exact functor $\varphi_* : \mathbf{S}(\mathbf{Y}) \to \mathbf{S}(\mathbf{X})$. If all of the transition morphisms $X_j \to X_i$ are local isomorphisms, the categories $\mathbf{S}(\mathbf{X})$ and $\mathbf{S}(\mathbf{Y})$ have enough injectives, and the high direct images $R^q\varphi_*(F)$ are defined by the sheaves $R^q\varphi_{i*}(F)$. If the morphism φ is separated, φ_* has a left exact subfunctor $\varphi_! : \mathbf{S}(\mathbf{Y}) \to \mathbf{S}(\mathbf{X})$ which are defined in the evident way and, in the above situation, the high direct image $R^q\varphi_!(F)$ is defined by the sheaf $R^q\varphi_{\mathcal{U}!}(F_{\mathcal{U}})$ on X. For example, φ_* is well defined for all morphisms in the category $B(\Sigma)$-$\mathcal{A}n$.

Proposition 1.1.3. *(Comparison Theorem for Cohomology with Compact Support) Let (X, Σ) be a Stein germ, and let $\varphi : \mathcal{Z} \to \mathcal{Y}$ be a compactifiable morphism between schemes finitely presented over $\mathcal{O}_X(\Sigma)$. Then for any étale abelian torsion sheaf $\mathcal{F}$ on $\mathcal{Z}$, there is a canonical isomorphism $(R\varphi_!\mathcal{F})^h \widetilde{\to} R\varphi_!^h\mathcal{F}^h$.*

Proof. We can shrink X and assume that it is a Stein space, the schemes $\mathcal{Z}$ and $\mathcal{Y}$ are base changes of schemes $\mathcal{Z}'$ and $\mathcal{Y}'$ finitely presented over $\mathcal{O}(X)$, the morphism φ is induced by a compactifiable morphism $\varphi' : \mathcal{Z}' \to \mathcal{Y}'$, and the sheaf $\mathcal{F}$ is defined by an abelian torsion sheaf $\mathcal{F}'$ on $\mathcal{Z}'$. It suffices therefore to show that the canonical homomorphism $(R^q\varphi'_!\mathcal{F}')^h \to R^q\varphi'^h_!\mathcal{F}'^h$ of sheaves on $\mathcal{Y}'^h$ is an isomorphism. For this it suffices to verify that this homomorphism induces an isomorphism of stalks of both sheaves at every point $y \in \mathcal{Y}'^h$. By the well known results on étale and classical cohomology, the stalks of the sheaves on the left and right hand sides are $H^q_c(\mathcal{Z}'_y, \mathcal{F}'_y)$ and $H^q_c(\mathcal{Z}'^h_y, \mathcal{F}'^h_y)$, respectively, and the classical comparison theorem for cohomology with compact support implies the required fact. □

Remarks 1.1.4. (i) We say that a Stein germ (X, Σ) (or a Stein compact Σ) is *noetherian* if the ring $\mathcal{O}_X(\Sigma)$ is noetherian. By a theorem of Frisch-Siu ([Fri67, (I,9)] and [Siu69]), a Stein compact Σ is noetherian if and only if it possesses the following property: if Y is a closed analytic subspace of an open neighborhood of Σ, then the set of connected components of the intersection $Y \cap \Sigma$ is finite.

(ii) One can prove the following analog of the generic comparison theorem [Ber93, 7.5.1] in which noetherian Stein compacts play the role of affinoid spaces. Suppose that $\mathcal{S}$ is a scheme of finite type over $\mathcal{O}_X(\Sigma)$, where (X, Σ) is a noetherian Stein germ, $f : \mathcal{Y} \to \mathcal{S}$ and $\varphi : \mathcal{Z} \to \mathcal{Y}$ are morphisms of finite type, and $\mathcal{F}$ is an étale constructible abelian (torsion) sheaf on $\mathcal{Z}$. Then there exists a dense open subset $\mathcal{U} \subset \mathcal{S}$ such that

(1) The sheaves $R^q\varphi_*\mathcal{F}\big|_{f^{-1}(\mathcal{U})}$ are constructible and almost all of them are equal to zero.
(2) The formation of the sheaves $R^q\varphi_*\mathcal{F}$ is compatible with any base change $\mathcal{S}' \to \mathcal{S}$ such that the image of $\mathcal{S}'$ is contained in $\mathcal{U}$.
(3) In (2), assume that $\mathcal{S}'$ is a scheme of finite type over $\mathcal{O}_{X'}(\Sigma')$, where (X', Σ') is a noetherian Stein germ, and that the morphism $\mathcal{S}' \to \mathcal{S}$ is the composition $\mathcal{S}' \to \mathcal{S} \otimes_{\mathcal{O}_X(\Sigma)} \mathcal{O}_{X'}(\Sigma') \to \mathcal{S}$ for a morphism of germs $(X', \Sigma') \to (X, \Sigma)$. Let φ' be the morphism $\mathcal{Z}' = \mathcal{Z} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{Y}' = \mathcal{Y} \times_{\mathcal{S}} \mathcal{S}'$, and let $\mathcal{F}'$ be the inverse image of $\mathcal{F}$ on $\mathcal{Z}'$. Then there is a canonical isomorphism
$$(R\varphi'_*\mathcal{F}')^h \widetilde{\to} R\varphi'^h_*\mathcal{F}'^h \ .$$

The proof is the same as that in *loc. cit.* which, in its turn, follows the proof of Deligne's generic theorem 1.9 from [SGA4$\frac{1}{2}$, Th. finitude]. If $\mathcal{S} = \mathrm{Spec}(\mathbf{C})$ is a point, the above fact gives the classical comparison theorem from [SGA4, Exp. XI]. Here is another case of application. Let $\varphi : \mathcal{Z} \to \mathcal{Y}$ be a morphism between schemes of finite type over the fraction field $\mathcal{K}$ of the local ring $\mathcal{O}_{\mathbf{C},0}$, and let $\mathcal{F}$ be a constructible sheaf on $\mathcal{Z}$. Then there is a canonical isomorphism $(R\varphi_*\mathcal{F})^h \widetilde{\to} R\varphi^h_*\mathcal{F}^h$.

1.2. **An example.** Suppose we are given a morphism of germs $(B, b) \to (\mathbf{C}, 0)$, where b is a point of a complex analytic space B. For an $\mathcal{O}_{B,b}$-scheme $\mathcal{Y}$, we set $\mathcal{Y}_\eta = \mathcal{Y} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathcal{K}$ (the *generic fiber* of $\mathcal{Y}$) , $\widetilde{\mathcal{Y}} = \mathcal{Y} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathbf{C}$ (the *special fiber* of $\mathcal{Y}$), and $\mathcal{Y}_s = \mathcal{Y} \otimes_{\mathcal{O}_{B,b}} \mathbf{C}$ (the *closed fiber* of $\mathcal{Y}$). Here $\mathcal{K}$ is the fraction field of $\mathcal{O}_{\mathbf{C},0}$. Of course, if $(B, b) = (\mathbf{C}, 0)$, then $\mathcal{Y}_s = \widetilde{\mathcal{Y}}$. In general, there are morphisms of schemes

$$\begin{tikzcd} \mathcal{Y}_\eta \arrow[r, "j"] & \mathcal{Y} & \mathcal{Y}_s \arrow[l, "i"'] \arrow[d] \\ & & \widetilde{\mathcal{Y}} \arrow[ul, "\tilde{i}"] \end{tikzcd}$$

By Construction 1.1.1, applied to the germ (B, b), there is an associated diagram of morphisms of $B(b)$-analytic spaces (which are also pro-analytic spaces over $\mathbf{C}(0)$)

$$\begin{tikzcd} \mathcal{Y}^h_\eta \arrow[r, "j^h"] & \mathcal{Y}^h & \mathcal{Y}^h_s \arrow[l, "i^h"'] \arrow[d] \\ & & \widetilde{\mathcal{Y}}^h \arrow[ul, "\tilde{i}^h"] \end{tikzcd}$$

Notice that $\mathcal{Y}^h_s$ is just the analytification of the scheme $\mathcal{Y}_s$ and that the vertical arrow induces a homeomorphism $\mathcal{Y}^h_s \widetilde{\to} |\widetilde{\mathcal{Y}}^h|$.

Furthermore, every subscheme $\mathcal{Z} \subset \mathcal{Y}_s$ defines a $\mathbf{C}$-germ $(\mathcal{Y}^h, \mathcal{Z}^h)$ which, in its turn, defines a pro-analytic space $\mathcal{Y}^h(\mathcal{Z}^h) = \text{"}\varprojlim\text{"} V$, where V runs through open neighborhoods of $\mathcal{Z}^h$ in $\mathcal{Y}^h$. The *generic fiber* of the latter is the pro-analytic space $\mathcal{Y}^h(\mathcal{Z}^h)_\eta = \text{"}\varprojlim\text{"} V_\eta$ over $\mathbf{C}^*$, where V_η is the preimage of $\mathbf{C}^*$ in V. There are canonical morphisms of pro-analytic spaces $\mathcal{Y}^h(\mathcal{Z}^h) \to \mathcal{Y}^h$ and $\mathcal{Y}^h(\mathcal{Z}^h)_\eta \to \mathcal{Y}^h_\eta$, which are isomorphisms if $\mathcal{Y}$ is proper over $\mathcal{O}_{B,b}$ and $\mathcal{Z} = \mathcal{Y}_s$.

On the other hand, the formal completion $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ of $\mathcal{Y}$ along a subscheme $\mathcal{Z} \subset \mathcal{Y}_s$ is a formal scheme of finite type over $\mathrm{Spf}(\widehat{\mathcal{O}}_{B,b})$, where $\widehat{\mathcal{O}}_{B,b}$ is the $\mathbf{m}_b$-adic completion of $\mathcal{O}_{B,b}$. This completion is a special $\widehat{\mathcal{O}}_{\mathbf{C},0}$-algebra and, therefore, $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ is a special formal scheme over $\widehat{\mathcal{K}}^\circ = \widehat{\mathcal{O}}_{\mathbf{C},0}$,where $\widehat{\mathcal{K}}$ is the completion of $\mathcal{K}$ with respect to a fixed discrete valuation. Notice that, for every open neighborhood $\mathcal{V}$ of $\mathcal{Z}$ in $\mathcal{Y}$ there are canonical isomorphisms $\mathcal{V}^h(\mathcal{Z}^h) \widetilde{\to} \mathcal{Y}^h(\mathcal{Z}^h)$ and $\widehat{\mathcal{V}}_{/\mathcal{Z}} \widetilde{\to} \widehat{\mathcal{Y}}_{/\mathcal{Z}}$. Recall (see [Ber06, §1.1]) that a strictly k-analytic space X is said to be rig-smooth if, for every connected strictly affinoid domain $V \subset X$, the sheaf of differentials Ω^1_V is locally free of rank $\dim(V)$. If $\mathrm{char}(k) = 0$, this is equivalent to the property that the local ring $\mathcal{O}_{X,x}$ of every point $x \in X$ with $[\mathcal{H}(x) : k] < \infty$ is regular. The following statement is a characterization of rig-smoothness of the generic fiber of $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ in simple complex analytic terms.

Theorem 1.2.1. *In the above situation, the following are equivalent:*

(a) *the $\widehat{\mathcal{K}}$-analytic space $(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta$ is rig-smooth;*
(b) *there is an open neighborhood $\mathcal{V}$ of $\mathcal{Z}$ in $\mathcal{Y}$ such that $\mathcal{V}_\eta$ is regular;*
(c) *the morphism $\mathcal{Y}^h(\mathcal{Z}^h)_\eta \to \mathbf{C}^*$ is smooth.*

The property (c) just tells that there is an open neighborhood V of $\mathcal{Z}^h$ in $\mathcal{Y}^h$ such that the induced morphism $V \to \mathbf{C}$ is smooth outside the preimage of zero.

Proof. First of all, we remark that, for every closed point $y \in \mathcal{Y}_s$, there is a canonical isomorphism $\widehat{\mathcal{O}}_{\mathcal{Y},y} \widetilde{\to} \widehat{\mathcal{O}}_{\mathcal{Y}^h,y}$. Since the local rings considered are excellent, it follows that regularity of the scheme $\mathrm{Spec}(\mathcal{O}_{\mathcal{Y},y})_\eta$ is equivalent to regularity of the scheme $\mathrm{Spec}(\mathcal{O}_{\mathcal{Y}^h,y})_\eta$. In particular, if the property (b) holds, then the schemes $\mathrm{Spec}(\mathcal{O}_{\mathcal{Y}^h,y})_\eta$ are regular for all closed points $y \in \mathcal{Z}$. Conversely, suppose the latter is true. Then the schemes $\mathrm{Spec}(\mathcal{O}_{\mathcal{Y},y})_\eta$ are regular for all closed points $y \in \mathcal{Z}$ and, therefore, they are contained in the regularity locus $\mathcal{U}$ of $\mathcal{Y}_\eta$. If now $\mathcal{V}$ is the complement of the Zariski closure of the set $\mathcal{Y}_\eta \backslash \mathcal{U}$ in $\mathcal{Y}$, then $\mathcal{V} \supset \mathcal{Y}_s$ and $\mathcal{V} \cap \mathcal{Y}_\eta = \mathcal{U}$, i.e., (b) holds.

(a)$\Longleftrightarrow$(b). Since $(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta \widetilde{\to} \pi^{-1}(\mathcal{Z})$, where π is the reduction map $\widehat{\mathcal{Y}}_\eta \to \mathcal{Y}_s$, the K-analytic space $(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta$ is rig-smooth if and only if the spaces $(\widehat{\mathcal{Y}}_{/\{z\}})_\eta$ are rig-smooth for all closed points $z \in \mathcal{Z}$. (Since the latter spaces have no boundary, rig-smoothness for them is equivalent to smoothness.) The above remark therefore reduces the situation to the case $\mathcal{Y} = \mathrm{Spec}(\mathcal{O}_{B,b})$ and $\mathcal{Z} = \mathcal{Y}_s = \{b\}$, and we have to show that $\widehat{\mathcal{Y}}_\eta$ is smooth if and only if the scheme $\mathcal{Y}_\eta$ is regular.

Till the end of the proof we set $K = \widehat{\mathcal{K}}$. Let $A = \mathcal{O}_{B,b}$. Then $\widehat{\mathcal{Y}} = \mathrm{Spf}(\widehat{A})$, where $\widehat{A}$ is the $\mathbf{m}_b$-adic completion of A. By a result of de Jong [deJ95, 7.1.9], the map $y \mapsto \mathbf{n}_y$ that takes a point $y \in \widehat{\mathcal{Y}}_\eta$ with $[\mathcal{H}(y) : K] < \infty$ to the preimage of $\mathbf{m}_y$ under the canonical homomorphism $\widehat{A} \otimes_{K^\circ} K \to \mathcal{O}_{\widehat{\mathcal{Y}}_\eta,y}$ is a bijection between the set of such points and the set of maximal ideals of $\widehat{A} \otimes_{K^\circ} K$, and this homomorphism induces an isomorphism between the $\mathbf{n}_y$-adic completion of $\widehat{A} \otimes_{K^\circ} K$ and the $\mathbf{m}_y$-adic completion of $\mathcal{O}_{\widehat{\mathcal{Y}}_\eta,y}$. We now notice that the above maximal ideals $\mathbf{n}_y$ of $\widehat{A} \otimes_{K^\circ} K$ correspond to the prime ideals $\mathfrak{p} \subset \widehat{A}$ which have coheight one and whose intersection with K° is zero. Moreover, the $\mathbf{n}_y$-adic completion of $\widehat{A} \otimes_{K^\circ} K$

coincides with the $\mathfrak{p}$-adic completion of the localization $(\widehat{A})_{\mathfrak{p}}$. This implies that the K-analytic space $\widehat{\mathcal{Y}}_\eta$ is rig-smooth if and only if the affine scheme $\mathrm{Spec}(\widehat{A})$ is regular at all points that correspond to the above prime ideals $\mathfrak{p} \subset \widehat{A}$. Since the ring A is excellent, the latter is equivalent to regularity of the affine scheme $\mathcal{Y}_\eta$.

(b)$\Longrightarrow$(c). Indeed, replacing $\mathcal{Y}$ by $\mathcal{V}$, we may assume that $\mathcal{Y}_\eta$ is regular. By Temkin's result on desingularization from [Tem08], there exists a blow-up $\varphi : \mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'_\eta \widetilde{\to} \mathcal{Y}_\eta$ and such that $\mathcal{Y}'$ is regular and the support of $\widetilde{\mathcal{Y}} = \mathcal{Y} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathbf{C}$ is a divisor with strict normal crossings. Given a closed point $y' \in \mathcal{Z}'$, the preimage of $\mathcal{Z}$ in $\mathcal{Y}'_s$, let $t_1, \dots, t_d$ be a system of regular parameters of $\mathcal{Y}'$ at y' such that $t_1, \dots, t_n$ for $1 \le n \le d$ define the irreducible components of $\widetilde{\mathcal{Y}}$ passing through y'. Then $z = t_1^{e_1} \cdot \ldots \cdot t_n^{e_n} u$ for some $e_i \ge 1$ and $u \in \mathcal{O}^*_{\mathcal{Y}',y}$. We can find an étale neighborhood $\psi : \mathcal{Y}'' \to \mathcal{Y}'$ of the point y' such that all of the functions $t_1, \dots, t_d, u$ are defined on $\mathcal{Y}''$ and the ring $\mathcal{O}(\mathcal{Y}'')$ contains an e_1-th root of u. If $y'' \in \psi^{-1}(y)$, it induces an isomorphism of complex analytic germs $(\mathcal{Y}''^h, y'') \widetilde{\to} (\mathcal{Y}^h, y)$. We set $t'_1 = \frac{t_1}{\sqrt[e_1]{u}}$, and $\mathcal{P} = \mathrm{Spec}(\mathcal{O}_{\mathbf{C},0}[T_1, \dots, T_d]/(T_1^{e_1} \cdot \ldots \cdot T_n^{e_n} - z))$. The homomorphism

$$\mathcal{O}_{\mathbf{C},0}[T_1, \dots, T_d]/(T_1^{e_1} \cdot \ldots \cdot T_n^{e_n} - z) \to \mathcal{O}(\mathcal{Y}'') : T_1 \mapsto t'_1,\ T_i \mapsto t_i \text{ for } 2 \le i \le d,$$

gives rise to a morphism $\chi : \mathcal{Y}'' \to \mathcal{P}$. If $p = \chi(y'')$, there is an induced isomorphism of completions $\widehat{\mathcal{O}}_{\mathcal{P},p} = \widehat{\mathcal{O}}_{\mathcal{P}^h,p} \widetilde{\to} \widehat{\mathcal{O}}_{\mathcal{Y}'',y''} = \widehat{\mathcal{O}}_{\mathcal{Y}''^h,y''}$ and, therefore, it induces an isomorphism of complex analytic germs $(\mathcal{Y}''^h, y'') \widetilde{\to} (\mathcal{P}^h, p)$. Since the morphism $\mathcal{P}^h_\eta \to \mathbf{C}^*$ is smooth, it follows that there exists an open neighborhood V_y of y in $\mathcal{Y}^h$ for which the morphism $V_y \cap \mathcal{Y}^h_\eta \to \mathbf{C}^*$ is smooth. Then the property (c) holds for the union $V = \bigcup V_y$ taken over all closed points $y \in \mathcal{Z}$.

(c)$\Longrightarrow$(a). By the remark at the beginning of the proof, it suffices to consider the case when $\mathcal{Y} = \mathrm{Spec}(\mathcal{O}_{B,b})$ and $\mathcal{Z} = \mathcal{Y}_s = \{b\}$, and we have to show that the space $\widehat{\mathcal{Y}}_\eta$ is rig-smooth.. Recall the definition of the Jacobian ideal $H_{A/R}$ of $A = \mathcal{O}_{B,b}$ over $R = \mathcal{O}_{\mathbf{C},0}$. Fix generators $f_1, \dots, f_n$ of the maximal ideal of A, and consider the associated surjective homomorphism $S = \mathcal{O}_{\mathbf{C} \times \mathbf{C}^n,0} \to A$ over R that takes T_i to f_i, $1 \le i \le n$. Let $g_1, \dots, g_m$ be generators of the kernel the latter surjection, and denote by Δ the matrix $(\frac{\partial g_i}{\partial T_j})_{1 \le i \le m, 1 \le j \le n}$ with coefficients in S. Furthermore, for a subset $L \subset \{1, \dots, m\}$, let H_L denote the ideal of S generated by the $r \times r$-minors of Δ whose rows correspond to the elements of L, where $r = |L|$. Let also J_L denote the ideal of S generated by g_i's with $i \in L$, and set $J = (g_1, \dots, g_m) = \mathrm{Ker}(S \to A)$. The Jacobian ideal of A over R is the ideal

$$H_{A/R} = \mathrm{rad}\left(\sum_L (J_L : J) H_L A\right) ,$$

where $(J_L : J) = \{x \in S \big| xJ \subset J_L\}$. It is well known that the ideal $H_{A/R}$ depends only on the homomorphism $R \to A$. Let V be an open neighborhood of the point b in B for which the latter homomorphism is induced by a morphism $V \to \mathbf{C}$ such that all elements from a finite system of generators of $H_{A/R}$ are defined over V. By the assumption, we can shrink V and assume that the morphism $V \to \mathbf{C}$ is smooth outside the preimage of zero. The Jacobian criterion of smoothness implies that the ideal $H_{A/R}$ contains a nonzero element of the maximal ideal of $R = \mathcal{O}_{\mathbf{C},0}$. It follows that the similar Jacobian ideal $H_{\widehat{A}/\widehat{R}}$ for the completions of R and A contains a nonzero element of the maximal ideal of $K^\circ = \widehat{R}$. Finally, the strictly K-analytic space $\widehat{\mathcal{Y}}_\eta$ can be covered by strictly affinoid domains X such that $X = \mathfrak{X}_\eta$ for an affine formal scheme $\mathfrak{X} = \mathrm{Spf}(D)$ of finite type over K° and the canonical embedding $X \to \widehat{\mathcal{Y}}_\eta$ is induced by a morphism of formal scheme $\mathfrak{X} \to \widehat{\mathcal{Y}}$. It follows that the Jacobian ideal H_{D/K° contains a nonzero element of the maximal ideal of K°, i.e., it is open in D. By [Tem08, Proposition 3.3.2], X is rig-smooth. This implies that $\widehat{\mathcal{Y}}_\eta$ is rig-smooth. □

Remark 1.2.2. Let $\mathfrak{X} = \mathrm{Spf}(A)$, where $A = \mathbf{C}[[T_1, \dots, T_n]]$ and $n \geq 1$. Each nonzero element f of the maximal ideal of A defines a homomorphism $\widehat{\mathcal{K}}^\circ = \mathbf{C}[[z]] \to A : z \mapsto f$ that makes $\mathfrak{X}$ a special formal scheme over $\widehat{\mathcal{K}}^\circ$. Since the ring A is regular, it follows that the $(n-1)$-dimensional $\widehat{\mathcal{K}}$-analytic space $\mathfrak{X}_\eta$ is rig-smooth. Furthermore, the number $\mu(f) = \dim_{\mathbf{C}}(A/J(f))$, where $J(f)$ is the ideal generated by the partial derivatives $\frac{\partial f}{\partial T_i}$, is said to be the Milnor number of f. If $\mu(f) < \infty$, then f is equivalent to a polynomial g, i.e., there exists an adic automorphism α of A over $\mathbf{C}$ with $\alpha(f) = g$. The polynomial g defines a morphism $\mathcal{Y} = \mathrm{Spec}(A) \to \mathrm{Spec}(\mathbf{C}[z])$ which is smooth outside the zero point 0 in its open neighborhood, and the automorphism α defines an isomorphism $\widehat{\mathcal{Y}}_{/\{0\}} \widetilde{\to} \mathfrak{X}$ over $\widehat{\mathcal{K}}^\circ$. If $n \geq 3$, there exists an element f of the maximal ideal of A which is not equivalent to a convergent power series from $\mathcal{O}_{\mathbf{C}^n,0}$ (see [Sh76]).

1.3. **Nearby and vanishing cycles functors.** In this subsection we recall the definition of the nearby and vanishing cycles functors in complex analytic geometry (see [SGA7, Exp. XIV]).

Let $\mathbf{C} \to \mathbf{C}^* : b \mapsto \exp(b) = e^b$ be the exponential map. It is a universal covering of $\mathbf{C}^*$. For an open disc D with center at zero, the preimage $\overline{D^*}$ of $D^* = D \backslash \{0\}$ in $\mathbf{C}$ (which has the form $\{b \in \mathbf{C} \big| \mathrm{Re}(b) < r\}$) is a universal covering of D^*. The fundamental group $\Pi = \pi_1(\mathbf{C}^*, t)$ does not depend on the choice of a point $t \in \mathbf{C}^*$, acts on $\mathbf{C}$, and the loop $[0,1] \to \mathbf{C}^* : a \mapsto te^{2\pi i a}$, which is a generator of Π, corresponds to the shift $b \mapsto b + 2\pi i$ of $\mathbf{C}$.

Furthermore, let $\mathcal{K}^{\mathrm{a}}$ be the field of functions meromorphic in some $\overline{D^*}$ and algebraic over $\mathcal{K}$, the fraction field of $\mathcal{O}_{\mathbf{C},0}$. It is an algebraic closure of $\mathcal{K}$, and it is generated over $\mathcal{K}$ by the functions $b \mapsto e^{\frac{b}{n}}$, $n \geq 1$. The action of the Galois group $G = \mathrm{Gal}(\mathcal{K}^{\mathrm{a}}/\mathcal{K})$ on those functions gives rise to an isomorphism $G \widetilde{\to} \varprojlim \mu_n$, where μ_n is the group of n-th roots of unity. The canonical action of the fundamental group Π on $\mathcal{K}^{\mathrm{a}}$ identifies it with a dense subgroup of G, and the above generator of Π corresponds to the element $\sigma = (e^{\frac{2\pi i}{n}})_{n \geq 1}$ of G.

We set $\mathbf{D} = \mathbf{C}(0) = \text{"}\varprojlim\text{"} D$ and $\mathbf{D}^* = \text{"}\varprojlim\text{"} D^*$, where D runs through open discs in $\mathbf{C}$ with center at zero. (Notice that $\mathbf{D} = \mathrm{Spec}(\mathcal{O}_{\mathbf{C},0})^h$ and $\mathbf{D}^* = \mathrm{Spec}(\mathcal{K})^h$.) For a pro-analytic space $\mathbf{X}$ over $\mathbf{D}$, we set $\mathbf{X}_\eta = \mathbf{X} \times_{\mathbf{D}} \mathbf{D}^*$ (the *generic fiber of* $\mathbf{X}$) and $\widetilde{\mathbf{X}} = \mathbf{X} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathbf{C}$ (the *special fiber of* $\mathbf{X}$), and denote by $\mathbf{X}_s$ the underlying topological space of $\widetilde{\mathbf{X}}$ (the *closed fiber of* $\mathbf{X}$). There are morphisms of pro-analytic spaces

$$\begin{tikzcd} \mathbf{X}_\eta \arrow[r, "j"] & \mathbf{X} & \mathbf{X}_s \arrow[l, "i"'] \arrow[d] \\ & & \widetilde{\mathbf{X}} \arrow[ul, "\tilde{i}"] \end{tikzcd}$$

Notice that if $\mathbf{X}$ is a $\mathbf{D}$-analytic space, then $\mathbf{X}_s \widetilde{\to} \widetilde{\mathbf{X}}$. The complex analytic *nearby cycles functor* is the functor $\Theta : \mathbf{T}(\mathbf{X}_\eta) \to \mathbf{T}(\mathbf{X}_s)$ from the category of sheaves on $\mathbf{X}_\eta$ to that of sheaves on $\mathbf{X}_s$ defined by $\Theta(F) = i^*(j_*(F))$. If $F^\cdot \in D(\mathbf{X}_\eta)$, one has $R\Theta(F^\cdot) = i^*(Rj_*(F^\cdot))$ in $D(\mathbf{X}_s)$.

Furthermore, we set $\overline{\mathbf{D}^*} = \text{"}\varprojlim\text{"}\overline{D^*}$ and $\mathbf{X}_{\overline{\eta}} = \mathbf{X}_\eta \times_{\mathbf{D}} \overline{\mathbf{D}^*}$. These are pro-topological spaces over $\mathbf{D}$ provided with an action of the group Π, and there are morphisms

$$\begin{tikzcd} \mathbf{X}_{\overline{\eta}} \arrow[d] \arrow[dr, "\overline{j}"] & & \\ \mathbf{X}_\eta \arrow[r, "j"] & \mathbf{X} & \mathbf{X}_s \arrow[l, "i"'] \end{tikzcd}$$

The complex analytic *vanishing cycles functor* $\Psi_\eta : \mathbf{T}(\mathbf{X}_\eta) \to \mathbf{T}_\Pi(\mathbf{X}_s)$ is defined by $\Psi_\eta(F) = i^*(\overline{j}_*\overline{F})$, where $\mathbf{T}_\Pi(\mathbf{X}_s)$ is the category of Π-sheaves on $\mathbf{X}_s$ (i.e., sheaves provided with an action of Π) and $\overline{F}$ is the pullback of F on $\mathbf{X}_{\overline{\eta}}$. If $F^\cdot \in D(\mathbf{X}_\eta)$, one has $R\Psi_\eta(F^\cdot) = i^*(R\overline{j}_*(\overline{F}^\cdot))$ in the derived category $D(\mathbf{X}_s(\Pi))$ of abelian sheaves on $\mathbf{X}_s$ provided with an action of Π. If $\mathcal{I}^\Pi$ denotes the functor that takes a Π-sheaf on $\mathbf{X}_s$ to the subsheaf of Π-invariant sections, then there is a canonical isomorphism $R\mathcal{I}^\Pi(R\Psi_\eta(F^\cdot)) \widetilde{\to} R\Theta(F^\cdot)$ and, in particular, for every $q \geq 1$, there is an exact sequence

$$0 \to R^{q-1}\Psi_\eta(F^\cdot)/(\sigma-1)R^{q-1}\Psi_\eta(F^\cdot) \to R^q\Theta(F^\cdot) \to R^q\Psi_\eta(F^\cdot)^\Pi \to 0 \ .$$

Example 1.3.1. Suppose we are given a morphism of germs $(B, b) \to (\mathbf{C}, 0)$ and a scheme $\mathcal{Y}$ of finite type over $\mathcal{O}_{B,b}$ (as in §1.2). If the above $\mathbf{X}$ is the analytification $\mathcal{Y}^h$ of $\mathcal{Y}$, which is a $B(b)$-analytic space over $\mathbf{D}$, then $\mathbf{X}_\eta$, $\widetilde{\mathbf{X}}$ and $\mathbf{X}_s$ are the analytifications $\mathcal{Y}^h_\eta$, $\widetilde{\mathcal{Y}}^h$ and $\mathcal{Y}^h_s$ of the corresponding objects of $\mathcal{Y}$. The above construction gives rise to nearby and vanishing cycles functors Θ and Ψ_η from the category of sheaves on $\mathcal{Y}^h_\eta$ to those of sheaves and Π-sheaves on $\mathcal{Y}^h_s$, respectively.

1.4. **Comparison with algebraic vanishing cycles.** Suppose we are given a morphism of germs $(B, b) \to (\mathbf{C}, 0)$ and a scheme $\mathcal{Y}$ of finite type over $\mathcal{O}_{B,b}$ as in Example 1.3.1. There are morphisms $\mathcal{Y}_\eta \stackrel{j}{\hookrightarrow} \mathcal{Y} \stackrel{i}{\leftarrow} \mathcal{Y}_s$ and $\mathcal{Y}_{\overline{\eta}} \stackrel{\overline{j}}{\to} \mathcal{Y} \stackrel{i}{\leftarrow} \mathcal{Y}_s$, where $\overline{j}$ is the composition of j with the canonical morphism $\mathcal{Y}_{\overline{\eta}} = \mathcal{Y}_\eta \otimes_{\mathcal{K}} \mathcal{K}^{\mathrm{a}} \to \mathcal{Y}_\eta$.

The algebraic geometry *nearby cycles functor* is the functor $\Theta : \mathbf{T}(\mathcal{Y}_\eta) \to \mathbf{T}(\mathcal{Y}_s)$ from the category of étale sheaves on $\mathcal{Y}_\eta$ to that of étale sheaves on $\mathcal{Y}_s$ defined by $\Theta(\mathcal{F}) = i^*j_*(\mathcal{F})$. If $\mathcal{F}^\cdot \in D(\mathcal{Y}_\eta)$, then $R\Theta(\mathcal{F}^\cdot) = i^*(Rj_*(\mathcal{F}^\cdot))$. The *vanishing cycles functor* is the functor $\Psi_\eta : \mathbf{T}(\mathcal{Y}_\eta) \to \mathbf{T}_G(\mathcal{Y}_s)$ to the category $\mathbf{T}_G(\mathcal{Y}_s)$ of étale G-sheaves on $\mathcal{Y}_s$ (i.e., étale sheaves on $\mathcal{Y}_s$ provided with a continuous action of the group G) defined by $\Psi_\eta(\mathcal{F}) = i^*\overline{j}_*(\overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ is the pullback of $\mathcal{F}$ on $\mathcal{Y}_{\overline{\eta}}$. If $\mathcal{F}^\cdot \in D(\mathcal{Y}_\eta)$, one has $R\Psi_\eta(\mathcal{F}^\cdot) = i^*(R\overline{j}_*(\overline{\mathcal{F}}^\cdot))$.

For a scheme $\mathcal{Z}$ and $d \geq 1$, let $D_c(\mathcal{Z}, \mathbf{Z}/d\mathbf{Z})$ denote the derived category of étale $\mathbf{Z}/d\mathbf{Z}$-modules on $\mathcal{Z}$ with constructible cohomology sheaves.

Theorem 1.4.1. *In the above situation, for any $\mathcal{F}^\cdot \in D^b_c(\mathcal{Y}_\eta, \mathbf{Z}/d\mathbf{Z})$ the complexes $R\Theta(\mathcal{F}^\cdot)$ and $R\Psi_\eta(\mathcal{F}^\cdot)$ have constructible cohomology, and there are canonical isomorphisms in $D^b(\mathcal{Y}^h_s)$ and $D^b(\mathcal{Y}^h_s(\Pi))$, respectively,*

$$(R\Theta(\mathcal{F}^\cdot))^h \widetilde{\to} R\Theta(\mathcal{F}^{h\cdot}) \text{ and } (R\Psi_\eta(\mathcal{F}^\cdot))^h \widetilde{\to} R\Psi_\eta(\mathcal{F}^{h\cdot}) \ .$$

Proof. Since the reasoning is the same for the nearby and vanishing cycles sheaves, we consider only the latter. We also notice that validity of the theorem for sheaves is equivalent to its validity for bounded below complexes of constructible sheaves of $\mathbf{Z}/d\mathbf{Z}$-modules. Replacing $\mathcal{Y}$ by the scheme theoretic closure of $\mathcal{Y}_\eta$, we may assume that $\mathcal{Y}_\eta$ is dense in $\mathcal{Y}$.

Step 1. Suppose we are given a proper morphism $\varphi : \mathcal{Y}' \to \mathcal{Y}$, and a complex of constructible sheaves $\mathcal{G}^\cdot$ on $\mathcal{Y}'_\eta$. *If the theorem is true for the pair* $(\mathcal{Y}', \mathcal{G}^\cdot)$*, then it is also true for the pair* $(\mathcal{Y}, R\varphi_{\eta *}(\mathcal{G}^\cdot))$. Indeed, since φ is proper, the complex $R\varphi_{\eta *}(\mathcal{G}^\cdot)$ has constructible cohomology sheaves, and one has

$$R\Psi_\eta(R\varphi_{\eta *}\mathcal{G}^\cdot) \widetilde{\to} R\varphi_{s *}(R\Psi_\eta \mathcal{G}^\cdot) \ .$$

It follows that the complex on the left hand side also has constructible cohomology sheaves and

$$(R\Psi_\eta(R\varphi_{\eta *}\mathcal{G}^\cdot))^h \overset{\alpha}{\to} R\varphi^h_{s *}(R\Psi_\eta \mathcal{G}^\cdot)^h \overset{\beta}{\to} R\varphi^h_{s *}(R\Psi_\eta \mathcal{G}^{\cdot h}) \overset{\gamma}{\to} R\Psi_\eta(R\varphi^h_{\eta *}\mathcal{G}^{\cdot h}) \ ,$$

where α is an isomorphism, by Proposition 1.1.3, β is an isomorphism, by the assumption, and γ is an isomorphism because φ^h is a proper map.

Step 2. *To prove the theorem, it suffices to find for each constructible sheaf of* $\mathbf{Z}/d\mathbf{Z}$*-modules* $\mathcal{F}$ *an embedding of* $\mathcal{F} \hookrightarrow \mathcal{G}$*, where* $\mathcal{G}$ *is a similar sheaf* $\mathcal{G}$ *for which the theorem holds.* Indeed, if this is true then, we can find for each $m \geq 1$ an exact sequence of constructible sheaves, $0 \to \mathcal{F} \to \mathcal{G}^0 \to \ldots \to \mathcal{G}^m$, such that the theorem is true for all of the sheaves $\mathcal{G}^i$. This easily implies validity of the theorem for $\mathcal{F}$.

Step 3. *We may assume that* $\mathcal{Y}$ *is irreducible and reduced, i.e., integral, and* $\mathcal{F}$ *is constant.* Indeed, by [SGA4, Exp. IX, 2.14(ii)], the sheaf $\mathcal{F}$ can be embedded in a finite direct sum of sheaves of the form $f_*\mathcal{G}$, where $f : \mathcal{Z}' \to \mathcal{X}_\eta$ is a finite morphism and $\mathcal{G}$ is constant. We may assume that all such $\mathcal{Z}'$ are reduced and, therefore, we can replace them by their normalizations and assume that they are irreducible. If $\mathcal{Z}$ is the normalization of $\mathcal{Y}$ in $\mathcal{Z}'$, we may assume that $\mathcal{Z}' = \mathcal{Z}_\eta$, where $\mathcal{Z}$ is irreducible, normal and finite over $\mathcal{Y}$. It remains to use Steps 1 and 2.

Step 4. *We may assume that the scheme* $\mathcal{Y}$ *is regular and the supports of* $\mathcal{Y}_s$ *and* $\widetilde{\mathcal{Y}}$ *are divisors with strict normal crossings.* Indeed, replacing $\mathcal{Y}$ by a blow-up, we may assume that the support of $\mathcal{Y}_s$ is a divisor. Since the scheme $\mathcal{Y}$ is excellent, we can apply the result of Temkin [Tem08, 1.1] for $\mathcal{Y}$ and its subscheme $\widetilde{\mathcal{Y}}$. It follows that there is a blow-up $\mathcal{Y}' \to \mathcal{Y}$ such that $\mathcal{Y}'_s$ and $\widetilde{\mathcal{Y}}$ are divisors with strict normal crossings. Step 1 implies that validity of theorem for the pair $(\mathcal{Y}, \mathcal{F})$ follows from its validity for the pair $(\mathcal{Y}', \mathcal{F}')$, where $\mathcal{F}'$ is the pullback of $\mathcal{F}$ on $\mathcal{Y}'_{\eta'}$.

Step 5. *The theorem is true.* Indeed, in the situation of Step 4 the required statement follows from the well known description of algebraic (and analytic) nearby and vanishing cycles sheaves which are easy consequences of the characteristic zero purity theorem [SGA4, Exp. XIX, 3.2]. □

Remark 1.4.2. Theorem 1.4.1 and the generic comparison theorem stated in Remark 1.1.4 can be used to prove the following fact. Let (X, Σ) be a Stein germ such that the dimension of X is at most one and the set of connected components of Σ is finite. (By the results mentioned at the beginning of §1.1, the latter is equivalent to the property that the Stein germ (X, Σ) is noetherian.) Given a morphism $\varphi : \mathcal{Z} \to \mathcal{Y}$ of schemes of finite type over $\mathcal{O}_X(\Sigma)$ and a constructible sheaf $\mathcal{F}$ on $\mathcal{Z}$, the complex $R\varphi_*(\mathcal{F})$ has constructible cohomology and there is a canonical isomorphism

$$(R\varphi_*\mathcal{F})^h \widetilde{\to} R\varphi^h_*\mathcal{F}^h \ .$$

1.5. **Vanishing cycles on log smooth analytic spaces.** In the complex pro-analytic spaces $\mathbf{X} = "\lim\limits_{\overleftarrow{I}}" X_i$, considered in this subsection, all of the transition morphisms $X_{i'} \to X_i$ are assumed to be open immersions and all X_i's are assumed to be of finite dimension. Notice that any morphism $\mathbf{Y} = "\lim\limits_{\overleftarrow{J}}" Y_j \to \mathbf{X} = "\lim\limits_{\overleftarrow{I}}" X_i$ between such pro-analytic spaces is defined (in the evident way) by a morphism of analytic spaces $Y_j \to X_i$ for some $i \in I$ and $j \in J$.

Basic notions of log geometry are naturally extended from analytic to such pro-analytic spaces. Namely, a *pre-log structure* on a pro-analytic space $\mathbf{X} = "\lim\limits_{\overleftarrow{I}}" X_i$ is a homomorphism of multiplicative monoids $\beta : M \to \mathcal{O}_\mathbf{X}$ which is induced by a pre-log structure $\beta_i : M_i \to \mathcal{O}_{X_i}$ on the complex analytic space X_i for some $i \in I$. A pre-log structure is said to be a *log structure* if $\beta^{-1}(\mathcal{O}^*_\mathbf{X}) \widetilde{\to} \mathcal{O}^*_\mathbf{X}$. A log pro-analytic space $(\mathbf{X}, \beta : M \to \mathcal{O}_\mathbf{X})$ as above is said to be *coherent* (resp. *fine*; resp. *fs*) if β is induced by a coherent (resp. fine; resp. fs) log structure $\beta_i : M_i \to \mathcal{O}_{X_i}$ for some $i \in I$. A morphism of log pro-analytic spaces $\mathbf{Y} \to \mathbf{X}$ is said to be *log smooth* if it is defined by a log smooth morphism $Y_j \to X_i$ for some $i \in I$ and $j \in J$. (Recall that a morphism of log analytic spaces $Y \to X$ is log smooth if locally in the topology of X and Y it admits a chart $(P \to \mathcal{O}(X), Q \to \mathcal{O}(Y), P \to Q)$ with finitely generated and integral monoids P and Q such that the induced morphism $Y \to X \times_{\mathrm{Spec}(P)^h} \mathrm{Spec}(Q)^h$ is a strict open immersion.)

For example, the pro-analytic space $\mathbf{D} = "\lim\limits_{\longleftarrow}" D$ is provided with the fs log-structure $M_\mathbf{D} = \mathcal{O}_\mathbf{D} \cap \mathcal{O}^*_{\mathbf{D}^*} \hookrightarrow \mathcal{O}_\mathbf{D}$. (Notice that $\mathbf{D} = \mathcal{D}^h$, where the scheme $\mathcal{D} = \mathrm{Spec}(R)$ with $R = \mathcal{O}_{\mathbf{C},0}$ is provided with the log structure that corresponds to the homomorphism of multiplicative monoids $R \backslash \{0\} \hookrightarrow R = \mathcal{O}(\mathcal{D})$.) We are interested here with *log analytic spaces over* $\mathbf{D}$, i.e., log pro-analytic spaces $\mathbf{X}$ provided with a morphism of log pro-analytic spaces $\mathbf{X} \to \mathbf{D}$. For such $\mathbf{X}$ the special and closed fibers $\widetilde{\mathbf{X}}$ and $\mathbf{X}_s$ are provided with the log structures $\widetilde{\beta} : \widetilde{M} = \widetilde{i}^{-1}(M) \to \mathcal{O}_{\widetilde{\mathbf{X}}}$ and $\beta_s : M_s = i^{-1}(M) \to \mathcal{O}_{\mathbf{X}_s}$, where $\widetilde{i}$ and i are the closed immersions $\widetilde{\mathbf{X}} \to \mathbf{X}$ and $\mathbf{X}_s \to \mathbf{X}$, respectively. They are also provided with the induced morphisms of log pro-analytic and analytic spaces $\widetilde{\mathbf{X}} \to \mathbf{D}_s$ and $\mathbf{X}_s \to \mathbf{D}_s$. By the way, $\mathbf{D}_s$ is an analytic log point which is provided with a homomorphism $P \to \mathbf{C}$ from the free monoid generated by the coordinate function z on the complex plane which goes to zero in $\mathbf{C}$. This log point is denoted by $\mathbf{pt}$, and the image of z in $M_\mathbf{pt}$ is denoted by the same z.

Log smoothness of the morphism $\mathbf{X} \to \mathbf{D}$ means that it is defined by a log smooth morphism $X \to D$, i.e., locally in the topology of X there is a fine chart $P \to \mathcal{O}(X)$ and an element $p \in P$ whose image in $\mathcal{O}(X)$ coincides with the image of z and such that the morphism of log analytic spaces $X \to \mathrm{Spec}(R[P]/(p-z))^h$ is a strict open immersion. Such a log structure on $\mathbf{X}$ is said to be *vertical* if it restriction to $\mathbf{X}_\eta$ is trivial. In this case one can find a local chart as above with the additional property that, for every $a \in P$, there exist $b \in P$ and $n \geq 1$ with

$ab = p^n$. If $\mathbf{X}$ is log smooth over $\mathbf{D}$, then $\widetilde{\mathbf{X}}$ is log smooth over $\mathbf{pt}$, but $\mathbf{X}_s$ is not log smooth over $\mathbf{pt}$ in general.

Recall that in [KN99] Kato and Nakayama constructed in a functorial way for every fs log analytic space (X, M_X) a topological space $X^{\log}$ and a proper surjective map $\tau : X^{\log} \to X$. The construction works for the class of fine and not necessarily saturated log analytic spaces. Recall the definition. Let X be a fine log analytic space. As a set, $X^{\log}$ is defined by

$$X^{\log} = \left\{(x, h_x)\big| x \in X, h_x \in \mathrm{Hom}(M^{gr}_{X,x}, S^1) \text{ with } h_x(f) = \frac{f(x)}{|f(x)|} \text{ for all } f \in \mathcal{O}^*_{X,x}\right\}$$

and τ is the canonical projection $(x, h_x) \mapsto x$. If $\beta : P_U \to \mathcal{O}_U$ is a chart over an open subset $U \subset X$, there is a bijection

$$\tau^{-1}(U) \widetilde{\to} \left\{(x, h) \in U \times \mathrm{Hom}(P^{gr}, S^1)\big|\beta(p)(x) = h(p)|\beta(p)(x)| \text{ for all } p \in P\right\}$$

that identifies $\tau^{-1}(U)$ with a closed subset of $U \times \mathrm{Hom}(P^{gr}, S^1)$, and the induced topology on $\tau^{-1}(U)$ does not depend on the choice of the chart on U. In this way, one gets the required topology on $X^{\log}$. If X is log smooth, $X^{\log}$ is a topological manifold with boundary. For every strict morphism of fine log analytic spaces $\varphi : Y \to X$, there is a canonical homeomorphism $Y^{\log} \widetilde{\to} Y \times_X X^{\log}$. (In particular, if X_{red} is the underlying reduced analytic space provided with the induced log structure, then $X^{\log}_{\mathrm{red}} \widetilde{\to} X^{\log}$.) For every point $x \in X$, there is a (non-canonical) homeomorphism $\tau^{-1}(x) \widetilde{\to} \mathrm{Hom}(M^{gr}_{X,x}/\mathcal{O}^*_{X,x}, S^1)$, where S^1 is the unit circle in $\mathbf{C}$. In particular, $\tau^{-1}(x)$ is homeomorphic to disjoint union of k copies of $(S^1)^l$, where k is the order of the torsion subgroup of $M^{gr}_{X,x}/\mathcal{O}^*_{X,x}$ and l is its rational rank. If X is log smooth, $X^{\log}$ is a topological manifold with boundary.

Examples 1.5.1. (i) (see [KN99,(1.2.1.1)]). Suppose $X = \mathrm{Spec}(\mathbf{C}[P])^h$ for a fine monoid P, and provide X with the log structure that corresponds to the homomorphism $P \to \mathbf{C}[P]$. Then there are homeomorphisms $X \widetilde{\to} \mathrm{Hom}(P, \mathbf{C}) : x \mapsto \chi_x$ and $X^{\log} \widetilde{\to} \mathrm{Hom}(P, \mathbf{R}_+ \times S^1) : (x, h_x) \mapsto (|\chi_x|, h_x\big|_P)$ that are included in the following commutative diagram in which the right vertical arrow is induced by the map $\mathbf{R}_+ \times S^1 \to \mathbf{C} : (t, a) \mapsto ta$

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \mathrm{Hom}(P, \mathbf{C}) \\ \uparrow{\scriptstyle\tau} & & \uparrow \\ X^{\log} & \xrightarrow{\sim} & \mathrm{Hom}(P, \mathbf{R}_+) \times \mathrm{Hom}(P, S^1) \end{array}$$

(ii) Consider the log complex plane $\mathbf{C}$ with the log structure generated by the coordinate function z. Then

$$\mathbf{C}^{\log} = \{(c, h) \in \mathbf{C} \times \mathrm{Hom}(P^{gr}, S^1)\big| c = h(z)|c|\} \widetilde{\to} \mathbf{R}_+ \times S^1 \ ,$$

where P is monoid freely generated by z, and the map takes a pair (c, h) to the pair $(|c|, h(z))$. In what follows we identify $\mathbf{C}^{\log}$ with $\mathbf{R}_+ \times S^1$ via the above map. Then the map $\mathbf{C}^{\log} \to \mathbf{C}$ takes (t, a) to ta. The exponential maps $\mathbf{C} \to \mathbf{C}^*$ and $i\mathbf{R} \to S^1 : b \mapsto \exp(b) = e^b$ are universal coverings, and they gives rise to the universal covering $\widetilde{\mathbf{C}^{\log}} = \mathbf{R}_+ \times i\mathbf{R} \to \mathbf{C}^{\log} : (t, b) \mapsto (t, e^b)$. We get a commutative diagram of maps

$$\begin{tikzcd}
& \overline{\mathbf{C}^{\log}} = \mathbf{R}_+ \times i\mathbf{R} \arrow[d] & \overline{\mathbf{pt}^{\log}} = i\mathbf{R} \arrow[l, "\overline{i^{\log}}"'] \arrow[d, "\exp"] \\
\mathbf{C} \arrow[ur, "\overline{j^{\log}}"] \arrow[d, "\exp"'] & \mathbf{C}^{\log} = \mathbf{R}_+ \times S^1 \arrow[d] & \mathbf{pt}^{\log} = S^1 \arrow[l, "i^{\log}"'] \arrow[d] \\
\mathbf{C}^* \arrow[ur, "j^{\log}"] \arrow[r, "j"] & \mathbf{C} & \mathbf{pt} = \{0\} \arrow[l, "i"']
\end{tikzcd}$$

Here $j^{\log}(c) = (|c|, \frac{c}{|c|})$ and $\overline{j^{\log}}(b) = (e^{\mathrm{Re}(b)}, i\mathrm{Im}(b))$.

(iii) For a fine log analytic space X over the log complex plane $\mathbf{C}$, there is an induced map $X^{\log} \to \mathbf{C}^{\log} : (x, h_x) \mapsto (|\varphi(x)|, h_x(z))$, where φ denotes the morphism $X \to \mathbf{C}$, and we set

$$\overline{X^{\log}} = X^{\log} \times_{\mathbf{C}^{\log}} \overline{\mathbf{C}^{\log}} = \{((x, h_x), (t, b)) \big| |\varphi(x)| = t \text{ and } h_x(z) = e^b\} \ .$$

The canonical map $\overline{X^{\log}} \to X^{\log} : ((x, h_x), (t, b)) \mapsto (x, h_x)$ is a topological covering map with the Galois group $\Pi = \pi_1(S^1)$ and the generator σ of Π acting by $((x, h_x), (t, b)) \mapsto ((x, h_x), (t, b + 2\pi i))$. In particular, if $D = D(0; p)$ is the open disc in $\mathbf{C}$ with center at zero of radius $p > 0$ and provided with the induced log structure, then $D^{\log}$ and $\overline{D^{\log}}$ can be identified is with $[0, p) \times S^1$ and $[0, p) \times i\mathbf{R}$, respectively.

For a fine log pro-analytic space $\mathbf{X} = \text{"}\varprojlim_I\text{"} X_i$, we define $\mathbf{X}^{\log} = \text{"}\varprojlim_I\text{"} X_i^{\log}$ as a pro-topological space. For example, the maps $D^{\log} \to D$ give rise to a map $\mathbf{D}^{\log} = \text{"}\varprojlim\text{"}([0, p[\times S^1) \to \mathbf{D} = \text{"}\varprojlim\text{"} D(0; p)$, and it identifies the fundamental group Π of $\mathbf{C}^*$ with those of the pro-topological space $\mathbf{D}^{\log}$ and of the topological space $(\mathcal{D}_s^h)^{\log} = \mathbf{pt}^{\log}$. The universal coverings $\overline{D^{\log}} \to D^{\log}$ from Example 1.5.1(ii) give rise to a universal covering $\overline{\mathbf{D}^{\log}} \to \mathbf{D}^{\log}$, and there is a Π-equivariant open embedding $\overline{\mathbf{D}^*} \hookrightarrow \overline{\mathbf{D}^{\log}}$. The complement of $\overline{\mathbf{D}^*}$ in $\overline{\mathbf{D}^{\log}}$ is the universal covering $\overline{\mathbf{pt}^{\log}} = i\mathbf{R}$ of $\mathbf{pt}^{\log} = S^1$.

For a vertical log pro-analytic space $\mathbf{X}$ over $\mathbf{D}$, we set $\overline{\mathbf{X}^{\log}} = \mathbf{X}^{\log} \times_{\mathbf{D}^{\log}} \overline{\mathbf{D}^{\log}}$. (If $\mathbf{X}$ is just a log analytic space X over $\mathbf{D}_s = \mathbf{pt}$, then $\overline{X^{\log}} = X^{\log} \times_{\mathbf{pt}^{\log}} \overline{\mathbf{pt}^{\log}}$.) The group $\Pi = \pi_1(\mathbf{C}^*)$ acts on $\overline{\mathbf{X}^{\log}}$ (through the action on $\overline{\mathbf{D}^{\log}}$). There is the following commutative diagram with cartesian squares

$$\begin{tikzcd}
& \overline{\mathbf{X}^{\log}} \arrow[d, "\nu'"] & \overline{\mathbf{X}_s^{\log}} \arrow[l, "\overline{i^{\log}}"'] \arrow[d, "\nu"] \\
\mathbf{X}_{\overline{\eta}} \arrow[ur, "\overline{j^{\log}}"] \arrow[d] & \mathbf{X}^{\log} \arrow[d, "\tau'"] & \mathbf{X}_s^{\log} \arrow[l, "i^{\log}"'] \arrow[d, "\tau"] \\
\mathbf{X}_\eta \arrow[ur, "j^{\log}"] \arrow[r, "j"] & \mathbf{X} & \mathbf{X}_s \arrow[l, "i"']
\end{tikzcd}$$

Since the restriction of the log structure to $\mathbf{X}_\eta$ is trivial, the map $\tau' : \mathbf{X}^{\log} \to \mathbf{X}$ is a homeomorphism over the open subset $\mathbf{X}_\eta$ and, therefore, it gives rise to compatible open embeddings $j^{\log} : \mathbf{X}_\eta \hookrightarrow \mathbf{X}^{\log}$ and $\overline{j^{\log}} : \mathbf{X}_{\overline{\eta}} \to \overline{\mathbf{X}^{\log}}$ over j. We denote by $\overline{\tau}$ and $\overline{\tau}'$ the induced maps $\overline{\mathbf{X}_s^{\log}} \to \mathbf{X}_s$ and $\overline{\mathbf{X}^{\log}} \to \mathbf{X}$, respectively, and by $\overline{j}$ the canonical map $\mathbf{X}_{\overline{\eta}} \to \mathbf{X}$.

Any Π-module Λ defines a locally constant sheaf on each of the pro-analytic spaces $\mathbf{D}^*$, $\mathbf{D}^{\log}$ and $\mathbf{pt}^{\log}$ (whose fundamental group is Π), and the pullback of the latter to $\mathbf{X}_\eta$, $\mathbf{X}^{\log}$ and $\mathbf{X}_s^{\log}$ is denoted by $\Lambda_{\mathbf{X}_\eta}$, $\Lambda_{\mathbf{X}^{\log}}$ and $\Lambda_{\mathbf{X}_s^{\log}}$, respectively. Its pullback to $\mathbf{X}_{\overline{\eta}}$, $\overline{\mathbf{X}^{\log}}$ and $\overline{\mathbf{X}_s^{\log}}$ is a Π-sheaf which is denoted by $\underline{\Lambda}_{\mathbf{X}_{\overline{\eta}}}$, $\underline{\Lambda}_{\overline{\mathbf{X}^{\log}}}$ and $\underline{\Lambda}_{\overline{\mathbf{X}_s^{\log}}}$, respectively. We also denote by $\underline{\Lambda}_{\mathbf{X}_s}$ the constant Π-sheaf on $\mathbf{X}_s$ associated to Λ.

Theorem 1.5.2. *Let* $\mathbf{X}$ *be a vertical log pro-analytic space* $\mathbf{X}$ *log smooth over* $\mathbf{D}$. *Then for any* $\Lambda^{\cdot} \in D^b(\Pi\text{-Mod})$, *there are canonical isomorphisms in* $D^b(\mathbf{X}_s)$ *and* $D^b(\mathbf{X}_s(\Pi))$, *respectively,*

$$R\mathcal{I}^{\Pi}(R\Psi_\eta(\Lambda^{\cdot}_{\mathbf{X}_\eta})) \widetilde{\to} R\Theta(\Lambda^{\cdot}_{\mathbf{X}_\eta}) \widetilde{\to} R\tau_*(\Lambda^{\cdot}_{\mathbf{X}_s^{\log}}) \ ,$$

$$R\Psi_\eta(\mathbf{Z}_{\mathbf{X}_\eta}) \otimes^{\mathbf{L}}_{\mathbf{Z}_{\mathbf{X}_s}} \underline{\Lambda}^{\cdot}_{\mathbf{X}_s} \widetilde{\to} R\Psi_\eta(\Lambda^{\cdot}_{\mathbf{X}_\eta}) \widetilde{\to} R\overline{\tau}_*(\underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}_s^{\log}}}) \ .$$

Lemma 1.5.3. *Let* (X, M_X) *be a log smooth analytic space, and let* $\varphi : X' \to X$ *be the normalization of* X *provided with the log structure* $M_{X'}$ *which is the saturation of the sheaf of monoids* $\varphi^*(M_X)$ *in* $\mathcal{O}_{X'}$. *Then* X' *is an fs log smooth analytic space, and the canonical map* $X'^{\log} \to X^{\log}$ *is a homeomorphism.*

We notice that, for a log smooth analytic space (X, M_X), the homomorphism of sheaves of monoids $M_X \to \mathcal{O}_X$ is injective.

Proof. The statement is local in X and, therefore, we may assume that $X = \mathrm{Spec}(\mathbf{C}[P])^h$ for a fine monoid P. Then $X' = \mathrm{Spec}(\mathbf{C}[P'])^h$, where P' is the saturation of P in P^{gr}, and the log structure $M_{X'} \to \mathcal{O}_{X'}$ is defined by the canonical homomorphism $P' \to \mathbf{C}[P']$. Since the monoid $\mathbf{R}_+$ is uniquely divisible, one has $\mathrm{Hom}(P', \mathbf{R}_+) \widetilde{\to} \mathrm{Hom}(P, \mathbf{R}_+)$. Furthermore, since $P'^{gr} = P^{gr}$, one also has $\mathrm{Hom}(P', S^1) \widetilde{\to} \mathrm{Hom}(P, S^1)$. By Example 1.5.1(i), one has $X'^{\log} \widetilde{\to} X^{\log}$. ☐

For a log analytic space X, let X^* denote the open subset at which the log structure is trivial. Then $(X^*)^{\log} = X^*$ and, therefore, there is a canonical open immersion $j^{\log} : X^* \hookrightarrow X^{\log}$ over the open immersion $j : X^* \hookrightarrow X$.

Corollary 1.5.4. *Let* X *be a log smooth analytic space. Then each point of* $X^{\log}$ *has a fundamental system of open neighborhoods* V *such that* $(j^{\log})^{-1}(V)$ *is nonempty and contractible.*

Proof. If the log structure on X is saturated, the statement is a result of Ogus ([Ogus03, 3.1.2]). If X is arbitrary, let X' be its normalization provided with the log structure as in Lemma 1.5.3. Then $X'^* \widetilde{\to} X^*$ and $X'^{\log} \widetilde{\to} X^{\log}$, and the general case of the statement follows from the result of Ogus. ☐

Proof. The first isomorphism for the functor Θ was already mentioned in §1.5. Furthermore, by Corollary 1.5.4, there is a canonical isomorphism $\Lambda^{\cdot}_{\mathbf{X}^{\log}} \widetilde{\to} Rj^{\log}_*(\Lambda^{\cdot}_{\mathbf{X}_\eta})$ and, therefore, $Rj_*(\Lambda^{\cdot}_{\mathbf{X}_\eta}) \widetilde{\to} R\tau'_*(\Lambda^{\cdot}_{\mathbf{X}^{\log}})$. Since the map $\tau' : \mathbf{X}^{\log} \to \mathbf{X}$ is proper, we get the second isomorphism for the functor Θ.

One has $R\Psi_\eta(\Lambda^{\cdot}_{\mathbf{X}_\eta}) = i^*(R\overline{j}_*(\underline{\Lambda}^{\cdot}_{\mathbf{X}_{\overline{\eta}}}))$. Since $Rj^{\log}_*(\Lambda^{\cdot}_{\mathbf{X}_\eta}) = \Lambda^{\cdot}_{\mathbf{X}^{\log}}$, it follows that $\overline{Rj^{\log}_*}(\underline{\Lambda}^{\cdot}_{\mathbf{X}_{\overline{\eta}}}) = \underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}^{\log}}}$ and, therefore, $R\Psi_\eta(\Lambda^{\cdot}_{\mathbf{X}_\eta}) = i^*(R\overline{\tau}'_*(\underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}^{\log}}}))$. Furthermore, one has $R\overline{\tau}'_*(\underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}^{\log}}})\widetilde{\to}R\tau'_*(R\nu'_*(\underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}^{\log}}}))$. Since the map τ' is proper, we get $R\Psi_\eta(\Lambda^{\cdot}_{\mathbf{X}_\eta}) = R\tau_*(i^{\log *}(R\nu'_*(\underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}^{\log}}})))$. The map ν' is not proper, but it is a base change of the topological covering map $\overline{\mathbf{D}^{\log}} \to \mathbf{D}^{\log}$ and, in particular, ν' and ν are also topological covering maps. It follows that $i^{\log *}(R\nu'_*(\underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}^{\log}}}))\widetilde{\to}R\nu_*(\underline{\Lambda}^{\cdot}_{\overline{\mathbf{X}^{\log}_s}})$ and, therefore,

$$R\Psi_\eta(\Lambda^{\cdot}_{\mathbf{X}_\eta})\widetilde{\to}R\tau_*(R\nu_*(\underline{\Lambda}_{\overline{\mathbf{X}^{\log}_s}}))\widetilde{\to}R\overline{\tau}_*(\underline{\Lambda}_{\overline{\mathbf{X}^{\log}_s}})\ .$$

This gives the second isomorphism for the functor Ψ_η. It follows also that in order to get the first isomorphism for Ψ_η, it suffices to show that, given a log smooth morphism $X \to \mathbf{pt}$, for any $\mathbf{Z}$-torsion free Π-module Λ and any $q \geq 0$, the canonical map $R^q\overline{\tau}_*(\mathbf{Z}_{\overline{X^{\log}}}) \otimes_{\mathbf{Z}} \underline{\Lambda}_X \to R^q\overline{\tau}_*(\underline{\Lambda}_{\overline{X^{\log}}})$ is an isomorphism. For this we can disregard the action of Π on Λ and even assume that it is trivial. The stalk of the sheaf on the left hand side at a point $x \in X$ is the inductive limit of the cohomology groups $H^q(\overline{\tau}^{-1}(U), \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$ taken on the open neighborhoods U of x, and that on the right hand side is the inductive limit of the groups $H^q(\overline{\tau}^{-1}(U), \Lambda)$. Since for sufficiently small U the space $\overline{\tau}^{-1}(U)$ is a connected topological manifold with boundary, it follows that $H^q(\overline{\tau}^{-1}(U), \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda\widetilde{\to}H^q(\overline{\tau}^{-1}(U), \Lambda)$, and we get the required isomorphism for Ψ_η. □

2. Distinguished formal schemes

2.1. **Uniformization of special formal schemes.** Let k be a non-Archimedean field with nontrivial discrete valuation. All formal schemes considered in this section are special formal schemes over k°, all morphisms between them are assumed to be over k°, and the étale topology on a special formal scheme is the Grothendieck topology which is generated in the usual way by the étale morphisms introduced in [Ber96b, §2].

Given an element $\gamma \in k^\circ \backslash \{0\}$ and integers $e_1, \ldots, e_m \geq 1$ with $m \geq 1$, we set

$$A^{(\gamma)}_{e_1,\ldots,e_m} = k^\circ[T_1, \ldots, T_m]/(T_1^{e_1} \cdot \ldots \cdot T_m^{e_m} - \gamma)\ .$$

Definition 2.1.1. (i) A scheme $\mathcal{X}$ of locally finite type and flat over k° is said to be *distinguished* (resp. *semistable*) if each point $x \in \mathcal{X}_s$ has an étale neighborhood $\mathcal{X}' \to \mathcal{X}$ that admits an étale morphism $\mathcal{X}' \to \mathrm{Spec}(A)$ with $A = A^{(\gamma)}_{e_1,\ldots,e_m}[T_{m+1}, \ldots, T_n]$ for $\gamma \in k^{\circ\circ}\backslash(k^{\circ\circ})^2$ (resp. $e_1 = \ldots = e_m = 1$).

(ii) A special formal scheme $\mathfrak{X}$ over k° is said to be *distinguished* (resp. *semistable*) if étale locally it is isomorphic to a formal scheme of the form $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$, where $\mathcal{Y}$ is a distinguished (resp. semistable) scheme over k° and $\mathcal{Z}$ is a union of some of the irreducible components of $\mathcal{Y}_s$.

Remarks 2.1.2. (i) Every semistable scheme $\mathcal{X}$ over k° is normal, the generic fiber $\mathcal{X}_\eta$ is smooth over k, and the closed fiber $\mathcal{X}_s$ is a divisor with normal crossings. Every distinguished scheme $\mathcal{X}$ over k° is regular and, therefore, $\mathcal{X}_\eta$ is also regular. The support of the closed fiber $\mathcal{X}_s$ of any distinguished scheme $\mathcal{X}$ is a divisor with normal crossings and, if $\mathrm{char}(k) = 0$, $\mathcal{X}_\eta$ is smooth over k.

(ii) It follows from (i) that a distinguished (resp. semistable) formal scheme $\mathfrak{X}$ is regular (resp. normal), and the generic fiber $\mathfrak{X}_\eta$ is regular (resp. rig-smooth).

If $\mathrm{char}(k) = 0$, then generic fiber of any distinguished formal scheme is also rig-smooth.

For a special formal scheme $\mathfrak{X}$ over k°, we denote by $\widetilde{\mathfrak{X}}$ the closed (formal) subscheme of $\mathfrak{X}$ defined by the ideal generated by $k^{\circ\circ}$. It is called the *special fiber of* $\mathfrak{X}$. A *closed fiber of* $\mathfrak{X}$ is a scheme $\mathfrak{X}_s$ of locally finite type over $\widetilde{k}$ which is defined by an ideal of definition of $\mathfrak{X}$ that contains $k^{\circ\circ}$. It is also a closed fiber of $\widetilde{\mathfrak{X}}$ and, if $\mathfrak{X}$ is of locally finite type over k°, then the supports of both coincide. (We will be interested only in the étale site of $\mathfrak{X}_s$ and, when $\widetilde{k} = \mathbf{C}$, in the underlying topological space of the complex analytification $\mathfrak{X}_s^h$, which do not depend on the choice of an ideal of definition.)

We say that a morphism $\mathfrak{X}' \to \mathfrak{X}$ of special formal schemes over k° is *proper* if it is of finite type and the induced morphism between their closed fibers $\mathfrak{X}'_s \to \mathfrak{X}_s$ is proper. An example of a proper morphism is the blow-up of $\mathfrak{X}$ with center at a coherent subsheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$. It is a morphism of finite type $\varphi : \mathfrak{Y} = \mathrm{Bl}_{\mathcal{I}}(\mathfrak{X}) \to \mathfrak{X}$ such that (1) $\mathcal{I}$ generates an invertible subsheaf of ideals of $\mathcal{O}_{\mathfrak{Y}}$, and (2) every morphism of special formal schemes $\mathfrak{Z} \to \mathfrak{X}$, such that $\mathcal{I}$ generates an invertible subsheaf of ideals of $\mathcal{O}_{\mathfrak{Z}}$, goes through a unique morphism $\mathfrak{Z} \to \mathfrak{Y}$. In this case, the ideal $\mathcal{I}$ as well as the corresponding closed formal subscheme of $\mathfrak{X}$ are called centers of the blow-up. Recall the construction of blow-up (see [Tem08, §2.1]).

For every open affine subscheme $\mathfrak{U} = \mathrm{Spf}(A)$ of $\mathfrak{X}$, the restriction of $\mathcal{I}$ to $\mathfrak{U}$ corresponds to an ideal $\mathbf{a} \subset A$. Let $\mathcal{V} = \mathrm{Bl}_{\mathbf{a}}(\mathcal{U}) \to \mathcal{U}$ be the algebraic geometry blow-up of the scheme $\mathcal{U} = \mathrm{Spec}(A)$ with center $\mathbf{a}$. Then $\mathfrak{V} = \mathrm{Bl}_{\mathbf{a}}(\mathfrak{U})$ is the formal completion of $\mathrm{Bl}_{\mathbf{a}}(\mathcal{U})$ with respect to the ideal of definition of $\mathfrak{U}$. The blow-ups $\mathrm{Bl}_{\mathbf{a}}(\mathfrak{U})$ are compatible on intersections of open affine subschemes of $\mathfrak{X}$, and so one can glue all of them, and in this way one gets the required blow-up $\mathrm{Bl}_{\mathcal{I}}(\mathfrak{X})$. For example, if $f_1, \dots, f_n$ are fixed generators of the ideal $\mathbf{a}$, then $\mathcal{V} = \mathrm{Bl}_{\mathbf{a}}(\mathcal{U})$ is obtained by gluing the affine schemes $\mathcal{V}^i = \mathrm{Spec}(A_i)$, $1 \le i \le n$, where A_i is the quotient of by the f_i-torsion of

$$A'_i = A[T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n]/(f_i T_j - f_j)_{j \ne i}$$

and, therefore, $\mathrm{Bl}_{\mathbf{a}}(\mathfrak{U})$ is obtained by gluing the affine formal schemes $\mathfrak{V}^i = \mathrm{Spf}(\widehat{A}_i)$, $1 \le i \le n$, where $\widehat{A}_i$ is the quotient by the f_i-torsion of $\widehat{A}'_i$, the $k^{\circ\circ}$-adic completion of A'_i. Recall also that the composition of two blow-ups is a blow-up.

Theorem 2.1.3. *Suppose that* $\mathrm{char}(\widetilde{k}) = 0$*, and let* $\mathfrak{X}$ *be a quasicompact reduced special formal scheme flat over* k°*. Then*

(i) *there exists a blow-up* $\mathfrak{Y} \to \mathfrak{X}$ *which induces an isomorphism over the regular locus of* $\mathfrak{X}_\eta$ *and such that* $\mathfrak{Y}$ *is distinguished over* k°*;*
(ii) *if* $\mathfrak{X}$ *is distinguished, there exists an integer* $e \ge 1$ *such that the normalization* $\mathfrak{X}'$ *of* $\mathfrak{X}\widehat{\otimes}_{k^\circ} k'^\circ$*, where* $k' = k(\sqrt[e]{1}, \sqrt[e]{\varpi})$ *for a generator* ϖ *of* $k^{\circ\circ}$*, is a semistable formal scheme over* k'°*.*

Proposition 2.1.4. *Suppose that* $\mathrm{char}(\widetilde{k}) = 0$*. Then a special formal scheme* $\mathfrak{X}$ *flat over* k° *is distinguished if and only if it possesses the following properties:*

(1) $\mathfrak{X}$ *is regular;*
(2) *the support of* $\widetilde{\mathfrak{X}}$ *is a divisor with normal crossings;*
(3) *the support of* $\mathfrak{X}_s$ *is a union of some of the irreducible components of* $\widetilde{\mathfrak{X}}$*.*

A closed (formal) subscheme $\mathfrak{Y}$ of a special formal scheme $\mathfrak{X}$ is said to be a *divisor with normal crossings* if, for every open affine subscheme $\mathrm{Spf}(A)$ of $\mathfrak{X}$, the closed subscheme of $\mathrm{Spec}(A)$ that is induced by $\mathfrak{Y}$ is a divisor with normal crossings. (The empty subscheme is considered as a divisor with normal crossings.) The property (3) has the similar meaning. Namely, for every open affine subscheme $\mathfrak{U} = \mathrm{Spf}(A)$ of $\mathfrak{X}$, $\mathfrak{U}_s$ is a union of some of the irreducible components of the scheme $\mathrm{Spec}(\widetilde{A})$, where $\widetilde{\mathfrak{U}} = \mathrm{Spf}(\widetilde{A})$.

Proof. The direct implication easily follows from the definition of a distinguished formal scheme. Suppose therefore that a special formal scheme $\mathfrak{X}$ possesses the properties (1)-(3). In order to show that $\mathfrak{X}$ is distinguished, we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine. We set $\mathcal{X} = \mathrm{Spec}(A)$, $\widetilde{\mathcal{X}} = \mathrm{Spec}(A/I)$, where $I = \{a \in A \big| a^n \in k^{\circ\circ}A \text{ for some } n \geq 1\}$, and $\mathcal{X}_s = \mathrm{Spec}(A/J)$, where J is the Jacobson radical of A. Since the required property is local in the étale topology, we may assume that $\widetilde{\mathcal{X}}$ and $\mathcal{X}_s$ are divisors with strict normal crossings.

Let ϖ be a generator of $k^{\circ\circ}$, let $\mathbf{x}$ be a closed point of $\mathcal{X}_s$, and let $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ be the irreducible components of $\widetilde{\mathcal{X}}$ that contain the point $\mathbf{x}$. One has $1 \leq n \leq d$, where d is the dimension of $\mathcal{X}$. We assume that the irreducible components of $\mathcal{X}_s$ are $\mathcal{Z}_1, \dots, \mathcal{Z}_m$ with $1 \leq m \leq n$. Furthermore, let $t_1, \dots, t_d$ be a regular system of parameters of $\mathcal{O}_{\mathcal{X},\mathbf{y}}$ such that each t_i for $1 \leq i \leq n$ defines $\mathcal{Z}_i$ in an open neighborhood of $\mathbf{x}$ in $\mathcal{X}$. Then $\varpi = t_1^{e_1} \cdot \ldots \cdot t_n^{e_n} u$ for $e_1, \dots, e_n \geq 1$ and $u \in \mathcal{O}^*_{\mathcal{X},\mathbf{x}}$. Let $\mathcal{X}' = \mathrm{Spec}(A')$ be an open affine neighborhood of the point $\mathbf{x}$ in $\mathcal{X}$ such that $t_1, \dots, t_d \in A'$ and $u \in A'^*$. If $\mathbf{a}'$ is the ideal of A' generated by the elements ϖ and $t_1 \cdot \ldots \cdot t_m$, then $\widehat{\mathcal{X}}' = \mathrm{Spf}(\widehat{A}')$, where $\widehat{A}'$ is the $\mathbf{a}'$-adic completion of A'. Since $\mathrm{char}(\widetilde{k}) = 0$, the special k°-algebra $A'' = A'[\sqrt[e_1]{u}]$ is finite étale over A', i.e., $\mathcal{X}'' = \mathrm{Spec}(A'') \to \mathcal{X}'$ is a finite étale morphism. We replace t_1 by the element $t_1 \cdot \sqrt[e_1]{u}$ of B'', and so we may assume that $\varpi = t_1^{e_1} \cdot \ldots \cdot t_n^{e_n}$ in A''. If $\mathbf{a}''$ is the ideal of A'' generated by the elements ϖ and $t_1 \cdot \ldots \cdot t_m$, then $\widehat{\mathcal{X}}'' = \mathrm{Spf}(\widehat{A}'')$, where $\widehat{A}''$ is the $\mathbf{A}''$-adic completion of A''. Notice that the induced morphism $\widehat{\mathcal{X}}'' \to \widehat{\mathcal{X}}'$ is also finite étale. Let $\mathbf{x}''$ be a preimage of the point $\mathbf{x}$ in $\mathcal{X}''_s$.

Let $B = k^\circ[T_1, \dots, T_d]/(T_1^{e_1} \cdot \ldots \cdot T_n^{e_n} - \varpi)$, and let $\widehat{B}$ be the $\mathbf{b}$-adic completion of B, where $\mathbf{b}$ is the ideal generated by the elements ϖ and $T_1 \cdot \ldots \cdot T_m$. *We claim that one can replace $\mathcal{X}''$ by an open neighborhood of $\mathbf{x}''$ so that the morphism of special formal schemes $\widehat{\mathcal{X}}'' \to \mathfrak{Y} = \mathrm{Spf}(\widehat{B})$, which is induced to the homomorphism $B \to A'' : T_i \mapsto t_i$, is étale.* Indeed, by [Ber15, Lemma 3.2.5], one can shrink $\mathcal{X}''$ so that the induced morphism $\widehat{\mathcal{X}}''_s \to \mathfrak{Y}_s = \mathrm{Spec}(\widetilde{k}[T_1, \dots, T_d]/(T_1 \cdot \ldots \cdot T_m))$ is étale. By [Ber96b, 2.1(i)], there exists an étale morphism $\mathfrak{Z} = \mathrm{Spf}(C) \to \mathfrak{Y}$ with $\widehat{\mathcal{X}}''_s \widetilde{\to} \mathfrak{Z}_s$ over $\mathfrak{Y}_s$. Since C is formally étale over $\widehat{B}$, the latter isomorphism is induced by a unique homomorphism $C \to \widehat{A}''$ over $\widehat{B}$ ([EGA4$_0$, 19.3.10]). From [Bou, Ch. III, §2, n° 11, Prop. 14] it follows that the homomorphism $C \to \widehat{A}''$ is surjective. Since both rings are regular of the same dimension, we get $C \widetilde{\to} \widehat{A}''$ and the claim follows. $\square$

It is a minor consequence of the proof of Proposition 2.1.4 that, given a distinguished $\mathfrak{X}$, for any generator ϖ of $k^{\circ\circ}$ one can always find étale morphisms as in Definition 2.1.1 with $\gamma = \varpi$.

Proof of Theorem 2.1.3. (i) First of all, we recall a result of de Jong. Let $\mathfrak{Y} = \mathrm{Spf}(A)$ be a special affine formal scheme over k°, and set $\mathcal{Y} = \mathrm{Spec}(A)$. By [deJ95, Lemma 7.1.9], the map $y \mapsto \mathbf{n}_y$ that takes a point $y \in \mathfrak{Y}_\eta$ with $[\mathcal{H}(y) : k] < \infty$ to the preimage of $\mathbf{m}_y$ under the canonical homomorphism $\mathcal{A} = A \otimes_{k^\circ} k \to \mathcal{O}_{\mathfrak{Y}_\eta, y}$ is a bijection between the set of such points y and the set of maximal ideals of $\mathcal{A}$. Furthermore, this homomorphism induces an isomorphism $\widehat{\mathcal{A}}_y \widetilde{\to} \widehat{\mathcal{O}}_{\mathfrak{Y}_\eta, y}$ between the $\mathbf{n}_y$-adic completion of $\mathcal{A}$ and the $\mathbf{m}_y$-adic completion of $\mathcal{O}_{\mathfrak{Y}_\eta, y}$. These facts imply that the regular locus of $\mathfrak{Y}_\eta$ coincides with the preimage of the regular locus of the affine scheme $\mathcal{Y}_\eta = \mathrm{Spec}(\mathcal{A})$.

By Temkin's Theorem 1.1.13 from [Tem18], there exists a blow-up $\varphi : \mathfrak{Y} \to \mathfrak{X}$ with the following properties:

(a) for any open affine formal subscheme $\mathrm{Spf}(A) \subset \mathfrak{X}$, the corresponding blow-up of the affine scheme $\mathrm{Spec}(A)$ is an isomorphism over its regular locus;
(b) $\mathfrak{Y}$ possesses the property (1)-(3) of Proposition 2.1.4.

It follows that the special formal scheme $\mathfrak{Y}$ is distinguished and the induced morphism $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is an isomorphism over the regular locus of $\mathfrak{X}_\eta$. This gives the statement (i).

(ii) Since $\mathfrak{X}$ is quasicompact, it has a finite étale covering by affine formal schemes that admit an étale morphism to an affine formal scheme of the form as in Definition 2.1.1. Let e be a positive integer divisible by all of the numbers e_i's that appear in the construction of those schemes, and let $k' = k(\sqrt[e]{1}, \sqrt[e]{\varpi})$ and $\mathfrak{X}'$ the normalization of the formal scheme $\mathfrak{X}\widehat{\otimes}_{k^\circ} k'^\circ$. Then the induced morphism of special formal schemes $\mathfrak{X}' \to \mathfrak{X}$ is finite and, since $\mathfrak{X}_\eta$ is rig-smooth, it follows that $\mathfrak{X}'_\eta \widetilde{\to} \mathfrak{X}_\eta \widehat{\otimes}_k k'$. *We claim that the special formal scheme $\mathfrak{X}'$ is semistable.*

Indeed, in order to prove the claim, we may replace k by k' and $\mathfrak{X}$ by $\mathfrak{X}\widehat{\otimes}_{k^\circ} k'^\circ$. Since the normalization commutes with completion and étale morphisms, it suffices to show that the normalization $\mathcal{X}' = \mathrm{Spec}(A')$ of the scheme $\mathcal{X} = \mathrm{Spec}(A)$ with $A = k^\circ[T_1, \dots, T_d]/(T_1^{e_1} \cdot \ldots \cdot T_n^{e_n} - \varpi^l)$ such that k contains all e-th roots of one and l is divisible by all of e_i's is semistable over k°.

We set $v = \mathrm{g.c.d.}(e_1, \dots, e_n)$, $e'_i = \frac{e_i}{v}$, $l' = \frac{l}{v}$, denote by t_i the image of T_i in A, and set $t = t_1^{e'_1} \cdot \ldots \cdot t_n^{e'_n}$. One has $t^v = (\varpi^{l'})^v$ and, therefore, $\left(\frac{t}{\varpi^{l'}}\right)^v = 1$. Let A'' be the subalgebra of A' generated over A by the element $\frac{t}{\varpi^{l'}}$. Then $\mathcal{X}'' = \mathrm{Spec}(A'')$ is a disjoint union of the schemes $\mathcal{X}'_\zeta = \mathrm{Spec}(A'_\zeta)$, where ζ is a v-th root of one and $A'_\zeta = k^\circ[T_1, \dots, T_d]/(T_1^{e'_1} \cdot \ldots \cdot T_n^{e'_n} - \zeta\varpi^{l'})$. If ζ_1 is an l'-root of ζ, then $\zeta\varpi^{l'} = (\zeta_1\varpi)^{l'}$. Replacing A by any of A'_ζ's, we reduce the situation to the case $v = 1$.

In the case $v = 1$, the group M^{gr} of the monoid M generated by the elements $t_1, \dots, t_n$ and ϖ has no torsion, and $t_1^{e_1} \cdot \ldots \cdot t_n^{e_n} = \varpi^l$. The algebra A is the ring of polynomials $k^\circ[M][T_{n+1}, \dots, T_d]$ over the monoid algebra $k^\circ[M]$. Let $\overline{M}$ be the saturation of M in M^{gr}, i.e., $\overline{M} = \{p \in M \big| p^k \in M \text{ for some } k \geq 1\}$.

Lemma 2.1.5. *There exist elements $s_1, \dots, s_n \in \overline{M}$ which together with the element ϖ generate the monoid $\overline{M}$ and are such that $s_1 \cdot \ldots \cdot s_n = \varpi^r$ for $r = \frac{l}{\mathrm{l.c.m.}(e_1, \dots, e_n)}$.*

Proof. We set $m = \mathrm{l.c.m.}(e_1, \dots, e_n)$ and $r = \frac{l}{m}$. If $q_i = \frac{m}{e_i}$, then $\mathrm{g.c.d}(q_1, \dots, q_n) = 1$ and, therefore, $\mathrm{g.c.d.}(\widehat{q}_1, \dots, \widehat{q}_n) = 1$, where $\widehat{q}_i = q_1 \cdot \ldots \cdot q_{i-1} \cdot q_{i+1} \cdot \ldots \cdot q_n$. Let N

be the submonoid of M generated by the elements $t_1, \dots, t_n$ and ϖ^r, and consider the homomorphism $\alpha : N \to \mathbf{Z}_+^n$ to the additive monoid $\mathbf{Z}_+^n$ that takes t_i to $q_i f_i$ and ϖ^r to $\sum_{i=1}^n f_i$, where $f_1, \dots, f_n$ is the canonical basis of $\mathbf{Z}^n$. *We claim that α induces an isomorphism $N^{gr} \widetilde{\to} \mathbf{Z}^n$.*

Indeed, it suffices to show that the subgroup of $\mathbf{Z}^n$ generated by the vectors $\alpha(t_1), \dots, \alpha(t_n), \alpha(\varpi^r)$ coincides with the whole group. This subgroup contains the $n+1$ subgroups generated by n of the above elements. We now notice that the index of the subgroup of $\mathbf{Z}^n$ generated by n linearly independent vectors equals (up to a sign) to the determinant of the matrix formed by the coordinates of those vectors. In our case the determinants that correspond to those n subgroups are $\widehat{q}_1, \dots, \widehat{q}_n, q_1 \cdot \ldots \cdot q_n$, and the claim follows.

The claim implies that α induces an isomorphism of monoids $\overline{N} \widetilde{\to} \mathbf{Z}_+^n$, where $\overline{N}$ is the saturation of N in N^{gr}. If $s_1, \dots, s_n$ are the preimages of the basis vectors $f_1, \dots, f_n$, we get $s_1 \cdot \ldots \cdot s_n = \varpi^r$. $\square$

The algebra $A'' = k'^\circ[\overline{M}][T_{n+1}, \dots, T_d]$ is integral over $A = k^\circ[M][T_{n+1}, \dots, T_d]$ and, therefore, it is embedded in A'. By Lemma 2.1.5, one has

$$A'' = k^\circ[S_1, \dots, S_n, T_{n+1}, \dots, T_d]/(S_1 \cdot \ldots \cdot S_n - \varpi^r) \ .$$

Since $\mathrm{Spec}(A'')$ is a semistable scheme over k'°, it is normal. It follows that $A'' = A'$, and the required fact follows. $\square$

Recall (see [Ber15, §3.3) that an augmented simplicial formal scheme $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ is said to be a *compact hypercovering of* $\mathfrak{X}$ if all of the morphisms $\mathfrak{Y}_n \to \mathfrak{X}$ are of finite type and the augmented k-analytic space $\mathfrak{Y}_{\bullet,\eta} \to \mathfrak{X}_\eta$ is a compact hypercovering of $\mathfrak{X}_\eta$. If in addition all of the morphisms $\mathfrak{Y}_n \to \mathfrak{X}$ are proper, it is called a *proper hypercovering of* $\mathfrak{X}$. Furthermore, a hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ is said to be *distinguished* if all formal schemes $\mathfrak{Y}_n$ are distinguished.

Corollary 2.1.6. *If* $\mathrm{char}(\widetilde{k}) = 0$, *every quasicompact special formal scheme $\mathfrak{X}$ over k° admits a distinguished proper hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$.* $\square$

Remarks 2.1.7. (i) In the construction of the functor $R\Psi_\eta^h$, we use a weaker fact that every special formal scheme over k° admits a distinguished compact hypercovering. Existence of such a hypercovering is proved in the same way but, instead of functorial desingularization from [Tem18], one can apply Temkin's result on desingularization from [Tem08] to affine schemes of the form $\mathrm{Spec}(A)$ with an integral special k°-algebra A.

(ii) In the situation of §1.2, assume that the scheme $\mathcal{Y}$ is flat over $\mathcal{O}_{\mathbf{C},0}$ and regular, and that the support of $\widetilde{\mathcal{Y}}$ is a divisor with normal crossings and the support of $\mathcal{Y}_s$ is the union of some of the irreducible components of $\widetilde{\mathcal{Y}}$. Proposition 2.1.4 then implies that the formal completion $\widehat{\mathcal{Y}}$ of $\mathcal{Y}$ along $\mathcal{Y}_s$ is a distinguished formal scheme over $\widehat{\mathcal{O}}_{\mathbf{C},0}$.

(iii) Temkin's Theorem 1.1.8 from [Tem18] implies that, in the situation of Theorem 2.1.3, there exists a blow-up $\mathfrak{Y} \to \mathfrak{X}$, which induces an isomorphism over the regular locus of $\mathfrak{X}_\eta$, and a finite extension k' over k such that the normalization $\mathfrak{Y}'$ of $\mathfrak{Y}\widehat{\otimes}_{k^\circ} k'^\circ$ is semistable and regular (i.e., for $\mathfrak{Y}'$ one always has $\gamma \in k'^{\circ\circ} \backslash (k'^{\circ\circ})^2$ *and* $e_1 = \ldots = e_m = 1$ in Definition 2.1.1).

2.2. **Log special formal schemes.** Basic notions of logarithmic geometry for schemes are naturally extended to special formal schemes. Namely, a *pre-log structure* on a special formal scheme $\mathfrak{X}$ is a homomorphism of étale sheaves of monoids $\beta : M \to \mathcal{O}_{\mathfrak{X}}$. A pre-log structure is said to be a *log structure* if $\beta^{-1}(\mathcal{O}^*_{\mathfrak{X}}) \widetilde{\to} \mathcal{O}^*_{\mathfrak{X}}$. If $\beta : M \to \mathcal{O}_{\mathfrak{X}}$ is a pre-log structure, there is a homomorphism $M \to M^a$ to a log structure on $\mathfrak{X}$ such that any homomorphism $M \to N$ to a log structure on $\mathfrak{X}$ goes through a unique homomorphism $M^a \to N$. If $\mathfrak{X}$ is provided with a log structure, it is said to be a *log special formal scheme*. For example, every special formal scheme $\mathfrak{X}$ can be provided with the *trivial* log structure for which $M = \mathcal{O}^*_{\mathfrak{X}}$. If necessary, the underlying formal scheme of a log special formal scheme $\mathfrak{X}$ is sometimes denoted by $\mathring{\mathfrak{X}}$. Given a log special formal scheme $\mathfrak{X}$, any morphism of special formal schemes $\varphi : \mathfrak{Y} \to \mathring{\mathfrak{X}}$, gives rise to a homomorphism $\varphi^{-1}(M_{\mathfrak{X}}) \to \mathcal{O}_{\mathfrak{Y}}$ from the inverse image of $M_{\mathfrak{X}}$. The sheaf of monoids for the corresponding log structure on $\mathfrak{Y}$ is denoted by $\varphi^*(M_{\mathfrak{X}})$.

A morphism of log special formal schemes $\mathfrak{Y} \to \mathfrak{X}$ is a pair consisting of a morphism $\varphi : \mathring{\mathfrak{Y}} \to \mathring{\mathfrak{X}}$ and a homomorphisms of sheaves of monoids $\varphi^{-1}(M_{\mathfrak{X}}) \to M_{\mathfrak{Y}}$ which is compatible with the homomorphism $\varphi^{-1}(\mathcal{O}_{\mathfrak{X}}) \to \mathcal{O}_{\mathfrak{Y}}$. It gives rise to a homomorphism of sheaves $\varphi^*(M_{\mathfrak{X}}) \to M_{\mathfrak{Y}}$. A morphism is called *strict* if the latter is an isomorphism, i.e., $\varphi^*(M_{\mathfrak{X}}) \widetilde{\to} M_{\mathfrak{Y}}$. The category of log special formal schemes admits finite inverse limits which are constructed in the same way as for schemes (see [Kato89, (1.6)]).

Example 2.2.1. Every special formal scheme $\mathfrak{X}$ flat over k° (e.g., $\mathrm{Spf}(k^\circ)$) is provided with the following log structure, called *canonical*: for an étale morphism $\mathfrak{U} \to \mathfrak{X}$, $M(\mathfrak{U})$ consists of all elements of $\mathcal{O}(\mathfrak{U})$ whose image in $\mathcal{O}(\mathfrak{U}_\eta)$ is invertible. Notice that any morphism of special formal schemes is the underlying morphism of log special formal schemes provided with the canonical log structures.

A *k°-log special formal scheme* is a log special formal scheme $\mathfrak{X}$ which is flat over k° and provided with a morphism of log formal schemes $\mathfrak{X} \to \mathrm{Spf}(k^\circ)$ in which the log structure on $\mathrm{Spf}(k^\circ)$ is canonical. A k°-log special formal scheme $\mathfrak{X}$ is said to be *vertical* if the localization of $M_{\mathfrak{X}}$ with respect to $k^\circ \backslash \{0\}$ is a sheaf of groups. For example, if $\mathfrak{X}$ is provided with the canonical log structure, it is a vertical k°-log special formal scheme.

A *k°-log scheme* is a log scheme $\mathcal{X}$ with $\mathring{\mathcal{X}}$ of locally finite type over k° provided with a morphism of log schemes $\mathcal{X} \to \mathrm{Spec}(k^\circ)$ in which the log structure on $\mathrm{Spec}(k^\circ)$ is canonical, i.e., defined by $k^\circ \backslash \{0\} \hookrightarrow k^\circ$. (A scheme of locally finite type over k° is a locally finite union of open affine subschemes $\mathrm{Spec}(A)$ with finitely generated k°-algebras A.)

If $\mathfrak{X}$ is a k°-log special formal scheme, its closed fiber $\mathfrak{X}_s$ is provided with the log structure $i^*(M_{\mathfrak{X}})$, where i is the closed immersion $\mathfrak{X}_s \to \mathfrak{X}$ (notice that $\mathfrak{X}_s$ can be considered as a special formal scheme over k°). It is easy to see that this log structure on $\mathfrak{X}_s$ is the homomorphism $M_{\mathfrak{X}}/\mathcal{O}^1_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}_s}$, where $\mathcal{O}^1_{\mathfrak{X}}$ is the subsheaf of $\mathcal{O}^*_{\mathfrak{X}}$ consisting of the local sections which are congruent to 1 modulo the ideal of definition of $\mathfrak{X}$ that defines $\mathfrak{X}_s$. In particular, this defines a log structure on the scheme $\mathrm{Spec}(\widetilde{k})$, which is the closed fiber of the formal scheme $\mathrm{Spf}(k^\circ)$. It is an algebraic log point associated to the field k, and it is denoted by $\mathrm{pt}_{k^\circ_1}$. Every generator ϖ of the maximal ideal $k^{\circ\circ}$ of k° gives rise to a chart $P \to M_{k^\circ_1} = M_{\mathrm{pt}_{k^\circ_1}} =$

$k^\circ\backslash\{0\}/k^1$, where P is a free monoid generated by ϖ and $k^1 = \{a \in k \big| |a-1| < 1\}$. A k_1°*-log scheme* is a scheme of locally finite type over $\widetilde{k}$ provided with a morphism to the log scheme $\mathrm{pt}_{k_1^\circ}$.

Examples 2.2.2. (i) Let $\mathcal{X}$ be a scheme of locally finite type over k°. Then any log structure $\beta : M_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$ gives rise to a log structure $\widehat{\beta} : M_{\widehat{\mathcal{X}}} \to \mathcal{O}_{\widehat{\mathcal{X}}}$ on the formal completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along its closed fiber $\mathcal{X}_s = \mathcal{X} \otimes_{k^\circ} \widetilde{k}$, which is the inverse image of the log structure β with respect to the canonical morphism of locally ringed spaces $\widehat{\mathcal{X}} \to \mathcal{X}$. Of course, of β is k°-log, then so is $\widehat{\beta}$. In this case, the canonical morphism of k_1°-log schemes $(\widehat{\mathcal{X}})_s \to \mathcal{X}_s$ (which is the identity on the underlying schemes) is an isomorphism. If in addition, the restriction of β to $\mathcal{X}_\eta$ is the trivial log structure, then $\widehat{\beta}$ is vertical over k°.

(ii) Given a log (resp. k°-log) special formal scheme $\mathfrak{X}$, the log structure $\beta : M_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ on $\mathfrak{X}$ gives rise to a log (resp. k°-log) structure $\widehat{\beta}_{/\mathcal{Y}} : M_{\mathfrak{X}_{/\mathcal{Y}}} \to \mathcal{O}_{\mathfrak{X}_{/\mathcal{Y}}}$ on the formal completion $\mathfrak{X}$ along a subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, which is the inverse image of β with respect to the morphism $\mathfrak{X}_{/\mathcal{Y}} \to \mathfrak{X}$. In particular, in the situation of (i), given a subscheme $\mathcal{Y} \subset \mathcal{X}_s$, the log (resp., k°-log) structure β gives rise to a log (resp. k°-log) structure $\widehat{\beta}_{/\mathcal{Y}} : M_{\widehat{\mathcal{X}}_{/\mathcal{Y}}} \to \mathcal{O}_{\widehat{\mathcal{X}}_{/\mathcal{Y}}}$. If β is k°-log, then the k_1°-log structure on $\mathfrak{X}_s = \mathcal{Y}$ is canonically isomorphic to the restriction of the k_1°-log structure of $\mathcal{X}_s$ to $\mathcal{Y}$.

(iii) Let $(B, b) \to (\mathbf{C}, 0)$ be a morphism of complex analytic germs, and let $\mathcal{Y}$ be a scheme of finite type over $\mathcal{O}_{B,b}$. As in (i), any log structure $\beta : M_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}$ on $\mathcal{Y}$ gives rise to a log structure $\widehat{\beta} : M_{\widehat{\mathcal{Y}}} \to \mathcal{O}_{\widehat{\mathcal{Y}}}$ on the special formal scheme $\widehat{\mathcal{Y}}$ over $\widehat{\mathcal{O}}_{\mathbf{C},0}$ (see §1.2).

As for schemes, a log structure on $\mathfrak{X}$ is said to be *coherent* if locally in the étale topology it is associated to a pre-log structure defined by a homomorphism $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ (called a *chart* of the log structure), where $P_{\mathfrak{X}}$ is the constant sheaf for a finitely generated monoid P. If such P is integral, the log structure is said to be *fine* and if, in addition, P is saturated, it is said to be *fine saturated* or, for brevity, *fs*. For example, the canonical log structure on $\mathrm{Spf}(k^\circ)$ is fs, and it is associated by the pre-log structure defined by a homomorphism $P \to k^\circ$, where P is a free monoid generated by one element which maps to a generator of $k^{\circ\circ}$. If a log structure on $\mathfrak{X}$ is associated to a chart $P_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$, then its inverse image on the closed fiber $\mathfrak{X}_s$ is associated to the induced chart $P_{\mathfrak{X}_s} \to \mathcal{O}_{\mathfrak{X}_s}$.

The category of fine log special formal schemes admits finite inverse limits which are constructed in the same way as for schemes (see [Kato89, (2.8)]). For example, if $\mathfrak{X}$ is a fine k°-log formal scheme and k' is a finite extension of k, the formal scheme $\mathfrak{X}\widehat{\otimes}_{k^\circ} k'^\circ$, considered as the fiber product in the category of fine log special formal schemes, is a fine k'°-log formal scheme.

In [Kato89, §3], Kato introduced the notion of a log smooth (resp. log étale) morphism between fine log schemes. He also proves that a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ is log smooth if and only if locally in the étale topology there exist a chart $(P_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}, Q_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}, P \to Q)$ of φ such that the kernel and the torsion of the cokernel (resp. the kernel and the cokernel) of the homomorphism of groups $P^{gr} \to Q^{gr}$ are finite of orders invertible in $\mathcal{X}$ and the induced morphism of schemes $\mathcal{Y} \to \mathcal{X} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ is étale.

Definition 2.2.3. A k°-log special formal scheme $\mathfrak{X}$ is said to be *k°-log smooth* (resp. *formally k°-log smooth*) if locally in the étale topology $\mathfrak{X}$ it is isomorphic to the formal completion $\widehat{\mathcal{X}}$ (resp. $\widehat{\mathcal{X}}_{/\mathcal{Y}}$) for a vertical log smooth morphism $\mathcal{X} \to \mathrm{Spec}(k^\circ)$ (resp. and a subscheme $\mathcal{Y} \subset \mathcal{X}_s$).

2.3. **Formal log smoothness of distinguished formal schemes.** Every scheme $\mathcal{X}$ flat over k° is provided with the following log structure called *canonical*: for an étale morphism $\mathcal{U} \to \mathcal{X}$, $M(\mathcal{U})$ consists of all elements of $\mathcal{O}(\mathcal{U})$ whose image in $\mathcal{O}(\mathcal{U}_\eta)$ is invertible. In the examples we really need, $\mathcal{X}$ is a noetherian excellent regular scheme in which the closed fiber $\widetilde{\mathcal{X}}$ is a divisor with normal crossings. In this case the canonical log structure on $\mathcal{X}$ is fs. It is trivial outside $\widetilde{\mathcal{X}}$ and, locally in the étale topology, it is associated with a chart $P_\mathcal{X} \to \mathcal{O}_\mathcal{X}$ for the monoid generated by the regular parameters at a point $x \in \widetilde{\mathcal{X}}$ which define the irreducible components of $\widetilde{\mathcal{X}}$ passing through x.

In the situation of Example 2.2.2(ii), the canonical log structure on $\mathcal{X}$ defines a log structure on the formal completion $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ along a subscheme $\mathcal{Y} \subset \mathcal{X}_s$ which maps in a natural way to the canonical log structure on the special formal scheme $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ over k°. Similarly, in the situation of Example 2.2.2(iii), the canonical log structure on $\mathcal{Y}$ defines a log structure on the formal completion $\widehat{\mathcal{Y}}$ which maps in a natural way to the canonical log structure on the special formal scheme $\widehat{\mathcal{Y}}$ over $\widehat{\mathcal{O}}_{\mathbf{C},0}$.

For example, any semistable (resp. distinguished) scheme $\mathcal{X}$ over k° (resp. with $\mathrm{char}(\widetilde{k}) = 0$) provided with the canonical log structure is smooth (resp. log smooth) over k° and, therefore, the formal completion $\widehat{\mathcal{X}}$ (resp. $\widehat{\mathcal{X}}_{/\mathcal{Y}}$) provided with the log structure induced from $\mathcal{X}$ are k°-log smooth (resp. formally k°-log smooth).

Theorem 2.3.1. *Suppose that a scheme $\mathcal{X}$ admits an étale morphism $\mathcal{X} \to \mathcal{T}$, and either*

1. $\mathcal{T} = \mathrm{Spec}(k^\circ[T_1, \dots, T_n]/(T_1 \cdot \ldots \cdot T_m - \varpi^l))$, $l \geq 1$, *or*
2. $\mathrm{char}(\widetilde{k}) = 0$ *and* $\mathcal{T} = \mathrm{Spec}(k^\circ[T_1, \dots, T_n]/(T_1^{e_1} \cdot \ldots \cdot T_m^{e_m} - \varpi))$, $e_i \geq 1$,

where $1 \leq m \leq n$ and ϖ is a generator of $k^{\circ\circ}$. We set $\mathfrak{X} = \widehat{\mathcal{X}}_{/\mathcal{Y}}$ for a closed subscheme $\mathcal{Y} \subset \mathcal{X}_s$, and denote by P the multiplicative submonoid of $\mathcal{O}(\mathfrak{X})$ generated by the images of the coordinate functions T_i for $1 \leq i \leq m$ and the element ϖ. Then the log structure associated to the chart $P_\mathfrak{X} \to \mathcal{O}_\mathfrak{X}$ coincides with the canonical log structure on $\mathfrak{X}$.

Proof. In the case (1), the above facts easily follow from results from [Ber99, §5], especially Theorem 5.3. Namely, we can shrink $\mathcal{X}$ so that the étale morphism $\mathcal{X} \to \mathcal{T}$ induces a homeomorphism of skeletons $S(\widehat{\mathcal{X}}) \widetilde{\to} S(\widehat{\mathcal{T}})$. The skeleton $S(\widehat{\mathcal{X}})$ is a polytope, its intersection with $\mathfrak{X}_\eta$ is the complement of a union of proper faces of $S(\widehat{\mathcal{X}})$ and, in particular, $S(\widehat{\mathcal{X}}) \cap \mathfrak{X}_\eta$ contains the interior of $S(\widehat{\mathcal{X}})$. There is a retraction map $\tau : \widehat{\mathcal{X}}_\eta \to S(\widehat{\mathcal{X}})$ and, for $x \in S(\widehat{\mathcal{X}})$, the fiber $\tau^{-1}(x)$ is an affinoid domain with the maximal point x. If $x \in S(\widehat{\mathcal{X}}) \cap \mathfrak{X}_\eta$, then $\tau^{-1}(x) \subset \mathfrak{X}_\eta$. It follows that, for every function $h \in \mathcal{O}(\mathfrak{X}_\eta)$ and every point $y \in \mathfrak{X}_\eta$, one has $|h(y)| \leq |h(\tau(y))|$. If now f is as above, then the restriction of the real valued function $x \mapsto |f(x)|$ to the interior of $S(\widehat{\mathcal{X}})$ is equal to the function $x \mapsto |g(x)|$ for some $g \in P$. This implies that $f = gu$ for $u \in \mathcal{O}(\mathfrak{X}_\eta)^*$ with the property $|u(y)| = 1$ for all $y \in \mathfrak{X}_\eta$. Since the ring $\mathcal{O}(\mathfrak{X})$ is normal, a theorem of de Jong [deJ95, 7.4.1] implies that $u \in \mathcal{O}(\mathfrak{X})$. For the same reason, one has $u^{-1} \in \mathcal{O}(\mathfrak{X})$ and, therefore, $u \in \mathcal{O}(\mathfrak{X})^*$.

In the case (2), let v be the greatest common divisor of $e_1, \dots, e_m$. If $e_i = vq_i$, then the k°-subalgebra of $\mathcal{O}(\mathcal{T})$ generated by the element $t_1^{q_1} \cdot \ldots \cdot t_m^{q_m}$ is the ring of integers k'° of the field $k' = k(\sqrt[v]{\varpi})$, i.e., $\mathcal{T}$ and $\mathcal{Y}$ can be considered as distinguished schemes over k'°. This reduces the situation to the case $v = 1$.

Let e be a positive integer divisible by all of the numbers e_i's, $\mathcal{X}'$ the normalization of $\mathcal{Y} \otimes_{k^\circ} k'^\circ$, where $k' = k(\sqrt[e]{1}, \sqrt[e]{\varpi})$, $\mathcal{Y}'$ the preimage of $\mathcal{Y}$ in $\mathcal{X}'_s$, $\mathfrak{X}' = \widehat{\mathcal{X}}'_{/\mathcal{Y}'}$, P' the submonoid of $\mathcal{O}(\mathfrak{X}')$ generated by the functions from P and the element $\pi = \sqrt[e]{\varpi}$, and $\overline{P}'$ the saturation of P' in P'^{gr}. By Theorem 2.1.3(ii) and the previous case, the formal scheme $\mathfrak{X}'$ is semistable over k'° and the lift of the function f to $\mathfrak{X}'$ is of the form gv with $g \in \overline{P}'$ and $v \in \mathcal{O}(\mathfrak{X}')^*$. Notice that each element of P'^{gr} has the form $h\pi^r$, where $h \in P$ and $r \in \mathbf{Z}$ and, therefore, $f = hu$, where $h \in P$ and $u = \pi^r v$. Since $\mathfrak{X}'_\eta$ is a finite Galois covering of $\mathfrak{X}_\eta$, it follows that $u \in \mathcal{O}(\mathfrak{X}_\eta)^*$ and the function $x \mapsto |u(x)|$ on $\mathfrak{X}_\eta$ is a constant equal to $|\pi|^r$. *It suffices to show that the latter number belongs to* $|k^*|$, *i.e., r is divisible by e.* Indeed, suppose this is true. Then replacing h by $h\varpi^{\frac{r}{e}}$ and u by $u\varpi^{-\frac{r}{e}}$, we may assume that $h \in P^{gr}$ and $u \in \mathcal{O}(\mathfrak{X})^*$. Since the element h belongs to $\overline{P}'$ and the monoid P is saturated in P^{gr}, it follows that $h \in P$.

In order to verify the required fact, we may replace $\mathcal{Y}$ by any closed point $\mathbf{y}$ whose image in $\mathcal{T}_s$ is the point $\mathbf{t}$ at which all of the coordinate functions are zero. Replacing k by a finite unramified extension, we may assume that the point $\mathbf{y}$ is $\widetilde{k}$-rational. Then $\mathfrak{X} = \widehat{\mathcal{X}}_{/\{\mathbf{y}\}} \widetilde{\to} \widehat{\mathcal{T}}_{/\{\mathbf{t}\}}$. We may therefore assume that $\mathcal{X} = \mathcal{T}$, and the generic fiber $\mathfrak{X}_\eta$ has the following description. Let Y be the closed analytic subspace of $\mathbf{A}^m$ defined by the equation $T_1^{e_1} \cdot \ldots \cdot T_m^{e_m} = \varpi$, $\mathcal{V}$ the open subset $\{y \in Y \big| |T_i(y)| < 1 \text{ for all } 1 \le i \le m\}$, and D the open unit polydisc with center at zero in $\mathbf{A}^{n-m}$. Then $\mathfrak{X}_\eta \widetilde{\to} \mathcal{V} \times D$. Notice that the zero of D defines a closed immersion $\mathcal{V} \to \mathfrak{X}_\eta : x \mapsto (x, 0)$, and so it suffices to verify the necessary fact for the restriction of the function u to $\mathcal{V}$ instead of $\mathfrak{X}_\eta$.

The space $\mathcal{V}$ can be described as follows. Since the greatest common divisor of $e_1, \dots, e_m$ is one, we can find integers $l_1, \dots, l_m$ with $\sum_{i=1}^m e_i l_i = 1$. If $\mathcal{T}'$ is the torus in the n-dimensional affine space defined by the equation $T_1^{e_1} \cdot \ldots \cdot T_m^{e_m} = 1$, then $\mathcal{T}'^{\mathrm{an}} \widetilde{\to} Y : x = (x_1, \dots, x_m) \mapsto (x_1 \varpi^{l_1}, \dots, x_m \varpi_m^{l_m})$. The preimage of $\mathcal{V}$ in $\mathcal{T}'^{\mathrm{an}}$ is the open subset $\mathcal{U} = \{x \in \mathcal{T}'^{\mathrm{an}} \big| |T_i(x)| < |\varpi|^{-l_i} \text{ for all } 1 \le i \le m\}$. The latter is the preimage of the open subset U of the skeleton $S(\mathcal{T}')$, defined by the same inequalities in $S(\mathcal{T}')$, with respect to the retraction map $\tau : \mathcal{T}'^{\mathrm{an}} \to S(\mathcal{T}')$. The explicit description of analytic functions on $\tau^{-1}(U)$ in terms of convergent Laurent power series in T_i's easily implies that, for every invertible analytic function u on $\tau^{-1}(U)$ with constant absolute value $|u(x)|$, $|u(x)|$ is an element of $|k^*|$. $\square$

Corollary 2.3.2. *Any semistable (resp. distinguished) formal scheme over k° (resp. with* $\mathrm{char}(\widetilde{k}) = 0$*) provided with the canonical log structure is fs formally k°-log smooth (resp. k°-log smooth).* $\square$

Corollary 2.3.3. *In the situation of Remark 2.1.7(ii), the inverse image of the canonical log structure on $\mathcal{Y}$ coincides with the canonical log structure on the distinguished formal scheme $\widehat{\mathcal{Y}}$ over $\widehat{\mathcal{O}}_{\mathbf{C},0}$.* $\square$

3. The field K and associated groupoids

3.1. **Groupoids G_K, Π_K, $\Pi_{K_r^\circ}$, and $\Pi_{\mathcal{K}}$.** In this section and till the end of the paper, the capital letter K is used for a non-Archimedean field with nontrivial discrete valuation and such that $\mathbf{C} \subset K^\circ$ and $\mathbf{C} \widetilde{\to} \widetilde{K}$. Each generator ϖ of the maximal ideal $K^{\circ\circ}$ of K° induces a homomorphism $\mathcal{O}_{\mathbf{C},0} \to K^\circ$ that takes the coordinate function z of $\mathbf{C}$ to ϖ. It gives rise to an isomorphism $\widehat{\mathcal{O}}_{\mathbf{C},0} \widetilde{\to} K^\circ$ and an embedding $\mathcal{K} \hookrightarrow K$ of the fraction field $\mathcal{K}$ of $\mathcal{O}_{\mathbf{C},0}$ whose image is dense in K. (The valuation on K induces a valuation on $\mathcal{K}$, which does not depend on the element ϖ.) Let $\exp : \mathbf{C} \to \mathbf{C}^*$ be the exponential map $b \mapsto e^b$, and let $\mathcal{K}^{\mathrm{a}}$ be the algebraic closure of $\mathcal{K}$ that consists of the functions meromorphic in some half plane $\{b \in \mathbf{C} \big| \mathrm{Re}(b) < r\}$ and algebraic over $\mathcal{K}$. The field $\mathcal{K}^{\mathrm{a}}$ is an algebraic closure of $\mathcal{K}$, and it is generated by the functions $b \mapsto e^{\frac{b}{n}}$. We denote by $K^{(\varpi)}$ the field $\mathcal{K}^{\mathrm{a}} \otimes_{\mathcal{K}} K$, which is an algebraic closure of K. Let G_K be the groupoid whose objects are the fields $K^{(\varpi)}$ for generators ϖ of $K^{\circ\circ}$ and in which the set of morphisms $\mathrm{Hom}_{G_K}(K^{(\varpi)}, K^{(\varpi')})$ is the profinite set of isomorphisms of fields $K^{(\varpi)} \widetilde{\to} K^{(\varpi')}$ over K. For example, $\mathrm{Hom}_{G_K}(K^{(\varpi)}, K^{(\varpi)})$ is canonically isomorphic to the Galois group G of K, which is in its turn canonically isomorphic to $\varprojlim_n \mu_n$. The dense subgroup of G generated by the element $\sigma = (e^{\frac{2\pi i}{n}})_{n \geq 1}$ is denoted by Π. The canonical functor from G_K to the étale fundamental groupoid of K is an equivalence of categories.

Applying the above construction to field $\widehat{\mathcal{K}}$, we get a groupoid $G_{\widehat{\mathcal{K}}}$. Let $G_{\mathcal{K}}$ be the full subcategory of the latter whose objects correspond to the fields $\widehat{\mathcal{K}}^{(\varpi)}$ for generators ϖ of $\mathcal{K}^{\circ\circ}$. One has $\widehat{\mathcal{K}}^{(\varpi)} = \mathcal{K}^{\mathrm{a}} \otimes_{\mathcal{K}} \widehat{\mathcal{K}}$, where the tensor product is taken with respect to the embedding $\mathcal{K} \hookrightarrow \widehat{\mathcal{K}} : z \mapsto \varpi$.

Furthermore, for $r \geq 1$, we set $K_r^\circ = K^\circ/(K^{\circ\circ})^r$. It is a finitely dimensional $\mathbf{C}$-vector space, and the exponential function $\exp(\beta) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!}$ is well defined on it. Since $K^\circ \widetilde{\to} \varprojlim K_r^\circ$, we can provide the $\mathbf{C}$-algebra K° with the topology of a projective limit of finitely dimensional $\mathbf{C}$-vector spaces, and the same exponential function is well defined on K°. It gives rise to an exact sequence of abelian groups $0 \to 2\pi i\mathbf{Z} \to K^\circ \to (K^\circ)^* \to 0$ and to isomorphisms $\mathbf{R} \widetilde{\to} \mathbf{R}_+^*$ and $K^{\circ\circ} \widetilde{\to} K^1 = \{u \in (K^\circ)^* \big| |u-1| < 1\}$. The inverse isomorphisms to the latter give rise to an isomorphism $\mathbf{R}_+^* \cdot K^1 \widetilde{\to} \mathbf{R} + K^{\circ\circ} : v = au \mapsto \log(v) = \log|a| + \log(u)$.

Let Π_K be the groupoid whose objects are generators of $K^{\circ\circ}$. If ϖ and ϖ' are two generators, then $\varpi' = \alpha\varpi$ for $\alpha \in (K^\circ)^*$, and the set of morphism $\mathrm{Hom}_{\Pi_K}(\varpi, \varpi')$ is the set of elements $\beta \in K^\circ$ with $\exp(\beta) = \alpha^{-1}$. Composition of morphisms corresponds to the addition operation in K°. For example, $\mathrm{Hom}_{\Pi_K}(\varpi, \varpi)$ is the subgroup $\mathbf{Z}(1) = 2\pi i\mathbf{Z} \subset i\mathbf{R}$, which is canonically isomorphic to the group Π under the homomorphism that takes $2\pi i$ to the element σ. By the previous paragraph, if $\alpha \in \mathbf{R}_+^* \cdot K^1$, there exists a unique element $\beta \in \mathbf{R} + K^{\circ\circ}$ with $\exp(\beta) = \alpha^{-1}$. It follows also that, given ϖ and $\varpi' = \alpha\varpi$ as above and $\alpha = av$ with $a \in S^1 = \{c \in \mathbf{C}^* \big| |c| = 1\}$ and $v \in \mathbf{R}_+^* \cdot K^1$, there is a one-to-one correspondence

$$\mathrm{Hom}_{\Pi_K}(\varpi, \varpi') \widetilde{\to} \{b \in i\mathbf{R} \big| e^b = a^{-1}\} : \beta \mapsto \mathrm{Im}(\beta(0))i \ ,$$

where $\beta(0)$ denotes the "constant coefficient" of β, i.e., the complex number with $\beta - \beta(0) \in K^{\circ\circ}$. There is a faithful functor $\Pi_K \to G_K$ constructed as follows.

The field $\mathcal{K}^{\mathrm{a}}$ is generated over $\mathcal{K}$ by the functions $b \mapsto e^{\frac{b}{n}}$, $n \geq 1$. If ϖ_n is the image of the latter function in $K^{(\varpi)}$, then $\varpi_1 = \varpi$ and $\varpi_{mn}^m = \varpi_n$ for all $m, n \geq 1$, and $K^{(\varpi)} = \bigcup_{n=1}^{\infty} K(\varpi_n)$. If ϖ' is another generator of $K^{\circ\circ}$, then $\varpi' = \alpha\varpi$ for $\alpha \in (K^\circ)^*$. If ϖ'_n is the image of the function $b \mapsto e^{\frac{b}{n}}$ in $K^{(\varpi')}$, then each isomorphism $\varphi : K^{(\varpi)} \widetilde{\to} K^{(\varpi')}$ over K corresponds to a sequence of elements $(\alpha_n)_{n\geq 1}$ in $(K^\circ)^*$ with $\alpha_1 = \alpha$ and $\alpha_{mn}^m = \alpha_n$, where $\varphi(\varpi_n) = \alpha_n^{-1}\varpi'_n$. The functor $\Pi_K \to G_K$ takes ϖ to the algebraic closure $K^{(\varpi)}$ of K and a morphism $\varphi : \varpi \to \varpi'$, which corresponds to an element $\beta \in K^\circ$ with $\exp(\beta) = \alpha^{-1}$, to the isomorphism $\varphi_{\overline{K}} : K^{(\varpi)} \widetilde{\to} K^{(\varpi')}$ with $\varphi_{\overline{K}}(\varpi_n) = \exp(\frac{\beta}{n})\varpi'_n$.

Applying the above construction to the field $\widehat{\mathcal{K}}$, we get a groupoid $\Pi_{\widehat{\mathcal{K}}}$ and a faithful functor $\Pi_{\widehat{\mathcal{K}}} \to G_{\widehat{\mathcal{K}}}$. Each $\varpi \in \Pi_K$, gives rise to isomorphisms of groupoids $\Pi_{\widehat{\mathcal{K}}} \widetilde{\to} \Pi_K$ and $G_{\widehat{\mathcal{K}}} \widetilde{\to} G_K$ (that take $z \in \Pi_{\widehat{\mathcal{K}}}$ to ϖ). We now notice that, for an element $\beta \in \mathcal{K}^\circ$, one has $\exp(\beta) \in (\mathcal{K}^\circ)^*$. This means that one can define a full subcategory $\Pi_{\mathcal{K}} \subset \Pi_{\widehat{\mathcal{K}}}$ whose objects are generators of the maximal ideal $\mathcal{K}^{\circ\circ}$ of $\mathcal{K}^\circ$. The category $\Pi_{\mathcal{K}}$ is a subgroupoid of $G_{\mathcal{K}}$.

In what follows, we will also use the following groupoids which are equivalent to Π_K (resp. $\Pi_{\mathcal{K}}$). Let pt_{K° (resp. $\mathrm{pt}_{\mathcal{K}^\circ}$) be the scheme $\mathrm{Spec}(K^\circ)$ (resp. $\mathrm{Spec}(\mathcal{K}^\circ)$) provided with the canonical log structure. Generators of the maximal ideal of K° (resp. $\mathcal{K}^\circ$) can be viewed as elements of the monoid $M_{K^\circ} = M_{\mathrm{pt}_{K^\circ}} = K^\circ \backslash \{0\}$ (resp. $M_{\mathcal{K}^\circ} = M_{\mathrm{pt}_{\mathcal{K}^\circ}} = \mathcal{K}^\circ \backslash \{0\}$) whose image in the quotient $M_{K^\circ}/(K^\circ)^*$ (resp. $M_{\mathcal{K}^\circ}/(\mathcal{K}^\circ)^*$), which is a free monoid of rank one, is the generator of the latter. For $r \geq 1$, we denote by $\mathrm{pt}_{K_r^\circ}$ (resp. $\mathrm{pt}_{\mathcal{K}_r^\circ}$) the scheme $\mathrm{Spec}(K_r^\circ)$ (resp. $\mathrm{Spec}(\mathcal{K}_r^\circ)$ provided with the log structure which is induced from that on pt_{K° (resp. $\mathrm{pt}_{\mathcal{K}^\circ}$). The groupoid we are going to introduce is associated to the log scheme $\mathrm{pt}_{K_r^\circ}$ (resp. $\mathrm{pt}_{\mathcal{K}_r^\circ}$) and denoted by $\Pi_{K_r^\circ}$ (resp. $\Pi_{\mathcal{K}_r^\circ}$). Since $\mathcal{K}_r^\circ = \widehat{\mathcal{K}}_r^\circ$, it suffices to define $\Pi_{K_r^\circ}$.

Objects of $\Pi_{K_r^\circ}$ are elements of the monoid $M_{K_r^\circ} = M_{\mathrm{pt}_{K_r^\circ}} = (K^\circ \backslash \{0\})/K^r$, where $K^r = \{\alpha \in K^\circ \big| \alpha - 1 \in (K^{\circ\circ})^r\}$, whose image in the quotient $M_{K_r^\circ}/(K_r^\circ)^*$ is the generator of the latter. There is a canonical surjection from the set of objects of Π_K to that of $\Pi_{K_r^\circ}$, and we define morphisms in $\Pi_{K_r^\circ}$ by

$$\mathrm{Hom}_{\Pi_{K_r^\circ}}(\varpi, \varpi') = \{\beta \in K_r^\circ \big| \exp(\beta) = \alpha^{-1}\} \ ,$$

where $\alpha \in (K_r^\circ)^*$ is such that $\varpi' = \alpha\varpi$. If ϖ and $\varpi' \in \Pi_K$, there is a canonical bijection $\mathrm{Hom}_{\Pi_K}(\varpi, \varpi') \widetilde{\to} \mathrm{Hom}_{\Pi_{K_r^\circ}}(\varpi, \varpi')$. Here and later the image of an object ϖ of Π_K (i.e., a generator of $K^{\circ\circ}$) in $\Pi_{K_r^\circ}$ is denoted by ϖ, but the image of the latter in K_r° is denoted by $\widetilde{\varpi}$. Of course, the canonical functors $\Pi_K \to \Pi_{K_r^\circ}$ and $\Pi_{\mathcal{K}} \to \Pi_{\mathcal{K}_r^\circ}$ are equivalences (but not isomorphisms) of categories. As above, each $\varpi \in \Pi_{K_r^\circ}$ gives rise to an isomorphism of groupoids $\Pi_{\mathcal{K}_r^\circ} \widetilde{\to} \Pi_{K_r^\circ} : z \mapsto \varpi$.

A groupoid $\mathcal{P}$ is called connected, if the set of morphisms between any two of its objects is nonempty. For example, all of the above groupoids are connected. All groupoids considered here are assumed to be connected (and small). A groupoid $\mathcal{P}$ is said to be *abelian* if the groups $G^{(P)} = \mathrm{Aut}(P)$ for $P \in \mathcal{P}$ are abelian. If $\mathcal{P}$ is abelian, then all of the groups $G^{(P)}$ are canonically isomorphic. For example, the groupoids G_K, Π_K and $\Pi_{K_r^\circ}$, as well as $G_{\mathcal{K}}$, $\Pi_{\mathcal{K}}$ and $\Pi_{\mathcal{K}_r^\circ}$, are abelian, and the latter groups for them are G and Π, respectively.

Remark 3.1.1. As explained above, each element $\varpi \in \Pi_K$ defines for each $n \geq 1$ elements $\varpi_n \in K^{(\varpi)}$ such that $\varpi_1 = \varpi$ and $\varpi_{mn}^m = \varpi^n$ for all $m, n \geq 1$. Thus, for every nonzero rational number λ one can define an element $\varpi^\lambda \in K^{(\varpi)}$ by

$\varpi^\lambda = (\varpi_n)^m$ if $\lambda = \frac{m}{n}$ with $n > 0$. For a morphism $\varphi : \varpi \to \varpi' = \alpha\varpi$ in Π_K which corresponds to an element $\beta \in K^\circ$ with $\exp(\beta) = \alpha^{-1}$, the image of ϖ^λ under the corresponding isomorphism $\varphi_{\overline{K}} : K^{(\varpi)} \widetilde{\to} K^{(\varpi')}$ is $\exp(\lambda\beta)\varpi'^\lambda$.

3.2. **$\mathcal{P}$-spaces.** Let $\mathcal{P}$ be a groupoid. The category of *$\mathcal{P}$-spaces* is, by definition, the category of contravariant functors $\mathcal{P} \mapsto \mathcal{T}op : P \mapsto X^{(P)}$ to the category of topological spaces $\mathcal{T}op$. (In the same way one defines $\mathcal{P}$-spaces in other geometric categories such as complex and non-Archimedean analytic spaces, schemes, formal schemes and so on.) For a morphism $g : P \to P'$, we denote by ${}^t g$ the induced morphism $X^{(P')} \to X^{(P)}$. We say that a $\mathcal{P}$-space X is *single* if the corresponding functor takes each $P \in \mathcal{P}$ to the same space. We say that a $\mathcal{P}$-space X is *univocal* if, for any pair $P, P' \in \mathcal{P}$, it takes each morphism $P \to P'$ to the same map $X^{(P')} \to X^{(P)}$. If X is single and univocal, it is called *strict*. We say that a $\mathcal{P}$-space X is *trivial* if it is strict and takes each morphism in $\mathcal{P}$ to the identity map.

Every $\mathcal{P}$-space X is isomorphic to a single $\mathcal{P}$-space. Indeed, fix an object P_0 of $\mathcal{P}$ and, for every object $P \in \mathcal{P}$, fix a morphism $\alpha_P : P_0 \to P$ in $\mathcal{P}$. We define a single $\mathcal{P}$-space Y as follows: it takes each P to $X^{(P_0)}$ and each morphism $\varphi : P \to P'$ to ${}^t(\alpha_{P'}^{-1} \circ \varphi \circ \alpha_P) : X^{(P_0)} \to X^{(P_0)}$. The correspondence $P \mapsto {}^t(\alpha_P)$ defines an isomorphism $X \widetilde{\to} Y$. Notice that if the $\mathcal{P}$-space X is univocal, the $\mathcal{P}$-space Y is trivial, and it does not depend on P_0 up to a canonical isomorphism. Conversely, any $\mathcal{P}$-space, which is isomorphic to a trivial $\mathcal{P}$-space, is univocal.

Suppose that the action of a groupoid $\mathcal{P}$ on a $\mathcal{P}$-space X is *free*, i.e., for every $P \in \mathcal{P}$, the action of the group $G^{(P)}$ on $X^{(P)}$ is free. Then the quotient spaces $G^{(P)} \backslash X^{(P)}$ are well defined and form a univocal $\mathcal{P}$-space denoted by $\mathcal{P} \backslash X$.

The following examples of $\mathcal{P}$-spaces (for $\mathcal{P} = \Pi_K$, $\Pi_{K_r^\circ}$, and $\Pi_{\mathcal{K}}$) play an important role in the paper.

Examples 3.2.1. (i) Given a K-analytic space X, the correspondence

$$\overline{X} : \varpi \mapsto X^{(\varpi)} = X \widehat{\otimes}_K \widehat{K^{(\varpi)}}$$

is G_K-space and, in particular, a Π_K-space.

(ii) Given an integer $r \geq 1$, we set $\mathbf{pt}_{K_r^\circ} = (\mathrm{pt}_{K_r^\circ})^h$ and $\mathbf{pt}_{\mathcal{K}_r^\circ} = (\mathrm{pt}_{\mathcal{K}_r^\circ})^h$. Notice that the monoids of both $\mathrm{pt}_{K_r^\circ}$ and $\mathbf{pt}_{K_r^\circ}$ (resp. $\mathrm{pt}_{\mathcal{K}_r^\circ}$ and $\mathbf{pt}_{\mathcal{K}_r^\circ}$) coincide. The monoid $M_{\mathcal{K}_r^\circ} = M_{\mathbf{pt}_{\mathcal{K}_r^\circ}}$ has a canonical element, the image of the coordinate function z (which is also denoted by z), and so the space $\mathbf{pt}^{\log}_{\mathcal{K}_r^\circ}$ can be identified with $\mathbf{pt}^{\log} = S^1$ for the logarithmic point $\mathbf{pt}$ from §1.5. Then the universal covering $\overline{\mathbf{pt}^{\log}} = i\mathbf{R}$ of $\mathbf{pt}^{\log}$ defines a universal covering of $\mathbf{pt}^{\log}_{\mathcal{K}_r^\circ}$ which is denoted by $\mathbf{pt}^{(z)}_{\mathcal{K}_r^\circ}$. Each object $\varpi \in \Pi_{K_r^\circ}$ defines a morphism of log analytic spaces $\mathbf{pt}_{K_r^\circ} \to \mathbf{pt}_{\mathcal{K}_r^\circ}$ which, in its turn, defines a map $\mathbf{pt}^{\log}_{K_r^\circ} \to \mathbf{pt}^{\log}_{\mathcal{K}_r^\circ} = S^1$. Namely, it takes a point of $\mathbf{pt}^{\log}_{K_r^\circ}$ which corresponds to a homomorphism $h : M^{gr}_{K_r^\circ} \to S^1$, such that $h(a) = \frac{a}{|a|}$ for all $a \in \mathbf{C}^*$ and $h(u) = 1$ for all $u \in K_r^\circ$ congruent to one modulo the maximal ideal of K_r° (i.e., $u(0) = 1$), to $h(\varpi) \in S^1$. (Notice that such a homomorphism h is completely determined by its value $h(\varpi)$.) We set

$$\mathbf{pt}^{(\varpi)}_{K_r^\circ} = \mathbf{pt}^{\log}_{K_r^\circ} \times_{S^1} i\mathbf{R} \ ,$$

i.e., a point of $\mathbf{pt}^{(\varpi)}_{K_r^\circ}$ is a pair $(h, b) \in \mathbf{pt}^{\log}_{K_r^\circ} \times i\mathbf{R}$ with $h(\varpi) = e^b$. Each morphism $\varphi : \varpi \to \varpi' = \alpha\varpi$ in $\Pi_{K_r^\circ}$, i.e., an element $\beta \in K^\circ$ with $\exp(\beta) = \alpha^{-1}$, gives rise to a continuous map ${}^t\varphi : \mathbf{pt}^{(\varpi')}_{K^\circ_r} \to \mathbf{pt}^{(\varpi)}_{K^\circ_r}$ that takes a point $(h,b) \in \mathbf{pt}^{(\varpi')}_{K^\circ_r} \times i\mathbf{R}$ to the point $(h, b + \mathrm{Im}(\beta(0))i) \in \mathbf{pt}^{(\varpi)}_{K^\circ_r} \times i\mathbf{R}$. Thus, the correspondence $\varpi \mapsto \mathbf{pt}^{(\varpi)}_{K^\circ_r}$ is a $\Pi_{K^\circ_r}$-space over the space $\mathbf{pt}^{\log}_{K^\circ_r}$, and it will be denoted by $\overline{\mathbf{pt}^{\log}_{K^\circ_r}}$. The action of $\Pi_{K^\circ_r}$ on the latter is free, and there is a canonical isomorphism $\Pi_{K^\circ_r}\backslash\overline{\mathbf{pt}^{\log}_{K^\circ_r}} \widetilde{\to} \mathbf{pt}^{\log}_{K^\circ_r}$. Of course, there are canonical isomorphisms of topological $\Pi_{K^\circ_{r+1}}$-spaces $\overline{\mathbf{pt}^{\log}_{K^\circ_{r+1}}} \widetilde{\to} \overline{\mathbf{pt}^{\log}_{K^\circ_r}}$. (In §9, these spaces will be endowed with non-isomorphic ringed structures.)

(iii) Let X be a fine log complex analytic space over $\mathbf{pt}_{K^\circ_r}$. Then the correspondence

$$\overline{X^{\log}} : \varpi \mapsto X^{(\varpi)} = X^{\log} \times_{\mathbf{pt}^{\log}_{K^\circ_r}} \mathbf{pt}^{(\varpi)}_{K^\circ_r} = X^{\log} \times_{S^1} i\mathbf{R}$$

is a $\Pi_{K^\circ_r}$-space. A point of $X^{(\varpi)}$ is a pair $((x,h_x),c) \in X^{\log} \times i\mathbf{R}$ with $h_x(\varpi) = e^c$. Each morphism $\varpi \to \varpi'$ as in (ii) gives rise to a map

$$X^{(\varpi')} \to X^{(\varpi)} : ((x,h_x),c) \mapsto ((x,h_x), c + \mathrm{Im}(\beta(0))i) \ .$$

As at the end of (ii), the action of $\Pi_{K^\circ_r}$ on $\overline{X^{\log}}$ is free, and there is a canonical isomorphism of $\Pi_{K^\circ_r}$-spaces $\Pi_{K^\circ_r}\backslash\overline{X^{\log}} \widetilde{\to} X^{\log}$.

(iv) Let $\mathfrak{X}$ be a distinguished formal scheme over K°. Recall that $\mathfrak{X}$ is a regular formal scheme. For an integer $r \geq 1$, let $\mathcal{J}_r$ be the ideal of definition of $\mathfrak{X}$ such that, for an open subset $\mathfrak{U} \subset \mathfrak{X}$, $\mathcal{J}_r(\mathfrak{U})$ consists of the element $f \in \mathcal{O}(\mathfrak{U})$ with $\mathrm{ord}_Y(f) \geq r \cdot \mathrm{ord}_Y(\varpi)$ for every irreducible component Y of the closed fiber of $\mathfrak{U}$, where $\mathrm{ord}_Y(f)$ is the order of f at the generic point of Y. We denote by $\mathfrak{X}_{s_r}$ the closed subscheme of $\mathfrak{X}$ defined by the ideal $\mathcal{J}_r$ and provided with the induced log structure. It is an fs log scheme of finite type over the log scheme $\mathrm{pt}_{K^\circ_r}$. The complex analytification $X = \mathfrak{X}^h_{s_r}$ of $\mathfrak{X}_{s_r}$ is an fs log complex analytic space over $\mathbf{pt}_{K^\circ_r}$. As in (iii), one gets a $\Pi_{K^\circ_r}$-space

$$\overline{X^{\log}} : \varpi \mapsto X^{(\varpi)} = X^{\log} \times_{\mathbf{pt}^{\log}_{K^\circ_r}} \mathbf{pt}^{(\varpi)}_{K^\circ_r} \ .$$

Of course, all these $\Pi_{K^\circ_r}$-spaces (for different r's) are canonically homeomorphic but in §9 they will be provided with an extra structure that depends on r.

(v) Let $\mathbf{D}$ and $\mathbf{D}^*$ be the log pro-analytic spaces "$\varprojlim_p$"$D(0;p)$ and "$\varprojlim_p$"$D^*(0;p)$ from §1.5, where $D(0;p)$ is the open disc of radius $p > 0$ with center at zero. As in (ii), one can construct for each generator ϖ of $\mathcal{K}^{\circ\circ}$ universal coverings $\mathbf{D}^{(\varpi)}$ of $\mathbf{D}^{\log}$ and $\mathbf{D}^{*(\varpi)}$ of $\mathbf{D}^*$. Namely, let $D = D(0;p)$ be an open disc at which ϖ is convergent and is invertible at D^*. Then it induces a morphism $D \to \mathbf{C} : c \mapsto \varpi(c)$ and a map $D^{\log} = [0,p) \times S^1 \to \mathbf{C}^{\log} = \mathbf{R}_+ \times S^1 : (t,a) \mapsto \left(t|\gamma(ta)|, a\frac{\gamma(ta)}{|\gamma(ta)|}\right)$, where $\gamma = \frac{\varpi}{z}$. We set

$$D^{*(\varpi)} = D^* \times_{\mathbf{C}^*} \mathbf{C} \text{ and } D^{(\varpi)} = D^{\log} \times_{\mathbf{C}^{\log}} \overline{\mathbf{C}^{\log}} \ ,$$

where the first fiber product is taken with respect to the exponential map $\mathbf{C} \to \mathbf{C}^* : b \mapsto e^b$. A point of $D^{*(\varpi)}$ is a pair $(c,b) \in D^* \times \mathbf{C}$ with $\varpi(c) = e^b$, and a point of $D^{(\varpi)}$ is a pair $((t,a),(s,b)) \in D^{\log} \times \overline{\mathbf{C}^{\log}}$ with $t|\gamma(ta)| = s$ and $a\frac{\gamma(ta)}{|\gamma(ta)|} = e^b$. Given a morphism $\varphi : \varpi \to \varpi' = \alpha\varpi$ in $\Pi_{\mathcal{K}}$, i.e., an element $\beta \in \mathcal{K}^\circ$ with $\exp(\beta) = \alpha^{-1}$, we can shrink the disc D so that ϖ' is also convergent at D and invertible at

D^*. Then there induced maps ${}^t\varphi : D^{*(\varpi')} \to D^{*(\varpi)}$ with ${}^t\varphi(c,b) = (c, b+\beta(c))$ and ${}^t\varphi : D^{(\varpi')} \to D^{(\varpi)}$ with ${}^t\varphi((t,a),(s,b)) = ((t,a),(\frac{s}{|\alpha(ta)|}, b + \mathrm{Im}(\beta(ta))i))$. Thus, the spaces $D^{(\varpi)}$ and $D^{*(\varpi)}$ define pro-topological $\Pi_{\mathcal{K}}$-spaces $\varpi \mapsto \mathbf{D}^{(\varpi)}$ and $\varpi \mapsto \mathbf{D}^{*(\varpi)}$ denoted by $\overline{\mathbf{D}^{\log}}$ and $\overline{\mathbf{D}^*}$, respectively. Furthermore, the maps $D^{*(\varpi)} \to D^{(\varpi)} : (c,b) \mapsto ((|c|, \frac{c}{|c|}), (e^{\mathrm{Re}(b)}, \mathrm{Im}(b)i))$ define an open immersion of pro-topological $\Pi_{\mathcal{K}}$-spaces $\overline{\mathbf{D}^*} \hookrightarrow \overline{\mathbf{D}^{\log}}$ and, for every $r \geq 1$, the maps $\mathbf{pt}_{\mathcal{K}_r^\circ} \to D^{(\varpi)} : (h,b) \mapsto ((0,h(z)),(0,b))$ define a closed immersion of pro-topological $\Pi_{\mathcal{K}}$-spaces $\overline{\mathbf{pt}^{\log}_{\mathcal{K}_r^\circ}} \to \overline{\mathbf{D}^{\log}}$. Both immersions are complementary to each other.

(vi) Each fine vertical log germ of a complex analytic space (Y,X) over $(\mathbf{C},0)$ defines pro-topological $\Pi_{\mathcal{K}}$-spaces $\overline{Y(X)^{\log}} : \varpi \mapsto Y(X)^{(\varpi)} = Y(X)^{\log} \times_{\mathbf{D}^{\log}} \mathbf{D}^{(\varpi)}$ and $Y(X)_{\overline{\eta}} : \varpi \mapsto Y(X)^{(\varpi)}_{\eta} = Y(X)_\eta \times_{\mathbf{D}^*} \mathbf{D}^{*(\varpi)}$.

Remark 3.2.2. As was mentioned in the previous subsection, each $\varpi \in \Pi_K$ defines isomorphisms of groupoids $\Pi_{\widehat{\mathcal{K}}} \widetilde{\to} \Pi_K$ and $G_{\widehat{\mathcal{K}}} \widetilde{\to} G_K$, and this identifies the categories of $\Pi_{\widehat{\mathcal{K}}}$ and Π_K-spaces as well as of $G_{\widehat{\mathcal{K}}}$ and G_K-spaces. Similarly, each $\varpi \in \Pi_{K_r^\circ}$ defines an isomorphism of groupoids $\Pi_{\mathcal{K}_r^\circ} \widetilde{\to} \Pi_{K_r^\circ}$, and this identifies the categories of $\Pi_{\mathcal{K}_r^\circ}$ and $\Pi_{K_r^\circ}$-spaces.

3.3. **$\mathcal{P}$-sheaves, $\mathcal{P}$-modules and $\mathcal{P}$-cosheaves.** Let $\mathcal{P}$ be a groupoid, and let X be a $\mathcal{P}$-space. A *$\mathcal{P}$-sheaf of sets on* X is a family of sheaves $F^{(P)}$ on $X^{(P)}$ for $P \in \mathcal{P}$ provided with a system of isomorphisms $g_F : ({}^tg)^{-1}(F^{(P)}) \widetilde{\to} F^{(P')}$ such that $(hg)_F = h_F \circ ({}^th)^{-1}(g_F)$ for all morphisms $g : P \to P'$ and $h : P' \to P''$. (The same definition works of $\mathcal{P}$-sheaves of rings, fields and so on.) The family of $\mathcal{P}$-sheaves of sets on X forms a category, which is denoted by $\mathbf{T}_{\mathcal{P}}(X)$. Given a morphism of $\mathcal{P}$-spaces $\varphi : Y \to X$ and $\mathcal{P}$-sheaves E on X and F on Y, the correspondences $P \mapsto (\varphi^{(P)})^{-1}(E^{(P)})$ and $P \mapsto (\varphi^{(P)})_*(F^{(P)})$ are $\mathcal{P}$-sheaves on Y and X, respectively. In the following subsection we show that $\mathbf{T}_{\mathcal{P}}(X)$ is equivalent to the category of sheaves on a site and, in particular, that it is a topos.

If X is a one point space, then the corresponding category of $\mathcal{P}$-sheaves is just the category of covariant functors from $\mathcal{P}$ to that of sets (resp. rings, fields and so on). Such an object is called a *$\mathcal{P}$-set* (a *$\mathcal{P}$-ring*, a *$\mathcal{P}$-field* and so on). If W is a $\mathcal{P}$-ring, a *W-module* is a covariant functor that takes an object $P \in \mathcal{P}$ to an $W^{(P)}$-module $\Lambda^{(P)}$ and a morphism $g : P \to P'$ to a homomorphism $g_\Lambda : \Lambda^{(P)} \to \Lambda^{(P')}$ which is compatible with the homomorphism $g_W : W^{(P)} \to W^{(P')}$. If $W = \mathbf{Z}$ considered as a trivial $\mathcal{P}$-ring, such an object is called a *$\mathcal{P}$-module*. The abelian category of W-modules is denoted by W-Mod, and its derived category is denoted by $D(W\text{-Mod})$. If $W = \mathbf{Z}$, they are denoted by $\mathcal{P}$-Mod and $D(\mathcal{P}\text{-Mod})$, respectively. We will also denote by $D_c(\mathcal{P}\text{-Mod})$ the full subcategory of complexes whose cohomology are finitely generated abelian groups.

A $\mathcal{P}$-set is called *single*, *univocal*, *strict* or *trivial* if it possesses the properties from the corresponding definitions for $\mathcal{P}$-spaces. One shows in the same way that any $\mathcal{P}$-set (resp. univocal $\mathcal{P}$-set) is isomorphic to a single (resp. trivial) $\mathcal{P}$-set.

Remarks 3.3.1. (i) Every $\mathcal{P}$-set Λ defines a $\mathcal{P}$-sheaf $\underline{\Lambda}_X$ on every $\mathcal{P}$-space X. Namely, for $P \in \mathcal{P}$, $\Lambda_X^{(P)}$ is the constant sheaf on $X^{(P)}$ associated to the set $\Lambda^{(P)}$ with the isomorphisms g_Λ (for morphisms $g : P \to P'$ in $\mathcal{P}$) defined in the evident way.

(ii) Let X be a trivial $\mathcal{P}$-space. Then for every open subset $U \subset X$ (resp. a point $x \in X$), the set of sections $F(U)$ (resp. the stalk F_x) is a $\mathcal{P}$-set. Namely, it takes each object $P \in \mathcal{P}$ to the set $F^{(P)}(U)$ (resp. the stalk $F_x^{(P)}$) and each morphism $g : P \to P'$ to the map $g_F : F^{(P)}(U) \to F^{(P')}(U)$ (resp. $F_x^{(P)} \to F_x^{(P')}$). We denote by $F^{\mathcal{P}}$ the sheaf on X whose set of sections over an open subset $U \subset X$ consists of families $(f^{(P)})_P$ of elements $f^{(P)} \in F^{(P)}(U)$ with $g_F(f^{(P)}) = f^{(P')}$ for all morphisms $g : P \to P'$ in $\mathcal{P}$. Notice that, for every $P \in \mathcal{P}$, the projection $(f^{(P)})_P \mapsto f^{(P)}$ gives rise to an isomorphism $F^{\mathcal{P}} \widetilde{\to} (F^{(P)})^{G^{(P)}}$. We will denote by $\mathcal{I}^{\mathcal{P}} = \mathcal{I}_X^{\mathcal{P}}$ the left exact functor that takes a $\mathcal{P}$-sheaf F to the sheaf $F^{\mathcal{P}}$.

(iii) Suppose that the action of a groupoid $\mathcal{P}$ on a $\mathcal{P}$-space X is free and we are given a homeomorphism $\mathcal{P}\backslash X \widetilde{\to} Y$ with a trivial $\mathcal{P}$-space Y. Let π denote the map $X \to Y$. Then for any $\mathcal{P}$-sheaf A on X, $\pi_*(A)$ is a $\mathcal{P}$-sheaf on Y, and so there is a well defined sheaf $\pi_*^{\mathcal{P}}(A) = (\pi_*(A))^{\mathcal{P}}$. Conversely, for a sheaf B on Y, $f^{-1}(B)$ is a $\mathcal{P}$-sheaf on X. It follows from [Gro57, §5.1] that $B \widetilde{\to} \pi_*^{\mathcal{P}}(\pi^{-1}(B))$ and $\pi^{-1}(\pi_*^{\mathcal{P}}(A)) \widetilde{\to} A$. This means that the correspondences $B \mapsto \pi^{-1}(B)$ and $A \mapsto \pi_*^{\mathcal{P}}(A)$ are inverse to each other and establish an equivalence between the category of sheaves on Y and that of $\mathcal{P}$-sheaves on X.

Examples 3.3.2. (i) In the situation of Example 3.2.1(iii), every $\Pi_{K_r^\circ}$-set Λ defines a $\Pi_{K_r^\circ}$-sheaf $\underline{\Lambda}_{\overline{X^{\log}}}$ on the $\Pi_{K_r^\circ}$-space $\overline{X^{\log}}$. If ν denotes the map $\overline{X^{\log}} \to X^{\log}$, then the latter sheaf gives rise to the locally constant sheaf $\Lambda_{X^{\log}} = \nu_*^{\Pi_{K_r^\circ}}(\underline{\Lambda}_{\overline{X^{\log}}})$ on $X^{\log}$. Notice that, if Λ is a trivial $\Pi_{K_r^\circ}$-set (e.g., $\Lambda = \mathbf{Z}$), the latter sheaf coincides with $\underline{\Lambda}_{X^{\log}}$. In general, they are different objects.

(ii) By the same construction, in the situation of Example 3.2.1(vi) every $\Pi_{\mathcal{K}}$-set Λ defines sheaves $\Lambda_{Y(X)_\eta}$ and $\Lambda_{Y(X)^{\log}}$ on the pro-topological $\Pi_{\mathcal{K}}$-spaces $Y(X)_\eta$ and $Y(X)^{\log}$, respectively.

If W is a $\mathcal{P}$-ring, its inverse image W_X on a $\mathcal{P}$-space X is a $\mathcal{P}$-ring on X, and sheaves of left modules over the latter are said to be *sheaves of W-modules on X*, or just *W-modules on X*. An object of the derived category of abelian $\mathcal{P}$-sheaves on X will be said to be a W-module, if it is provided with a homomorphism from W to the $\mathcal{P}$-ring of endomorphism ring of the object. For example, any complex of sheaves of W-modules $E^\cdot$ on X is a W-module in the derived category of $\mathcal{P}$-sheaves. Furthermore, any quasi-isomorphism of complexes of abelian $\mathcal{P}$-sheaves $E^\cdot \to F^\cdot$ (from the above $E^\cdot$) provides $F^\cdot$ with the structure of a W-module in the derived category of abelian $\mathcal{P}$-sheaves.

Examples 3.3.3. (i) The field K (resp. $\mathcal{K}$) can be considered as a strict Π_K-field (resp. $\Pi_{\mathcal{K}}$-field) which will be denoted by $\underline{K}$ (resp. $\underline{\mathcal{K}}$). Namely, one associates to each morphism $\varpi \to \varpi'$ in Π_K (resp. $\Pi_{\mathcal{K}}$) the automorphism that takes $f(\varpi)$ to $f(\varpi')$ for $f = \sum_n a_n T^n \in \mathbf{C}((T))$ (resp. $f = \sum_n a_n z^n \in \mathcal{K}$). In the same way one provides the ring of integers K° (resp. $\mathcal{K}^\circ$) and its quotients K_r° (resp. $\mathcal{K}_r^\circ$), $r \geq 1$, with the structure of a strict Π_K and $\Pi_{K_r^\circ}$-ring (resp. $\Pi_{\mathcal{K}}$ and $\Pi_{\mathcal{K}_r^\circ}$-ring) which will be denoted by $\underline{K}^\circ$ and $\underline{K}_r^\circ$ (resp. $\underline{\mathcal{K}}^\circ$ and $\underline{\mathcal{K}}_r^\circ$). Since $\mathcal{K}_r^\circ = \widehat{\mathcal{K}}_r^\circ$, $\underline{\mathcal{K}}_r^\circ$ is also a strict $\Pi_{\widehat{\mathcal{K}}}$-ring.

(ii) Let W_K (resp. $W_{\mathcal{K}}$) be the algebra of $\mathbf{C}$-linear endomorphisms of K (resp. $\mathcal{K}$) generated by multiplications by elements of K (resp. $\mathcal{K}$) and derivations $\frac{\partial}{\partial \varpi}$ for generators ϖ of the maximal ideal $K^{\circ\circ}$ (resp. $\mathcal{K}^{\circ\circ}$). If ϖ is a fixed generator, each element of W_K (resp. $W_{\mathcal{K}}$) has a unique representation in the form $g_n \frac{\partial^n}{\partial \varpi^n} + g_{n-1} \frac{\partial^{n-1}}{\partial \varpi^{n-1}} + \ldots + g_1 \frac{\partial}{\partial \varpi} + g_0$ with $n \geq 0$ and $g_i \in K$ (resp. $\mathcal{K}$). It can be considered as a strict Π_K-ring (resp. $\Pi_{\mathcal{K}}$-ring) which will be denoted by $\underline{W}_K$ (resp.

$\underline{W}_{\mathcal{K}}$). Namely, one associates to each morphism $\varpi \to \varpi'$ in Π_K (resp. $\Pi_{\mathcal{K}}$) the automorphism that takes $f(\varpi)$ to $f(\varpi')$ as in (i) and $\frac{\partial}{\partial\varpi}$ to $\frac{\partial}{\partial\varpi'}$. Notice that $\underline{K}$ (resp. $\underline{\mathcal{K}}$) is a $\underline{W}_K$-module (resp. $\underline{W}_{\mathcal{K}}$-module).

(iii) For a generator ϖ of $K^{\circ\circ}$ (resp. $\mathcal{K}^{\circ\circ}$), let δ_ϖ denote the derivation $\varpi\frac{\partial}{\partial\varpi}$ on K (resp. $\mathcal{K}$). Then $\delta_\varpi(\varpi^j) = j\varpi^j$ for all $j \geq 0$ and $\delta_\varpi = (1 + \frac{\delta_\varpi(\alpha)}{\alpha})\delta_{\varpi'}$ for $\varpi' = \alpha\varpi$ with $\alpha \in (K^\circ)$ (resp. $(\mathcal{K}^\circ)^*$). In particular, δ_ϖ preserves the subring K° (resp. $\mathcal{K}^\circ$) and all of its ideals. We denote by W_{K° (resp. $W_{\mathcal{K}^\circ}$) the K°-subalgebra of W_K (resp. $\mathcal{K}^\circ$-subalgebra of $W_{\mathcal{K}}$) generated by all of the operators δ_ϖ. This algebra is isomorphic to the algebra of noncommutative polynomials over K_r° (resp. $\mathcal{K}_r^\circ$) in one variable δ_ϖ and the relations $\delta_\varpi \cdot g - g \cdot \delta_\varpi = \delta_\varpi(g)$ for $g \in K^\circ$ (resp. $\mathcal{K}^\circ$). It can be considered as a strict Π_K-ring (resp. $\Pi_{\mathcal{K}}$-ring) which will be denoted by $\underline{W}_{K^\circ}$ (resp. $\underline{W}_{\mathcal{K}^\circ}$). Namely, one associates to each morphism $\varpi \to \varpi'$ in Π_K (resp. $\Pi_{\mathcal{K}}$) the automorphism that takes $f(\varpi)$ to $f(\varpi')$ as in (i) and δ_ϖ to $\delta_{\varpi'}$. Notice that $\underline{K}^\circ$ (resp. $\underline{\mathcal{K}}^\circ$) is a $\underline{W}_{K^\circ}$-module (resp. $\underline{W}_{\mathcal{K}^\circ}$-module).

(iv) For $r \geq 1$, let $W_{K_r^\circ}$ (resp. $W_{\mathcal{K}_r^\circ}$) be the quotient of W_{K° (resp. $W_{\mathcal{K}^\circ}$) by the ideal generated by $(K^{\circ\circ})^r$ (resp. $(\mathcal{K}^{\circ\circ})^r$)ABE9@$W_K$, $W_{\mathcal{K}}$, W_{K°, $W_{\mathcal{K}^\circ}$, $W_{K_r^\circ}$, $W_{\mathcal{K}_r^\circ}$: the algebras associated to K (and so on)—). This algebra is isomorphic to the algebra of noncommutative polynomials over K_r° (resp. $\mathcal{K}_r^\circ$) in one variable δ_ϖ and the relation $\delta_\varpi \cdot \widetilde{\varpi} - \widetilde{\varpi} \cdot \delta_\varpi = \widetilde{\varpi}$. If $r = 1$, the algebra $W_{K_1^\circ}$ is in fact commutative, and all of the elements δ_ϖ are equal. As in (iii), one provides $W_{K_r^\circ}$ (resp. $W_{\mathcal{K}_r^\circ}$) with the structure of a strict Π_K-ring (resp. $\Pi_{\mathcal{K}}$-ring), and it will be denoted by $\underline{W}_{K_r^\circ}$ (resp. $\underline{W}_{\mathcal{K}_r^\circ}$). Since $\mathcal{K}_r^\circ = \widehat{\mathcal{K}}_r^\circ$, one has $\underline{W}_{\mathcal{K}_r^\circ} = \underline{W}_{\widehat{\mathcal{K}}_r^\circ}$. Notice that $\underline{K}_r^\circ$ (resp. $\underline{\mathcal{K}}_r^\circ$) is a $\underline{W}_{K_r^\circ}$-module (resp. $\underline{W}_{\mathcal{K}_r^\circ}$-module). Notice also that any $\underline{W}_{K_r^\circ}$-module (resp. $\underline{W}_{\mathcal{K}_r^\circ}$-module) can be also considered as a $\underline{W}_{K^\circ}$-module (resp. $\underline{W}_{\mathcal{K}^\circ}$-module).

A first example of a $\underline{W}_{K_r^\circ}$-module on a $\Pi_{K_r^\circ}$-space will be considered in §4.4. Other examples of sheaves of $\underline{W}_{K_r^\circ}$-modules and examples of complexes of $\Pi_{K_r^\circ}$-sheaves, which are $\underline{W}_{K_r^\circ}$-modules in the derived category of such complexes, will be considered in §9.

Remarks 3.3.4. (i) The field $\mathcal{K}$ (resp. $\widehat{\mathcal{K}}$) can be considered as a trivial $\Pi_{\mathcal{K}}$-ring (resp. Π_K-ring), and there is an isomorphism of $\Pi_{\mathcal{K}}$-fields $\mathcal{K}\widetilde{\to}\underline{\mathcal{K}}$ (resp. Π_K-fields $\widehat{\mathcal{K}}\widetilde{\to}\underline{K}$). Namely, it takes $\varpi \in \Pi_{\mathcal{K}}$ (resp. $\varpi \in \Pi_{\widehat{\mathcal{K}}}$) to the isomorphism $\mathcal{K}\widetilde{\to}\mathcal{K} : z \mapsto \varpi$ (resp. $\widehat{\mathcal{K}}\widetilde{\to}K : z \mapsto \varpi$). This isomorphism identifies the categories of $\mathcal{K}$ and $\underline{\mathcal{K}}$-vector spaces (resp. $\widehat{\mathcal{K}}$ and $\underline{K}$-vector spaces), which are $\Pi_{\mathcal{K}}$-modules (resp. Π_K-modules). Similarly, $\mathcal{K}_r^\circ$ can be considered as a trivial $\Pi_{\mathcal{K}_r^\circ}$-ring (resp. $\Pi_{K_r^\circ}$-ring), and there is an isomorphism of $\Pi_{\mathcal{K}_r^\circ}$-rings $\mathcal{K}_r^\circ\widetilde{\to}\underline{\mathcal{K}}_r^\circ$ (resp. $\Pi_{K_r^\circ}$-rings $\mathcal{K}_r^\circ\widetilde{\to}\underline{K}_r^\circ$).

(ii) As in (i), $W_{\mathcal{K}}$ (resp. $W_{\widehat{\mathcal{K}}}$) can be considered as a trivial $\Pi_{\mathcal{K}}$-ring (resp. Π_K-ring), and there is an isomorphism of $\Pi_{\mathcal{K}}$-rings $W_{\mathcal{K}}\widetilde{\to}\underline{W}_{\mathcal{K}}$ (resp. Π_K-rings $W_{\widehat{\mathcal{K}}}\widetilde{\to}\underline{W}_K$) that takes z to ϖ and $\frac{\partial}{\partial z}$ to $\frac{\partial}{\partial\varpi}$ and identifies $W_{\mathcal{K}}$ and $\underline{W}_{\mathcal{K}}$-modules (resp. $W_{\widehat{\mathcal{K}}}$ and $\underline{W}_K$-modules). Similarly, $W_{\mathcal{K}_r^\circ}$ can be considered as a trivial $\Pi_{\mathcal{K}_r^\circ}$-ring (resp. $\Pi_{K_r^\circ}$-ring), and there is an isomorphism of $\Pi_{\mathcal{K}_r^\circ}$-rings $W_{\mathcal{K}_r^\circ}\widetilde{\to}\underline{W}_{\mathcal{K}_r^\circ}$ (resp. $\Pi_{K_r^\circ}$-rings $W_{\mathcal{K}_r^\circ}\widetilde{\to}\underline{W}_{K_r^\circ}$).

Recall that a precosheaf of sets on a topological space X is a covariant functor $U \mapsto \Upsilon(U)$ from the category of open subsets of X to that of sets. A precosheaf is called a cosheaf if $\Upsilon(\emptyset) = \emptyset$ and, for any open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of an open subset $U \subset X$, one has $\Upsilon(\mathcal{U}) \widetilde{\to} \Upsilon(U)$, where $\Upsilon(\mathcal{U})$ is the set of equivalence classes on $\coprod_{i\in I} \Upsilon(U_i)$ with respect to the equivalence relation induced by the two canonical maps to it from the set $\coprod_{i,j\in I} \Upsilon(U_i \cap U_j)$. For example, given a continuous map of locally connected topological spaces $\varphi : Y \to X$, the correspondence $U \mapsto \pi_0(\varphi^{-1}(U))$ is a cosheaf of sets.

A $\mathcal{P}$-*cosheaf* of sets on a $\mathcal{P}$-space X is a family of cosheaves $\Upsilon^{(P)}$ on $X^{(P)}$ for $P \in \mathcal{P}$ provided with a compatible system of bijections $\Upsilon^{(P')}(({}^t g)^{-1}(U)) \widetilde{\to} \Upsilon^{(P)}(U)$ for all morphisms $g : P \to P'$ and all open subsets $U \subset X^{(P)}$. Given a $\mathcal{P}$-cosheaf Υ on X, for any $\mathcal{P}$-sheaf F on X the correspondence $U \mapsto F^\Upsilon(U)$ that takes an open subset U to the $\mathcal{P}$-set of maps $\Upsilon(U) \to F(U)$ is a $\mathcal{P}$-sheaf on X, denoted by F^Υ.

Example 3.3.5. For an fs log complex analytic space X over $\mathbf{pt}_{K_r^\circ}$, let τ and $\overline{\tau}$ denote the maps $X^{\log} \to X$ and $\overline{X^{\log}} \to X$, respectively. The correspondence $U \mapsto \pi_0(\overline{\tau}^{-1}(U))$ is a $\Pi_{K_r^\circ}$-cosheaf on the trivial $\Pi_{K_r^\circ}$-space X, denoted by $\overline{\pi}_{0,X}$. If Λ is a $\Pi_{K_r^\circ}$-module, there is a canonical isomorphism of $\Pi_{K_r^\circ}$-modules $\underline{\Lambda}_X^{\overline{\pi}_{0,X}} \widetilde{\to} \overline{\tau}_*(\underline{\Lambda}_{\overline{X^{\log}}})$. In §4, the cosheaf $\overline{\pi}_{0,X}$ will be described for a class of log analytic spaces in terms of their logarithmic structure.

3.4. **The category $\mathbf{T}_\mathcal{P}(X)$ as a topos.** Let $X(\mathcal{P})$ denote a pair consisting of a groupoid $\mathcal{P}$ and a $\mathcal{P}$-space X. If $\mathcal{P}$ is the trivial groupoid, then a $\mathcal{P}$-space is just a topological space. The pairs $X(\mathcal{P})$ form a category in which a morphism $\overline{\varphi} : X'(\mathcal{P}') \to X(\mathcal{P})$ consists of a functor $\nu_\varphi : \mathcal{P}' \to \mathcal{P}$ and a functor morphism $\varphi : X' \to X \circ \nu_\varphi$. The latter is a compatible family of continuous maps $\varphi_{P'} : X'^{(P')} \to X^{(\nu_\varphi P')}$ for all $P' \in \mathcal{P}'$. If $\mathcal{P}'$ is a subcategory of $\mathcal{P}$ and ν_φ is the canonical embedding, such a morphism is said to be a $\mathcal{P}'$-morphism.

Let $\acute{\mathrm{E}}\mathrm{t}(X(\mathcal{P}))$ denote the category of $\mathcal{P}$-morphisms $U(\mathcal{P}) \to X(\mathcal{P})$ such that all of the underlying maps $U^{(P)} \to X^{(P)}$ are local homeomorphisms. We denote by $X(\mathcal{P})_{\acute{e}t}$ the Grothendieck topology on $\acute{\mathrm{E}}\mathrm{t}(X(\mathcal{P}))$ generated by the pretopology for which the set of coverings of $(U(\mathcal{P}) \to X(\mathcal{P})) \in \acute{\mathrm{E}}\mathrm{t}(X(\mathcal{P}))$ consists of the families $\{U_i(\mathcal{P}) \overset{\overline{f}_i}{\to} U(\mathcal{P})\}_{i\in I}$ with $\bigcup_{i\in I} f_{i,P}(U_i^{(P)}) = U^{(P)}$ for all $P \in \mathcal{P}$, and we denote by $X(\mathcal{P})_{\widetilde{\acute{e}t}}$ the category of sheaves on $X(\mathcal{P})_{\acute{e}t}$ (the étale topos of $X(\mathcal{P})$). For example, $X_{\widetilde{\acute{e}t}}$ is the category of sheaves on the topological space X.

For a $\mathcal{P}$ space, we denote by $X^{(\mathcal{P})}$ the topological space $\coprod_{P\in\mathcal{P}} X^{(P)}$. Every $\mathcal{P}$-sheaf F can be considered as a sheaf on $X^{(\mathcal{P})}$. On the other hand, if $(U(\mathcal{P}) \to X(\mathcal{P})) \in \acute{\mathrm{E}}\mathrm{t}(X(\mathcal{P}))$, then $(U^{(\mathcal{P})} \to X^{(\mathcal{P})}) \in \acute{\mathrm{E}}\mathrm{t}(X^{(\mathcal{P})})$ and a covering in $\acute{\mathrm{E}}\mathrm{t}(X(\mathcal{P}))$ gives rise to a covering in $\acute{\mathrm{E}}\mathrm{t}(X^{(\mathcal{P})})$. This means that there is a morphism of sites $b : X^{(\mathcal{P})}_{\acute{e}t} \to X(\mathcal{P})_{\acute{e}t}$.

Proposition 3.4.1. *The inverse image functor for the morphism of sites* $b : X^{(\mathcal{P})}_{\acute{e}t} \to X(\mathcal{P})_{\acute{e}t}$ *gives rise to an equivalence of categories* $X(\mathcal{P})_{\widetilde{\acute{e}t}} \widetilde{\to} \mathbf{T}_\mathcal{P}(X)$.

Proof. Step 1. For $P \in \mathcal{P}$ and an open subset $U \subset X^{(P)}$, we introduce as follows a $\mathcal{P}$-space $\widetilde{U}$. It takes $P' \in \mathcal{P}$ to $\widetilde{U}^{(P')} = \coprod {}^t g(U)$, where the disjoint union is taken over all morphisms $g : P' \to P$. For a morphism $h : P'' \to P'$ in $\mathcal{P}$. For a morphism $h : P'' \to P'$ in $\mathcal{P}$, the induced map ${}^t h : X^{(P')} \to X^{(P'')}$ takes ${}^t g(U)$ to ${}^t h({}^t g(U)) = {}^t(gh)(U)$ and, therefore, it induces a map $\widetilde{U}^{(P')} \to \widetilde{U}^{(P'')}$, i.e., $\widetilde{U}$ is a $\mathcal{P}$-space. The identity morphism $P \to P$ defines a map $U \to \widetilde{U}^{(P)}$ which possesses the following universal property: any continuous map $U \to V$ to a $\mathcal{P}$-space V extends in a unique way to a morphism $\widetilde{U}(\mathcal{P}) \to V(\mathcal{P})$. Notice that, by the construction, the induced morphism $\widetilde{U}(\mathcal{P}) \to X(\mathcal{P})$ is a morphism in the category $\text{Ét}(X(\mathcal{P}))$.

Step 2. For a sheaf $\mathcal{F}$ on $X(\mathcal{P})$ and an open subset $U \subset X^{(P)}$ for $P \in \mathcal{P}$, we set $F^{(P)}(U) = \mathcal{F}(\widetilde{U}(\mathcal{P}))$. By universality of $\widetilde{U}(\mathcal{P})$, the sheaf $(b^*\mathcal{F})\big|_{X^{(P)}}$ is associated to the presheaf $U \mapsto F^{(P)}(U)$. *We claim that* $F^{(P)} \widetilde{\to} (b^*\mathcal{F})\big|_{X^{(P)}}$. Indeed, for this it suffices to verify that, given an open covering $\{U_i\}_{i\in I}$ of U, one has

$$F^{(P)}(U) \widetilde{\to} \mathrm{Ker}(\prod_i F^{(P)}(U_i) \rightrightarrows \prod_{i,j} F^{(P)}(U_i \cap U_j)) \ .$$

But this follows from the easy facts that $\{\widetilde{U}_i(\mathcal{P})\}_{i\in I}$ is a covering of $\widetilde{U}(\mathcal{P})$ in $X(\mathcal{P})_{\text{ét}}$ and that $(\widetilde{U_i \cap U_j})^{(P)} = \widetilde{U}_i^{(P)} \cap \widetilde{U}_j^{(P)}$ in $\widetilde{U}^{(P)}$ for all $i, j \in I$ and $P \in \mathcal{P}$.

Step 3. *We claim that the correspondence $P \mapsto F^{(P)}$ is a $\mathcal{P}$-sheaf on X.* (It will be denoted by $\widetilde{\mathcal{F}}$.) Indeed, for a morphism $g : P' \to P$ and an open subset $U \subset X^{(P)}$, the composition of the map $({}^t g)^{-1} : {}^t g(U) \widetilde{\to} U$ with the map $U \to \widetilde{U}$ is induced by a morphism $(\widetilde{{}^t gU})(\mathcal{P}) \widetilde{\to} \widetilde{U}(\mathcal{P})$. We get a map

$$F^{(P')}({}^t gU) = \mathcal{F}((\widetilde{{}^t gU})(\mathcal{P})) \widetilde{\to} \mathcal{F}(\widetilde{U}(\mathcal{P})) = F^{(P)}(U) \ .$$

This defines an isomorphism of sheaves $g_F : ({}^t g)^{-1}(F^{(P')}) \widetilde{\to} F^{(P)}$, and the isomorphisms defined in this way possess the required properties.

Step 4. Let F be a $\mathcal{P}$-sheaf on X. For $(V(\mathcal{P}) \to X(\mathcal{P})) \in \text{Ét}(X(\mathcal{P}))$ one has $b_*F(V(\mathcal{P})) = F(V^{(\mathcal{P})})$. An element of the latter is a collection of sections $f_P \in F^{(P)}(V^{(P)})$ for $P \in \mathcal{P}$. We define a sheaf $\overline{F}$ on $X(\mathcal{P})_{\text{ét}}$ by

$$\overline{F}(V(\mathcal{P})) = \{(f_P)_{P\in\mathcal{P}} \in F(V^{(\mathcal{P})}) \big| g_F(f_{P'}) = f_P \text{ for all } g : P' \to P \ \text{ in } \mathcal{P}\} \ .$$

We claim that $\widetilde{\overline{F}} \widetilde{\to} F$. Indeed, if U is an open subset of $X^{(P)}$ for some $P \in \mathcal{P}$, we have $\widetilde{\overline{F}}^{(P)}(U) = \overline{F}(\widetilde{U}(\mathcal{P}))$. An element of the latter is a collection of sections $f_{P'} \in F^{(P')}(\widetilde{U}^{(P')})$ for $P' \in \mathcal{P}$ with the property that $h_F(f_{P''}) = f_{P'}$ for all morphisms $h : P'' \to P'$ in $\mathcal{P}$. Since $\widetilde{U}^{(P')} = \coprod {}^t g(U)$, where the disjoint union is taken over all morphisms $g : P' \to P$, the section $f_{P'}$ is a collection of elements $f_{P',g} \in F^{(P')}({}^t gU)$ for $g \in \mathrm{Hom}(P', P)$. The above condition implies that $h_F(f_{P'',gh}) = f_{P',g}$ for all morphisms $h : P'' \to P'$ in $\mathcal{P}$. This implies that the sections $f_{P'}$ are completely determined by the element $f_{P,\mathrm{Id}_P} \in F(U)$ and, therefore, $\overline{F}(\widetilde{U}(\mathcal{P})) = F(U)$.

Step 5. *For $\mathcal{F} \in X(\mathcal{P})_{\widetilde{\text{ét}}}$, one has $\mathcal{F} \widetilde{\to} \overline{\widetilde{\mathcal{F}}}$.* Indeed, each object of $\text{Ét}(X(\mathcal{P}))$ can be covered by objects of the form $\widetilde{U}(\mathcal{P})$ for an open subset $U \subset X^{(P)}$ with $P \in \mathcal{P}$, and we have

$$\mathcal{F}(\widetilde{U}(\mathcal{P})) = \widetilde{\mathcal{F}}^{(P)}(U) = \overline{\widetilde{\mathcal{F}}}(\widetilde{U}(\mathcal{P})) \ . \qquad \square$$

In what folows, Proposition 3.4.1 is used in order to apply usual sheaf constructions to $\mathcal{P}$-sheaves.

Suppose we are given a $\mathcal{P}$-morphism $X'(\mathcal{P}) \to X(\mathcal{P})$. It gives rise to a commutative diagram of morphisms of sites

$$\begin{tikzcd} X'(\mathcal{P})_{\text{ét}} \arrow[r, "\overline{\varphi}"] & X(\mathcal{P})_{\text{ét}} \\ X'^{(\mathcal{P})}_{\text{ét}} \arrow[u, "b'"] \arrow[r, "\varphi"] & X^{(\mathcal{P})}_{\text{ét}} \arrow[u, "b"'] \end{tikzcd}$$

Furthermore, let W be a $\mathcal{P}$-ring. For an W-modules F on X', let $R\overline{\varphi}_*(F)$ be the higher direct image of F in the derived category of W-modules on X.

Corollary 3.4.2. *In the above situation, for any W-module F on X' there is a canonical isomorphism in the derived category of abelian sheaves on $X^{(\mathcal{P})}$*

$$b^*(R\overline{\varphi}_*F)\widetilde{\to}R\varphi_*(b'^*F) \ .$$

Proof. It suffices to verify that $b^*(R^q\overline{\varphi}_*F)\widetilde{\to}R^q\varphi_*(b'^*F)$ for all $q \geq 0$. If $q=0$, for every open subset $U \subset X^{(P)}$, $P \in \mathcal{P}$, one has

$$(b^*\overline{\varphi}_*F)(U) = \overline{\varphi}_*F(\widetilde{U}(\mathcal{P})) = F((X' \times_X \widetilde{U})(\mathcal{P}))$$

Since $X' \times_X \widetilde{U} = \widetilde{U}'$, where $U' = X'^{(P)} \times_{X^{(P)}} U$, the latter coincides with

$$F(\widetilde{U}'(\mathcal{P})) = (b'^*F)(U') = (\varphi_*b'^*F)(U) \ .$$

Thus, it remains to show that every W-module F on X' can be embedded in a W-module F' on X' with $R^q\overline{\varphi}_*(F')=0$ and $R^q\varphi_*(b'^*F')=0$ for all $q \geq 1$. For this we notice that the family of morphisms $x_{\widetilde{\text{ét}}} \to X'(\mathcal{P})_{\widetilde{\text{ét}}}$ for points $x \in X'^{(\mathcal{P})}$ is a conservative family of points of the topos $X'(\mathcal{P})_{\widetilde{\text{ét}}}$. This means that, if X'^d is the space $X'^{(\mathcal{P})}$ provided with the discrete topology and k is the morphism $X'^d_{\text{ét}} \to X'(\mathcal{P})_{\text{ét}}$, then for any sheaf F on $X'(\mathcal{P})_{\text{ét}}$ the canonical morphism of sheaves $F \to k_*k^*(F)$ is injective. By [SGA4, Exp. XVII, 6.4.2], for abelian F the sheaf $k_*k^*(F)$ on $X'(\mathcal{P})_{\text{ét}}$ is flabby. One has $k = b \circ l$, where l is the canonical map $X'^d \to X'$, and it is easy to see that there is a canonical isomorphism of sheaves $b'^*(k_*k^*(F))\widetilde{\to}l_*l^*(b'^*F)$. This implies that the sheaf $b'^*(k_*k^*(F))$ is flabby, and the required fact follows. □

Example 3.4.3. In the situation of Example 3.2.1(iii), the constant sheaf $(\underline{K}^\circ_r)_{\overline{X^{\log}}}$ is a sheaf of $\underline{W}_{K^\circ_r}$-modules on $\overline{X^{\log}}$. Corollary 3.4.2 implies that

$$R\overline{\tau}_*(\underline{K}^\circ_r)_{\overline{X^{\log}}} = R\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{K}^\circ_r$$

is a complex of sheaves of $\underline{W}_{K^\circ_r}$-modules on the trivial $\Pi_{K^\circ_r}$-space X, where $\overline{\tau}$ denotes the map $\overline{X^{\log}} \to X$. In particular, $R^q\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{K}^\circ_r$ are sheaves of $\underline{W}_{K^\circ_r}$-modules on $\overline{X^{\log}}$.

If X is a trivial $\mathcal{P}$-space, the left exact functor $\mathcal{I}^{\mathcal{P}} : \mathbf{T}_{\mathcal{P}}(X) \to \mathbf{T}(X)$ gives rise to an exact functor

$$R\mathcal{I}^{\mathcal{P}} : D^+(X(\mathcal{P})) \to D^+(X) \ .$$

Since for every $P \in \mathcal{P}$ the projection $(f^{(P)})_P \mapsto f^{(P)}$ gives rise to an isomorphism $F^{\mathcal{P}}\widetilde{\to}(F^{(P)})^{G^{(P)}}$, it also induces an isomorphism of functors $R\mathcal{I}^{\mathcal{P}}\widetilde{\to}R\mathcal{I}^{G^{(P)}}$.

The following statement will be applied in the situation of Example 3.2.1(iii) to the maps $\overline{\tau} : \overline{X^{\log}} \xrightarrow{\nu} X^{\log} \xrightarrow{\tau} X$.

Proposition 3.4.4. *Suppose that the action of a groupoid $\mathcal{P}$ on a $\mathcal{P}$-space $\overline{Y}$ is free, and we are given an isomorphism $\mathcal{P}\backslash\overline{Y}\widetilde{\to}Y$ and a continuous map $\tau : Y \to X$ with a trivial $\mathcal{P}$-space Y. Let $\overline{\tau}$ denote the induced map $\overline{Y} \to X$. Then for every $F^\cdot \in D^+(Y)$, there is a canonical isomorphism*

$$R\tau_*(F^\cdot)\widetilde{\to}R\mathcal{I}^{\mathcal{P}}(R\overline{\tau}_*(\overline{F}^\cdot)) ,$$

where $\overline{F}^\cdot$ is the pullback of $F^\cdot$ on $\overline{Y}$.

Recall that the quotient $\mathcal{P}$-space $\mathcal{P}\backslash\overline{Y}$ is univocal and, therefore, it is isomorphic to a trivial $\mathcal{P}$-space.

Proof. One has $\overline{\tau} = \tau \circ \nu$, where ν is the induced map $\overline{Y} \to Y$. Since for every injective $\mathcal{P}$-sheaf A on $\overline{Y}$ the $\mathcal{P}$-sheaf $\nu_*(A)$ is also injective, it follows that $F^\cdot\widetilde{\to}R\mathcal{I}^{\mathcal{P}}_Y(R\nu_*(\overline{F}^\cdot))$ and, therefore, $R\tau_*(F^\cdot)\widetilde{\to}R\tau_*(R\mathcal{I}^{\mathcal{P}}_Y(R\nu_*(\overline{F}^\cdot)))$. We now notice that there is an isomorphism of functors $\tau_* \circ \mathcal{I}^{\mathcal{P}}_Y\widetilde{\to}\mathcal{I}^{\mathcal{P}}_X \circ \tau_*$. Since the functor $\mathcal{I}^{\mathcal{P}}_Y$ takes injective $\mathcal{P}$-sheaves to flabby sheaves (see [Gro57, Proposition 5.1.3]), it follows that there is an isomorphism of functors $R\tau_* \circ R\mathcal{I}^{\mathcal{P}}_Y\widetilde{\to}R\mathcal{I}^{\mathcal{P}}_X \circ R\tau_*$, and we get the required isomorphism. □

3.5. **Distinguished W_R-modules.** Let R be either K°_r for $1 \le r < \infty$, or K°, or $\mathcal{K}^\circ$. In the latter two cases we set $r = \infty$. Let Π_R denote the corresponding groupoid (where $\Pi_{K^\circ} = \Pi_K$ and $\Pi_{\mathcal{K}^\circ} = \Pi_{\mathcal{K}}$) and, for $\varpi \in \Pi_R$, let $\sigma^{(\varpi)}$ be the automorphism of ϖ that corresponds to the number $2\pi i$. We denote by $R^{\circ\circ}$ the maximal ideal of R (it coincides with $K^{\circ\circ} \cdot R$, if $r < \infty$ or $R = K^\circ$, and with $\mathcal{K}^{\circ\circ} \cdot R$ if $R = \mathcal{K}^\circ$). The algebra R defines a Π_R-algebra $\underline{W}_R$. The abelian category of left $\underline{W}_R$-modules is denoted by $\underline{W}_R$-Mod. Since the ring R is commutative, all $\underline{W}_R$-modules have the canonical structure of an $\underline{R}$-bimodule (but the left action of $\underline{W}_R$ and the right action of $\underline{R}$ do not commute).

For a left $\underline{W}_R$-module D, a complex number λ and an element $\varpi \in \Pi_R$, we set $D^{(\varpi)}_\lambda = \{x \in D^{(\varpi)} \big| (\delta_\varpi - \lambda)^n(x) = 0$ for some $n \ge 1\}$. If λ is fixed, the correspondence $\varpi \mapsto D^{(\varpi)}_\lambda$ is a Π_R-submodule of D denoted by D_λ. For a subset $I \subset \mathbf{C}$, we set $D_I = \oplus_{\lambda\in I} D_\lambda$. We also denote by $\widetilde{D}$ the Π_R-module $D/(R^{\circ\circ} \cdot D)$.

Definition 3.5.1. A left $\underline{W}_R$-module D is said to be *distinguished* if it possesses the following properties:

(1) D is free of finite rank over R;
(2) there exists a finite subset $I = I(D) \subset \mathbf{Q} \cap [0, 1))$ such that the canonical map $D \to \widetilde{D}$ induces an isomorphism of Π_R-modules $D_I\widetilde{\to}\widetilde{D}$;
(3) for $\varpi \in \Pi_R$, the actions of $\sigma^{(\varpi)}$ and δ_ϖ on $D^{(\varpi)}$ are related by the equality $\sigma^{(\varpi)} = \exp(-2\pi i\delta_\varpi)$.

Remarks 3.5.2. (i) It follows from (2) that each element $x \in D^{(\varpi)}$ has a unique presentation in the form $\sum_{n\ge0} x_n\varpi^n$ with $x_n \in D^{(\varpi)}_I$. If $R = K^\circ_r$, the sum is finite (and one should write $\widetilde{\widetilde{\varpi}}$ instead of ϖ). (If $R = \mathcal{K}^\circ$, the sum is convergent, i.e., there exists $\rho > 0$ with $\sum_{n\ge0} ||x_n||\rho^n < \infty$, where $||\ ||$ is a fixed norm on the finitely generated $\mathbf{C}$-vector space $D^{(\varpi)}_I$.) It follows also that if $x \in D^{(\varpi)}_\lambda\backslash\{0\}$ for some $\lambda \in \mathbf{C}$, then $x \in \varpi^n D^{(\varpi)}_\mu$ for some $\mu \in I$ and $n \ge 0$ (in particular, $\lambda = \mu+n$).

(ii) The operator $\exp(-2\pi i\delta_\varpi)$ in (3) is well defined by the usual Taylor decomposition of the exponent function. It takes the above element x to the sum $\sum_{n\ge0} \exp(-2\pi i\delta_\varpi)(x_n)\varpi^n$.

(iii) If D is a distinguished $\underline{W}_R$-module, then for any $1 \le r' < r$, $D' = D/(R^{\circ\circ})^{r'}D$ is an distinguished $\underline{W}_{K^\circ_{r'}}$-module.

The full subcategory of $\underline{W}_R$-Mod consisting of distinguished $\underline{W}_R$-modules is denoted by $\underline{W}_R$-Dist. It follows from the definition that, for $D \in \underline{W}_R$-Dist, the operator $\sigma^{(\varpi)} = \exp(-2\pi i \delta_\varpi)$ on $\widetilde{D}$ is quasi-unipotent and, therefore, the correspondence $D \mapsto \widetilde{D}$ gives rise to a functor $\underline{W}_R$-Dist $\to \mathbf{C}\Pi_R$-Qun, where $k\Pi_R$-Qun for a field k denotes the category of quasi-unipotent Π_R-modules of finite dimension over k.

It is easy to construct a functor in the opposite direction which is left adjoint and inverse to the above one. For this we recall that the exponential map $N \mapsto \exp(N)$ on the set of nilpotent operators on a finitely dimensional vector space over a field of characteristic zero gives rise to a bijection with the set of unipotent operators, and the inverse map is given by the logarithmic map $U \mapsto \log(U)$. We extend the latter to the set of quasi-unipotent operators by $\log(E) = \frac{1}{n}\log(E^n)$, where n is a positive integer for which the operator E^n is unipotent. Suppose now that the ground field is $\mathbf{C}$. Given a quasi-unipotent operator E ona $\mathbf{C}$-vector space V, let $E = E_s \cdot E_u$ be its multiplicative Jordan decomposition, i.e., a unique decomposition of E as a product of commuting semisimple and unipotent operators E_s and E_u, respectively. In some basis $x_1, \dots, x_n$ of V, one has $E_s(x_j) = e^{-2\pi i \lambda_j} x_j$ for $\lambda_j \in \mathbf{Q} \cap [0,1)$, and we define an operator $\mathrm{Log}(E_s)$ by $\mathrm{Log}(E_s)(x_j) = -2\pi i \lambda_j x_j$ for all $1 \le j \le n$. This operator does not depend on the choice of the basis, and we set $\mathrm{Log}(E) = \mathrm{Log}(E_s) + \log(E)$. Notice that the latter is the additive Jordan decomposition of the operator $\mathrm{Log}(E)$, and one has $E = \exp(\mathrm{Log}(E))$. If F is an operator whose eigenvalues are imaginary numbers $-2\pi i \lambda$ with $\lambda \in \mathbf{Q} \cap [0,1)$ and $F = F_s + F_n$ is its additive Jordan decomposition, then $F_s = \mathrm{Log}(\exp(F_s))$ and, therefore, $F = \mathrm{Log}(\exp(F))$.

Let $V \in \mathbf{C}\Pi_R$-Qun. Then the tensor product $V \otimes_{\mathbf{C}} R$ is provided with the structure of a $\underline{W}_R$-module as follows. First of all, if $\varphi : \varpi \to \varpi'$ is a morphism in Π_R, then the corresponding $\mathbf{C}$-linear isomorphism $V^{(\varpi)} \otimes_{\mathbf{C}} R \widetilde{\to} V^{(\varpi')} \otimes_{\mathbf{C}} R$ is induced by the isomorphisms $\varphi_V : V^{(\varpi)} \to V^{(\varpi')}$ and $\varphi_R : R \widetilde{\to} R$. Furthermore, each nonzero element $x \in V^{(\varpi)} \otimes_{\mathbf{C}} R$ is represented in a unique way in the form $\sum_{n \ge 0} x_n \varpi^n$ for $x_n \in V^{(\varpi)}$ (as in Remark 3.5.2(ii), if $r < \infty$, one should write $\widetilde{\varpi}$ instead of ϖ). Then

$$\delta_\varpi \left(\sum_{n \ge 0} x_n \varpi^n \right) = \sum_{n \ge 0} \left(-\frac{1}{2\pi i} \mathrm{Log}(\sigma^{(\varpi)})(x_n) + n x_n \right) \varpi^n \ .$$

It is easy to see that this provides $V \otimes_{\mathbf{C}} R$ with the structure of a $\underline{W}_R$-module, and it will be denoted by $V \otimes_{\mathbf{C}} \underline{R}$. The correspondence $V \mapsto V \otimes_{\mathbf{C}} \underline{R}$ is functorial in V. The following proposition is trivial.

Proposition 3.5.3. *(i) The functor* $\underline{W}_R$-Dist $\to \mathbf{C}\Pi_R$-Qun $: D \mapsto \widetilde{D}$ *is an equivalence of categories;*

(ii) the functor $\mathbf{C}\Pi_R$-Qun $\to \underline{W}_R$-Dist $: V \mapsto V \otimes_{\mathbf{C}} \underline{R}$ *is left adjoint and inverse to that from (i).* $\square$

The notion of a distinguished $\underline{W}_R$-module is naturally extended to sheaves on a trivial Π_R-space X. Namely, we say that a sheaf of $\underline{W}_R$-modules $\mathcal{D}$ on X is *distinguished* if it is provided with a Π_R-subsheaf of $\mathbf{C}$-vectors spaces $\mathcal{D}_{\mathcal{I}} \subset \mathcal{D}$ such that, for every $\varpi \in \Pi_R$, $\mathcal{D}_{\mathcal{I}}^{(\varpi)}$ is preserved under the action of δ_ϖ and, for every point $x \in X$, $\mathcal{D}_x$ is a distinguished $\underline{W}_R$-module with $(\mathcal{D}_x)_{I(\mathcal{D}_x)} = (\mathcal{D}_{\mathcal{I}})_x$. In the same way one defines quasi-unipotent sheaves of $\mathbf{C}\Pi_R$-modules on X. The following statement is a trivial extension of Proposition 3.5.3.

Proposition 3.5.4. *(i) The functor $\mathcal{D} \mapsto \widetilde{\mathcal{D}} = \mathcal{D}/(R^{\circ\circ} \cdot \mathcal{D})$ is an equivalence between the category of distinguished $\underline{W}_R$-modules on X and that of quasi-unipotent $\mathbf{C}\Pi_R$-modules on X;*

(ii) the functor $\mathcal{V} \mapsto \mathcal{V} \otimes_{\mathbf{C}} \underline{R}$ is left adjoint and inverse to that from (i);

(iii) for every distinguished $\underline{W}_R$-module $\mathcal{D}$ on X, there is a canonical isomorphism

$$\widetilde{\mathcal{D}} \otimes_{\mathbf{C}} \underline{R} \widetilde{\to} \mathcal{D} . \qquad \square$$

Remark 3.5.5. Let $V \in \mathbf{Q}\Pi_R$-Qun. Then for every $\varpi \in \Pi_R$, $\log(\sigma^{(\varpi)})$ is a nilpotent $\mathbf{Q}$-linear operator on V. By the above, the tensor product $V \otimes_{\mathbf{Q}} \underline{R}$ has the structure of a $\underline{W}_R$-module. In particular, the operator δ_ϖ acts on the $\mathbf{C}$-vector space $V_{\mathbf{C}} = V \otimes_{\mathbf{Q}} \mathbf{C}$, and one has $\delta_\varpi = -\frac{1}{2\pi i}\mathrm{Log}(\sigma^{(\varpi)})$. It follows that $N_{\mathbf{C}}^{(\varpi)} = -\frac{1}{2\pi i}\log(\sigma^{(\varpi)})$, where $N_{\mathbf{C}}^{(\varpi)}$ denotes the nilpotent part from the additive Jordan decomposition of the operator δ_ϖ. Since $\log(\sigma^{(\varpi)})$ is defined on V, $N_{\mathbf{C}}^{(\varpi)}$ is induced by a nilpotent $\mathbf{Q}$-linear operator $N^{(\varpi)} : V \to V(-1) = V \otimes_{\mathbf{Z}} \mathbf{Z}(-1)$, where $\mathbf{Z}(-1) = \frac{1}{2\pi i}\mathbf{Z} \subset \mathbf{C}$. *We claim that, for any morphism $\varphi : \varpi \to \varpi'$ in Π_R, one has $\varphi_{V(-1)} \circ N^{(\varpi)} = N^{(\varpi')} \circ \varphi_V$.* Indeed, it suffices to verify the claim for the operators $N_{\mathbf{C}}^{(\varpi)}$ on the $\mathbf{C}$-vector spaces $V_{\mathbf{C}}$, i.e., $\varphi_{V_{\mathbf{C}}} \circ N_{\mathbf{C}}^{(\varpi)} = N_{\mathbf{C}}^{(\varpi')} \circ \varphi_{V_{\mathbf{C}}}$. Since $N_{\mathbf{C}}^{(\varpi)} = -\frac{1}{2\pi i}\log(\sigma^{(\varpi)})$, the latter follows from the equality $\varphi_{V_{\mathbf{C}}} \circ \sigma^{(\varpi)} = \sigma^{(\varpi')} \circ \varphi_{V_{\mathbf{C}}}$.

4. Distinguished log complex analytic spaces

4.1. **Definition and properties.** In this section, R is either K_r° for $1 \le r < \infty$, or $\mathcal{K}^\circ = \mathcal{O}_{\mathbf{C},0}$ (in the latter case we set $r = \infty$), and we denote by $R^{\circ\circ}$ the maximal ideal of R. The ring R gives rise to a log space $\mathbf{pt}_R$, which is the log point $\mathbf{pt}_{K_r^\circ}$, if $r < \infty$, and the log germ $(\mathbf{C}, 0)$, if $r = \infty$. We also consider both log spaces as one point spaces provided with the log structure defined by the homomorphism of monoids $M_R = R\backslash\{0\} \to R$.

Given integers $m, e_1, \dots, e_m \ge 1$ and an element $\varpi \in \Pi_R$, equal to z for $r = \infty$, we set

$$A_{e_1,\dots,e_m} = R[T_1, \dots, T_m]/(T_1^{e_1} \cdot \ldots \cdot T_m^{e_m} - \widetilde{\varpi}) .$$

The monoid freely generated by the coordinate functions $T_1, \dots, T_m$ defines an fs log structure on the scheme $\mathcal{Y} = \mathrm{Spec}(A_{e_1,\dots,e_m})$ and a log smooth morphism of log spaces $\mathcal{Y}^h \to \mathbf{pt}_R$.

Definition 4.1.1. (i) $r < \infty$: A log complex analytic space X over $\mathbf{pt}_R$ is said to be *distinguished* if every point $x \in X$ has an open neighborhood U which admits a strict open immersion over $\mathbf{pt}_R$ in the log space $Z = \mathrm{Spec}(B)^h$, where

$$B = A_{e_1,\dots,e_m}[T_{m+1}, \dots, T_n]/(T_1^{re_1} \cdot \ldots \cdot T_\mu^{re_\mu}),\ 1 \le \mu \le m \le n,$$

the log structure on Z is generated by that of $\mathcal{Y}$ as above.

(ii) $r = \infty$: A log germ (Y, X) of a complex analytic space over $(\mathbf{C}, 0)$ is said to be *distinguished* if each point $x \in X$ has an open neighborhood V in Y that admits a strict open immersion over $(\mathbf{C}, 0)$ in the log space $Z = \mathrm{Spec}(B)^h$, where

$$B = A_{e_1,\dots,e_m}[T_{m+1}, \dots, T_n]$$

such that $X \cap V$ is the preimage of the closed analytic subspace defined by the equation $T_1 \cdot \ldots \cdot T_\mu = 0$ with $1 \le \mu \le m \le n$.

Notice also that, for any point $x \in X$ one can find a strict open immersion as in Definition 4.1.1 such that all of the coordinate functions T_i are equal to zero at x.

Examples 4.1.2. (i) Let (Y, X) be a distinguished log germ over $(\mathbf{C}, 0)$. Given $r \ge 1$, let X_r be the closed analytic subspace of Y whose intersection with the chart V as in Definition 4.1.1(ii) is defined by the ideal generated by z^r and $T_1^{re_1} \cdot \ldots \cdot T_\mu^{re_\mu}$. The subspace X_r provided with the induced log structure is a distinguished log analytic space over $\mathbf{pt}_{\mathcal{K}_r^\circ}$. The support of the closed analytic subspace X_r in Y coincides with X. Given a generator ϖ of $K^{\circ\circ}$, one can consider X_r as a distinguished log analytic space over $\mathbf{pt}_{K_r^\circ}$ with respect to the isomorphism $\widehat{\mathcal{K}}^\circ \widetilde{\to} K^\circ : z \mapsto \varpi$. Notice that any distinguished log analytic space over $\mathbf{pt}_{K_r^\circ}$ is locally of the form X_r for any generator ϖ of $K^{\circ\circ}$ and a distinguished log germ (Y, X) over $(\mathbf{C}, 0)$.

(ii) Let $\mathfrak{X}$ be a distinguished formal scheme over K°. *Then $\mathfrak{X}^h_{s_r}$ is a distinguished log analytic space over* $\mathbf{pt}_{K_r^\circ}$. Indeed, let $\mathbf{x}$ be a closed point of $\mathfrak{X}_s$, and let $\widehat{\mathcal{X}}_{/\mathcal{Y}} \to \mathfrak{X}$ be an étale neighborhood of $\mathbf{x}$ such that $\mathcal{X}$ is a distinguished scheme over K° and $\mathcal{Y}$ the union of some of the irreducible components of $\mathcal{X}_s$. Let $\mathcal{J}_r$ be the coherent sheaf of ideals on $\mathcal{X}$ such that, for every open subset $\mathcal{U} \subset \mathcal{X}$, $\mathcal{J}_r(\mathcal{U})$ is generated by the elements $f \in \mathcal{O}(\mathcal{U})$ with $\mathrm{ord}_{\mathcal{Z}}(f) \ge r \cdot \mathrm{ord}_{\mathcal{Z}}(z)$ for each irreducible component $\mathcal{Z}$ of $\mathcal{U} \cap \mathcal{Y}$, where $\mathrm{ord}_{\mathcal{Z}}(f)$ is the order of f at the generic point of $\mathcal{Z}$. If $\mathcal{Y}_r$ the closed subscheme of $\mathcal{X}$ defined by the ideal $\mathcal{J}_r$ and provided with the induced log structure, then $\mathcal{Y}_r^h$ is a distinguished log analytic space over $\mathbf{pt}_{K_r^\circ}$. The above morphism gives rise to an étale morphism $\mathcal{Y}_r^h \to \mathfrak{X}^h_{s_r}$, which induces an isomorphism from an open neighborhood of a point $\mathbf{x}' \in \mathcal{Y}$ over $\mathbf{x}$ in $\mathcal{Y}_r^h$ and an open neighborhood of $\mathbf{x}$ in $\mathfrak{X}^h_{s_r}$.

(iii) Let X be a distinguished log analytic space over $\mathbf{pt}_{K_r^\circ}$. Given $1 \le r' \le r$, let $X_{r'}$ denote the closed analytic subspace defined by the ideal generated by $\widetilde{\varpi}^l$ and $T_1^{r'e_1} \cdot \ldots \cdot T_\mu^{r'e_\mu}$ on each chart V as in Definition 4.1.1(i). Then $X_{r'}$ is a distinguished log analytic space over $\mathbf{pt}_{K_{r'}^\circ}$, and canonical morphism $X_{r'} \to X$ is an exact closed immersion of log analytic spaces.

In this section we study distinguished log analytic spaces over $\mathbf{pt}_R$ from Definition 4.1.1(i) and log germs over $(\mathbf{C}, 0)$ from Definition 4.1.1(ii). The results obtained have similar formulation but slighly different interpretation. In order to consider them simulteneously, in the case $r = \infty$ we view X (from a distinguished log germ (Y, X)) as a topological space provided with the sheaf of local rings $\mathcal{O}_X = i^{-1}(\mathcal{O}_{Y(X)})$ and the log structure $M_X = i^{-1}(M_{Y(X)}) \to \mathcal{O}_X$, where i is the map $X \to Y(X)$. Other sheaves on X considered here are always induced from $Y(X)$ (as the sheaves $\mathcal{O}_X$ and M_X). We also denote by $X^{\log}$ and $\overline{X^{\log}}$ the preimage of X in $Y(X)^{\log}$ and $\overline{Y(X)^{\log}}$, respectively. Notice that, for every $r \ge 1$, there is a canonical exact closed immersion of log spaces $X_r \to X$, which induces a homeomorphism between the underlying topological spaces as well as homeomorphisms $X_r^{\log} \widetilde{\to} X^{\log}$ and $\overline{X_r^{\log}} \widetilde{\to} \overline{X^{\log}}$.

Thus, we are back to the general situation when $1 \le r \le \infty$. We study the maps of Π_R-spaces $\nu : \overline{X^{\log}} = X^{\log} \times_{\mathbf{pt}_R} \overline{\mathbf{pt}_R^{\log}} \to X^{\log}$, $\tau : X^{\log} \to X$ and $\overline{\tau} = \tau \circ \nu : \overline{X^{\log}} \to X$. We also denote by $\tau^{(\varpi)}$, $\varpi \in \Pi_R$, the restriction of $\overline{\tau}$ to $X^{(\varpi)}$.

Lemma 4.1.3. *Each point $x \in X$ has a fundamental system of open neighborhoods U such that there are compatible strong deformation retractions of U to x, of $\tau^{-1}(U)$ to $\tau^{-1}(x)$, and of $\overline{\tau}^{-1}(U)$ to $\overline{\tau}^{-1}(x)$.*

Proof. By the remark in Example 4.1.2(i), we may assume that we are given a distinguished log germ (Y, X) over $(\mathbf{C}, 0)$, and it suffices to show that each point $x \in X$ has a fundamental system of open neighborhoods U of x in Y which preserves the intersection $U \cap X$ and lifts to strong deformation retractions of $\tau^{-1}(U)$ to $\tau^{-1}(x)$ and of $\overline{\tau}^{-1}(U)$ to $\overline{\tau}^{-1}(x)$, where τ and $\overline{\tau}$ are the maps $Y^{\log} \to Y$ and $\overline{Y^{\log}} \to Y$, respectively. Thus, we may assume that Y is the affine space $\mathbf{C}^n$ provided with the log structure generated by the coordinate functions $T_1, \ldots, T_m$, $1 \le m \le n$, as in Definition 4.1.1(ii), X is the union of μ hyperplanes defined by the equations $T_i = 0$ for $1 \le i \le \mu \le m$, and x is the zero point in $\mathbf{C}^n$.

There is a homeomorphism $(\mathbf{R}_+^m \times (S^1)^m) \times \mathbf{C}^{n-m} \widetilde{\to} (\mathbf{C}^n)^{\log}$, and the projection from the latter to $\mathbf{C}^n$ is as follows

$$(\mathbf{C}^n)^{\log} \to \mathbf{C}^n : ((r, a), c) \mapsto (ra, c) \ ,$$

where $r = (r_1, \ldots, r_m)$, $a = (a_1, \ldots, a_m)$, and $c = (c_{m+1}, \ldots, c_n)$. One also has

$$\overline{(\mathbf{C}^n)^{\log}} = \{(((r, a), c), b) \in (\mathbf{C}^n)^{\log} \times i\mathbf{R} \big| \prod_{j=1}^{m} a_j^{e_j} = e^b\} \ .$$

If U is an open neighborhood of zero in $\mathbf{C}^n$ with the property that, for each point $y \in U$, the interval $\{ty \big| t \in [0, 1]\}$ lies in U, then the map $\Phi_U : U \times [0, 1] \to U$ that takes a pair (y, t) to the point $(1-t)y$ is a strong deformation retraction of U to the zero point 0, and this map Φ_U lifts to deformation retractions of $\tau^{-1}(U)$ to $\tau^{-1}(0) : (((r, a), c), t) \mapsto (((1-t)r, a), (1-t)c)$ and of $\overline{\tau}^{-1}(U)$ to $\overline{\tau}^{-1}(0)$. Notice also that Φ_U preserves the intersection of U with each of the hyperplanes in X. $\square$

Corollary 4.1.4. *Let (Y, X) be a distinguished log germ over $(\mathbf{C}, 0)$. Then for every point $x \in X$, there are canonical isomorphisms*

$$R^q\Theta(\Lambda_{Y(X)_\eta})_x \widetilde{\to} H^q(\tau^{-1}(x), \Lambda) \text{ and } R^q\Psi_\eta(\Lambda_{Y(X)_\eta})_x \widetilde{\to} H^q(\overline{\tau}^{-1}(x), \Lambda) \ .$$

Proof. By Theorem 1.5.2, the left hand sides are the inductive limits of the groups $H^q(\tau^{-1}(U), \Lambda)$ and $H^q(\overline{\tau}^{-1}(U), \Lambda)$, and they coincide with the right hand sides since $\tau^{-1}(x)$ and $\overline{\tau}^{-1}(x)$ are strong deformation retractions of $\tau^{-1}(U)$ and $\overline{\tau}^{-1}(U)$, respectively, for sufficiently small U's. $\square$

Notice that the first isomorphism holds in the more general setting of Theorem 1.5.2 because the map $\tau : X^{\log} \to X$ is proper.

Corollary 4.1.5. *Let Z be a closed analytic subspace of X provided with the induced log structure with respect to which it is also distinguished over $\mathbf{pt}_R$. Then for any Π_R-module Λ, there is a canonical isomorphism*

$$R\overline{\tau}_*(\underline{\Lambda}_{\overline{X^{\log}}})\big|_Z \widetilde{\to} R\overline{\tau}_{Z*}(\underline{\Lambda}_{\overline{Z^{\log}}}) \ . \qquad \square$$

Recall (see Example 3.3.5) that $\overline{\pi}_{0,X}$ denotes the Π_R-cosheaf $U \mapsto \pi_0(\overline{\tau^{-1}(U)}) = \pi_0(\overline{U^{\log}})$. The purpose of the following two subsection is to describe it in terms of the logarithmic structure on X.

4.2. **Description of the cosheaf $\overline{\pi}_{0,X}$.** Let M_X^{gr} be the sheaf of abelian groups associated to the sheaf of monoids M_X. It contains the sheaf $\mathcal{O}_X^*$, and we set $\overline{M}_X^{gr} = M_X^{gr}/\mathcal{O}_X^*$. The latter is a sheaf of finitely generated abelian groups. For example, $\overline{M}_R^{gr}$ is canonically isomorphic to $\mathbf{Z}$. Let $\overline{M}_X^{(tors)}$ denote the torsion subsheaf of $\overline{M}_X^{gr}$. Finally, we set

$$\overline{M}_{X/R} = \mathrm{Coker}(\overline{M}_R^{gr} \to \overline{M}_X^{gr})$$

and denote by $\overline{M}_{X/R}^{(tors)}$ the torsion subsheaf of $\overline{M}_{X/R}$.

Proposition 4.2.1. *For every nonempty connected open subset $U \subset X$, the following is true:*

(i) *the group $\overline{M}_{X/R}^{(tors)}(U)$ is finite cyclic (of order e_U);*
(ii) *given a covering of U by nonempty connected open subsets $\{U_i\}_{i\in I}$, one has $e_U = \mathrm{g.c.d.}(e_{U_i})_{i\in I}$;*
(iii) *for every nonempty connected open subset $V \subset U$, the canonical homomorphism $\overline{M}_{X/R}^{(tors)}(U) \to \overline{M}_{X/R}^{(tors)}(V)$ is injective;*
(iv) *there is a unique generator $\overline{m}_U$ of $\overline{M}_{X/R}^{(tors)}(U)$ with the property that its restriction to a sufficiently small connected open neighborhood V of every point of U lifts to an element $m \in M(V)$ such that m^{e_U} is an element of M_R whose image in $\overline{M}_R^{gr} \widetilde{\to} \mathbf{Z}$ is one.*

Proof. Step 1. By Definition 4.1.1, every point $x \in X$ has a connected open neighborhood U that admits a strict open immersion in a log space of the form from that definition (with a fixed $\varpi \in \Pi_R$) and such that x is its zero point. (We call such U a special open neighborhood of U.) If P is the free monoid generated by elements $v_1, \dots, v_m$, the log structure on U is defined by the chart $P \to \mathcal{O}(U) : v_i \mapsto T_i$. Let $P_{/u}$ denote the quotient of P^{gr} by the subgroup generated by the element $u = v_1^{e_1} \cdot \dots \cdot v_m^{e_m}$. Since $P^* = \{1\}$, one has $P \widetilde{\to} \overline{M}_{X,x}$ and $P_{/u} \widetilde{\to} \overline{M}_{X/R,x}$, and these isomorphisms go through a homomorphism $P \to M_X(U)$. In particular, $P_{/u}^{(tors)} \widetilde{\to} \overline{M}_{X/R,x}^{(tors)}$, where $P_{/u}^{(tors)}$ is the torsion subgroup of $P_{/u}$. The group $P_{/u}^{(tors)}$ is cyclic of order $e_U = \mathrm{g.c.d.}(e_1, \dots, e_m)$ generated by the image of the element $v = v_1^{e'_1} \cdot \dots \cdot v_m^{e'_m}$, where $e'_i = \frac{e_i}{e_U}$.

Step 2. *For any point $x' \in U$, the induced homomorphism $P_{/u}^{(tors)} \to \overline{M}_{X/R,x'}^{(tors)}$ is injective.* Indeed, suppose that for $1 \le i \le m$ the coordinate function T_i is zero at x' for only $1 \le i \le \nu$ or $\gamma + 1 \le i \le m$, where $1 \le \nu \le \mu \le \gamma \le m$. If P'' is the localization of P with respect to the elements $v_{\nu+1}, \dots, v_\gamma$, then $P''/P''^* \widetilde{\to} \overline{M}_{X,x'}$. The quotient $P' = P''/P''^*$ is isomorphic to the free monoid generated by the elements $v_1, \dots, v_\nu, v_{\gamma+1}, \dots, v_m$, and the image of u in P' is the element $u' = v_1^{e_1} \cdot \dots \cdot v_\nu^{e_\nu} \cdot v_{\gamma+1}^{e_{\gamma+1}} \cdot \dots \cdot v_m^{e_m}$. This implies the claim. This also implies the following facts:

(1) the group $\overline{M}_{X/R,x'}^{(tors)}$ is of order $\mathrm{g.c.d.}(e_1, \dots, e_\nu, e_{\gamma+1}, \dots, e_m)$, and one has $P'^{(tors)}_{/u'} \widetilde{\to} \overline{M}_{X/R,x'}^{(tors)}$;
(2) for any special open neighborhood V of x' in U at which the element $v_{\nu+1}, \dots, v_\gamma$ are invertible, one has $P'^{(tors)}_{/u'} \widetilde{\to} \overline{M}_{X/R}^{(tors)}(V)$ and the homomorphism $\overline{M}_{X/R}^{(tors)}(U) \to \overline{M}_{X/R}^{(tors)}(V)$ is injective.

Step 3. *The canonical homomorphism* $P^{(tors)}_{/u} \to \overline{M}^{(tors)}_{X/R}(U)$ *is a bijection.* Indeed, for a special open subset $U' \subset U$, as at the end of Step 2, we set $G(U') = P'^{(tors)}$. It suffices to show that the value of the sheaf, associated with the presheaf G, at U coincides with $G(U)$. Suppose we are given a covering $\{U_i\}_{i\in I}$ of U by nonempty special open subsets. By Step 2, all of the homomorphisms $G(U) \to G(U_i)$ are injective and, if $x \in U_{i_0}$, then $G(U) \widetilde{\to} G(U_{i_0})$. Let $\{g_i\}_{i\in I}$ be a system of elements $g_i \in G(U_i)$ with $g_i\big|_{U_i\cap U_j} = g_j\big|_{U_i\cap U_j}$ for all $i, j \in I$. We claim that for the element $g \in G(U)$ with $g\big|_{U_{i_0}} = g_{i_0}$, one has $g\big|_{U_i} = g_i$ for all $i \in I$. Indeed, if $U_{i_0} \cap U_i \neq$, take a nonempty special open subset V from the intersection. Then $g\big|_V = g_{i_0}\big|_V = g_i\big|_V$ and, therefore, $(g\big|_{U_i} - g_i)\big|_V = 0$. This implies that $g\big|_{U_i} = g_i$. If $i \in I$ is arbitrary, we can find a finite sequence $i_1, \dots, i_p = i$ with $U_{i_q} \cap U_{i_{q+1}} \neq \emptyset$ for all $0 \le q \le p-1$ and, by induction on q, we get $g_{U_i} = g_i$. It follows that the group $\overline{M}^{(tors)}_{X/R}(U)$ is cyclic of order e_U and it has a unique generator $\overline{m}_U$ which lifts to an element $m \in M(U)$ such that m^{e_U} is an element of M_R whose image in $\overline{M}^{gr}_R \widetilde{\to} \mathbf{Z}$ is one.

Step 4. Let now U be an arbitrary nonempty connected open subset of X. *We claim that the group* $\overline{M}^{(tors)}_X(U)$ *is finite cyclic, and the support of any of its nontrivial elements coincides with* U*.* Assume that the support $\mathrm{Supp}(g)$ of a nontrivial element $g \in \overline{M}^{(tors)}_X(U)$ is smaller than U. Let x be a from the topological boundary of $\mathrm{Supp}(g)$ in U, and let U' be a special open neighborhood of x in U. Since $g\big|_{U'} \neq 1$, Steps 2 and 3 imply that the image of g in $\overline{M}^{(tors)}_{X,x'}$ is nontrivial for every point $x' \in U'$, i.e., $U' \subset \mathrm{Supp}(g)$, which contradicts the assumption. Thus, $\mathrm{Supp}(g) = U$. If now U' is a nonempty special open subset of U, the latter implies that the homomorphism $\overline{M}^{(tors)}_X(U) \to \overline{M}^{(tors)}_X(U')$ is injective, and the claim follows. This implies the statements (i) and (iii).

Step 5. *The statements (ii) and (iv) are true.* Indeed, take a covering $\{U_i\}_{i\in I}$ of U by nonempty special open subsets. It suffices to show that

(1) the group $\overline{M}^{(tors)}_{X/R}(U)$ is of order $e_U = \mathrm{g.c.d.}(e_{U_i})_{i\in I}$;
(2) there is a unique generator $\overline{m}_U$ of $\overline{M}^{(tors)}_{X/R}(U)$ whose restriction to each U_i coincides with $\overline{m}^{k_i}_{U_i}$ for $k_i = \frac{e_{U_i}}{e_U}$.

First of all, since all of the homomorphisms $\overline{M}^{(tors)}_{X/R}(U) \to \overline{M}^{(tors)}_{X/R}(U_i)$ are injective, it follows that the order of $\overline{M}^{(tors)}_{X/R}(U)$ divides e_U. Furthermore, if V is a nonempty special open subset of $U_i \cap U_j$, the restrictions of the elements $\overline{m}^{k_i}_{U_i}$ and $\overline{m}^{k_j}_{U_j}$ to V coincide since the e_U-th powers of them are elements whose images in $\overline{M}^{gr}_R \widetilde{\to} \mathbf{Z}$ are one. This means that the elements $\overline{m}^{k_i}_{U_i}$ are compatible on intersections $U_i \cap U_j$ and, therefore, there exists a unique element $\overline{m}_U \in \overline{M}^{(tors)}_{X/R}(U)$ of order e_U with $\overline{m}_U\big|_{U_i} = \overline{m}^{k_i}_{U_i}$ for all $i \in I$. This implies the required statements (1) and (2). $\square$

For a nonempty connected open subset $U \subset X$, let k_U be the maximal positive integer with the property that there exists $m \in M(U)$ such that m^{k_U} lies in M_R and its image in $\overline{M}^{gr}_R \widetilde{\to} \mathbf{Z}$ is one. It is clear that k_U is a divisor of e_U, and if U is sufficiently small, then $k_U = e_U$. Furthermore, for $\varpi \in \Pi_R$ we set

$$\Upsilon^{(\varpi)}(U) = \{m \in M(U) \big| m^{k_U} = \varpi\} \ .$$

The set $\Upsilon^{(\varpi)}(U)$ is a principal homogeneous space for the group μ_{k_U} of k_U-th roots of one (acting by multiplication). Each morphism $\varphi : \varpi \to \varpi'$ in $\Pi_{K_r^\circ}$, i.e., $\varpi' = \alpha\varpi$ and φ is represented by an element $\beta \in K_r^\circ$ with $\exp(\beta) = \alpha^{-1}$, gives rise to a bijective map

$$\Upsilon^{(\varpi')}(U) \to \Upsilon^{(\varpi)}(U) : m' \mapsto \exp\left(\frac{\beta}{k_U}\right) m' \ .$$

For example, the generator $\sigma^{(\varpi)}$ of $\Pi_R^{(\varpi)} = \mathrm{Aut}_{\Pi_R}(\varpi)$ takes each $m \in \Upsilon^{(\varpi)}(U)$ to the element $e^{\frac{2\pi i}{k_U}} m$. This makes the correspondence $\varpi \mapsto \Upsilon^{(\varpi)}(U)$ a finite Π_R-space, which is denoted by $\overline{\Upsilon}(U)$. Finally, for an element $m \in \Upsilon^{(\varpi)}(U)$, we set (see Example 3.2.1(iii))

$$U^{(\varpi)}(m) = \{((x, h_x), c) \in U^{(\varpi)} \big| h_x(m) = e^{\frac{c}{k_U}}\} \ .$$

Proposition 4.2.2. *The correspondence $m \mapsto U^{(\varpi)}(m)$ gives rise to an isomorphism of finite Π_R-spaces $\overline{\Upsilon}(U) \widetilde{\to} \pi_0(\overline{U^{\log}})$.*

Proof. Step 1. *For every element $m \in \Upsilon^{(\varpi)}(U)$, the open and closed set $U^{(\varpi)}(m)$ is nonempty.* Indeed, let $((x, h_x), c) \in U^{(\varpi)}$. Since $h_x(\varpi) = e^c$, it follows that for every $m \in \Upsilon^{(\varpi)}(U)$ one has $h_x(m) = \zeta e^{\frac{c}{k_U}}$ for a k_U-root of one ζ. Moreover, multiplication by k_U-roots of one acts transitively on the set $\Upsilon^{(\varpi)}(U)$. This implies the claim. It follows that k_U divides the number $n = |\pi_0(U^{(\varpi)})|$.

Step 2. *The number n divides k_U.* Indeed, the element ϖ gives rise to homeomorphisms $\mathbf{pt}_R^{\log} \widetilde{\to} S^1 : h \mapsto h(\varpi)$ and $\mathbf{pt}^{(\varpi)} \widetilde{\to} i\mathbf{R} : (h, b) \mapsto b$. The exponential map $\mathbf{pt}^{(\varpi)} = i\mathbf{R} \to \mathbf{pt}_R^{\log} = S^1 : b \mapsto e^b$ is the composition of the map $i\mathbf{R} \to S^1 : b \mapsto e^{\frac{b}{n}}$ and the map $S^1 \to S^1 : a \mapsto a^n$. Since $|\pi_0(U^{(\varpi)})| = n$, the induced map

$$U^{(\varpi)} \to Y = U^{\log} \times_{S^1} S^1$$

gives rise to a bijection $\pi_0(U^{(\varpi)}) \widetilde{\to} \pi_0(Y)$. It follows that $|\pi_0(Y)| = n$ and, therefore, the projection $Y \to U^{\log}$ induces a homeomorphism of each connected component of Y onto $U^{\log}$. This implies that this projection has a section $U^{\log} \to Y : (x, h_x) \mapsto ((x, h_x), f(x, h_x))$ for a continuous map $f : U^{\log} \to S^1$ with $h_x(\varpi) = f(x, h_x)^n$.

Furthermore, we can find a covering $\{U_i\}_{i \in I}$ of U by connected open subsets such that all $k_{U_i} = e_{U_i} = |\pi_0(\overline{U_i^{\log}})|$. The latter implies that the number n divides all of the numbers e_{U_i} and, in particular, n divides e_U. Take elements $m_i \in \Upsilon^{(\varpi)}(U_i)$. Then for every point $x \in U_i$, one has $h_x(m_i)^{\frac{e_{U_i}}{n}} = c_i f(x, h_x)$ for a n-th root of one c_i. Since $U_i^{\log}$ is connected, it does not depend on the point x. We set $m_i' = c_i^{-1} m_i^{\frac{e_{U_i}}{n}}$. Then for every pair $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, one has $h_x(m_i') = h_x(m_j')$ for all points $(x, h_x) \in (U_i \cap U_j)^{\log}$. On the other hand $\frac{m_i'}{m_j'}$ is an element of $M^{gr}(U_i \cap U_j)$ whose n-th power is one. This implies that its restriction to each connected component W of $U_i \cap U_j$ is a n-root of one ζ, i.e., $m_i'\big|_W = \zeta m_j'\big|_W$ and, therefore, $h_x(m_i') = \zeta h_x(m_j')$ for all points $(x, h_x) \in W^{\log}$. This implies that $\zeta = 1$, i.e., $m_i'\big|_{U_i \cap U_j} = m_j'\big|_{U_i \cap U_j}$. Thus, there exists an element $m \in M(U)$ with

$m\big|_{U_i} = m'_i$ for all $i \in I$, and one has $m^n = \varpi$. The claim follows, and it implies that the correspondence $m \mapsto U^{(\varpi)}(m)$ gives rise to a bijection $\Upsilon^{(\varpi)}(U)\widetilde{\to}\pi_0(U^{(\varpi)})$.

Step 3. *The statement of the proposition is true.* Indeed, given a morphism $\varphi : \varpi \to \varpi' = \alpha\varpi$, i.e., an element $\beta \in R$ with $\exp(\beta) = \alpha^{-1}$, the induced map $\Upsilon^{(\varpi')}(U) \to \Upsilon^{(\varpi)}(U)$ takes an element m' to the element $m = \exp(\frac{\beta}{k_U})m'$, and a point $((x, h_x), c) \in U^{(\varpi')}$ to the point $((x, h_x), c + \mathrm{Im}(\beta(0))i) \in U^{(\varpi)}$. If the former point lies in $U^{(\varpi')}(m')$, then $h_x(m') = e^{\frac{c}{k_U}}$. It follows that

$$h_x(m) = h_x(\exp(\frac{\beta}{k_U})m') = e^{\frac{\mathrm{Im}(\beta(0))i}{k_U}} h_x(m') = e^{\frac{c+\mathrm{Im}(\beta(0))i}{k_U}}$$

and, therefore, the latter point lies in $U^{(\varpi)}(m)$. This implies the claim. $\square$

Proposition 4.2.2 implies that, for any pair of nonempty connected open subsets $U \subset V$, k_V divides k_U. We can therefore define a map

$$\Upsilon^{(\varpi)}(U) \to \Upsilon^{(\varpi)}(V) : m \mapsto m^{\frac{k_U}{k_V}} .$$

(There exists a unique element of $\Upsilon^{(\varpi)}(V)$ whose restriction to U is $m^{\frac{k_U}{k_V}}$, and it is denoted here in the same way.) This map is compatible with the canonical map $\pi_0(U^{(\varpi)}) \to \pi_0(V^{(\varpi)})$. Thus, if we extend the definition of to arbitrary open subsets $U \subset X$ by $\Upsilon^{(\varpi)}(U) = \coprod_{i\in\pi_0(U)} \Upsilon^{(\varpi)}(U_i)$, where $\{U_i\}_{i\in\pi_0(U)}$ is the set of connected components of U, then the correspondence $U \mapsto \Upsilon^{(\varpi)}(U)$ is a cosheaf of sets, denoted by $\Upsilon^{(\varpi)}_X$, and the family of them is a Π_R-cosheaf of sets, denoted by $\overline{\Upsilon}_X$.

Corollary 4.2.3. *The above construction gives rise to an isomorphism of* Π_R*-cosheaves of sets*

$$\overline{\Upsilon}_X \widetilde{\to} \overline{\pi}_{0,X} . \qquad \square$$

Remark 4.2.4. Here is an example of a connected distinguished log space X over the log point $\mathbf{pt}$ whose space $\overline{X^{\log}}$ is also connected (i.e., $k_X = 1$) but $e_X = 3$. Consider the affine algebraic curves $\mathcal{X}_i = \mathrm{Spec}(A_i)$, $0 \le i \le 2$, where A_i is the quotient of the ring of polynomials in two variables $\mathbf{C}\left[\frac{T_0}{T_i}, \frac{T_1}{T_i}, \frac{T_2}{T_i}\right]$ by the ideal generated by the element $\left(\frac{T_0}{T_i} \cdot \frac{T_1}{T_i} \cdot \frac{T_2}{T_i}\right)^3$, and provide $\mathcal{X}_i$ with the log structure generated by the variables. Furthermore, let ζ be a nontrivial cubic root of one and, for $0 \le i \ne j \le 2$, let $\mathcal{X}_{ij} = \mathrm{Spec}(A_{ij})$ denote the open subset of $\mathcal{X}_i$ where the function $\frac{T_j}{T_i}$ is invertible. We construct a connected log algebraic curve $\mathcal{X}$ by gluing the log curves $\mathcal{X}_i$'s along the following isomorphisms $A_{10}\widetilde{\to}A_{01} : (\frac{T_0}{T_1}, \frac{T_2}{T_1}) \mapsto (\zeta\frac{T_1}{T_0}, \frac{T_2}{T_0})$, $A_{20}\widetilde{\to}A_{02} : (\frac{T_0}{T_2}, \frac{T_1}{T_2}) \mapsto (\zeta\frac{T_2}{T_0}, \frac{T_1}{T_0})$, and $A_{21}\widetilde{\to}A_{12} : (\frac{T_0}{T_2}, \frac{T_1}{T_2}) \mapsto (\frac{T_0}{T_1}, \zeta\frac{T_2}{T_1})$. There is a morphism of log analytic spaces $X = \mathcal{X}^h \to \mathbf{pt}$ that takes a fixed generating element α for $\mathbf{pt}$ to the element $\left(\frac{T_0}{T_i} \cdot \frac{T_1}{T_i} \cdot \frac{T_2}{T_i}\right)^3$ in $M(\mathcal{X}_i)$. Then $\overline{M}^{(tors)}(X)$ is a cyclic group of order three generated by the image of the element $\frac{T_0}{T_i} \cdot \frac{T_1}{T_i} \cdot \frac{T_2}{T_i}$, and the corresponding cocycle $\{\zeta_{ij}\}_{0\le i,j\le 2}$ on the open covering $\{\mathcal{X}^h_i\}_{0\le i\le 2}$ of X is defined by the following values for $i < j$: $\zeta_{01} = \zeta_{02} = \zeta_{12} = \zeta^2$. This cocycle is not a coboundary because the equality $\zeta_{01} \cdot \zeta_{12} = \zeta_{02}$ does not hold.

4.3. **Description of the sheaves $R^q\overline{\tau}_*(\Lambda_{\overline{X^{\log}}})$.** For a Π_R-sheaf F on X, let F^Υ denote the Π_R-sheaf on X whose set of sections over an open subset $U \subset X$ is the Π_R-set of maps $\overline{\Upsilon}(U) \to F(U)$. Of course, if F is an abelian Π_R-sheaf, then so is F^Υ and, for $q \in \mathbf{Z}$, we set $F(q) = F \otimes_{\mathbf{Z}} \mathbf{Z}(q)_X$. By Corollary 4.2.3, for any Π_R-module Λ there is a canonical isomorphism of abelian Π_R-sheaves $\underline{\Lambda}_X^\Upsilon \widetilde{\to} \overline{\tau}_*(\underline{\Lambda}_{\overline{X^{\log}}})$.

We now set

$$\overline{M}_{X/R}^{(nont)} = \overline{M}_{X/R}/\overline{M}_{X/R}^{(tors)} .$$

Theorem 4.3.1. *For every $q \geq 0$ and every Π_R-module Λ without torsion (as an abelian group), there is an isomorphism of Π_R-modules on X*

$$R^q\overline{\tau}_*(\underline{\Lambda}_{\overline{X^{\log}}}) \widetilde{\to} \underline{\Lambda}(-q)_X^\Upsilon \otimes_{\mathbf{Z}} \bigwedge^q \overline{M}_{X/R}^{(nont)} .$$

We use a construction from the proof of [KN99], Lemma (1.5)]. For a topological space T, let $\mathcal{R}_T$ (resp. $\mathcal{S}_T$) denote the abelian sheaf of continuous functions on T with values in $i\mathbf{R}$ (resp. $S^1 = \{a \in \mathbf{C}^* \big| |a| = 1\}$). Notice that the exponential map $b \mapsto \exp(b)$ represents $\mathcal{R}_T$ as an extension of $\mathcal{S}_T$ by the constant sheaf $\mathbf{Z}(1)_T$. We now apply this to the Π_R-space $\overline{X^{\log}}$. The homomorphism of sheaves $\tau^{-1}(M_X^{gr}) \to \mathcal{S}_{X^{\log}}$ that takes $m \in M_X^{gr}$ to the function $(x, h_x) \mapsto h_x(m)$ induces a homomorphism of Π_R-sheaves $\overline{\tau}^{-1}(M_X^{gr}) \to \mathcal{S}_{\overline{X^{\log}}}$ which gives rise to an extension $\mathcal{L}_{\overline{X^{\log}}}$ of $\overline{\tau}^{-1}(M_X^{gr})$ by $\mathbf{Z}(1)_{\overline{X^{\log}}}$. The restriction of the above homomorphism to the Π_R-subsheaf $\overline{\tau}^{-1}(\mathcal{O}_X^*)$ is the homomorphism $f \mapsto \frac{f}{|f|}$ from the latter to $\mathcal{S}_{\overline{X^{\log}}}$, and it lifts to the homomorphism $\overline{\tau}^{-1}(\mathcal{O}_X) \to \mathcal{R}_{\overline{X^{\log}}} : f \mapsto \mathrm{Im}(f)i$. Thus, we get a commutative diagram of homomorphisms of abelian Π_R-sheaves with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}(1)_{\overline{X^{\log}}} \arrow[r] & \mathcal{R}_{\overline{X^{\log}}} \arrow[r, "\exp"] & \mathcal{S}_{\overline{X^{\log}}} \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{Z}(1)_{\overline{X^{\log}}} \arrow[r] \arrow[u, equal] & \mathcal{L}_{\overline{X^{\log}}} \arrow[r, "\exp"] \arrow[u] & \overline{\tau}^{-1}(M_X^{gr}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbf{Z}(1)_{\overline{X^{\log}}} \arrow[r] \arrow[u, equal] & \overline{\tau}^{-1}(\mathcal{O}_X) \arrow[r, "\exp"] \arrow[u] & \overline{\tau}^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Notice that there is a canonical isomorphism $\nu^{-1}(\mathcal{L}_{X^{\log}}) \widetilde{\to} \mathcal{L}_{\overline{X^{\log}}}$, where ν is the topological covering map $\overline{X^{\log}} \to X^{\log}$ and $\mathcal{L}_{X^{\log}}$ is the abelian sheaf on $X^{\log}$ introduced in [KN99, (1.4)] (and denoted there just by $\mathcal{L}$) .

Examples 4.3.2. (i) Consider the log space $\mathbf{pt}_R$. Then for every $\varpi \in \Pi_R$, the homomorphism of groups of global sections $\mathcal{L}_R^{(\varpi)} = \mathcal{L}(\mathbf{pt}_R^{(\varpi)}) \to M_R^{gr}$ is surjective. Indeed, the pair consisting of the function $\mathbf{pt}_R^{(\varpi)} \to i\mathbf{R} : (h, c) \mapsto c$ in $\mathcal{R}(\mathbf{pt}_R^{(\varpi)})$ and the element ϖ in $\tau^{(\varpi)-1}(M_R^{gr})(\mathbf{pt}_R^{(\varpi)})$ defines an element $\log(\varpi) \in \mathcal{L}_R^{(\varpi)}$ with $\exp(\log(\varpi)) = \varpi$, and the surjectivity claim follows from that of the exponential map $\exp : R \to R^*$. Furthermore, for a morphism $\varpi \to \varpi' = \alpha\varpi$, i.e., an element $\beta \in R$ with $\exp(\beta) = \alpha^{-1}$, the corresponding map $\mathcal{L}_R^{(\varpi)} \to \mathcal{L}_R^{(\varpi')}$ takes $\log(\varpi)$ to $\log(\varpi') + \beta$. The lift of $\log(\varpi)$ to $\mathcal{L}(X^{(\varpi)})$ will be denoted in the same way by $\log(\varpi)$.

(ii) For a connected open subset $U \subset X$ and elements $\varpi \in \Pi_R$ and $m \in \Upsilon^{(\varpi)}(U)$, the pair consisting of the function $U^{(\varpi)}(m) \to i\mathbf{R} : ((x, h_x), c) \mapsto \frac{c}{k_U}$ in $\mathcal{R}(U^{(\varpi)}(m))$ and the element m in $\overline{\tau}^{-1}(M_X^{gr})(U^{(\varpi)}(m))$ defines an element of $\mathcal{L}(U^{(\varpi)}(m))$ denoted by $\log(m)$ with $\exp(\log(m)) = m$. Notice that the restriction of $\log(\varpi)$ from (i) to $U^{(\varpi)}(m)$ coincides with $k_U \cdot \log(m)$. For a morphism $\varpi \to \varpi' = \alpha\varpi$, i.e., an element $\beta \in K_r^\circ$ with $\exp(\beta) = \alpha^{-1}$, the corresponding map $\mathcal{L}(U^{(\varpi)}) \to \mathcal{L}(U^{(\varpi')})$ takes $\log(m)$ to $\log(m') + \frac{\beta}{k_U}$, where m' is the preimage of m with respect to the canonical map $\Upsilon^{(\varpi')}(U) \to \Upsilon^{(\varpi)}(U)$.

Proof of Theorem 4.3.1. Since Λ has no torsion, the tensor product the second row of the above diagram with $\underline{\Lambda}_{\overline{X}^{\log}}$ is exact. Applying to it the left exact functor $\overline{\tau}_*$, we get a homomorphism

$$\begin{aligned}\psi : \underline{\Lambda}_X^\Upsilon \otimes_{\mathbf{Z}} M_X^{gr} \to \overline{\tau}_*(\underline{\Lambda}_{\overline{X}^{\log}}) \otimes_{\mathbf{Z}} \overline{\tau}_*(\overline{\tau}^{-1} M_X^{gr}) \to \\ \to \overline{\tau}_*(\underline{\Lambda}_{\overline{X}^{\log}} \otimes_{\mathbf{Z}} \overline{\tau}^{-1} M_X^{gr}) \to R^1\overline{\tau}_*(\underline{\Lambda}(1)_{\overline{X}^{\log}}) \ .\end{aligned}$$

The homomorphism ψ is clearly trivial on the subgroup $\mathcal{O}_X^* \subset M_X^{gr}$, i.e., ψ goes through a homomorphism from $\underline{\Lambda}_X^\Upsilon \otimes_{\mathbf{Z}} \overline{M}_X^{gr}$. Furthermore, since $\exp(\log(\varpi)) = \varpi$ for all $\varpi \in \Pi_R$, ψ is trivial on the image of the homomorphism $\overline{M}_R^{gr} \to \overline{M}_X^{gr}$, i.e., it goes through a homomorphism from $\underline{\Lambda}_X^\Upsilon \otimes_{\mathbf{Z}} \overline{M}_{X/R}$. Finally, if U is a sufficiently small nonempty connected open subset of X, then $k_U = e_U$ and, therefore, the image of an element $m \in \Upsilon^{(\varpi)}(U)$ in $\overline{M}^{gr}(U)$ generates the subgroup $\overline{M}^{(tors)}(U)$. Since $\exp(\log(m)) = m$, it follows that ψ goes through a homomorphism from $\underline{\Lambda}_X^\Upsilon \otimes_{\mathbf{Z}} \overline{M}_{X/R}^{(nont)}$.

Thus, ψ gives rise to a homomorphism

$$\underline{\Lambda}(-1)_X^\Upsilon \otimes_{\mathbf{Z}} \overline{M}_{X/R}^{(nont)} \to R^1\overline{\tau}_*(\underline{\Lambda}_{\overline{X}^{\log}}) \ .$$

Using the cup product, one gets a homomorphism from the sheaf on the left hand side to that on the right hand side for arbitrary $q \geq 1$. It is an isomorphism since it induces isomorphisms on stalks of both sheaves. $\square$

The following statement is an analog of [SGA7, Exp. 1, 3.3] (see also [Nak98, 3.5]).

Corollary 4.3.3. *Given a morphism of germs $(B, b) \to (\mathbf{C}, 0)$, let $\mathcal{Y}$ be a scheme of finite type over $\mathcal{O}_{B,b}$ such that $\mathcal{Y}$ is regular, flat over $\mathcal{O}_{\mathbf{C},0}$, the support of the special fiber $\widetilde{\mathcal{Y}}$ is the divisor with normal crossings, and that of the closed fiber $\mathcal{Y}_s$ is a union of some of the irreducible components of $\widetilde{\mathcal{Y}}$. We provide $\mathcal{Y}_s^h$ with the log structure $M_{\mathcal{Y}_\eta^h}$ induced by the canonical log structure on $\mathcal{Y}$. Then there are canonical isomorphisms of Π-modules*

$$R^q\Psi_\eta(\mathbf{Z}_{\mathcal{Y}_\eta^h}) \widetilde{\to} \mathbf{Z}(-q)_{\mathcal{Y}_s^h}^\Upsilon \otimes_{\mathbf{Z}_{\mathcal{Y}_s^h}} \bigwedge^q \overline{M}_{\mathcal{Y}_s^h/\mathbf{pt}}^{(nont)} \ .$$

Proof. By Corollary 2.3.3, the log structure $M_{\mathcal{Y}_s^h}$ coincides with that induced by the canonical log structure on the distinguished formal scheme $\widehat{\mathcal{Y}}$. It follows that the log space $\mathcal{Y}_s^h$ is distinguished and, therefore, the required fact follows from Theorems 1.5.2 and 4.3.1. $\square$

4.4. **A sheaf of W_R-algebras $\mathcal{C}_X$.** Let U be a nonempty connected open subset of X. For $\varpi \in \Pi_R$, let $t_U^{(\varpi)}$ be the image in $\mathcal{O}(X)$ of an element $m_U^{(\varpi)} \in \Upsilon^{(\varpi)}(U)$ (the latter is defined up to a multiplication by k_U-th root of one). Then $(t_U^{(\varpi)})^{k_U} = \widetilde{\varpi}$. For $\lambda = \frac{j}{k_U}$ with $0 \le j < rk_U$, let $\mathcal{C}_\lambda^{(\varpi)}(U)$ denote the $\mathbf{C}$-vector subspace of $\mathcal{O}(U)$ generated by the element $(t_U^{(\varpi)})^j$. Given a morphism $\varphi : \varpi \to \varpi' = \alpha\varpi$ in Π_R, i.e., an element $\beta \in R$ with $\exp(\beta) = \alpha^{-1}$, the multiplication by the element $\exp(-\lambda\beta) \in R$ induces an isomorphism $\mathcal{C}_\lambda^{(\varpi)}(U) \widetilde{\to} \mathcal{C}_\lambda^{(\varpi')}(U)$. If a rational number $0 \le \lambda < r$ is not of the form $\frac{j}{k_U}$ with $0 \le j < rk_U$, we set $\mathcal{C}_\lambda^{(\varpi)}(U) = 0$. By Proposition 4.2.1, for any bigger connected open subset V the restriction homomorphism $\mathcal{O}(V) \to \mathcal{O}(U)$ induces an isomorphism $\mathcal{C}_\lambda^{(\varpi)}(V) \widetilde{\to} \mathcal{C}_\lambda^{(\varpi)}(U)$. It follows that the spaces $\mathcal{C}_\lambda^{(\varpi)}(U)$ define a (non-single) sheaf of Π_R-modules $\mathcal{C}_{X,\lambda}^{(\varpi)}$ of dimension at most one over $\mathbf{C}$. It follows also that the direct sum $\mathcal{C}(U) = \oplus_\lambda \mathcal{C}_\lambda^{(\varpi)}(U)$ is a local R-algebra, which is free of finite rank over R, and it does not depend on the choice of ϖ.

The isomorphisms $\mathcal{C}(U) \widetilde{\to} \mathcal{C}(U)$ that correspond to the above morphisms $\varphi : \varpi \to \varpi'$ define the structure of an $\underline{R}$-ring on $\mathcal{C}(U)$. The $\mathbf{C}$-linear operators $\delta_\varpi : \mathcal{C}_\lambda^{(\varpi)}(U) \to \mathcal{C}_\lambda^{(\varpi)}(U)$, defined by $\delta_\varpi((t_U^{(\varpi)})^j) = \frac{j}{k_U}(t_U^{(\varpi)})^j$, provide $\mathcal{C}(U)$ with the structure of a $\underline{W}_R$-module. The algebras $\mathcal{C}(U)$ form a sheaf of local Artinian R-algebras $\mathcal{C}_X = \oplus_{\lambda \in \mathbf{Q} \cap [0,r)} \mathcal{C}_{X,\lambda}$, which is in fact a sheaf of distinguished $\underline{W}_R$-modules on X .

Theorem 4.4.1. *There is a canonical isomorphism of sheaves of distinguished $\underline{W}_R$-modules on X*

$$\mathcal{C}_X \widetilde{\to} \overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R} \ .$$

Proof. Let U be a connected open subset of X, and let $\varpi \in \Pi_R$. Given an element $m = m_U^{(\varpi)} \in \Upsilon^{(\varpi)}(U)$, a basis of the free R-module $\mathcal{C}(U)$ is formed by the elements t_m^j for $0 \le j \le k_U - 1$, where t_m is the image of m in $\mathcal{C}(U)$. We define a homomorphism of free R-modules of the same rank

$$\mu_{U,m}^{(\varpi)} : \mathcal{C}(U) \to \mathrm{Hom}(\Upsilon^{(\varpi)}(U), \mathbf{C}) \otimes_{\mathbf{C}} R = \mathrm{Hom}(\Upsilon^{(\varpi)}(U), R)$$

by $\mu_{U,m}^{(\varpi)}(t_m^j)(m') = \left(\frac{m}{m'}\right)^j$, where for elements $m, m' \in \Upsilon^{(\varpi)}(U)$, $\frac{m}{m'}$ denotes the k_U-th root of one ζ such that $m = \zeta m'$. If $m'' \in \Upsilon^{(\varpi)}(U)$, then $t_{m''} = \left(\frac{m''}{m}\right) t_m$ and, therefore, one has

$$\mu_{U,m}^{(\varpi)}(t_{m''}^j)(m') = \left(\frac{m''}{m}\right)^j \mu_{U,m}^{(\varpi)}(t_m^j)(m') = \left(\frac{m''}{m'}\right)^j = \mu_{U,m''}^{(\varpi)}(t_{m''}^j)(m') \ .$$

This means that the homomorphism $\mu_{U,m}^{(\varpi)}$ does not depend on the choice of m. We can therefore denote it by $\mu_U^{(\varpi)}$. The matrix of the R-linear operator $\mu_U^{(\varpi)}$ is a Vandermonde one and, therefore, $\mu_U^{(\varpi)}$ is an isomorphism.

If V is a bigger connected open subset, then the map $\Upsilon^{(\varpi)}(U) \to \Upsilon^{(\varpi)}(V)$ takes m to $n = m^{\frac{k_U}{k_V}}$ and m' to $n' = m'^{\frac{k_U}{k_V}}$, and one has $t_n\big|_U = t_m^{\frac{k_U}{k_V}}$. We get

$$\mu_V^{(\varpi)}(t_n^j)(n') = \left(\frac{n}{n'}\right)^j = \left(\frac{m}{m'}\right)^{\frac{jk_U}{k_V}} = \mu_U^{(\varpi)}(t_n^j\big|_U)(m') \ .$$

This means that the isomorphisms $\mu_U^{(\varpi)}$ and $\mu_V^{(\varpi)}$ are compatible, and we get an isomorphism of sheaves $\mu^{(\varpi)} : \mathcal{C}_X \widetilde{\to} \overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R}$. We have to verify that it gives rise to an isomorphism of sheaves of $\underline{W}_R$-modules.

First of all, it is an isomorphism of K_r°-modules, by the construction. Furthermore, set $\gamma_j = \mu_U^{(\varpi)}(t_m^j)$. By the same construction, one has $\gamma_j(m') = \left(\frac{m}{m'}\right)^j$. Since $\sigma^{(\varpi)}(m') = e^{\frac{2\pi i}{k_U}} m'$, it follows that $\sigma^{(\varpi)}(\gamma_j) = e^{-\frac{2\pi i j}{k_U}} \gamma_j$, i.e., the elements γ_j, which generate the free R-module $\mathrm{Hom}(\Upsilon^{(\varpi)}(U), R)$ are eigenvectors with eigenvalues $e^{-\frac{2\pi i j}{k_U}}$, respectively. By the construction of the operator δ_ϖ, one gets $\delta_\varpi(\gamma_j) = \frac{j}{k_U}\gamma_j$. Since $\delta_\varpi(t_m^j) = \frac{j}{k_U} t_m^j$, it follows that $\mu^{(\varpi)}$ is an isomorphism of sheaves of W_R-modules.

Suppose now we are given a morphism $\varphi : \varpi \to \varpi' = \alpha\varpi$, i.e., an element $\beta \in R$ with $\exp(\beta) = \alpha^{-1}$. The corresponding map $\Upsilon^{(\varpi')}(U) \to \Upsilon^{(\varpi)}(U)$ is induced by multiplication by the element $\exp\left(\frac{\beta}{k_U}\right) \in R^*$. It follows that the homomorphism $\mathcal{C}^{(\varpi)}(U) \to \mathcal{C}^{(\varpi')}(U)$ takes t_m to $t_{\gamma m}$, where $\gamma = \exp\left(-\frac{\beta}{k_U}\right)$, and therefore one has

$$\mu_U^{(\varpi')}(t_{\gamma m}^j)(\gamma m') = \left(\frac{\gamma m}{\gamma m'}\right)^j = \left(\frac{m}{m'}\right)^j = \mu_U^{(\varpi)}(t_m^j)(m') \ ,$$

i.e., the isomorphism considered is a map of Π_R-sheaves. $\square$

Theorem 4.3.1 implies that the Π_K-sheaves $R^q\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}})$ are of the type considered in Proposition 3.5.4, and so the latter implies that their tensor products with the constant sheaf associated to R are sheaves of distinguished $\underline{W}_R$-modules. Using Theorem 4.4.1 together with the isomorphism $\mathcal{C}_X(q) \widetilde{\to} \mathcal{C}_X : f \otimes (2\pi i)^q \mapsto (2\pi i)^q f$, we get the following fact.

Proposition 4.4.2. *There are canonical isomorphisms of sheaves of distinguished $\underline{W}_R$-modules*

$$R^q\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R} \widetilde{\to} \mathcal{C}_X \otimes_{\mathbf{Z}} \bigwedge^q \overline{M}_{X/R}^{(nont)} \ . \qquad \square$$

5. The analytification of vanishing cycles for log smooth formal schemes

5.1. **Formulation of results.** The purpose of this section is to show that, for an fs formally K°-log smooth special formal scheme $\mathfrak{X}$ and any finite discrete $\mathbf{Z}/n\mathbf{Z}[G]$-module Λ, the analytifications of the complexes $R\Theta(\Lambda_{\mathfrak{X}_\eta})$ and $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$, as defined in [Ber96b] and [Ber15], are described in the same way as in Theorem 1.5.2.

In fact instead of working with discrete Galois modules, we work here with discrete G_K-modules. The latter are covariant functors $\Lambda : K^{(\varpi)} \mapsto \Lambda^{(\varpi)}$ from G_K to the category of abelian groups such that for any pair of finite tuples $(a_i)_{1\le i\le n} \in (\Lambda^{(\varpi)})^n$ and $(a_i')_{1\le i\le n} \in (\Lambda^{(\varpi')})^n$ the set of morphisms $K^{(\varpi)} \widetilde{\to} K^{(\varpi')}$ with $\varphi_\Lambda(a_i) = a_i'$ for all $1 \le i \le n$ is open in $\mathrm{Hom}_{G_K}(K^{(\varpi)}, K^{(\varpi')})$. The category of discrete G_K-modules is denoted by G_K-Mod . Of course, for any ϖ the restriction functor $\Lambda \mapsto \Lambda^{(\varpi)}$ is an equivalence of categories, but an inverse functor is not canonical.

Any discrete G_K-module $\Lambda : \varpi \mapsto \Lambda^{(\varpi)}$ gives rise to an étale abelian G_K-sheaf $\Lambda_K : \varpi \mapsto \Lambda_K^{(\varpi)}$ on $\mathrm{Spec}(K)$. Namely, $\Lambda_K^{(\varpi)}$ is the étale abelian sheaf that associates to an étale morphism $\mathcal{X} \to \mathrm{Spec}(K)$ the group $\Lambda_K^{(\varpi)}(\mathcal{X})$ of maps $\mathcal{X} \otimes_K K^{(\varpi)} \to \Lambda^{(\varpi)}$

invariant under the action of the Galois group G. The pullback of Λ_K to any scheme $\mathcal{X}$ over K and a K-analytic space X is denoted by $\Lambda_{\mathcal{X}}$ and Λ_X, respectively.

If $\mathfrak{X}$ is a special formal scheme over K°, the nearby and vanishing cycles functors Θ and Ψ_η from [Ber96b] and [Ber15] are naturally extended to the category of étale abelian G_K-sheaves on $\mathfrak{X}_\eta$ and take values in the category of étale abelian G_K-sheaves on $\mathfrak{X}_s$. Namely, the functor Θ takes an étale abelian G_K-sheaf $F : \varpi \mapsto F^{(\varpi)}$ to the functor on G_K whose value at ϖ is $\Theta(F^{(\varpi)})$ with evident homomorphisms $\Theta(F^{(\varpi)}) \to \Theta(F^{(\varpi')})$ for morphisms $\varpi \to \varpi'$ in G_K. Notice that the G_K-sheaf $\Theta(F)$ is univocal and, in particular, it is isomorphic to a trivial $\mathcal{P}$-sheaf. Similarly, the functor Ψ_η takes F to the functor on G_K whose value at ϖ is $\Psi_\eta(F^{(\varpi)})$ constructed using the algebraic closure $K^{(\varpi)}$ of K, and each morphism $\varpi \to \varpi'$ induced the evident homomorphism $\Psi_\eta(F^{(\varpi)}) \to \Psi_\eta(F^{(\varpi')})$.

Notice that there is a natural faithful functor G_K-Mod$\to \Pi_K$-Mod. In particular, in the situation of Example 3.2.1(iii) every G_K-module Λ defines Π_K-sheaves $\Lambda_{X^{\log}}$ and $\underline{\Lambda}_{\overline{X^{\log}}}$ on $X^{\log}$ and $\overline{X^{\log}}$, respectively.

The derived category of complexes of discrete $\mathbf{Z}/n\mathbf{Z}[G_K]$-modules with finite cohomology modules is denoted by $D_c(\mathbf{Z}/n\mathbf{Z}[G_K]\text{-Mod})$.

Theorem 5.1.1. *Let $\mathfrak{X}$ be a formally K°-log smooth special formal scheme, and set $X = \mathfrak{X}_s^h$. Then for any $\Lambda^\cdot \in D_c^+(\mathbf{Z}/n\mathbf{Z}[G_K]\text{-Mod})$, there are canonical isomorphisms of complexes of Π_K-sheaves*

$$R\Theta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h \widetilde{\to} R\tau_*(\Lambda^\cdot_{X^{\log}}) \text{ and } R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h \widetilde{\to} R\overline{\tau}_*(\underline{\Lambda}^\cdot_{\overline{X^{\log}}}) \ .$$

Remark 5.1.2. The proof of Theorem 5.1.1 is based on log étale cohomology developed by Kazuya Kato and his collaborators for fs log schemes. We refer to [Ill02] for a survey of log étale cohomology.

5.2. **Kummer étale morphisms of log special formal schemes.** Recall (see [Ill02, 1.6]) that a morphism of fs log schemes $\mathcal{Y} \to \mathcal{X}$ is said to be Kummer étale if locally in the étale topology it admits a chart $P \to \mathcal{O}(\mathcal{X})$ and $Q \to \mathcal{O}(\mathcal{Y})$ with fs monoids P and Q such that (1) the homomorphism $P \to Q$ is injective and $P = Q \cap P^{gr}$; (2) the cokernel of the homomorphism $P^{gr} \to Q^{gr}$ is finite of order invertible on $\mathcal{Y}$; (3) the induced morphism of schemes $\mathcal{Y} \to \mathcal{X} \otimes_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q])$ is étale. If both schemes are of locally finite type over $\mathbf{C}$, then the induced map $(\mathcal{Y}^h)^{\log} \to (\mathcal{X}^h)^{\log}$ is a local homeomorphism. Kummer étale morphisms to an fs log scheme $\mathcal{X}$ give rise to a Kummer étale site $\mathcal{X}_{k\acute{e}t}$ of $\mathcal{X}$ and, if $\mathcal{X}$ is of locally finite type over $\mathbf{C}$, there is a morphism of sites $(\mathcal{X}^h)^{\log} \to \mathcal{X}_{k\acute{e}t}$.

Let k be a non-Archimedean field with nontrivial discrete valuation. A morphism of fs k°-log special formal schemes $\mathfrak{Y} \to \mathfrak{X}$ is said to be *Kummer étale* if it is of locally finite type and, for any ideal of definition $\mathcal{J}$ of $\mathfrak{X}$, the morphism of log schemes $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{J}\mathcal{O}_{\mathfrak{Y}}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is Kummer étale. The following is an analog of [Ber96b, Proposition 2.1].

Proposition 5.2.1. *Let $\mathfrak{X}$ be an fs k°-log special formal scheme. Then*

(i) *the correspondence $\mathfrak{Y} \mapsto \mathfrak{Y}_s$ gives rise to an equivalence between the category of fs k°-log special formal schemes Kummer étale over $\mathfrak{X}$ and the category of fs k_1°-log schemes Kummer étale over $\mathfrak{X}_s$;*

(ii) *If $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is a Kummer étale morphism, then $\varphi_\eta(\mathfrak{Y}_\eta) = \pi^{-1}(\varphi_s(\mathfrak{Y}_s))$ and, in particular, $\varphi_\eta(\mathfrak{Y}_\eta)$ is a closed analytic domain in $\mathfrak{X}_\eta$;*

(iii) *if the k°-log structures on $\mathfrak{X}$ and $\mathfrak{Y}$ are vertical, then for any Kummer étale morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ the induced morphism of k-analytic spaces $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is quasi-étale.*

Proof. (i) Since Kummer étale morphisms are log étale, fully faithfulness of the functor follows from the definition of log étale morphisms (see [Kato89, 3.3]). Therefore, in order to show that it is essentially surjective, it suffices to construct a lifting of a Kummer étale morphism $f : \mathcal{Y} \to \mathfrak{X}_s$ locally in the étale topology. We may therefore assume that the log structures on $\mathfrak{X}$ and $\mathcal{Y}$ are defined by charts $P \to \mathcal{O}(\mathfrak{X})$ and $Q \to \mathcal{O}(\mathcal{Y})$ and the morphism f is defined by an injective homomorphism of fs monoids $P \to Q$ such that (a) the image of P contains the image of a generator ϖ of the maximal ideal $k^{\circ\circ}$ of k°, (b) the cokernel of the homomorphism $P^{gr} \to Q^{gr}$ is finite of orders prime to $\mathrm{char}(\widetilde{k})$, (c) P coincides with the preimage of Q in P^{gr} with respect the latter homomorphism, and (d) the induced morphism of schemes $\mathcal{Y} \to \mathcal{X}' = \mathfrak{X}_s \otimes_{\mathrm{Spec}(\widetilde{k}[P])} \mathrm{Spec}(\widetilde{k}[Q])$ is étale. The scheme $\mathcal{X}'$ is the closed fiber $\mathfrak{X}'_s$ of the special formal scheme $\mathfrak{X}' = \mathfrak{X} \times_{\mathrm{Spf}(k^\circ\{P\})} \mathrm{Spf}(k^\circ\{Q\})$ and, by [Ber96b, 2.1(i)], the morphism $\mathcal{Y} \to \mathfrak{X}'_s$ lifts to an étale morphism $\mathfrak{Y} \to \mathfrak{X}'$. If we provide $\mathfrak{Y}$ with the log structure defined by the induced homomorphism $Q \to \mathcal{O}(\mathfrak{Y})$, we get the required Kummer étale morphism $\mathfrak{Y} \to \mathfrak{X}$.

(ii) By [Ber96b, 2.1(ii)], the required property holds for the étale morphism $\mathfrak{Y} \to \mathfrak{X}'$ (with $\mathfrak{X}'$ from the proof of (i)). It suffices therefore to verify this property for the morphism $\mathfrak{X}' \to \mathfrak{X}$ which is a base change of the morphism $\mathrm{Spf}(k^\circ\{Q\}) \to \mathrm{Spf}(k^\circ\{P\})$. Since the latter morphism is finite and surjective, then so is the induced morphism of k-affinoid spaces $\mathcal{M}(k\{Q\}) \to \mathcal{M}(k\{P\})$, and the required fact follows.

(iii) By [Ber96b, 2.3(iii)], the morphism $\mathfrak{Y}_\eta \to \mathfrak{X}'_\eta$ is quasi-étale. Let p be an element of P whose image in $\mathcal{O}(\mathfrak{X})$ coincides with the image of ϖ. Then $\mathfrak{X}' = \mathfrak{X} \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B)$, where $A = k^\circ\{P\}/(p - \varpi)$ and $B = k^\circ\{Q\}/(p - \varpi)k^\circ\{Q\}$. In particular, the morphism $\mathfrak{X}'_\eta \to \mathfrak{X}_\eta$ is a base change of the morphism of k-affinoid spaces $\mathcal{M}(B) \to \mathcal{M}(A)$. By the assumption, the monoids P and Q are vertical. It follows that their images in A and B consist of invertible elements and coincide with the images of P^{gr} and Q^{gr}, respectively. This implies that the morphism $\mathcal{M}(B) \to \mathcal{M}(A)$ is étale and, therefore, the morphism $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is quasi-étale. $\square$

Let $\mathfrak{X}$ be an fs vertical k°-log special formal scheme. We fix a functor $\mathfrak{U}_s \mapsto \mathfrak{U}$ from the category of fs k_1°-log schemes Kummer étale over $\mathfrak{X}_s$ to the category of fs k°-log special formal scheme Kummer étale over $\mathfrak{X}$, which is inverse to that of Proposition 5.2.1(i). By the proposition, the composition of the functor $\mathfrak{U}_s \mapsto \mathfrak{U}$ with the functor $\mathfrak{U} \mapsto \mathfrak{U}_\eta$ induces a morphism of sites $\nu^{\log} : \mathfrak{X}_{\eta_{q\text{ét}}} \to \mathfrak{X}_{sk\text{ét}}$, which is an analog of the morphism of sites $\nu : \mathfrak{X}_{\eta_{q\text{ét}}} \to \mathfrak{X}_{s\text{ét}}$ from [Ber96b, §2]. In this way we get a commutative diagram of morphisms of sites

$$\begin{tikzcd}
\mathfrak{X}_{\eta_{\text{ét}}} & \mathfrak{X}_{\eta_{q\text{ét}}} \arrow[l, "\mu"'] \arrow[r, "\nu"] \arrow[dr, "\nu^{\log}"'] & \mathfrak{X}_{s\text{ét}} \\
 & & \mathfrak{X}_{sk\text{ét}} \arrow[u, "\varepsilon"']
\end{tikzcd}$$

The nearby cycles functor from [Ber96b] is the functor $\Theta : \mathfrak{X}^{\sim}_{\eta_{\text{ét}}} \to \mathfrak{X}^{\sim}_{s\text{ét}}$, defined by $\Theta(F) = \nu_*(\mu^* F)$, and the log nearby cycles functor is the functor $\Theta^{\log} : \mathfrak{X}^{\sim}_{\eta_{\text{ét}}} \to$

$\mathfrak{X}^{\sim}_{s\,k\text{ét}}$, defined by $\Theta^{\log}(F) = \nu_*^{\log}(\mu^* F)$. They are analogs of the usual (from [SGA7]) and logarithmic (from [Nak98]) algebraic geometry functors. Namely, for an fs vertical k°-log scheme $\mathcal{X}$, there are canonical morphisms of schemes $\mathcal{X}_\eta \stackrel{j}{\hookrightarrow} \mathcal{X} \stackrel{i}{\longleftarrow} \mathcal{X}_s$ and of log schemes $\mathcal{X}_\eta \stackrel{j^{\log}}{\hookrightarrow} \mathcal{X} \stackrel{i^{\log}}{\longleftarrow} \mathcal{X}_s$. The above functors Θ and $\Theta^{\log}$ are analogs of the functors $\mathcal{X}^{\sim}_{\eta\,\text{ét}} \to \mathfrak{X}^{\sim}_{s\,\text{ét}} : \mathcal{F} \mapsto i^*(j_*\mathcal{F})$ and $\mathcal{X}^{\sim}_{\eta\,\text{ét}} \to \mathfrak{X}^{\sim}_{s\,k\text{ét}} : \mathcal{F} \mapsto i^{\log *}(j_*^{\log}\mathcal{F})$, which will be denoted Θ and $\Theta^{\log}$, respectively, as well.

The following is a straightforward generalization of [Ber94, 4.1 and 4.2].

Lemma 5.2.2. *Let $\mathfrak{X}$ be an fs vertical k°-log special formal scheme, and let F be an étale sheaf on $\mathfrak{X}_\eta$. Then*

(i) *if $\mathfrak{Y}_s$ is Kummer étale over $\mathfrak{X}_s$, then $\Theta^{\log}(F)(\mathfrak{Y}_s) = F(\mathfrak{Y}_\eta)$;*

(ii) *if F is abelian, then the sheaf $R^q\Theta^{\log}(F)$ is associated to the presheaf $\mathfrak{Y}_s \mapsto H^q(\mathfrak{Y}_\eta, F)$;*

(iii) *if F is abelian soft, then the sheaf $\Theta^{\log}(F)$ is flabby.* □

Corollary 5.2.3. *(i) For a Kummer étale morphism $\mathfrak{Y} \to \mathfrak{X}$ and an étale abelian sheaf on $\mathfrak{X}_\eta$, one has $R^q\Theta^{\log}(F)\big|_{\mathfrak{Y}_s} \widetilde{\to} R^q\Theta^{\log}(F\big|_{\mathfrak{Y}_\eta})$;*

(ii) for a morphism of fs vertical k°-log special formal schemes $\varphi : \mathfrak{Y} \to \mathfrak{X}$ and $F^\cdot \in D^+(\mathfrak{Y}_\eta)$, one has $R\Theta^{\log}(R\varphi_{\eta *}(F)) \widetilde{\to} R\varphi_{s*}(R\Theta^{\log}(F))$. □

5.3. **Nearby cycles of formally log smooth formal schemes.** We turn back to our field K. Every discrete G_K-module Λ defines an étale G_K-sheaf Λ_K on $\mathrm{Spec}(K)$. Given a generator ϖ of $K^{\circ\circ}$, the Kummer étale sheaf $\Theta^{\log}(\Lambda_K^{(\varpi)})$ on the algebraic log point $\mathrm{pt}_{K_1^\circ}$ is denoted by $\Lambda^{(\varpi)}_{K_1^\circ}$. Furthermore, each morphism $\varpi \to \varpi'$ in G_K gives rise to a morphism $\Lambda^{(\varpi)}_{K_1^\circ} \to \Lambda^{(\varpi')}_{K_1^\circ}$, and so the correspondence $\varpi \mapsto \Lambda^{(\varpi)}_{K_1^\circ}$ is a Kummer étale G_K-sheaf on $\mathrm{pt}_{K_1^\circ}$. The pull back of the latter to the Kummer étale site $\mathcal{X}_{k\text{ét}}$ of a log scheme $\mathcal{X}$ over $\mathrm{pt}_{K_1^\circ}$ is denoted by $\Lambda_{\mathcal{X}_{k\text{ét}}}$.

Theorem 5.3.1. *Let $\mathfrak{X}$ be an fs formally K°-log smooth special formal scheme, and $\Lambda^\cdot \in D_c^+(\mathbf{Z}/n\mathbf{Z}[G_K]\text{-Mod})$. Then there is a canonical isomorphism of complexes of Kummer étale G_K-sheaves*

$$\Lambda^\cdot_{\mathfrak{X}_{s\,k\text{ét}}} \widetilde{\to} R\Theta^{\log}(\Lambda^\cdot_{\mathfrak{X}_\eta}) \ .$$

Proof. First of all, it suffices to show that $\Lambda^{(\varpi)}_{\mathfrak{X}_{s\,k\text{ét}}} \widetilde{\to} \Theta^{\log}(\Lambda^{(\varpi)}_{\mathfrak{X}_\eta})$ and $R^q\Theta^{\log}(\Lambda^{(\varpi)}_{\mathfrak{X}_\eta}) = 0$ for any $q \geq 1$, any finite discrete $\mathbf{Z}/n\mathbf{Z}[G_K]$-modules Λ, and any fixed ϖ. We may therefore drop ϖ in the superscript. Furthermore, for any $m \geq 1$ the morphism $\mathrm{Spf}(K_m^\circ) \to \mathrm{Spf}(K^\circ)$, where K_m is the extension of K in $K^{(\varpi)}$ of degree m, is Kummer étale and, therefore, so is its base change to $\mathfrak{X}$. Since the statement is local in the Kummer étale topology, this reduces the situation to the case when the action of G on Λ is trivial. Finally, for the same reason, we may assume that $\mathfrak{X}$ is of the form $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ for a log smooth morphism of schemes $\mathcal{X} \to \mathrm{Spec}(K^\circ)$ with trivial log structure on $\mathcal{X}_\eta$ and a subscheme $\mathcal{Y} \subset \mathcal{X}_s$ (see Definition 2.2.3). We may also assume that the log structure on $\mathcal{X}$ is defined by a chart $P_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ for an fs monoid P with $P^* = \{1\}$ such that, for every $a \in P$ there exist $b \in P$ and $m \geq 1$ with $ab = \varpi^m$.

In order to verify the required property, we use the following facts on the usual functor Θ (in the above situation):

(1) $\Lambda(-q)_{\mathcal{X}_s} \otimes_{\mathbf{Z}} \bigwedge^q \overline{M}^{gr}_{\mathcal{X}_s} \widetilde{\to} R^q\Theta(\Lambda_{\mathcal{X}_\eta})$, where $M_{\mathcal{X}_s} \to \mathcal{O}_{\mathcal{X}_s}$ is the log structure induced from that on $\mathcal{X}$ and $\overline{M}^{gr}_{\mathcal{X}_s} = M^{gr}_{\mathcal{X}_s}/\mathcal{O}^*_{\mathcal{X}_s}$ ([Nak98, (2.0.2)]);
(2) $R\Theta(\Lambda_{\mathcal{X}_\eta})\big|_{\mathcal{Y}} \widetilde{\to} R\Theta(\Lambda_{\mathfrak{X}_\eta})$ ([Ber96b, 3.1]);
(3) there is a spectral sequence $E_2^{p,q} = H^p(\mathfrak{X}_s, R^q\Theta(\Lambda_{\mathfrak{X}_\eta})) \Longrightarrow H^{p+q}(\mathfrak{X}_\eta, \Lambda)$ functorial in $\mathfrak{X}$ ([Ber96b, 2.2]).

We also use the fact that any Kummer étale morphism $\mathfrak{Y} \to \mathfrak{X}$ is locally in the Kummer étale topology is of the form $\widehat{\mathcal{X}}'_{/\mathcal{Y}'} \to \mathfrak{X} = \widehat{\mathcal{X}}_{/\mathcal{Y}}$ for a Kummer étale morphism $\mathcal{X}' \to \mathcal{X}$, where $\mathcal{Y}'$ is the preimage of $\mathcal{Y}$ in $\mathcal{X}'_s$.

By Lemma 5.2.2(i), if $\mathfrak{Y}_s$ is Kummer étale over $\mathfrak{X}_s$ then $\Theta^{\log}(\Lambda_{\mathfrak{X}_\eta})(\mathfrak{Y}_s) = H^0(\mathfrak{Y}_\eta, \Lambda)$. If $\mathfrak{Y} = \widehat{\mathcal{X}}'_{/\mathcal{Y}'}$ as above, then $\Lambda_{\mathcal{X}_s} \widetilde{\to} \Theta(\Lambda_{\mathcal{X}_\eta})$, by (1), and therefore $\Lambda_{\mathcal{Y}} \widetilde{\to} \Theta(\Lambda_{\mathfrak{X}_\eta})$, by (2). This implies that $H^0(\mathfrak{Y}_s, \Lambda) = H^0(\mathfrak{Y}_\eta, \Lambda)$.

Furthermore, by Lemma 5.2.2(ii), the sheaf $R^m\Theta^{\log}(\Lambda_{\mathfrak{X}_\eta})$ for $m \geq 1$ is associated to the presheaf $\mathfrak{Y}_s \mapsto H^n(\mathfrak{Y}_\eta, \Lambda)$. We therefore have to show that, given a Kummer étale morphism $\mathcal{X}' \to \mathcal{X}$, there exists a Kummer étale covering $\{\mathcal{X}^{(i)} \to \mathcal{X}'\}_{i \in I}$ such that the induced homomorphisms $H^m((\widehat{\mathcal{X}}'_{/\mathcal{Y}'})_\eta, \Lambda) \to H^m((\widehat{\mathcal{X}}^{(i)}_{/\mathcal{Y}^{(i)}})_\eta, \Lambda)$ are zero for all $m \geq 1$ and $i \in I$. By the spectral sequence (3) applied to $\widehat{\mathcal{X}}'_{/\mathcal{Y}'}$, each group $H^m((\widehat{\mathcal{X}}'_{/\mathcal{Y}'})_\eta, \Lambda)$ has a decreasing filtration $F^{0,m}(\widehat{\mathcal{X}}'_{/\mathcal{Y}'}) = H^m((\widehat{\mathcal{X}}'_{/\mathcal{Y}'})_\eta, \Lambda) \supset F^{1,m} \supset \ldots \supset F^{m,m} \supset F^{m+1,m} = 0$ functorial in $\widehat{\mathcal{X}}'_{/\mathcal{Y}'}$ and such that each quotient $F^{p,m}/F^{p+1,m}$ is isomorphic to a subquotient of $E_2^{p,m-p} = H^p(\mathcal{Y}', R^{m-p}\Theta(\Lambda_{(\widehat{\mathcal{X}}'_{/\mathcal{Y}'})_\eta}))$. Thus, it suffices to show that, given $\mathcal{X}' \to \mathcal{X}$ as above, there exists a Kummer étale covering $\{\mathcal{X}^{(i)} \to \mathcal{X}'\}_{i \in I}$ such that the above homomorphism takes $F^{p,m}(\widehat{\mathcal{X}}'_{/\mathcal{Y}'})$ in $F^{p+1,m}(\widehat{\mathcal{X}}^{(i)}_{/\mathcal{Y}^{(i)}})$ for all $0 \leq p \leq m$ and all $i \in I$. (If so, we can iterate this construction.) In order to show the latter, it suffices to verify that, for every pair (p,q) with $p+q \geq 1$, there exists a Kummer étale covering as above for which all of the homomorphisms $E_2^{p,q}(\widehat{\mathcal{X}}'_{/\mathcal{Y}'}) \to E_2^{p,q}(\widehat{\mathcal{X}}^{(i)}_{/\mathcal{Y}^{(i)}})$ are zero.

First of all, $E_2^{p,0} = H^p(\mathcal{Y}', \Lambda)$, and so the required fact is true for $q = 0$ (with an étale covering of $\mathcal{X}'$). If $q \geq 1$, we set $\mathcal{X}'' = \mathcal{X}' \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{\frac{1}{n}}]$, where $P \to P^{\frac{1}{n}}$ is the homomorphism $P \to P : a \mapsto a^n$. Then $f : \mathcal{X}'' \to \mathcal{X}'$ is a Kummer étale covering and, by (1), the homomorphism $f_s^{-1}(R^q\Theta(\Lambda_{(\widehat{\mathcal{X}}'_{/\mathcal{Y}'})_\eta})) \to R^q\Theta(\Lambda_{(\widehat{\mathcal{X}}''_{/\mathcal{Y}''})_\eta})$ is zero, and so is the homomorphism $E_2^{p,q}(\widehat{\mathcal{X}}'_{/\mathcal{Y}'}) \to E_2^{p,q}(\widehat{\mathcal{X}}''_{/\mathcal{Y}''})$. $\square$

Corollary 5.3.2. *In the situation of Theorem 5.3.1, there is a canonical isomorphism* $R\Theta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \widetilde{\to} R\varepsilon_*(\Lambda^\cdot_{\mathfrak{X}_{s\,\text{két}}})$. $\square$

5.4. **Proof of Theorem 5.1.1.** Consider first the case when $\mathfrak{X}$ is fs. By Corollary 5.3.2, there is a canonical isomorphism $R\Theta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \widetilde{\to} R\varepsilon_*(\Lambda^\cdot_{\mathfrak{X}_{s\,\text{két}}})$. It follows that $R\Theta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h \widetilde{\to} (R\varepsilon_*(\Lambda^\cdot_{\mathfrak{X}_{s\,\text{két}}}))^h$. It suffices therefore to show that the canonical homomorphism $(R\varepsilon_*(\Lambda^\cdot_{\mathfrak{X}_{s\,\text{két}}}))^h \to R\tau_*(\Lambda^\cdot_{X^{\log}})$, induced by the morphism of sites $X^{\log} \to \mathfrak{X}_{s\,\text{két}}$, is an isomorphism. For this we may assume that $\Lambda^\cdot$ is a just finite discrete G_K-module Λ, and it suffices to verify isomorphism between q-th cohomology groups of both complexes. By [Nak98, (2.0.2)] and [KN99, (1.5)], there are canonical and compatible isomorphisms

$$R^q\varepsilon_*(\Lambda_{\mathfrak{X}_{s\,\mathrm{k\acute{e}t}}})\widetilde{\to}\Lambda_{\mathfrak{X}_s}(-q)\otimes_{\mathbf{Z}}\bigwedge^q\overline{M}^{gr}_{\mathfrak{X}_s} \text{ and}$$

$$R^q\tau_*(\Lambda_{X^{\log}})\widetilde{\to}\Lambda_X(-q)\otimes_{\mathbf{Z}}\bigwedge^q\overline{M}^{gr}_X\ ,$$

and the required fact for the functor Θ follows.

In order to prove the required fact for the functor Ψ_η, we fix a generator ϖ. The induced homomorphism $\mathcal{O}_{\mathbf{C},0}\to K^\circ : z\mapsto\varpi$ gives rise to an embedding of algebraically closed fields $\mathcal{K}^{\mathrm{a}}\to K^{(\varpi)}$. We consider first the ϖ-th part of the G_K-module Λ and do not write the superscript ϖ in notations. Let $\Lambda_{\mathfrak{X}_\eta}\to F^{\cdot}$ be a resolution of $\Lambda_{\mathfrak{X}_\eta}$ by soft sheaves F^i (see [Ber94, §3]), and let K_m be the extension of K in $K^{(\varpi)}$ of degree $m\ge 1$. Then the pullbacks F^i_m of F^i's are soft sheaves on $\mathfrak{X}_{\eta_m}$, where $\eta_m=\eta_{K_m}$, and, therefore, $\Lambda_{\mathfrak{X}_{\eta_m}}\to F^{\cdot}_m$ is a soft resolution of $\Lambda_{\mathfrak{X}_{\eta_m}}$. By [Ber96b, 2.2(iii)], one has $R\Theta^{K_m}(\Lambda_{\mathfrak{X}_{\eta_m}})=\Theta^{K_m}(F^{\cdot}_m)$ and, by [Ber15, 3.1.6(ii)], there is a canonical isomorphism $\varinjlim\Theta^{K_m}(F^{\cdot}_m)\widetilde{\to}R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$. By the previous paragraph, for each $m\ge 1$ there is a canonical isomorphism $\Theta^{K_m}(F^{\cdot}_m)^h\widetilde{\to}R\tau_{m*}(\Lambda_{X_m^{\log}})$, where X_m is the analytification of the closed fiber of $\mathfrak{X}\widehat{\otimes}_{K^\circ}K^\circ_m$ with the induced log structure and τ_m denotes the map $X_m^{\log}\to X$. The composition of the latter with the canonical homomorphism $R\tau_{m*}(\Lambda_{X_m^{\log}})\to R\overline{\tau}_*(\Lambda_{X^{(\varpi)}})$ gives a homomorphism $\Theta^{K_m}(F^{\cdot}_m)^h\to R\overline{\tau}_*(\Lambda_{X^{(\varpi)}})$. In this way we get a canonical homomorphism $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})^h\to R\overline{\tau}_*(\Lambda_{\overline{X^{(\varpi)}}})$, and *we claim that the latter is an isomorphism.*

Indeed, since the claim is local in the étale topology of $\mathfrak{X}$, we may assume that $\mathfrak{X}$ is of the form $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, where $\mathcal{X}$ is a log smooth scheme of finite type over $\mathcal{O}_{\mathbf{C},0}$ and $\mathcal{Y}$ is a subscheme of $\mathcal{X}_s$. By [Ber96b, 3.1], one has $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})=R\Psi_\eta(\Lambda_{\mathcal{X}_\eta})\big|_{\mathcal{Y}}$ and, by Theorem 1.4.1, $R\Psi_\eta(\Lambda_{\mathcal{X}_\eta})^h\widetilde{\to}R\Psi_\eta(\Lambda_{\mathcal{X}^h_\eta})$. Hence, the claim follows from Theorem 1.5.2. The above construction is functorial with respect to $\varpi\in\Pi_K$, and the fs case of the theorem follows.

In the general case we need the following fact related to Lemma 1.5.3.

Lemma 5.4.1. *Let $\mathfrak{X}$ be a formally K°-log smooth special formal scheme, and let $\varphi:\mathfrak{X}'\to\mathfrak{X}$ be the normalization of $\mathfrak{X}$ with the log structure $M_{\mathfrak{X}'}$ which is the saturation of $\varphi^*(M_{\mathfrak{X}})$ in $\mathcal{O}_{\mathfrak{X}'}$. Then $\mathfrak{X}'$ is an fs formally K°-log smooth special formal scheme and, for $X=\mathfrak{X}^h_s$ and $X'=\mathfrak{X}'^h_s$ provided with the induced log structures, the canonical map $X'^{\log}\to X^{\log}$ is a homeomorphism.*

Proof. The statement is local in the étale topology of $\mathfrak{X}$, and so we may assume that $\mathfrak{X}$ is the formal completion $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$, where $\mathcal{Y}$ is the log scheme $\mathrm{Spec}(\mathbf{C}[P])$ for a fine monoid P, the morphism of log schemes $\mathcal{Y}\to\mathrm{Spec}(K^\circ))$ is defined by a chart $Q\to P:\varpi\mapsto p$ for a free monoid Q generated by $\varpi\in\Pi_K$ and an element $p\in P$ such that the localization of P with respect to it is a group, and $\mathcal{Z}$ is a closed subscheme of $\mathcal{Y}_s=\mathrm{Spec}(\mathbf{C}[P]/(p))$. Then $\mathfrak{X}'$ is the formal completion $\widehat{\mathcal{Y}}'_{/\mathcal{Z}'}$, where $\mathcal{Y}'=\mathrm{Spec}(\mathbf{C}[P'])$ for the saturation P' of P in P^{gr} and $\mathcal{Z}'$ is the preimage of $\mathcal{Z}$ in $\mathcal{Y}'_s$. This implies the first statement. Since $X^{\log}$ and $X'^{\log}$ are the preimages of $X=\mathcal{Z}^h$ and $X'=\mathcal{Z}'^h$ in $(\mathcal{Y}^h)^{\log}$ and $(\mathcal{Y}'^h)^{\log}$, respectively, in order to prove the second statement it suffices to prove that the canonical map $(\mathcal{Y}'^h)^{\log}\to(\mathcal{Y}^h)^{\log}$ is a homeomorphism, but this follows from Lemma 1.5.3. $\square$

Let $\mathfrak{X}'$ be the normalization of $\mathfrak{X}$ as in Lemma 5.4.1. Then by the previous case, one has $R\Theta(\Lambda^{\cdot}_{\mathfrak{X}'_\eta})^h \widetilde{\to} R\tau'_*(\Lambda^{\cdot}_{X'^{\log}})$ and $R\Psi_\eta(\Lambda^{\cdot}_{\mathfrak{X}'_\eta})^h \widetilde{\to} R\overline{\tau}'_*(\underline{\Lambda}^{\cdot}_{\overline{X'^{\log}}})$, where $X' = \mathfrak{X}'^h_s$, and τ' and $\overline{\tau}'$ are the canonical maps $X'^{\log} \to X'$ and $\overline{X'^{\log}} \to X'$, respectively. On the other hand, by [Ber96b, 2.3(ii)], there are canonical isomorphisms $R\Theta(\Lambda^{\cdot}_{\mathfrak{X}_\eta}) \widetilde{\to} R\varphi_{s*}(R\Theta(\Lambda^{\cdot}_{\mathfrak{X}'_\eta}))$ and $R\Psi_\eta(\Lambda^{\cdot}_{\mathfrak{X}_\eta}) \widetilde{\to} R\varphi_{s*}(R\Psi_\eta(\Lambda^{\cdot}_{\mathfrak{X}'_\eta}))$. This implies that

$$R\Theta(\Lambda^{\cdot}_{\mathfrak{X}_\eta})^h \widetilde{\to} R\varphi^h_{s*}(R\tau'_*(\Lambda^{\cdot}_{X'^{\log}})) \text{ and } R\Psi_\eta(\Lambda^{\cdot}_{\mathfrak{X}_\eta})^h \widetilde{\to} R\varphi^h_{s*}(R\overline{\tau}'_*(\underline{\Lambda}^{\cdot}_{\overline{X'^{\log}}})) \ .$$

Finally, by Lemma 5.4.1, there are canonical homeomorphisms $\alpha : X'^{\log} \widetilde{\to} X^{\log}$ and $\overline{\alpha} : \overline{X'^{\log}} \widetilde{\to} \overline{X^{\log}}$. Since $\varphi^h_s \circ \tau' = \tau \circ \alpha$ and $\varphi^h_s \circ \overline{\tau}' = \overline{\tau} \circ \overline{\alpha}$, we get the required isomorphisms. $\square$

6. Complex analytic vanishing cycles for formal schemes

6.1. **Construction and first properties.** We fix, for every special formal scheme $\mathfrak{X}$ over K°, a distinguished compact hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ which exists, by Corollary 2.1.6. (We do not require that this hypercovering is proper.) The formal schemes $\mathfrak{Y}_n$ provided with the canonical log structure form a simplicial object in the category of fs log special formal schemes. It follows that the complex analytic spaces $Y_n = \mathfrak{Y}^h_{n,s}$ provided with the induced log structures form a simplicial fs log complex analytic space $Y_\bullet = (Y_n)_{n\geq 0}$, and there is an associated augmented simplicial topological space $a^{\log} : Y^{\log}_\bullet = (Y^{\log}_n)_{n\geq 0} \to \mathfrak{X}^h_s$. We set

$$R\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta}) = Ra^{\log}_*(\mathbf{Z}_{Y^{\log}_\bullet}) \ .$$

If $\tau_\bullet$ denotes the map of simplicial topological spaces $Y^{\log}_\bullet \to Y_\bullet$, then $a^{\log} = a^h_s \circ \tau_\bullet$ and, therefore, one also has

$$R\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta}) = Ra^h_{s*}(R\tau_{\bullet *}(\mathbf{Z}_{Y^{\log}_\bullet})) \ .$$

Furthermore, the fs log analytic spaces Y_n are over the log point $\mathbf{pt}_{K^\circ_1}$, and there is an associated augmented simplicial topological Π_K-space $\overline{a}^{\log} : \overline{Y^{\log}_\bullet} = (\overline{Y^{\log}_n})_{n\geq 0} \to \mathfrak{X}^h_s$. (Here $\mathfrak{X}^h_s$ is considered as a trivial Π_K-space.) We set

$$R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}) = R\overline{a}^{\log}_*(\mathbf{Z}_{\overline{Y^{\log}_\bullet}}) \ .$$

If $\overline{\tau}_\bullet$ denotes the map of simplicial topological Π_K-spaces $\overline{Y^{\log}_\bullet} \to Y_\bullet$ (with trivial action of Π_K on $Y_\bullet$), then $\overline{a}^{\log} = a^h_s \circ \overline{\tau}_\bullet$ and, therefore, one also has

$$R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}) = Ra^h_{s*}(R\overline{\tau}_{\bullet *}(\mathbf{Z}_{\overline{Y^{\log}_\bullet}})) \ .$$

Theorem 6.1.1. *The following is true:*

(i) *the complexes $R\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ and $R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ do not depend on the choice of the hypercovering up to a canonical isomorphism, and are functorial in $\mathfrak{X}$;*

(ii) *there is a canonical isomorphism $R\mathcal{I}^{\Pi_K}(R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})) \widetilde{\to} R\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$;*

(iii) *the sheaves $R^q\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ and $R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ are constructible and equal to zero for $q > 2\dim(\mathfrak{X}_\eta) + 1$ and $q > 2\dim(\mathfrak{X}_\eta)$, respectively;*

(iv) *the action of Π_K on $R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ is quasi-unipotent.*

Remarks 6.1.2. (i) Functoriality in (i) means that each morphism of special formal schemes $\varphi : \mathfrak{Y} \to \mathfrak{X}$ gives rise to morphisms

$$\theta^h(\varphi) : \varphi_s^{h*}(R\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta})) \to R\Theta^h(\mathbf{Z}_{\mathfrak{Y}_\eta}) \text{ and}$$

$$\theta^h_\eta(\varphi) : \varphi_s^{h*}(R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})) \to R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{Y}_\eta})$$

Furthermore, if φ is the identity morphism $\mathfrak{X} \to \mathfrak{X}$, then so is the morphism $\theta^h_\eta(\varphi)$ and, given a second morphism $\psi : \mathfrak{Z} \to \mathfrak{Y}$, one has $\theta^h_\eta(\varphi \circ \psi) = \theta^h_\eta(\psi) \circ \psi_s^{h*}(\theta^h_\eta(\varphi))$ (and the same for the morphisms $\theta^h(\varphi)$).

(ii) Recall (see [Ver76, §2]) that an abelian sheaf F on the analytification $\mathcal{Y}^h$ of a scheme $\mathcal{Y}$ of locally finite type over $\mathbf{C}$ is said to be (algebraically) *constructible* if, for every open subscheme $\mathcal{Y}' \subset \mathcal{Y}$ of finite type over $\mathbf{C}$, there is a decreasing sequence of Zariski closed subschemes $\mathcal{Z}_0 = \mathcal{Y}' \supset \mathcal{Z}_1 \supset \ldots \supset \mathcal{Z}_n = \emptyset$ such that the restriction of F to each complex analytic space $\mathcal{Z}_i^h \backslash \mathcal{Z}_{i+1}^h$ is a locally constant sheaf whose stalks are finitely generated abelian groups. For example, the analytification $\mathcal{F}^h$ of an étale abelian constructible sheaf $\mathcal{F}$ on $\mathcal{Y}$ is a constructible sheaf on $\mathcal{Y}^h$ (whose stalks are finite abelian groups). Recall also that, given a morphism $\varphi : \mathcal{Z} \to \mathcal{Y}$ between schemes of finite type over $\mathbf{C}$ and a constructible sheaf F on $\mathcal{Z}^h$, the sheaves $R^q\varphi^h_*(F)$ are constructible ([Ver76, 2.4.2]). If F is an abelian Π_K-sheaf on $\mathcal{Y}^h$, we say that the action of Π_K on it is *quasi-unipotent* if, for every open subscheme $\mathcal{Y}' \subset \mathcal{Y}$ of finite type over $\mathbf{C}$, there exist $m, n \geq 1$ such that, for every $\varpi \in \Pi_K$, the element $(\sigma^{(\varpi)m} - 1)^n$ acts as zero on the sheaf $F\big|_{\mathcal{Y}'^h}$.

Lemma 6.1.3. *In the situation of Theorem 6.1.1, for any* $\Lambda^\cdot \in D^+_c(\mathbf{Z}/n\mathbf{Z}[G_K]\text{-Mod})$, *there are canonical isomorphisms*

$$R\Theta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h \widetilde{\to} Ra^{\log}_*(\Lambda^\cdot_{Y^{\log}_\bullet}) \textit{ and } R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h \widetilde{\to} R\overline{a}^{\log}_*(\Lambda^\cdot_{\overline{Y^{\log}_\bullet}}) \ .$$

Proof. By Theorem 5.1.1, for every $m \geq 0$ there are canonical isomorphisms

$$R\Theta(\Lambda^\cdot_{\mathfrak{Y}_{m,\eta}})^h \widetilde{\to} R\tau_{m*}(\Lambda^\cdot_{Y^{\log}_m}) \text{ and } R\Psi_\eta(\Lambda^\cdot_{\mathfrak{Y}_{m,\eta}})^h \widetilde{\to} R\overline{\tau}_{m*}(\Lambda^\cdot_{\overline{Y^{\log}_m}})$$

and, therefore, the statement follows from [Ber15, 1.2.2(ii) and 3.3.2]. $\square$

Proof of Theorem 6.1.1. In most of the proof we consider only the functor Ψ^h_η because the same reasoning is applied to Θ^h.

(iii) (except the second part for Θ^h) and (iv). We may assume that the formal scheme $\mathfrak{X}$ is quasicompact. By Theorem 4.3.1, for every $m \geq 1$ the sheaves $R^q\overline{\tau}_{m*}(\mathbf{Z}_{\overline{Y^{\log}_m}})$ are constructible, and the action of a sufficiently large power of $\sigma^{(\varpi)}$'s on them is trivial. It follows that the sheaves $R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ are constructible and the action of Π_K on them is quasi-unipotent.

Consider now for every $n \geq 1$ the exact sequence $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ which gives rise to exact sequences in the category of algebraically constructible sheaves on $\mathfrak{X}^h_s$

$$(*_{\mathfrak{Y}}) \qquad 0 \to R^q\overline{a}^{\log}_*(\mathbf{Z}_{\overline{Y^{\log}_\bullet}})_n \to R^q\overline{a}^{\log}_*((\mathbf{Z}/n\mathbf{Z})_{\overline{Y^{\log}_\bullet}}) \to {}_nR^{q+1}\overline{a}^{\log}_*(\mathbf{Z}_{\overline{Y^{\log}_\bullet}}) \to 0 \ ,$$

where for an abelian sheaf F we denoted by F_n and ${}_nF$ the cokernel and kernel of the multiplication by n on F. By Lemma 6.1.3, the sheaf in the middle is the analytification of the constructible sheaf $R^q\Psi_\eta((\mathbf{Z}/n\mathbf{Z})_{\mathfrak{X}_\eta})$ on $\mathfrak{X}_s$. Since the latter are zero for $q > 2\dim(\mathfrak{X}_\eta)$, it follows that $R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}) = 0$ for the same q's, and we get the statement (iii).

(ii) and the second part of (iii) for Θ^h. By Proposition 3.4.4, there is a canonical isomorphism

$$R\tau_{\bullet *}(\mathbf{Z}_{Y^{\log}_\bullet}) \widetilde{\to} R\mathcal{I}^{\Pi_K}(R\overline{\tau}_{\bullet *}(\mathbf{Z}_{\overline{Y^{\log}_\bullet}})) \ .$$

We now notice that, given an augmented topological space $b : Z_\bullet \to X$ provided with the trivial action of a discrete group Π, there is an isomorphism of functors $Rb_* \circ R\mathcal{I}^{\Pi}_{Z_\bullet} \widetilde{\to} R\mathcal{I}^{\Pi}_X \circ Rb_*$, and the statement (ii) follows. Furthermore, since $R\mathcal{I}^{\Pi_K} \widetilde{\to} R\mathcal{I}^{\Pi}$, for every $q \geq 1$ there is an exact sequence

$$0 \to R^{q-1}\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})/(\sigma - 1)R^{q-1}\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}) \to R^q\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta}) \to R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})^{\Pi_K} \to 0 \ .$$

This implies that second part of (iii) for Θ^h.

(i) It suffices to verify the following fact in the case when $\mathfrak{X}$ is quasicompact. Suppose we are given a commutative diagram of distinguished compact hypercoverings of $\mathfrak{X}$

$$\begin{tikzcd} \mathfrak{Y}_\bullet \arrow[r, "a"] & \mathfrak{X} \\ \mathfrak{Z}_\bullet \arrow[u, "\varphi"] \arrow[ur, "b"'] & \end{tikzcd}$$

Then there is a canonical isomorphism (with $Z_\bullet = \mathfrak{Z}^h_{\bullet s}$).

$$R\overline{a}^{\log}_*(\mathbf{Z}_{\overline{Y^{\log}_\bullet}}) \widetilde{\to} R\overline{b}^{\log}_*(\mathbf{Z}_{\overline{Z^{\log}_\bullet}}) \ ,$$

For this we consider the homomorphism of the exact sequences $(*_{\mathfrak{Y}}) \to (*_{\mathfrak{Z}})$ as above. The homomorphism between the middle terms is an isomorphism, by Lemma 6.1.3. Moreover, all of the sheaves considered are constructible and zero for $q > 2\dim(\mathfrak{X}_\eta)$. The induction from $q = 2\dim(\mathfrak{X}_\eta)$ to $q = 0$ shows that the homomorphisms between the first and third terms are also isomorphisms. The required facts follow. $\square$

We now can extend as follows the definition of vanishing cycles complexes to exact functors

$$R\Theta^h : D^b(\Pi_K\text{-Mod}) \to D^b(\mathfrak{X}^h_s) \text{ and } R\Psi^h_\eta : D^b(\Pi_K\text{-Mod}) \to D^b(\mathfrak{X}^h_s(\Pi_K))$$

for a special formal schemes $\mathfrak{X}$ over K° with $\dim(\mathfrak{X}_\eta) < \infty$ (e.g., for quasicompact $\mathfrak{X}$). For $\Lambda^\cdot \in D^b(\Pi_K\text{-Mod})$, one defines

$$R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) = R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}) \otimes^{\mathbf{L}}_{\mathbf{Z}_{\mathfrak{X}^h_s}} \underline{\Lambda}^\cdot_{\mathfrak{X}^h_s} \text{ and } R\Theta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) = R\mathcal{I}^{\Pi_K}(R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})) \ .$$

By Theorem 6.1.1, these complexes are functorial in $\mathfrak{X}$ and, in particular, any morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ defines morphisms $\theta^h(\varphi, \Lambda^\cdot)$ and $\theta^h_\eta(\varphi, \Lambda^\cdot)$ similar to those in Remark 6.1.2(i). If $\Lambda^\cdot \in D^b_c(\Pi_K\text{-Mod})$, then the above complexes lie in $D^b_c(\mathfrak{X}^h_s)$ and $D^b_c(\mathfrak{X}^h_s(\Pi_K))$, respectively.

The following corollaries of Theorem 6.1.1 are formulated for an arbitrary complex $\Lambda^\cdot \in D^b(\Pi_K\text{-Mod})$, but it suffices to verify them only for $\Lambda^\cdot = \mathbf{Z}$.

Corollary 6.1.4. *Given a morphism of finite type $\varphi : \mathfrak{Y} \to \mathfrak{X}$ with $\mathfrak{Y}_\eta \widetilde{\to} \mathfrak{X}_\eta$, there are canonical isomorphisms*

$$R\Theta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) \widetilde{\to} R\varphi^h_{s*}(R\Theta^h(\Lambda^\cdot_{\mathfrak{Y}_\eta})) \text{ and } R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \widetilde{\to} R\varphi^h_{s*}(R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{Y}_\eta})) \ .$$

Proof. Let $b : \mathfrak{Z}_\bullet \to \mathfrak{Y}$ be a distinguished compact hypercovering of $\mathfrak{Y}$. Since $\mathfrak{Y}_\eta \widetilde{\to} \mathfrak{X}_\eta$, the composition $a = \varphi \circ b : \mathfrak{Z}_\bullet \to \mathfrak{X}$ is a distinguished compact hypercovering of $\mathfrak{X}$, and we have (with $Z = \mathfrak{Z}^h_s$)

$$R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}))\widetilde{\to} R\overline{a}^{\log}_*(\mathbf{Z}_{\overline{Z^{\log}_\bullet}})\widetilde{\to} R\varphi^h_{s*}(R\overline{b}^{\log}_*(\mathbf{Z}_{\overline{Z^{\log}_\bullet}})) = R\varphi^h_{s*}(R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{Y}_\eta})) \ .$$

The same holds for the functor Θ. □

The nearby cycles and vanishing cycles functors $R\Theta^h$ and $R\Psi^h_\eta$ are extended component wise to simplicial formal schemes.

Corollary 6.1.5. *Given a compact hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$, there are canonical isomorphisms*

$$R\Theta^h(\Lambda^\cdot_{\mathfrak{X}_\eta})\widetilde{\to} Ra^h_{s*}(R\Theta^h(\Lambda^\cdot_{\mathfrak{Y}_{\bullet\eta}})) \text{ and } R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})\widetilde{\to} Ra^h_{s*}(R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{Y}_{\bullet\eta}})) \ .$$

Proof. One can find a distinguished compact hypercovering $b : \mathfrak{Z}_\bullet \to \mathfrak{X}$ that refines a, and has $R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})\widetilde{\to} Rb^h_{s*}(R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{Z}_{\bullet\eta}}))$. The required statement follows therefore from the fact that the canonical morphism $Ra^h_{s*}(R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{Y}_{\bullet\eta}})) \to Rb^h_{s*}(R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{Z}_{\bullet\eta}}))$ is an isomorphism. This fact is verified using the reasoning from the proof of Theorem 6.1.1. □

Corollary 6.1.6. *Let $\mathfrak{X}$ be a formally K°-log smooth special formal scheme, and let X be the analytification $\mathfrak{X}^h_s$ provided with the induced log structure. Then there are canonical isomorphisms*

$$R\tau_*(\Lambda^\cdot_{X^{\log}})\widetilde{\to} R\Theta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) \text{ and } R\overline{\tau}_*(\Lambda^\cdot_{\overline{X^{\log}}})\widetilde{\to} R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \ .$$

Proof. First of all, if $\mathfrak{X}$ is distinguished, this follows from Theorem 6.1.1. Furthermore, if $\mathfrak{X}$ is arbitrary, its generic fiber $\mathfrak{X}_\eta$ is regular and, by Theorem 2.1.3(i), there exists a blow-up $\varphi : \mathfrak{Y} \to \mathfrak{X}$ with distinguished $\mathfrak{Y}$ and $\mathfrak{Y}_\eta \widetilde{\to} \mathfrak{X}_\eta$. By Corollary 6.1.4, there is a canonical isomorphism $R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})\widetilde{\to} R\varphi^h_{s*}(R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{Y}_\eta}))$ and, by the previous case, we get $R\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})\widetilde{\to} R\varphi^h_{s*}(R\overline{\tau}_*(\mathbf{Z}_{\overline{Y^{\log}}}))$, where $Y = \mathfrak{Y}^h_s$. Thus, we have to show that the canonical morphism $R\overline{\tau}_*(\mathbf{Z}_{\overline{X^{\log}}}) \to R\varphi^h_{s*}(R\overline{\tau}_*(\mathbf{Z}_{\overline{Y^{\log}}}))$ is an isomorphism. By the reasoning from the proof of Theorem 6.1.1, it suffices to verify the above fact for the group $\mathbf{Z}/n\mathbf{Z}$ instead of $\mathbf{Z}$. By Theorem 5.1.1, this is equivalent to the fact that the canonical homomorphism $R\Psi_\eta((\mathbf{Z}/n\mathbf{Z})_{\mathfrak{X}_\eta}) \to R\varphi_{s*}(R\Psi_\eta((\mathbf{Z}/n\mathbf{Z})_{\mathfrak{Y}_\eta})$ is an isomorphism. The latter follows from [Ber96b, 2.3(ii)]. The same reasoning is applicable to the functor $R\Theta^h$. □

Here is the first comparison statement.

Theorem 6.1.7. *Let $\mathfrak{X}$ be a special formal scheme over K°. Then for any $\Lambda^\cdot \in D^b_c(\mathbf{Z}/n\mathbf{Z}[G_K]\text{-Mod})$, there are canonical isomorphisms*

$$R\Theta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h\widetilde{\to} R\Theta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) \textit{ and } R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h\widetilde{\to} R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}) \ .$$

Proof. Since $R\Theta(\Lambda^\cdot_{\mathfrak{X}_\eta}) = R\mathcal{I}^{G_K}(R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}))$ (see [Ber15, 3.1.7]) and $R\Theta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) = R\mathcal{I}^{\Pi_K}(R\Psi^h_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta}))$, it suffices to construct the second isomorphism. By Corollary 2.1.6, there exists a distinguished *proper* hypercovering $a : \mathfrak{Y}_\bullet \to \mathfrak{X}$ and, by Lemma 6.1.3, one has $R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h\widetilde{\to} R\overline{a}^{\log}_*(\Lambda^\cdot_{\overline{Y^{\log}_\bullet}})$, where $Y_n = \mathfrak{Y}^h_{n,s}$. Furthermore, since

$\overline{a}^{\log} = a_s^h \circ \overline{\tau}_\bullet$, where $\overline{\tau}_\bullet$ is the map of simplicial topological spaces $\overline{Y_\bullet^{\log}} \to Y_\bullet$, one has $R\overline{a}_*^{\log}(\Lambda^\cdot_{\overline{Y_\bullet^{\log}}}) \widetilde{\to} Ra_{s*}^h(R\overline{\tau}_{\bullet *}(\Lambda^\cdot_{\overline{Y_\bullet^{\log}}}))$, and since each $\overline{\tau}_m$ is a composition of a topological covering map $\overline{Y_m^{\log}} \to Y_m^{\log}$ and a proper map $Y_m^{\log} \to Y_m$, one has $R\overline{\tau}_{\bullet *}(\Lambda^\cdot_{\overline{Y_\bullet^{\log}}}) \widetilde{\to} R\overline{\tau}_{\bullet *}(\mathbf{Z}_{\overline{Y_\bullet^{\log}}}) \otimes^{\mathbf{L}}_{\mathbf{Z}} \underline{\Lambda}^\cdot_{Y_\bullet}$. Finally, since the hypercovering $a_s^h : Y_\bullet \to \mathfrak{X}_s^h$ is proper, we get

$$R\Psi_\eta(\Lambda^\cdot_{\mathfrak{X}_\eta})^h \widetilde{\to} R\Psi_\eta^h(\mathbf{Z}_{\mathfrak{X}_\eta}) \otimes^{\mathbf{L}}_{\mathbf{Z}} \underline{\Lambda}^\cdot_{\mathfrak{X}_s^h} = R\Psi_\eta^h(\Lambda^\cdot_{\mathfrak{X}_\eta}) \ . \qquad \square$$

6.2. **Invariance under formally smooth morphisms.** Let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of special formal schemes over k°, where k is a non-Archimedean field with discrete valuation. We say that φ is *smooth* if every point of $\mathfrak{Y}$ has an étale neighborhood $\mathfrak{Y}' \to \mathfrak{Y}$ such that the induced morphism $\mathfrak{Y}' \to \mathfrak{X}$ is a composition of an étale morphism $\mathfrak{Y}' \to \mathfrak{X} \times \mathfrak{Z}$ and the projection $\mathfrak{X} \times \mathfrak{Z} \to \mathfrak{X}$, where $\mathfrak{Z}$ is the n-dimensional formal affine space $\mathrm{Spf}(k^\circ\{T_1, \dots, T_n\})$. We say that φ is *formally smooth* if locally in the étale topology of $\mathfrak{Y}$ it is a composition of morphisms of the form $\mathfrak{Z}_{/\mathcal{Y}} \to \mathfrak{Z}$ for subschemes $\mathcal{Y} \subset \mathfrak{Z}_s$ and of smooth morphisms.

Theorem 6.2.1. *Let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be a formally smooth morphism between special formal schemes over K°. Then $\theta^h(\varphi, \Lambda^\cdot)$ and $\theta_\eta^h(\varphi, \Lambda^\cdot)$ are isomorphisms for all $\Lambda^\cdot \in D^b(\Pi_K\text{-Mod})$.*

First of all, in order to prove the above statement, it suffices to consider the case when $\Lambda^\cdot = \mathbf{Z}$. Furthermore, since the sheaves $R^q\Theta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ and $R^q\Psi_\eta^h(\mathbf{Z}_{\mathfrak{X}_\eta})$ are constructible, the situation is reduced to the case $\Lambda^\cdot = \mathbf{Z}/n\mathbf{Z}$. Thus, by the Comparison Theorem 6.1.7, Theorem 6.2.1 follows from the following statement in which k is a non-Archimedean field with nontrivial discrete valuation, and G is the Galois group $\mathrm{Gal}(k^{\mathrm{a}}/k)$ (for a fixed algebraic closure k^{a} of k).

Theorem 6.2.2. *Suppose that $\mathrm{char}(\widetilde{k}) = 0$, and let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be a formally smooth morphism between special formal schemes over k°. Then $\theta(\varphi, \Lambda^\cdot)$ and $\theta_\eta(\varphi, \Lambda^\cdot)$ are isomorphisms for all $\Lambda^\cdot \in D_c^b(\mathbf{Z}/n\mathbf{Z}[G]\text{-Mod})$.*

Proof. It suffices to consider the case when $\Lambda^\cdot$ is a finite discrete G-module Λ. By [Ber96b, 2.3(i)], the required fact is true if the morphism φ is étale. Thus, in order to prove the theorem, it suffices to consider the two cases when (a) φ is of the form $\mathfrak{X}_{/\mathcal{Y}} \to \mathfrak{X}$ for a subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, and (b) φ is the projection $\mathfrak{X} \times \mathfrak{Z} \to \mathfrak{X}$, where $\mathfrak{Z}$ is the n-dimensional formal affine space $\mathrm{Spf}(k^\circ\{T_1, \dots, T_n\})$.

(a) Let $a : \mathfrak{Z}_\bullet \to \mathfrak{X}$ be a distinguished *proper* hypercovering of $\mathfrak{X}$. If $\mathcal{Y}_n$ is the preimage of $\mathcal{Y}$ in $\mathfrak{Z}_{n,s}$, then $\mathfrak{Z}_{\bullet/\mathcal{Y}_\bullet} \to \mathfrak{X}_{/\mathcal{Y}}$ is a distinguished proper hypercovering of $\mathfrak{X}_{/\mathcal{Y}}$. By the definition of the vanishing cycles complexes, we have

$$R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta}) = Ra_{s*}(R\Psi_\eta(\Lambda_{\mathfrak{Z}_{\bullet\eta}})) \text{ and } R\Psi_\eta(\Lambda_{(\mathfrak{X}_{/\mathcal{Y}})_\eta}) = Ra_{s*}(R\Psi_\eta(\Lambda_{(\mathfrak{Z}_{\bullet/\mathcal{Y}_\bullet})_\eta})) \ .$$

The proper base change theorem for schemes implies that

$$R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})\big|_{\overline{\mathcal{Y}}} = Ra_{s*}(R\Psi_\eta(\Lambda_{\mathfrak{Z}_{\bullet\eta}})\big|_{\overline{\mathcal{Y}}_\bullet}) \ .$$

Since the special formal schemes $\mathfrak{Z}_n$ are locally algebraic, the comparison theorem [Ber96b, 3.1] implies that

$$R\Psi_\eta(\Lambda_{\mathfrak{Z}_{\bullet\eta}})\big|_{\overline{\mathcal{Y}}_\bullet} = R\Psi_\eta(\Lambda_{(\mathfrak{Z}_{\bullet/\mathcal{Y}_\bullet})_\eta}) \ ,$$

and the required fact follows. The same reasoning holds from the functor Θ.

(b) Let $\mathfrak{Y} = \mathfrak{X} \times \mathfrak{Z}$. Since all of the sheaves considered are constructible, it suffices to show that, for every closed point $\overline{\mathbf{y}} \in \mathfrak{Y}_{\overline{s}}$, one has $R\Theta(\Lambda_{\mathfrak{X}_\eta})_{\overline{\mathbf{x}}} \widetilde{\to} R\Theta(\Lambda_{\mathfrak{Y}_\eta})_{\overline{\mathbf{y}}}$ (resp. $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})_{\overline{\mathbf{x}}} \widetilde{\to} R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})_{\overline{\mathbf{y}}}$), where $\overline{\mathbf{x}}$ is the image of $\overline{\mathbf{y}}$ in $\mathfrak{X}_{\overline{s}}$. Replacing k by a finite unramified extension, we may assume that the images $\mathbf{x}$ and $\mathbf{y}$ of the points $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ in $\mathfrak{X}_s$ and $\mathfrak{Y}_s$, respectively, are $\widetilde{k}$-rational. By (a), it suffices to show that $R\Gamma(\pi^{-1}(\mathbf{x}), \Lambda) \widetilde{\to} R\Gamma(\pi^{-1}(\mathbf{y}), \Lambda)$ (resp. $R\Gamma(\overline{\pi^{-1}(\mathbf{x})}, \Lambda) \widetilde{\to} R\Gamma(\overline{\pi^{-1}(\mathbf{y})}, \Lambda)$), where π denotes the reduction maps $\mathfrak{X}_\eta \to \mathfrak{X}_s$ and $\mathfrak{Y}_\eta \to \mathfrak{Y}_s$, and $\overline{X} = X \widehat{\otimes}_k \widehat{k^{\mathrm{a}}}$. Since the morphism φ is smooth, it induces an isomorphism $\pi^{-1}(\mathbf{y}) \widetilde{\to} \pi^{-1}(\mathbf{x}) \times D$, where D is the open unit disc with center at zero in an affine space, and the required fact follows from acyclicity of the canonical projection $\pi^{-1}(\mathbf{x}) \times D \to \pi^{-1}(\mathbf{x})$ ([Ber93, 7.4.2]). $\square$

6.3. **Comparison theorem.** Suppose we are given a morphism of germs $(B, b) \to (\mathbf{C}, 0)$, an $\mathcal{O}_{B,b}$-scheme $\mathcal{X}$, and a subscheme $\mathcal{Y} \subset \mathcal{X}_s$. Every $\Pi_{\widehat{\mathcal{K}}}$-module Λ can be viewed as a $\Pi_{\mathcal{K}}$-module and, therefore, it gives rise to a locally constant sheaf $\Lambda_{\mathcal{X}^h_\eta}$ on the pro-analytic space $\mathcal{X}^h_\eta$ (see Example 3.3.2(ii)). Since $\mathcal{X}^h_\eta$ is a pro-analytic $\Pi_{\mathcal{K}}$-space (see Example 3.2.1(vi)), values of the complex analytic vanishing cycles functor Ψ_η are abelian $\Pi_{\mathcal{K}}$-sheaves on $\mathcal{X}^h_s$. Furthermore, the formal completion $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ is a special formal scheme over $\widehat{\mathcal{K}}^\circ = \widehat{\mathcal{O}}_{\mathbf{C},0}$, and $R\Psi^h_\eta(\Lambda_{(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta})$ is a complex of abelian $\Pi_{\widehat{\mathcal{K}}}$-sheaves on $\mathcal{Y}^h$.

Theorem 6.3.1. *In the above situation, for any $\Lambda^\cdot \in D^b(\Pi_{\widehat{\mathcal{K}}}\text{-Mod})$ there are canonical isomorphisms*

$$R\Theta(\Lambda^\cdot_{\mathcal{X}^h_\eta})\big|_{\mathcal{Y}^h} \widetilde{\to} R\Theta^h(\Lambda^\cdot_{(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta}) \text{ and } R\Psi_\eta(\Lambda^\cdot_{\mathcal{X}^h_\eta})\big|_{\mathcal{Y}^h} \widetilde{\to} R\Psi^h_\eta(\Lambda^\cdot_{(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta}) \ .$$

Proof. Theorem 6.2.1 reduces the situation to the case $\mathcal{Y} = \mathcal{X}_s$, and since the complexes of nearby cycles are expressed from those of vanishing cycles (see §1.3 and §6.1), it suffices to prove the required fact only for the latter. Consider first the case $\Lambda^\cdot = \mathbf{Z}$. By Temkin's theorem on desingularization from [Tem08], there exists a proper hypercovering $a : \mathcal{Y}_\bullet \to \mathcal{X}$ of $\mathcal{X}$ such that each scheme $\mathcal{Y}_n$ is regular and the supports of the subschemes $\mathcal{Y}_{n,s}$ and $\widetilde{\mathcal{Y}}_n$ are divisors with strict normal crossings. Then there are canonical isomorphisms

$$R\Psi_\eta(\mathbf{Z}_{\mathcal{X}^h_\eta}) \widetilde{\to} Ra^h_{s*}(R\Psi_\eta(\mathbf{Z}_{\mathcal{Y}^h_{\bullet\eta}})) \ .$$

By Theorem 1.5.2, one has

$$R\Psi_\eta(\mathbf{Z}_{\mathcal{Y}^h_{\bullet\eta}}) \widetilde{\to} R\overline{\tau}_{\bullet *}(\mathbf{Z}^\cdot_{\overline{Y^{\log}_\bullet}})$$

Since $\widehat{a} : \widehat{\mathcal{Y}}_\bullet \to \widehat{\mathcal{X}}$ is a proper hypercovering of $\widehat{\mathcal{X}}$, and all of the formal schemes $\widehat{\mathcal{Y}}_n$ are distinguished, the required isomorphisms (for $\Lambda^\cdot = \mathbf{Z}$) follow from the construction in §6.1. If $\Lambda^\cdot$ is arbitrary, they follow from Theorem 1.5.2 and the definition in §6.1. $\square$

7. Continuity theorems

7.1. **Formulation of results.** The first theorem is an easy consequence of previous results. Recall that the group of automorphisms of a special formal scheme $\mathfrak{X}$ trivial modulo an ideal of definition $\mathcal{J}$ is denoted (in [Ber96b]) by $\mathcal{G}_{\mathcal{J}}(\mathfrak{X})$.

Theorem 7.1.1. *Let $\mathcal{J}$ be the square of the maximal ideal of definition of $\mathfrak{X}$. Then for every Π_K-module Λ and every $q \geq 0$, the group $\mathcal{G}_{\mathcal{J}}(\mathfrak{X})$ acts trivially on the sheaves $R^q\Psi^h_\eta(\Lambda_{\mathfrak{X}_\eta})$.*

Proof. It suffices to show that, for every point $x \in \mathfrak{X}^h_s$ and every $q \geq 0$, the group $\mathcal{G}_{\mathcal{J}}(\mathfrak{X})$ acts trivially on the stalk $R^q\Psi^h_\eta(\Lambda_{\mathfrak{X}_\eta})_x$. By Theorem 6.2.1, the latter coincides with $R^q\Psi^h_\eta(\Lambda_{\mathfrak{Y}_\eta})$ for the affine formal scheme $\mathfrak{Y} = \mathfrak{X}_{/\{x\}}$. This reduces the situation to the case $\mathfrak{X} = \mathfrak{Y}$. If the Π_K-module Λ is torsion, the statement follows from the fact that the group $\mathcal{G}_{\mathcal{J}}(\mathfrak{X})$ is uniquely divisible (see [Ber94, Lemma 8.7]). Suppose now that Λ has no torsion. It is then flat over $\mathbf{Z}$ and, therefore, $R^q\Psi^h_\eta(\Lambda_{\mathfrak{X}_\eta}) = R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}) \otimes_{\mathbf{Z}} \underline{\Lambda}_{\mathfrak{X}^h_s}$. This reduces the situation to the case $\Lambda = \mathbf{Z}$. Since $R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ is a finitely generated abelian group and, for every $n \geq 1$, its quotient by the subgroup of elements divisible by n embeds in the finite group $R^q\Psi_\eta((\mathbf{Z}/n\mathbf{Z})_{\mathfrak{X}_\eta})$, it suffices to show that the action of $\mathcal{G}_{\mathcal{J}}(\mathfrak{X})$ on the latter is trivial. But this follows from the previous case. Finally, if Λ is arbitrary, let $\Lambda^{(tors)}$ be the torsion Π_K-submodule of Λ, and denote by A and B the image and cokernel of the homomorphism $R^q\Psi^h_\eta(\Lambda^{(tors)}_{\mathfrak{X}_\eta}) \to R^q\Psi^h_\eta(\Lambda_{\mathfrak{X}_\eta})$. Since B embeds in $R^q\Psi^h_\eta(\Lambda^{(nont)}_{\mathfrak{X}_\eta})$, where $\Lambda^{(nont)} = \Lambda/\Lambda^{(tors)}$, the group $\mathcal{G}_{\mathcal{J}}(\mathfrak{X})$ acts trivially on A and B. It follows that its image in the automorphism group of $R^q\Psi^h_\eta(\Lambda_{\mathfrak{X}_\eta})$ embeds in the torsion group $\mathrm{Hom}(B, A)$, and the same fact on unique divisibility of $\mathcal{G}_{\mathcal{J}}(\mathfrak{X})$ implies that the image is trivial. $\square$

In the following theorems, the formal schemes considered are assumed to be quasicompact special over K°.

Theorem 7.1.2. *Given $\mathfrak{X}$ with rig-smooth generic fiber, there exists $n \geq 1$ such that, for every Π_K-module Λ which is either finite or has no $\mathbf{Z}$-torsion, every $\mathfrak{Y}$* of finite type over K°, *every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ which are congruent modulo $(K^{\circ\circ})^n$, and every q, one has $\theta^{h,q}_\eta(\varphi, \Lambda) = \theta^{h,q}_\eta(\psi, \Lambda)$.*

Theorem 7.1.3. *Given $\mathfrak{X}$ and $\mathfrak{Y}$ with rig-smooth generic fibers, there exists an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ such that, for every Π_K-module Λ which is either finite or has no $\mathbf{Z}$-torsion, every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ which are congruent modulo $\mathcal{J}$, and every q, one has $\theta^{h,q}_\eta(\varphi, \Lambda) = \theta^{h,q}_\eta(\psi, \Lambda)$.*

Theorem 7.1.2 and 7.1.3 are deduced from the following Theorems 7.1.4 and 7.1.5, respectively, in which k is an arbitrary non-Archimedean field with nontrivial discrete valuation and $\mathrm{char}(\widetilde{k}) = 0$, G is the Galois group $\mathrm{Gal}(k^{\mathrm{a}}/k)$ for a fixed algebraic closure k^{a} of k, and the formal schemes considered are quasicompact special over k°.

Theorem 7.1.4. *Given $\mathfrak{X}$ with rig-smooth generic fiber, there exists $n \geq 1$ such that, for every finite discrete G-module Λ, every $\mathfrak{Y}$* of finite type over k°, *every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ which are congruent modulo $(k^{\circ\circ})^n$, and every q, one has $\theta^q_\eta(\varphi, \Lambda) = \theta^q_\eta(\psi, \Lambda)$.*

Theorem 7.1.5. *Given $\mathfrak{X}$ and $\mathfrak{Y}$ with rig-smooth generic fibers, there exists an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ such that, for every finite discrete G-module Λ, every pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ which are congruent modulo $\mathcal{J}$, and every q, one has $\theta^q_\eta(\varphi, \Lambda) = \theta^q_\eta(\psi, \Lambda)$.*

If Λ in Theorems 7.1.2 and 7.1.3 are finite, the required statements follow directly from the corresponding Theorems 7.1.4 and 7.1.5. If Λ has no $\mathbf{Z}$-torsion then, as in the proof of Theorem 7.1.1, the statements are reduced to the case $\Lambda^{\cdot} = \mathbf{Z}$, which follows from the torsion case $\Lambda = \mathbf{Z}/n\mathbf{Z}$ with $n \geq 1$.

7.2. **Proof of Theorem 7.1.4.** Let ϖ be a generator of the maximal ideal $k^{\circ\circ}$ of k°. Instead of the letter n, which will be used for a purpose different from that in the formulation, we will use the letter l.

Step 1. *The theorem is true with* $l = 3$ *if* $\mathfrak{X}$ *is distinguished.* In the first substep 1.1, we do not assume that $\mathrm{char}(\widetilde{k}) = 0$.

Substep 1.1. Let $\mathfrak{A}^1 = \mathrm{Spf}(k^\circ\{T\})$ be the formal affine line over k°, and let 0 and 1 be the k°-points of $\mathfrak{A}^1$ which correspond to the homomorphisms $k^\circ\{T\} \to k^\circ$ that take T to 0 and 1, respectively. A *homotopy* between two morphisms of special formal schemes over k°, $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$, is a morphism $\Phi : \mathfrak{Y} \times \mathfrak{A}^1 \to \mathfrak{X}$ such that $\Phi(\cdot, 0) = \varphi$ and $\Phi(\cdot, 1) = \psi$ (cf. [MW68, 2.7]).

Suppose $\mathfrak{X} = \mathrm{Spf}(A)$, where $A = k^\circ\{T_1, \dots, T_n\}/(T_1^{e_1} \cdot \ldots \cdot T_m^{e_m} - \varpi)$, $1 \leq m \leq n$, and $e_i \geq 1$ for all $1 \leq i \leq m$, and suppose that at least one of the integers e_i is not divisible by $\mathrm{char}(\widetilde{k})$. Let also $\mathfrak{Y}$ be a special formal scheme flat over k°. *We claim that, given two morphisms* $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ *congruent modulo* ϖ^3*, there exists a homotopy* $\Phi : \mathfrak{Y} \times \mathfrak{A}^1 \to \mathfrak{X}$ *between them which is trivial modulo* ϖ^2, i.e., it coincides modulo ϖ^2 with the composition of the projection $\mathfrak{Y} \times \mathfrak{A}^1 \to \mathfrak{Y}$ and φ. (The latter property implies that, for any subscheme $\mathcal{Z} \subset \mathfrak{X}_s$ that contains $\varphi_s(\mathfrak{Y}_s) = \psi_s(\mathfrak{Y}_s)$, Φ induces a homotopy between the induced morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}_{/\mathcal{Z}}$.)

Indeed, the two morphisms from the claim are defined by the elements $f_i = \varphi^*(T_i)$ and $g_i = \psi^*(T_i)$, $1 \leq i \leq n$. Since $\mathfrak{Y}$ is flat over k°, it follows that, for every $1 \leq i \leq n$, one has $g_i - f_i = \varpi^3 u_i$ with $u_i \in \mathcal{O}(\mathfrak{Y})$. Suppose that e_1 is not divisible by $\mathrm{char}(\widetilde{k})$. For $2 \leq i \leq n$, we set $H_i = f_i + \varpi^3 u_i T \in \mathcal{O}(\mathfrak{Y} \times \mathfrak{A}^1)$, and we have

$$f_1^{e_1} H_2^{e_2} \cdot \ldots \cdot H_m^{e_m} = f_1^{e_1}(f_2 + \varpi^3 u_2 T)^{e_2} \cdot \ldots \cdot (f_m + \varpi^3 u_m T)^{e_m} = \varpi(1 + \varpi^2 v T) \ ,$$

where $v \in \mathcal{O}(\mathfrak{Y} \times \mathfrak{A}^1)$. Since e_1 is not divisible by $\mathrm{char}(\widetilde{k})$, there exists an element $\alpha = \sqrt[e_1]{1 + \varpi^2 v T}$ congruent to one modulo ϖ^2. Then the element $H_1 = f_1 \alpha^{-1}$ is congruent to g_1 modulo ϖ^2, and one has

$$H_1^{e_1} \cdot H_2^{e_2} \cdot \ldots \cdot H_m^{e_m} = \varpi \ .$$

This means that there is a well defined homomorphism $A \to \mathcal{O}(\mathfrak{Y} \times \mathfrak{A}^1) : T_i \mapsto H_i$, $1 \leq i \leq n$. We are going to show that the induced morphism $\Phi : \mathfrak{Y} \times \mathfrak{A}^1 \to \mathfrak{X}$ is a homotopy between φ and ψ. By the construction, one has $H_i(0) = f_i$ for all $1 \leq i \leq n$, i.e., $\Phi(\cdot, 0) = \varphi$, and $H_i(1) = g_i$ for all $2 \leq i \leq n$. Since $g_1^{e_1} \cdot g_2^{e_2} \cdot \ldots \cdot g_m^{e_m} = \varpi$, $H_1(1)^{e_1} \cdot g_2^{e_2} \cdot \ldots \cdot g_m^{e_m} = \varpi$, and the homomorphism $\mathcal{O}(\mathfrak{Y}) \to \mathcal{O}(\mathfrak{Y}) \otimes_{k^\circ} k$ is injective, we get $H_1(1)^{e_1} = g_1^{e_1}$. The latter implies that $H_1(1) = g_1 \zeta$ for an e_1-th root of one ζ. Since H_1 is congruent to g_1 modulo ϖ^2, it follows that $\zeta = 1$, i.e., $H(1) = g_1$ and, therefore, $\Phi(\cdot, 1) = \psi$. This implies the claim.

Substep 1.2. *The claim of Step 1 is true if* $\mathfrak{X}$ *is the same as in Substep 1.1.* Indeed, suppose we are given a special formal scheme $\mathfrak{Y}$ (not necessarily of finite type) over k°, and two morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ that coincide modulo ϖ^3. We are going to show that $\theta_\eta^q(\varphi, \Lambda) = \theta_\eta^q(\psi, \Lambda)$ for all Λ and all q. First of all, since the sheaves considered are constructible, it suffices to show that, for every closed point $\overline{\mathbf{y}} \in \mathfrak{Y}_{\overline{s}}$, the homomorphisms $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})_{\overline{\mathbf{x}}} \to R^q\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})_{\overline{\mathbf{y}}}$ induced by φ

and ψ coincide, where $\overline{\mathbf{x}}$ is the image of $\overline{\mathbf{y}}$ in $\mathfrak{X}_{\overline{s}}$. Replacing the field k by a finite unramified extension, we may assume that the points $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ are over $\widetilde{k}$-rational points $\mathbf{x} \in \mathfrak{X}_s$ and $\mathbf{y} \in \mathfrak{Y}_s$, respectively. Furthermore, by Theorem 6.2.2, one has $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})\big|_{\{\overline{\mathbf{x}}\}} \widetilde{\to} R^q\Psi_\eta(\Lambda_{(\mathfrak{X}_{/\{\mathbf{x}\}})_\eta})$ and $R^q\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})\big|_{\overline{\{\mathbf{y}\}}} \widetilde{\to} R^q\Psi_\eta(\Lambda_{(\mathfrak{Y}_{/\{\mathbf{y}\}})_\eta})$. We may therefore replace $\mathfrak{X}$ by $\mathfrak{X}_{/\{\mathbf{x}\}}$ and $\mathfrak{Y}$ by $\mathfrak{Y}_{/\{\mathbf{y}\}}$ and assume that $\mathfrak{X}_s = \{\mathbf{x}\}$ and $\mathfrak{Y}_s = \{\mathbf{y}\}$. In this case, the sheaves considered are just finite discrete G-modules.

We set $\mathfrak{Z} = \mathfrak{Y} \times \mathfrak{A}^1$ and denote by p the canonical projection $\mathfrak{Z} \to \mathfrak{Y}$ and by i and j the morphisms $\mathfrak{Y} \to \mathfrak{Z} : y \mapsto (y, 0)$ and $(y, 1)$, respectively. It follows from Substep 1.1 that there exists a homotopy $\Phi : \mathfrak{Z} \to \mathfrak{X}$ between φ and ψ. By Theorem 6.2.2, applied to the projection p, $R^q\Psi_\eta(\Lambda_{\mathfrak{Z}_\eta})$ is the constant sheaf on the affine line $\mathfrak{A}^1_s$ over $\widetilde{k}$ associated to the G-module $R^q\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})$ and, therefore, $\theta^q_\eta(\Phi, \Lambda)$ is just a homomorphism between constant sheaves on $\mathfrak{A}^1_s$ associated to a homomorphism between finite discrete G-modules. Since $p \circ i = p \circ j = 1_{\mathfrak{Y}}$, the required fact follows.

Substep 1.3. *The claim of Step 1 is true.* Indeed, by Substep 1.2, it suffices to verify the following two facts:

(1) *given an étale morphism $f : \mathfrak{X}' \to \mathfrak{X}$, if the statement is true for $\mathfrak{X}$ (with some l), it is true for $\mathfrak{X}'$ (with the same l) and, if f is surjective, the converse is also true (with the same l);*

(2) *if $\mathfrak{X} = \mathfrak{Z}_{/\mathcal{Y}}$ for a subscheme $\mathcal{Y} \subset \mathfrak{Z}_s$, if the statement is true for $\mathfrak{Z}$, it is also true for $\mathfrak{X}$ (with the same l).*

(1) By [Ber96b, 2.3(i)], one has $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})\big|_{\mathfrak{X}'_{\overline{s}}} \widetilde{\to} R\Psi_\eta(\Lambda_{\mathfrak{X}'_\eta})$, and this immediately implies the direct implication. Conversely, assume that f is surjective and the statement is true for $\mathfrak{X}'$ with an integer $l \geq 1$. Given two morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ that coincide modulo ϖ^l, we set $\mathfrak{Y}' = \mathfrak{X}' \times_{\mathfrak{X},\varphi} \mathfrak{Y}$, $\mathfrak{Y}'' = \mathfrak{X}' \times_{\mathfrak{X},\psi} \mathfrak{Y}$, and denote by φ' and ψ'' the induced morphisms from $\mathfrak{Y}'$ and $\mathfrak{Y}''$ to $\mathfrak{X}'$, respectively. The canonical isomorphism $\mathfrak{Y}'_s \widetilde{\to} \mathfrak{Y}''_s$ over $\mathfrak{Y}_s$, induces an isomorphism $\mathfrak{Y}' \widetilde{\to} \mathfrak{Y}''$ over $\mathfrak{Y}$. Let ψ' be the composition of the latter isomorphism with ψ''. We get two morphisms $\varphi', \psi' : \mathfrak{Y}' \to \mathfrak{X}'$ that coincide modulo ϖ^l and are compatible with φ and ψ, respectively. By the assumption, we have $\theta^q_\eta(\varphi', \Lambda) = \theta^q_\eta(\psi', \Lambda)$. Since $R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})\big|_{\mathfrak{X}'_{\overline{s}}} \widetilde{\to} R\Psi_\eta(\Lambda_{\mathfrak{X}'_\eta})$ and $R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})\big|_{\mathfrak{Y}'_{\overline{s}}} \widetilde{\to} R\Psi_\eta(\Lambda_{\mathfrak{Y}'_\eta})$ and the étale morphisms $\mathfrak{X}'_s \to \mathfrak{X}_s$ and $\mathfrak{Y}'_s \to \mathfrak{Y}_s$ are surjective, we get $\theta^q_\eta(\varphi, \Lambda) = \theta^q_\eta(\psi, \Lambda)$.

(2) By Theorem 6.2.2, one has $R\Psi_\eta(\Lambda_{\mathfrak{Z}_\eta})\big|_{\overline{\mathcal{Y}}} \widetilde{\to} R\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$, and the required fact follows.

Step 2. *The theorem is true in the general case.*

Substep 2.1 (a little digression). Suppose $\mathfrak{Z}$ is a reduced formal scheme flat and of finite type over k°. If $\mathrm{Spf}(B)$ is an open affine subscheme of $\mathfrak{Z}$ and $\mathcal{B} = B \otimes_{k^\circ} k$, then $\mathcal{B}^\circ = \{g \in \mathcal{B} \big| |g(y)| \leq 1 \text{ for all } y \in \mathcal{M}(\mathcal{B})\}$ is finite over B and coincides with the integral closure of B in $\mathcal{B}$ (see [BGR, 6.4.1/6]). Furthermore, if $C = B_{\{f\}}$ for an element $f \in B$ and $\mathcal{C} = C \otimes_{k^\circ} k$, then $\mathcal{C}^\circ = (\mathcal{B}^\circ)_{\{f\}}$. We can therefore glue all of the affine formal schemes $\mathrm{Spf}(\mathcal{B}^\circ)$ so that we get a finite morphism of formal schemes $\mathfrak{Z}' \to \mathfrak{Z}$ with $\mathfrak{Z}'_\eta \widetilde{\to} \mathfrak{Z}_\eta$ and $B = \mathcal{B}^\circ$ for every open affine subscheme $\mathrm{Spf}(B) \subset \mathfrak{Z}'$, where $\mathcal{B} = B \otimes_{k^\circ} k$. We will say that $\mathfrak{Z}'$ is *the integral closure of $\mathfrak{Z}$ in $\mathfrak{Z}_\eta$*.

Substep 2.2. In order to prove the theorem, we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$ are reduced affine and flat over k°. Since $\mathfrak{X}_\eta$ is regular, there exists a blow-up $\alpha : \mathfrak{X}' \to \mathfrak{X}$ with distinguished $\mathfrak{X}'$ and $\mathfrak{X}'_\eta \widetilde{\to} \mathfrak{X}_\eta$ (see Theorem 2.1.3). The ideal $\mathbf{a} \subset A$, which is the center of the blow-up, contains the element ϖ^l for some $l \ge 1$. We are going to show that the theorem is true with the number $2l + 3$.

Let $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$ be two morphisms which are congruent modulo ϖ^{2l+3}. We set $\mathfrak{Y}''' = \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$, where the fiber product is taken with respect to the morphism φ. Furthermore, let $\mathfrak{Y}''$ be the closed formal subscheme of $\mathfrak{Y}'''$ with the same underlying space and whose structural sheaf is the quotient of that of $\mathfrak{Y}'''$ by the k°-torsion. Finally, let $\mathfrak{Y}'$ be the integral closure of $\mathfrak{Y}''$ in $\mathfrak{Y}''_\eta$ (see Substep 2.1), and denote by φ' the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$. Since $\mathfrak{X}'_\eta \widetilde{\to} \mathfrak{X}_\eta$ and $\mathfrak{Y}'_\eta \widetilde{\to} \mathfrak{Y}'''_\eta$, it follows that $\mathfrak{Y}'_\eta \widetilde{\to} \mathfrak{Y}_\eta$. *We claim that the morphism $\psi_\eta : \mathfrak{Y}'_\eta = \mathfrak{Y}_\eta \to \mathfrak{X}_\eta = \mathfrak{X}'_\eta$ extends to a morphism $\psi' : \mathfrak{Y}' \to \mathfrak{X}'$ which is congruent to φ' modulo ϖ^3.*

Indeed, suppose the ideal $\mathbf{a}$ is generated by elements $f_0 = \varpi^l, f_1, \dots, f_n$. Then $\mathfrak{X}' = \bigcup_{i=0}^n \mathfrak{X}^i$ with $\mathfrak{X}^i = \mathrm{Spf}(A_i)$, where A_i is the quotient of A'_i by the k°-torsion and

$$A'_i = A\{T_0, \dots, T_{i-1}, T_{i+1}, \dots, T_n\}/(f_i T_0 - f_0, \dots, f_i T_n - f_n) \ .$$

Then $\mathfrak{X}^i_\eta = \{x \in \mathfrak{X}_\eta \big| |f_j(x)| \le |f_i(x)| \text{ for } j \ne i\}$. (It is a strictly affinoid subdomain of $\mathfrak{X}_\eta$.) The preimage $\mathfrak{Y}^i$ of $\mathfrak{X}^i$ is an open affine subscheme of $\mathfrak{Y}'$. Let $\mathfrak{Y}^i = \mathrm{Spf}(B_i)$. Then $\mathfrak{Y}^i_\eta = \mathcal{M}(\mathcal{B}_i)$ for $\mathcal{B}_i = B_i \otimes_{k^\circ} k$, and one has $B_i = \mathcal{B}^\circ_i$. By the assumption, one has $\psi^*(f_i) - \varphi^*(f_i) = \varpi^{2l+3} g_i$ with $g_i \in B$ for all $0 \le i \le n$. This easily implies that $\psi_\eta(\mathfrak{Y}^i_\eta) \subset \mathfrak{X}^i_\eta$ for all $0 \le i \le n$. It follows that the morphism ψ_η gives rise to homomorphism $A_i \to \mathcal{B}_i$ whose images lie in B_i and, therefore, it extends to a morphism $\psi' : \mathfrak{Y}' \to \mathfrak{X}'$. It remains to verify that ψ' is congruent to φ' modulo ϖ^3.

Since $B_i = \mathcal{B}^\circ_i$, it suffices to show that $|(\psi^*(f) - \varphi^*(f))(y)| \le |\varpi|^3$ for all $0 \le i \le n$ and all $f \in A_i$. The k°-subalgebra of A_i, generated by the elements $\frac{f_j}{f_i}$ with $j \ne i$, is dense. Since the image of $\mathfrak{Y}^i_\eta$ in $\mathfrak{X}^i_\eta$ is compact, it follows that it suffices to verify the above inequality only for the elements $\frac{f_j}{f_i}$ with $j \ne i$. Notice that $|f_i(x)| \ge |\varpi|^l$ for all points $x \in \mathfrak{X}^i_\eta$. It follows that $\frac{1}{\varphi^*(f_i)}, \frac{1}{\psi^*(f_i)} \in \frac{1}{\varpi^l} B_i$. We therefore have

$$\psi^*\left(\frac{f_j}{f_i}\right) - \varphi^*\left(\frac{f_j}{f_i}\right) = \frac{\varpi^{2l+3}(g_j \varphi^*(f_i) - g_i \varphi^*(f_j))}{\varphi^*(f_i)\psi^*(f_i)} \in \varpi^3 B_i \ ,$$

and the claim follows.

Substep 2.3. *One has $\theta^q_\eta(\varphi, \Lambda) = \theta^q_\eta(\psi, \Lambda)$.* Indeed, by Substep 2.2, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{\alpha} & \mathfrak{X} \\ {\scriptstyle\varphi'}\Big\uparrow\Big\uparrow{\scriptstyle\psi'} & & {\scriptstyle\varphi}\Big\uparrow\Big\uparrow{\scriptstyle\psi} \\ \mathfrak{Y}' & \xrightarrow{\beta} & \mathfrak{Y} \end{array}$$

Since $\mathfrak{Y}'_\eta \widetilde{\to} \mathfrak{Y}_\eta$, one has $R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta}) \widetilde{\to} R\beta_{\overline{s}*}(R\Psi_\eta(\Lambda_{\mathfrak{Y}'_\eta}))$ and, therefore, the required equality is equivalent to the equality $\theta^q_\eta(\varphi\beta, \Lambda) = \theta^q_\eta(\psi\beta, \Lambda)$ which is equivalent, by commutativity of the above diagram, to the equality $\theta^q_\eta(\alpha\varphi', \Lambda) = \theta^q_\eta(\alpha\psi', \Lambda)$. The left hand side of the latter is the composition $\theta^q_\eta(\varphi', \Lambda) \circ \varphi'^*_{\overline{s}}(\theta^q_\eta(\alpha, \Lambda))$, and the right hand side is the composition $\theta^q_\eta(\psi', \Lambda) \circ \psi'^*_{\overline{s}}(\theta^q_\eta(\alpha, \Lambda))$. Since $\varphi'_{\overline{s}} = \psi'_{\overline{s}}$, the required equality follows from the equality $\theta^q_\eta(\varphi',\Lambda)=\theta^q_\eta(\psi',\Lambda)$, which is a consequence of Substep 2.2 and Step 1. $\square$

7.3. **Proof of Theorem 7.1.5.** First of all, we can replace k by the completion of the maximal unramified extension, and so we may assume that the residue field $\widetilde{k}$ is algebraically closed. We also fix a generator ϖ of the maximal ideal $k^{\circ\circ}$ of k°.

Step 1. Let $\beta:\mathfrak{Z}\to\mathfrak{Y}$ be a morphism of finite type such that the theorem is true for the pair $(\mathfrak{X},\mathfrak{Z})$, and suppose that either (1) $\mathfrak{Z}_\eta\widetilde{\to}\mathfrak{Y}_\eta$, or (2) β is a covering in the étale topology of $\mathfrak{Y}$. *Then the theorem is true for the pair* $(\mathfrak{X},\mathfrak{Y})$. Indeed, let $\mathcal{J}$ be an ideal of definition of $\mathfrak{Z}$ such that, for every Λ and every pair of morphisms $\varphi',\psi':\mathfrak{Z}\to\mathfrak{X}$, which are congruent modulo $\mathcal{J}$, one has $\theta^q_\eta(\varphi,\Lambda)=\theta^q_\eta(\psi,\Lambda)$. Let $\mathcal{I}$ be an ideal of definition of $\mathfrak{Y}$ which generates an ideal of definition of $\mathfrak{Z}$ contained in $\mathcal{J}$, and suppose we are given two morphisms $\varphi,\psi:\mathfrak{Y}\to\mathfrak{X}$, which are congruent modulo $\mathcal{I}$.

(1) Given an étale morphism $\mathfrak{X}'\to\mathfrak{X}$, and let $\mathfrak{Y}'$ and $\mathfrak{Y}''$ be its base changes with respect to the morphisms φ and ψ, respectively. Since $\varphi_s=\psi_s$, there is a canonical isomorphism $\mathfrak{Y}'_s\widetilde{\to}\mathfrak{Y}''_s$ which lifts to a unique isomorphism $\mathfrak{Y}'\widetilde{\to}\mathfrak{Y}''$. In this way we get two morphisms $\mathfrak{Y}'\to\mathfrak{X}'$ which are compatible with the morphisms φ and ψ, respectively, and they induce two homomorphisms $H^q(\mathfrak{X}'_{\overline{\eta}},\Lambda)=R^q\Gamma(\mathfrak{X}'_{\overline{\eta}},\Lambda)\to H^q(\mathfrak{Y}'_{\overline{\eta}},\Lambda)$. The equality $\theta^q_\eta(\varphi,\Lambda)=\theta^q_\eta(\psi,\Lambda)$ is equivalent to the property that the latter two homomorphisms always coincide for any étale morphism $\mathfrak{X}'\to\mathfrak{X}$.

We apply the above remark to the morphisms $\varphi',\psi':\mathfrak{Z}\to\mathfrak{X}$, induced by φ and ψ, respectively. By the construction of $\mathcal{I}$, the two morphisms φ' and ψ' are congruent modulo $\mathcal{J}$. It follows that the two homomorphisms $H^q(\mathfrak{X}'_{\overline{\eta}},\Lambda)\to H^q(\mathfrak{Z}'_{\overline{\eta}},\Lambda)$, induced by φ' and ψ', coincide, where $\mathfrak{Z}'=\mathfrak{Z}\times_{\mathfrak{X},\varphi'}\mathfrak{X}'$. Since $\mathfrak{Z}'\widetilde{\to}\mathfrak{Z}\times_{\mathfrak{Y}}\mathfrak{Y}'$, where $\mathfrak{Y}'=\mathfrak{Y}\times_{\mathfrak{X},\varphi}\mathfrak{X}'$, it follows that $\mathfrak{Z}'_\eta\widetilde{\to}\mathfrak{Y}'_\eta$ and, therefore, the two homomorphisms $H^q(\mathfrak{X}'_{\overline{\eta}},\Lambda)\to H^q(\mathfrak{Y}'_{\overline{\eta}},\Lambda)$, induced by φ and ψ, coincide. This implies that the theorem is true for the pair $(\mathfrak{X},\mathfrak{Y})$.

(2) The assumption implies that the two morphisms from $(\varphi\beta)^*_s(R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta}))$ to $R^q\Psi_\eta(\Lambda_{\mathfrak{Z}_\eta})$, induced by φ and ψ, coincide. Since $R^q\Psi_\eta(\Lambda_{\mathfrak{Z}_\eta})=\beta^*_s(R^q\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta}))$ and β is a covering in the étale topology of $\mathfrak{Y}$, it follows that the two morphisms $\varphi^*_s(R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta}))\to R^q\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})$, induced by φ and ψ, also coincide.

Since $\mathfrak{Y}_\eta$ is rig-smooth, we can apply Theorem 2.1.3 to $\mathfrak{Y}$. The above statement (1) then implies that, in order to prove the theorem, it suffices to consider the case when $\mathfrak{Y}$ is distinguished, and (2) implies that it suffices to find an étale neighborhood of every point of $\mathfrak{Y}_s$ in $\mathfrak{Y}$ for which the theorem is true (with $\mathfrak{X}$). We may therefore assume that $\mathfrak{Y}$ is affine and there is an étale morphism $\mathfrak{Y}\to\mathrm{Spf}(\widehat{C})$, where $\widehat{C}$ is the adic completion of $C=k^\circ\{T_1,\dots,T_n\}/(T_1^{e_1}\cdot\ldots\cdot T_m^{e_m}-\varpi)$ with respect to the ideal generated by $T_1\cdot\ldots\cdot T_v$, where $1\le v\le m\le n$, and $e_i\ge 1$ for all $1\le i\le m$. In this case, the ideal $\mathbf{b}\subset\mathcal{O}(\mathfrak{Y})$ generated by the elements $T_1\cdot\ldots\cdot T_v$ and ϖ is an ideal of definition of $\mathfrak{Y}$. Suppose the conclusion of Theorem 7.1.4 holds for the formal scheme $\mathfrak{X}$ with an integer $l\ge 1$. We are going to show that the conclusion of Theorem 7.1.5 for the pair $(\mathfrak{X},\mathfrak{Y})$ with the ideal $\mathbf{b}^{l_1}$, where $l_1=l(e_1+\ldots+e_m)$.

Step 2. Since the sheaves $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ and $R^q\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})$ are constructible, in order to prove the above fact, it suffices to show that for any Λ as in the theorem and any pair of morphisms $\varphi, \psi : \mathfrak{Y} \to \mathfrak{X}$, which are congruent modulo $\mathbf{b}^{l_1}$, the two homomorphisms $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})_{\mathbf{x}} \to R^q\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})_{\mathbf{y}}$, induced by φ and ψ, coincide for all $q \geq 0$ and all closed points $\mathbf{y} \in \mathfrak{Y}_s$, where $\mathbf{x} = \varphi_s(\mathbf{y})$. Recall that, by Theorem 6.2.1, there is a canonical isomorphism $R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})_{\mathbf{x}} \widetilde{\to} R\Psi_\eta(\Lambda_{\mathfrak{Y}'_\eta})$, where $\mathfrak{Y}' = \mathfrak{Y}_{/\{\mathbf{y}\}}$. Thus, the required fact is reduced to the verification of the following statement: given a closed point $\mathbf{y} \in \mathfrak{Y}_s$ and two morphisms $\varphi', \psi' : \mathfrak{Y}' = \mathfrak{Y}_{/\{\mathbf{y}\}} \to \mathfrak{X}$ which are congruent modulo $\mathbf{b}'^{l_1}$, where $\mathbf{b}'$ is the maximal ideal of definition of $\mathfrak{Y}'$, one has $\theta^q_\eta(\varphi', \Lambda) = \theta^q_\eta(\psi', \Lambda)$ for all Λ as in the theorem. Furthermore, since $\mathfrak{Y}' \widetilde{\to} \mathfrak{Z}_{/\{\mathbf{z}\}}$, where $\mathfrak{Z} = \mathrm{Spf}(C)$ with C from Step 1 and $\mathbf{z}$ is the image of $\mathbf{y}$ in $\mathfrak{Z}$, we may replace $\mathfrak{Y}$ by $\mathfrak{Z}$, i.e., $\mathfrak{Y} = \mathrm{Spf}(C)$ (we do not need the morphisms φ and ψ anymore).

Step 3. Suppose that $T_i(\mathbf{y}) = 0$ for $1 \leq i \leq u$ and $T_i(\mathbf{y}) \neq 0$ for $u+1 \leq i \leq m$. If $T_i(\mathbf{y}) = 0$ for some $m+1 \leq i \leq n$, we can replace such T_i by $T_i - 1$, and so we may assume that $T_i(\mathbf{y}) \neq 0$ precisely for $u+1 \leq i \leq n$. Then we may replace $\mathfrak{Y}$ by the open affine subscheme defined by the inequality $T_{u+1} \cdot \ldots \cdot T_n \neq 0$, i.e., we may replace C by the localization $C_{\{T_{u+1} \cdot \ldots \cdot T_n\}}$. Furthermore, the homomorphism

$$B = k^\circ\{T_1, \ldots, T_u, T^{\pm 1}_{u+1}, \ldots, T^{\pm 1}_n\}/(T_1^{e_1} \cdot \ldots \cdot T_u^{e_u} \cdot T_{u+1} \cdot \ldots \cdot T_m - \varpi) \longrightarrow C$$

that takes each T_i with $u+1 \leq i \leq m$ to $T_i^{e_i}$ and is identical on the other coordinate functions, gives rise to an étale morphism $\mathfrak{Y} \to \mathfrak{Z} = \mathrm{Spf}(B)$. Then we have again $\mathfrak{Y}' \widetilde{\to} \mathfrak{Z}_{/\{\mathbf{z}\}}$, where $\mathbf{z}$ is the image of the point $\mathbf{y}$ in $\mathfrak{Z}_s$, and so we may replace $\mathfrak{Y}$ by $\mathfrak{Z}$, i.e., we may assume that $\mathfrak{Y} = \mathrm{Spf}(B)$ with the above B.

Step 4. For every $u+1 \leq i \leq n$, the element $T_i(\mathbf{y})$ is congruent to $a_i \in (k^\circ)^*$. Replacing such T_i by $T_i a_i^{-1}$, we may assume that $T_i(\mathbf{y}) = 1$ for all $u+1 \leq i \leq n$. Then the maximal ideal of definition $\mathbf{b}'$ of $\mathfrak{Y}'$ is generated by the elements ϖ, T_i for $1 \leq i \leq u$, and $T_i - 1$ for $u+1 \leq i \leq n$, and one has and $\mathfrak{Y}' = \mathrm{Spf}(\widehat{B})$, where $\widehat{B}$ is the $\mathbf{b}'$-adic completion of B. Since each T_i with $u+1 \leq i \leq m$ is congruent to one in $\widehat{B}$, the latter ring contains an e_1-th root of their product $T_{u+1} \cdot \ldots \cdot T_m$. Thus, we can replace T_1 by its product with an invertible element of $\widehat{B}$ so that

$$\widehat{B} \widetilde{\to} k^\circ[[T_1, \ldots, T_u, S_{u+1}, \ldots, S_n]]/(T_1^{e_1} \cdot \cdots \cdot T_u^{e_u} - \varpi) ,$$

where $S_i = T_i - 1$. At this moment we may replace the letter u by m.

Step 5. From the above description of $\widehat{B}$ it follows that there is an isomorphism $\mathfrak{Y}'_\eta \widetilde{\to} Z \times D^{n-m}$, where

$$Z = \{x \in \mathbf{G}^m_{\mathrm{m}} \big| T_1^{e_1}(x) \cdot \cdots \cdot T_m^{e_m}(x) = \varpi \text{ and } |T_i(x)| < 1 \text{ for all } 1 \leq i \leq m\}$$

and D^{n-m} is the open unit polydisc in $\mathbf{A}^{n-m}$ with centre at zero. Notice that the projection $\mathfrak{Y}'_\eta \to Z$ gives rise to isomorphisms

$$H^q(\overline{Z}, \Lambda) \widetilde{\to} H^q(\mathfrak{Y}'_{\overline{\eta}}, \Lambda) = R^q\Psi_\eta(\Lambda_{\mathfrak{Y}'_\eta})$$

for all Λ as in the theorem.

Let $e =$g.c.d.$(e_1, \ldots, e_m)$, and k' a finite extension of k in k^{a} that contains an element ϖ' with $\varpi'^e = \varpi$. Then $Z \widehat{\otimes}_k k'$ is a disjoint union $\coprod_{\xi \in \mu_e} Z^{(\xi)}$ with

$$Z^{(\xi)} = \{x \in \mathbf{G}^m_{\mathrm{m},k'} \big| T_1^{e'_1}(x) \cdot \cdots \cdot T_m^{e'_m}(x) = \xi\varpi' \text{ and } |T_i(x)| < 1 \text{ for all } 1 \leq i \leq m\},$$

where $e'_i = \frac{e_i}{e}$ and, therefore, $\mathfrak{Y}'_{\overline{\eta}} \widetilde{\to} \coprod_{\xi\in\mu_e} Y^{(\xi)}$, where $Y^{(\xi)} = \overline{Z^{(\xi)}} \times \overline{D}^{n-m}$ and $\overline{Z^{(\xi)}} = Z^{(\xi)}\widehat{\otimes}_{k'}\widehat{k^{\mathrm{a}}}$. All of the k'-analytic spaces $Z^{(\xi)}$ are isomorphic, and we are going to describe them.

Let $\mathcal{T}$ be the kernel of the homomorphism of algebraic tori $G^m_{\mathrm{m},k'} \to G_{\mathrm{m},k'}$: $(x_1,\dots,x_m) \mapsto x_1^{e'_1}\cdot\dots\cdot x_m^{e'_m}$. It is a split torus of dimension $m-1$. Furthermore, we can find integers $p_1,\dots,p_m$ with $\sum_{i=1}^m e'_i p_i = 1$. Then the shift $G^m_{\mathrm{m},k'} \to G^m_{\mathrm{m},k'} : (x_1,\dots,x_m) \mapsto (\frac{x_1}{(\xi\varpi')^{p_1}},\dots,\frac{x_m}{(\xi\varpi')^{p_m}})$ takes $Z^{(\xi)}$ to the open subset $\{x\in\mathcal{T}^{\mathrm{an}}\big||t_i(x)| < |\varpi'|^{-p_i}$ for all $1\le i\le m\}$, where $t_i = \frac{T_i}{(\xi\varpi')^{p_i}}$. The latter is the preimage $\tau^{-1}(\mathcal{P})$ of an open convex subset $\mathcal{P}$ of the skeleton $S(\mathcal{T})$ of $\mathcal{T}$ with respect to the retraction map $\tau : \mathcal{T}^{\mathrm{an}} \to S(\mathcal{T})$.

We set $r = |\varpi|^{\frac{1}{e_1+\dots+e_m}}$ and $V = \{y\in\mathfrak{Y}'_\eta\big||g(y)|\le r$ for all $g\in\mathbf{b}'\}$. One has $V\widehat{\otimes}_k k' = \coprod_{\xi\in\mu_p} V^{(\xi)}$, where $V^{(\xi)} = (V\widehat{\otimes}_k k')\cap Y^{(\xi)}$. For every $\xi\in\mu_e$, there is an isomorphism $V^{(\xi)}\widetilde{\to} U\times E^{n-m}_{k'}(0;r)$, where $E^{n-m}_{k'}(0;r)$ is the closed polydisc in $D^{n-m}_{k'}$ of radius r with center at zero and $U = \tau^{-1}(z)$, where z is the point of $S(\mathcal{T})$ with $|T_i(z)| = r$ for all $1\le i\le m$, i.e., U is a poly-annulus with all internal and external poly-radii equal to r.

We claim that, for any Λ, there is a canonical isomorphism of cohomology groups $H^q(\mathfrak{Y}'_{\overline{\eta}},\Lambda)\widetilde{\to} H^q(\overline{V},\Lambda)$. (Notice that the group on the left hand side is $R^q\Psi_\eta(\Lambda_{\mathfrak{Y}'_\eta})$.) Indeed, this follows from [Ber96b, 3.3], which implies that $H^q(\overline{Z^{(\xi)}},\Lambda)\widetilde{\to} H^q(\overline{U},\Lambda)$ (and both of these groups are q-th exterior powers of $\Lambda(-1)$).

Step 6. *The theorem is true.* Indeed, suppose we are given two morphisms $\varphi',\psi' : \mathfrak{Y}' \to \mathfrak{X}$, which are congruent modulo $\mathbf{b}'^{l_1}$ with l_1 as in Step 1. Since both of them go through morphisms to $\mathfrak{X}' = \mathfrak{X}_{/\{\mathbf{x}\}}$, where $\mathbf{x} = \varphi'_s(\mathbf{y})$, it suffices to show that the homomorphisms $H^q(\mathfrak{X}'_{\overline{\eta}},\Lambda) \to H^q(\overline{V},\Lambda)$, induced by φ' and ψ', coincide.

Since $V = \mathcal{M}(\mathcal{C})$ is strictly k-affinoid, we can find an affine formal scheme $\mathfrak{V}$ flat and of finite type over k° with $\mathfrak{V}_\eta = V$. We may also assume that $\mathfrak{V}$ is normal. Then $\mathfrak{V} = \mathrm{Spf}(\mathcal{C}^\circ)$, where $\mathcal{C}^\circ = \{g\in\mathcal{C}\big||g(y)|\le 1$ for all $y\in V\}$. It follows that the canonical immersion $V\to\mathfrak{Y}'_\eta$ is induced by a morphism of formal schemes $\mathfrak{V}\to\mathfrak{Y}'$. Since φ' and ψ' are congruent modulo $\mathbf{b}'^{l_1}$, one has $\varphi'^*(f) - \psi'^*(f)\in\mathbf{b}'^{l_1}$ for all functions $f\in\mathcal{O}(\mathfrak{X}')$. It follows that $|(\varphi'^*(f)-\psi'^*(f))(y)|\le r^{l_1} = |\varpi|^l$ for all points $y\in V$. The latter implies that the restriction of the function $\varphi'^*(f)-\psi'^*(f)$ to V lies in the ideal of $\mathcal{C}^\circ$ generated by ϖ^l, i.e., the morphisms $\mathfrak{V}\to\mathfrak{X}$ induced by φ' and ψ' are congruent modulo ϖ^l. By our choice of l, the two homomorphisms $H^q(\mathfrak{X}'_{\overline{\eta}},\Lambda) \to H^q(\overline{V},\Lambda)$, induced by φ' and ψ', coincide. $\square$

8. Integral cohomology of restricted analytic spaces

8.1. **Construction and first properties.** As in §0.5, we introduce the category K-$\widehat{\mathcal{A}n}$ of *restricted K-analytic spaces*, which is the localization of the category quasicompact special formal schemes flat over K° with respect to *admissible proper* morphisms , i.e., proper morphisms $\mathfrak{Y}\to\mathfrak{X}$ that induce an isomorphism between the generic fibers $\mathfrak{Y}_\eta\widetilde{\to}\mathfrak{X}_\eta$. Its objects are denoted by $\widehat{X}$, $\widehat{Y}$ and so on. The quasicompact special formal schemes flat over K° which give rise to $\widehat{X}$ are said to be *formal models of* $\widehat{X}$. There is an evident faithful (but not fully faithful) functor K-$\widehat{\mathcal{A}n} \to K$-$\mathcal{A}n : \widehat{X}\mapsto X$ so that the generic fiber functor $\mathfrak{X}\mapsto\mathfrak{X}_\eta$ goes through it. Raynaud theory implies that, if $\widehat{Y} \in K\text{-}\widehat{\mathcal{A}n}$ is such that the strictly K-analytic space Y is compact, then for any $\widehat{X} \in K\text{-}\widehat{\mathcal{A}n}$ there is a canonical bijection $\mathrm{Hom}_{K\text{-}\widehat{\mathcal{A}n}}(\widehat{Y}, \widehat{X}) \widetilde{\to} \mathrm{Hom}_{K\text{-}\mathcal{A}n}(Y, X)$. In particular, the above functor gives rise to an equivalence between the full subcategory of $K\text{-}\widehat{\mathcal{A}n}$ formed by formal schemes flat and of finite type over K° and the category of compact strictly K-analytic spaces. We say that a restricted K-analytic space $\widehat{X}$ is *rig-smooth* if the K-analytic space X is rig-smooth. For such $\widehat{X}$, the family of distinguished formal models of $\widehat{X}$ is cofinal in that of all formal models

We fix for every restricted K-analytic space $\widehat{X}$ a formal model $\mathfrak{X}$. Given $\Lambda^{\cdot} \in D^b(\Pi_K\text{-Mod})$, we define complexes of Π_K-modules

$$R\Gamma(\widehat{X}, \Lambda^{\cdot}) = R\Gamma(\mathfrak{X}^h_s, R\Theta^h(\Lambda^{\cdot}_{\mathfrak{X}_\eta})) \text{ and } R\Gamma(\overline{\widehat{X}}, \Lambda^{\cdot}) = R\Gamma(\mathfrak{X}^h_s, R\Psi^h_\eta(\Lambda^{\cdot}_{\mathfrak{X}_\eta})) \ .$$

For a Π_K-module Λ, we also define Π_K-modules

$$H^q(\widehat{X}, \Lambda) = R^q\Gamma(\widehat{X}, \Lambda) \text{ and } H^q(\overline{\widehat{X}}, \Lambda) = R^q\Gamma(\overline{\widehat{X}}, \Lambda) \ .$$

For $\varpi \in \Pi_K$, the corresponding complex and group are denoted by $R\Gamma(\widehat{X}^{(\varpi)}, \Lambda^{\cdot})$ and $H^q(\widehat{X}^{(\varpi)}, \Lambda)$. If X is compact, then $\widehat{X}^{(\varpi)}$ can be viewed as the $\widehat{K^{(\varpi)}}$-analytic space $X^{(\varpi)}$, and $\overline{\widehat{X}}$ can be viewed as a Π_K-space $\varpi \mapsto X^{(\varpi)}$.

Theorem 8.1.1. *The following is true:*

(i) *the complexes* $R\Gamma(\widehat{X}, \Lambda^{\cdot})$ *and* $R\Gamma(\overline{\widehat{X}}, \Lambda^{\cdot})$ *do not depend on the choice of a model up to a canonical isomorphism, and are functorial in* $\widehat{X}$*;*

(ii) *there are canonical isomorphisms*

$$R\Gamma(\overline{\widehat{X}}, \mathbf{Z}) \otimes^{\mathbf{L}}_{\mathbf{Z}} \Lambda^{\cdot} \widetilde{\to} R\Gamma(\overline{\widehat{X}}, \Lambda^{\cdot}) \text{ and } R\Gamma(\widehat{X}, \Lambda^{\cdot}) \widetilde{\to} RI^{\Pi_K}(R\Gamma(\overline{\widehat{X}}, \Lambda^{\cdot})) \ ,$$

where I^{Π_K} *is the functor* $\Pi_K\text{-Mod} \to \mathcal{A}b : \Lambda \mapsto \Lambda^{\Pi_K}$*;*

(iii) $H^q(\widehat{X}, \mathbf{Z})$ *and* $H^q(\overline{\widehat{X}}, \mathbf{Z})$ *are finitely generated abelian groups equal to zero for* $q > 2\dim(X) + 1$ *and* $q > 2\dim(X)$*, respectively;*

(iv) *the action of* Π *on* $H^q(\overline{\widehat{X}}, \mathbf{Z})$ *is quasi-unipotent; if* $\widehat{X}$ *is rig-smooth, there exists* $p \geq 1$ *such that, for every* $q \geq 0$*, the action of the element* $(\sigma^p - 1)^{q+1}$ *on* $H^q(\overline{\widehat{X}}, \mathbf{Z})$ *is zero;*

(v) *if* $\Lambda^{\cdot} \in D^b_c(\mathbf{Z}/n\mathbf{Z}[G_K]\text{-Mod})$*, there are canonical isomorphisms*

$$R\Gamma(\widehat{X}, \Lambda^{\cdot}) \widetilde{\to} R\Gamma(X_{\text{ét}}, \Lambda^{\cdot}) \text{ and } R\Gamma(\overline{\widehat{X}}, \Lambda^{\cdot}) \widetilde{\to} R\Gamma(\overline{X}_{\text{ét}}, \Lambda^{\cdot}) \ .$$

Remarks 8.1.2. (i) The subscript ét in (v) means that the corresponding complexes are considered with respect to the étale site. They are also viewed as complexes of Π_K-modules and, in particular, the second isomorphism is the isomorphism $R\Gamma(\widehat{X}^{(\varpi)}, \Lambda^{(\varpi)\cdot}) \widetilde{\to} R\Gamma(X^{(\varpi)}_{\text{ét}}, \Lambda^{(\varpi)\cdot})$ for each $\varpi \in \Pi_K$.

(ii) By Theorem 8.1.1(i), one can define the cohomology groups $H^q(\widehat{X}, \Lambda)$ and $H^q(\overline{\widehat{X}}, \Lambda)$ canonically as projective limits of the groups $R^q\Gamma(\mathfrak{X}^h_s, R\Theta^h(\Lambda_{\mathfrak{X}_\eta}))$ and $R^q\Psi^h_\eta(\mathfrak{X}^h_s, R\Theta^h(\Lambda_{\mathfrak{X}_\eta}))$, respectively, taken over formal models $\mathfrak{X}$ of $\widehat{X}$.

Proof. (i) Let $\widehat{X}$ and $\widehat{Y}$ be restricted K-analytic spaces with formal models $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, and suppose we are given a morphism $\varphi : \widehat{Y} \to \widehat{X}$. By the definition, there exists a proper morphism $b : \mathfrak{Y}' \to \mathfrak{Y}$ with $\mathfrak{Y}'_\eta \widetilde{\to} \mathfrak{Y}_\eta$ and a morphism $\psi : \mathfrak{Y}' \to \mathfrak{X}$ which gives rise to the morphism φ. Since $\mathfrak{Y}'_\eta \widetilde{\to} \mathfrak{Y}_\eta$, Corollary 6.1.4

implies that $R\Theta^h(\Lambda^{\cdot}_{\mathfrak{Y}_\eta})\widetilde{\to}Rb^h_{s*}(R\Theta^h(\Lambda^{\cdot}_{\mathfrak{Y}'_\eta}))$ and $R\Psi^h_\eta(\Lambda^{\cdot}_{\mathfrak{Y}_\eta})\widetilde{\to}Rb^h_{s*}(R\Psi^h_\eta(\Lambda^{\cdot}_{\mathfrak{Y}'_\eta}))$. It follows that $R\Gamma(\mathfrak{Y}_\eta,\Lambda^{\cdot})\widetilde{\to}R\Gamma(\mathfrak{Y}'_\eta,\Lambda^{\cdot})$ and $R\Gamma(\mathfrak{Y}_{\overline{\eta}},\Lambda^{\cdot})\widetilde{\to}R\Gamma(\mathfrak{Y}'_{\overline{\eta}},\Lambda^{\cdot})$ and, therefore, the morphism φ induces morphisms $R\Gamma(\widehat{X},\Lambda^{\cdot})\to R\Gamma(\widehat{Y},\Lambda^{\cdot})$ and $R\Gamma(\overline{\widehat{X}},\Lambda^{\cdot})\to R\Gamma(\overline{\widehat{Y}},\Lambda^{\cdot})$, which do not depend on the choice of the morphism b. This implies the required statement.

(ii) follows from the corresponding properties of the functors $R\Theta^h$ and $R\Psi^h_\eta$ introduced in §6.1.

(v) follows from Theorem 6.1.7.

(iii) That the groups considered are finitely generated follows from Theorem 6.1.1(iii) and [Ver76, 2.4.2]. That they are zero for $q>2\dim(X)+1$ and $q>2\dim(X)$, respectively, follows from (iv) and the additional fact that the same holds for the Π_K-modules $\mathbf{Z}/n\mathbf{Z}$, $n\geq 1$.

(iv) Quasi-unipotence of the action follows from the similar fact for the sheaves $R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta})$ in Theorem 6.1.1(iv). If $\widehat{X}$ is rig-smooth, one can find a distinguished model $\mathfrak{X}$. Theorem 4.3.1 implies that, for such $\mathfrak{X}$, the group Π acts on the above sheaves through a finite quotient, and the required fact follows from the spectral sequence $E_2^{p,q}=H^p(\mathfrak{X}_s,R^q\Psi^h_\eta(\mathbf{Z}_{\mathfrak{X}_\eta}))\Longrightarrow H^{p+q}(\overline{\widehat{X}},\mathbf{Z})$. □

Corollary 8.1.3. *For every prime l, there are canonical Π_K-equivariant isomorphisms*

$$H^q(\overline{\widehat{X}},\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{Z}_l\widetilde{\to}H^q(\overline{X}_{\text{ét}},\mathbf{Z}_l)=\varprojlim H^q(\overline{X}_{\text{ét}},\mathbf{Z}/l^n\mathbf{Z})\ . \qquad \square$$

The above functors are naturally extended to functors $\widehat{Y}_\bullet\mapsto H^q(\widehat{Y}_\bullet,\Lambda^{\cdot})$ and $\widehat{Y}_\bullet\mapsto H^q(\overline{\widehat{Y}}_\bullet,\Lambda^{\cdot})$ on the category of simplicial restricted K-analytic spaces $\widehat{Y}_\bullet$. The following statement easily follow from Corollary 6.1.5.

Corollary 8.1.4. *Given a compact hypercovering $a:\widehat{Y}_\bullet\to\widehat{X}$, there are canonical isomorphisms $H^q(\widehat{X},\mathbf{Z})\widetilde{\to}H^q(\widehat{Y}_\bullet,\mathbf{Z})$ and $H^q(\overline{\widehat{X}},\mathbf{Z})\widetilde{\to}H^q(\overline{\widehat{Y}}_\bullet,\mathbf{Z})$ and, in particular, there are spectral sequences $E_1^{p,q}=H^q(\widehat{Y}_p,\mathbf{Z})\Longrightarrow H^{p+q}(\widehat{X},\mathbf{Z})$ and $E_1^{p,q}=H^q(\overline{\widehat{Y}}_p,\mathbf{Z})\Longrightarrow H^{p+q}(\overline{\widehat{X}},\mathbf{Z})$.* □

Corollary 8.1.5. *Given a finite covering of a compact strictly K-analytic space X by compact strictly analytic subdomains, $\mathcal{U}=\{U_i\}_{i\in I}$, there are Leray spectral sequences $E_2^{p,q}=\check{H}^p(\mathcal{U},\mathcal{H}^q(\mathbf{Z}))\Longrightarrow H^{p+q}(X,\mathbf{Z})$ and $E_2^{p,q}=\check{H}^p(\mathcal{U},\overline{\mathcal{H}}^q(\mathbf{Z}))\Longrightarrow H^{p+q}(\overline{X},\mathbf{Z})$, where $\mathcal{H}^q(\mathbf{Z})$ and $\overline{\mathcal{H}}^q(\mathbf{Z})$ are the presheaves $U\mapsto H^q(U,\mathbf{Z})$ and $U\mapsto H^q(\overline{U},\mathbf{Z})$ on the category of compact strictly analytic subdomains of X.* □

Suppose we are given a morphism of germs of complex analytic spaces $(B,b)\to(\mathbf{C},0)$, a separated scheme $\mathcal{Y}$ of finite type over $\mathcal{O}_{B,b}$ and flat over $\mathcal{O}_{\mathbf{C},0}$, and a subscheme $\mathcal{Z}\subset\mathcal{Y}_s$. The formal completion $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$ of $\mathcal{Y}$ along $\mathcal{Z}$ as a special formal scheme over $\widehat{\mathcal{K}}^\circ$. The scheme $\mathcal{Y}$ also defines a pro-analytic space $\mathcal{Y}^h$ over $\mathbf{C}$ and a pro-topological space $\overline{\mathcal{Y}^h}$ over $\overline{\mathbf{C}}$ (see §1.4).

Theorem 8.1.6. *In the above situation, there are canonical isomorphisms*

$$H^q(\mathcal{Y}^h(\mathcal{Z}^h_s)_\eta,\mathbf{Z})\widetilde{\to}H^q((\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta,\mathbf{Z})\text{ and }H^q(\mathcal{Y}^h(\mathcal{Z}^h_s)_{\overline{\eta}},\mathbf{Z})\widetilde{\to}H^q((\widehat{\mathcal{Y}}_{/\mathcal{Z}})_{\overline{\eta}},\mathbf{Z})\ .$$

Recall that the groups on the left hand sides are the inductive limits $\varinjlim H^q(V_\eta,\mathbf{Z})$ and $\varinjlim H^q(V_{\overline{\eta}},\mathbf{Z})$ taken over open neighborhoods of $\mathcal{Z}^h$ in (a representative of) $\mathcal{Y}^h$, where V_η is the preimage of $\mathbf{C}^*$ in V and $V_{\overline{\eta}} = V_\eta \times_{\mathbf{C}^*} \mathbf{C}$ with the fiber product taken with respect to the exponential map $\mathbf{C} \to \mathbf{C}^*$.

Proof. Comparison Theorem 6.3.1 implies that there are canonical isomorphisms $R^q\Gamma(\mathcal{Z}^h, R\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta})) \widetilde{\to} H^q((\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta, \mathbf{Z})$ (resp. $R^q\Gamma(\mathcal{Z}^h, R\Psi_\eta(\mathbf{Z}_{\mathcal{Y}^h_\eta})) \widetilde{\to} H^q((\widehat{\mathcal{Y}}_{/\mathcal{Z}})_{\overline{\eta}}, \mathbf{Z})$). Furthermore, since $\mathcal{Y}$ is separated, each representative of $\mathcal{Y}^h$ is a paracompact topological space. It follows that $\mathcal{Z}^h$ has a fundamental system of open paracompact neighborhoods in $\mathcal{Y}^h$ and, by [Gro57, §3.10], one has

$$R^q\Gamma(\mathcal{Z}^h, R\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta})) = \varinjlim R^q\Gamma(V, Rj_*\mathbf{Z}_{\mathcal{Y}^h_\eta}) = \varinjlim H^q(V_\eta, \mathbf{Z}) \ .$$

This gives the first isomorphism. The second isomorphism is established in a similar way. For this we use a construction from [SGA7, Exp. XIV].

Let also $\overline{\mathbf{C}}$ denote the set $\mathbf{C} \cup \{\infty\}$ provided with the topology which extends that on $\mathbf{C}$ and such that a fundamental system of open neighborhoods of ∞ is formed by the sets $\{z \in \mathbf{C} \big| \mathrm{Re}(z) < r\} \cup \{\infty\}$, $r \in \mathbf{R}$. Then the exponential map $\mathbf{C} \to \mathbf{C}^*$ extends to a continuous map $\overline{\mathbf{C}} \to \mathbf{C}$ that takes ∞ to zero, and the action of $\pi_1(\mathbf{C}^*)$ on $\mathbf{C}$ extends to a continuous action on $\overline{\mathbf{C}}$. It is easy to see that the space $\overline{\mathbf{C}}$ is homeomorphic to the subset $\{0\} \cup \{z \in \mathbf{C} \big| \mathrm{Re}(z) > 0\} \subset \mathbf{C}$. In particular, it is metrizable. Given a pro-analytic space $\mathbf{X}$ over $\mathbf{D}$, we set $\overline{\mathbf{X}} = \mathbf{X} \times_{\mathbf{C}} \overline{\mathbf{C}}$. Then the last diagram in §1.3 can be complemented as follows

$$\begin{tikzcd}
\mathbf{X}_{\overline{\eta}} \arrow[r, "\widetilde{j}"] \arrow[d] \arrow[dr, "\overline{j}"] & \overline{\mathbf{X}} \arrow[d] & \\
\mathbf{X}_\eta \arrow[r, "j"] & \mathbf{X} & \mathbf{X}_s \arrow[l, "i"] \arrow[ul, "\overline{i}"']
\end{tikzcd}$$

Here $\widetilde{j}$ is an open immersion, and the complement of its image is $\overline{i}(\mathbf{X}_s)$. Notice that, for any point $x \in \mathbf{X}_s$, each open neighborhood of the point $\overline{i}(x)$ in $\overline{\mathbf{X}}$ contains the preimage of an open neighborhood of the point $i(x)$ in $\mathbf{X}$. It follows that, for any abelian sheaf F on $\mathbf{X}_\eta$, there are canonical isomorphisms $\overline{i}^*(R\widetilde{j}_*(F)) \widetilde{\to} R\Psi_\eta(F)$.

Applying the above construction to the pro-analytic space $\mathcal{Y}^h$, we get a pro-topological space $\overline{\mathcal{Y}^h}$. Since representatives of $\mathcal{Y}^h$ are metrizable, then so are representatives of $\overline{\mathcal{Y}^h}$. It follows that $\mathcal{Z}^h$ has a fundamental system of open paracompact neighborhoods $\mathcal{V}$ in $\overline{\mathcal{Y}^h}$ and, therefore, $R^q\Gamma(\mathcal{Z}^h, R\Psi_h(\mathbf{Z}_{\mathcal{Y}^h})) = \varinjlim R^q\Gamma(\mathcal{V}, R\widetilde{j}_*\mathbf{Z}_{\mathcal{Y}^h})$. Since each open neighborhood of $\mathcal{Z}^h$ in $\overline{\mathcal{Y}^h}$ contains the preimage of an open neighborhoods of $\mathcal{Z}^h$ in $\mathcal{Y}^h$, the latter group coincides with $\varinjlim H^q(V_{\overline{\eta}}, \mathbf{Z})$ as in the formulation. $\square$

Corollary 8.1.7. *For every proper scheme $\mathcal{Y}$ over $\mathcal{K}$, there are functorial isomorphisms*

$$H^q(\mathcal{Y}^h, \mathbf{Z}) \widetilde{\to} H^q(\mathcal{Y}^{\mathrm{an}}, \mathbf{Z}) \text{ and } H^q(\overline{\mathcal{Y}^h}, \mathbf{Z}) \widetilde{\to} H^q(\overline{\mathcal{Y}^{\mathrm{an}}}, \mathbf{Z}) \ ,$$

where $\overline{\mathcal{Y}^h} = \mathcal{Y}^h \times_{\mathbf{C}^*} \mathbf{C}$.

Proof. We can find an open embedding $\mathcal{Y} \hookrightarrow \mathcal{Y}'$ in a proper scheme $\mathcal{Y}'$ over $\mathcal{O}_{\mathbf{C},0}$ for which $\mathcal{Y} = \mathcal{Y}'_\eta$ and $\mathcal{Y}^{\mathrm{an}} = \widehat{\mathcal{Y}}'_\eta$, and the inductive limit in Theorem 8.1.6 can be taken over the preimages of open neighborhoods of zero in $\mathbf{C}$. This gives the required isomorphisms. $\square$

Remark 8.1.8. An example of an admissible proper morphism is an *admissible blow-up* , i.e., a blow-up with the property that the restriction of its center $\mathcal{I}$ to every open quasicompact subscheme contains a nonzero element of $K^{\circ\circ}$. It would be interesting to know if the family of admissible blow-ups $\mathfrak{X}' \to \mathfrak{X}$ for a quasicompact special formal scheme $\mathfrak{X}$ is cofinal in that of all admissible proper morphisms. This is true if $\mathfrak{X}$ is of finite type over K°. In general, this would imply that K-$\widehat{\mathcal{A}n}$ coincides with the localization of the category of quasicompact special formal schemes with respect to admissible formal blow-ups. Notice that the canonical functor from the latter category to K-$\mathcal{A}n$ goes through the category of uniformly rigid spaces introduced by Kappen [Kap12]

8.2. **Compatibility with integral cohomology of algebraic varieties.** Suppose we are given a morphism of germs $(B, b) \to (\mathbf{C}, 0)$, and set $\mathcal{T} = \mathrm{Spec}(\mathcal{O}_{B,b})$ and $\mathcal{T}_\eta = \mathcal{T} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathcal{K}$. The formal completion $\widehat{\mathcal{T}} = \mathrm{Spf}(\widehat{\mathcal{O}}_{B,b})$ is a special formal scheme over $\widehat{\mathcal{K}}^\circ = \widehat{\mathcal{O}}_{\mathbf{C},0}$.

Every scheme $\mathcal{X}$ of finite type over $\mathcal{T}_\eta$ defines a complex pro-analytic space $\mathcal{X}^h$ over $\mathbf{D}^*$. One sets $\overline{\mathcal{X}^h} = \mathcal{X}^h \times_{\mathbf{D}^*} \overline{\mathbf{D}^*}$ (it is a $\Pi_{\mathcal{K}}$-space). The base change $\mathcal{X} \otimes_{\mathcal{O}_{B,b}} \widehat{\mathcal{O}}_{B,b}$ is a scheme of finite type over $\mathrm{Spec}(\widehat{\mathcal{O}}_{B,b} \otimes_{\widehat{\mathcal{K}}^\circ} \widehat{\mathcal{K}})$ and, therefore, it defines a strictly $\widehat{\mathcal{K}}$-analytic space $\mathcal{X}^{\mathrm{an}}$ over $\widehat{\mathcal{T}}_\eta$, which will be called the *(non-Archimedean) analytification* of $\mathcal{X}$ (see [Ber15, §3.2]).

Theorem 8.2.1. *Every morphism $\varphi : Y \to \mathcal{X}^{\mathrm{an}}$ from a compact strictly $\widehat{\mathcal{K}}$-analytic space Y to the analytification $\mathcal{X}^{\mathrm{an}}$ of a separated scheme $\mathcal{X}$ of finite type over $\mathcal{T}_\eta$ gives rise to homomorphisms*

$$H^q(\mathcal{X}^h, \mathbf{Z}) \to H^q(Y, \mathbf{Z}) \text{ and } H^q(\overline{\mathcal{X}^h}, \mathbf{Z}) \to H^q(\overline{Y}, \mathbf{Z})$$

functorial in Y and $\mathcal{X}$.

Remark 8.2.2. Functoriality in Y and $\mathcal{X}$ means that, given a morphism of compact strictly $\widehat{\mathcal{K}}$-analytic spaces $Y' \to Y$ and a morphism of schemes $\mathcal{X} \to \mathcal{X}'$ compatible with a morphism of germs $(B, b) \to (B', b')$ over $(\mathbf{C}, 0)$, where $\mathcal{X}'$ is a separated scheme of finite type over $\mathcal{T}'_\eta$ and $\mathcal{T}' = \mathrm{Spec}(\mathcal{O}_{B',b'})$, the following diagrams are commutative

$$\begin{array}{ccc} H^q(\mathcal{X}^h, \mathbf{Z}) & \longrightarrow & H^q(Y, \mathbf{Z}) \\ \uparrow & & \downarrow \\ H^q(\mathcal{X}'^h, \mathbf{Z}) & \longrightarrow & H^q(Y', \mathbf{Z}) \end{array} \qquad \begin{array}{ccc} H^q(\overline{\mathcal{X}^h}, \mathbf{Z}) & \longrightarrow & H^q(\overline{Y}, \mathbf{Z}) \\ \uparrow & & \downarrow \\ H^q(\overline{\mathcal{X}'^h}, \mathbf{Z}) & \longrightarrow & H^q(\overline{Y}', \mathbf{Z}) \end{array}$$

The vertical arrows here are the canonical ones, the upper horizontal arrows correspond to the morphism $\varphi : Y \to \mathcal{X}^{\mathrm{an}}$, and the lower arrows correspond to the induced morphism $Y' \to \mathcal{X}'^{\mathrm{an}}$.

Let k be a non-Archimedean field with nontrivial discrete valuation, R a Henselian discrete valuation ring whose completion is k°, S a local noetherian flat R-algebra with residue field $\widetilde{k}$, and $\mathcal{K}$ the fraction field of R (e.g., $R = \mathcal{O}_{\mathbf{C},0}$ and $S = \mathcal{O}_{B,b}$ as above). For a scheme of $\mathcal{X}$ of finite type over S, the formal completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along the closed fiber $\mathcal{X}_s$ (defined by the maximal ideal of S) is a special formal scheme over k°, whose generic fiber $\widehat{\mathcal{X}}_\eta$ is a paracompact strictly k-analytic space. We set $\mathcal{X}_\eta = \mathcal{X} \otimes_R \mathcal{K}$, and denote by $\mathcal{X}_\eta^{\mathrm{an}}$ the analytification of the scheme $\mathcal{X}_\eta \otimes_S \widehat{S}$ (defined in [Ber15, §3.2]). There is a canonical morphism $\widehat{\mathcal{X}}_\eta \to \mathcal{X}_\eta^{\mathrm{an}}$. If $\mathcal{X}$ is separated over S, it identifies the former with a closed analytic subdomain of the latter and, if $\mathcal{X}$ is proper over S, then $\widehat{\mathcal{X}}_\eta \widetilde{\to} \mathcal{X}_\eta^{\mathrm{an}}$. If $\mathcal{X}$ is a scheme of finite type over $S \otimes_R \mathcal{K}$, then $\mathcal{X}_\eta = \mathcal{X}$ and we write $\mathcal{X}^{\mathrm{an}}$ instead of $\mathcal{X}_\eta^{\mathrm{an}}$.

Lemma 8.2.3. *Let $\mathcal{X}$ be a separated scheme of finite type over $S \otimes_R \mathcal{K}$, and Σ a compact subset of $\mathcal{X}^{\mathrm{an}}$ such that the subset $\Sigma_0 = \{x \in \Sigma \big| [\mathcal{H}(x) : k] < \infty\}$ is dense in Σ. Then*

(i) *there exists an open embedding $\mathcal{X} \hookrightarrow \mathcal{Y}$ in a separated scheme of finite type over S such that $\mathcal{X} = \mathcal{Y}_\eta$ and $\Sigma \subset \widehat{\mathcal{Y}}_\eta$;*

(ii) *given a homomorphism $S' \to S$ from a similar local R-algebra S', a separated scheme $\mathcal{X}'$ of finite type over $S' \otimes_R \mathcal{K}$, a morphism $\varphi : \mathcal{X} \to \mathcal{X}'$ compatible with the homomorphism $S' \to S$, and an open embedding $\mathcal{X}' \hookrightarrow \mathcal{Y}'$ in a separated scheme of finite type over S' with $\mathcal{X}' = \mathcal{Y}'_\eta$ and $\varphi^{\mathrm{an}}(\Sigma) \subset \widehat{\mathcal{Y}}'_\eta$, there exist separated morphisms of finite type $\mathcal{Y}'' \to \mathcal{Y}$ and $\varphi' : \mathcal{Y}'' \to \mathcal{Y}'$ such that $\mathcal{Y}''_\eta \widetilde{\to} \mathcal{Y}_\eta = \mathcal{X}$, $\varphi'_\eta = \varphi$, and $\Sigma \subset \widehat{\mathcal{Y}}''_\eta$.*

Proof. (i) Step 1. By the Nagata compactification theorem (see [Con07]), there exists an open embedding $\mathcal{X} \hookrightarrow \mathcal{Z}$ in a proper scheme $\mathcal{Z}$ over S flat over R. One has $\widehat{\mathcal{Z}}_\eta = \mathcal{Z}_\eta^{\mathrm{an}}$ and $\Sigma \cap (\mathcal{Z}_\eta \backslash \mathcal{X})^{\mathrm{an}} = \emptyset$. It suffices therefore to verify the following statement. *Given a separated scheme $\mathcal{X}$ of finite type over S, a compact subset $\Sigma \subset \widehat{\mathcal{X}}_\eta$, and a Zariski closed subset $\mathcal{Y} \subset \mathcal{X}_\eta$ with $\mathcal{Y}^{\mathrm{an}} \cap \Sigma = \emptyset$, there exists a blow-up $\mathcal{X}' \to \mathcal{X}$ with $\mathcal{X}'_\eta \widetilde{\to} \mathcal{X}_\eta$ and $\Sigma \subset \widehat{\mathcal{Z}}_\eta$, where $\mathcal{Z}$ is the complement of the Zariski closure of $\mathcal{Y}$ in $\mathcal{X}'$.*

Step 2. *The statement is true if $\mathcal{X} = \mathrm{Spec}(A)$ is an affine scheme.* Indeed let elements $g_1, \dots, g_n \in A$ generate the ideal of $\mathcal{Y}$ in $A \otimes_R \mathcal{K}$. We can find $l \geq 1$ such that the closed analytic domain $W = \{x \in \mathcal{X}_\eta^{\mathrm{an}} \big| |g_i(x)| \leq |\varpi|^l \text{ for all } 1 \leq i \leq n\}$ has empty intersection with Σ, where ϖ is a generator of the maximal ideal of R. Let $f : \mathcal{X}' \to \mathfrak{X}$ be the blow-up of $\mathcal{X}$ whose center is the ideal of A generated by the elements $\varpi^l, g_1, \dots, g_n$. One of the open affine subschemes from the construction of $\mathcal{X}'$ is $\mathcal{W} = \mathrm{Spec}(B)$, where B is the quotient of $A[T_1, \dots, T_n]/(\varpi^l T_i - g_i)_{1 \leq i \leq n}$ by the k°-torsion . Since $\widehat{\mathcal{W}}_\eta = W$, it follows that $\pi'(\Sigma) \cap \mathcal{W}_s = \emptyset$, where π' is the reduction map $\widehat{\mathcal{X}}'_\eta \to \mathcal{X}'_s$. But $\mathcal{W}_s$ contains the intersection $\mathcal{Y}' \cap \mathcal{X}'_s$, where $\mathcal{Y}'$ is the Zariski closure of $\mathcal{Y}$ in $\mathcal{X}'$. Thus, if $\mathcal{Z}$ is the complement of $\mathcal{Y}'$ in $\mathcal{X}'$, then $\pi'(\Sigma) \subset \mathcal{Z}_s$ and, therefore, $\Sigma \subset \widehat{\mathcal{Z}}_\eta$.

Step 3. *The statement is true for arbitrary $\mathcal{X}$.* Indeed, let $\{\mathcal{X}^i\}_{i \in I}$ be a finite open affine covering of $\mathcal{X}$. By Step 2, for every $i \in I$ there exists a blow-up $\mathcal{X}'^i \to \mathcal{X}^i$ with $\mathcal{X}'^i_\eta \widetilde{\to} \mathcal{X}^i_\eta$ and such that $\Sigma \cap \widehat{\mathcal{X}}^i_\eta \subset \widehat{\mathcal{Z}}^i_\eta$, where $\mathcal{Z}^i = \mathcal{X}'^i \backslash \mathcal{Y}^i$ and $\mathcal{Y}^i$ is the Zariski closure of $\mathcal{Y} \cap \mathcal{X}^i_\eta$ in $\mathcal{X}'^i$. For every $i \in I$, the center of the i-th blow-up can be extended to a coherent subsheaf of ideals $\mathcal{J}_i \subset \mathcal{O}_\mathcal{X}$ that contains a nonzero element of $k^{\circ\circ}$. Let $f_i : \mathcal{X}'^i \to \mathcal{X}$ be the blow-up with center $\mathcal{J}_i$. We can find a blow-up $f : \mathcal{X}' \to \mathcal{X}$ whose center contains a nonzero element of $k^{\circ\circ}$ and such that, for every $i \in I$, one has $f = f_i \circ g_i$, where g_i is a morphism $\mathcal{X}' \to \mathcal{X}'^i$. *We claim that $\mathcal{X}'$ possesses the required property.*

Indeed, that property is equivalent to the fact that $\pi'(\Sigma) \cap (\mathcal{Y}' \cap \mathcal{X}'_s) = \emptyset$, where π' is the reduction map $\widehat{\mathcal{X}}'_\eta \to \mathcal{X}'_s$ and $\mathcal{Y}'$ is the Zariski closure of $\mathcal{Y}$ in $\mathcal{X}'$. Suppose there exists a point $x \in \Sigma$ with $\pi'(x) \in \mathcal{Y}' \cap \mathcal{X}'_s$. One has $x \in \Sigma \cap \widehat{\mathcal{X}}^i_\eta$ for some $i \in I$.

Then $\pi'^i(x) \in \mathcal{Y}'^i \cap \mathcal{X}'^i_s$, where π'^i is the reduction map $\widehat{\mathcal{X}}'^i_\eta \to \mathcal{X}'^i_s$ and $\mathcal{Y}'^i$ is the Zariski closure of $\mathcal{Y}$ in $\mathcal{X}'^i$. Since $\mathcal{X}''^i$ is an open subscheme of $\mathcal{X}'^i$, the intersection $\mathcal{Y}'^i \cap \mathcal{X}''^i$ coincides with the Zariski closure of $\mathcal{Y} \cap \mathcal{X}^i_\eta$ in $\mathcal{X}''^i$, i.e., with $\mathcal{Y}^i$, and we get $\pi'^i(x) \in \mathcal{Y}^i \cap \mathcal{X}''^i_s$. This contradicts the assumption $\Sigma \cap \widehat{\mathcal{X}}^i_\eta \subset \widehat{\mathcal{Z}}^i_\eta$.

(ii) Consider the graph morphism $\Gamma_\varphi : \mathcal{X} \to \mathcal{X} \times_{\mathrm{Spec}(S')} \mathcal{X}' = (\mathcal{Y} \times_{\mathrm{Spec}(S')} \mathcal{Y}')_\eta$. *We claim that the closure $\mathcal{Y}''$ of $\Gamma_\varphi(\mathcal{X})$ in $\mathcal{Y} \times_{\mathrm{Spec}(S')} \mathcal{Y}'$ and the induced morphisms $\mathcal{Y}'' \to \mathcal{Y}$ and $\varphi' : \mathcal{Y}'' \to \mathcal{Y}'$ possess the required properties.*

Indeed, by the construction, $\mathcal{X} = \mathcal{Y}''_\eta$ and $\psi'_\eta = \varphi$. It remains to verify that $\Sigma \subset \widehat{\mathcal{Y}}''_\eta$. Since the subset $\widehat{\mathcal{Y}}''_\eta$ is closed in $\mathcal{Y}''^{\mathrm{an}}_\eta$, it suffices to show that it contains all points $x \in \Sigma_0$. The field $\mathcal{H}(x)$ of such a point x is the completion of a finite extension $\mathcal{K}'$ of $\mathcal{K}$. The integral closure R' of R in $\mathcal{K}'$ is a Henselian discrete valuation ring. Since $x \in \widehat{\mathcal{Y}}_\eta$ and $\varphi^{\mathrm{an}}(x) \in \widehat{\mathcal{Y}}'_\eta$, there are associated morphisms $\mathrm{Spec}(R') \to \mathcal{Y}$ and $\mathrm{Spec}(R') \to \mathcal{Y}'$, which give rise to a morphism $\mathrm{Spec}(R') \to \mathcal{Y} \times_{\mathrm{Spec}(S')} \mathcal{Y}'$. The image of $\mathrm{Spec}(\mathcal{K}')$ under the latter lies in $\Gamma_\varphi(\mathcal{X})$. It follows that the image of the closed point of $\mathrm{Spec}(R')$ lies in $\mathcal{Y}''_s$. This implies that $x \in \widehat{\mathcal{Y}}''_\eta$. □

Proof of Theorem 8.2.1. By Lemma 8.2.3(i), there exists an open embedding $\mathcal{X} \hookrightarrow \mathcal{Y}$ in a separated scheme $\mathcal{Y}$ of finite type over $\mathcal{T}$ and flat over $\mathcal{K}^\circ$ such that $\mathcal{X} = \mathcal{Y}_\eta$ and $\varphi(Y) \subset \widehat{\mathcal{Y}}_\eta$. Comparison Theorem 6.3.1 implies that there is a canonical isomorphism $R\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta}) \widetilde{\to} R\Theta^h(\mathbf{Z}_{\widehat{\mathcal{Y}}_\eta})$ and, therefore, the morphism $Y \to \widehat{\mathcal{Y}}_\eta$ induced by φ gives rise to a homomorphism

$$R^q\Gamma(\mathcal{Y}^h_s, R\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta})) \widetilde{\to} R^q\Gamma(\mathcal{Y}^h_s, R\Theta^h(\mathbf{Z}_{\widehat{\mathcal{Y}}_\eta})) = H^q(\widehat{\mathcal{Y}}_\eta, \mathbf{Z}) \to H^q(Y, \mathbf{Z}) \ .$$

Furthermore, the spectral sequence

$$E_2^{p,q} = H^p(\mathcal{Y}^h_s, R^q\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta})) \Longrightarrow R^{p+q}\Gamma(\mathcal{Y}^h_s, R\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta}))$$

gives rise to a homomorphism $E_2^{0,q} = H^0(\mathcal{Y}^h_s, R^q\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta})) \to R^q\Gamma(\mathcal{Y}^h_s, R\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta}))$. The composition of the canonical map $H^q(\mathcal{X}^h, \mathbf{Z}) \to H^0(\mathcal{Y}^h_s, R^q\Theta(\mathbf{Z}_{\mathcal{Y}^h_\eta}))$ with the above two homomorphisms gives the required homomorphism $H^q(\mathcal{X}^h, \mathbf{Z}) \to H^q(Y, \mathbf{Z})$. That it does not depend on the choice of the open embedding $\mathcal{X} \hookrightarrow \mathcal{Y}$ easily follows from Lemma 8.2.3(ii). That this homomorphism is functorial in Y is trivial. Functoriality in $\mathcal{X}$ also easily follows from Lemma 8.2.3(ii). The homomorphism $H^q(\overline{\mathcal{X}^h}, \mathbf{Z}) \to H^q(\overline{Y}, \mathbf{Z})$ is constructed in the same way. □

8.3. **Compatibility with cohomology of the underlying topological space.** Given a K-analytic space X, there are morphisms of sites $X_{\text{ét}} \to |X|$ and $\overline{X}_{\text{ét}} \to |\overline{X}|$, where $|X|$ and $|\overline{X}|$ denote the underlying topological spaces and Π_K-spaces of X and $\overline{X}$, respectively. It follows that, for any abelian group (resp. discrete G_K-module) Λ, there are canonical homomorphisms from $H^q(|X|, \Lambda) \to H^q(X_{\text{ét}}, \Lambda)$ (resp. $H^q(|\overline{X}|, \Lambda) \to H^q(\overline{X}_{\text{ét}}, \Lambda)$) and, for finite Λ's, the groups on the right hand side coincide with the groups $H^q(X, \Lambda)$ (resp. $H^q(\overline{X}, \Lambda)$).

Theorem 8.3.1. *For every restricted K-analytic space $\widehat{X}$ and every abelian group (resp. Π_K-module) Λ, there are canonical homomorphisms*

$$H^q(|X|, \Lambda) \to H^q(\widehat{X}, \Lambda) \ (\textit{resp. } H^q(|\overline{X}|, \Lambda) \to H^q(\overline{\widehat{X}}, \Lambda))$$

which are functorial in Λ *and* X *and, for finite* Λ*'s, coincide with the above homomorphisms.*

Proof. We construct the second homomorphism since the first one is constructed in the same way. Let Λ be a Π_K-module.

Step 1. Suppose that $\widehat{X}$ comes from a formal scheme of the form $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$, where $\mathcal{Y}$ is a strictly semistable scheme over K° and $\mathcal{Z}$ is a union of some of the irreducible components of $\mathfrak{X}_s$. As in the proof of [Ber00, Lemma 4.1], one deduces from results of [Ber99, §5] that there is a canonical isomorphism $H^q(|\overline{X}|,\Lambda)\widetilde{\to}H^q_{\mathrm{Zar}}(\mathcal{Z},\Lambda)$. Furthermore, the canonical homomorphism $\underline{\Lambda}_{\mathcal{Z}^h} \to R\overline{\tau}_*(\underline{\Lambda}_{\overline{(\mathcal{Z}^h)^{\log}}})$ gives rise to a homomorphism

$$H^q(\mathcal{Z}^h,\Lambda) \to H^q(\mathcal{Z}^h, R\Psi^h_\eta(\Lambda_{(\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta})) = H^q(\overline{\widehat{X}},\Lambda) \ .$$

Thus, the canonical homomorphism $H^q_{\mathrm{Zar}}(\mathcal{Z}_s,\Lambda) \to H^q(\mathcal{Z}^h,\Lambda)$ gives rise to the required homomorphism which is functorial in Λ and $\widehat{X}$.

Step 2. Suppose that $\widehat{X}'$ be a restricted K'-analytic space for a finite extension K' of K, and $\widehat{X}$ is the space $\widehat{X}'$ but considered as a restricted K-analytic space. Then $\overline{\widehat{X}}\widetilde{\to}\overline{\widehat{X}'} \times \mathrm{Hom}_K(K',K^{\mathrm{a}})$ with the induced action of the Galois group of K. Step 1 implies that there are isomorphisms $H^q(|\overline{X}|,\Lambda)\widetilde{\to}H^q(\overline{\widehat{X}},\Lambda)$ which are also functorial on Λ and $\widehat{X}$.

Step 3. The functor $\widehat{X} \mapsto H^q(|\overline{X}|,\Lambda)$ is naturally extended to the category of simplicial restricted K-analytic spaces. Thus, if $\widehat{Y}_\bullet$ is a simplicial restricted K-analytic space such that each $\widehat{Y}_n$ is a finite disjoint union of spaces from Step 2, then there are canonical homomorphisms $H^q(|\overline{Y}_{\bullet\eta}|,\Lambda) \to H^q(\overline{\widehat{Y}}_{\bullet\eta},\Lambda)$ which are functorial in Λ and $\widehat{Y}_\bullet$.

Step 4: Let $\widehat{X}$ be a restricted K-analytic space, and let $\mathfrak{X}$ be an arbitrary formal model of X. By Temkin's results from [Tem08] (or Theorem 2.1.3), there exists a compact hypercovering $a:\widehat{Y}_\bullet \to \widehat{X}$ with $\widehat{Y}_\bullet$ as in Step 3. Then there are canonical isomorphisms

$$H^q(|\overline{X}|,\Lambda) \to H^q(|\overline{Y}_{\bullet\eta}|,\Lambda) \to H^q(\overline{\widehat{Y}}_{\bullet\eta},\Lambda) = H^q(\overline{\widehat{X}},\Lambda) \ ,$$

which are easily verified to be functorial in Λ and $\widehat{X}$. ☐

9. Differential forms on distinguished log spaces and germs

9.1. **Complexes $\omega^\cdot_X$ and $\omega^\cdot_{X/R}$.** Recall that, given a morphism of log complex analytic spaces $\varphi : X \to B$, one defines a coherent sheaf of relative logarithmic differentials $\omega^1_{X/B}$ as follows: it is the $\mathcal{O}_X$-module which the quotient of $\Omega^1_{X/B} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} M^{gr}_X)$ by the $\mathcal{O}_X$-submodule generated by local sections of the form $(d\beta(m),0)-(0,\beta(m)\otimes m)$ and $(0,1\otimes n)$ with m and n local sections of M_X and $\varphi^{-1}(M_B)$, respectively. The image of a local section m of M^{gr}_X under the homomorphism $M^{gr}_X \to \omega^1_X$ that takes $m \in M^{gr}_X$ to $(0,1\otimes m)$ is denoted by $d\log(m)$, and one has $d\log(f) = \frac{df}{f}$ for a local section f of $\mathcal{O}^*_X$. If φ is log étale, then $\omega^1_{X/B}=0$.

Notice that homomorphisms of $\mathcal{O}_X$-modules $\omega^1_{X/B} \to \mathcal{O}_X$ are in one-to-one correspondence with $\varphi^{-1}(\mathcal{O}_B)$-linear *log derivations* on $\mathcal{O}_X$, i.e., pairs $(\partial,\overline{\partial})$ consisting of a $\varphi^{-1}(\mathcal{O}_B)$-linear derivation $\partial : \mathcal{O}_X \to \mathcal{O}_X$ and a homomorphism $\overline{\partial} : M_X^{gr} \to \mathcal{O}_X$ (to the sheaf of additive groups $\mathcal{O}_X$) such that $\partial(\beta(m)) = \beta(m)\overline{\partial}(m)$ and $\overline{\partial}(n) = 0$ for all local sections m of M_X and n of $\varphi^{-1}(M_B)$. The $\varphi^{-1}(\mathcal{O}_B)$-linear log derivations of $\mathcal{O}_X$ form a sheaf of Lie $\varphi^{-1}(\mathcal{O}_B)$-algebras $\mathcal{D}er_{X/B}$ with respect to the Lie bracket $[(\partial_1,\overline{\partial}_1),(\partial_2,\overline{\partial}_2)] = ([\partial_1,\partial_2],[\overline{\partial}_1,\overline{\partial}_2])$, where $[\partial_1,\partial_2]$ is defined in the usual way and $[\overline{\partial}_1,\overline{\partial}_2](m) = \partial_1(\overline{\partial}_2(m)) - \partial_2(\overline{\partial}_1(m))$ for local sections m of M_X.

Let $\omega^q_{X/B}$ be the q-th exterior power of $\omega^1_{X/B}$ over $\mathcal{O}_X$. The direct sum $\omega^{\cdot}_{X/B} = \oplus_{q=0}^{\infty}\omega^q_{X/B}$ is a differential graded algebra. If the log structures on X and B are trivial, then $\omega^{\cdot}_{X/B}$ is the usual de Rham differential graded algebra $\Omega^{\cdot}_{X/B}$. The *q-th de Rham cohomology sheaf* $\mathcal{H}^q_{\mathrm{dR}}(X/B)$ is the sheaf on B which is the q-th hypercohomology of the complex $\omega^{\cdot}_{X/B}$. If $B = \mathrm{Spec}(\mathbf{C})^h$ provided with the trivial log structure, the complex and the de Rham cohomology are denoted by $\omega^{\cdot}_X$ and $H^q_{\mathrm{dR}}(X)$, respectively. The classical Poincaré lemma is extended to log spaces as follows: if $\varphi^*(M_B)\widetilde{\to}M_X$ and the morphism of the underlying complex analytic spaces is smooth, then for every point $x \in X$, the canonical morphism of complexes $\omega^{\cdot}_{B,b} \to \omega^{\cdot}_{X,x}$ is a quasi-isomorphism, where $b = \varphi(x)$.

The definition of the relative de Rham complex extends in the evident way to morphisms of log pro-analytic spaces in which all of the transition morphisms are open immersions.

Till the end of this section, X is a distinguished log analytic space over $\mathbf{pt}_R$, where R is from §4, i.e., R is either K_r° for $1 \le r < \infty$, or $\mathcal{K}^\circ = \mathcal{O}_{\mathbf{C},0}$ (in the latter case we set $r = \infty$). Recall also that, if $r = \infty$, X comes from a distinguished log germ (Y, X) over $(\mathbf{C}, 0)$ from Definition 4.1.1(ii), and it is provided with the sheaf of local rings $\mathcal{O}_X = i^{-1}(\mathcal{O}_{Y(X)})$ and the log structure $M_X = i^{-1}(M_{Y(X)})$, where i is the map $X \to Y(X)$. We also set $\omega^{\cdot}_X = i^{-1}(\omega^{\cdot}_{Y(X)})$ and $\omega^{\cdot}_{X/R} = i^{-1}(\omega^{\cdot}_{Y(X)/\mathbf{C}(0)})$, and denote by $H^q_{\mathrm{dR}}(X)$ and $H^q_{\mathrm{dR}}(X/R)$ their cohomology groups with respect to the functor of global sections. Notice that if X has a fundamental system of open paracompact neighborhoods in Y, then the above groups are just the de Rham cohomology groups of the pro-analytic space $Y(X)$, $H^q_{\mathrm{dR}}(Y(X))$ and $H^q_{\mathrm{dR}}(Y(X)/\mathbf{C}(0))$, respectively, and one has

$$H^q_{\mathrm{dR}}(X) = \varinjlim H^q_{\mathrm{dR}}(V) \text{ and } H^q_{\mathrm{dR}}(X/R) = \varinjlim H^q_{\mathrm{dR}}(V/\mathbf{C}) ,$$

where V runs through open neighborhoods of X in Y and the logarithmic structure on the complex plane $\mathbf{C}$ is is generated by the coordinate function z.

The sheaf $\omega^1_{\mathbf{pt}_R}$, which will be denoted by ω^1_R, is a free R-module of rank one with generator $d\log(\varpi)$ for each $\varpi \in \Pi_R$. If ϖ' is another element of Π_R, then $\varpi' = \alpha\varpi$ for $\alpha \in R^*$, and one has $d\log(\varpi') = (1 + \frac{\delta_\varpi(\alpha)}{\alpha})d\log(\varpi)$. The Lie algebra $\mathcal{D}er_R$ of $\mathbf{C}$-linear log derivations on R coincides with the Lie subalgebra of W_R (from Example 3.3.3(iii)) formed by homogeneous elements of degree one.

The sheaves of $\mathcal{O}_X$-modules ω^q_X and $\omega^q_{X/R}$ are locally free, and there is an exact sequence of complexes

$$(*) \qquad 0 \to \omega^1_R \otimes_R \omega^{\cdot}_{X/R}[-1] \xrightarrow{f} \omega^{\cdot}_X \to \omega^{\cdot}_{X/R} \to 0 .$$

Here ω^1_R is considered as a complex in degree one, the homomorphism f takes the element $d\log(\varpi) \otimes \eta$ for a local section η of $\omega^{q-1}_{X/R}$ to the element $d\log(\varpi) \wedge \overline{\eta}$ for a local section $\overline{\eta}$ of ω^{q-1}_X that lifts η. The exact sequence $(*)$ induces a connecting homomorphism

$$\nabla : H^q_{\rm dR}(X/R) \to \omega^1_R \otimes_R H^q_{\rm dR}(X/R)$$

called the *Gauss-Manin connection.* That ∇ is a connection, i.e., $\nabla(\gamma x) = d\gamma \otimes x + \gamma\nabla(x)$ for all $\gamma \in R$ and $x \in H^q_{\rm dR}(X/R)$, follows from the facts it coincides with the differential $d_1^{0,q}$ of the spectral sequence $E_1^{p,q} = R^{p+q}\varphi_*(\mathrm{gr}^p) \Longrightarrow R^{p+q}\varphi_*(\omega^\cdot_X)$ of the filtered object

$$F^0 = \omega^\cdot_X \supset F^1 = \omega^1_R \otimes_R \omega^\cdot_{X/R}[-1] \supset F^2 = 0$$

(see [EGA3, Ch. 0, 13.6.4]), the filtration is compatible with the exterior product, i.e., $F^i \wedge F^j \subset F^{i+j}$, and the sequence of functors $R^q\varphi_*$ is multiplicative (see [KO68]).

For each element $\varpi \in \Pi_R$, the composition of ∇ with the isomorphism $\chi_\varpi : \omega^1_R \widetilde{\to} R : d\log(\varpi) \mapsto 1$ is a homomorphism

$$\delta_\varpi : H^q_{\rm dR}(X/R) \to H^q_{\rm dR}(X/R)$$

so that $\nabla(x) = \delta_\varpi(x) \otimes d\log(\varpi)$ for $x \in H^q_{\rm dR}(X/R)$. One has $\delta_\varpi\widetilde{\varpi} - \widetilde{\varpi}\delta_\varpi = \widetilde{\varpi}$. If ϖ' is another element of Π_R as above, then $\delta_\varpi = (1 + \frac{\delta_\varpi(\alpha)}{\alpha})\delta_{\varpi'}$. Thus, the homomorphisms δ_ϖ give rise to the structure of a left W_R-module on the de Rham cohomology groups $H^q_{\rm dR}(X/R)$.

The exact sequence $(*)$ gives rise to the similar long exact sequence for cohomology sheaves of the complexes and, in particular, to a similar homomorphism of sheaves

$$\nabla : \mathcal{H}^q(\omega^\cdot_{X/R}) \to \omega^1_R \otimes_R \mathcal{H}^q(\omega^\cdot_{X/R}) \ .$$

which is easily seen to possess the similar property $\nabla(\gamma x) = d(\gamma) \otimes x + \gamma\nabla(x)$ for all $\gamma \in R$ and all local sections x of $\mathcal{H}^q(\omega^\cdot_{X/R})$. Again, for each element $\varpi \in \Pi_R$ the composition of ∇ with the isomorphism $\chi_\varpi : \omega^1_R \widetilde{\to} R : d\log(\varpi) \mapsto 1$ gives a homomorphism

$$\delta_\varpi : \mathcal{H}^q(\omega^\cdot_{X/R}) \to \mathcal{H}^q(\omega^\cdot_{X/R}) \ ,$$

and all these homomorphisms give rise to the structure of a sheaf of W_R-modules on $\mathcal{H}^q(\omega^\cdot_{X/R})$.

We now notice that the above operators δ_ϖ on the groups $H^q_{\rm dR}(X/R)$ and the sheaves $\mathcal{H}^q(\omega^\cdot_{X/R})$ are induced by endomorphisms $\widetilde{\delta}_\varpi$ of the complex $\omega^\cdot_{X/R}$ in the derived category of complexes of sheaves of $\mathbf{C}$-vector spaces. Namely, $\widetilde{\delta}_\varpi$, as a morphism in the derived category, is defined by the canonical quasi-isomorphism $C(f)^\cdot \to \omega^\cdot_{X/R}$ and the morphism of complexes $\widetilde{\delta}_\varpi : C(f)^\cdot \to \omega^\cdot_{X/R}$, which is the composition of the canonical morphism $-\delta(f) : C(f)^\cdot \to \omega^1_R \otimes_R \omega^\cdot_{X/R}$ and the isomorphism $\omega^1_R \widetilde{\to} R : d\log(\varpi) \mapsto 1$. It follows that the spectral sequence

$$(**) \qquad E_2^{p,q} = H^p(X, \mathcal{H}^q(\omega^\cdot_{X/R})) \Longrightarrow H^{p+q}_{\rm dR}(X/R)$$

is compatible with the action of the operators $\widetilde{\delta}_\varpi$. We will show in §9.4 that the operators $\widetilde{\delta}_\varpi$ define a homomorphism from W_R to the endomorphism ring of $\omega^\cdot_R$ in the derived category of sheaves of $\mathbf{C}$-vector spaces on X.

Proposition 9.1.1. *The homomorphism $M^{gr}_X \to \omega^1_X : m \mapsto d\log(m)$ gives rise to isomorphisms of sheaves of $\mathbf{C}$-vector spaces*

$$\mathbf{C}_X \otimes_{\mathbf{Z}} \bigwedge^q \overline{M}^{gr}_X \widetilde{\to} \mathcal{H}^q(\omega^\cdot_X)$$

and of W_R-modules

$$\mathcal{C}_X \otimes_{\mathbf{Z}} \bigwedge^q \overline{M}^{(nont)}_{X/R} \widetilde{\to} \mathcal{H}^q(\omega^{\cdot}_{X/R}) \ .$$

The proposition is an easy consequence of Lemma 9.1.4 which gives a local description of the complexes $\omega^{\cdot}_X$ and $\omega^{\cdot}_{X/R}$ (and also includes an analog of Lemma 17 from [HoAt55]). For this we recall the following classical construction.

Let A be a commutative $\mathbf{C}$-algebra provided with p pairwise commuting $\mathbf{C}$-linear maps $D_1, \ldots, D_p : A \to A$. One associates with these objects a complex of $\mathbf{C}$-vector spaces $\mathrm{K}^{\cdot}_A(D_1, \ldots, D_p)$ with $\mathrm{K}^q_A(D_1, \ldots, D_p) = \bigwedge^q_A(A^p)$ and the differential defined by

$$d(fl_{j_1} \wedge \cdots \wedge l_{j_q}) = \sum_{i=1}^{p} D_i(f) l_i \wedge l_{j_1} \wedge \cdots \wedge l_{j_q} \ .$$

It is called the *Koszul complex* on A with operators $D_1, \ldots, D_p$. If $D_1 = \ldots = D_p = 0$, this complex (with zero differentials) will be denoted by $\mathrm{K}^{\cdot}_A(0^p)$. Notice that if one of the maps is bijective, the complex $\mathrm{K}^{\cdot}_A(D_1, \ldots, D_p)$ is exact. Indeed, suppose D_i is bijective. We define a $\mathbf{C}$-linear map $F_i : \mathrm{K}^q_A(D_1, \ldots, D_p) \to \mathrm{K}^{q-1}_A(D_1, \ldots, D_p)$ that takes $fl_{j_1} \wedge \cdots \wedge l_{j_q}$ with $j_1 < \ldots < j_q$ to zero, if $i \notin \{j_1, \ldots, j_q\}$, and to $D_i^{-1}(f) l_{j_1} \wedge \ldots \wedge \widehat{l}_{j_k} \wedge \ldots \wedge l_{j_q}$, if $i = j_k$. Then $F_i \circ d + d \circ F_i = \mathrm{Id}$.

Construction 9.1.2. Suppose that A is a commutative $\mathbf{C}$-algebra which is embedded in the $\mathbf{C}$-vector space of formal power series of the form $f = \sum_{\mathbf{k} \in \mathbf{Z}^n} a_{\mathbf{k}} T^{\mathbf{k}}$ with coefficients in $\mathbf{C}$ and such that, if f as above lies in the image of A, then the latter contains all of the monomials $T^{\mathbf{k}}$ with $a_{\mathbf{k}} \neq 0$ (see examples of such A's below). Suppose we are given a tuple of functions $\delta = (\delta_1, \ldots, \delta_p)$ on $\mathbf{k} \in \mathbf{Z}^n$ with values in $\mathbf{C}$. For $1 \leq i \leq p+1$, let $A^{(i)}_\delta$ denote the $\mathbf{C}$-vector subspace of A whose nonzero elements are f's as above in which the sum is taken over the tuples $\mathbf{k}$ with the property $\delta_j(\mathbf{k}) = 0$ for all $1 \leq j \leq i-1$ and $\delta_i(\mathbf{k}) \neq 0$. (If $i = p+1$, the latter condition is empty.) Then there is an isomorphism of $\mathbf{C}$-vector spaces $\oplus_{i=1}^{p+1} A^{(i)}_\delta \widetilde{\to} A$. Finally, suppose we are given p pairwise commutating $\mathbf{C}$-linear maps $D_1, \ldots, D_p : A \to A$ such that, if $f = \sum_{\mathbf{k} \in \mathbf{Z}^n} a_{\mathbf{k}} T^{\mathbf{k}}$ lies in the image of A, one has $D_i(f) = \sum_{\mathbf{k} \in \mathbf{Z}^n} a_{\mathbf{k}} \delta_i(\mathbf{k}) T^{\mathbf{k}}$ for each $1 \leq i \leq p$. Then for every $1 \leq i \leq p$, D_i induces an injective $\mathbf{C}$-linear operator $A^{(i)}_\delta \to A^{(i)}_\delta$, and we assume that this operator is bijective. (This amounts to convergence of the formal power series $D_i^{-1}(T_j)$, $1 \leq j \leq p$, and will always hold in our examples.) Then one can define subcomplexes $E^{\cdot}_{\delta,1}, \ldots, E^{\cdot}_{\delta,m+1}$ of $\mathrm{K}^{\cdot}_A(D_1, \ldots, D_p)$ in which

$$E^q_{\delta,i} = \{\omega = \sum_{\mathbf{j}} f_{\mathbf{j}} l_{j_1} \wedge \ldots \wedge l_{j_q} \big| f_{\mathbf{j}} \in A^{(i)}_\delta\} \ ,$$

and there is an isomorphism of complexes $\oplus_{i=1}^{p+1} E^{\cdot}_{\delta,i} \widetilde{\to} \mathrm{K}^{\cdot}_A(D_1, \ldots, D_s)$. Since the restriction of each D_i to $A^{(i)}_\delta$ for $1 \leq i \leq p$ is a bijection, one can define $\mathbf{C}$-linear maps $F_i : E^q_{\delta,i} \to E^{q-1}_{\delta,i}$ (as above) with $F_i \cdot d + d \cdot F_i = \mathrm{Id}$. This means that the complexes $E^{\cdot}_{\delta,1}, \ldots, E^{\cdot}_{\delta,p}$ are acyclic and, therefore, there is a canonical quasi-isomorphism

$$E^{\cdot}_{\delta,p+1} \widetilde{\to} \mathrm{K}^{\cdot}_A(D_1, \ldots, D_p) \ .$$

Examples 9.1.3. Here are some of the examples of $\mathbf{C}$-algebras to which Construction 9.1.2 will be applied in this and the following sections with the field $K = \widehat{\mathcal{K}}$.

(1) A is the local ring $\mathcal{O}_{\mathcal{X}^h,x}$, where $\mathcal{X}$ is the log scheme $\mathrm{Spec}(C_r)$ with

$$C_r = K_r^\circ[T_1, \dots, T_n]/(T_1^{e_1} \cdot \dots \cdot T_m^{e_m} - z, T_1^{re_1} \cdot \dots \cdot T_\nu^{re_\nu}), \text{ if } r < \infty,$$

$$\text{and } C_\infty = \mathcal{K}^\circ[T_1, \dots, T_n]/(T_1^{e_1} \cdot \dots \cdot T_m^{e_m} - z), \text{ if } r = \infty,$$

$1 \le \nu \le m \le n$, the log structure on $\mathcal{X}$ is generated by the coordinate functions $T_1, \dots, T_m$, the morphism of log schemes $\mathcal{X} \to \mathrm{pt}_R$ is defined by the homomorphism $z \mapsto T_1^{e_1} \cdot \dots \cdot T_m^{e_m}$, and x is the zero point of $\mathcal{X}^h$, i.e., $t_i(x) = 0$ for all $1 \le i \le n$, where t_i is the image of T_i in C_r. (If $r < \infty$, z is a fixed generator of $R^{\circ\circ}$.) Each element of A has a unique representation as a power series $f = \sum_{\mathbf{k} \in \mathbf{Z}_+^n} a_{\mathbf{k}} t^{\mathbf{k}}$ over $\mathbf{C}$ taken over tuples $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{Z}_+^n$ with the property that, if $r < \infty$, then $k_i < re_i$ for some $1 \le i \le \nu$, and such that f is convergent at each point from the intersection of $\mathcal{X}^h$ with a small ball in $\mathbf{C}^n$ with center at zero. Notice that the local ring $\mathcal{O}_{X,x}$ for a distinguished log analytic space (for $r < \infty$) or germ (for $r = \infty$) X over $\mathbf{pt}_R$ is of the above form A.

(2) B is the localization of A from (1) with respect to powers of the element $t_1 \cdot \dots \cdot t_\mu$ with $1 \le \mu < \nu$. Each element of B has a unique representation as a power series $f = \sum_{\mathbf{k} \in \mathbf{Z}^n} a_{\mathbf{k}} t^{\mathbf{k}}$ taken over tuples $\mathbf{k} \in \mathbf{Z}^\mu \times \mathbf{Z}_+^{n-\mu}$ such that $(t_1 \cdot \dots \cdot t_\mu)^l f \in A$ for some $l \ge 0$. If $\mathcal{X}'$ is the spectrum of the localization of C_r with respect to powers of the element $t_1 \cdot \dots \cdot t_\mu$ and j denotes the open immersion $\mathcal{X}' \hookrightarrow \mathcal{X}$, then B is the stalk at x of the analytification $(j_*\mathcal{O}_{\mathcal{X}'})^h$ of the sheaf $j_*\mathcal{O}_{\mathcal{X}'}$.

(3) B' is the stalk at x of the sheaf $j_*^h\mathcal{O}_{\mathcal{X}'^h}$ for $\mathcal{X}'$ from (2). Each element of B' has a unique representation as a power series $f = \sum_{\mathbf{k} \in \mathbf{Z}^n} a_{\mathbf{k}} t^{\mathbf{k}}$ taken over tuples $\mathbf{k} \in \mathbf{Z}^\mu \times \mathbf{Z}_+^{n-\mu}$ with the property that, if $r < \infty$, then $k_i < re_i$ for some $\mu + 1 \le i \le \nu$ and such that f is convergent at each point from the intersection of $\mathcal{X}'^h$ with a small ball in $\mathbf{C}^n$ with center at zero.

In the situation of examples (1)-(3), we set $e = \text{g.c.d.}(e_1, \dots, e_m)$, $e'_i = \frac{e_i}{e}$ for $1 \le i \le m$, and we denote by ϱ the image of the element $T_1^{e'_1} \cdot \dots \cdot T_m^{e'_m}$ in A. Notice that $\varrho^e = z$, and ϱ generates the R-algebra $\mathcal{C}_{\mathcal{X}^h,x}$.

Lemma 9.1.4. *In the examples (1)-(3), the following is true:*

(i) *the map* $l_{j_1} \wedge \dots \wedge l_{j_q} \mapsto d\log(T_{j_1}) \wedge \dots \wedge d\log(T_{j_q})$, $1 \le j_1 < \dots < j_q \le m$, *induces quasi-isomorphisms of complexes*

$$\mathrm{K}^\cdot_{\mathbf{C}}(0^m) \widetilde{\to} \omega^\cdot_{\mathcal{X}^h,x} \widetilde{\to} (j_*\omega^\cdot_{\mathcal{X}'})^h_x \widetilde{\to} (j_*^h\omega^\cdot_{\mathcal{X}'^h})_x \ ;$$

(ii) *the map* $\varrho^k l_{j_1} \wedge \dots \wedge l_{j_q} \mapsto \varrho^k d\log(T_{j_1}) \wedge \dots \wedge d\log(T_{j_q})$, $0 \le k \le e-1$ *and* $1 \le j_1 < \dots < j_q \le m-1$, *induces quasi-isomorphisms of complexes*

$$\mathrm{K}^\cdot_{R[\varrho]}(0^{m-1}) \widetilde{\to} \omega^\cdot_{\mathcal{X}^h/R,x} \widetilde{\to} (j_*\omega^\cdot_{\mathcal{X}'/R})^h_x \widetilde{\to} (j_*^h\omega^\cdot_{\mathcal{X}'^h/R})_x \ .$$

We notice that the complexes $(j_*\omega^\cdot_{\mathcal{X}'})^h_x$ and $(j_*\omega^\cdot_{\mathcal{X}'/R})^h_x$ depend only on the complex analytic germs $(\mathcal{X}^h, x)$ and $(\mathcal{Y}^h, x)$, where $\mathcal{Y} = \mathcal{X} \backslash \mathcal{X}'$. Indeed, if $\mathcal{J}$ is the subsheaf of ideals of $\mathcal{O}_{\mathcal{X}^h}$ with support $\mathcal{Y}^h$, then $(j_*\omega^\cdot_{\mathcal{X}'})^h_x$ and $(j_*\omega^q_{\mathcal{X}/R})^h_x$ coincide with the stalks at x of the sheaves $\varinjlim_n \mathcal{H}om(\mathcal{J}^n, \omega^q_{\mathcal{X}^h})$ and $\varinjlim_n \mathcal{H}om(\mathcal{J}^n, \omega^q_{\mathcal{X}/R})$, respectively.

Proof. Let U be one of the rings A, B, or B'.

(i) Each of the complexes on the right hand side is naturally isomorphic to the Koszul complex

$$\mathrm{K}^{\cdot}_U\left(T_1\frac{\partial}{\partial T_1},\dots,T_m\frac{\partial}{\partial T_m},\frac{\partial}{\partial T_{m+1}},\dots,\frac{\partial}{\partial T_n}\right) .$$

The classical Poincaré lemma implies that the latter complex is quasi-isomorphic to the Koszul complex $\mathrm{K}^{\cdot}_{U'}(T_1\frac{\partial}{\partial T_1},\dots,T_m\frac{\partial}{\partial T_m})$ of the similar ring U' with $n=m$. We may therefore assume that $n=m$.

Since $T_i\frac{\partial(T^{\mathbf{k}})}{\partial T_i}=k_iT^{\mathbf{k}}$, we can apply Construction 9.1.2 for the tuple of functions $\delta=(\delta_1,\dots,\delta_m)$ with $\delta_i(\mathbf{k})=k_i$. The $\mathbf{C}$-linear maps $T_i\frac{\partial}{\partial T_i}:U^i_\delta\to U^i_\delta$ are bijective and, therefore, the complex considered is quasi-isomorphic to the subcomplex $E^{\cdot}_{\delta,m+1}$. It remains to notice that $\mathrm{K}^{\cdot}_{\mathbf{C}}(0^m)\widetilde{\to}E^{\cdot}_{\delta,m+1}$.

(ii) Let $F^{\cdot}_U$ and $G^{\cdot}_U$ denote the complexes that corresponds to U in (i) and (ii), respectively. The U-module G^1_U is the quotient of F^1_U by the U-submodule generated by the one-form $d\log(z)=\sum_{i=1}^m e_id\log(T_i)$, and, in particular, it is a free U-module of rank $n-1$ with generators $d\log(T_1),\dots,d\log(T_{m-1}),dT_{m+1},\dots,dT_n$. For $1\le i\le m-1$, we set $D_i=T_i\frac{\partial}{\partial T_i}-\frac{e_i}{e_m}T_m\frac{\partial}{\partial T_m}$. Then for any $f\in U$, one has

$$\sum_{i=1}^{m-1}D_i(f)d\log(T_i)+\sum_{i=m+1}^{n}\frac{\partial f}{\partial T_i}dT_i=df-\frac{1}{e_m}T_m\frac{\partial f}{\partial T_m}d\log(z) .$$

This implies that there is a canonical isomorphism of complexes

$$\mathrm{K}^{\cdot}_U\left(D_1,\dots,D_{m-1},\frac{\partial}{\partial T_{m+1}},\dots,\frac{\partial}{\partial T_n}\right)\widetilde{\to}G^{\cdot}_U .$$

As in (i), the Poincaré lemma reduces the situation to the case $n=m$.

One has $D_i(T^{\mathbf{k}})=\delta_i(\mathbf{k})T^{\mathbf{k}}$ for $\delta_i(\mathbf{k})=k_i-k_m\frac{e_i}{e_m}$, and the corresponding map $D_i:U^{(i)}_\delta\to U^{(i)}_\delta$ is bijective. We can therefore apply Construction 9.1.2. It follows that the canonical map $E^{\cdot}_{\delta,m}\to\mathrm{K}^{\cdot}_U(D_1,\dots,D_{m-1})$ is a quasi-isomorphism. If $\mathbf{k}$ is a tuple as above with $k_i=k_m\frac{e_i}{e_m}$ for all $1\le i\le m$, then $\mathbf{k}=(le'_1,\dots,le'_m)$ for some $l\ge0$. If $r<\infty$, one even has $l\le re-1$ and, therefore, $K^{\cdot}_{R[\varrho]}(0^{m-1})\widetilde{\to}E^{\cdot}_{\delta,m}$. If $r=\infty$, the number l is not bounded from above, but the series $\sum_{l=0}^\infty a_l\varrho^l$, which appear in the decomposition of elements of $E^q_{\delta,m}$ are convergent in a small neighborhood of zero in $\mathbf{C}^m$. This implies that $K^{\cdot}_{R[\varrho]}(0^{m-1})\widetilde{\to}E^{\cdot}_{\delta,m}$. □

Proof of Proposition 9.1.1. In order to show that the homomorphisms constructed are isomorphisms, we may assume that X and x are from Example 9.1.3(1) with $K=\widehat{\mathcal{K}}$. Both isomorphisms follows from Lemma 9.1.4. It remains to show that the second isomorphism is a homomorphism of W_R-modules. By the above description, each cohomology class in $\mathcal{H}^q(\omega^{\cdot}_{X/R})_x$ is represented by a $\mathbf{C}$-linear combination of elements of the form $\xi=\varrho^i\eta$ with $\eta=d\log(T_{j_1})\wedge\ldots\wedge d\log(T_{j_q})$ and $\varrho^i\in\mathcal{C}_{X,\lambda,x}$ for $\lambda=\frac{i}{e}<r$. One has $d\xi=\lambda\varrho^id\log(\varpi)\wedge\eta$. The form on the right hand side is the image of element $d\log(\varpi)\otimes\lambda\xi\in(\omega^1_R\otimes_R\omega^{\cdot}_{X/R}[-1])^{q+1}_x$. It follows that $\delta_\varpi(\xi)=\lambda\xi$. □

Corollary 9.1.5. *In the situation of Proposition 9.1.1, eigenvalues of the* $\mathbf{C}$*-linear operators* δ_ϖ *on* $H^q_{\mathrm{dR}}(X/R)$ *are rational numbers in the interval* $[0,r)$.

Proof. Let λ be a complex number which does not lie in $\mathbf{Q}\cap[0,r)$. By (i), the operator $\delta_\varpi-\lambda$ is invertible on all of the sheaves $\mathcal{H}^q(\omega^{\cdot}_{X/R})$. This implies that the operator $\widetilde{\delta}_\varpi-\lambda$ is invertible on all of the $\mathbf{C}$-vectors spaces $E_2^{p,q}$ from the spectral sequence $(**)$ and, therefore, it is invertible on the groups $H^q_{\mathrm{dR}}(X/R)$. □

9.2. **Complexes $\omega^{\cdot}_{X^{\log}}$ and $\overline{\omega}^{\cdot}_{\overline{X^{\log}}}$.** Recall that, by [KN99, §3], the space $X^{\log}$ is provided with a sheaf of differential graded $\mathbf{C}$-algebras $\omega^{\cdot}_{X^{\log}}$ (denoted there by $\omega^{\cdot,\log}_X$). The same construction provides the space $\overline{X^{\log}}$ with a sheaf of differential graded Π_R-algebras $\omega^{\cdot}_{\overline{X^{\log}}}$. Namely, consider the exact sequence of abelian Π_R-sheaves

$$0\longrightarrow\overline{\tau}^{-1}(\mathcal{O}_X)\xrightarrow{\mu}\mathcal{L}_{\overline{X^{\log}}}\longrightarrow\overline{\tau}^{-1}(\overline{M}^{gr}_X)\longrightarrow 0\ ,$$

where $\mathcal{L}_{\overline{X^{\log}}}$ is the abelian Π_R-sheaf on $\overline{X^{\log}}$ introduced in §4.3. One defines a Π_R-algebra $\mathcal{O}_{\overline{X^{\log}}}$ by

$$\mathcal{O}_{\overline{X^{\log}}}=(\overline{\tau}^{-1}(\mathcal{O}_X)\otimes_{\mathbf{Z}}\mathrm{Sym}_{\mathbf{Z}}(\mathcal{L}_{\overline{X^{\log}}}))/\mathcal{J}\ ,$$

where $\mathcal{J}$ is the sheaf of ideals generated by sections of the form $f\otimes1-1\otimes\mu(f)$ for local section f of $\overline{\tau}^{-1}(\mathcal{O}_X)$. The canonical homomorphism $\mathcal{L}_{\overline{X^{\log}}}\to\mathcal{O}_{\overline{X^{\log}}}$ is universal among homomorphisms from $\mathcal{L}_{\overline{X^{\log}}}$ to $\overline{\tau}^{-1}(\mathcal{O}_X)$-algebras. Since $\nu^{-1}(\mathcal{L}_{X^{\log}})\widetilde{\to}\mathcal{L}_{\overline{X^{\log}}}$, it follows that $\nu^{-1}(\mathcal{O}_{X^{\log}})\widetilde{\to}\mathcal{O}_{\overline{X^{\log}}}$. One defines

$$\omega^{\cdot}_{\overline{X^{\log}}}=\mathcal{O}_{\overline{X^{\log}}}\otimes_{\overline{\tau}^{-1}(\mathcal{O}_X)}\overline{\tau}^{-1}(\omega^{\cdot}_X)\ .$$

The restrictions of the above Π_R-sheaves to $X^{(\varpi)}$ are denoted by $\mathcal{O}_{X^{(\varpi)}}$ and $\omega^{\cdot}_{X^{(\varpi)}}$, respectively, and for a morphism $\varphi:\varpi\to\varpi'$ in Π_R, the corresponding isomorphism $({}^t\varphi)^{-1}(\omega^{\cdot}_{X^{(\varpi)}})\widetilde{\to}\omega^{\cdot}_{X^{(\varpi')}}$ is denoted by $\varphi_{\overline{\omega}}$. For example, if $\varpi'=\alpha\varpi$ and φ corresponds to an element $\beta\in R$ with $\exp(\beta)=\alpha^{-1}$, $\varphi_{\overline{\omega}}$ takes $\log(\varpi)$ to $\log(\varpi')+\beta$ (see Example 4.3.2(i)). This is consistent with the fact that $\varphi_{\overline{\omega}}$ takes $d\log(\varpi)$ to itself since $d\log(\varpi)=d\log(\varpi')-\frac{d\alpha}{\alpha}$.

Notice that the Poincaré lemma implies the following fact: given a smooth morphism $\varphi:X'\to X$ with $\varphi^*(M_X)\widetilde{\to}M_{X'}$, for every point $\overline{y}'\in\overline{X'^{\log}}$, the canonical morphisms of complexes $\omega^{\cdot}_{X^{\log},y}\to\omega^{\cdot}_{X'^{\log},y'}$ and $\omega^{\cdot}_{\overline{X^{\log}},\overline{y}}\to\omega^{\cdot}_{\overline{X'^{\log}},\overline{y}'}$ are quasi-isomorphisms, where y, y' and $\overline{y}$ are the images of the point $\overline{y}'$ in $X^{\log}$, $X'^{\log}$ and $\overline{X^{\log}}$, respectively.

We are going to introduce a bigger complex of sheaves of $\underline{R}$-modules on $\overline{X^{\log}}$

$$\overline{\omega}^{\cdot}_{\overline{X^{\log}}}=\oplus_{\lambda\in\mathbf{Q}_+}\omega^{\cdot}_{\overline{X^{\log}},\lambda}\ ,$$

where each $\omega^{\cdot}_{\overline{X^{\log}},\lambda}$ is a complex of sheaves of Π_R-modules. The q-th component of the restriction of the latter to $X^{(\varpi)}$ in essence coincides with the subsheaf $\widetilde{\widetilde{\varpi}}^{[\lambda]}\omega^q_{X^{(\varpi)}}$ of $\omega^q_{X^{(\varpi)}}$, where $[\lambda]$ is the integral part of λ, but its differential is different so that it is convenient to denote it by $\varpi^{-\lambda}\widetilde{\widetilde{\varpi}}^{[\lambda]}\omega^{\cdot}_{X^{(\varpi)}}$. (By the way, this implies that $\overline{\omega}^{\cdot}_{\overline{X^{\log}}}=0$ for $\lambda\ge r$.) Namely, it is defined by

$$d(\varpi^{-\lambda}\eta)=\varpi^{-\lambda}(-\lambda d\log(\varpi)\wedge\eta+d\eta)$$

for a local section η of $\widetilde{\widetilde{\varpi}}^{[\lambda]}\omega^q_{X^{(\varpi)}}$ (e.g., $d(\varpi^{-\lambda}\widetilde{\widetilde{\varpi}}^{[\lambda]})=\varpi^{-\lambda}([\lambda]-\lambda)\widetilde{\widetilde{\varpi}}^{[\lambda]}d\log(\varpi)$). If $\varphi:\varpi\to\varpi'$ is a morphism in Π_R as above, then the corresponding isomorphism $\varphi_{\overline{\omega}}:({}^t\varphi)^{-1}(\varpi^{-\lambda}\widetilde{\widetilde{\varpi}}^{[\lambda]}\omega^q_{X^{(\varpi)}})\widetilde{\to}\varpi'^{-\lambda}\widetilde{\widetilde{\varpi}}'^{[\lambda]}\omega^q_{X^{(\varpi')}}$ is defined by

$$\varphi_{\overline{\omega}}(\varpi^{-\lambda}\eta)=\varpi'^{-\lambda}\exp(-\lambda\beta)\varphi_{\overline{\omega}}(\eta)\ .$$

The element $\varphi_{\overline{\varpi}}(d(\varpi^{-\lambda}\eta))$ is equal to $\varpi'^{-\lambda}\exp(-\lambda\beta)$ multiplied

$$\begin{aligned}\varphi_{\overline{\varpi}}(-\lambda d\log(\varpi)\wedge\eta+d\eta) = -\lambda\left(d\log(\varpi')-\frac{d\alpha}{\alpha}\right)\wedge\varphi_{\overline{\varpi}}(\eta)+d\varphi_{\overline{\varpi}}(\eta) = \\ = \quad -\lambda d\log(\varpi')\wedge\varphi_{\overline{\varpi}}(\eta)+\lambda\frac{d\alpha}{\alpha}\wedge\varphi_{\overline{\varpi}}(\eta)+d\varphi_{\overline{\varpi}}(\eta)\ .\end{aligned}$$

On the other hand, the element $d\varphi_{\overline{\varpi}}(\eta)$ is equal to $\varpi'^{-\lambda}$ multiplied by

$$\begin{aligned}&-\lambda\exp(-\lambda\beta)d\log(\varpi')\wedge\varphi_{\overline{\varpi}}(\eta)+d(\exp(-\lambda\beta)\varphi_{\overline{\varpi}}(\eta)) = \\ = \quad &\exp(-\lambda\beta)\left(-\lambda d\log(\varpi')\wedge\varphi_{\overline{\varpi}}(\eta)+\lambda\frac{d\alpha}{\alpha}\wedge\varphi_{\overline{\varpi}}(\eta)+d\varphi_{\overline{\varpi}}(\eta)\right)\ .\end{aligned}$$

This means that $\omega^{\cdot}_{\overline{X^{\log}},\lambda}$ is a complex of sheaves of Π_R-modules. If $\lambda=0$, it coincides with $\omega^{\cdot}_{\overline{X^{\log}}}$.

We now provide the sheaves $\overline{\omega}^q_{X^{(\varpi)}} = \oplus_{\lambda\in\mathbf{Q}_+}\varpi^{-\lambda}\widetilde{\varpi}^{[\lambda]}\omega^q_{X^{(\varpi)}}$ with a different structure of a R-module so that the differentials between them commute with the action of R and the complex $\overline{\omega}^{\cdot}_{\overline{X^{\log}}}$ becomes a complex of sheaves of $\underline{R}$-modules. Namely, for $\varpi\in\Pi_R$ we define

$$\widetilde{\varpi}\cdot(\varpi^{-\lambda}\eta)=\varpi^{-(\lambda+1)}(\widetilde{\varpi}\eta)$$

for a local section η of $\widetilde{\varpi}^{[\lambda]}\omega^q_{X^{(\varpi)}}$ as above. One has

$$\begin{aligned}d(\widetilde{\varpi}\cdot(\varpi^{-\lambda}\eta)) &= \varpi^{-(\lambda+1)}(-(\lambda+1)d\log(\varpi)\wedge(\widetilde{\varpi}\eta)+d(\widetilde{\varpi}\eta) = \\ &= \varpi^{-(\lambda+1)}(\widetilde{\varpi}(-\lambda d\log(\varpi)\wedge\eta+d\eta))=\widetilde{\varpi}\cdot d(\varpi^{-\lambda}\eta)\ .\end{aligned}$$

This means that the endomorphism of multiplication by $\widetilde{\varpi}$ commutes with the differential. Furthermore, given a morphism $\varphi:\varpi\to\varpi'$ in Π_R as above, the element $\varphi_{\overline{\varpi}}(\widetilde{\varpi}\cdot(\varpi^{-\lambda}\eta))$ is equal to

$$\varphi_{\overline{\varpi}}(\varpi^{-(\lambda+1)}(\widetilde{\varpi}\eta))=\varpi'^{-(\lambda+1)}\exp(-(\lambda+1)\beta)\widetilde{\varpi}\varphi_{\overline{\varpi}}(\eta)\ .$$

Since $\exp(-\beta)\varpi=\alpha\varpi=\varpi'$, that element is equal to

$$\varpi'^{-(\lambda+1)}\widetilde{\varpi}'\exp(-\lambda\beta)\varphi_{\overline{\varpi}}(\eta)=\widetilde{\varpi}'\cdot\varphi_{\overline{\varpi}}(\varpi^{-\lambda}\eta)\ .$$

Thus, $\overline{\omega}^{\cdot}_{\overline{X^{\log}}}$ is a complex of sheaves of $\underline{R}$-modules on $\overline{X^{\log}}$.

There is a canonical morphism of complexes of sheaves of $\underline{R}$-modules on $\overline{X^{\log}}$

$$h_\lambda:\overline{\tau}^{-1}(\mathcal{C}_{X,\lambda})\to\omega^{\cdot}_{\overline{X^{\log}},\lambda}\ ,$$

where $\overline{\tau}^{-1}(\mathcal{C}_{X,\lambda})$ is considered as a complex in degree zero. Namely, by the definition of $\mathcal{C}_{X,\lambda}$ (see §4.2), if U is a connected open subset of X and $\lambda\neq\frac{j}{k_U}$ for $0\le j<rk_U$, then $\mathcal{C}^{(\varpi)}_\lambda(U)=0$ for all $\varpi\in\Pi_R$. Suppose $\lambda=\frac{j}{k_U}$ for $0\le j<rk_U$. Then $\mathcal{C}^{(\varpi)}_\lambda(U)$ is the one-dimensional $\mathbf{C}$-vector space generated by the element $t^j=\widetilde{\varpi}^{[\lambda]}t^p$ with $t^{k_U}=\widetilde{\varpi}$ and $p=j-k_U\cdot\lambda$. We define a homomorphism $\mathcal{C}^{(\varpi)}_\lambda(U)\to\varpi^{-\lambda}\varpi^{[\lambda]}\mathcal{O}_{X^{(\varpi)}}(U)$ by sending t^j to $\varpi^{-\lambda}t^j$. One has

$$d(\varpi^{-\lambda}t^j)=\varpi^{-\lambda}\left(-\lambda t^jd\log(\varpi)+jt^jd\log(t)\right)=0$$

and, therefore, h_λ is a morphism of complexes. Furthermore, for a morphism $\varphi:\varpi\to\varpi'$ in Π_R as above, the corresponding homomorphism $\mathcal{C}^{(\varpi)}_\lambda(U)\to\mathcal{C}^{(\varpi')}_\lambda(U)$

is induced by multiplication by $\exp(-\lambda\beta)$ which is compatible with the similar homomorphism for the sheaf of Π_R-modules $\varpi^{-\lambda}\varpi^{[\lambda]}\mathcal{O}_{\overline{X^{\log}}}$. Thus, h_λ is a morphism of complexes of sheaves of R-modules. Finally, one has

$$h_\lambda(\widetilde{\varpi}t^j) = h_\lambda(t^{j+k_U}) = \varpi^{-(\lambda+1)}(\widetilde{\varpi}t^j) = \widetilde{\varpi}\cdot h_\lambda(t^j)$$

and, therefore, the morphism $h : \overline{\tau}^{-1}(\mathcal{C}_X) \to \overline{\omega}^{\cdot}_{\overline{X^{\log}}}$ induced by h_λ's is morphism of complexes of $\underline{R}$-modules on $\overline{X^{\log}}$.

Proposition 9.2.1. *The morphism h is a quasi-isomorphism.*

This statement implies that the complex $\overline{\omega}^{\cdot}_{\overline{X^{\log}}}$, as an object of the derived category of sheaves of $\underline{R}$-modules, has the structure of a $\underline{W}_R$-module.

Proof. Step 1. Since $\nu^{-1}(\omega^{\cdot}_{X^{\log}})\widetilde{\to}\omega^{\cdot}_{\overline{X^{\log}}}$, it suffices to show that, for every point $y \in X^{\log}$, there is a canonical quasi-isomorphism $\mathcal{C}^{(\varpi)}_{X,\lambda,x}\widetilde{\to}\varpi^{-\lambda}\omega^{\cdot}_{X^{\log},y}$, where $x = \tau(y)$. We may therefore assume that $X = \mathrm{Spec}(B)^h$ with B as in Definition 4.1.1 and x is the zero point in X, i.e., $T_i(x) = 0$ for all $1 \le i \le n$. (We use notations from that definition). By the Poincaré lemma, we may even assume that $n = m$. Notice that for any connected open neighborhood V of x one has $k_V = e_V = e = \text{g.c.d.}(e_1, \dots, e_m)$. We set $A = \mathcal{O}_{X,x}$. By [KN99, (3.3)], if we fix elements of $\mathcal{L}_{X^{\log},y}$ whose images under the exponential map $\mathcal{L}_{X^{\log},y} \to \overline{M}^{gr}_{X,x}$ are the generators $T_1, \dots, T_m$ of $P(X)$, we get an isomorphism $R[S_1, \dots, S_m]\widetilde{\to}\mathcal{O}_{X^{\log},y}$. It follows that, for every $q \ge 0$, one has

$$\omega^q_{X^{\log},y} = A \otimes_{\mathbf{C}} \Omega^q_{\mathbf{C}[S_1,\dots,S_m]/\mathbf{C}}$$

with $d\varpi = \sum_{i=1}^m e_i dS_i$ and $dT_i = T_i dS_i$ for $1 \le i \le m$. Elements of the $\mathbf{C}$-algebra A are represented as convergent power series $\sum_{\mathbf{k}} a_{\mathbf{k}} T^{\mathbf{k}}$, where $a_{\mathbf{k}} \in \mathbf{C}$ and the sum is taken over the tuples $\mathbf{k} = (k_1, \dots, k_m) \in \mathbf{Z}^m_+$ with $k_i < re_i$ for some $1 \le i \le \mu$. For such $\mathbf{k}$, one has

$$d(\underline{\varpi}^{-\lambda}T^{\mathbf{k}}) = \underline{\varpi}^{-\lambda}T^{\mathbf{k}}\sum_{i=1}^m (k_i - \lambda e_i)\, dS_i \ .$$

Notice that $k_i - \lambda e_i = 0$ for all $1 \le i \le m$ if and only if $\lambda = \frac{j}{e}$ for some $0 \le j < re$, an in this case $k_i = je'_i$ for all $1 \le i \le m$, where $e'_i = \frac{e_i}{e}$.

Step 2. We set $V^q = \Omega^q_{\mathbf{C}[S_1,\dots,S_m]/\mathbf{C}}$ and, for a tuple of complex numbers $\mathbf{p} = (p_1, \dots, p_m)$, define a differential $d_{\mathbf{p}} : V^q \to V^{q+1}$ by

$$d_{\mathbf{p}}\omega = -\left(\sum_{i=1}^m p_i dS_i\right) \wedge \omega + d\omega \ .$$

Each element $\omega \in \omega^q_{X^{\log},y}$ is a convergent sum $\sum_{\mathbf{k}} T^{\mathbf{k}}\omega_{\mathbf{k}}$ with $\max_{\mathbf{k}}\{\deg(\omega_{\mathbf{k}})\} < \infty$, where the degree of $\sum_{\mathbf{j}} f_{\mathbf{j}} dS_{j_1}\wedge\ldots\wedge dS_{j_q} \in V^q$ is the maximum of degrees of nonzero $f_{\mathbf{j}}$'s. Set $\mathbf{e} = (e_1, \dots, e_m)$, Then there is a morphism of complexes

$$(V^{\cdot}, d_{\lambda\mathbf{e}-\mathbf{k}}) \to \varpi^{-\lambda}\omega^{\cdot}_{X^{\log},y} : \eta \mapsto \varpi^{-\lambda}T^{\mathbf{k}}\eta$$

such that $(T^{\mathbf{k}}\eta)_{\mathbf{k}'} = \delta_{\mathbf{k},\mathbf{k}'}\eta$. Furthermore, the correspondence $\omega \mapsto \omega_{\mathbf{k}}$ defines a morphism of the same complexes in the opposite direction.

Thus, in order the prove proposition, it suffices to construct, for every nonzero tuple $\mathbf{p}$, a system of $\mathbf{C}$-linear maps $F^q_{\mathbf{p}} : V^q \to V^{q-1}$ with $d^{q-1}_{\mathbf{p}} \circ F^q_{\mathbf{p}} + F^{q+1}_{\mathbf{p}} \circ d^q_{\mathbf{p}} = \mathrm{Id}$

and such that, for every $\eta \in V^q$, one has $\deg(F^q_{\mathbf{p}}(\eta)) \le \deg(\eta)$ and, for every $\omega \in \omega^q_{X^{\log},y}$ such that $\omega_{\mathbf{k}} = 0$ for $\mathbf{k}$ with $\lambda\mathbf{e} - \mathbf{k} = 0$ (as at the end of Step 2), the sum $\sum_{\mathbf{k}} T^{\mathbf{k}} F^q_{\lambda\mathbf{e}-\mathbf{k}}(\omega_{\mathbf{k}})$ is convergent.

Step 3. Let $|\mathbf{p}|$ denote the Euclidean length of a nonzero tuple $\mathbf{p} \in \mathbf{C}^m$. Then the tuple $\frac{\mathbf{p}}{|\mathbf{p}|}$ lies on the unit sphere in $\mathbf{R}^m$ and, therefore, there exists an orthogonal $(m \times m)$-matrix D that takes it to the tuple $\mathbf{p}_0 = (1, 0, \dots, 0)$, and for the matrix $C = \frac{1}{|\mathbf{r}|} D$ one has $\mathbf{p} \cdot C = \mathbf{p}_0$. Notice that all entries c_{ij} of the matrix C are of length at most $|\mathbf{p}|^{-1}$. Consider the automorphism φ of the $\mathbf{C}$-algebra $\mathbf{C}[S_1, \dots, S_m]$ which is induced by the linear transformation $\varphi(S_i) = \sum_{j=1}^m c_{ij} S_j$. It gives rise to an isomorphism of complexes $\Phi^{\cdot} : (V^{\cdot}, d_{\mathbf{p}}) \widetilde{\to} (V^{\cdot}, d_{\mathbf{p}_0})$. The latter is isomorphic to the tensor product $V^{\cdot}_1 \otimes_{\mathbf{C}} \Omega^{\cdot}_{\mathbf{C}[S_2,\dots,S_m]/\mathbf{C}}$, where $V^{\cdot}_1$ is the complex constructed for the ring of polynomials $\mathbf{C}[S_1]$ and the tuple 1. The required homotopy for $\mathbf{C}[S_1]$, i.e. a $\mathbf{C}$-linear map $F_1 : V^1_1 = \mathbf{C}[S_1]dS_1 \to V^0_1 = \mathbf{C}[S_1]$, is given by the formula

$$F_1(S_1^n dS_1) = -S_1^n - \sum_{i=1}^n (-1)^i n(n-1) \cdot \ldots \cdot (n-i+1) S_1^{n-i}$$

It induces a homotopy $F^q_{\mathbf{p}_0} : (V^q, d_{\mathbf{p}_0}) \to (V^{q-1}, d_{\mathbf{p}_0})$ which, in its turn, induces a homotopy $F^q_{\mathbf{p}} = (\Phi^{q-1})^{-1} \circ F^q_{\mathbf{p}_0} \circ \Phi^q : (V^q, d_{\mathbf{p}}) \to (V^{q-1}, d_{\mathbf{p}})$ that satisfies the required properties. ☐

Corollary 9.2.2. *There is a canonical isomorphism of complexes of sheaves of* $\mathbf{C}$*-vector spaces*

$$R\tau_*(\mathbf{C}_{X^{\log}}) \widetilde{\to} \omega^{\cdot}_X \ .$$

Proof. By Proposition 9.2.1 applied to $\lambda = 0$, there is a canonical quasi-isomorphism $\mathbf{C}_{X^{\log}} \widetilde{\to} \omega^{\cdot}_{X^{\log}}$. It gives rise to an isomorphism $R\tau_*(\mathbf{C}_{X^{\log}}) \widetilde{\to} R\tau_*(\omega^{\cdot}_{X^{\log}})$. Thus, it remains to show that the canonical morphism of complexes $\omega^{\cdot}_X \to R\tau_*(\omega^{\cdot}_{X^{\log}})$ is a quasi-isomorphism.

The quasi-isomorphism $\mathbf{C}_{X^{\log}} \widetilde{\to} \omega^{\cdot}_{X^{\log}}$ gives an exact sequence of sheaves

$$0 \to \mathbf{C}_{X^{\log}} \to \mathcal{O}_{X^{\log}} \to \mathrm{Ker}(\omega^1_{X^{\log}} \xrightarrow{d} \omega^2_{X^{\log}}) \to 0$$

which, in its turn, gives rise to an injective morphism $\mathcal{H}^1(\tau_*(\omega^{\cdot}_{X^{\log}})) \to R^1\tau_*(\mathbf{C}_{X^{\log}})$. Its composition with canonical morphism $\mathcal{H}^1(\omega^1_X) \to \mathcal{H}^1(\tau_*(\omega^{\cdot}_{X^{\log}}))$ gives a homomorphism $\varphi : \mathcal{H}^1(\omega^1_X) \to R^1\tau_*(\mathbf{C}_{X^{\log}})$.

Furthermore, by [KN99, Lemma (1.5)], for all $q \ge 0$ there are canonical isomorphisms $\bigwedge^q R^1\tau_*(\mathbf{C}_{X^{\log}}) \widetilde{\to} R^q\tau_*(\mathbf{C}_{X^{\log}})$. It follows that $\bigwedge^q \mathcal{H}^1(R\tau_*(\omega^{\cdot}_{X^{\log}})) \widetilde{\to} \mathcal{H}^q(R\tau_*(\omega^{\cdot}_{X^{\log}}))$. Finally, by Proposition 9.1.1, one has $\bigwedge^q \mathcal{H}^1(\omega^{\cdot}_X) \widetilde{\to} \mathcal{H}^q(\omega^{\cdot}_X)$. Thus, in order to prove the required statement, it suffices to verify that φ is an isomorphism.

By Proposition 9.1.1 and [KN99, Lemma (1.5)], there are canonical isomorphisms $f : \mathbf{C}_X \otimes_{\mathbf{Z}_X} \overline{M}^{gr}_X \widetilde{\to} \mathcal{H}^1(\omega^{\cdot}_X)$ and $g : \mathbf{C}(-1)_X \otimes_{\mathbf{Z}_X} \overline{M}^{gr}_X \widetilde{\to} R^1\tau_*(\mathbf{C}_{X^{\log}})$, where $\mathbf{C}(-1) = \mathbf{C} \otimes_{\mathbf{Z}} \frac{1}{2\pi i}\mathbf{Z}$. The homomorphism $a \otimes \frac{1}{2\pi i} n \mapsto \frac{an}{2\pi i}$ identifies the latter with $\mathbf{C}$. Thus, verification of the required statement amounts to commutativity of the following diagram

$$\begin{array}{ccc} \mathbf{C}_X \otimes_{\mathbf{Z}_X} \overline{M}^{gr}_X & \xrightarrow{f} & \mathcal{H}^1(\omega^{\cdot}_X) \\ \Vert & & \downarrow{\scriptstyle\varphi} \\ \mathbf{C}_X \otimes_{\mathbf{Z}_X} \overline{M}^{gr}_X & \xrightarrow{g} & R^1\tau_*(\mathbf{C}_{X^{\log}}) \end{array}$$

This fact easily follows from the construction of f, g and φ. $\square$

Corollary 9.2.3. *For every distinguished formal scheme $\mathfrak{X}$ over K°, there is a compatible system of canonical isomorphisms*

$$R\Theta^h(\mathbf{C}_{\mathfrak{X}_\eta})\widetilde{\to}\omega^{\cdot}_{\mathfrak{X}^h_{s_r}} .$$

Proof. By the definition of $R\Theta^h$, the complex on the left hand side of the isomorphism in Corollary 9.2.2 is $R\Theta^h(\mathbf{C}_{\mathfrak{X}_\eta})$, and the required fact follows. $\square$

9.3. **Complexes $L^{\cdot}_X$.** For $\lambda \in \mathbf{Q}_+$ and $\varpi \in \Pi_R$, let $L^{(\varpi)q}_{X,\lambda}$ denote the subsheaf of $\tau^{(\varpi)}_*(\varpi^{-\lambda}\widetilde{\varpi}^{[\lambda]}\omega^q_{X^{(\varpi)}})$ with local sections of the form

$$\eta = \varpi^{-\lambda}\sum_{l=0}^{p}(\log\varpi)^l\eta_l \ ,$$

where $\eta_0,\dots,\eta_p$ are local sections of the subsheaf $\widetilde{\varpi}^{[\lambda]}\omega^q_X$ of ω^q_X. One has

$$d\eta = \varpi^{-\lambda}\sum_{l=0}^{p}(\log\varpi)^l\left(d\log(\varpi)\wedge(-\lambda\eta_l+(l+1)\eta_{l+1})+d\eta_l\right) \ .$$

This means that $d(L^{(\varpi)q}_{X,\lambda}) \subset L^{(\varpi)q+1}_{X,\lambda}$ and, therefore, there is a well defined subcomplex $L^{(\varpi)\cdot}_{X,\lambda}$ of $\tau^{(\varpi)}_*(\varpi^{-\lambda}\widetilde{\varpi}^{[\lambda]}\omega^q_{X^{(\varpi)}})$, and $L^{(\varpi)\cdot}_X = \oplus_{\lambda\in\mathbf{Q}_+}L^{(\varpi)\cdot}_{X,\lambda}$ is a subcomplex of $\tau^{(\varpi)}_*(\overline{\omega}^{\cdot}_{X^{(\varpi)}})$. One also has

$$\widetilde{\varpi}\cdot\eta = \varpi^{-(\lambda+1)}\sum_{l=0}^{p}(\log\varpi)^l\widetilde{\varpi}\eta_l \ .$$

This means that the endomorphism of multiplication by $\widetilde{\varpi}$ on $\overline{\omega}^q_{X^{(\varpi)}}$ takes $L^{(\varpi)q}_X$ to itself, and so $L^{(\varpi)\cdot}_X$ is a complex of sheaves of R-modules. We introduce $\mathbf{C}$-linear operators $\delta_\varpi : L^{(\varpi)q}_{X,\lambda} \to L^{(\varpi)q}_{X,\lambda}$ by

$$\delta_\varpi(\eta) = \varpi^{-\lambda}\sum_{l=0}^{p}(\log\varpi)^l\left(\lambda\eta_l-(l+1)\eta_{l+1}\right) \ .$$

One has

$$\begin{aligned}
\delta_\varpi(\widetilde{\varpi}\cdot\eta) &= \delta_\varpi\left(\varpi^{-(\lambda+1)}\sum_{l=0}^{p}(\log\varpi)^l\widetilde{\varpi}\eta_l\right) = \\
&= \varpi^{-(\lambda+1)}\sum_{l=0}^{p}(\log\varpi)^l\left((\lambda+1)\widetilde{\varpi}\eta_l-(l+1)\widetilde{\varpi}\eta_{l+1}\right) = \\
&= (\widetilde{\varpi}\cdot\delta_\varpi+\widetilde{\varpi})(\eta) \ .
\end{aligned}$$

This means that the operators δ_ϖ make each $L^{(\varpi)q}_X$ a sheaf of W_R-modules. We notice that this operator δ_ϖ *commutes* with the canonical action of R on $L^{(\varpi)q}_{X,\lambda}$ (which takes the above η to $\varpi^{-\lambda}\sum_{l=0}^{p}(\log\varpi)^l\widetilde{\varpi}\eta_l$).

Finally, one has $\delta_\varpi(d\eta) = d(\delta_\varpi\eta)$ since both sides are equal to

$$\varpi^{-\lambda}\sum_{l=0}^{p}(\log\varpi)^l \qquad (d\log(\varpi)\wedge(-\lambda^2\eta_l + 2(l+1)\eta_{l+1} - (l+1)(l+2)\eta_{l+2}) + +\lambda d\eta_l - (l+1)d\eta_{l+1})$$

Thus, $L_X^{(\varpi)\cdot}$ is a complex of sheaves of W_R-modules.

Let now $\varphi : \varpi \to \varpi' = \alpha\varpi$ be a morphism in Π_R given by an element $\beta \in R$ with $\exp(\beta) = \alpha^{-1}$. Then the corresponding homomorphism $\varphi_{\overline{\omega}}$ from §9.2 takes the above q-form η to

$$\varpi'^{-\lambda}\exp(-\lambda\beta)\sum_{l=0}^{p}(\log(\varpi') + \beta)^l\eta_l \ ,$$

which is a local section of $L_{X,\lambda}^{(\varpi')q}$. This implies that φ gives rise to $\mathbf{C}$-linear morphisms of complexes $\varphi_{L_\lambda} : L_{X,\lambda}^{(\varpi)\cdot} \to L_{X,\lambda}^{(\varpi')\cdot}$ and $\varphi_L : L_X^{(\varpi)\cdot} \to L_X^{(\varpi')\cdot}$. It follows from the definition of the multiplication by $\widetilde{\varpi}$ that $\widetilde{\varpi}'\cdot\varphi_L = \varphi_L\cdot\widetilde{\varpi}$ and, therefore, the collections of complexes $L_{X,\lambda}^{(\varpi)\cdot}$ and $L_X^{(\varpi)\cdot}$ form subcomplexes of sheaves of Π_R-modules $L^\cdot_{X,\lambda} \subset \overline{\tau}_*(\omega^\cdot_{\overline{X^{\log}},\lambda})$ and of $\underline{R}$-modules $L^\cdot_X \subset \overline{\tau}_*(\overline{\omega}^\cdot_{\overline{X^{\log}}})$. *We claim that* $\delta_{\varpi'}\circ\varphi_{L_\lambda} = \varphi_{L_\lambda}\circ\delta_\varpi$.

Indeed, setting $\alpha^\lambda = \exp(-\lambda\beta)$, we have

$$\begin{aligned}\varphi_{L_\lambda}(\eta) &= \varpi'^{-\lambda}\alpha^\lambda\sum_{l=0}^{p}(\log(\varpi')+\beta)^l\eta_l = \varpi'^{-\lambda}\alpha^\lambda\sum_{l=0}^{p}\sum_{j=0}^{l}\binom{l}{j}(\log\varpi')^j\beta^{l-j}\eta_l = \\ &= \varpi'^{-\lambda}\alpha^\lambda\sum_{j=0}^{p}(\log\varpi')^j\left(\sum_{l=j}^{p}\binom{l}{j}\beta^{l-j}\eta_l\right) .\end{aligned}$$

If we set $\xi_j = \sum_{l=j}^{p}\binom{l}{j}\beta^{l-j}\eta_l$, we get

$$\delta_{\varpi'}(\varphi_{L_\lambda}(\eta)) = \varpi'^{-\lambda}\alpha^\lambda\sum_{j=0}^{p}(\log\varpi')^j\,(\lambda\xi_j - (j+1)\xi_{j+1})$$

On the other hand, we have

$$\begin{aligned}\varphi_{L_\lambda}(\delta_\varpi(\eta)) &= \varpi'^{-\lambda}\alpha^\lambda\sum_{l=0}^{p}(\log(\varpi')+\beta)^l(\lambda\eta_l - (l+1)\eta_{l+1}) = \\ &= \varpi'^{-\lambda}\alpha^\lambda\sum_{j=0}^{p}(\log\varpi')^j\left(\sum_{l=j}^{p}\binom{l}{j}\beta^{l-j}(\lambda\eta_l - (l+1)\eta_{l+1})\right) = \\ &= \varpi'^{-\lambda}\alpha^\lambda\sum_{j=0}^{p}(\log\varpi')^j\left(\lambda\xi_j - \sum_{l=j}^{p}(l+1)\binom{l}{j}\beta^{l-j}\eta_{l+1}\right) .\end{aligned}$$

Since $(l+1)\binom{l}{j} = (j+1)\binom{l+1}{j+1}$, it follows that

$$\sum_{l=j}^{p}(l+1)\binom{l}{j}\beta^{l-j}\eta_{l+1} = (j+1)\sum_{l=j+1}^{p}\binom{l}{j+1}\beta^{l-j-1}\eta_l = (j+1)\xi_{j+1} .$$

The claim follows and, therefore, $L^\cdot_X$ is a complex of sheaves of $\underline{W}_R$-modules.

We notice that there is a canonical isomorphism of $\underline{W}_R$-sheaves

$$\mathcal{C}^{\cdot}_X \widetilde{\to} \mathrm{Ker}(L^0_X \xrightarrow{d} L^1_X) \ .$$

It gives rise to a commutative diagram of morphisms of complexes of sheaves on $\overline{X^{\log}}$

$$\begin{tikzcd} \overline{\tau}^{-1}(\mathcal{C}^{\cdot}_X) \arrow[r] \arrow[d] & \overline{\tau}^{-1}(L^{\cdot}_X) \arrow[dl] \\ \overline{\omega}^{\cdot}_{\overline{X^{\log}}} & \end{tikzcd}$$

Since the $\underline{W}_R$-module structure on the complex at the bottom is defined by the left arrow quasi-isomorphism, it follows that $\overline{\tau}^{-1}(L^{\cdot}_X) \to \overline{\omega}^{\cdot}_{\overline{X^{\log}}}$ is a morphism of $\underline{W}_R$-modules in the derived category of complexes of $\underline{R}$-sheaves on $\overline{X^{\log}}$, and it induces a morphism of $\underline{W}_R$-modules $L^{\cdot}_X \to R\overline{\tau}_*(\overline{\omega}^{\cdot}_{\overline{X^{\log}}})$ in the similar derived category on X.

9.4. **A quasi-isomorphism $L^{\cdot}_X \widetilde{\to} \omega^{\cdot}_{X/R}$.** First of all, we construct, for every $\varpi \in \Pi_R$, a morphism of complexes of sheaves of R-modules $\psi^{(\varpi)} : L^{(\varpi)}_X \to \omega^{\cdot}_{X/R}$. If η is a local section of $L^{(\varpi)q}_{X,\lambda}$ as above, then $\psi^{(\varpi)}(\eta) = \overline{\eta}_0$, where $\overline{\xi}$ denotes the image of a local section ξ of ω^q_X in $\omega^q_{X/R}$. This morphism is clearly R-linear. Moreover, since $(d\eta)_0 = d\log(\varpi) \wedge (-\lambda\eta_0 + \eta_1) + d\eta_0$, it follows that $(\overline{d\eta})_0 = d\overline{\eta}_0$, i.e., $\psi^{(\varpi)}$ is really a morphism of complexes.

Proposition 9.4.1. *(i) The morphisms $\psi^{(\varpi)}$ are quasi-isomorphisms and, in particular, they define a $\underline{W}_R$-module structure on the complex $\omega^{\cdot}_{X/R}$ (considered as an object of the derived category of complexes of sheaves of R-modules);*

(ii) the morphisms δ_ϖ on $L^{(\varpi)}_X$ give rise to the morphisms $\widetilde{\delta}_\varpi$ on $\omega^{\cdot}_{X/R}$ (introduced in §9.1) and, in particular, they induce the Gauss-Manin connection on the de Rham cohomology groups $H^q_{\mathrm{dR}}(X/R)$.

Proof. Step 1. In order to prove (i), we have to show that, for every point $x \in X$, the map $\oplus_\lambda \mathcal{H}^q(L^{(\varpi)\cdot}_{X,\lambda,x}) \to \mathcal{H}^q(\omega^{\cdot}_{X/R,x})$ induced by $\psi^{(\varpi)}$ is a bijection. We can therefore assume that $X = \mathcal{X}^h$ for $\mathcal{X} = \mathrm{Spec}(B)^h$ with B as in Step 1 from the proof of Proposition 9.2.1, x the zero point, and $n = m$.

Step 2. The $\mathbf{C}$-vector space $L^{(\varpi)q}_{X,\lambda,x}$ is generated elements of the form

$$\varpi^{-\lambda}\widetilde{\widetilde{\varpi}}^{[\lambda]}(\log\varpi)^l f d\log(T_{j_1}) \wedge \ldots \wedge d\log(T_{j_q}) \ ,$$

where $1 \le j_1 < \ldots < j_q \le m$, $l \ge 0$, and $f \in A = \mathcal{O}_{X_{r-[\lambda]},x}$. The latter is a convergent power series $\sum_{\mathbf{k}} a_{\mathbf{k}} T^{\mathbf{k}}$ taken over $\mathbf{k} \in \mathbf{Z}^m_+$ with the property that $k_i < (r-[\lambda])e_i$ for some $1 \le i \le \mu$. Notice that the differential $d(\varpi^{-\lambda}\widetilde{\widetilde{\varpi}}^{[\lambda]}(\log\varpi)^l T^{\mathbf{k}})$ is equal to

$$\varpi^{-\lambda}\widetilde{\widetilde{\varpi}}^{[\lambda]}T^{\mathbf{k}}\left(\sum_{i=1}^m \left((k_i - (\lambda - [\lambda])e_i)(\log\varpi)^l + le_i(\log\varpi)^{l-1}\right) d\log(T_i)\right) \ .$$

Let $\delta = (\delta_1, \ldots, \delta_m)$ be the tuple of functions with $\delta_i = k_i - (\lambda - [\lambda])e_i$ and, for $1 \le i \le m+1$, let $C^{\cdot}_{\lambda,i}$ be the subcomplex of $L^{(\varpi)\cdot}_{X,\lambda,x}$ such that $C^q_{\lambda,i}$ consists of $\mathbf{C}$-linear combinations of the above elements with $f \in A^{(i)}_\delta$ (see Construction 9.1.2).

There is an isomorphism of complexes

$$\bigoplus_{i=1}^{m+1} C^{\cdot}_{\lambda,i} \widetilde{\to} L^{(\varpi)\cdot}_{X,\lambda} \ .$$

Step 3. For $l \geq 0$, let $C^{\cdot}_{\lambda,i,l}$ be the subcomplex of $C^{\cdot}_{\lambda,i}$ consisting of forms in which the degree in $\log(\varpi)$ is at most l. One has $C^{\cdot}_{\lambda,i,0} = E^{\cdot}_{\delta,i}$, $C^{\cdot}_{\lambda,i} = \bigcup_{l=0}^{\infty} C^{\cdot}_{\lambda,i,l}$, and there are exact sequences of complexes

$$0 \to C^{\cdot}_{\lambda,i,l} \to C^{\cdot}_{\lambda,i,l+1} \to E^{\cdot}_{\delta,i} \to 0 \ .$$

Thus, if $1 \leq i \leq m$, the complex $E^{\cdot}_{\delta,i}$ is exact, and from the above exact sequence follow that all of the complexes $C^{\cdot}_{\lambda,i,l}$ are exact and, therefore, the complex $C^{\cdot}_{\lambda,i}$ is exact, i.e., there is a canonical quasi-isomorphism complexes $C^{\cdot}_{\lambda,m+1} \widetilde{\to} L^{(\varpi)\cdot}_{\lambda,x}$. The complex $C^{\cdot}_{\lambda,m+1}$ is generated by the elements as above with sums $\sum_{\mathbf{k}} a_{\mathbf{k}} T^{\mathbf{k}}$ taken over tuples $\mathbf{k} \in \mathbf{Z}^m_+$ with the property that $k_i = (\lambda - [\lambda])e_i$ for all $1 \leq i \leq m$. Notice that such a tuple exists only for λ's of the form $[\lambda] + \frac{p}{e}$ with $0 \leq p < e$. In particular, if λ is not of this form, then the complex $L^{\cdot}_{X,\lambda,x}$ is acyclic.

Step 4. Suppose $\lambda = [\lambda] + \frac{p}{e}$ with $0 \leq p < e$. Then for the above tuples $\mathbf{k}$, one has $k_i = pe'_i$, $1 \leq i \leq m$. It follows that each element of $C^q_{\lambda,m+1}$ is of the form

$$\eta = \varpi^{-\lambda} \widetilde{\varpi}^{[\lambda]} t^p \sum_{j=0}^{l} (\log \varpi)^j \xi_j \ ,$$

where t denotes the image of $T_1^{e'_1} \cdot \ldots \cdot T_m^{e'_m}$ in A, and ξ_j are $\mathbf{C}$-linear combination of the q-forms $d\log(T_{j_1}) \wedge \ldots \wedge d\log(T_{j_q})$. Notice that

$$d\eta = \underline{\varpi}^{-\lambda} \varpi^{[\lambda]} t^p \sum_{j=0}^{l} j (\log \varpi)^{j-1} d\log(\varpi) \wedge \xi_j \ .$$

It follows that $d\eta = 0$ if and only if $d\log(\varpi) \wedge \xi_j = 0$ for all $1 \leq j \leq l$. We also notice that, since $\psi^{(\varpi)}(\eta) = \varpi^{[\lambda]} t^p \xi_0$, Proposition 9.1.1 implies that the map considered in Step 1 is a surjection, and it remains to verify that the map $\psi^{(\varpi)} : \mathcal{H}^q(C^{\cdot}_{\lambda,m+1}) \to \mathcal{H}^q(\omega^{\cdot}_{X/R,x})$ is an injection.

Step 5. Suppose that for the above element η, one has $d\eta = 0$ and $\psi^{(\varpi)}(\eta) = 0$. It follows that $\xi_0 = 0$ and, therefore,

$$\eta = d\left(\varpi^{-\lambda} \widetilde{\varpi}^{[\lambda]} t^p \sum_{j=1}^{k} \frac{1}{j+1} (\log \varpi)^{j+1} \chi_j \right) \ ,$$

where χ_j is a $(q+1)$-form of the same kind with $\xi_j = d\log(\varpi) \wedge \chi_j$. (Existence of such χ_j's follows from the fact that the Koszul complex $\mathrm{K}_{\mathbf{C}}(D_1, \ldots, D_m)$ for the $\mathbf{C}$-linear maps $D_i : \mathbf{C} \to \mathbf{C}$ of multiplication by e_i is exact.) Thus, the map $\mathcal{H}^q(L^{(\varpi)\cdot}_{X,\lambda,x}) \to \mathcal{H}^q(\omega^{\cdot}_{X/R,x})$ is injective, and (i) is proved.

Step 6. Let $C(f)^{\cdot}$ be the cone of the morphism f from the exact sequence of complexes $(*)$ in 9.1. In order to prove (ii), it suffices to construct a morphism of complexes $\gamma^{(\varpi)} : L_X^{(\varpi)\cdot} \to C(f)^\cdot$ that makes the following diagram commutative

$$\begin{tikzcd} L_X^{(\varpi)\cdot} \arrow[r, "\gamma^{(\varpi)}"] \arrow[dr, "\psi^{(\varpi)}"'] & C(f)^\cdot \arrow[d, "\widetilde{\delta}_\varpi"] \\ & \omega^\cdot_{X/R} \end{tikzcd}$$

Recall that, for a local section $\eta = \underline{\varpi}^{-\lambda}\sum_{l=0}^{l}(\log\varpi)^l\eta_l$ of $L^{(\varpi)q}$, one has $\psi^{(\varpi)}(\eta) = \overline{\eta}_0$. Recall also that $C(f)^q = (\omega^1_R \otimes_R \omega^q_{X/R}) \oplus \omega^q_X$, and $\widetilde{\delta}_\varpi(d\log(\varpi)\otimes\xi, \chi) = \lambda\xi - \overline{\chi}$. We define a $\mathbf{C}$-linear homomorphism of sheaves $\gamma^{(\varpi)} : L_X^{(\varpi)q} \to C(f)^q$ by

$$\gamma^{(\varpi)}(\eta) = (d\log(\varpi) \otimes (-\lambda\overline{\eta}_0 + \overline{\eta}_1), \eta_0)\ .$$

We see that $\psi^{(\varpi)}(\eta) = \widetilde{\delta}_\varpi(\gamma^{(\varpi)}(\eta))$, and we have to verify that $\gamma^{(\varpi)}$ is a morphism of complexes. For this we recall that $(d\eta)_0 = d\log(\varpi) \wedge (-\lambda\eta_0 + \eta_1) + d\eta_0$ and, in particular, $\overline{(d\eta)}_0 = d\overline{\eta}_0$, and notice that $(d\eta)_1 = d\log(\varpi) \wedge (-\lambda\eta_1 + 2\eta_2) + d\eta_1$ and, in particular, $\overline{(d\eta)}_1 = d\overline{\eta}_1$. It follows that

$$\gamma^{(\varpi)}(d\eta) = (d\log(\varpi) \otimes (-\lambda d\overline{\eta}_0 + d\overline{\eta}_1), d\log(\varpi) \wedge (-\lambda\eta_0 + \eta_1) + d\eta_0) = d(\gamma^{(\varpi)}\eta)\ .$$

This implies the required fact. ☐

Corollary 9.4.2. *The W_R-module structure on the de Rham cohomology groups $H^q_{\mathrm{dR}}(X/R)$ is the restriction of the $\underline{W}_R$-module structure induced by that on the complex $\omega^\cdot_{X/R}$.* ☐

We say that a R-linear endomorphism M of a sheaf of R-modules F on X is *locally nilpotent* if, for every section $f \in F(U)$ over an open subset $U \subset X$ and every point $x \in U$, there exist an open neighborhood U' of x in U and an integer $n \geq 1$ with $M^n(f\big|_{U'}) = 0$. For such M, the exponent $\exp(M) = \sum_{n=0}^\infty \frac{M^n}{n!}$ is a well defined R-linear automorphism of F. More generally, for any element $\beta \in R$ the exponent $\exp(N) = \sum_{n=0}^\infty \frac{N^n}{n!}$ of the operator $N = \beta\cdot\mathrm{Id}_F + M$ is well defined, and it is in fact equal to $\exp(\beta)\cdot\exp(M)$. Indeed, for the above local section f, let $l \geq 0$ be an integer with $M^{l+1}(f\big|_{U'}) = 0$. Setting $g = f\big|_{U'}$, one has

$$\begin{aligned} \exp(N)(g) &= \sum_{n=0}^\infty \frac{1}{n!}(\beta\cdot\mathrm{Id} + M)^n(g) = \sum_{n=0}^\infty \frac{1}{n!}\left(\sum_{j=0}^{l}\binom{n}{j}M^j(\beta^{n-j}g)\right) = \\ &= \sum_{j=0}^{l}\frac{1}{j!}\left(\sum_{n=0}^\infty \frac{\beta^n}{n!}\right)M^j(g) = \exp(\beta)\cdot\exp(M)(g)\ . \end{aligned}$$

An example of such N is the R-linear endomorphism δ_ϖ acting on the sheaf $L^{(\varpi)q}_{X,\lambda}$. (The action of R on the latter sheaf is the canonical one.) Indeed, for a local section $\eta = \varpi^{-\lambda}\sum_{l=0}^{p}(\log\varpi)^l\eta_l$ of that sheaf, one has

$$\delta_\varpi(\eta) = \varpi^{-\lambda}\sum_{l=0}^{p}(\log\varpi)^l\,(\lambda\eta_l - (l+1)\eta_{l+1})$$

and, therefore, $\delta_\varpi = \lambda\mathrm{Id} + M$, where M is the locally nilpotent R-linear endomorphism of $L^{(\varpi)q}_{X,\lambda}$, defined by $M(\eta) = \sum_{l=0}^{p}(l+1)\eta_{l+1}$. A more general example of such an endomorphism N is the product $\beta\delta_\varpi$ for $\beta \in R$ (with respect to the canonical R-module structure on $L^{(\varpi)q}_{X,\lambda}$). Notice that the automorphism $\exp(\beta\delta_\varpi)$ extends naturally to the sheaf $L^{(\varpi)q}_X = \bigoplus_{\lambda\in\mathbf{Q}_+} L^{(\varpi)q}_{X,\lambda}$.

Proposition 9.4.3. *Given a morphism $\varphi : \varpi \to \varpi' = \alpha\varpi$ in Π_R, i.e., an element $\beta \in R$ with $\exp(\beta) = \alpha^{-1}$, and $q \geq 0$, the following diagram (in which $e^{-\beta\delta_\varpi} = \exp(-\beta\delta_\varpi)$) is commutative*

$$\begin{tikzcd} L^{(\varpi)q}_X \arrow[r, "e^{-\beta\delta_\varpi}"] \arrow[d, "\varphi^q_L"] & L^{(\varpi)q}_X \arrow[d, "\psi^{(\varpi)}"] \\ L^{(\varpi')q}_X \arrow[r, "\psi^{(\varpi')}"] & \omega^q_{X/R} \end{tikzcd}$$

Proof. It suffices to verify commutativity of the diagram on each of the sheaves $L^{(\varpi)q}_{X,\lambda}$. For a local section $\eta = \varpi^{-\lambda}\sum_{l=0}^\infty (\log\varpi)^l\eta_l$ of $L^{(\varpi)q}_{X,\lambda}$ (the sum is in fact finite), one has $-\beta\delta_\varpi(\eta) = -\lambda\beta\eta + M(\eta)$, where M is the locally nilpotent operator with

$$M(\eta) = \varpi^{-\lambda}\beta\sum_{l=0}^\infty (\log\varpi)^l(l+1)\eta_{l+1}\ ,$$

and therefore $\exp(-\beta\delta_\varpi) = \exp(-\lambda\beta)\exp(M)$. One has

$$\begin{aligned} \exp(M)(\eta) &= \varpi^{-\lambda}\sum_{n=0}^\infty \frac{\beta^n}{n!}\left(\sum_{l=0}^\infty (\log\varpi)^l(l+1)\cdot\ldots\cdot(l+n)\eta_{l+n}\right) = \\ &= \varpi^{-\lambda}\sum_{j=0}^\infty\left(\sum_{l=0}^\infty \binom{j}{l}(\log\varpi)^l\cdot\beta^{j-l}\right)\eta_j = \\ &= \varpi^{-\lambda}\sum_{j=0}^\infty (\log(\varpi)+\beta)^j\eta_j\ . \end{aligned}$$

It follows that $\psi^{(\varpi)}(\exp(-\beta\delta_\varpi)(\eta)) = \exp(-\lambda\beta)\sum_{j=0}^\infty\beta^j\eta_j$. On the other hand, one has

$$\varphi^q_{L_\lambda}(\eta) = \varpi'^{-\lambda}\exp(-\lambda\beta)\sum_{j=0}^\infty (\log(\varpi')+\beta)^j\eta_j$$

and, therefore, $\psi^{(\varpi')}(\varphi_{L_\lambda}(\eta)) = \exp(-\lambda\beta)\sum_{j=0}^\infty\beta^j\eta_j$. The required fact follows. ∎

In the situation of Proposition 9.4.3, the isomorphisms φ^q_L are induced by an isomorphism of complexes $\varphi_L : L^{(\varpi)\cdot}_X \to L^{(\varpi')\cdot}_X$ and, in their turn, give rise to an automorphism φ_ω of the complex $\omega^\cdot_{X/R}$ (in the derived category of complexes of sheaves of $\mathbf{C}$-vector spaces). But the automorphisms $\exp(-\beta\delta_\varpi)$ do not commute with the differential of the complex $L^{(\varpi)\cdot}_X$ unless $\alpha\in\mathbf{C}^*$. In the latter case $\beta\in\mathbf{C}$, and we denote in the same way by $\exp(-\beta\delta_\varpi)$ the induced automorphism of the complexes $L^{(\varpi)\cdot}_X$ and $\omega^\cdot_{X/R}$.

Corollary 9.4.4. *In the situation of Proposition 9.4.3, assume that $\alpha\in\mathbf{C}^*$. Then the automorphisms φ_ω and $\exp(-\beta\delta_\varpi)$ of the complex $\omega^\cdot_{X/R}$ coincide.* ∎

For example, let $\sigma^{(\varpi)}$ denote the generator of the automorphism group of ϖ (in Π_R) that corresponds to the number $2\pi i$. It gives rise to a R-linear automorphism of the complex $\omega^{\cdot}_{X/R}$, and the statement of Corollary 9.4.4 can be written as the equality $\sigma^{(\varpi)} = \exp(-2\pi i\delta_\varpi)$.

If $r = 1$, the assumption of Corollary 9.4.4 holds for all morphisms in the category $\Pi_{K_1^\circ}$. In this case one also has $W_{K_1^\circ} = \mathbf{C}[\delta_\varpi]$, and the element δ_ϖ does not depend on ϖ. Thus, if δ denotes the operator induced by δ_ϖ on $\omega^{\cdot}_{X/K_1^\circ}$, one has $\varphi_\omega = \exp(-\beta\delta)$. In particular, the action of the groupoid $\Pi_{K_1^\circ}$ on $\omega^{\cdot}_{X/K_1^\circ}$ is completely determined by the operator δ.

9.5. **An isomorphism $R\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} R \widetilde{\to} \omega^{\cdot}_{X/R}$.** By Theorem 4.4.1, there is a canonical isomorphism of sheaves of $\underline{W}_R$-modules on X

$$\chi : \mathcal{C}_X \widetilde{\to} \overline{\tau}_*(\underline{R}_{\overline{X^{\log}}}) = \overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R}$$

which induces a morphism of complexes of sheaves of $\underline{W}_R$-modules on X

$$f : R\overline{\tau}_*(\overline{\tau}^{-1}(\mathcal{C}_X)) \to R\overline{\tau}_*(\underline{R}_{\overline{X^{\log}}}) = R\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R} \ .$$

By Proposition 9.2.1, there is an isomorphism of $\underline{W}_R$-modules in the derived category

$$g : R\overline{\tau}_*(\overline{\tau}^{-1}(\mathcal{C}_X)) \widetilde{\to} R\overline{\tau}_*(\overline{\omega}^{\cdot}_{\overline{X^{\log}}}) \ .$$

We construct a morphism $\theta : L^{\cdot}_X \to R\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R}$ in the derived category as the following composition of the homomorphisms

$$L^{\cdot}_X \to R\overline{\tau}_*(\overline{\omega}^{\cdot}_{\overline{X^{\log}}}) \stackrel{g^{-1}}{\to} R\overline{\tau}_*(\overline{\tau}^{-1}(\mathcal{C}_X)) \stackrel{f}{\to} R\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R} \ .$$

Proposition 9.5.1. *The morphism θ is an isomorphism in the derived category of complexes of sheaves of $\mathbf{C}$-vector spaces, and it gives rise to an isomorphism of $\underline{W}_R$-modules*

$$R\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_{\mathbf{C}} \underline{R} \widetilde{\to} \omega^{\cdot}_{X/R}$$

Proof. It suffices to prove that, for every point $x \in X$ and every integer $q \geq 0$, φ induces an isomorphism $\mathcal{H}^q(L^{\cdot}_{X,x}) \widetilde{\to} R^q\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}})_x \otimes_{\mathbf{C}} R$ and, for this, it suffices to verify commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{H}^q(\omega^{\cdot}_{X/R,x}) \arrow[d, "\psi_x^{-1}"] & \mathcal{C}_{X,x} \otimes_{\mathbf{Z}} \bigwedge^q \overline{M}^{(nont)}_{X,x} \arrow[l, "u"'] \arrow[r, "v"] & R^q\overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}})_x \otimes_{\mathbf{C}} R \\
\mathcal{H}^q(L^{\cdot}_{X,x}) \arrow[r] & R^q\overline{\tau}_*(\overline{\omega}^{\cdot}_{\overline{X^{\log}}})_x \arrow[r, "g_x^{-1}"] & R^q\overline{\tau}_*(\overline{\tau}^{-1}(\mathcal{C}_X))_x \arrow[u, "f_x"']
\end{tikzcd}$$

where u is the second isomorphism of Proposition 9.1.1, and v is induced by the isomorphism of Theorem 4.3.1.

We may assume that $X = \mathcal{X}^h$ for $\mathcal{X} = \mathrm{Spec}(B)$ with B as in Step 1 from the proof of Proposition 9.2.1. We set $e = \text{g.c.d.}(e_1, \ldots, e_m)$ and denote by t the image of the element $T_1^{e'_1} \cdot \ldots \cdot T_m^{e'_m}$ in $\mathcal{O}(X)$, where $e'_i = \frac{e_i}{e}$. Furthermore, the group $\overline{M}^{(nont)}_{X,x}$ is freely generated by the images of the coordinate functions $T_1, \ldots, T_{m-1}$ and, in particular, its q-th external power is zero for $q \geq m$. We may therefore assume that $q \leq m-1$. Each element of the tensor product in the first row is a $\mathbf{C}$-linear combination of elements of the form $\gamma = t^j T_{i_1} \wedge \ldots \wedge T_{i_q}$. It suffices to check commutativity on these elements. After a permutation of coordinates, we may assume that $\gamma = t^j T_1 \wedge \ldots \wedge T_q$. Then $u(\gamma)$ is represented by the element $t^j d\log(T_1) \wedge \ldots \wedge d\log(T_q)$, and so $\psi_x^{-1}(u(\gamma))$ is represented by the element $\varpi^{-\frac{j}{e}} t^j d\log(T_1) \wedge \ldots \wedge d\log(T_q)$ of $\mathcal{H}^q(L^{\cdot})_x$ that maps to $\mathcal{H}^q(\overline{\tau}_* \overline{\omega}^{\cdot}_{\overline{X^{\log}}})_x$ which, in its turn, maps to $R^q\overline{\tau}_*(\overline{\omega}^{\cdot}_{\overline{X^{\log}}})_x$.

On the other hand, there is a canonical homomorphism of sheaves

$$\overline{\tau}_*(\overline{\tau}^{-1}(\mathcal{C}_X)) \otimes_{\mathbf{Z}_X} \bigwedge^q \overline{M}^{(nont)}_{X,x} \to R^q\overline{\tau}_*(\overline{\tau}^{-1}(\mathcal{C}_X))$$

and the image of the element $\eta = t^j T_1 \wedge \ldots \wedge T_q$ from the stalk at x of the sheaf on the left hand side in the stalk of that on the right hand side goes under the map g_x to the class of $\varpi^{-\frac{j}{e}} t^j d\log(T_1) \wedge \ldots \wedge d\log(T_q)$ in $R^q\overline{\tau}_*(\overline{\omega}^{\cdot}_{\overline{X^{\log}}})_x$. Thus, commutativity of the above diagram follows from the fact that both maps v and f_x are induces by the same isomorphism $\chi : \mathcal{C}_X \widetilde{\to} \overline{\tau}_*(\mathbf{C}_{\overline{X^{\log}}}) \otimes_C R$. $\square$

Corollary 9.5.2. *For every distinguished formal scheme $\mathfrak{X}$ over K°, there is a compatible system of canonical isomorphisms of $\underline{W}_{K^\circ_r}$-modules in the derived category*

$$R\Psi^h_\eta(\mathbf{C}_{\mathfrak{X}_\eta}) \otimes_{\mathbf{C}} K^\circ_r \widetilde{\to} \omega^{\cdot}_{\mathfrak{X}^h_{s_r}/K^\circ_r} .$$

Proof. By the definition of $R\Psi^h_\eta$, the complex on the left hand side of the isomorphism in Proposition 9.5.1 is $R\Psi^h_\eta(\mathbf{C}_{\mathfrak{X}_\eta})$, and the required fact follows. $\square$

10. Comparison with de Rham cohomology

10.1. **Formulation of results.** Let k be a non-Archimedean field (whose valuation is not assumed to be nontrivial). For a morphism of k-analytic spaces $\varphi : Y \to X$, we consider the sheaf of relative one-differential forms $\Omega^1_{Y/X}$ as a sheaf in the G-topology on Y (it is denoted by Ω_{Y_G/X_G} in [Ber93, §1.4]). Its exterior powers $\Omega^q_{Y/X}$ form a relative de Rham complex $\Omega^{\cdot}_{Y/X}$, and the higher direct images of the latter with respect to the morphism φ are called *de Rham cohomology sheaves* and denoted by $\mathcal{H}^q_{\mathrm{dR}}(Y/X)$. (They are also considered in the G-topology of X.) We are in fact interested only in the following situation.

Let X be a rig-smooth K-analytic space. The de Rham complex and de Rham cohomology of the canonical morphism $X \to \mathcal{M}(K)$ are denoted by $\Omega^{\cdot}_{X/K}$ and $H^q_{\mathrm{dR}}(X/K)$, respectively. (For example, if $\mathcal{X}$ is a smooth scheme of finite type over K then, by a theorem of Kiehl [Kie67], there is a canonical isomorphism $H^q_{\mathrm{dR}}(\mathcal{X}/K) \widetilde{\to} H^q_{\mathrm{dR}}(\mathcal{X}^{\mathrm{an}}/K)$.) Furthermore, X can be also considered as a $\mathbf{C}$-analytic space for the field $\mathbf{C}$ provided with the trivial valuation. The de Rham complex and de Rham cohomology of the canonical morphism $X \to \mathcal{M}(\mathbf{C})$ are denoted by $\Omega^{\cdot}_X$ and $H^q_{\mathrm{dR}}(X)$, respectively.

For example, for the morphism $\mathcal{M}(K) \to \mathcal{M}(\mathbf{C})$, one has $\Omega^0_K = K$ and Ω^1_K is a one dimensional K-vector space generated by the one form $d\log(\varpi) = \frac{d\varpi}{\varpi}$ for any generator ϖ of the maximal ideal $K^{\circ\circ}$ of K°. In particular, $H^0_{\mathrm{dR}}(K) = \mathbf{C}$ and $H^1_{\mathrm{dR}}(K)$ is a one-dimensional $\mathbf{C}$-vector space with a canonical generator, the image of $d\log(\varpi)$ which does not depend on the choice of ϖ.

Furthermore, consider the exact sequence of complexes

$$0 \to \Omega^1_K \otimes_K \Omega^{\cdot}_{X/K}[-1] \xrightarrow{f} \Omega^{\cdot}_X \to \Omega^{\cdot}_{X/K} \to 0 .$$

As in §9.1, one shows that this exact sequence gives rise to a connection

$$\nabla : H^q_{\mathrm{dR}}(X/K) \to \Omega^1_K \otimes_K H^q_{\mathrm{dR}}(X/K)$$

called the *Gauss-Manin connection*. . For a generator ϖ of $K^{\circ\circ}$, the composition of the latter with the isomorphism $\Omega^1_K \widetilde{\to} K : d\log(\varpi) \mapsto 1$, gives rise to $\mathbf{C}$-linear endomorphisms

$$\delta_\varpi : H^q_{\mathrm{dR}}(X/K) \to H^q_{\mathrm{dR}}(X/K) \ ,$$

which provide the $\mathbf{C}$-vector spaces $H^q_{\mathrm{dR}}(X/K)$ with an action of the algebra W_K.

Furthermore, let k be a non-Archimedean field with discrete valuation which is not assumed to be nontrivial. Given a morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ of special formal schemes over k°, the sheaf of relative differential one-forms $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}$ is the conormal sheaf of the diagonal immersion $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$. It is a coherent $\mathcal{O}_{\mathfrak{X}}$-module which gives rise to the sheaf of relative differential one-forms $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}$. (If $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$, then $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}$ is the sheaf associated to the finite A-module I/I^2, where I is the kernel of the multiplication homomorphism $A\widehat{\otimes}_B A \to A$.)

Furthermore, suppose that $\varphi : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of fine log special formal schemes over k°. The sheaf of relative logarithmic differential one-forms $\omega^1_{\mathfrak{X}/\mathfrak{Y}}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module which is the quotient of $\Omega^1_{\mathfrak{X}/\mathfrak{Y}} \oplus (\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbf{Z}} M^{gr}_{\mathfrak{X}})$ by the $\mathcal{O}_{\mathfrak{X}}$-submodule generated by local sections of the form $(d\beta(m), 0) - (0, \beta(m) \otimes m)$ and $(0, 1 \otimes n)$ with m a local section of $M_{\mathfrak{X}}$ and n the image of a local section of $M_{\mathfrak{Y}}$ in $M_{\mathfrak{X}}$. The image of a local section m of $M^{gr}_{\mathfrak{X}}$ under the homomorphism $M^{gr}_{\mathfrak{X}} \to \omega^1_{\mathfrak{X}/\mathfrak{Y}} : m \mapsto (0, 1 \otimes m)$ is denoted by $d\log(m)$. The exterior powers of $\omega^1_{\mathfrak{X}/\mathfrak{Y}}$ form a relative log de Rham complex $\omega^\cdot_{\mathfrak{X}/\mathfrak{Y}}$. The *log de Rham cohomology sheaves* $\mathcal{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{Y})$ of the morphism φ are the cohomology sheaves of the complex $R\varphi_*(\omega^\cdot_{\mathfrak{X}/\mathfrak{Y}})$. If both formal schemes $\mathfrak{X}$ and $\mathfrak{Y}$ are of finite type over k° and their log structures are vertical, then $\omega^\cdot_{\mathfrak{X}/\mathfrak{Y}} \otimes_{k^\circ} k = \Omega^\cdot_{\mathfrak{X}_\eta/\mathfrak{Y}_\eta}$ and, therefore,

$$\mathcal{H}^q_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{Y}) \otimes_{k^\circ} k = \mathcal{H}^q_{\mathrm{dR}}(\mathfrak{X}_\eta/\mathfrak{Y}_\eta) \ .$$

Let us turn back to our field K, and let $\mathfrak{X}$ be a quasicompact distinguished special formal scheme over K° provided with the canonical log structure. The de Rham complex and de Rham cohomology groups of the canonical morphism $\mathfrak{X} \to \mathrm{Spf}(K^\circ)$ will be denoted by $\omega^\cdot_{\mathfrak{X}/K^\circ}$ and $H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ)$, respectively. By the previous paragraph, if $\mathfrak{X}$ is of finite type over K°, then $\omega^\cdot_{\mathfrak{X}/K^\circ} \otimes_{K^\circ} K = \Omega^\cdot_{\mathfrak{X}_\eta/K}$ and $H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \otimes_{K^\circ} K = H^q_{\mathrm{dR}}(\mathfrak{X}_\eta/K)$. The log formal scheme $\mathfrak{X}$ can be also considered as a log special formal scheme over the field $\mathbf{C}$ provided with the trivial valuation and trivial log structure. The corresponding de Rham complex and de Rham cohomology groups are denoted by $\omega^\cdot_{\mathfrak{X}}$ and $H^q_{\mathrm{dR}}(\mathfrak{X})$, respectively.

For example, for the morphism $\mathrm{Spf}(K^\circ) \to \mathrm{Spf}(\mathbf{C})$, one has $\omega^0_{K^\circ} = K^\circ$ and $\omega^1_{K^\circ}$ is a free K°-module of rank one generated by the one form $d\log(\varpi)$ for any generator ϖ of $K^{\circ\circ}$. In particular, $\omega^1_{K^\circ} \otimes_{K^\circ} K = \Omega^1_K$, $H^0_{\mathrm{dR}}(K^\circ) = \mathbf{C}$ and $H^1_{\mathrm{dR}}(K^\circ)$ is a one-dimensional $\mathbf{C}$-vector space with a canonical generator, the image of $d\log(\varpi)$ which does not depend on the choice of ϖ.

As above (and §9.1), one defines the *Gauss-Manin connection*

$$\nabla : H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \to \omega^1_{K^\circ} \otimes_{K^\circ} H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \ ,$$

which gives rise to the W_{K°-module structure on the de Rham cohomology groups $H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ)$ and, in particular, to $\mathbf{C}$-linear endomorphisms $\delta_\varpi : H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \to H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ)$.

Recall that $H^q(\mathfrak{X}_{\overline{\eta}}, \mathbf{C})$ are quasi-unipotent Π_K-modules of finite dimension over $\mathbf{C}$ and, by the construction from §3.5, the tensor products $H^q(\mathfrak{X}_{\overline{\eta}}, \mathbf{C}) \otimes_{\mathbf{C}} \underline{K}^\circ$ is provided with the structure of a distinguished $\underline{W}_K$-module.

Theorem 10.1.1. *Let $\mathfrak{X}$ be a quasicompact distinguished special formal scheme over K°. Then*

(i) *there is a canonical isomorphism of finitely generated $\mathbf{C}$-vector spaces*

$$H^q(\mathfrak{X}_\eta, \mathbf{C}) \widetilde{\to} H^q_{\mathrm{dR}}(\mathfrak{X}) \ ;$$

(ii) *the groups $H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ)$ have the structure of a distinguished $\underline{W}_{K^\circ}$-module, and there are canonical isomorphisms of distinguished $\underline{W}_{K^\circ}$-modules*

$$H^q(\mathfrak{X}_{\overline{\eta}}, \mathbf{C}) \otimes_{\mathbf{C}} \underline{K}^\circ \widetilde{\to} H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \ .$$

We notice that, if $\mathcal{X}$ is a proper distinguished log scheme over K°, Theorem (4.1.5) from [EGA3] implies that there are canonical isomorphisms

$$H^q_{\mathrm{dR}}(\mathcal{X}/K^\circ) \widetilde{\to} H^q_{\mathrm{dR}}(\widehat{\mathcal{X}}/K^\circ) \ .$$

Theorem 10.1.1 implies that, for any admissible proper morphism between quasicompact distinguished log special formal schemes $\mathfrak{X}' \to \mathfrak{X}$, there are canonical isomorphisms

$$H^q_{\mathrm{dR}}(\mathfrak{X}) \widetilde{\to} H^q_{\mathrm{dR}}(\mathfrak{X}') \text{ and } H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \widetilde{\to} H^q_{\mathrm{dR}}(\mathfrak{X}'/K^\circ) \ .$$

This allows us to define de Rham cohomology groups of a rig-smooth restricted K-analytic space as follows.

For a rig-smooth restricted K-analytic space $\widehat{X}$, we define

$$H^q_{\mathrm{dR}}(\widehat{X}) = \varprojlim H^q_{\mathrm{dR}}(\mathfrak{X}) \text{ and } H^q_{\mathrm{dR}}(\widehat{X}/K^\circ) = \varprojlim H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \ ,$$

where the projective limits are taken over distinguished formal models $\mathfrak{X}$ of $\widehat{X}$. Notice that all transition homomorphisms in these projective systems are isomorphisms.

Corollary 10.1.2. *Let $\widehat{X}$ be a rig-smooth restricted K-analytic space. Then*

(i) *there is a canonical isomorphism of finitely generated $\mathbf{C}$-vector spaces*

$$H^q(\widehat{X}, \mathbf{C}) \widetilde{\to} H^q_{\mathrm{dR}}(\widehat{X}) \ ;$$

(ii) *the groups $H^q_{\mathrm{dR}}(\widehat{X}/K^\circ)$ have the structure of a distinguished $\underline{W}_{K^\circ}$-module, and there are canonical isomorphisms of distinguished $\underline{W}_{K^\circ}$-modules*

$$H^q(\overline{\widehat{X}}, \mathbf{C}) \otimes_{\mathbf{C}} \underline{K}^\circ \widetilde{\to} H^q_{\mathrm{dR}}(\widehat{X}/K^\circ) \ . \qquad \square$$

Here is a consequence of Corollary 10.1.2 for compact rig-smooth K-analytic spaces. For this we say that a $\underline{W}_K$-module D is *distinguished* if it is isomorphic to the tensor product $D^\circ \otimes_{K^\circ} \underline{K}$ for a distinguished $\underline{W}_{K^\circ}$-module D°. It is easy to see that the functor $D^\circ \mapsto D^\circ \otimes_{K^\circ} \underline{K}$ from the category of distinguished $\underline{W}_{K^\circ}$-modules to that of distinguished $\underline{W}_K$-modules is an equivalence of categories.

Corollary 10.1.3. *Let X be a compact rig-smooth K-analytic space. Then*

(i) *there are canonical isomorphisms of finitely generated* $\mathbf{C}$*-vector spaces*

$$H^q(X,\mathbf{C})\widetilde{\to}H^q_{\mathrm{dR}}(X) \ ;$$

(ii) *the groups* $H^q_{\mathrm{dR}}(X/K)$ *have the structure of a distinguished* $\underline{W}_K$*-module, and there are canonical isomorphisms of distinguished* $\underline{W}_K$*-modules*

$$H^q(\overline{X},\mathbf{C})\otimes_{\mathbf{C}}\underline{K}\widetilde{\to}H^q_{\mathrm{dR}}(X/K) \ . \qquad \square$$

Suppose now we are given a separated distinguished scheme $\mathcal{X}$ of finite type over $\mathcal{K}^\circ = \mathcal{O}_{\mathbf{C},0}$ and a closed subscheme $\mathcal{Y}\subset\mathcal{X}_s$ which is a union of some of the irreducible components of $\mathcal{X}_s$. Then $(\mathcal{X}^h,\mathcal{Y}^h)$ is a distinguished log germ over $(\mathbf{C},0)$ in the sense of Definition 4.1.1(ii). It gives rise to a logarithmic space structure on $\mathcal{Y}^h$ and was an object of study of the previous section in the case $r=\infty$. Instead of the notation $H^q_{\mathrm{dR}}(\mathcal{Y}^h)$ and $H^q(\mathcal{Y}^h/K^\circ_\infty)$ for the corresponding de Rham cohomology groups used in §9.1, we denote them by $H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h))$ and $H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h)/\mathcal{K}^\circ)$, respectively. By Corollary 9.4.2, the latter groups are provided with the structure of a $\underline{W}_{\mathcal{K}^\circ}$-module.

Theorem 10.1.4. *In the above situation, the following is true:*

(i) *there are canonical isomorphisms*

$$H^q(\mathcal{X}^h(\mathcal{Y}^h)_\eta,\mathbf{C})\widetilde{\to}H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h))\widetilde{\to}H^q_{\mathrm{dR}}(\widehat{\mathcal{X}}_{/\mathcal{Y}}) \ ;$$

(ii) *there are canonical isomorphisms*

$$H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h)/\mathcal{K}^\circ)\otimes_{\mathcal{K}^\circ}\widehat{\mathcal{K}}^\circ\widetilde{\to}H^q_{\mathrm{dR}}(\widehat{\mathcal{X}}_{/\mathcal{Y}}/\widehat{\mathcal{K}}^\circ) \ ;$$

(iii) *the* $\underline{W}_{\mathcal{K}^\circ}$*-structure on the groups* $H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h)/\mathcal{K}^\circ)$ *is distinguished, and there are canonical isomorphisms of distinguished* $\underline{W}_{\mathcal{K}^\circ}$*-modules*

$$H^q(\mathcal{X}^h(\mathcal{Y}^h)_{\overline{\eta}},\mathbf{C})\otimes_{\mathbf{C}}\underline{\mathcal{K}}^\circ\widetilde{\to}H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h)/\mathcal{K}^\circ) \ ,$$

which induce the isomorphisms of Theorem 10.1.1(ii) for $\widehat{\mathcal{X}}_{/\mathcal{Y}}$.

Notice that, if $\mathcal{X}$ is proper over $\mathcal{K}^\circ$, GAGA implies that there are canonical isomorphisms

$$H^q_{\mathrm{dR}}(\mathcal{X}/\mathcal{K}^\circ)\widetilde{\to}H^q_{\mathrm{dR}}(\mathcal{X}^h/\mathcal{K}^\circ) \ .$$

Theorem 10.1.1 will be proved in §10.4 using results from §9 and §§10.2-10.3, and Theorem 10.1.4 will be proved in §10.5.

10.2. **Comparison of algebraic and analytic de Rham cohomology.**

Theorem 10.2.1. *Let* $\mathfrak{X}$ *be a quasicompact distinguished special formal scheme of* K°*. Then for every* $r\geq 1$*, there are canonical isomorphisms*

$$H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r})\widetilde{\to}H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r}) \text{ and } H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}/K^\circ_r)\widetilde{\to}H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r}/K^\circ_r) \ .$$

Proof. We use the reasoning from the proof of Grothendieck's theorem [Gro66].

Step 1. *The statement is true if there exists an open immersion* $\mathfrak{X}\hookrightarrow\mathfrak{X}'=\widehat{\mathcal{Y}}_{/\mathcal{Z}}$*, where* $\mathcal{Y}$ *is a proper distinguished scheme over* K° *and* $\mathcal{Z}$ *is a union of irreducible components of* $\mathcal{Y}_s$ *such that* $\mathcal{Z}\backslash\mathfrak{X}_s=\mathcal{Z}\cap\mathcal{W}$*, where* $\mathcal{W}$ *is a union of some of the other irreducible components of* $\mathcal{Y}_s$*.*

Indeed, in this case $\mathfrak{X}'_{s_r}$ is a proper log scheme over K°_r, the open immersion $j:\mathfrak{X}_{s_r}\hookrightarrow\mathfrak{X}'_{s_r}$ is strict, and the complement of $\mathfrak{X}_{s_r}$ is locally defined by one equation. For every $q\geq 0$, the coherent sheaves $\omega^q_{\mathfrak{X}_{s_r}}$ and $\omega^q_{\mathfrak{X}_{s_r}/K^\circ_r}$ are the restrictions to

$\mathfrak{X}_{s_r}$ of the coherent sheaves $\omega^q_{\mathfrak{X}'_{s_r}}$ and $\omega^q_{\mathfrak{X}'_{s_r}/K^\circ_r}$, respectively. Since the morphism of schemes j is affine, it follows that $R^pj_*(\mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on $\mathfrak{X}_{s_r}$ and any $p \geq 1$ and, therefore, the de Rham cohomology groups $H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r})$ and $H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}/K^\circ_r)$ are the q-th hypercohomology groups of the complexes $j_*\omega^q_{\mathfrak{X}_{s_r}}$ and $j_*\omega^q_{\mathfrak{X}_{s_r}/K^\circ_r}$, respectively. Since the scheme $\mathfrak{X}'_{s_r}$ is proper, GAGA implies that

$$H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}) \widetilde{\to} \mathbf{H}^q(\mathfrak{X}'^h_{s_r}, (j_*\omega^\cdot_{\mathfrak{X}_{s_r}})^h) \text{ and } H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}/K^\circ_r) \widetilde{\to} \mathbf{H}^q(\mathfrak{X}'^h_{s_r}, (j_*\omega^\cdot_{\mathfrak{X}_{s_r}/K^\circ_r})^h) \ .$$

On the other hand, since the complement of $\mathfrak{X}_{s_r}$ is locally defined by one equation, each point of $\mathfrak{X}'^h_{s_r}$ has a fundamental system of open Stein neighborhoods whose intersections with $\mathfrak{X}^h_{s_r}$ is a Stein space. It follows that $R^pj^h_*(F) = 0$ for any coherent sheaf F on $\mathfrak{X}^h_{s_r}$ and any $p \geq 1$ and, therefore, one has

$$H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r}) \widetilde{\to} \mathbf{H}^q(\mathfrak{X}'^h_{s_r}, j^h_*\omega^\cdot_{\mathfrak{X}^h_{s_r}}) \text{ and } H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r}/K^\circ_r) \widetilde{\to} \mathbf{H}^q(\mathfrak{X}'^h_{s_r}, j^h_*\omega^\cdot_{\mathfrak{X}^h_{s_r}/K^\circ_r}) \ .$$

Thus, in order to verify the claim, it suffices to show that there are quasi-isomorphisms of complexes

$$(j_*\omega^\cdot_{\mathfrak{X}_{s_r}})^h \widetilde{\to} j^h_*\omega^\cdot_{\mathfrak{X}^h_{s_r}} \text{ and } (j_*\omega^\cdot_{\mathfrak{X}_{s_r}/K^\circ_r})^h \widetilde{\to} j^h_*\omega^\cdot_{\mathfrak{X}^h_{s_r}/K^\circ_r} \ .$$

This is a purely local complex analytic fact which follows from Lemma 9.1.4.

Step 2. Let $\mathfrak{X}$ be an arbitrary quasicompact distinguished formal scheme over K°. *Then each point of $\mathfrak{X}$ has an étale affine neighborhood which satisfies the assumptions of Step 1.* Indeed, by Definition 2.1.1(ii), each point of $\mathfrak{X}$ has an étale neighborhood of the form $\widehat{\mathcal{Y}}_{/\mathcal{Z}}$, where $\mathcal{Y}$ is an affine distinguished scheme over K° and $\mathcal{Z}$ is a union of irreducible components of $\mathcal{Y}_s$. First of all, replacing $\mathcal{Y}$ by an étale neighborhood, we may assume that all of the irreducible components of the support of $\mathcal{Y}_s$ are smooth. Furthermore, take an open immersion $\mathcal{Y} \hookrightarrow \mathcal{Y}'$ in an integral projective scheme over K°. After replacing $\mathcal{Y}'$ by a blow-up, we may assume that $\mathcal{Y}'_s \backslash \mathcal{Y}_s$ is a union of irreducible components of $\mathcal{Y}'_s$. By Temkin's theorem [Tem08, 1.1], there exists a blow-up $\mathcal{Y}'' \to \mathcal{Y}'$ whose center is disjoint from $\mathcal{Y}$. The scheme $\mathcal{Y}''$ is proper and distinguished, the morphism $\mathcal{Y}'' \to \mathcal{Y}'$ is an isomorphism over $\mathcal{Y}$ and, in particular, there is an open immersion $\mathcal{Y} \hookrightarrow \mathcal{Y}''$, and the complement of $\mathcal{Y}_s$ in $\mathcal{Y}''_s$ is a union of irreducible components of $\mathcal{Y}''_s$. The claim follows.

Step 3. *The theorem is true for $\mathfrak{X}$.* Indeed, by Step 2, there exists an étale hypercovering $\mathfrak{Y}_\bullet \to \mathfrak{X}$ such that each $\mathfrak{Y}_n$, $n \geq 0$, is a finite disjoint union of formal schemes which satisfy the assumptions of Step 1. By Step 1, the required statement is true for all $\mathfrak{Y}_n$'s. Since the de Rham cohomology groups considered are expressed in terms of the schemes and their complex analytifications related to $\mathfrak{Y}_n$'s, the claim follows. □

Corollary 10.2.2. *In the situation of Theorem 10.2.1, the de Rham cohomology groups $H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r})$ and $H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}/K^\circ_r)$ have finite dimension over* $\mathbf{C}$.

Proof. By Corollaries 9.2.3 and 9.5.2, the cohomology sheaves of the complexes $\omega^\cdot_{\mathfrak{X}^h_{s_r}}$ and $\omega^\cdot_{\mathfrak{X}^h_{s_r}/K^\circ}$ coincide with the nearby and vanishing cycles sheaves of $\mathfrak{X}$, and they are constructible sheaves of $\mathbf{C}$-vector spaces on $\mathfrak{X}_{s_r}$, by Theorem 6.1.1(iii). This implies that the de Rham cohomology groups of the log analytic space $\mathfrak{X}^h_{s_r}$ have finite dimension over $\mathbf{C}$. The required fact therefore follows from the isomorphism of §9.5. $\square$

10.3. **de Rham cohomology as a projective limit.**

Theorem 10.3.1. *Let $\mathfrak{X}$ be a quasicompact distinguished special formal scheme of K°. Then there are canonical isomorphisms*

$$H^q_{\mathrm{dR}}(\mathfrak{X}) \widetilde{\to} \lim_{\overleftarrow{r}} H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}) \text{ and } H^q_{\mathrm{dR}}(\mathfrak{X}/K^\circ) \widetilde{\to} \lim_{\overleftarrow{r}} H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}/K^\circ_r) \ .$$

The following proposition and lemma are slight modifications of Theorem (4.5) and Lemma (4.6) from Hartshorne's paper [Har75]. All complexes $F^\cdot$ considered here are assumed to be such that $F^q = 0$ for $q < 0$.

Proposition 10.3.2. *Let $\{F^\cdot_r\}_{r\ge 1}$ is a projective system of complexes of abelian sheaves on a topological space X, and set $F^\cdot = \lim_{\overleftarrow{r}} F^\cdot_r$. Let also T be a functor from the category of abelian sheaves to that of abelian groups that commutes with direct products. Assume that there is a base $\mathcal{B}$ of the topology of X such that for each $U \in \mathcal{B}$*

1. *the homomorphisms $F^q_{r+1}(U) \to F^q_r(U)$ are surjective for all $q \ge 0$ and $r \ge 1$;*
2. *$H^p(U, F^q_r) = 0$ for all $p > 0$, $q \ge 0$ and $r \ge 1$.*

Then for each $p \in \mathbf{Z}$, there is an exact sequence

$$0 \to \lim_{\overleftarrow{r}}{}^{(1)} \ R^{p-1}T(F^\cdot_r) \to R^pT(F^\cdot) \stackrel{\alpha_p}{\to} \lim_{\overleftarrow{r}} R^pT(F^\cdot_r) \to 0 \ .$$

In particular, if for some p, the system $\{R^{p-1}T(F^\cdot_r)\}_{r\ge1}$ satisfies the Mittag-Leffler condition (ML)*, then α_p is an isomorphism.*

Lemma 10.3.3. *Given a morphism of complexes of abelian sheaves $\alpha^\cdot : G^\cdot \to F^\cdot$ and an injective resolution $\varphi^\cdot : F^\cdot \to I^\cdot$, there exists an injective resolution $\psi^\cdot : G^\cdot \to J^\cdot$ and a commutative diagram*

$$\begin{tikzcd} F^\cdot \arrow[r, "\varphi^\cdot"] & I^\cdot \\ G^\cdot \arrow[u, "\alpha^\cdot"] \arrow[r, "\psi^\cdot"] & J^\cdot \arrow[u, "\beta^\cdot"'] \end{tikzcd}$$

with the property that, for every p, there is an isomorphism $J^p \widetilde{\to} I^p \oplus K^p$ such that β^p is the projection onto the first summand.

Proof. For a complex of abelian sheaves $K^\cdot$ and a homomorphism $\gamma : K^0 \to L$, there is a complex $K^\cdot_\gamma$ with $K^0_\gamma = L$ and a quasi-isomorphism of complexes $\gamma^\cdot : K^\cdot \to K^\cdot_\gamma$ with $\gamma^0 = \gamma$ which possess the universal property that, for any pair consisting of a morphism of complexes $\delta^\cdot : K^\cdot \to P^\cdot$ and a homomorphism $L \to P^0$ whose composition with γ coincides with δ^0, $\delta^\cdot$ goes through a unique morphism of complexes $K^\cdot_\gamma \to P^\cdot$. (The complex $K^\cdot_\gamma$ is constructed as follows: $K^0_\gamma = L$ and, for $i \ge 1$, K^i_γ is the cokernel of the homomorphism $K^{i-1} \to K^i \oplus K^{i-1}_\gamma : (x \mapsto (d^{i-1}_K(x), -\gamma^{i-1}(x)).)$

Let $\chi : G^0 \to K^0$ be an embedding in an injective sheaf. Then the sheaf $J^0 = I^0 \oplus K^0$ is also injective, and denote by ψ^0 the homomorphism $G^0 \to J^0 : x \mapsto (\alpha^0(\varphi^0(x)), \psi(x))$. The canonical projection $\beta^0 : J^0 \to I^0$ gives rise to a morphism of complexes $G^{\cdot}_{\psi^0} \to F^{\cdot}_{\varphi^0}$. Application of the same procedure to the induced morphism of truncated complexes $\sigma_{\geq 1}(G^{\cdot}_{\psi^0}) \to \sigma_{\geq 1}(F^{\cdot}_{\varphi^0})$ and the injective resolution $\sigma_{\geq 1}(F^{\cdot}_{\varphi^0}) \to \sigma_{\geq 1}(I^{\cdot})$ gives an inductive procedure for constructing the required injective resolution of $G^{\cdot}$. □

Proof of Proposition 10.3.2. Step 1. By Lemma 10.3.3, applied inductively to morphisms of complexes $F^{\cdot}_{r+1} \to F^{\cdot}_r$ we can find a compatible system of injective resolutions $\beta^{\cdot}_r : F^{\cdot}_r \to I^{\cdot}_r$ such that $I^p_{r+1} \widetilde{\to} I^p_r \oplus K^p_r$ and $\beta^{\cdot}_r$ is the projection onto the first summand. Then all of the sheaves I^p from the projective limit of complexes $I^{\cdot} = \varprojlim_r I^{\cdot}_r$ are injective. We are going to show that the canonical morphism $F^{\cdot} \to I^{\cdot}$ is a quasi-isomorphism.

Step 2. *For every $U \in \mathcal{B}$ and every $r \geq 1$, the morphism $F^{\cdot}_r(U) \to I^{\cdot}_r(U)$ is a quasi-isomorphism.* Indeed, since $F^{\cdot}_r \to I^{\cdot}_r$ is an injective resolution, it induces an isomorphism of hypercohomology groups $\mathbf{H}^p(U, F^{\cdot}_r) \widetilde{\to} \mathbf{H}^p(U, I^{\cdot}_r)$. But the spectral sequence $E_1^{p,q} = H^q(U, F^p_r) \Longrightarrow \mathbf{H}^{p+q}(U, F^{\cdot}_r)$ and the condition (2) imply that $\mathbf{H}^p(U, F^{\cdot}_r) = F^p(U)$ for all $p \geq 0$. Since one also has $\mathbf{H}^p(U, I^{\cdot}_r) = I^p_r(U)$ for all $p \geq 0$, the claim follows.

Step 3. *For every $U \in \mathcal{B}$, the morphism $F^{\cdot}(U) \to I^{\cdot}(U)$ is a quasi-isomorphism.* Indeed, by the condition (1), all of the homomorphisms $F^p_{r+1}(u) \to F^p_r(U)$ are surjective and, by the construction of the sheaves I^q_r the same is true for them. We can therefore apply Proposition (4.4) from [Har75], and we get a homomorphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_r{}^{(1)}\ H^{p-1}(F^{\cdot}_r(U)) & \longrightarrow & H^p(F^{\cdot}(U)) & \longrightarrow & \varprojlim_r H^p(F^{\cdot}_r(U)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim_r{}^{(1)}\ H^{p-1}(I^{\cdot}_r(U)) & \longrightarrow & H^p(I^{\cdot}(U)) & \longrightarrow & \varprojlim_r H^p(I^{\cdot}_r(U)) & \longrightarrow & 0
\end{array}$$

By Step 1, the left and right vertical arrows are isomorphisms and, therefore, so is the middle one. This implies that $F^{\cdot} \to I^{\cdot}$ is an injective resolusion of $F^{\cdot}$.

Step 4. *The proposition is true.* Indeed, one has $R^pT(F^{\cdot}_r) = H^p(T(I^{\cdot}_r))$ and, by Step 3, one also has $R^pT(F^{\cdot}) = H^p(T(I^{\cdot}))$. Since the functor T commutes with direct products, one has $T(I^{\cdot}) = \varprojlim_r T(I^{\cdot}_r))$, and since I^p_r ia a direct summand of I^p_{r+1}, the homomorphisms $T(I^p_{r+1}) \to T(I^p_r)$ are surjective. The required fact now follows from the same Proposition (4.4) from [Har75]. □

Proof of Theorem 10.3.1. We apply Proposition 10.3.2 to formal scheme $\mathfrak{X}$ which coincides, as a topological space, with each $\mathfrak{X}_{s_r}$. The base $\mathcal{B}$ consists of open affine subschemes. The sheaves $\omega^q_{\mathfrak{X}_{s_r}}$ and $\omega^q_{\mathfrak{X}_{s_r}/K^\circ_r}$ are coherent on $\mathcal{X}_{s_r}$ and, therefore, the condition (2) is satisfied. That (1) holds follows the from the same coherence and the construction of those sheaves, which implies surjectivity of the canonical homomorphisms from $(r+1)$-th sheaf to r-th one. Furthermore, the functor T is the functor of global sections and, finally, the Mittag-Leffler condition is satisfied, by Corollary 10.2.2. This implies Theorem 10.3.1. □

10.4. **Proof of Theorem 10.1.1.** Step 1. By the definition of the functor $R\Theta$ and Corollary 9.2.2, there is a compatible system of canonical isomorphisms in the derived category

$$R\Theta(\mathbf{C}_{\mathfrak{X}_\eta})\widetilde{\rightarrow}\omega^{\cdot}_{\mathfrak{X}^h_{s_r}} \ ,$$

and it gives rise to a compatible system of isomorphisms of finitely dimensional $\mathbf{C}$-vector spaces $H^q(\mathfrak{X}_\eta, \mathbf{C})\widetilde{\rightarrow}H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r})$. By Theorem 10.2.1, the group on the right hand side of the latter isomorphism is canonically isomorphic to $H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r})$ and, therefore, the statement (i) follows from Theorem 10.3.1.

Step 2. Similarly, by the definition of the functor $R\Psi^h_\eta$ and Proposition 9.5.1, there is a compatible system of isomorphisms of $\underline{W}_{K^\circ_r}$-modules in the derived category

$$R\Psi^h_\eta(\mathbf{C}_{\mathfrak{X}_\eta})\otimes_{\mathbf{C}} K^\circ_r\widetilde{\rightarrow}\omega^{\cdot}_{\mathfrak{X}^h_{s_r}/K^\circ_r} \ ,$$

and it gives rise to a compatible system of isomorphisms of $\underline{W}_{K^\circ_r}$-modules

$$H^q(\mathfrak{X}_{\overline{\eta}}, \mathbf{C})\otimes_{\mathbf{C}} \underline{K}^\circ_r\widetilde{\rightarrow}H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r}/K^\circ_r) \ ,$$

which are free K°_r-modules of finite rank. Recall that the $\underline{W}_{K^\circ_r}$-module structure on $\omega^{\cdot}_{\mathfrak{X}^h_{s_r}/K^\circ_r}$ comes from the quasi-isomorphism with the complex of sheaves of $\underline{W}_{K^\circ}$-modules $L^{\cdot}_{\mathfrak{X}^h_{s_r}}$ (see Proposition 9.4.1). The latter is a direct product of complexes $L^{\cdot}_{\mathfrak{X}^h_{s_r},\lambda}$ taken over $\lambda \in \mathbf{Q}_+ \cap [0, r)$, and the restriction of the operator δ_ϖ to its ϖ-part is a sum of the operator of multiplication by λ and a locally nilpotent operator. Furthermore, by Proposition 9.4.3, one has $\sigma^{(\varpi)} = \exp(-2\pi i\delta_\varpi)$. All this implies that $H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r}/K^\circ_r)$ are distinguished $\underline{W}_{K^\circ_r}$-modules.

Step 3. The isomorphism $H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}/K^\circ_r)\widetilde{\rightarrow}H^q_{\mathrm{dR}}(\mathfrak{X}^h_{s_r}/K^\circ_r)$ of Theorem 10.2.1 provides the group on the left hand side with the structure of a $\underline{W}_{K^\circ_r}$-module. In this way we get an isomorphism of distinguished $\underline{W}_{K^\circ_r}$-modules

$$H^q(\mathfrak{X}_{\overline{\eta}}, \mathbf{C})\otimes_{\mathbf{C}} \underline{K}^\circ_r\widetilde{\rightarrow}H^q_{\mathrm{dR}}(\mathfrak{X}_{s_r}/K^\circ_r) \ ,$$

and the statement (ii) follows from Theorem 10.3.1. $\square$

10.5. **Proof of Theorem 10.1.4.** Step 1. Consider the commutative diagram, in which the horizontal arrows are isomorphisms, provided by Corollary 9.2.2, and the left vertical arrow is an isomorphism, by Theorem 8.1.6,

$$\begin{tikzcd}
H^q(\mathcal{X}^h(\mathcal{Y}^h)_\eta, \mathbf{C}) \arrow[r] \arrow[d] & H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h)) \arrow[d] \\
H^q((\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta, \mathbf{C}) \arrow[r] & H^q_{\mathrm{dR}}(\widehat{\mathcal{X}}_{/\mathcal{Y}})
\end{tikzcd}$$

It follows that the right vertical arrow is an isomorphism, and this gives the statement (i).

Step 2. Consider the similar commutative diagram, in which the horizontal arrows are isomorphisms, provided by Corollary 9.2.2 and Theorem 10.1.1,

$$\begin{tikzcd} H^q(\mathcal{X}^h(\mathcal{Y}^h)_{\overline{\eta}}, \mathbf{C}) \otimes_{\mathbf{C}} \mathcal{K}^\circ \arrow[r] \arrow[d] & H^q_{\mathrm{dR}}(\mathcal{X}^h(\mathcal{Y}^h)/\mathcal{K}^\circ) \arrow[d] \\ H^q((\widehat{\mathcal{X}}_{/\mathcal{Y}})_{\overline{\eta}}, \mathbf{C}) \otimes_{\mathbf{C}} \widehat{\mathcal{K}}^\circ \arrow[r] & H^q_{\mathrm{dR}}(\widehat{\mathcal{X}}_{/\mathcal{Y}}/\widehat{\mathcal{K}}^\circ) \end{tikzcd}$$

By Theorem 8.1.6, one has $H^q(\mathcal{X}^h(\mathcal{Y}^h)_{\overline{\eta}}, \mathbf{C}) \widetilde{\to} H^q((\widehat{\mathcal{X}}_{/\mathcal{Y}})_{\overline{\eta}}, \mathbf{C})$, and the statement (ii) follows.

Step 3. The upper and lower horizontal arrows in the above diagram are compatible homomorphisms of $\underline{W}_{\mathcal{K}^\circ}$ and $\underline{W}_{\widehat{\mathcal{K}}^\circ}$-modules, respectively, by the construction of §9.4. This implies the statement (iii). $\square$

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Index of Notations

Index of Terminology

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