ON THE COMPARISON THEOREM FOR ÉTALE COHOMOLOGY OF NON-ARCHIMEDEAN ANALYTIC SPACES

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ABSTRACT

Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of finite type between schemes of locally finite type over a non-Archimedean field k, and let $\mathcal{F}$ be an étale constructible sheaf on $\mathcal{Y}$. In [Ber2] we proved that if the torsion orders of $\mathcal{F}$ are prime to the characteristic of the residue field of k then the canonical homomorphisms $(R^q\varphi_*\mathcal{F})^{\rm an} \to R^q\varphi_*^{\rm an}\mathcal{F}^{\rm an}$ are isomorphisms. In this paper we extend the above result to the class of sheaves $\mathcal{F}$ with torsion orders prime to the characteristic of k.

Introduction

In [Ber2] (see also [Ber3]), an étale cohomology theory for non-Archimedean analytic spaces has been constructed. In particular, the following two comparison theorems have been proved. Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism between schemes of locally finite type over a non-Archimedean field k, and let $\mathcal{F}$ be an étale abelian torsion sheaf on $\mathcal{Y}$. The comparison theorem for cohomology with compact support ([Ber2], 7.1.4) states that if the morphism φ is compactifiable, then there are canonical isomorphisms

$$(!) \qquad (R^q\varphi_!\mathcal{F})^{\rm an} \stackrel{\sim}{\to} R^q\varphi_!^{\rm an}\mathcal{F}^{\rm an} \ .$$

The comparison theorem ([Ber2], 7.5.3) states that if φ is of finite type and $\mathcal{F}$ is constructible with torsion orders prime to char$(\widetilde{k})$, where $\widetilde{k}$ is the residue field of k, then there are canonical isomorphisms

$$(*) \qquad (R^q\varphi_*\mathcal{F})^{\rm an} \stackrel{\sim}{\to} R^q\varphi_*^{\rm an}\mathcal{F}^{\rm an} \ .$$

The latter comparison theorem does not say anything on p-torsion sheaves when char$(k) = 0$ and char$(\widetilde{k}) = p > 0$. But the evidence that the isomorphism $(*)$ should be true also in such a situation

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has been provided by the p-adic Riemann existence theorem, proved by W. Lütkebohmert in [Lu2]. It implies straightforwardly that $H^1(\mathcal{Y}, \mathbf{Z}/n\mathbf{Z}) \widetilde{\to} H^1(\mathcal{Y}^{\mathrm{an}}, \mathbf{Z}/n\mathbf{Z})$ for arbitrary n prime to char(k).

The main purpose of this paper is to prove that the isomorphism $(*)$ really takes place without any restriction on the torsion orders of $\mathcal{F}$ in the case when k is of characteristic zero. The proof is given in §3 and follows the proof of the comparison theorem of M. Artin and A. Grothendieck ([SGA4], Exp. XVI, 4.1). Using Hironaka's theorem on resolution of singularities, the weak base change theorem ([Ber2], 5.3.1) and the comparison theorem for cohomology with compact support, the situation is reduced to the case when $\mathcal{X}$ is smooth, φ is an open immersion, and $\mathcal{F} = \Lambda_{\mathcal{Y}}$, where $\Lambda = \mathbf{Z}/n\mathbf{Z}$. In this case, the isomorphism $(*)$ for $q = 0$ follows from the p-adic Riemann extension theorem, proved by W. Lütkebohmert in [Lu1], and the verification of $(*)$ for $q \geq 1$ is reduced to the case when $\mathcal{Z} := \mathcal{X}\backslash\mathcal{Y}$ is also smooth. If i denotes the closed immersion $\mathcal{Z} \to \mathcal{X}$, then $(*)$ is equivalent to the fact that the canonical homomorphism

$$(?) \qquad (R^q i^! \Lambda_{\mathcal{X}})^{\mathrm{an}} \to R^q i^{\mathrm{an}!} \Lambda_{\mathcal{X}^{\mathrm{an}}}$$

is an isomorphism. The latter is deduced from the cohomological purity theorem proved in §2. Using a result of W. Lütkebohmert from [Lu2], we prove that the affine space is universally acyclic, and deduce from this that if (Y, X) is a smooth S-pair of codimension c, then $R^q i^! \Lambda_X = 0$ for $q \neq 2c$ and $R^{2c} i^! \Lambda_X$ is locally isomorphic to Λ_Y. (In particular, the both sheaves in (?) are locally isomorphic.) Furthermore, we construct an isomorphism $R^{2c} i^! \Lambda_X(c) \widetilde{\to} \Lambda_Y$ and establish its properties which guarantee that (?) is an isomorphism. For this we use the Verdier duality theorem, proved in §1, and the trace mapping $R^{2d}\varphi_! \Lambda_Y(d) \to \Lambda_X$ constructed in [Ber2], §7.2, for any separated smooth morphism $\varphi : Y \to X$ of pure dimension d and any n prime to char(k). (In [Ber2], the trace mapping was used only for n prime to char($\widetilde{k}$).)

Throughout the paper we fix a non-Archimedean field k, a positive integer n, and we set $\Lambda = \mathbf{Z}/n\mathbf{Z}$. (As in [Ber1]-[Ber3], the valuation on k is not assumed to be nontrivial.)

§1. Verdier Duality

1.1. Theorem. *Let $\varphi : Y \to X$ be a Hausdorff morphism of finite dimension between k-analytic spaces. Then there is an exact functor*

$$R\varphi^! : D^+(X, \Lambda) \to D^+(Y, \Lambda)$$

and, for any $G^\cdot \in D^-(Y, \Lambda)$ and $F^\cdot \in D^+(X, \Lambda)$, a functorial isomorphism

$$R\varphi_*(\underline{\mathcal{H}om}(G^\cdot, R\varphi^! F^\cdot)) \widetilde{\to} \underline{\mathcal{H}om}(R\varphi_! G^\cdot, F^\cdot) \ .$$

It is clear that Theorem 1.1 will be proved if we construct the functor $R\varphi^!$ and prove the following

1.2. Corollary. *There is a functorial isomorphism*

$$\underline{\mathrm{Hom}}(G^{\cdot}, R\varphi^! F^{\cdot}) \tilde{\to} \underline{\mathrm{Hom}}(R\varphi_! G^{\cdot}, F^{\cdot}) \ . \qquad \blacksquare$$

Proof. Let $d = \dim(\varphi)$. We say that a sheaf $L \in \mathbf{S}(Y, \Lambda)$ is *strongly $\varphi_!$-acyclic* if, for any separated étale morphism $g : V \to Y$, the sheaf $L_{V/Y} = g_!(L\big|_V)$ is $\varphi_!$-acyclic.

1.3. Lemma. *If a sheaf $L \in \mathbf{S}(Y, \Lambda)$ is flat strongly $\varphi_!$-acyclic, then for any $G \in \mathbf{S}(Y, \Lambda)$ the sheaf $L \otimes G$ is $\varphi_!$-acyclic.*

Proof. Take a resolution of G

$$\ldots \to G_1 \to G_0 \to G \to 0$$

whose members are of the form $\oplus_i \Lambda_{V_i/Y}$, where $V_i \to Y$ are separated étale morphisms. Tensoring it with L, we get an exact sequence

$$\ldots \stackrel{d_2}{\to} L \otimes G_1 \stackrel{d_1}{\to} L \otimes G_0 \stackrel{d_0}{\to} L \otimes G \to 0$$

whose members are of the form $L \otimes (\oplus_i \Lambda_{V_i/Y}) = \oplus_i L_{V_i/Y}$. Since the functor $\varphi_!$ commutes with direct sums, all the sheaves $L \otimes G_m$ are $\varphi_!$-acyclic. It follows that for $q \geq 1$ one has

$$R^q \varphi_!(L \otimes G) \tilde{\to} R^{q+2d} \varphi_!(\mathrm{Ker}\ d_{2d-1}) = 0$$

because $R^q \varphi_! = 0$ for $q > 2d$, by [Ber2], 5.3.8. $\blacksquare$

For a flat strongly $\varphi_!$-acyclic sheaf $L \in \mathbf{S}(Y, \Lambda)$, we denote by $\varphi_!^L$ the following functor

$$\mathbf{S}(Y, \Lambda) \to \mathbf{S}(X, \Lambda) : G \mapsto \varphi_!(L \otimes G) \ .$$

1.4. Lemma. *The functor $\varphi_!^L$ is exact and has a right adjoint functor $\varphi_L^! : \mathbf{S}(X, \Lambda) \to \mathbf{S}(Y, \Lambda)$. The functor $\varphi_L^!$ takes injectives to injectives.*

Proof. Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of sheaves on Y. Since L is flat, the sequence $0 \to L \otimes G' \to L \otimes G \to L \otimes G'' \to 0$ is also exact. By Lemma 1.3, $R^1 \varphi_!(L \otimes G') = 0$, and therefore the sequence $0 \to \varphi_!^L(G') \to \varphi_!^L(G) \to \varphi_!^L(G'') \to 0$ is exact. Furthermore, we claim that for any $F \in \mathbf{S}(X, \Lambda)$ the contravariant functor

$$\mathbf{S}(Y, \Lambda) \to \mathcal{A}b : G \mapsto \mathrm{Hom}(\varphi_!^L(G), F)$$

is representable. Indeed, for this it suffices to verify that this functor takes inductive limits to projective limits (see [SGA4], Exp. XVIII, 3.1.3). But this follows from the facts that the functor $\varphi_!^L$ is exact and that the tensor product functor and the functor $\varphi_!$ take direct sums to direct sums. If $\varphi^!_L(F)$ denotes a sheaf which represents the functor considered, then the correspondence $F \mapsto \varphi^!_L(F)$ is a functor right adjoint to $\varphi_!^L$. The last statement of the lemma follows from the fact that the functor $\varphi_!^L$ is exact. ∎

1.5. Proposition. *Any flat sheaf $G \in \mathbf{S}(Y,\Lambda)$ has a resolution*

$$0 \to G \to L^0 \to L^1 \to \ldots \to L^{2d} \to 0$$

in which all L^i are flat strongly $\varphi_!$-acyclic sheaves.

Proof. 1. Recall the construction of the Godement resolution from [SGA4], Exp. XVII, §4.2, adopted to our situation. Suppose we are given a set I, a surjective map $\sigma : I \to Y$ and, for each $i \in I$, an algebraically closed non-Archimedean field K_i over $\mathcal{H}(\sigma(i))$. These data define a morphism of analytic spaces over k, $\nu : \mathcal{Y} \to Y$, where $\mathcal{Y}$ is the disjoint union of $\mathcal{M}(K_i)$ over all $i \in I$. For a sheaf $G \in \mathbf{S}(Y,\Lambda)$, let $\mathcal{C}^{\cdot}(G)$ denote the right resolution of G constructed as follows:

(a) $\mathcal{C}^0(G) = \nu_*\nu^*(G)$, and $\varepsilon = d^{-1} : G \to \mathcal{C}^0(G)$ is the adjunction morphism;

(b) if $m \geq 0$, then $\mathcal{C}^{m+1}(G) = \mathcal{C}^0(\text{Coker } d^{m-1})$, and d^m is the composition $d^m : \mathcal{C}^m(G) \to \text{Coker } d^{m-1} \to \mathcal{C}^0(\text{Coker } d^{m-1})$.

By *loc. cit.*, 4.2.3, one has:

(i) $\mathcal{C}^m(G)$ is a flabby sheaf;

(ii) the functor $G \mapsto \mathcal{C}^m(G)$ is exact;

(iii) the fibre of the complex $\mathcal{C}^{\cdot}(G)$ at a point $y \in Y$ is a canonically split resolution of G_y.

1.6. Lemma. *The sheaves $\mathcal{C}^m(G)$ are strongly $\varphi_!$-acyclic.*

Proof. It suffices to verify the statement for $m = 0$. We have to show that $R^q(\varphi g)_!(\mathcal{C}^0(G)\big|_V) = 0$, $q \geq 1$, for any separated étale morphism $g : V \to Y$. Replacing the set I by another one, we may replace Y by V, and so we have to show that $R^q\varphi_!(\mathcal{C}^0(G)) = 0$, $q \geq 1$. Since the statement is local in the étale topology of X and the sheaf $R^q\varphi_!(\mathcal{C}^0(G))$ is associated with the presheaf $(U \stackrel{f}{\to} X) \mapsto H^q_{\mathcal{C}_\varphi(f)}(Y \times_X U, \mathcal{C}^0(G))$, where $\mathcal{C}_\varphi$ is the φ-family of supports defined in [Ber2], 5.1.3, it suffices to show that in the case of paracompact X one has $H^q_\Phi(Y, \mathcal{C}^0(G)) = 0$ for all $q \geq 1$, where $\Phi = \mathcal{C}_\varphi(\text{Id})$. For this we use the spectral sequence $E_2^{p,q} = H^p_\Phi(|Y|, R^q\pi_*(\mathcal{C}^0(G))) \Longrightarrow H^{p+q}_\Phi(Y, \mathcal{C}^0(G))$, where π is the morphism of sites $Y_{\text{ét}} \to |Y|$. The sheaf $\mathcal{C}^0(G)$ is flabby, and therefore $R^q\pi_*(\mathcal{C}^0(G)) = 0$ for $q \geq 1$, by [Ber2], 4.2.5. Furthermore, from the construction of $\mathcal{C}^0(G)$ it follows that the sheaf

$\pi_*(\mathcal{C}^0(G))$ is flasque in the sense of [God]. Since the family of supports Φ is paracompactifying, it follows that the latter sheaf is Φ-soft, and therefore $H^p_\Phi(|Y|, \pi_*(\mathcal{C}^0(G))) = 0$ for all $p \geq 1$. ∎

2. Suppose now that G is flat. We set $L^m = \mathcal{C}^m(G)$ for $0 \leq m \leq 2d-1$, and $L^{2d} = \mathrm{Ker}(d^{2d})$. From 1(iii) it follows that all the sheaves $L^0, \dots, L^{2d}$ are flat. Let $V \to Y$ be a separated étale morphism. By Lemma 1.6, the sheaves $L^0, \dots, L^{2d-1}$ are strongly $\varphi_!$-acyclic, and therefore $R^q\varphi_!(L^{2d}_{V/Y}) \tilde{\to} R^{q+2d}\varphi_!(G_{V/Y}) = 0$ for all $q \geq 1$, i.e., L^{2d} is a strongly $\varphi_!$-acyclic sheaf. ∎

We fix a flat strongly $\varphi_!$-acyclic resolution of the constant sheaf Λ_Y

$$0 \to \Lambda_Y \to L^0 \to L^1 \to \dots \to L^{2d} \to 0 \ .$$

For a complex $G^\cdot \in C^-(Y, \Lambda)$, let $\varphi_!^{L^\cdot}(G^\cdot)$ denote the complex $\varphi_!(L^\cdot \otimes G^\cdot)$. Furthermore, for a complex $F^\cdot \in C^+(X, \Lambda)$, let $\varphi^!_{L^\cdot}(F^\cdot)$ denote the simple complex associated with the double complex $K^{p,q} = \varphi^!_{L^{-p}}(F^q)$. It follows that there is a functorial isomorphism

$$\mathrm{Hom}^\cdot(G^\cdot, \varphi^!_{L^\cdot}(F^\cdot)) \tilde{\to} \mathrm{Hom}^\cdot(\varphi_!^{L^\cdot}(G^\cdot), F^\cdot) \ .$$

We now define the functor $R\varphi^! : D^+(X, \Lambda) \to D^+(Y, \Lambda)$ as follows. Let $F^\cdot \to I^\cdot$ be an injective resolution of a complex $F^\cdot \in C^+(X, \Lambda)$. We set

$$R\varphi^! F^\cdot = \varphi^!_{L^\cdot}(I^\cdot) \ .$$

It is easy to see that $R\varphi^! F^\cdot$ does not depend (up to a canonical isomorphism) on the choice of the resolution $I^\cdot$ and that, for $G^\cdot \in D^-(Y, \Lambda)$ and $F^\cdot \in D^+(X, \Lambda)$, there is a functorial isomorphism $\underline{\mathrm{Hom}}(G^\cdot, R\varphi^! F^\cdot) \tilde{\to} \underline{\mathrm{Hom}}(R\varphi_! G^\cdot, F^\cdot)$. Theorem 1.1 is proved. ∎

1.7. Remarks. (i) From the construction of $R\varphi^!$ it follows that if the cohomology sheaves of a complex $F^\cdot \in D^+(X, \Lambda)$ are trivial at dimensions $< q$, then the cohomology sheaves of the complex $R\varphi^! F^\cdot \in D^+(Y, \Lambda)$ are trivial at dimensions $< q - 2d$.

(ii) If $\psi : Z \to Y$ is a similar morphism, then the canonical isomorphism of functors $R(\varphi\psi)_! \tilde{\to} R\varphi_! \circ R\psi_!$ induces an isomorphism of functors $R\psi^! \circ R\varphi^! \tilde{\to} R(\varphi\psi)^!$.

(iii) Suppose that $d = 0$. Then $R\varphi^!$ is actually the right derived functor of a left exact functor $\varphi^! : \mathbf{S}(X, \Lambda) \to \mathbf{S}(Y, \Lambda)$ defined as follows

$$\Gamma(V, \varphi^!(F)) = \mathrm{Hom}(\varphi_!(\Lambda_{V/Y}), F) \ .$$

Moreover, $\varphi^!$ is right adjoint to $\varphi_!$. If φ is étale, then $\varphi^! = \varphi^*$. If φ is a quasi-immersion ([Ber2], §4.3) such that $\varphi(Y)$ is closed in X, then $\varphi^!$ is the functor of sections with supports in $\varphi(Y)$ (defined in [Ber2], §5.1.1), and the sheaves $R^q\varphi^!(F)$ were denoted in [Ber2] by $\mathcal{H}^q_Y(X, F)$.

The complex $R\varphi^!\Lambda_X$ is said to be the *dualizing complex* of the morphism φ and is denoted by $T^{\cdot}_{Y/X}$ (if $X = \mathcal{M}(k)$, it is denoted by $T^{\cdot}_Y$). By Remark 1.7(i), $H^q(T^{\cdot}_{Y/X}) = 0$ for $q < -2d$.

Let $\varphi : Y \to X$ be a separated smooth morphism of pure dimension d, and *assume that* n *is prime to* char(k). In [Ber2], §7.2, we constructed a canonical homomorphism of sheaves (the trace mapping)

$$\mathrm{Tr}_\varphi : R^{2d}\varphi_!\Lambda_Y(d) \to \Lambda_X \ .$$

Recall also that if the fibres of φ are non-empty, then Tr_φ is an epimorphism and if, in addition, the geometric fibres of φ are connected and n is prime to char($\widetilde{k}$), then Tr_φ is an isomorphism. By Theorem 1.1, the trace mapping induces a morphism of complexes $t_\varphi : \Lambda_Y \to T^{\cdot}_{Y/X}(-d)[-2d]$ or, equivalently, a homomorphism of sheaves $c_\varphi = H^0(t_\varphi) : \Lambda_Y \to H^{-2d}(T^{\cdot}_{Y/X}(-d))$. The image of 1 under c_φ is called the fundamental class of φ, and so t_φ and c_φ will be called the *fundamental class mappings*. By Poincaré Duality Theorem ([Ber2], 7.3.1), if n is prime to char($\widetilde{k}$), then t_φ (and therefore c_φ) is an isomorphism. We claim that in the general case (when n is prime only to char(k)) the homomorphism c_φ is injective. Indeed, to verify this, it suffices to assume that n is a prime integer. The set of points over which the homomorphism c_φ is not injective is open, and so decreasing Y we may assume that the morphism t_φ is zero. Furthermore, since a smooth morphism is an open map ([Ber2], 3.7.4), we can decrease X and assume that φ is surjective. In this case the vanishing of t_φ contradicts to the surjectivity of the trace mapping Tr_φ. The following proposition lists properties of the fundamental class mappings which follow straightforwardly from the properties of the trace mappings established in [Ber2], §7.2.

1.8. Proposition. *The fundamental class mappings t_φ have the following properties and are uniquely determined by them:*

(a) t_φ are compatible with base change, i.e., given a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow f' & & \uparrow f \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

the following diagram is commutative

$$\begin{array}{ccccc} f'^*(T^{\cdot}_{Y/X})(-d)[-2d] & & \longrightarrow & & T^{\cdot}_{Y'/X'}(-d)[-2d] \\ & f'^*(t_\varphi) \nwarrow & & \nearrow t_{\varphi'} & \\ & & \Lambda_{Y'} & & \end{array}$$

(b) t_φ are compatible with composition, i.e., given a separated smooth morphism $\psi : Z \to Y$ of pure dimension e, the following diagram is commutative

$$\begin{array}{ccc} T^{\cdot}_{Z/Y}(-e)[-2e] & \xrightarrow{R\psi^!(t_\varphi)(-e)[-2e]} & T^{\cdot}_{Z/X}(-d-e)[-2d-2e] \\ & t_\psi \nwarrow \quad \nearrow t_{\varphi\psi} & \\ & \Lambda_Z & \end{array}$$

(c) if φ is étale, then t_φ is the identity map $\Lambda_Y \widetilde{\to} T^{\cdot}_{Y/X} = \Lambda_Y$;

(d) if $\varphi : \mathcal{Y} \to \mathcal{X}$ is a separated smooth morphism of pure dimension d between schemes of locally finite type over $\mathrm{Spec}(\mathcal{A})$, where $\mathcal{A}$ is a k-affinoid algebra, then the following diagram is commutative

$$\begin{array}{ccc} (T^{\cdot}_{\mathcal{Y}/\mathcal{X}}(-d)[-2d])^{\mathrm{an}} & \longrightarrow & T^{\cdot}_{\mathcal{Y}^{\mathrm{an}}/\mathcal{X}^{\mathrm{an}}}(-d)[-2d] \\ & (t_\varphi)^{\mathrm{an}} \nwarrow \quad \nearrow t_{\varphi^{\mathrm{an}}} & \\ & \Lambda_{\mathcal{Y}^{\mathrm{an}}} & \end{array}$$

(Recall that, by Poincaré Duality for schemes, t_φ is an isomorphism.) ∎

§2. Cohomological Purity Theorem

In this section the integer n is assumed to be prime to $\mathrm{char}(k)$.

Let S be a k-analytic space. Recall ([Ber2], §7.4) that a smooth S-pair (Y, X) is a commutative diagram of morphisms of k-analytic spaces

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & g \searrow \quad \swarrow f & \\ & S & \end{array}$$

where f and g are smooth, and i is a closed immersion. The codimension of (Y, X) at a point $y \in Y$ is the codimension at y of the fibre Y_s in X_s, where $s = g(y)$. Given a smooth S-pair (Y, X), we denote by j the open immersion $U := X \backslash Y \hookrightarrow S$ and by h the induced morphism $U \to S$. Recall also ([Ber2], §1.5) that a k-analytic space is said to be good if each point of it has an affinoid neighborhood.

2.1. Theorem. *Let (Y, X) be a smooth S-pair of codimension c, and assume that S is good. Then*

(i) for any abelian sheaf F on X which is locally isomorphic (in the étale topology) to a sheaf of the form f^*G, where G is an étale Λ_S-module, one has $R^q i^! F = 0$ for $q \neq 2c$ and $i^*F \otimes R^{2c} i^! \Lambda_X \widetilde{\to} R^{2c} i^! F$.

(ii) there is a canonical isomorphism $R^{2c}i^!\Lambda_X(c)\widetilde{\to}\Lambda_Y$ such that if g is of pure dimension e, then the following diagram is commutative

$$\begin{array}{ccc} R^{2c}i^!\Lambda_X(c) & \stackrel{H^{2c}(Ri^!\circ t_\varphi)(c)}{\longrightarrow} & H^{-2e}(T^{\cdot}_{Y/S}(-e)) \\ & \searrow \quad \nearrow c_g & \\ & \Lambda_Y & \end{array}$$

2.2. Lemma *(Universal acyclicity of the affine space). Let X be a k-analytic space, and let φ be the canonical projection $\varphi : X \times \mathbf{A}^d \to X$. Then for any étale Λ_X-module F one has $F\widetilde{\to}\varphi_*\varphi^*F$ and $R^q\varphi_*(\varphi^*F) = 0$ for all $q \geq 1$.*

Proof. We may assume that $d = 1$. The isomorphism $F\widetilde{\to}\varphi_*\varphi^*F$ follows from [Ber2], 7.3.2. Since the sheaf $R^q\varphi_*(\varphi^*F)$ is associated with the presheaf $(U \to X) \mapsto H^q(\mathbf{A}^1_U, \varphi^*F)$, where $\mathbf{A}^1_U = U \times \mathbf{A}^1$, it suffices to show that $H^q(X, F)\widetilde{\to}H^q(\mathbf{A}^1_X, \varphi^*F)$ under the assumption that X is paracompact.

Take a number $r > 1$ and denote by φ_m the canonical projection $Y_m := X \times E(0, r^m) \to X$, where $E(0, r^m)$ is the closed disc in $\mathbf{A}^1$ of radius r^m with center at zero. The paracompact k-analytic space $\mathbf{A}^1$ is a union of the increasing sequence of the closed k-analytic domains Y_m. From [Ber2], 5.3.8 and 6.1.3, it follows that $R^q\varphi_{m*}(\varphi_m^*F) = 0$ for $q \geq 2$. If n is prime to $\mathrm{char}(\widetilde{k})$, then $R^1\varphi_{m*}(\varphi_m^*F) = 0$, and therefore $H^q(X, F)\widetilde{\to}H^q(Y_m, \varphi_m^*F)$ and $H^q(X, F)\widetilde{\to}H^q(\mathbf{A}^1_X, \varphi^*F)$ for all $q \geq 1$, by [Ber2], 6.3.12.

Assume now that $\mathrm{char}(k) = 0$, $p := \mathrm{char}(\widetilde{k}) > 0$ and $n = p^d$ for some $d \geq 1$. By Lütkebohmert's Theorem ([Lu2], 2.1), there exists a constant $0 < \varepsilon < 1$ depending only on p and d such that for any algebraically closed non-Archimedean field K with $\mathrm{char}(K) = 0$ and $\mathrm{char}(\widetilde{K}) = p$ and for any $R > 0$ the following holds. Any finite étale covering of the closed disc $E(0, R)$ over K of degree at most p^d splits over $E(0, \varepsilon R)$. The latter implies that for any Λ-module M the restriction homomorphism $H^1(E(0, R), M) \to H^1(E(0, \varepsilon R), M)$ is zero. If we now choose the number r so that $\varepsilon r > 1$, then [Ber2], 5.3.1, implies that the canonical homomorphism $R^1\varphi_{m+1*}(\varphi_{m+1}^*F) \to R^1\varphi_{m*}(\varphi_m^*F)$ is zero. Using the spectral sequence $E_2^{p,q} = H^p(X, R^q\varphi_{m*}(\varphi_m^*F)) \Longrightarrow H^{p+q}(Y_m, \varphi_m^*F)$ and the fact that $R^q\varphi_{m*}(\varphi_m^*F) = 0$ for $q \geq 2$, we get that the image of $H^q(Y_{m+1}, \varphi_{m+1}^*F)$ in $H^q(Y_m, \varphi_m^*F)$ coincides with the image of $H^q(X, F)$. By [Ber2], 6.3.12, one has $H^q(X, F)\widetilde{\to}H^q(\mathbf{A}^1_X, \varphi^*F)$ for all $q \geq 1$. The lemma is proved. ∎

Proof of Theorem 2.1. To construct the isomorphism (ii), it suffices to show that the canonical homomorphism $R^{2c}i^!\Lambda_X(c) \to H^{-2e}(T^{\cdot}_{Y/S}(-e))$ identifies the first sheaf with the image of Λ_Y under the injective homomorphism c_g. Furthermore, since the formation of $Ri^!$ commutes with any étale base change, we can apply Proposition 3.5.9 from [Ber2] (where the assumption that S is good is used) and assume that (Y, X) is the pair $(\mathbf{A}^{d-c}_S, \mathbf{A}^d_S)$ and F is of the form f^*G.

Step 1. *(i) is true and the sheaf $R^{2c}i^!\Lambda_X(c)$ is isomorphic to Λ_Y (here S is not necessarily good).*

Consider first the case $c = 1$. Using Proposition 1.8(b), we may replace S by $\mathbf{A}^{d-1}_S$ and assume that Y is the zero section in the affine line $X = \mathbf{A}^1_S$. After that we may assume that $X = \mathbf{P}^1_S$ and Y is the section at infinity. Consider the spectral sequence

$$E_2^{p,q} = R^p f_*(R^q j_*(h^*G)) \Longrightarrow R^{p+q}h_*(h^*G) \ .$$

First of all, we claim that $f^*G \widetilde{\to} j_*(h^*G)$. Indeed, let F' be the sheaf defined by the exact sequence

$$0 \to f^*G \to j_*(h^*G) \to F' \to 0 \ .$$

By [Ber2], 5.3.1, $R^1 f_*(f^*G) = 0$ and, by [Ber2], 7.3.2, $G \widetilde{\to} f_*(f^*G) \widetilde{\to} h_*(h^*G)$. It follows that $f_*F' = 0$, and therefore $F' = 0$. Thus, $E_2^{p,0} = R^p f_*(f^*G) = 0$ for $p \neq 0, 2$, and, by [Ber2], 5.3.9, $G(-1) \widetilde{\to} G \otimes R^2 f_*\Lambda_X \widetilde{\to} E_2^{2,0} = R^2 f_*(f^*G)$.

Furthermore, since the supports of the sheaves $R^q j_*(h^*G)$ for $q \geq 1$ are contained in Y and g is an isomorphism, then $E_2^{p,q} = 0$ for $p \geq 1$ and $q \geq 1$ and $E_2^{0,q} = g_*i^*(R^q j_*(h^*G))$ for $q \geq 1$. By Lemma 2.2, $R^q h_*(h^*G) = 0$ for $q \geq 1$, and therefore the spectral sequence implies that $E_2^{0,q} = 0$ for $q \geq 2$ and $E_2^{0,1} \widetilde{\to} E_2^{2,0}$. It follows that $R^q j_*(h^*G) = 0$ for $q \geq 2$ and $R^1 j_*(h^*G) \widetilde{\to} i_*(g^*G)(-1)$. Step 1 for $c = 1$ now follows from [Ber2], 5.2.7.

The case $c > 1$ is verified by induction. Let $c = a + b$, where $a, b > 0$. We set $Z = \mathbf{A}^{d-b}_S$ and denote by μ (resp. ν) the closed immersion $Y \to Z$ (resp. $Z \to X$). Consider the spectral sequence

$$E_2^{p,q} = R^p\mu^!(R^q\nu^! f^*G) \Longrightarrow R^{p+q}i^!(f^*G) \ .$$

By induction, $R^q\nu^! f^*G = 0$ for $q \neq 2b$ and $R^{2b}\nu^! f^*G \widetilde{\to} \nu^* f^*G(-b)$. Similarly, $E_2^{p,2b} = 0$ for $p \neq 2b$ and

$$g^*G(-c) = g^*G(-b) \otimes \Lambda_Y(-a) \widetilde{\to} R^{2a}\mu^!(R^{2b}\nu^! f^*G) = E_2^{2a,2b} \ .$$

Step 1 follows.

Step 2. *(ii) is true.*

Since S is good, we can shrink it and assume that $S = \mathcal{M}(\mathcal{A})$ is k-affinoid. Then our pair (Y, X) is the analytification of the smooth $\mathcal{S}$-pair $(\mathcal{Y}, \mathcal{X}) = (A^{d-c}_{\mathcal{S}}, A^d_{\mathcal{S}})$, where $\mathcal{S} = \mathrm{Spec}(\mathcal{A})$. By

Poincaré Duality for schemes, the fundamental class mapping $\Lambda_{\mathcal{Y}} \to T^{\cdot}_{\mathcal{Y}/\mathcal{S}}(-e)[-2e]$, $e = d - c$, is an isomorphism. Using Proposition 1.8(d), we get that the image of $R^{2c}i^!\Lambda_X(c)$ in $H^{-2e}(T^{\cdot}_{Y/X}(-e))$ contains the image of Λ_Y under the injective homomorphism c_g. Since, by Step 1, $R^{2c}i^!\Lambda_X(c)$ is isomorphic to Λ_Y, the required statement follows. ∎

In the situation of Theorem 2.1, it implies the same corollaries as [Ber2], 7.4.6-7.4.8. In §3, the following corollary will be used.

2.3. Corollary. *Suppose that $\mathcal{S}$ is a scheme of locally finite type over* Spec$(\mathcal{A})$*, where $\mathcal{A}$ is a k-affinoid algebra, $(\mathcal{Y}, \mathcal{X})$ is a smooth $\mathcal{S}$-pair, j is the open immersion $\mathcal{U} = \mathcal{X} \backslash \mathcal{Y} \hookrightarrow \mathcal{X}$, and $\mathcal{F}$ is an abelian sheaf on $\mathcal{X}$ which is locally isomorphic to a sheaf of the form $f^*\mathcal{G}$, where $\mathcal{G}$ is an étale $\Lambda_{\mathcal{S}}$-module. Then for any $q \geq 0$ there is a canonical isomorphism*

$$(R^q j_*(\mathcal{F}|_{\mathcal{U}}))^{\mathrm{an}} \tilde{\to} R^q j^{\mathrm{an}}_*(\mathcal{F}^{\mathrm{an}}|_{\mathcal{U}^{\mathrm{an}}}) \ .$$

Proof. Using Corollary 5.2.7 from [Ber2] and its analog for schemes, it suffices to verify that $(R^q i^! \mathcal{F})^{\mathrm{an}} \tilde{\to} R^q i^{\mathrm{an}!}(\mathcal{F}^{\mathrm{an}})$. But the latter follows from Theorem 2.1, its analog for schemes and Proposition 1.8(d). ∎

§3. The Comparison Theorem

3.1. Theorem. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of finite type between schemes of locally finite type over k, and let $\mathcal{F}$ be a constructible abelian sheaf on $\mathcal{Y}$ with torsion orders prime to* char(k)*. Then for any $q \geq 0$ there is a canonical isomorphism*

$$(R^q \varphi_* \mathcal{F})^{\mathrm{an}} \tilde{\to} R^q \varphi^{\mathrm{an}}_* \mathcal{F}^{\mathrm{an}} \ .$$

Proof. If the torsion orders of $\mathcal{F}$ are prime to char$(\widetilde{k})$, the theorem is proved in [Ber2], 7.5.3. We assume therefore that char$(k) = 0$. We may assume that $\mathcal{X}$ and $\mathcal{Y}$ are of finite type, reduced and separated and that $\mathcal{F}$ is an étale Λ_Y-module for some $n \geq 1$, where $\Lambda = \mathbf{Z}/n\mathbf{Z}$. The theorem is proved by induction on dim$(\mathcal{Y})$. It is evidently true when dim$(\mathcal{Y}) = 0$. Assume that it is true when dim$(\mathcal{Y}) \leq d - 1$, where $d \geq 1$, and prove it when dim$(\mathcal{Y}) = d$.

Step 1. *The theorem is true if $\mathcal{X}$ is smooth, φ is an open immersion, and $\mathcal{F}$ is constant.*

We may assume that $\mathcal{Y}$ is everywhere dense in $\mathcal{X}$ and $\mathcal{F} = \Lambda_{\mathcal{Y}}$. From GAGA ([Ber1], 3.4.4) it follows that $\mathcal{Y}^{\mathrm{an}}$ is everywhere dense in $\mathcal{X}^{\mathrm{an}}$.

Case $q = 0$. By [SGA4], Exp. XVI, 3.2, one has $\Lambda_{\mathcal{X}} \tilde{\to} \varphi_* \Lambda_{\mathcal{Y}}$. The isomorphism $\Lambda_{\mathcal{X}^{\mathrm{an}}} \tilde{\to} \varphi_*^{\mathrm{an}} \Lambda_{\mathcal{Y}^{\mathrm{an}}}$ follows from the fact that the complement of a closed k-analytic subspace in a connected normal k-analytic space is connected. This fact follows from the Riemann Extension Theorem proved by Lütkebohmert ([Lu1], see also [Ber1], 3.3.15).

Case $q \geq 1$. We define an integer $l(\mathcal{Y}, \mathcal{X})$ as the length m of the sequence of open subschemes $\mathcal{Y}_0 = \mathcal{Y} \subset \mathcal{Y}_1 \subset \ldots \subset \mathcal{Y}_m = \mathcal{X}$ such that $\mathcal{Y}_{i+1} \backslash \mathcal{Y}_i$ is the set of regular points of the reduced closed subscheme $\mathcal{X} \backslash \mathcal{Y}_i$. By Corollary 2.3, Step 1 is true if $l(\mathcal{Y}, \mathcal{X}) \leq 1$. Assume that it is true when $l(\mathcal{Y}, \mathcal{X}) \leq m - 1$, where $m \geq 2$, and prove it when $l(\mathcal{Y}, \mathcal{X}) = m$. We set $\mathcal{Z} = \mathcal{Y}_1$ (in the above sequence) and denote by μ (resp. ν) the open immersion $\mathcal{Y} \hookrightarrow \mathcal{Z}$ (resp. $\mathcal{Z} \hookrightarrow \mathcal{X}$). Consider the morphism of spectral sequences

$$\begin{array}{ccccc} {}'E_2^{p,q} & = & (R^p \nu_* (R^q \mu_* \Lambda_{\mathcal{Y}}))^{\mathrm{an}} & \Longrightarrow & (R^{p+q} \varphi_* \Lambda_{\mathcal{Y}})^{\mathrm{an}} \\ & & \downarrow & & \downarrow \\ {}''E_2^{p,q} & = & R^p \nu_*^{\mathrm{an}} (R^q \mu_*^{\mathrm{an}} \Lambda_{\mathcal{Y}^{\mathrm{an}}}) & \Longrightarrow & R^{p+q} \varphi_*^{\mathrm{an}} \Lambda_{\mathcal{Y}^{\mathrm{an}}} \end{array}$$

One has $\Lambda_{\mathcal{Z}} \tilde{\to} \mu_* \Lambda_{\mathcal{Y}}$ and $\Lambda_{\mathcal{Z}^{\mathrm{an}}} \tilde{\to} \mu_*^{\mathrm{an}} \Lambda_{\mathcal{Y}^{\mathrm{an}}}$. Since $l(\mathcal{Z}, \mathcal{X}) = m - 1$ then, by induction, ${}'E_2^{p,0} \tilde{\to} {}''E_2^{p,0}$ for all $p \geq 0$. Furthermore, $\mathcal{Z}' := \mathcal{Z} \backslash \mathcal{Y}$ is open everywhere dense in the reduced closed subscheme $\mathcal{X}' := \mathcal{X} \backslash \mathcal{Y}$. The sheaves $R^q \mu_* \Lambda_{\mathcal{Y}}$ (resp. $R^q \mu_*^{\mathrm{an}} \Lambda_{\mathcal{Y}^{\mathrm{an}}}$) for $q \geq 1$ are concentrated on $\mathcal{Z}'$ (resp. $\mathcal{Z}'^{\mathrm{an}}$). Since $\dim(\mathcal{Z}') < d$ then, by induction, ${}'E_2^{p,q} \tilde{\to} {}''E_2^{p,q}$ for all $p \geq 0$ and $q \geq 1$. Step 1 follows.

Step 2. *The theorem is true if φ is an open immersion and $\mathcal{F}$ is constant.*

Let $f : \mathcal{X}' \to \mathcal{X}$ be a resolution of singularities of $\mathcal{X}$, i.e., a proper surjective birational morphism with smooth $\mathcal{X}'$. Then there is a commutative diagram with cartesian squares

$$\begin{array}{ccccc} \mathcal{U} & \overset{j}{\hookrightarrow} & \mathcal{Y} & \overset{\varphi}{\hookrightarrow} & \mathcal{X} \\ \uparrow h & & \uparrow f' & & \uparrow f \\ \mathcal{U}' & \overset{j'}{\hookrightarrow} & \mathcal{Y}' & \overset{\varphi'}{\hookrightarrow} & \mathcal{X}' \end{array}$$

where $\mathcal{U}$ is an open everywhere dense subset of $\mathcal{Y}$ and h is an isomorphism. By Step 1, the theorem is true for the pair $(\varphi', \Lambda_{\mathcal{Y}'})$. From the Comparison Theorem for cohomology with compact support ([Ber2], 7.1.4) it follows that the theorem is true for $(f\varphi', \Lambda_{\mathcal{Y}'})$. Let i (resp. i') denote the closed immersion $\mathcal{Z} := \mathcal{Y} \backslash \mathcal{U} \to \mathcal{Y}$ (resp. $\mathcal{Z}' := \mathcal{Y}' \backslash \mathcal{U}' \to \mathcal{Y}'$). Since $\dim(\mathcal{Z}') < d$ then, by induction, the theorem is true for $(f\varphi', i'^* \Lambda_{\mathcal{Z}'})$. From the exact sequence $0 \to j'_! \Lambda_{\mathcal{U}'} \to \Lambda_{\mathcal{Y}'} \to i'^* \Lambda_{\mathcal{Z}'} \to 0$ it follows that the theorem is true for $(\varphi f', j'_! \Lambda_{\mathcal{U}'})$. Furthermore, by the Proper Base Change Theorems for schemes and analytic spaces ([Ber2], 5.3.1), one has $R^q f'_*(j'_! \Lambda_{\mathcal{U}'}) = 0$ and $R^q f'^{\mathrm{an}}_*(j'^{\mathrm{an}}_! \Lambda_{\mathcal{U}'^{\mathrm{an}}}) = 0$ for all $q \geq 1$. Since $f'_*(j'_! \Lambda_{\mathcal{U}'}) = j_! \Lambda_{\mathcal{U}}$ and $f'^{\mathrm{an}}_*(j'^{\mathrm{an}}_! \Lambda_{\mathcal{U}'^{\mathrm{an}}}) = j_!^{\mathrm{an}} \Lambda_{\mathcal{U}^{\mathrm{an}}}$, it follows that $R^q \varphi_*(j_! \Lambda_{\mathcal{U}}) \tilde{\to} R^q (\varphi f')_*(j'_! \Lambda_{\mathcal{U}'})$ and $R^q \varphi_*^{\mathrm{an}}(j_!^{\mathrm{an}} \Lambda_{\mathcal{U}^{\mathrm{an}}}) \tilde{\to} R^q (\varphi f')_*^{\mathrm{an}}(j'^{\mathrm{an}}_! \Lambda_{\mathcal{U}'^{\mathrm{an}}})$, and therefore the theorem is true for $(\varphi, j_! \Lambda_{\mathcal{U}})$. Finally, since $\dim(\mathcal{Z}) < d$ then, by induction, the theorem is true for $(\varphi, i^*\Lambda_{\mathcal{Z}})$. From the exact sequence $0 \to j_!\Lambda_{\mathcal{U}} \to \Lambda_{\mathcal{Y}} \to i^*\Lambda_{\mathcal{Z}} \to 0$ it follows that the theorem is true for $(\varphi, \Lambda_{\mathcal{Y}})$.

Step 3. *The theorem is true if $\mathcal{F}$ is constant.*

We may assume that $\mathcal{X}$ and $\mathcal{Y}$ are affine. Then we can represent the morphism φ as a composition of an open immersion $j : \mathcal{Y} \hookrightarrow \overline{\mathcal{Y}}$ with a proper morphism $\overline{\varphi} : \overline{\mathcal{Y}} \to \mathcal{Y}$. By Step 2, the theorem is true for $(j, \Lambda_{\mathcal{Y}})$ and, by the Comparison Theorem for cohomology with compact support, the theorem is true for $(\overline{\varphi}, R^q j_* \Lambda_{\mathcal{U}})$. It follows that the theorem is true for $(\varphi, \Lambda_{\mathcal{Y}})$.

Step 4. *The theorem is true in the general case.*

We can embed any constructible sheaf $\mathcal{F}$ in a finite direct sum of sheaves of the form $f_*\Lambda_{\mathcal{Y}'}$, where $f : \mathcal{Y}' \to \mathcal{Y}$ is a finite morphism. By Step 4, the theorem is true for $(\varphi f, \Lambda_{\mathcal{Y}'})$. It follows that the theorem is true for $(\varphi, \mathcal{F})$. ∎

3.2. Corollary. *Let $\mathcal{X}$ be a scheme of locally finite type over k, and let $\mathcal{F}$ be a constructible abelian sheaf on $\mathcal{X}$ with torsion orders prime to* $\mathrm{char}(k)$. *Then for any $q \geq 0$ there is a canonical isomorphism $H^q(\mathcal{X}, \mathcal{F}) \xrightarrow{\sim} H^q(\mathcal{X}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$.* ∎

3.3. Corollary. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a compactifiable morphism between schemes of locally finite type over k, and let $\mathcal{F}^\cdot \in D_c^b(\mathcal{X}, \Lambda)$ with n prime to* $\mathrm{char}(k)$. *Assume that either n is prime to* $\mathrm{char}(\widetilde{k})$ *or φ is a closed immersion. Then there is a canonical isomorphism*

$$(R\varphi^! \mathcal{F}^\cdot)^{\mathrm{an}} \xrightarrow{\sim} R\varphi^{\mathrm{an}!} \mathcal{F}^{\mathrm{an}\cdot} .$$

Proof. Suppose first that n is prime to $\mathrm{char}(\widetilde{k})$. Since the statement is local with respect to $\mathcal{Y}$, we may assume that φ is the composition $\mathcal{Y} \xrightarrow{i} \mathcal{X}' \xrightarrow{\psi} \mathcal{X}$, where i is a closed immersion and ψ is smooth. By Poincaré Duality for schemes and for analytic spaces ([Ber2], 7.3.1), the statement is true for ψ. Thus, in both cases we may assume that $\varphi = i$ is a closed immersion. Let j be the open immersion $\mathcal{X} \backslash \mathcal{Y} \hookrightarrow \mathcal{X}$. Then there is a morphism of exact triangles

$$\begin{array}{ccccccccc} \to & i_*^{\mathrm{an}}(Ri^! \mathcal{F}^\cdot)^{\mathrm{an}} & \to & \mathcal{F}^{\mathrm{an}\cdot} & \to & (Rj_*j^*\mathcal{F}^\cdot)^{\mathrm{an}} & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & i_*^{\mathrm{an}}(Ri^{\mathrm{an}!}\mathcal{F}^{\mathrm{an}\cdot}) & \to & \mathcal{F}^{\mathrm{an}\cdot} & \to & Rj_*^{\mathrm{an}} j^{\mathrm{an}*}\mathcal{F}^{\mathrm{an}\cdot} & \to \end{array}$$

The third vertical arrow is an isomorphism, by Theorem 3.1. The statement follows. ∎

3.4. Corollary. *Let $\mathcal{X}$ be a scheme of locally finite type over k. Then for all $\mathcal{F}^\cdot \in D_c^-(\mathcal{X}, \Lambda)$ and $\mathcal{G}^\cdot \in D_c^+(\mathcal{X}, \Lambda)$ with n prime to* $\mathrm{char}(k)$ *there is a canonical isomorphism*

$$(\underline{\mathcal{H}om}(\mathcal{F}^\cdot, \mathcal{G}^\cdot))^{\mathrm{an}} \xrightarrow{\sim} \underline{\mathcal{H}om}(\mathcal{F}^{\mathrm{an}\cdot}, \mathcal{G}^{\mathrm{an}\cdot}) .$$

In particular, the canonical functor $D^b_c(\mathcal{X},\Lambda)\to D^b(\mathcal{X}^{\mathrm{an}},\Lambda)$ *is fully faithful.*

Proof. It suffices to verify the statement when $\mathcal{F}^{\cdot}=\mathcal{F}$ is a constructible sheaf and $\mathcal{X}$ is of finite type, separated and connected. If $\mathcal{F}$ is constant, the statement follows from Corollary 3.2. If $\mathcal{F}$ is locally constant, then there is a finite étale morphism $\varphi:\mathcal{X}'\to\mathcal{X}$ such that $\mathcal{F}'=\mathcal{F}\big|_{\mathcal{X}'}$ is constant. Since $\mathcal{F}$ is embedded in $\varphi_*\mathcal{F}'$, the statement is easily reduced to the case of $\mathcal{F}'$ on $\mathcal{X}'$. In the general case, we can find an open everywhere dense subset $\mathcal{U}\subset\mathcal{X}$ such that $\mathcal{F}\big|_{\mathcal{U}}$ is locally constant. Let j (resp. i) be the open (resp. closed) immersion $\mathcal{U}\hookrightarrow\mathcal{X}$ (resp. $\mathcal{X}\backslash\mathcal{U}\to\mathcal{X}$). Consider the exact sequence $0\to j_!(\mathcal{F}\big|_{\mathcal{U}})\to\mathcal{F}\to i_*i^*\mathcal{F}\to 0$. Since $\underline{\mathrm{Hom}}(j_!(\mathcal{F}\big|_{\mathcal{U}}),\mathcal{G}^{\cdot})\widetilde{\to}\underline{\mathrm{Hom}}(\mathcal{F}\big|_{\mathcal{U}},\mathcal{G}^{\cdot}\big|_{\mathcal{U}})$, then the statement is true for the sheaf $j_!(\mathcal{F}\big|_{\mathcal{U}})$. Furthermore, since $\underline{\mathrm{Hom}}(i_*i^*\mathcal{F},\mathcal{G}^{\cdot})\widetilde{\to}\underline{\mathrm{Hom}}(i^*\mathcal{F},Ri^!\mathcal{G}^{\cdot})$ and $\dim(\mathcal{X}\backslash\mathcal{U})<\dim(\mathcal{X})$, then, by induction and Corollary 3.3, the statement is true for $i_*i^*\mathcal{F}$. It follows that it is true for $\mathcal{F}$. ∎

3.5. Remark. Corollary 3.3 is not true if n is a power of $p=\mathrm{char}(\widetilde{k})>0$, $\mathrm{char}(k)=0$ and φ is not a closed immersion. Indeed, assume that k is algebraically closed. Then $\Lambda_{P^1}(1)[2]\widetilde{\to}T^{\cdot}_{P^1}$, where P^1 is the algebraic projective line. But the dualizing complex $T^{\cdot}_{\mathbf{P}^1}$ is more complicated. For example, if D is an open disc in $\mathbf{P}^1$, then the group $H^2_c(D,\mu_p)$ is very big (see [Ber2], 6.2.10), and therefore $T^{\cdot}_{\mathbf{P}^1}\big|_D\widetilde{\to}T^{\cdot}_D$ is not isomorphic to $\Lambda_D(1)[2]$. It would be interesting to understand the structure of the dualizing complexes in this situation.

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