INTEGRATION OF ONE-FORMS ON P-ADIC ANALYTIC SPACES

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Introduction

One of the basic facts of complex analysis is the exactness of the de Rham complex of *sheaves* of analytic differential forms on a smooth complex analytic space. In its turn, its proof is based on the fact that every point of such a space has an open neighborhood isomorphic to an open polydisc, which reduces the verification of the exactness to the classical Poincaré lemma. The latter states that the de Rham complex of *spaces* of analytic differential forms on an open polydisc is exact. Its proof actually works over any non-Archimedean field k of characteristic zero as well, and so it implies also that the de Rham complex of sheaves of analytic differential forms on a smooth k-analytic space (as introduced in [Ber1] and [Ber2]) is exact at every point that admits a fundamental system of étale neighborhoods isomorphic to an open polydisc. One can show (Corollary 2.3.3) that a point x of a smooth k-analytic space possesses the above property if and only if the non-Archimedean field $\mathcal{H}(x)$, associated with the point x, possesses the property that its residue field $\widetilde{\mathcal{H}(x)}$ is algebraic over $\widetilde{k}$ and the group $|\mathcal{H}(x)^*|/|k^*|$ is torsion.

It is a distinctive feature of non-Archimedean analytic spaces that the subset X_{st} of points with the latter property does not coincide with the whole space X. Notice that X_{st} contains the set $X_0 = \{x \in X \big| [\mathcal{H}(x) : k] < \infty\}$ (the underlying space of X in rigid analytic geometry) and, in particular, the set of k-rational points $X(k) = \{x \in X \big| \mathcal{H}(x) = k\}$. Although X is locally arcwise connected, the topology induced on X_{st} is totally disconnected and, if the valuation on k is nontrivial, X_{st} is dense in X. Moreover, if X is smooth, X_{st} is precisely the set of points at which the de Rham complex is exact and, in fact, for every point $x \notin X_{st}$ there is a closed one-form, defined in an open neighborhood of x, that has no primitive at any étale neighborhood of x.

We now recall that a locally analytic function is a map $f : X(k) \to k$ such that, for every point $x \in X(k)$, there is an analytic function g defined on an open neighborhood U of x with $f(y) = g(y)$ for all $y \in U(k)$. It is clear that the local behavior of such a function does not determine its global behavior. For example, if its differential is zero, the function is not necessarily constant. On the other hand, for a long time number theorists have been using very natural locally analytic functions possessing certain properties that make them look like analytic ones. An example of such a function (for $X = \mathbf{G}_\mathrm{m} = \mathbf{A}^1 \backslash \{0\}$) is a homomorphism $k^* \to k$ from the multiplicative to the additive group of k which extends the homomorphism $a \mapsto \log(a)$ on the subgroup $k^1 = \{a \in k^* \big| |a-1| < 1\}$, where $\log(T)$ is the usual logarithm defined by the power series $-\sum_{i=1}^{\infty} \frac{(1-T)^i}{i}$ (convergent on k^1).

Let us assume (till the end of the introduction) that k is a closed subfield of $\mathbf{C}_p$, the completion of the algebraic closure $\overline{\mathbf{Q}}_p$ of the field of p-adic numbers $\mathbf{Q}_p$. Then such a homomorphism is uniquely determined by its value at p, and the homomorphism, whose value at p is an element $\lambda \in k$, is denoted by $\log^\lambda(T)$ and is called a branch of the logarithm. One of the properties we had in mind states that, if X is an open annulus in $\mathbf{A}^1$ with center at zero and the differential of a locally analytic function on $X(k)$ of the form $\sum_{i=0}^n f_i \log^\lambda(T)^i$ with $f_i \in \mathcal{O}(X)$ is equal to zero in an open subset of $X(k)$, then the function is a constant and, in fact, $f_0 \in k$ and $f_i = 0$ for all $1 \le i \le n$. Notice also that every one-form on X with coefficients of the above form has a primitive which is a locally analytic function of the same form.

It was an amazing discovery of R. Coleman ([Col1], [CoSh]) more than twenty years ago that there is a way to construct primitives of analytic one-forms and their iterates in the class of locally analytic functions on certain smooth k-analytic curves, called by him basic wide opens (they are closely related to basic curves considered here), such that the primitives are defined up to a constant. Namely, given a branch of the logarithm $\log^\lambda(T)$, he constructed for every such curve X an $\mathcal{O}(X)$-algebra $A(X)$ of locally analytic functions filtered by free $\mathcal{O}(X)$-modules of finite rank $A^0(X) \subset A^1(X) \subset \dots$ with $dA^i(X) \subset A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ and such that

(a) $A^0(X) = \mathcal{O}(X)$;

(b) every function from $A(X)$ with zero differential is a constant;

(c) $A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X) \subset dA^{i+1}(X)$;

(d) $A^{i+1}(X)$ is generated over $\mathcal{O}(X)$ by primitives of one-forms from $A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$;

(e) if X is an open annulus with center at zero, then $\log^\lambda(T) \in A^1(X)$;

(f) for a morphism $X' \to X$ and a function $f \in A^i(X)$, one has $\varphi^*(f) \in A^i(X')$.

Moreover, if a function in $A(X)$ is equal to zero on a nonempty open subset of $X(k)$, it is equal to zero everywhere. Thus, if ω is a one-form in $A(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$, then for any pair of points $x, y \in X(k)$ one can define an integral $\int_x^y \omega \in k$. It follows also that, given a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with a unipotent integrable connection, the kernel of the induced connection on the $\mathcal{O}(X)$-module $\mathcal{F}(X) \otimes_{\mathcal{O}(X)} A(X)$ is a vector space of dimension equal to the rank of $\mathcal{F}$ and, therefore, for any pair of points $x, y \in X(k)$ one can define a parallel transport $\mathrm{T}_{x,y} : \mathcal{F}_x^\nabla \xrightarrow{\sim} \mathcal{F}_y^\nabla$.

Since then there were several attempts to extend Coleman's work to higher dimensions. R. Coleman himself ([Col2]) constructed an integral $\int_x^y \omega$ of a closed analytic one-form ω on the analytification $\mathcal{X}^{\mathrm{an}}$ of a connected projective scheme $\mathcal{X}$ with good reduction. Yu. Zarhin ([Zar]) and P. Colmez ([Colm]) constructed similar integrals for arbitrary connected smooth $\mathcal{X}$ (see Remark 9.1.2(ii)). A. Besser ([Bes]) constructed iterated integrals on the generic fiber $\mathfrak{X}_\eta$ of a connected smooth formal scheme $\mathfrak{X}$, which is an open subscheme of a formal scheme $\mathfrak{Y}$ of finite type over k° such that the Zariski closure of the closed fiber $\mathfrak{X}_s$ in $\mathfrak{Y}_s$ is proper (see Remark 8.1.5(ii)). V.

Vologodsky ([Vol]) constructed a parallel transport $\mathrm{T}_{x,y}$ on $\mathcal{X}^{\mathrm{an}}$ for arbitrary connected smooth $\mathcal{X}$ (see Remark 9.4.4). All of these constructions gave additional evidence for a certain phenomenon of local analytic nature which was already present in the work of Coleman and is described as follows.

Let us call a naive analytic function on a smooth k-analytic space X a map that associates to every point $x \in X_{st}$ an element $f(x) \in \mathcal{H}(x)$ such that there is an analytic function g defined at an open neighborhood U of x with $f(y) = g(y)$ for all $y \in U_{st}$. This class of functions is better than that of locally analytic ones since, for every closed subfield $k \subset k' \subset \mathbf{C}_p$ and every naive analytic function f on X, one can define the pullback of f on $X\widehat{\otimes}_k k'$. For example, the locally analytic function $\log^\lambda(T)$ is the restriction to k^* of a natural naive analytic function $\mathrm{Log}^\lambda(T)$ on the multiplicative group $\mathbf{G}_{\mathrm{m}}$, and in fact for a basic curve X all locally analytic functions in Coleman's algebra $A(X)$ are restrictions to $X(k)$ of natural naive analytic functions on X. If now $\mathfrak{n}(X)$ denotes the space of naive analytic functions on X, the correspondence $U \mapsto \mathfrak{n}(U)$ is a sheaf of $\mathcal{O}_X$-algebras denoted by $\mathfrak{n}_X$. Coleman's work was actually evidence for the fact that, for a fixed branch of the logarithm, every smooth k-analytic space X is provided with an $\mathcal{O}_X$-subalgebra of $\mathfrak{n}_X$ whose associated de Rham complex is exact and in which the kernel of the first differential coincides with the sheaf of constant analytic functions $\mathfrak{c}_X = \mathrm{Ker}(\mathcal{O}_X \xrightarrow{d} \Omega^1_X)$.

The purpose of this work is to show that such an $\mathcal{O}_X$-subalgebra $\mathcal{S}_X$ of $\mathfrak{n}_X$ exists and is unique with respect to certain very natural properties. More precisely, $\mathcal{S}_X$ is a filtered $\mathcal{O}_X$-algebra with $d\mathcal{S}^i_X \subset \mathcal{S}^i_X \otimes_{\mathcal{O}_X} \Omega^1_X$ for all $i \geq 0$, and the properties are similar to (a)-(f) from above. Although we are not yet able to prove the exactness of the whole de Rham complex for $\mathcal{S}_X$, we show that the de Rham complex is exact at Ω^1 and the kernel of the first differential coincides with $\mathfrak{c}_X$. In particular, under a certain natural assumption (which is automatically satisfied if $k = \mathbf{C}_p$) one can define an integral $\int_\gamma \omega$ of a closed one-form $\omega \in (\mathcal{S}_X \otimes_{\mathcal{O}_X} \Omega^1_X)(X)$ along a path $\gamma : [0,1] \to X$ with ends in $X(k)$. Furthermore, the extended class of functions contains a full set of local solutions of all unipotent differential equations and, in fact, a coherent $\mathcal{O}_X$-module with an integrable connection has a full set of local horizontal sections in the étale topology if and only if it is locally unipotent in the étale topology (such a module is called here locally quasi-unipotent). As a consequence, we construct parallel transport along a path and an étale path of local horizontal sections of locally unipotent and locally quasi-unipotent modules, respectively. In comparison with the previous constructions mentioned above, both integral and parallel transport depend nontrivially on the homotopy class of a path and not only on its ends.

The filtered $\mathcal{O}(X)$-algebra $A(X)$, constructed by Coleman for a basic curve X, appears here in the following way. One can show that for such X the group $H^1(X, \mathfrak{c}_X)$ is zero and, therefore, every one-form $\omega \in (\mathcal{S}^i_X \otimes_{\mathcal{O}_X} \Omega^1_X)(X)$ has a primitive in $\mathcal{S}^{i+1}(X)$. Then $A^0(X) = \mathcal{O}(X)$ and, for $i \geq 0$, $A^{i+1}(X)$ is generated over $\mathcal{O}(X)$ by the primitives of all one-forms $\omega \in A^i(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ in $\mathcal{S}^{i+1}(X)$. The algebra $\mathcal{S}(X)$ is in fact much bigger than $A(X)$. For example, if X is an open annulus, then $A(X) = \mathcal{O}(X)[\log^\lambda(T)]$, but the $\mathcal{O}(X)$-modules $\mathcal{S}^i(X)/\mathcal{S}^{i-1}(X)$ are of infinite rank for all $i \geq 1$. By the way, the latter is even true for the projective line $\mathbf{P}^1$ (see Lemma 8.5.2).

In fact the sheaves $\mathcal{S}_X$ are constructed in a more general setting, and the reason for that is as follows. Let X be the Tate elliptic curve which is the quotient of $\mathbf{G}_{\mathrm{m}}$ by the discrete subgroup generated by an element $q \in k^*$ with $|q| \neq 1$, and let ω be the invariant one-form on X whose preimage on $\mathbf{G}_{\mathrm{m}}$ is $\frac{dT}{T}$. The curve X is homotopy equivalent to a circle, and the only reasonable value for the integral of ω along a loop $\gamma : [0,1] \to X$ with end in $X(k)$, whose class generates the fundamental group of X, should be $\mathrm{Log}^\lambda(q)$ (up to a sign). But for every $q \in k^*$ with $|q| \neq 1$ there exist λ's in k with $\mathrm{Log}^\lambda(q) \neq 0$ as well as those with $\mathrm{Log}^\lambda(q) = 0$.

A natural way to resolve the above problem is to consider the universal logarithm, i.e., the one whose value at p is a variable. Such a universal logarithm was already used in the work of P. Colmez and V. Vologodsky mentioned above, and it can be specialized to any of the branches of the logarithm whose values at p are elements of k. But the properties of the sheaves $\mathcal{S}_X$, whose construction is based on the universal logarithm, do not seem to easily imply the same properties of the similar sheaves whose construction is based on a classical branch of the logarithm. Thus, to consider all possible branches of the logarithm simultaneously, we proceed as follows. Fix a filtered k-algebra K, i.e., a commutative k-algebra provided with an exhausting filtration by k-vector spaces $K^0 \subset K^1 \subset \dots$ with $K^i \cdot K^j \subset K^{i+j}$. Furthermore, define a filtered $\mathcal{O}_X$-algebra of naive analytic functions $\mathfrak{N}^K_X$ in the same way as $\mathfrak{n}_X$ but starting with the filtered algebra $\mathcal{O}^K_X = \mathcal{O}_X \otimes_k K$ instead of $\mathcal{O}_X$ and, for an element $\lambda \in K^1$, define a logarithmic function $\mathrm{Log}^\lambda(T)$, which is an element of $\mathfrak{N}^{K,1}(\mathbf{G}_{\mathrm{m}})$. In the similar way we define the $\mathcal{O}_X$-algebras of naive analytic q-forms $\Omega^q_{\mathfrak{N}^K,X}$. One has $\mathfrak{N}^K_X \otimes_{\mathcal{O}_X} \Omega^q_X \xrightarrow{\sim} \Omega^q_{\mathfrak{N}^K,X}$ and, in particular, $\mathfrak{N}^K_X$ is a filtered $\mathcal{D}_X$-algebra (the latter notion is defined in §1.3). For a $\mathcal{D}_X$-submodule $\mathcal{F}$ of $\mathfrak{N}^K_X$, let $\Omega^q_{\mathcal{F},X}$ denote the image of the canonical injective homomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^q_X \to \Omega^q_{\mathfrak{N}^K,X}$.

The main result (Theorem 1.6.1) states that, given K and $\lambda \in K^1$, there is a unique way to provide every smooth k-analytic space X with a filtered $\mathcal{D}_X$-subalgebra $\mathcal{S}^\lambda_X \subset \mathfrak{N}^K_X$ so that the following is true:

(a) $\mathcal{S}^{\lambda,0}_X = \mathcal{O}_X \otimes_k K^0$;

(b) $\mathrm{Ker}(\mathcal{S}^{\lambda,i}_X \xrightarrow{d} \Omega^1_{\mathcal{S}^{\lambda,i},X}) = \mathfrak{c}_X \otimes_k K^i$;

(c) $\mathrm{Ker}(\Omega^1_{\mathcal{S}^{\lambda,i},X} \xrightarrow{d} \Omega^2_{\mathcal{S}^{\lambda,i},X}) \subset d\mathcal{S}^{\lambda,i+1}_X$;

(d) $\mathcal{S}_X^{\lambda,i+1}$ is generated by the local sections f for which df is a local section of $\Omega^1_{\mathcal{S}^{\lambda,i},X}$;

(e) $\mathrm{Log}^\lambda(T) \in \mathcal{S}^{\lambda,1}(\mathbf{G}_{\mathrm{m}})$;

(f) for a morphism $\varphi : X' \to X$ and a function $f \in \mathcal{S}^{\lambda,i}(X)$, one has $\varphi^*(f) \in \mathcal{S}^{\lambda,i}(X')$.

In Theorem 1.6.2 we list several properties of the sheaves $\mathcal{S}_X^\lambda$. Among them is the uniqueness property which tells that, if X is connected, then for any nonempty open subset $\mathcal{U} \subset X$ the restriction map $\mathcal{S}^\lambda(X) \to \mathcal{S}^\lambda(\mathcal{U})$ is injective. The sheaves $\mathcal{S}_X^\lambda$ are functorial in the best possible sense involving an embedding of the ground fields $k \hookrightarrow k'$, a morphism $X' \to X$ and a homomorphism of filtered algebras $K \to K'$ over that embedding. If one is given only a homomorphism of filtered k-algebras $K \to K' : \lambda \mapsto \lambda'$, there is a canonical isomorphism $\mathcal{S}_X^\lambda \otimes_K K' \xrightarrow{\sim} \mathcal{S}_X^{\lambda'}$. In particular, if $\mathcal{S}_X$ denotes the sheaf constructed for the universal logarithm, then the canonical homomorphism $k[\mathrm{Log}(p)] \to K : \mathrm{Log}(p) \mapsto \lambda$ gives rise to an isomorphism $\mathcal{S}_X \otimes_{k[\mathrm{Log}(p)]} K \xrightarrow{\sim} \mathcal{S}_X^\lambda$.

Theorems 1.6.1 and 1.6.2 are proven in §7. The proof is based on preliminary results obtained in §§1-6 and having an independent interest. Since the formulation of the main result actually makes sense for an arbitrary non-Archimedean field of characteristic zero, the preparatory part of the proof in §§1-5 is done over fields as general as possible, and the assumption that k is a closed subfield of $\mathbf{C}_p$ is only made beginning with §6. In §8, further properties of the sheaves $\mathcal{S}_X^\lambda$ are established and, in §9, they are used for a construction of the integral and parallel transport along a path. A detailed summary of each section is given at its beginning.

There are many natural questions on the sheaves $\mathcal{S}_X^\lambda$ one may ask. Here are some of them.

(1) Is the de Rham complex associated to $\mathcal{S}_X^\lambda$ exact? We believe this is true.

(2) Does the extended class of functions contain local primitives of relative closed one-forms with respect to an arbitrary smooth morphism $\varphi : Y \to X$? Again, we believe this is true. It is in fact enough to consider morphisms of dimension one, and the positive answer to this question would imply the positive answer to (1) and to the relative version of (1).

(3) Are the sheaves of rings $\mathcal{S}_K^\lambda$ coherent for reasonable K (e.g., $K = k[\mathrm{Log}(p)]$ or $K = k$)? Like (1) and (2), we believe this is true. Notice that, for a point $x \in X$ whose field $\widetilde{\mathcal{H}(x)}$ is transcendent over $\widetilde{k}$, the stalk $\mathcal{S}_{X,x}^\lambda$ is a non-Noetherian ring.

(4) What are the cohomology groups $H^q(X, \mathcal{S}_X^{\lambda,n})$ and $H^q(X, \mathcal{S}_X^\lambda)$?

(5) Assume that k is finite over $\mathbf{Q}_p$ and a p-adic group G acts continuously on a smooth k-analytic space X (e.g., $G = \mathrm{PGL}_d(k)$ and X is the projective space $\mathbf{P}^{d-1}$ or the Drinfeld half-plane $\Omega^d \subset \mathbf{P}^{d-1}$). What are the representations of G on the space of global sections $\mathcal{S}^\lambda(X)$?

(6) Let X and Y be smooth k-analytic spaces, and assume there is a morphism of germs of analytic spaces $\varphi : (Y, Y_{st}) \to (X, X_{st})$ (see [Ber2, §3.4]) which takes local sections of $\mathcal{S}_X^\lambda$ to those

of $\mathcal{S}_Y^\lambda$. Is it true that φ is induced by a morphism of analytic spaces $Y \to X$?

The answer to (6) would shed light on the following philosophical question. What does the existence of the sheaves $\mathcal{S}_X^\lambda$ mean? If the answer to (6) is negative, it would mean that smooth p-adic analytic spaces can be considered as objects of a category with bigger sets of morphisms in the same way as complex analytic spaces can be considered as real analytic or differentiable manifolds. On the other hand, if the answer to (6) is positive, it could mean that complex analytic functions have at least two p-adic analogs, namely, genuine analytic ones and functions from the broader class provided by the sheaves $\mathcal{S}_X^\lambda$. This reminds us of the similar phenomenon with the topology of a complex analytic space whose p-adic reincarnation is two-faced. It appears as the usual topology of a p-adic analytic space as well as the stronger étale topology of the space. Besides, the existence of the sheaves $\mathcal{S}_X^\lambda$ is somehow related to the fact that smooth p-adic analytic spaces are not locally simply connected in the étale topology. In any case, we hope what is done in this paper will be useful for understanding the p-adic Hodge theory in terms of p-adic analytic geometry.

Acknowledgements

As it is clear from the above and the text which follows, this paper is motivated by and based on the work and ideas of R. Coleman. In its very first version I constructed local primitives of closed analytic one-forms (or, equivalently, the sheaves $\mathcal{S}_X^{\lambda,1}$) on smooth analytic spaces defined over finite extentions of $\mathbf{Q}_p$. The construction of all of the sheaves $\mathcal{S}_X^{\lambda,n}$ was done after I borrowed the idea of using unipotent isocrystals from A. Besser's work [Bes]. Finally, I am very grateful to O. Gabber for providing a key fact (Lemma 5.5.1) which allowed me to extend the whole theory for arbitrary closed subfields of $\mathbf{C}_p$.

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§1. Naive analytic functions and formulation of the main result

After recalling some notions and notation, we give a precise definition of the sheaves of naive analytic functions $\mathfrak{N}^K_X$. We then recall the definition of a $\mathcal{D}_X$-module, introduce a related notion of a D_X-module, and establish a simple relation between the de Rham complexes of a D_X-module and of its pullback under a so called discoid morphism $Y \to X$. In §1.4, we introduce the logarithmic function $\mathrm{Log}^\lambda(T) \in \mathfrak{N}^{K,1}(\mathbf{G}_\mathrm{m})$ and a filtered D_X-module $L^\lambda(X)$ which is generated over $\mathcal{O}(X)$ by the logarithms $\mathrm{Log}^\lambda(f)$ of invertible analytic functions on X. Furthermore, given a so called semi-annular morphism $Y \to X$, we establish a relation between the de Rham complexes of certain D_X-modules and of the D_Y-modules which are generated by the pullbacks of the latter and the logarithms of invertible analytic functions on Y. It implies the exactness of the de Rham complex of the spaces of differential forms with coefficients in the D_X-module $L^\lambda(X)$ on a semi-annular analytic space X. In §1.6, we formulate the main result on existence and uniqueness of the sheaves $\mathcal{S}^\lambda_X$ and list their basic properties.

1.1. Preliminary remarks and notation. In this paper we work in the framework of non-Archimedean analytic spaces in the sense of [Ber1] and [Ber2]. A detailed definition of these spaces is given in [Ber2, §1], and an abbreviated one is given in [Ber6, §1]. We only recall that the affinoid space associated with an affinoid algebra $\mathcal{A}$ is the set of all bounded multiplicative seminorms on $\mathcal{A}$. It is a compact space with respect to the evident topology, and it is denoted by $\mathcal{M}(\mathcal{A})$.

Let k be a non-Archimedean field with a nontrivial valuation. All of the k-analytic spaces considered in the paper are assumed to be Hausdorff. For example, any separated k-analytic space is Hausdorff and, for the class of the spaces which are good in the sense of [Ber2, §1.2], the converse is also true.

Although in this paper we are mostly interested in smooth k-analytic spaces (in the sense of [Ber2, §3.5]), we have to consider more general strictly k-analytic spaces. Among them, of special interest are strictly k-analytic spaces smooth in the sense of rigid geometry (for brevity we call them rig-smooth). Namely, a strictly k-analytic space X is *rig-smooth* if for any connected strictly affinoid domain V the sheaf of differentials Ω^1_V is locally free of rank $\dim(V)$. Such a space is smooth at all points of its interior (see [Ber4, §5]). In particular, a k-analytic space X is smooth if and only if it is rig-smooth and has no boundary (in the sense of [Ber2, §1.5]). An intermediate class between smooth and rig-smooth spaces is that of k-analytic spaces locally embeddable in a smooth space (see [Ber7, §9]). For example, it follows from R. Elkik's results (see [Ber7, 9.7]) that any rig-smooth k-affinoid space is locally embeddable in a smooth space. If X is locally embeddable in a smooth space, the sheaf of differential one-forms Ω^1_X is locally free in the usual topology of X. (If X rig-smooth, the sheaf of differential one-forms is locally free in the more strong G-topology X_G, the Grothendieck topology formed by strictly analytic subdomains of X, see [Ber2, §1.3].) Recall that for any strictly k-analytic space X the set $X_0 = \{x \in X \big| [\mathcal{H}(x) : k] < \infty\}$ is dense in X ([Ber1, 2.1.15]). For a point $x \in X_0$, the field $\mathcal{H}(x)$ coincides with the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathbf{m}_x$ of the local ring $\mathcal{O}_{X,x}$.

1.1.1. Lemma. *Let X be a connected rig-smooth k-analytic space. Then every nonempty Zariski open subset $X' \subset X$ is dense and connected, and one has $\mathfrak{c}(X) \xrightarrow{\sim} \mathfrak{c}(X')$.*

Recall that $\mathfrak{c}(X)$ is the space of global sections of the sheaf of constant analytic functions $\mathfrak{c}_X$ defined in [Ber9, §8]. If the characteristic of k is zero, then $\mathfrak{c}_X = \mathrm{Ker}(\mathcal{O}_X \xrightarrow{d} \Omega^1_X)$.

Proof. We may assume that the space $X = \mathcal{M}(\mathcal{A})$ is strictly k-affinoid, and we may replace X' by a smaller subset of the form $X_f = \{x \in X \big| f(x) \neq 0\}$ with f a non-zero element of $\mathcal{A}$. Such a subset is evidently dense in X. We now notice that X_f is the analytification of the affine scheme $\mathcal{X}_f = \mathrm{Spec}(\mathcal{A}_f)$ over $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$. Since $\mathcal{X}_f$ is connected, from [Ber2, Corollary 2.6.6] it follows that X_f is connected. To prove the last property, we can replace k by $\mathfrak{c}(X)$, and so we may assume that $\mathfrak{c}(X) = k$. By [Ber9, Lemma 8.1.4], the strictly k'-affinoid space $X\widehat{\otimes}k'$ is connected for any finite extension k' of k and, therefore, the same is true for the space $X_f\widehat{\otimes}k' = (X\widehat{\otimes}k')_f$. The latter implies that $\mathfrak{c}(X_f) = k$. ∎

Recall that in [Ber1, §9.1] we introduced the following invariants of a point x of a k-analytic space X. The first one is the number $s(x) = s_k(x)$ equal to the transcendence degree of $\widetilde{\mathcal{H}(x)}$ over $\widetilde{k}$, and the second one is the number $t(x) = t_k(x)$ equal to the dimension of the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$. One has $s(x) + t(x) \leq \dim_x(X)$, and if x' is a point of $X\widehat{\otimes}\widehat{k}^{\mathrm{a}}$ over k then $s(x') = s(x)$ and $t(x') = t(x)$. Moreover, the functions $s(x)$ and $t(x)$ are additive in the sense that, given a morphism of k-analytic spaces $\varphi : Y \to X$, one has $s(y) = s(x) + s_{\mathcal{H}(x)}(y)$ and $t(y) = t(x) + t_{\mathcal{H}(x)}(y)$, where $x = \varphi(y)$. Let X_{st} denote the set of points $x \in X$ with $s(x) = t(x) = 0$. This set contains X_0 and, in particular, if X is strictly k-analytic, X_{st} is dense in X. By [Ber1, §9], the topology on X_{st} induced from that on X is totally disconnected. If k' is a closed subfield of the completion $\widehat{k}^{\mathrm{a}}$ of an algebraic closure of k and $X' = X\widehat{\otimes}k'$, then the image of X'_{st} under the canonical map $X' \to X$ is contained in X_{st}. (Notice that if k' is not finite over k the latter fact is not true for the sets X'_0 and X_0.)

For an étale sheaf F on a k-analytic space X, we denote by F_x the stalk at a point $x \in X$ of the restriction of F to the usual topology of X and, for a section $f \in F(X)$, we denote by f_x its image in F_x. Furthermore, a *geometric point of* X is a morphism (in the category of analytic spaces over k) of the form $\overline{x} : \mathbf{p}_{\mathcal{H}(\overline{x})} \to X$, where $\mathbf{p}_{\mathcal{H}(\overline{x})}$ is the spectrum of an algebraically closed non-Archimedean field $\mathcal{H}(\overline{x})$ over k. The *stalk* $F_{\overline{x}}$ of an étale sheaf F at $\overline{x}$ is the stalk of its pullback with respect to the morphism $\overline{x}$, i.e., the inductive limit of $F(Y)$ taken over all pairs (φ, α) consisting of an étale morphism $\varphi : Y \to X$ and a morphism $\alpha : \mathbf{p}_{\mathcal{H}(\overline{x})} \to Y$ over $\overline{x}$. Notice that, if $G_{\overline{x}/x}$ is the Galois group of the separable closure of $\mathcal{H}(x)$ in $\mathcal{H}(\overline{x})$ over $\mathcal{H}(x)$, there is a discrete action of $G_{\overline{x}/x}$ on $F_{\overline{x}}$ and, by [Ber2, Proposition 4.2.2], one has $F_x = F_{\overline{x}}^{G_{\overline{x}/x}}$. (Recall that in [Ber2] we denoted by F_x the pullback of F under the canonical morphism $\mathbf{p}_{\mathcal{H}(x)} \to X$, which can be also identified with a discrete $G_{\overline{x}/x}$-set.)

Given a finite extension k' of k, the ground field extension functor from the category of strictly k-analytic space to that of strictly k'-analytic ones $X \mapsto X\widehat{\otimes}k'$ has a left adjoint functor $X' \mapsto X$ which associates with a strictly k'-analytic space X' the same space considered as a strictly k-analytic one (see [Ber9, §7.1]). The essential image of the latter functor consists of the strictly k-analytic spaces X for which there exists an embedding of k' to the ring of analytic functions $\mathcal{O}(X)$. The canonical morphism $X' \to X$ in the category of analytic spaces over k (see [Ber2, §1.4]) gives rise to an isomorphism of locally ringed spaces, to bijection $X'_0 \stackrel{\sim}{\to} X_0$ and $X'_{st} \stackrel{\sim}{\to} X_{st}$ and to isomorphisms of étale sites $X'_{\text{ét}} \stackrel{\sim}{\to} X_{\text{ét}}$ and of étale topoi $X'^{\sim}_{\text{ét}} \stackrel{\sim}{\to} X^{\sim}_{\text{ét}}$. For an étale sheaf F on X, we denote by F' the corresponding étale sheaf on X'.

1.2. The sheaf of naive analytic functions. Let X be a strictly k-analytic space. For an étale sheaf F on X, we define a presheaf $\widetilde{F}$ as follows. Given an étale morphism $Y \to X$, we set $\widetilde{F}(Y) = \varinjlim F(\mathcal{V})$, where the direct limit is taken over open neighborhoods $\mathcal{V}$ of Y_{st} in Y. The proof of the following lemma is trivial.

1.2.1. Lemma. *Assume that an étale sheaf F on X possesses the following property: for any étale morphism $Y \to X$ and any open neighborhood $\mathcal{V}$ of Y_{st} in Y, the canonical map $F(Y) \to F(\mathcal{V})$ is injective. Then the following is true:*

(i) the presheaf $\widetilde{F}$ is a sheaf;

(ii) the canonical morphism of sheaves $F \to \widetilde{F}$ is injective and gives rise to a bijection of stalks $F_{\overline{x}} \stackrel{\sim}{\to} \widetilde{F}_{\overline{x}}$ for any geometric point $\overline{x}$ of X over a point $x \in X_{st}$;

(iii) the sheaf $\widetilde{F}$ possesses the stronger property that, given an étale morphism $Y \to X$ and an open neighborhood $\mathcal{V}$ of Y_{st} in Y, the canonical map $\widetilde{F}(Y) \to \widetilde{F}(\mathcal{V})$ is bijective;

(iv) one has $\widetilde{\widetilde{F}} = \widetilde{F}$. ∎

1.2.2. Remarks. (i) The property of the sheaf F implies that for any element $f \in \widetilde{F}(X)$

there exists a unique maximal open subset $X_{st} \subset \mathcal{U} \subset X$ such that f comes from $F(\mathcal{U})$.

(ii) The assumption of Lemma 1.2.1 for an étale *abelian* sheaf F on X is equivalent to the property that, for any étale morphism $Y \to X$ and any element $f \in F(Y)$, the intersection $\mathrm{Supp}(f) \cap Y_{st}$ is dense in the support $\mathrm{Supp}(f)$ of f. For the sheaves we are going to consider even the smaller intersection $\mathrm{Supp}(f) \cap Y_0$ is dense in $\mathrm{Supp}(f)$.

Let K be a *filtered* k-algebra K, i.e., a commutative k-algebra with unity provided with an increasing sequence of k-vector subspaces $K^0 \subset K^1 \subset K^2 \subset \dots$ such that $K^i{\cdot}K^j \subset K^{i+j}$ and $K = \cup_{i=0}^{\infty} K^i$. Given a strictly k-analytic space X, we set $\mathcal{O}_X^{K,i} = \mathcal{O}_X \otimes_k K^i$. Notice that if the number of connected components of X is finite, then $\mathcal{O}^{K,i}(X) = \mathcal{O}(X) \otimes_k K^i$. The sheaf $\mathcal{O}_X^K = \mathcal{O}_X \otimes_k K$ is an example of a *filtered $\mathcal{O}_X$-algebra* which is defined as a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ provided with an increasing sequence of $\mathcal{O}_X$-modules $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \mathcal{A}^2 \subset \dots$ such that $\mathcal{A}^i{\cdot}\mathcal{A}^j \subset \mathcal{A}^{i+j}$ and $\mathcal{A} = \varinjlim \mathcal{A}^i$. If X is reduced, we set $\mathcal{C}_X^{K,i} = \mathfrak{c}_X \otimes_k K^i$ and $\mathcal{C}_X^K = \mathfrak{c}_X \otimes_k K$.

The sheaf of $\mathfrak{N}^{K,i}$*-analytic functions* $\mathfrak{N}_X^{K,i}$ is the sheaf $\widetilde{F}$ associated to $F = \mathcal{O}_X^{K,i}$ (which evidently satisfies the assumption of Lemma 1.2.1). The inductive limit $\mathfrak{N}_X^K = \varinjlim \mathfrak{N}_X^{K,i}$ is a sheaf of filtered $\mathcal{O}_X$-algebras. Notice that for every function $f \in \mathfrak{N}^K(X)$ there exists a unique maximal open subset $X_{st} \subset \mathcal{U} \subset X$, called the *analyticity set of* f, such that f comes from $\mathcal{O}^K(\mathcal{U})$. Furthermore, assume we are given a non-Archimedean field k', a strictly k'-analytic space X', a filtered k'-algebra K', and a morphism of analytic spaces $X' \to X$ and a homomorphism of filtered algebras $K \to K'$ over an isometric embedding of fields $k \hookrightarrow k'$. If the analyticity set of a function $f \in \mathfrak{N}^K(X)$ contains the image of X'_{st} in X, then there is a well defined function $\varphi^*(f) \in \mathfrak{N}^{K'}(X')$. For example, if $k' \subset \widehat{k}^{\mathrm{a}}$, then the image of X'_{st} is contained in X_{st} and the latter property is true for all local sections of $\mathfrak{N}_X^K$.

If X is reduced, elements of $\mathfrak{N}^{K,i}(X)$ can be interpreted as the maps f that take a point $x \in X_{st}$ to an element $f(x) \in \mathcal{H}(x) \otimes_k K^i$ and such that every point from X_{st} admits an open neighborhood $X' \subset X$ and an analytic function $g \in \mathcal{O}^{K,i}(X')$ with $f(x) = g(x)$ for all $x \in X'_{st}$. In particular, if $K = k$, $\mathfrak{N}_X^K$ is the sheaf $\mathfrak{n}_X$ introduced in the introduction.

More generally, the sheaf of $\mathfrak{N}^{K,i}$*-differential q-form* $\Omega^q_{\mathfrak{N}^{K,i},X}$, $q \geq 0$, is the sheaf $\widetilde{F}$ associated to $F = \Omega_X^q \otimes_k K^i$ (which also satisfies the assumption of Lemma 1.2.1). We also set $\Omega^q_{\mathfrak{N}^K,X} = \varinjlim \Omega^q_{\mathfrak{N}^{K,i},X}$. If X is locally embeddable in a smooth space, the sheaf Ω_X^1 is locally free over $\mathcal{O}_X$ and, therefore, there is a canonical isomorphism of filtered $\mathcal{O}_X$-modules $\mathfrak{N}_X^K \otimes_{\mathcal{O}_X} \Omega_X^q \xrightarrow{\sim} \Omega^q_{\mathfrak{N}^K,X}$.

Assume now that X is reduced, and that $\mathfrak{c}(X)$ contains a finite extension k' of k. Let X' be the same X considered as a strictly k'-analytic space, and let K' be the filtered k'-algebra

$K \otimes_k k'$. Then the sheaf $(\mathcal{O}_X^K)'$ on X' that corresponds to $\mathcal{O}_X^K$ coincides with $\mathcal{O}_{X'}^{K'}$. It follows that $(\mathfrak{N}_X^K)' = \mathfrak{N}_{X'}^{K'}$ and $(\mathcal{C}_X^K)' = \mathcal{C}_{X'}^{K'}$.

1.3. $\mathcal{D}_X$-modules and D_X-modules. Till the end of this section, the field k is assumed to be of characteristic zero.

Let X be a smooth k-analytic space. A $\mathcal{D}_X$*-module* on X is an étale $\mathcal{O}_X$-module $\mathcal{F}$ provided with an integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1$. For such a $\mathcal{D}_X$-module $\mathcal{F}$, the subsheaves of horizontal sections $\mathcal{F}^\nabla = \mathrm{Ker}(\nabla)$ and of closed one forms $(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1)^{\mathrm{cl}} = \mathrm{Ker}(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1 \stackrel{\nabla}{\to} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^2)$ are étale sheaves of modules over $\mathfrak{c}_X$. If $\mathcal{F} = \mathcal{O}_X$, the former is $\mathfrak{c}_X$ and the latter is denoted by $\Omega_X^{1,\mathrm{cl}}$.

A $\mathcal{D}_X$*-algebra* is an étale commutative $\mathcal{O}_X$-algebra $\mathcal{A}$ which is also a $\mathcal{D}_X$-module whose connection ∇ satisfies the Leibniz rule $\nabla(f\cdot g) = f\nabla(g) + g\nabla(f)$. If in addition $\mathcal{A}$ is a filtered $\mathcal{O}_X$-algebra such that all $\mathcal{A}^i$ are $\mathcal{D}_X$-submodules of $\mathcal{A}$, then $\mathcal{A}$ is said to be a *filtered* $\mathcal{D}_X$*-algebra*. For example, the sheaves of naive analytic functions $\mathfrak{N}_X^{K,i}$ provided with the canonical differential are $\mathcal{D}_X$-modules, and $\mathfrak{N}_X^K$ is a filtered $\mathcal{D}_X$-algebra. Given a non-Archimedean field k', a smooth k'-analytic space X' and a morphism $\varphi : X' \to X$ over an isometric embedding of fields $k \hookrightarrow k'$, we denote by $\varphi^*(\mathcal{F})$ the $\mathcal{D}_{X'}$-module $\varphi^{-1}(\mathcal{F}) \otimes_{\varphi^{-1}(\mathcal{O}_X)} \mathcal{O}_{X'}$, where $\varphi^{-1}(\mathcal{F})$ denotes the pullback of $\mathcal{F}$ as a sheaf of abelian groups.

1.3.1. Lemma. *Let $\mathcal{F}$ be a $\mathcal{D}_X$-module, and assume that the sheaf $\mathcal{F}^\nabla$ possesses the following property: for any étale morphism $Y \to X$, the support* $\mathrm{Supp}(f)$ *of any nonzero element $f \in \mathcal{F}^\nabla(Y)$ is not contained in a nowhere dense Zariski closed subset of Y. Then*

(i) for any étale morphism $Y \to X$ with connected Y and the property that the $\mathcal{O}_Y$-module Ω_Y^1 is free over a nonempty Zariski open subset of Y, the canonical map $\mathcal{F}^\nabla(Y) \otimes_{\mathfrak{c}(Y)} \mathcal{O}(Y) \to \mathcal{F}(Y)$ is injective;

(ii) the canonical morphism of $\mathcal{D}_X$-modules $\mathcal{F}^\nabla \otimes_{\mathfrak{c}_X} \mathcal{O}_X \to \mathcal{F}$ is injective.

Notice that the assumption on Ω_Y^1 in (i) is always satisfied if Y admits a flat quasifinite morphism ([Ber2, §3.2]) to the analytification of a smooth scheme over k.

Proof. The statement (ii) trivially follows from (i). To verify (i), we may assume that X is connected and the $\mathcal{O}_X$-module Ω_X^1 is free over a nonempty Zariski open subset of X, and it suffices to show that if $f_1, \ldots, f_n$ are elements of $\mathcal{F}^\nabla(X)$ linearly independent over $\mathfrak{c}(X)$ and $g_1, \ldots, g_n$ are elements of $\mathcal{O}(X)$ with $f_1 g_1 + \ldots + f_n g_n = 0$, then $g_1 = \ldots = g_n = 0$. Assume that $g_n \neq 0$. By Lemma 1.1.1 and the property on $\mathcal{F}^\nabla$, we may replace X by the Zariski open subset of X

over which g_n is invertible and the $\mathcal{O}_X$-module Ω^1_X is free. We may therefore assume that $g_n = 1$ and, in particular, this immediately implies the statement for $n = 1$. Assume that $n \geq 2$ and the statement is true for $n-1$. We have $\sum_{i=1}^{n-1} f_i \otimes dg_i = 0$. Let $\omega_1, \ldots, \omega_m \in \Omega^1(X)$ be a basis of Ω^1_X over $\mathcal{O}_X$, and let $dg_i = \sum_{j=1}^m h_{ij}\omega_j$ with $h_{ij} \in \mathcal{O}(X)$. It follows that, for every $1 \leq j \leq m$, one has $\sum_{i=1}^{n-1} f_i h_{ij} = 0$, and the induction hypothesis implies that $h_{ij} = 0$ for all i, j. We get $dg_i = 0$, i.e., $g_i \in \mathfrak{c}(X)$ for all $1 \leq i \leq n$, and the elements $f_1, \ldots, f_n$ are linearly dependent over $\mathfrak{c}(X)$, which contradicts the assumption. ∎

It is well known that any coherent $\mathcal{O}_X$-module which admits a connection is locally free over $\mathcal{O}_X$ (see [Bor, Ch. III]) and, in particular, an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module $\mathcal{F}$ is always a locally free $\mathcal{O}_X$-module and the canonical morphism of $\mathcal{D}_X$-modules $\mathcal{F}^\nabla \otimes_{\mathfrak{c}_X} \mathcal{O}_X \to \mathcal{F}$ is injective. For example, the structural sheaf $\mathcal{O}_X$ provided with the canonical differential is an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module. A finite direct sum of copies of the $\mathcal{D}_X$-module $\mathcal{O}_X$ is called a *trivial* $\mathcal{D}_X$-module. A $\mathcal{D}_X$-module $\mathcal{F}$ is said to be *unipotent* if there is a sequence of $\mathcal{D}_X$-submodules $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \ldots \subset \mathcal{F}^n = \mathcal{F}$ such that all of the quotients $\mathcal{F}^i/\mathcal{F}^{i-1}$ are isomorphic to the trivial $\mathcal{D}_X$-module $\mathcal{O}_X$. Such a $\mathcal{D}_X$-module $\mathcal{F}$ is automatically $\mathcal{O}_X$-coherent.

The *de Rham cohomology groups* $H^q_{\mathrm{dR}}(X, \mathcal{F})$ *of a* $\mathcal{D}_X$*-module* $\mathcal{F}$ are the hypercohomology groups of the complex $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_X$ with respect to the functor of global sections on X. If $\mathcal{F} = \mathcal{O}_X$, they are called the *de Rham cohomology groups of* X and denoted by $H^q_{\mathrm{dR}}(X)$. Notice that since the characteristic of k is zero it does not matter if we calculate the groups in the étale or the usual topology of X (see [Ber2, §4.2]). There are two spectral sequences that converge to the de Rham cohomology groups. The term $E_2^{p,q}$ in the first one is $H^p(H^q(X, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_X))$. It follows that, if $\mathcal{F}$ is $\mathcal{O}_X$-coherent and X is a union of an increasing sequence of affinoid subdomains such that each of them is Weierstrass in the next one, then $H^q(X, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^p_X) = 0$ for all $p \geq 0$ and $q \geq 1$ and, therefore, $H^q_{\mathrm{dR}}(X, \mathcal{F})$ coincide with the cohomology groups of the complex $(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_X)(X)$. The term $E_2^{p,q}$ in the second spectral sequence is $H^p(X, (\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^q_X)^{\mathrm{cl}}/\nabla(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{q-1}_X))$, and it gives rise to an exact sequence

$$0 \to H^1(X, \mathcal{F}^\nabla) \to H^1_{\mathrm{dR}}(X, \mathcal{F}) \to H^0(X, (\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X)^{\mathrm{cl}}/\nabla(\mathcal{F})) \to H^2(X, \mathcal{F}^\nabla) \ .$$

Given a $\mathcal{D}_X$-submodule $\mathcal{F}$ of $\mathfrak{N}^K_X$, we denote by $\Omega^q_{\mathcal{F},X}$ the image of the canonical injective homomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^q_X \to \Omega^q_{\mathfrak{N}^K,X}$. If $\mathcal{G}$ is another $\mathcal{D}_X$-submodule of $\mathfrak{N}^K_X$, we denote by $\mathcal{F} \cdot \mathcal{G}$ and $\mathcal{F} + \mathcal{G}$ the $\mathcal{O}_X$-submodules of $\mathfrak{N}^K_X$ locally generated by products and sums of local sections of $\mathcal{F}$ and $\mathcal{G}$, respectively. They are also $\mathcal{D}_X$-submodules of $\mathfrak{N}^K_X$ as well as their intersection $\mathcal{F} \cap \mathcal{G}$.

Assume we are given a non-Archimedean field k', a smooth k'-analytic space X', a filtered k'-algebra K', and a morphism of analytic spaces $\varphi : X' \to X$ and a homomorphism of filtered algebras $K \to K'$ over an isometric embedding of fields $k \hookrightarrow k'$. Assume also we are given $\mathcal{D}_X$-submodule $\mathcal{F}$ of $\mathfrak{N}_X^K$ such that the pullback $\varphi^*(f)$ is well defined for all local sections f of $\mathcal{F}$ (for example, this assumption is true if k' is a subfield of $\widehat{k}^{\mathrm{a}}$.). In this case we denote by $\varphi^{\#}(\mathcal{F})$ the image of the canonical homomorphism of $\mathcal{D}_{X'}$-modules $\varphi^*(\mathcal{F}) \to \mathfrak{N}_{X'}^{K'}$, i.e., the $\mathcal{O}_{X'}$-submodule of $\mathfrak{N}_{X'}^{K'}$ generated by the local sections $\varphi^*(f)$, where f is a local section of $\mathcal{F}$. (We want to note that even under the assumptions that k' is finite over k and $K' = K \otimes_k k'$ we do not know if the canonical homomorphism $\varphi^*(\mathfrak{N}_X^K) \to \mathfrak{N}_{X'}^{K'}$ is injective.)

Let X be a rig-smooth k-analytic space. A *D_X-module* on X is an $\mathcal{O}(X)$-submodule $M \subset \mathfrak{N}^K(X)$ such that $dM \subset \Omega^1_M(X)$, where $\Omega^q_M(X)$, $q \geq 0$, denotes the image of $M \otimes_{\mathcal{O}(X)} \Omega^q(X)$ in $\Omega^q_{\mathfrak{N}^K}(X)$. For example, $\mathcal{O}^K(X)$ is a D_X-module and, if X is smooth and Ω^1_X is a free $\mathcal{O}_X$-module then, for every $\mathcal{D}_X$-submodule $\mathcal{F}$ of $\mathfrak{N}^K$, $\mathcal{F}(X)$ is a D_X-module. For D_X-modules M and N on X, we denote by $M \cdot N$ and $M + N$ the $\mathcal{O}(X)$-submodule of $\mathfrak{N}^K(X)$ generated by products and sums of elements of M and N, respectively. They are also D_X-modules. Given a morphism of rig-smooth k-analytic spaces $\varphi : Y \to X$ and a D_X-module M on X, we denote by $\varphi^{\#}(M)$ the $\mathcal{O}(Y)$-submodule of $\mathfrak{N}^K(Y)$ generated by the functions $\varphi^*(f)$ for $f \in M$. It is a D_Y-module on Y.

An open subset Y of the affine line $\mathbf{A}^1_X = X \times \mathbf{A}^1$ over X is said to be *discoid over* X if every point $y \in Y$ has an open neighborhood in Y of the form $\mathcal{U} \times D$, where $\mathcal{U}$ is an open neighborhood of the image of y in X and D is an open disc in $\mathbf{A}^1$ with center at zero. A morphism $\varphi : Y \to X$ is said to be *discoid of dimension* 1 if there is an isomorphism of Y with an open subset of $\mathbf{A}^1_X$ which is discoid over X. A morphism $\varphi : Y \to X$ is said to be *discoid of dimension* n if it is a composition of n discoid morphisms of dimension 1.

1.3.2. Proposition. *Let $\varphi : Y \to X$ be a surjective discoid morphism of rig-smooth k-analytic spaces. Then for any D_X-module M on X the canonical homomorphism of complexes $\Omega^{\cdot}_M(X) \to \Omega^{\cdot}_{\varphi^{\#}M}(Y)$ is a homotopy equivalence.*

Proof. We may assume that $Y \subset \mathbf{A}^1_X$. We say that a sequence $\{\omega_j\}_{j\geq 0}$ of elements of $\Omega^n_M(X)$, $n \geq 0$, is *φ-bounded* if there exist elements $\{f_i\}_{1\leq i\leq m} \subset M$ and $\{\omega_{ij}\}_{1\leq i\leq m, j\geq 0} \subset \Omega^n(X)$ such that, for any $j \geq 0$, one has $\omega_j = \sum_{i=1}^m f_i\omega_{ij}$, and, for any affinoid domain $U \subset X$ and any closed disc $E \subset \mathbf{A}^1$ of radius $t > 0$ with center at zero for which $U \times E \subset Y$ and for any $1 \leq i \leq m$, one has $||\omega_{ij}||_U t^j \to 0$ as $j \to \infty$. Here $||\ ||_U$ is a Banach norm on the finite $\mathcal{A}$-module of one-differentials $\Omega^1_{\mathcal{A}}$, where $U = \mathcal{M}(\mathcal{A})$. (Proposition 2.1.5 from [Ber1] implies that the equivalence class of the

Banach norm is uniquely defined.)

It follows easily from the definition of $\varphi^\# M$ that each element $f \in \varphi^\# M$ (resp. $\omega \in \Omega^n_{\varphi^\# M}(Y)$ for $n \geq 1$) has a unique representation as a sum $\sum_{j=0}^\infty T^j g_j$ (resp. $\sum_{j=0}^\infty T^j \eta_j + \sum_{j=0}^\infty T^j \xi_j \wedge dT$), where the sequences $\{g_j\}_{j\geq 0} \subset M$ (resp. $\{\eta_j\}_{j\geq 0} \subset \Omega^n_M(X)$ and $\{\xi_j\}_{j\geq 0} \subset \Omega^{n-1}_M(X)$) are φ-bounded. Moreover, such a sum always defines an element of $\varphi^\# M$ (resp. $\Omega^n_{\varphi^\# M}(Y)$).

Let I denote the homomorphism from the formulation, and let R denote the homomorphism of complexes $\Omega^\cdot_{\varphi^\# M}(Y) \to \Omega^\cdot_M(X)$ that takes the above element $f \in \varphi^\# M$ (resp. $\omega \in \Omega^n_{\varphi^\# M}(Y)$ for $n \geq 1$) to the element $g_0 \in M$ (resp. $\eta_0 \in \Omega^n_M(X)$). One clearly has $R \circ I = 1$. Let B be the homomorphism $\Omega^n_{\varphi^\# M}(Y) \to \Omega^{n-1}_{\varphi^\# M}(Y)$, $n \geq 1$, that takes the above element ω to the element $(-1)^{n-1} \sum_{j=0}^\infty T^{j+1} \frac{\xi_j}{j+1}$. (The sequence $\{\frac{\xi_j}{j+1}\}_{j\geq 0}$ is φ-bounded, and so the latter sum is an element of $\Omega^{n-1}_{\varphi^\# M}(Y)$.) It is easy to verify that $B \circ d + d \circ B = 1 - I \circ R$, and the lemma follows. ∎

1.3.3. Corollary. *Any unipotent $\mathcal{D}_D$-module on an open polydisc D with center at zero is trivial.*

Proof. Let $\mathcal{F}$ be a unipotent $\mathcal{D}_D$-module of rank n. *We claim that the following sequence is exact*

$$\mathcal{F}(D) \xrightarrow{\nabla} (\mathcal{F} \otimes_{\mathcal{O}_D} \Omega^1_D)(D) \xrightarrow{\nabla} (\mathcal{F} \otimes_{\mathcal{O}_D} \Omega^2_D)(D) \xrightarrow{\nabla} \dots$$

and $\mathcal{F}^\nabla(D)$ is a vector space over k of dimension n. Indeed, if $n = 1$, the claim follows from Proposition 1.3.2 and is easily extended by induction for all n. It follows that $\mathcal{F}^\nabla$ is a free $\mathfrak{c}_D$-module of rank n and, therefore, there is an isomorphism of $\mathcal{D}_D$-modules $\mathcal{F}^\nabla \otimes_{\mathfrak{c}_D} \mathcal{O}_D \xrightarrow{\sim} \mathcal{F}$. ∎

In particular, the following sequence is exact (Classical Poincaré Lemma):

$$0 \longrightarrow k \longrightarrow \mathcal{O}(D) \xrightarrow{d} \Omega^1(D) \xrightarrow{d} \Omega^2(D) \xrightarrow{d} \dots$$

1.4. Logarithms. The first example of a closed one-form which has no a primitive in the class of analytic functions is provided by the one-form $\frac{dT}{T}$ on the analytic multiplicative group $\mathbf{G}_\mathrm{m} = \mathbf{A}^1 \backslash \{0\}$. In comparison with the classical situation, the space $\mathbf{G}_\mathrm{m}$ is simply-connected and, in fact, $H^1(\mathbf{G}_\mathrm{m}, \mathfrak{c}_{\mathbf{G}_\mathrm{m}}) = 0$. This means that, if an integration theory we are looking for exists, the one-form $\frac{dT}{T}$ must have a primitive f in a bigger class of functions on the whole space $\mathbf{G}_\mathrm{m}$. Let us normalize it by the condition $f(1) = 0$. Furthermore, let m (resp p_i) denote the multiplication morphism (resp. the projection to the i-th coordinate) $\mathbf{G}_\mathrm{m} \times \mathbf{G}_\mathrm{m} \to \mathbf{G}_\mathrm{m}$. One has $m^*(\frac{dT}{T}) = p_1^*(\frac{dT}{T}) + p_2^*(\frac{dT}{T})$. Again, if we want the primitive to behave functorially, it should satisfy the relation $m^*(f) = p_1^*(f) + p_2^*(f)$ which in the usual form is written as $f(a \cdot b) = f(a) + f(b)$.

A *branch of the logarithm* (*over* K) is an $\mathfrak{N}^{K,1}$-analytic function f on $\mathbf{G}_{\rm m}$ such that $df = \frac{dT}{T}$ and $m^*(f) = p_1^*(f) + p_2^*(f)$. It is clear that the restriction of such a function f to $D(1;1)$, the open disc with center at one and of radius one, coincides with the usual logarithm $\log(T)$ defined by the convergent power series $-\sum_{i=1}^{\infty} \frac{(1-T)^i}{i}$. Let us try to understand in simpler terms what a branch of the logarithm is.

First of all, such a function f gives rise to a homomorphism of abelian $k^* \to K^1$ whose restriction to the subgroup $k^* \cap D(1;1)$ coincides with $\log(T)$. More generally, if k' is a finite extension of k, the pullback of f to $\mathbf{G}_{\rm m}\widehat{\otimes}k'$ defines a similar homomorphism $k'^* \to K \otimes_k k'$, and these homomorphisms for different k''s are compatible. Thus, f defines a $\mathrm{Gal}(k^{\rm a}/k)$-equivariant homomorphism of abelian groups $\lambda_f : k^{{\rm a}*} \to K^1 \otimes_k k^{\rm a}$ whose restriction to the open unit disc with center at one coincides with $\log(T)$. A homomorphism $\lambda : k^{{\rm a}*} \to K^1 \otimes_k k^{\rm a}$ with the latter properties will be called a *logarithmic character with values in* K.

1.4.1. Lemma. *(i) The correspondence $f \mapsto \lambda_f$ gives rise to a bijection between the set of branches of the logarithm over K and the set of logarithmic characters with values in K;*

(ii) the analyticity set of any branch of the logarithm f is the complement of the set $S(\mathbf{G}_{\rm m}) = \{p(E(0;r)) \big| r > 0\} \subset \mathbf{G}_{\rm m}$, where $p(E(0;r))$ is the maximal point of the closed disc $E(0;r)$ of radius r with center at zero.

Proof. (i) That the map considered is injective is trivial. Let $\lambda : k^{{\rm a}*} \to K^1 \otimes_k k^{\rm a}$ be a logarithmic character. For a point $x \in (\mathbf{G}_{\rm m})_0$, let D be the open disc of radius $|T(x)|$ with center at x, i.e., the open set $\{y \in \mathbf{G}_{\rm m} \big| |P(y)| < |T(x)|^n\}$, where $P(T) = T^n + a_1T^{n-1} + \ldots + a_n$ is the monic polynomial which generates the maximal ideal of $k[T]$ that corresponds to x. Furthermore, let k' be a finite Galois extension of k such that all points from $\varphi^{-1}(x)$ are k'-rational, where φ is the morphism $\mathbf{G}_{\rm m}\widehat{\otimes}_k k' \to \mathbf{G}_{\rm m}$. Then $\varphi^{-1}(D) = \coprod_{i=1}^m D_i$, where each D_i is of the form $D(b; |T(b)|)$ for some point $b \in \varphi^{-1}(x)$. Let f_i be the k'-analytic function $\lambda(b) + \log(\frac{T}{b})$ on D_i. The properties of λ and $\log(T)$ imply that f_i does not depend on the choice of the point $b \in \varphi^{-1}(x) \cap D_i$. It follows that the analytic function on $\varphi^{-1}(D)$, defined by the family $\{f_i\}_{1 \le i \le m}$, is invariant under the action of the Galois group of k' over k and, therefore, it is the pullback of some analytic function f_D on D. It is easy to see that the $\mathfrak{N}^{K,1}$-analytic function on $\mathbf{G}_{\rm m}$, defined by the family $\{f_D\}$, is a branch of the logarithm and $\lambda_f = \lambda$.

(ii) By the construction, the complement of the analyticity set of f is contained in the set $S(\mathbf{G}_{\rm m})$. To show that they coincide, it suffices to prove that there is no a connected open subset $\mathcal{U} \subset \mathbf{G}_{\rm m}$ which is strictly bigger than $D(1;1)$ and such that $f\big|_{\mathcal{U}} \in \mathcal{O}(\mathcal{U})$, and to verify the latter we may assume that the field k is algebraically closed. Assume such a subset $\mathcal{U}$ exists. Recall ([Ber1, §4]) that the topological boundary of $D(1;1)$ in $\mathbf{G}_{\mathrm{m}}$ is the point $x = p(E(1;1))$ and that the set $\mathcal{U}$ is arc-wise connected. It follows that $x \in \mathcal{U}$. Let V be a connected affinoid neighborhood of x in $\mathcal{U}$. One has $V = E(1;r) \backslash \coprod_{i=1}^n D(a_i; r_i)$, where $r > 1$, $0 < r_i < 1$ and $a_1, \dots, a_n \in k^*$ are such that $|a_i - 1| = 1$ and $|a_i - a_j| = 1$ for all $1 \le i, j \le n$. We see that the set V contains all but finitely many roots of unity of degree prime to the characteristic of the residue field $\widetilde{k}$, i.e., the function f has infinitely many zeroes on V, which is impossible. ∎

Given a logarithmic character $\lambda : k^{\mathrm{a}*} \to K^1 \otimes_k k^{\mathrm{a}}$, the corresponding branch of the logarithm is denoted by Log^λ. Given an embedding of non-Archimedean fields $k \hookrightarrow k'$ such that the subfield of the elements of k' algebraic over k is dense in k', the pullback of Log^λ under the induced morphism $\mathbf{G}_{\mathrm{m}} \widehat{\otimes}_k k' \to \mathbf{G}_{\mathrm{m}}$ is a branch of the logarithm on $\mathbf{G}_{\mathrm{m}} \widehat{\otimes}_k k'$ over $K' = K \otimes_k k'$. The corresponding logarithmic character $(k'^{\mathrm{a}})^* \to K'^1 \otimes_{k'} k'^{\mathrm{a}} = K^1 \otimes_k k'^{\mathrm{a}}$ is the unique one that extends λ. (Notice that for any element $\alpha \in (k'^{\mathrm{a}})^*$ there exist $n \ge 1$ and $\beta, \gamma \in (k'^{\mathrm{a}})^*$ such that β is algebraic over k, $|\gamma - 1| < 1$ and $\alpha^n = \beta\gamma$.)

Let now X be a rig-smooth k-analytic space. An invertible analytic function $f \in \mathcal{O}(X)^*$ defines a morphism $f : X \to \mathbf{G}_{\mathrm{m}}$, and we denote by $\mathrm{Log}^\lambda(f)$ the $\mathfrak{N}^{K,1}$-analytic function $f^*\mathrm{Log}^\lambda$ on X. If $f \in \mathcal{O}(X)$ is such that $|f(x) - 1| < 1$ for all $x \in X$, then $\mathrm{Log}^\lambda(f)$ is analytic (it coincides with $\log(f) = f^* \log$). Given $f, g \in \mathcal{O}(X)^*$, one has $\mathrm{Log}^\lambda(fg) = \mathrm{Log}^\lambda(f) + \mathrm{Log}^\lambda(g)$. One also has $d\mathrm{Log}^\lambda(f) = \frac{df}{f}$. In particular, for every $n \ge 0$ the $\mathcal{O}(X)$-submodule of $\mathfrak{N}^K(X)$, generated by elements of the form $\alpha \mathrm{Log}^\lambda(f_1) \cdot \ldots \cdot \mathrm{Log}^\lambda(f_i)$ with $0 \le i \le n$, $\alpha \in K^{n-i}$ and $f_j \in \mathcal{O}(X)^*$ is a D_X-module. It will be denoted by $L^{\lambda,n}(X)$. The D_X-module $L^\lambda(X) = \cup_{n=0}^n L^{\lambda,n}(X)$ is a filtered $\mathcal{O}(X)$-algebra. If X is smooth, the $\mathcal{O}_X$-submodule of $\mathfrak{N}_X^{K,n}$ associated to the presheaf $U \mapsto L^{\lambda,n}(U)$ is a $\mathcal{D}_X$-submodule denoted by $\mathcal{L}_X^{\lambda,n}$. One also has a $\mathcal{D}_X$-submodule $\mathcal{L}_X^\lambda = \varinjlim \mathcal{L}_X^{\lambda,n}$ of $\mathfrak{N}_X^K$ which is a filtered $\mathcal{O}_X$-subalgebra.

1.4.2. Lemma. *Let X be a connected open subset of $\mathbf{G}_{\mathrm{m}}$ which contains a nonempty open annulus with center at zero, and let f be a function from $L^\lambda(X)$ of the form $\sum_{i=0}^n f_i \mathrm{Log}^\lambda(T)^i$ with $f_i \in \mathcal{O}^K(X)$. If $f\big|_{\mathcal{U}} = 0$ for a nonempty open subset $\mathcal{U} \subset X$, then $f_i = 0$ for all $0 \le i \le n$.*

Proof. Let $\{e_\mu\}_\mu$ be a basis of the k-vector space K, and set $f_i = \sum_\mu f_{i,\mu} e_\mu$. Notice that the latter representation is unique and that we have to show that $f_{i,\mu} = 0$ for all i and μ. If $n = 0$, the required fact follows from the uniqueness property of analytic functions on the affine line. Assume $n \ge 1$ and that the fact is true for $n-1$, and let l be the number of μ's with $f_{n,\mu} \ne 0$. The fact is evidently true if $l = 0$, and so assume that $l \ge 1$ and that it is true for $l-1$, and let ν be such that

$f_{n,\nu} \neq 0$. If $f_{n,\nu}$ is not a constant, we replace X by the open subset where it does not vanish and then replace f by $f/f_{n,\nu}$, and so we may assume that $f_{n,\nu}$ is a nonzero constant. Notice that X is still connected and contains a nonempty open annulus with center at zero. Consider the derivative

$$f' = \sum_{\mu} \left(f'_{n,\mu} \mathrm{Log}^{\lambda}(T)^n + \sum_{i=0}^{n-1} \left(f'_{i,\mu} + \frac{i+1}{T} f_{i+1,\mu} \right) \mathrm{Log}^{\lambda}(T)^i \right) e_{\mu} \ .$$

The induction hypothesis implies that $f'_{n-1,\nu} + \frac{n}{T} f_{n,\nu} = 0$, which is impossible since $f_{n,\nu}$ is a nonzero constant and X contains a nonempty open annulus with center at zero. ∎

1.4.3. Examples. (i) Let k be a non-Archimedean field over $\mathbf{Q}_p$ such that the residue field $\widetilde{k}$ is algebraic over $\mathbf{F}_p$ and the group $|k^*|/|\mathbf{Q}_p^*|$ is torsion. Then a logarithmic character $\lambda : (k^{\mathrm{a}})^* \to K^1 \otimes_k k^{\mathrm{a}}$ is uniquely determined by its value at p which belongs to K^1 and, therefore, in this case, it will be identified with that element of K^1. Let k_{Log} denote the ring of polynomials $k[\mathrm{Log}(p)]$ (in the variable $\mathrm{Log}(p)$). It is a filtered k-algebra in which k^i_{Log} is the subspace of polynomials of degree at most i. We write $\mathfrak{N}^i_X$, $\mathfrak{N}_X$, $\mathcal{C}^i_X$ and $\mathcal{C}_X$ instead of $\mathfrak{N}^{k_{\mathrm{Log}},i}_X$, $\mathfrak{N}^{k_{\mathrm{Log}}}_X$, $\mathcal{C}^{k_{\mathrm{Log}},i}_X$ and $\mathcal{C}^{k_{\mathrm{Log}}}_X$, respectively, and denote by Log the corresponding $\mathfrak{N}_X$-analytic function. Notice that the $\mathfrak{N}^K$-analytic function Log^{λ} is the image of Log under the homomorphism of the sheaves $\mathfrak{N}_{\mathbf{G}_{\mathrm{m}}} \to \mathfrak{N}^K_{\mathbf{G}_{\mathrm{m}}}$ which corresponds to the homomorphism of filtered k-algebras $k_{\mathrm{Log}} \to K$ that takes $\mathrm{Log}(p)$ to λ. The main example of a field with the above property, considered in the paper, is a closed subfield of $\mathbf{C}_p$, the completion of an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. (Recall that, by the Ax-Sen-Tate theorem (see [Ax]), every closed subfield k of $\mathbf{C}_p$ coincides with the closure of $k \cap \overline{\mathbf{Q}}_p$ in $\mathbf{C}_p$.) Furthermore, if k possesses the above property and X is a k-analytic space, then the field $\mathcal{H}(x)$ of every point $x \in X_{st}$ possesses the same property.

(ii) Let k be the field of Laurent power series $\mathbf{C}((T))$. Each element of k^* has a unique representation in the form aT^ng with $a \in \mathbf{C}^*$, $n \in \mathbf{Z}$ and $|g-1| < 1$ (i.e., $g = 1 + \sum_{i=1}^{\infty} a_i T^i$). It follows easily that any logarithmic character $\lambda : (k^{\mathrm{a}})^* \to K^1 \otimes_k k^{\mathrm{a}}$ is uniquely determined by the element $\lambda(T)$ and the induced homomorphism $\mathbf{C}^* \to K^1$. A natural class of the latter consists of homomorphisms of the form $a \mapsto \log|a|$, where log is a branch of the real logarithm.

1.4.4. Remark. If an object depends on the algebra K, we indicate K in its notation (e.g., $\mathcal{C}^K_X$, $\mathfrak{N}^K_X$). If it also depends on a logarithmic character λ (or an element $\lambda \in K^1$ as in Example 1.4.3(i)), we only indicate λ having in mind the algebra K it is related to (e.g., $\mathcal{L}^{\lambda}_X$, $\mathcal{L}^{\lambda,n}_X$). If, in Example 1.4.3(i), $K = k_{\mathrm{Log}}$ with $K^i = k^i_{\mathrm{Log}}$ and $\lambda = \mathrm{Log}(p)$, we omit the reference to K and λ in the notations. In principle, such a system of notations is ambiguous, but since we use λ only as a variable and do not use its place for a concrete value which is an integer, a notation like $\mathcal{L}^1_X$ should be understood as the first member of the filtration on $\mathcal{L}_X$ (for $K = k_{\mathrm{Log}}$ and $\lambda = \mathrm{Log}(p)$) and not as the whole $\mathcal{O}^K$-algebra associated to $\lambda = 1$.

1.5. Logarithmic Poincaré Lemma. Let X be a k-analytic space. An open subset $Y \subset X \times \mathbf{G}_{\mathrm{m}}$ is said to be *annular over* X if it has connected fibers over X and every point $y \in Y$ has an open neighborhood in Y of the form $\mathcal{U} \times B$, where $\mathcal{U}$ is an open neighborhood of the image of y in X and B is an open annulus in $\mathbf{G}_{\mathrm{m}}$ with center at zero. A morphism $\varphi : Y \to X$ is said to be *annular of dimension* 1 if there is an isomorphism of Y with an open subset of $X \times \mathbf{G}_{\mathrm{m}}$ which is annular over X. A morphism $\varphi : Y \to X$ is said to be *semi-annular of dimension* (m, n) if it is a composition (in any order) of m annular and n discoid morphisms of dimension 1. If $n = 0$, it is called *annular of dimension* m. If $X = \mathcal{M}(k)$, the k-analytic space Y itself is said to be semi-annular, annular or discoid. If Y is of dimension (m, n), then every point of it has an open neighborhood isomorphic to a direct product of an open poly-annulus of dimension m and an open disc of dimension n.

In this subsection we consider the following situation. Let $\varphi : Y \to X$ be a semi-annular morphism between rig-smooth k-analytic spaces, and assume we are given an increasing sequence of D_X-modules $P^0 \subset P^1 \subset P^2 \subset \ldots \subset \mathfrak{N}^K(X)$ with $L^{\lambda,i}(X) \subset P^i$ and $P^i{\cdot}L^{\lambda,j}(X) \subset P^{i+j}$ for all $i, j \geq 0$. We set $M^n = \sum_{i=0}^n \varphi^\#(P^i){\cdot}L^{\lambda,n-i}(Y)$. Our purpose is to relate the complexes $\Omega^\cdot_{P^n}(X)$ and $\Omega^\cdot_{M^n}(Y)$. For this we denote by I_{M^n} and I_{P^n} the canonical morphisms of complexes $\Omega^\cdot_{M^n}(Y) \to \Omega^\cdot_{M^{n+1}}(Y)$ and $\Omega^\cdot_{P^n}(X) \to \Omega^\cdot_{P^{n+1}}(X)$, respectively, and by J_n the canonical morphism $\Omega^\cdot_{P^n}(X) \to \Omega^\cdot_{M^n}(Y)$. We have a commutative diagram of morphisms of complexes

$$\begin{array}{ccc} \Omega^\cdot_{M^n}(Y) & \xrightarrow{I_{M_n}} & \Omega^\cdot_{M^{n+1}}(Y) \\ \uparrow J_n & & \uparrow J_{n+1} \\ \Omega^\cdot_{P^n}(X) & \xrightarrow{I_{P_n}} & \Omega^\cdot_{P^{n+1}}(X) \end{array}$$

1.5.1. Proposition. *Assume that φ is surjective. Then there is a compatible system of morphisms $R_n : \Omega^\cdot_{M^n}(Y) \to \Omega^\cdot_{P^n}(X)$ such that the composition $R_n \circ J_n$ is identical on $\Omega^\cdot_{P^n}(X)$ and the morphisms I_{M^n} and $J_{n+1} \circ I_{P^n} \circ R_n$ are homotopy equivalent.*

Proof. It suffices to consider the cases when Y is a discoid subset of $X \times \mathbf{A}^1$ or an annular subset of $Y \times \mathbf{G}_{\mathrm{m}}$. In the first case, every invertible analytic function on Y is of the form $f{\cdot}g$ with $f \in \mathcal{O}(X)^*$ and $g \in \mathcal{O}(Y)^*$ such that $|g(y) - 1| < 1$ for all $y \in Y$. This implies that $L^{\lambda,i}(Y) = \varphi^\#(L^{\lambda,i}(X))$ and $M^i = \varphi^\#(P^i)$, and the required fact follows from Proposition 1.3.2 (with R constructed in its proof). Thus, assume Y is an annular subset of $X \times \mathbf{G}_{\mathrm{m}}$.

We say that a sequence $\{\omega_j\}_{-\infty<j<\infty}$ of elements of $\Omega^q_{P^n}(X)$ is *φ-bounded* if there exist elements $\{f_i\}_{1\leq i\leq m} \subset P^n$ and $\{\omega_{ij}\}_{1\leq i\leq m, -\infty<j<\infty} \subset \Omega^q(X)$ such that, for any $-\infty < j < \infty$, one has

$\omega_j = \sum_{i=1}^m f_i\omega_{ij}$, and, for any affinoid domain $U \subset X$ and any closed annulus $A \subset \mathbf{G}_\mathrm{m}$ with center at zero whose both radii are $t > 0$ and for which $U \times A \subset Y$, one has $||\omega_{ij}||_U t^j \to 0$ as $j \to \pm\infty$, where $||\ ||_U$ is as in the proof of Proposition 1.3.2. Notice that every analytic function on Y has a unique representation in the form $\sum_{j=-\infty}^{\infty} T^j f_j$ where $\{f_j\}_{-\infty<j<\infty}$ is a φ-bounded sequence of elements of $\mathcal{O}(X)$. Notice also that every invertible analytic function on Y is of the form $T^m fg$ with $m \in \mathbf{Z}$, $f \in \mathcal{O}(X)^*$ and $g \in \mathcal{O}(Y)^*$ such that $|g(y) - 1| < 1$ for all $y \in Y$. It follows that every element $f \in M^n$ (resp. $\omega \in \Omega^q_{M^n}(Y)$ for $q \geq 1$) can be uniquely represented as a sum $\sum_{i=0}^n \sum_{j=-\infty}^{\infty} T^j \mathrm{Log}^\lambda(T)^i g_{ij}$ (resp. $\sum_{i=0}^n \sum_{j=-\infty}^{\infty} T^j \mathrm{Log}^\lambda(T)^i \eta_{ij} + \sum_{i=0}^n \sum_{j=-\infty}^{\infty} T^j \mathrm{Log}^\lambda(T)^i \xi_{ij} \wedge dT$), where for each $0 \leq i \leq n$ the sequences $\{g_{ij}\}_{-\infty<j<\infty} \subset P^{n-i}$ (resp. $\{\eta_{ij}\}_{-\infty<j<\infty} \subset \Omega^q_{P^{n-i}}(X)$ and $\{\xi_{ij}\}_{-\infty<j<\infty} \subset \Omega^{q-1}_{P^{n-i}}(X)$) are φ-bounded.

Let R_n be the morphism $\Omega^\cdot_{M^n}(Y) \to \Omega^\cdot_{P^n}(X)$ that takes an element $f \in M^n$ (resp. $\omega \in \Omega^q_{M^n}(Y)$ for $q \geq 1$), represented in the above form, to $g_{00} \in P^n$ (resp. $\eta_{00} \in \Omega^q_{P^n}(X)$). Furthermore, let $\mathcal{Z}^\cdot_n$ be the subcomplex of $\Omega^\cdot_{M^n}(Y)$ such that $\mathcal{Z}^0_n$ (resp. $\mathcal{Z}^q_n$ for $q \geq 1$) consists of the sums $\sum_{i=0}^n \mathrm{Log}^\lambda(T)^i g_i$ with $g_i \in P^{n-i}$ (resp. $\sum_{i=0}^n \mathrm{Log}^\lambda(T)^i \eta_i + \sum_{i=0}^n \mathrm{Log}^\lambda(T)^i \xi_i \wedge \frac{dT}{T}$ with $\eta_i \in \Omega^q_{P^{n-i}}(X)$ and $\xi_i \in \Omega^{q-1}_{P^{n-i}}(X)$).

1.5.2. Lemma. *The canonical morphism $\mathcal{Z}^\cdot_n \to \Omega^\cdot_{M^n}(Y)$ is a homotopy equivalence.*

Proof. Let N denote the morphism considered, and let Res denote the morphism in the opposite direction that takes an element $f \in M^n$ (resp. $\Omega^q_{M^n}(Y)$ for $q \geq 1$), represented in the above form, to $\sum_{i=0}^n \mathrm{Log}^\lambda(T)^i g_{i0}$ (resp. $\sum_{i=0}^n \mathrm{Log}^\lambda(T)^i \eta_{i0} + \sum_{i=0}^n \mathrm{Log}^\lambda(T)^i \xi_{i,-1} \wedge \frac{dT}{T}$). One evidently has $Res \circ N = 1$. To prove the lemma, we have to construct a k-linear map $C : \Omega^\cdot_{M^n}(Y) \to \Omega^\cdot_{M^n}(Y)$ of degree -1 with $C \circ d + d \circ C = 1 - N \circ Res$.

First of all, for integers $i \geq 0$ and $j \neq -1$ we introduce as follows a polynomial in one variable

$$h_{i,j}(T) = \frac{1}{j+1}T^i - \frac{i}{(j+1)^2}T^{i-1} + \frac{i(i-1)}{(j+1)^3}T^{i-2} - \ldots + (-1)^i \frac{i!}{(j+1)^{i+1}}$$

One has $h'_{i,j} + (j+1)h_{i,j} = T^i$ and, therefore, $T^{j+1}h_{i,j}(\mathrm{Log}^\lambda(T))$ is a primitive of $T^j \mathrm{Log}^\lambda(T)^i$. If $i \geq 1$, one also has $(j+1)h_{i,j} + ih_{i-1,j} = T^i$. We now define for an element $\omega \in \Omega^q_{M^n}(Y)$, $q \geq 1$, represented in the above form, the element $C(\omega)$ as follows

$$C(\omega) = (-1)^{q-1} \sum_{i=0}^n \sum_{\substack{j=-\infty \\ j\neq -1}}^{\infty} T^{j+1} h_{i,j}(\mathrm{Log}^\lambda(T))\xi_{ij}$$

It is easy to check that the sum on the right hand side is a well defined element of $\Omega^{q-1}_{M^n}(Y)$, and that $C \circ d + d \circ C = 1 - N \circ Res$. ∎

Consider now the following diagram of morphisms of complexes

$$\begin{array}{ccc} \mathcal{Z}^{\cdot}_n & \stackrel{I'_{M_n}}{\longrightarrow} & \mathcal{Z}^{\cdot}_{n+1} \\ \downarrow R'_n & & \uparrow J'_{n+1} \\ \Omega^{\cdot}_{P^n}(X) & \stackrel{I_{P_n}}{\longrightarrow} & \Omega^{\cdot}_{P^{n+1}}(X) \end{array}$$

where I'_{M^n} denotes the canonical morphism $\mathcal{Z}_n \to \mathcal{Z}_{n+1}$, R'_n denotes the restriction of R_n to $\mathcal{Z}^{\cdot}_n$, and J'_n denotes the canonical morphism $\Omega^{\cdot}_{P^n}(X) \to \mathcal{Z}^{\cdot}_n$. The proposition follows from the following lemma.

1.5.3. Lemma. *The morphisms I'_{M^n} and $J'_{n+1} \circ I_{P^n} \circ R'_n$ are homotopy equivalent.*

Proof. To prove the lemma, we have to construct a k-linear map $C : \mathcal{Z}^{\cdot}_n \to \mathcal{Z}^{\cdot}_{n+1}$ of degree -1 with $C \circ d + d \circ C = I'_{M^n} - J'_{n+1} \circ I_{P^n} \circ R'_n$. If $\omega = \sum_{i=0}^n \mathrm{Log}^\lambda(T)^i \eta_i + \sum_{i=0}^n \mathrm{Log}^\lambda(T)^i \xi_i \wedge \frac{dT}{T} \in \mathcal{Z}^q_n$, $q \geq 1$, we define

$$C(\omega) = (-1)^{q-1} \sum_{i=0}^n \mathrm{Log}^\lambda(T)^{i+1} \frac{\xi_i}{i+1}$$

The required equality is easily verified. ∎

1.5.4. Corollary. *Let X be a semi-annular k-analytic space. Then the morphism of complexes $\Omega^{\cdot}_{L^{\lambda,n}}(X) \to \Omega^{\cdot}_{L^{\lambda,n+1}}(X)$ is homotopy equivalent to zero and, in particular, there is an exact sequence*

$$0 \longrightarrow K \longrightarrow L^\lambda(X) \stackrel{d}{\longrightarrow} \Omega^1_{L^\lambda}(X) \stackrel{d}{\longrightarrow} \Omega^2_{L^\lambda}(X) \stackrel{d}{\longrightarrow} \dots$$ ∎

The following consequences of Proposition 1.5.1 are formulated in the form convenient for applications in §3 and §7, respectively.

Let X be a semi-annular k-analytic space. Given a subgroup $G \subset \mathcal{O}(X)^*$ with $\mathcal{O}(X)^* = G{\cdot}k^*$ and $\mathrm{Log}^\lambda(\alpha) \in k^*$ for all $\alpha \in G \cap k^*$, let $L^\lambda_0(X)$ denote the filtered $\mathcal{O}_X$-subalgebra of $L^\lambda(X)$ generated over $\mathcal{O}(X)$ by the logarithms of functions from G . One can easily see that $L^\lambda_0(X)$ is a filtered D_X-algebra, and the canonical homomorphism of filtered D_X-algebras $L^\lambda_0(X) \otimes_k K \to L^\lambda(X)$ is an isomorphism.

1.5.5. Corollary. *In the above situation, the morphism of complexes $\Omega^{\cdot}_{L_0^{\lambda,n}}(X) \to \Omega^{\cdot}_{L_0^{\lambda,n+1}}(X)$ is homotopy equivalent to zero and, in particular, there is an exact sequence*

$$0 \longrightarrow k \longrightarrow L^\lambda_0(X) \stackrel{d}{\longrightarrow} \Omega^1_{L^\lambda_0}(X) \stackrel{d}{\longrightarrow} \Omega^2_{L^\lambda_0}(X) \stackrel{d}{\longrightarrow} \dots$$ ∎

Let X be a smooth k-analytic space whose sheaf of one-differentials Ω^1_X is free over $\mathcal{O}_X$, and assume we are given an increasing sequence of $\mathcal{D}_X$-submodules $\mathcal{S}^0 \subset \mathcal{S}^1 \subset \dots \subset \mathcal{S}^n \subset \mathfrak{N}^K_X$

with the properties that $\mathcal{L}_X^{\lambda,i} \subset \mathcal{S}^i$ and $\mathcal{S}^i{\cdot}\mathcal{L}_X^{\lambda,j} \subset \mathcal{S}^{i+j}$ for all $0 \le i \le n$ and $0 \le j \le n-i$. Notice that the assumption on Ω^1_X guarantees that $\mathcal{S}^i(X)$ are D_X-modules. Furthermore, let $\varphi : Y \to X$ be a surjective semi-annular morphism. Consider the following D_Y-modules: $M = \sum_{i=0}^n \varphi^\#(\mathcal{S}^i(X)){\cdot}L^{\lambda,n-i}(Y)$ and $N = \sum_{i=0}^n \varphi^\#(\mathcal{S}^i(X)){\cdot}L^{\lambda,n+1-i}(Y)$.

1.5.6. Corollary. *There is a morphism of complexes $R : \Omega^\cdot_M(Y) \to \Omega^\cdot_{\mathcal{S}^n}(X)$ whose composition with the canonical morphism $\Omega^\cdot_{\mathcal{S}^n}(X) \to \Omega^\cdot_M(Y)$ is identical on $\Omega^\cdot_{\mathcal{S}^n}(X)$ and such that the morphisms I and $J \circ R$ from the following diagram are homotopy equivalent*

$$\begin{tikzcd} \Omega^\cdot_M(Y) \arrow[rr, "I"] \arrow[dr, "R"'] & & \Omega^\cdot_N(Y) \\ & \Omega^\cdot_{\mathcal{S}^n}(X) \arrow[ur, "J"'] & \end{tikzcd}$$

Proof. It is enough to apply Proposition 1.5.1 to the increasing sequence of D_X-modules $P^0 = \mathcal{S}^0(X) \subset \ldots \subset P^n = \mathcal{S}^n(X) \subset P^{n+1} \subset \ldots$, where $P^i = \sum_{j=0}^n \mathcal{S}^j(X){\cdot}L^{\lambda,i-j}(X)$ for $i \ge n$, and to notice that $M^n = M$ and $M^{n+1} = N$. ∎

1.6. Formulation of the main results

1.6.1. Theorem. *Given a closed subfield $k \subset \mathbf{C}_p$, a filtered k-algebra K and an element $\lambda \in K^1$, there is a unique way to provide every smooth k-analytic space X with a filtered $\mathcal{D}_X$-subalgebra $\mathcal{S}_X^\lambda \subset \mathfrak{N}_X^K$ so that the following is true:*

(a) $\mathcal{S}_X^{\lambda,0} = \mathcal{O}_X^{K,0}$;

(b) $\mathrm{Ker}(\mathcal{S}_X^{\lambda,i} \xrightarrow{d} \Omega^1_{\mathcal{S}^{\lambda,i},X}) = \mathcal{C}_X^{K,i}$;

(c) $\mathrm{Ker}(\Omega^1_{\mathcal{S}^{\lambda,i},X} \xrightarrow{d} \Omega^2_{\mathcal{S}^{\lambda,i},X}) \subset d\mathcal{S}_X^{\lambda,i+1}$;

(d) $\mathcal{S}_X^{\lambda,i+1}$ is generated by the local sections f for which df is a local section of $\Omega^1_{\mathcal{S}^{\lambda,i},X}$;

(e) $\mathrm{Log}^\lambda(T) \in \mathcal{S}^{\lambda,1}(\mathbf{G}_\mathrm{m})$;

(f) for any a morphism of smooth k-analytic spaces $\varphi : X' \to X$, one has $\varphi^\#(\mathcal{S}_X^{\lambda,i}) \subset \mathcal{S}_{X'}^{\lambda,i}$.

In the following theorem k is a closed subfield of $\mathbf{C}_p$, K is a filtered k-algebra, λ is an element of K^1, and X is a smooth k-analytic space.

1.6.2. Theorem. *(i) If X is connected, then for any nonempty open subset $\mathcal{U} \subset X$ the canonical map $\mathcal{S}^\lambda(X) \to \mathcal{S}^\lambda(\mathcal{U}) : f \mapsto f\big|_\mathcal{U}$ is injective;*

(ii) if $\mathfrak{c}(X)$ contains a finite extension k' of k and X' is X considered as a strictly k'-analytic space, then $(\mathcal{S}_X^\lambda)' = \mathcal{S}_{X'}^{\lambda'}$, where λ' is the element $\lambda \otimes 1$ of $K' = K \otimes_k k'$;

(iii) given a closed subfield k' of $\mathbf{C}_p$, a smooth k'-analytic space X', a morphism $\varphi : X' \to X$ over an isometric embedding $k \hookrightarrow k'$, a filtered k'-algebra K', and a homomorphism of filtered

algebras $K \to K'$ over the embedding $k \hookrightarrow k'$ that takes λ to an element $\lambda' \in K'^1$, one has $\varphi^\#(\mathcal{S}_X^{\lambda,i}) \subset \mathcal{S}_{X'}^{\lambda',i}$;

(iv) in the situation of (iii), if $k' = k$ and $X' = X$, then $\mathcal{S}_X^\lambda \otimes_K K' \widetilde{\to} \mathcal{S}_X^{\lambda'}$;

(v) in the situation of (iii), if $K' = K \otimes_k k'$ and $\lambda' = \lambda \otimes 1$, then $\varphi^*(\mathcal{S}_X^{\lambda,i}) \widetilde{\to} \varphi^\#(\mathcal{S}_X^{\lambda,i})$ and, if in addition $X' = X \widehat{\otimes}_k k'$, then $\varphi^\#(\mathcal{S}_X^{\lambda,i}) \widetilde{\to} \mathcal{S}_{X'}^{\lambda',i}$;

(vi) for any geometric point $\overline{x}$ of X, the stalk $\mathcal{S}_{X,\overline{x}}^{\lambda,i}$ (resp. $\mathcal{S}_{X,\overline{x}}^\lambda$) is a free $\mathcal{O}_{X,\overline{x}}$-module (resp. $\mathcal{O}_{X,\overline{x}}^K$-module).

Theorems 1.6.1 and 1.6.2(i)-(iii) will be proved in §7. In §8, we shall prove the statements (iv)-(vi) and establish more properties of the sheaves $\mathcal{S}_X^\lambda$.

1.6.3. Remarks. (i) The property (iv) of Theorem 1.6.2 implies that there is an isomorphism of filtered $\mathcal{D}_X$-algebras $\mathcal{S}_X \otimes_{k_{\mathrm{Log}}} K \widetilde{\to} \mathcal{S}_X^\lambda$ with respect to the homomorphism $k_{\mathrm{Log}} \to K$ that takes $\mathrm{Log}(p)$ to λ. In §8.3, we shall construct, for every geometric point $\overline{x}$ over a point $x \in X$, a $G_{\overline{x}/x}$-invariant filtered D-subalgebra $\mathcal{E}_{X,\overline{x}}^\lambda \subset \mathcal{S}_{X,\overline{x}}^\lambda$ which depends functorially on $(k, X, \overline{x}, K, \lambda)$ and such that each $\mathcal{E}_{X,\overline{x}}^{\lambda,i}$ is a free $\mathcal{O}_{X,\overline{x}}$-module of at most countable rank, the homomorphism $k_{\mathrm{Log}} \to K : \mathrm{Log}(p) \mapsto \lambda$ gives rise to an isomorphism $\mathcal{E}_{X,\overline{x}} \widetilde{\to} \mathcal{E}_{X,\overline{x}}^\lambda$ and, if $f_1, \ldots, f_t$ are elements of $\mathcal{O}_{X,x}^*$ for which $|f_1(x)|, \ldots, |f_t(x)|$ form a basis of the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$, then there is an isomorphism of filtered D-algebras $\mathcal{E}_{X,\overline{x}}^\lambda[T_1, \ldots, T_t] \otimes_k K \widetilde{\to} \mathcal{S}_{X,\overline{x}}^\lambda : T_i \mapsto \mathrm{Log}^\lambda(f_i)$.

(ii) The formulation of the main result makes sense for an arbitrary non-Archimedean field k of characteristic zero if λ is considered as a logarithmic character with values in K. We believe it is true at least for the field $\mathbf{C}((T))$ provided with a logarithmic character as in Example 1.4.3(ii).

§2. Étale neighborhoods of a point in a smooth analytic space

In this section we describe a fundamental system of étale neighborhoods of a point x in a smooth k-analytic space X. If $s(x) = \dim(X)$, we use a result of J. de Jong [deJ3] to show that there is a morphism $\varphi : \mathfrak{X}_\eta \to X$ from the generic fiber of a so called proper marked formal scheme $\mathfrak{X}$ over the ring of integers k'° of a finite extension k' of k such that φ is étale in an open neighborhood of the generic point of $\mathfrak{X}$ which is a unique preimage of x in $\mathfrak{X}_\eta$. (This morphism φ is in fact étale everywhere if $\dim(X) = 1$.) In §2.2, we recall a result from [Ber2] on the local structure of smooth analytic curves, and apply it in §2.3 to the case $s(x) < \dim(X)$. Namely, shrinking X one can find a morphism $\varphi : X \to Y$ to a smooth k-analytic space Y of dimension $\dim(X) - 1$ such that $s(y) = s(x)$ for $y = \varphi(x)$ and, if $t(y) = t(x)$ (resp. $t(y) < t(x)$), there is an étale neighborhood $Y' \to Y$ of y and an open neighborhood X' of a preimage of x in $X \times_Y Y'$ such that X' is isomorphic to a direct product of Y' and an open disc (resp. an open annulus). (Such X' is called a Y-split étale neighborhood of x over the étale morphism $Y' \to Y$.) In §2.4, we introduce so called smooth basic curves and show that they are precisely the generic fibers of proper marked formal schemes of dimension one. We also introduce affinoid basic curves and establish a property of morphisms from them to the generic fibers of formal schemes.

2.1. Étale neighborhoods of a point with $s(x) = \dim(X)$. Recall that for any formal scheme $\mathfrak{X}$ locally finitely presented over k° one can define a reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ such that the preimage of a closed (resp. open) subset of $\mathfrak{X}_s$ is open (resp. closed) in $\mathfrak{X}_\eta$ (see [Ber3, §3]). If $\mathfrak{X}$ is pluri-nodal over k° (in the sense of [Ber7, §1]) then, by [Ber7, Corollary 1.7], the generic point of an irreducible component $\mathcal{Y}$ of $\mathfrak{X}_s$ has a unique preimage x in $\mathfrak{X}_\eta$ and $\widetilde{\mathcal{H}(x)} = \widetilde{k}(\mathcal{Y})$. In particular, one has $s(x) = \dim_x(\mathfrak{X}_\eta)$ and $t(x) = 0$. The point x will be called the *generic point* of $\mathcal{Y}$ in $\mathfrak{X}_\eta$. Notice that such a point cannot lie in a nowhere dense Zariski closed subset of $\mathfrak{X}_\eta$. Examples of pluri-nodal formal schemes are nondegenerate strictly poly-stable formal schemes, which were introduced in [Ber9, §4] and whose definition will be recalled in §3, and an example of the latter is any base change of a strictly semi-stable formal scheme, i.e., $\mathfrak{X}\widehat{\otimes}_{k_0^\circ} k^\circ$, where k_0 is a closed subfield of k on which the valuation is discrete and $\mathfrak{X}$ is a strictly semi-stable formal scheme over k_0°. (An example of a strictly semi-stable formal scheme over k_0° is the formal completion $\widehat{\mathcal{X}}$ of a strictly semi-stable scheme $\mathcal{X}$ over k_0° along its closed fiber $\mathcal{X}_s$.)

Let k_0 be a non-Archimedean field with a nontrivial discrete valuation, and let k be an extension of k_0 which is a closed subfield of the completion $\widehat{k_0^{\mathrm{a}}}$ of an algebraic closure k_0^{a} of k_0. A k_0-*special* formal scheme over k° is a formal scheme $\mathfrak{X}$ isomorphic to $\mathfrak{X}'\widehat{\otimes}_{k_0'^\circ} k^\circ$, where k_0' is a subfield of k

finite over k_0 and $\mathfrak{X}'$ is a special formal scheme over ${k'_0}^\circ$ in the sense of [Ber5] (recall that special formal schemes were introduced in [Ber5] only over fields with discrete valuation). Such a formal scheme $\mathfrak{X}$ has a closed fiber $\mathfrak{X}_s$, which is a scheme of locally finite type over $\widetilde{k}$, and a generic fiber $\mathfrak{X}'_\eta$, which is a strictly k-analytic space, and there is a reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$. Of course, one has $\mathfrak{X}_s = \mathfrak{X}'_s \otimes_{\widetilde{k'_0}} \widetilde{k}$ and $\mathfrak{X}_\eta = \mathfrak{X}'_\eta \widehat{\otimes}_{k'_0} k$.

A k_0-special formal scheme $\mathfrak{X}$ over k° is said to be *marked* if it is isomorphic to a formal scheme of the form $\widehat{\mathcal{X}'}_{/\mathcal{Y}'} \widehat{\otimes}_{{k'_0}^\circ} k^\circ$, where k'_0 is a subfield of k finite over k_0, $\mathcal{X}'$ is a nondegenerate strictly poly-stable separated scheme of finite type over ${k'_0}^\circ$, $\mathcal{Y}'$ is an irreducible component of the closed fiber $\mathcal{X}'_s$ such that the scheme $\mathcal{Y} = \mathcal{Y}' \otimes_{\widetilde{k'_0}} \widetilde{k}$ is also irreducible, and $\widehat{\mathcal{X}'}_{/\mathcal{Y}'}$ is the formal completion of $\mathcal{X}'$ along $\mathcal{Y}'$. The closed fiber $\mathfrak{X}_s$ of such $\mathfrak{X}$ coincides with $\mathcal{Y}$ and the generic fiber $\mathfrak{X}_\eta$ is a strictly analytic subdomain of the analytification $\mathcal{X}_\eta^{\mathrm{an}}$ of the generic fiber $\mathcal{X}_\eta$ of the nondegenerate strictly poly-stable scheme $\mathcal{X} = \mathcal{X}' \otimes_{{k'_0}^\circ} k^\circ$ over k°. The generic point of $\mathcal{Y}$ in $\mathfrak{X}_\eta$ will be called the *generic point of* $\mathfrak{X}$ and denoted by $\sigma = \sigma_{\mathfrak{X}}$. If in addition the scheme $\mathcal{Y}$ is proper, we say that $\mathfrak{X}$ is *proper marked.* If $\mathfrak{X}$ is proper marked, its generic fiber $\mathfrak{X}_\eta$ is an open subset of $\mathcal{X}_\eta^{\mathrm{an}}$ and, therefore, it is a connected smooth k-analytic space. Recall also that if $\mathcal{X}$ is proper over k° then $\widehat{\mathcal{X}}_\eta = \mathcal{X}_\eta^{\mathrm{an}}$. We say that a k_0-special formal scheme $\mathfrak{X}$ over k° is *strongly marked* if the above scheme $\mathcal{X}'$ is a strictly semi-stable projective over $k_0'^\circ$. Of course, in this case $\mathfrak{X}$ is proper marked.

If the field k'_0 and the schemes $\mathcal{X}'$ and $\mathcal{Y}'$ are not important in our consideration of a marked formal scheme $\mathfrak{X}$ as above, we say, by abuse of language, that $\mathfrak{X}$ is the formal completion $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ of the scheme $\mathcal{X}$ along the irreducible component $\mathcal{Y}$. (Recall again that the object $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ was defined in [Ber5] only in the case when the valuation on k is discrete.)

Let X be a smooth k-analytic space, and x a point of X with $s(x) = \dim(X)$. A *marked* (resp. *strongly marked*) *neighborhood of* x is a morphism of strictly k-analytic spaces $\varphi : \mathfrak{X}_\eta \to X$ which is étale outside $\mathcal{Z}^{\mathrm{an}} \cap \mathfrak{X}_\eta$ and such that $\varphi(\sigma_{\mathfrak{X}}) = x$, where $\mathfrak{X}$ is a proper marked (resp. strongly marked) formal scheme over ${k'}^\circ$, k' is a finite extension of k, and $\mathcal{Z}$ is a nowhere dense Zariski closed subset of $\mathcal{X}_\eta$ (from the above definition). If $\dim(X) = 1$, we require in addition that the morphism is étale everywhere, i.e., one can take $\mathcal{Z} = \emptyset$. Furthermore, given two marked neighborhoods $\varphi' : \mathfrak{X}'_\eta \to X$ and $\varphi'' : \mathfrak{X}''_\eta \to X$ of the point x with $\mathfrak{X}' = \widehat{\mathcal{X}}'_{/\mathcal{Y}'}$ and $\mathfrak{X}'' = \widehat{\mathcal{X}}''_{/\mathcal{Y}''}$ and the corresponding fields k' and k'', we say that the latter *refines* the former, if there is a morphism of formal schemes $\mathfrak{X}'' \to \mathfrak{X}'$ over an embedding of fields $k' \hookrightarrow k''$ which takes the generic point of $\mathfrak{X}''$ to that of $\mathfrak{X}'$ and is étale outside $\mathcal{Z}^{\mathrm{an}} \cap \mathfrak{X}''_\eta$, where $\mathcal{Z}$ is a nowhere dense Zariski closed subset of $\mathcal{X}''_\eta$.

2.1.1. Proposition. *Assume that the characteristic of k_0 is zero, and let X be a smooth k-analytic space. Then*

(i) every point $x \in X$ with $s(x) = \dim(X)$ admits a strongly marked neighborhood;

(ii) given two marked neighborhoods of the point x, there exists a third one which is strongly marked and refines both of them.

2.1.2. Lemma. *Let k be a non-Archimedean field of characteristic zero with a nontrivial discrete valuation, X a smooth k-analytic space, and x a point of X with $s(x) = \dim(X)$. Given a morphism of strictly k-analytic spaces $X \to \mathfrak{Y}_\eta$, where $\mathfrak{Y}$ is a special formal scheme over k°, there exists a finite extension k' of k, a strictly semi-stable projective scheme $\mathcal{X}$ over k'°, a geometrically irreducible component $\mathcal{Y}$ of $\mathcal{X}_s$, a morphism of strictly k-analytic spaces $\varphi : \mathfrak{X}_\eta \to X$ with $\mathfrak{X} = \widehat{\mathcal{X}}_{/\mathcal{Y}}$ and a morphism of formal schemes $\mathfrak{X} \to \mathfrak{Y}$ such that the following is true:*

(i) the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{X}_\eta & \xrightarrow{\varphi} & X \\ & \searrow \quad \swarrow & \\ & \mathfrak{Y}_\eta & \end{array}$$

(ii) φ is étale outside $\mathcal{Z}^{\rm an} \cap \mathfrak{X}_\eta$, where $\mathcal{Z}$ is a Zariski closed proper subset of $\mathcal{X}_\eta$, and, if $\dim(X) = 1$, φ is étale everywhere;

(ii) $\varphi(\sigma_{\mathfrak{X}}) = x$.

Proof. Step 1. First of all, we may assume that X is an open subset in $\mathcal{X}^{\rm an}$, where $\mathcal{X} = \mathrm{Spec}(B)$ is a smooth irreducible affine scheme over k. Since $s(x) = n = \dim(X)$, the image $\mathbf{x}$ of x in $\mathcal{X}$ is the generic point of $\mathcal{X}$. Furthermore, we may assume that there are elements $f_1, \dots, f_n \in B$ with $|f_i(x)| = 1$ such that their image in $\widetilde{\mathcal{H}(x)}$ are algebraically independent over $\widetilde{k}$. This means that if f denotes the morphism from $\mathcal{X}$ to the n-dimensional affine space defined by $f_1, \dots, f_n$ then the image of the point x under the morphism $f^{\rm an} : \mathcal{X}^{\rm an} \to \mathbf{A}^n$ is the maximal point of the closed unit polydisc E with center at zero. Let $f_{n+1}, \dots, f_m$ be elements of B such that $|f_i(x)| \le 1$, the elements $f_1, \dots, f_m$ generate B over k, and the strictly affinoid domain $Z = \{z \in \mathcal{X}^{\rm an} \big| |f_i(z)| \le 1, 1 \le i \le m\}$ is contained in X. Since $f^{\rm an}(x)$ is the maximal point of E, it follows that the image of x under the reduction map $\pi : Z \to \widetilde{Z}$ is the generic point of an irreducible component of $\widetilde{Z}$. (If $Z = \mathcal{M}(\mathcal{A})$, then $\widetilde{Z} = \mathrm{Spec}(\widetilde{\mathcal{A}})$, where $\widetilde{\mathcal{A}} = \mathcal{A}^\circ / \mathcal{A}^{\circ\circ}$.) Furthermore, we can find numbers $a_i \in \sqrt{|k^*|}$ bigger than one such that the strictly affinoid domain $W = \{z \in \mathcal{X}^{\rm an} \big| |f_i(z)| \le a_i, 1 \le i \le m\}$ is also contained in X. Notice that W is a neighborhood of the point x.

Step 2. Let α denote the induced morphism of strictly k-analytic spaces $W \to \mathfrak{Y}_\eta$. Since the space $\mathfrak{Y}_\eta$ is Hausdorff (see [Ber5, §1]), one has $\alpha(W) \subset \cup_{i=1}^l \mathfrak{Y}^i_\eta$, where $\mathfrak{Y}^i$ are open affine subschemes of $\mathfrak{Y}$, and the intersection of $\alpha(W)$ with each $\mathfrak{Y}^i_\eta$ is compact. It follows that there is a strictly affinoid subdomain $Y_i \subset \mathfrak{Y}^i_\eta$ that contains $\alpha(W) \cap \mathfrak{Y}^i_\eta$ and, therefore, $W_i = \alpha^{-1}(\mathfrak{Y}^i_\eta) = \alpha^{-1}(Y_i)$ is a strictly affinoid subdomain of W. One has $W = \cup_{i=1}^l W_i$.

Step 3. By Lemma 9.4 from [Ber7], there is an open embedding of $\mathcal{X}$ in $\mathcal{Y}_\eta$, where $\mathcal{Y}$ is an integral scheme proper and flat over k°, open subschemes $\mathcal{Z} \subset \mathcal{W} \subset \mathcal{Y}_s$, and a closed subscheme $\mathcal{V} \subset \mathcal{Y}_s$, which is a union of irreducible components of $\mathcal{Y}_s$, such that $Z = \pi^{-1}(\mathcal{Z})$, $W = \pi^{-1}(\mathcal{W})$, and $\pi(x) \in \mathcal{V} \subset \mathcal{W}$. Making a finite number of additional blow-ups as in the proof of [Ber7, Lemma 9.4], we may also assume that there are open subschemes $\mathcal{W}_i \subset \mathcal{W}$ with $W_i = \pi^{-1}(\mathcal{W}_i)$ and $\mathcal{W} = \cup_{i=1}^n \mathcal{W}_i$.

Step 4. By de Jong's Theorem 6.5 from [deJ3] (resp. the semi-stable reduction theorem for curves if $\dim(X) = 1$), there is a finite extension k' of k, a projective strictly semi-stable scheme $\mathcal{Y}'$ over k'°, and a proper, dominant and generically finite morphism $\varphi : \mathcal{Y}' \to \mathcal{Y}$ (resp. with the property $\mathcal{Y}'_\eta \xrightarrow{\sim} \mathcal{Y}_\eta \otimes_k k'$). Since k is of characteristic zero, the morphism φ is étale outside a closed proper subset of $\mathcal{Y}'_\eta$ (resp. étale everywhere). If it is necessary, we may replace the field k' by a finite unramifield extension and assume that all of the irreducible components of $\mathcal{Y}'_s$ are geometrically irreducible. Let x' be a point from the preimage of x in $\mathcal{Y}'^{\mathrm{an}}_\eta$. *We claim that $\pi(x')$ is the generic point of an irreducible component $\mathcal{P}$ of $\mathcal{Y}'_s$.* Indeed, let $\mathfrak{Z} = \mathrm{Spf}(C)$ be an open affine subscheme of $\widehat{\mathcal{Y}}'$ with $\pi(x') \in \mathfrak{Z}_s \subset \varphi^{-1}(\mathcal{Z})$. By [Ber7, Proposition 1.4], one has $C \xrightarrow{\sim} \mathcal{C}^\circ$, where $\mathcal{C} = C \otimes_{k^\circ} k$. It follows that the morphism of strictly k-affinoid spaces $\mathfrak{Z}_\eta \to Z$ gives rise to a morphism of schemes of same dimension $\mathfrak{Z}_s = \widetilde{\mathfrak{Z}}_\eta \to \widetilde{Z}$. Since $\pi(x)$ is the generic point of an irreducible component of $\widetilde{Z}$, the claim follows. Notice that $\mathcal{P} \subset \varphi^{-1}(\mathcal{V}) \subset \varphi^{-1}(\mathcal{W})$.

Step 5. It remains to show that there is a well defined morphism of special formal schemes $\widehat{\mathcal{Y}}'_{/\mathcal{P}} \to \mathfrak{Y}$ compatible with the canonical morphism between their generic fibers. For this it suffices to show that there is a well defined morphism $\widehat{\mathcal{Y}}'_{/\mathcal{W}'} \to \mathfrak{Y}$ with the same property, where $\mathcal{W}' = \varphi^{-1}(\mathcal{W})$ or, equivalently, that there is a system of compatible morphisms $\widehat{\mathcal{Y}}'_{/\mathcal{W}'_i} \to \mathfrak{Y}^i$. Let $\mathfrak{Y}^i = \mathrm{Spf}(A)$, and let $\mathfrak{T} = \mathrm{Spf}(B)$ be an open affine subscheme of $\widehat{\mathcal{Y}}'_{/\mathcal{W}'_i}$. The generic fiber $\mathfrak{T}_\eta$ is the strictly k-affinoid space $\mathcal{M}(\mathcal{B})$, where $\mathcal{B} = B \otimes_{k'^\circ} k'$. The canonical morphism of strictly k-analytic spaces $\mathfrak{T}_\eta \to \mathfrak{Y}^i_\eta$ is induced by a homomorphism $A \to \mathcal{B}$. The image of A is contained in $\mathcal{B}^\circ$ and again, by [Ber7, Proposition 1.4], one has $B \xrightarrow{\sim} \mathcal{B}^\circ$. The homomorphisms $A \to B$ constructed in this way give rise to the required morphism of formal schemes $\widehat{\mathcal{Y}}'_{/\mathcal{W}'_i} \to \mathfrak{Y}^i$. ∎

2.1.3. Lemma. *Let $k_0 \subset k \subset \widehat{k}^{\mathrm{a}}_0$ be non-Archimedean fields, and assume that, if k is not finite over k_0, k is perfect. Then* *(i) given a k_0-analytic space X, any étale morphism $\varphi : Y \to X\widehat{\otimes}_{k_0}k$ is, locally on Y, a base change of an étale morphism $Y' \to X\widehat{\otimes}_{k_0}k'_0$ for some finite extension k'_0 of k_0 in k;*

(ii) given a k_0-affinoid (resp. strictly k_0-affinoid) space X, any rational affinoid (resp. strictly affinoid) subdomain of $X\widehat{\otimes}_{k_0}k$ is a base change of a rational affinoid (resp. strictly affinoid) subdomain of $X\widehat{\otimes}_{k_0}k'_0$ for some finite extension k'_0 of k_0 in k.

Proof. (i) Let y be a point of Y, x its image in $X\widehat{\otimes}_{k_0}k$ and, for a finite extension k'_0 of k_0 in k, let $x_{k'_0}$ be the image of x in $X\widehat{\otimes}_{k_0}k'_0$. Since k is perfect, the Ax-Sen-Tate theorem ([Ax]) implies that the subfield $k \cap k_0^{\mathrm{a}}$ is dense in k. It follows that the subfield $\cup_{k'_0}\mathcal{H}(x_{k'_0})$ is dense in $\mathcal{H}(x)$ and, therefore, one can find such k'_0 that the finite separable extension $\mathcal{H}(y)/\mathcal{H}(x)$ is induced by a finite separable extension of $\mathcal{H}(x_{k'_0})$ of the same degree. The required fact now follows from [Ber2, Proposition 3.4.1].

(ii) Let $X = \mathcal{M}(\mathcal{A})$, and let Y be a rational affinoid subdomain of $X\widehat{\otimes}_{k_0}k = \mathcal{M}(\mathcal{A}\widehat{\otimes}_{k_0}k)$. By the definition, $Y = \{x \in X\widehat{\otimes}_{k_0}k \big| |f_i(x)| \le a_i|g(x)| \text{ for all } 1 \le i \le n\}$, where $f_1, \dots, f_n, g$ are elements of $\mathcal{A}\widehat{\otimes}_{k_0}k$ without common zeroes on $X\widehat{\otimes}_{k_0}k$, and $a_1, \dots, a_n$ are positive numbers. (If X and Y are strictly affinoid, one may assume that $a_i = 1$ for all $1 \le i \le n$.) It follows that the element g is invertible on Y and, therefore, $|g(x)| \ge b$ for all $x \in Y$ and some $b \in |k_0^*|$. Thus, if Z is the Laurent subdomain of $X\widehat{\otimes}_{k_0}k$ defined by the inequality $|g(x)| \ge b$, Y is the Weierstrass subdomain in Z defined by the inequalities $|\frac{f_i}{g}(x)| \le a_i$, $1 \le i \le n$. Since the subfield $k \cap k_0^{\mathrm{a}}$ is dense in k, the subalgebra $\cup_{k'_0}\mathcal{A}\widehat{\otimes}_{k_0}k'_0$ is dense in $\mathcal{A}\widehat{\otimes}_{k_0}k$ and, therefore, there is a finite extension k'_0 of k_0 in k and an element $g' \in \mathcal{A}\widehat{\otimes}_{k_0}k'_0$ with $||g' - g|| < b$. It follows that $Z = Z'\widehat{\otimes}_{k'_0}k$, where Z' is the Laurent subdomain of $X\widehat{\otimes}_{k_0}k'_0$ defined by the inequality $|g'(x)| \ge b$. Replacing k_0 by k'_0, X by Z' and f_i by $\frac{f_i}{g}$, we may assume that Y is the Weierstrass subdomain of $X\widehat{\otimes}_{k_0}k$ defined by the inequalities $|f_i(x)| \le a_i$, $1 \le i \le n$. For the same reason, there is a finite extension k'_0 of k_0 in k and elements $f_i \in \mathcal{A}\widehat{\otimes}_{k_0}k'_0$ with $||f'_i - f_i|| < a_i$. It follows that $Y = Y'\widehat{\otimes}_{k'_0}k$, where Y' is the Weierstrass subdomain of $X\widehat{\otimes}_{k_0}k'_0$ defined by the inequalities $|f'_i(x)| \le a_i$, $1 \le i \le n$. ∎

Proof of Proposition 2.1.1. (i) Shrinking X, we may assume that there is an étale morphism from X to the affine space $\mathbf{A}_k^n$ over k. Since $\mathbf{A}_k^n = \mathbf{A}_{k_0}^n\widehat{\otimes}_{k_0}k$, Lemma 2.1.3(i) implies that we can shrink X and find a finite extension k'_0 of k_0 in k such that $X = X'\widehat{\otimes}_{k'_0}k$ for some smooth k'_0-analytic space X'. By Lemma 2.1.2 applied to the point x', the image of x in X' and the morphism $X' \to \mathfrak{Y}_\eta$ with $\mathfrak{Y} = \mathrm{Spf}(k'^{\circ}_0)$, we can find a finite extension k''_0 of k'_0 in k_0^{a}, a strictly semi-stable projective scheme $\mathcal{X}$ over k''°_0, a geometrically irreducible component $\mathcal{Y}$ of $\mathcal{X}_s$ and a morphism of strictly k-analytic spaces $\varphi' : \mathfrak{X}'_\eta \to X'$ with $\mathfrak{X}' = \widehat{\mathcal{X}}_{/\mathcal{Y}}$ étale outside $\mathcal{Z}^{\mathrm{an}} \cap \mathfrak{X}'_\eta$ and such that

$\varphi'(\sigma_{\mathfrak{X}'}) = x'$. Let k' be the composite of k and k_0'' in $\widehat{k}_0^{\mathrm{a}}$ and $\varphi : \mathfrak{X}_\eta \to X$ the induced morphism with $\mathfrak{X} = \mathfrak{X}'\widehat{\otimes}_{k_0''^\circ} k'^\circ$. Since $\mathcal{Y}$ is geometrically irreducible, the closed fiber $\mathfrak{X}_s$ is irreducible and, therefore, $\varphi(\sigma_{\mathfrak{X}}) = x$, i.e., the morphism φ possesses the required properties.

(ii) Increasing the field k, we may assume that we are given two marked neighborhoods $\varphi' : \mathfrak{X}'_\eta \to X$ and $\varphi'' : \mathfrak{X}''_\eta \to X$ of the point x with $\mathfrak{X}'$ and $\mathfrak{X}''$ proper marked over k°. By Lemma 2.1.3, we can increase the field k_0 and find an open neighborhood of the point x in X of the form $Y\widehat{\otimes}_{k_0} k$, where Y is a smooth k_0-analytic space, and two étale neighborhoods $Y' \to Y$ and $Y'' \to Y$ of the image of x in Y whose base change to k are isomorphic to the restrictions of φ' and φ'' to some open neighborhoods of the points $\sigma_{\mathfrak{X}'}$ and $\sigma_{\mathfrak{X}''}$, respectively. In what follows we identify $Y\widehat{\otimes}_{k_0} k$, $Y'\widehat{\otimes}_{k_0} k$ and $Y''\widehat{\otimes}_{k_0} k$ with open subsets of X, $\mathfrak{X}'_\eta$ and $\mathfrak{X}''_\eta$, respectively. Let Z' and Z'' be strictly affinoid neighborhoods of the images y' and y'' of the points $\sigma_{\mathfrak{X}'}$ and $\sigma_{\mathfrak{X}''}$ in Y' and Y'', respectively. We can find a rational strictly affinoid covering $\{W'_i\}_{1\le i\le m}$ of $Z'\widehat{\otimes}_{k_0} k$ (resp. $\{W''_j\}_{1\le j\le l}$ of $Z''\widehat{\otimes}_{k_0} k$) with $W'_i \subset \mathfrak{X}'^i_\eta$ (resp. $W''_j \subset \mathfrak{X}''^j_\eta$) for some open affine subschemes $\mathfrak{X}'^i \subset \mathfrak{X}'$ (resp. $\mathfrak{X}''^j \subset \mathfrak{X}''$). By Lemma 2.1.3(ii), we can increase the field k_0 so that $W'_i = Z'_i\widehat{\otimes}_{k_0} k$ (resp. $W''_j = Z''_j\widehat{\otimes}_{k_0} k$) for rational strictly affinoid subdomains $Z'_i \subset Z'$ (resp. $Z''_j \subset Z''$). We now can construct a formal scheme $\mathfrak{Z}'$ (resp. $\mathfrak{Z}''$) of finite type over k_0° and its open covering $\{\mathfrak{Z}'^i\}_{1\le i\le m}$ (resp. $\{\mathfrak{Z}''^j\}_{1\le j\le l}$) with $\mathfrak{Z}'_\eta = Z'$ and $\mathfrak{Z}'^i_\eta = Z'_i$ (resp. $\mathfrak{Z}''_\eta = Z''$ and $\mathfrak{Z}''^j_\eta = Z''_j$). It follows from the construction that the canonical embedding $Z'\widehat{\otimes}_{k_0} k \to \mathfrak{X}'_\eta$ (resp. $Z''\widehat{\otimes}_{k_0} k \to \mathfrak{X}''_\eta$) is induced by a morphism of formal schemes $\mathfrak{Z}'\widehat{\otimes}_{k_0^\circ} k^\circ \to \mathfrak{X}'$ (resp. $\mathfrak{Z}''\widehat{\otimes}_{k_0^\circ} k^\circ \to \mathfrak{X}''$). Finally, let z be a point of $Z' \times_Y Z''$ over the points y' and y'', and let Z be an open neighborhood of z in $Z' \times_Y Z''$ which is also open in $Y' \times_Y Y''$. By Lemma 2.1.2, applied to the pair (Z, z) and the canonical morphism of strictly F-analytic spaces $Z \to (\mathfrak{Z}' \times \mathfrak{Z}'')_\eta$, there exists a finite extension F' of F, a strongly marked formal scheme $\mathfrak{Y}$ over $k_0'^\circ$, a morphism of strictly k-analytic spaces $\mathfrak{Y}_\eta \to Z$, and a morphism of formal schemes $\mathfrak{Y} \to \mathfrak{Z}' \times \mathfrak{Z}''$ over k_0° for which the properties (i)-(iii) are true. If k' is the composite of k and k_0' in $\widehat{k}_0^{\mathrm{a}}$ and $\mathfrak{X} = \mathfrak{Y}\widehat{\otimes}_{k'^\circ} k'^\circ$, it follows that the induced morphism $\mathfrak{X}_\eta \to X$ is a strongly marked neighborhood of the point x which refines the marked neighborhoods we started from. ∎

2.2. The local structure of a smooth analytic curve. In this subsection we recall a result from [Ber2, §3.6], which will be used in the following two subsections. It describes the local structure of a smooth k-analytic curve, where k is an arbitrary non-Archimedean field. I am very grateful to Brian Conrad for pointing out that a part of the proof of [Ber2, Proposition 3.6.1] does not work if the field k is non-perfect and that its formulation should be slightly changed if, in addition, the valuation on k is trivial (see also Remark 2.2.2).

Recall that the affine line $\mathbf{A}^1$ is the set of all multiplicative seminorm on the ring of polynomials $k[T]$ that extend the valuation on k. Here is a description of points of $\mathbf{A}^1$ from [Ber1], 1.1.4] in the case when the field k is algebraically closed.

First of all, every element $a \in k$ defines a multiplicative seminorm $k[T] \to \mathbf{R}_+ : f \mapsto |f(a)|$. The corresponding point of $\mathbf{A}^1$ is said to be of type (1). Furthermore, every closed disc $E = E(a,\rho) = \{x \in \mathbf{A}^1 \big| |T(x)| \le \rho\}$ defines a multiplicative seminorm $f \mapsto |f|_E = \max_n |a_n|\rho^n$, where $\sum_{n=0}^{\infty} a_n (T-a)^n$ is the expansion of f with center at a. The corresponding point of $\mathbf{A}^1$ is denoted by $p(E)$ and is said to be of type (2) if $\rho \in |k|$ and of type (3) if $\rho \notin |k|$. Finally, let $\mathcal{E} = \{E^{(\rho)}\}$ be a family of embedded discs in $\mathbf{A}^1$, i.e., $E^{(\rho)}$ is a closed disc of radius ρ and $E^{(\rho)} \supset E^{(\rho')}$ if $\rho > \rho'$. Then $\mathcal{E}$ defines a multiplicative seminorm $f \mapsto |f|_{\mathcal{E}} = \inf |f|_{E^{(\rho)}}$. The corresponding point of $\mathbf{A}^1$ is denoted by $p(\mathcal{E})$. Let σ denote the intersection of all $E^{(\rho)}$. If $\sigma \cap k$ is nonempty, then σ is a point $a \in k$ and $p(\mathcal{E}) = a$, or else σ is a closed disc E and $p(\mathcal{E}) = p(E)$. If $\sigma \cap k$ is empty, we obtain a new point which is said to be of type (4). Each point of $\mathbf{A}^1$ is of one of the above types. We list some properties of points $x \in \mathbf{A}^1$:

(1) if $x = a$ is of type (1), then $\mathcal{H}(x) = k$ and a basis of open neighborhoods of x is form by the open discs $D(a,\rho)$;

(2) if $x = p(E(a,\rho))$ is of type (2), then $\widetilde{\mathcal{H}(x)} \xrightarrow{\sim} \widetilde{k}(T)$, $|\mathcal{H}(x)| = |k|$, and a basis of open neighborhoods of x is form by open sets of the form $D(a,r) \backslash \coprod_{i=1}^n E(a_i, r_i)$ with $r_i < \rho < r$, $|a_i - a| \le \rho$, and $|a_i - a_j| = \rho$ for $i \ne j$;

(3) if $x = p(E(a,\rho))$ is of type (3), then $\widetilde{\mathcal{H}(x)} = \widetilde{k}$, $|\mathcal{H}(x)^*|$ is generated by $|k^*|$ and ρ, and a basis of open neighborhoods of x is formed by the open annuli $B(a;r,R)$ with $r < \rho < R$;

(4) if $x = p(\mathcal{E})$ is of type (4), where $\mathcal{E} = \{E(a_\rho, \rho)\}$, then $\widetilde{\mathcal{H}(x)} = \widetilde{k}$ and $|\mathcal{H}(x)| = |k|$ (i.e., $\mathcal{H}(x)$ is an immediate extension of k), and a basis of open neighborhoods of x is formed by the open discs $D(a_\rho, \rho)$.

If the field k is not necessarily algebraically closed, the type of a point $x \in \mathbf{A}^1$ is, by definition, the type of some (and therefore all) of its preimages in $\mathbf{A}^1 \widehat{\otimes} \widehat{k}^{\mathrm{a}}$. If x is of type (1) or (4), then $\widetilde{\mathcal{H}(x)}$ is algebraic over $\widetilde{k}$ and $\sqrt{|\mathcal{H}(x)^*|} = \sqrt{|k^*|}$ and, in particular, $s(x) = t(x) = 0$. If x is of type (2), then $\widetilde{\mathcal{H}(x)}$ is finitely generated of transcendence degree one over $\widetilde{k}$ and the group $|\mathcal{H}(x)^*|/|k^*|$ is finite and, in particular, $s(x) = 1$ and $t(x) = 0$. If x is of type (3), then $\widetilde{\mathcal{H}(x)}$ is finite over $\widetilde{k}$ and the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$ is of dimension one and, in particular, $s(x) = 0$ and $t(x) = 1$.

Let now X be a smooth k-analytic curve. For each point $x \in X$ there is an étale morphism from an open neighborhood of x to the affine line $\mathbf{A}^1$. The type of the image of x in $\mathbf{A}^1$, which does not depend on the choice of the étale morphism, is said to be the *type of* x. Notice that each point x from $X_0 = \{x \in X \big| [\mathcal{H}(x) : k] < \infty\}$ is of type (1) and its local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring. In all other cases, $\mathcal{O}_{X,x}$ is a field. Here is a corrected version of Proposition 3.6.1 from [Ber2] (see also Remark 2.2.2).

2.2.1. Proposition. *Assume that the field k is perfect or its valuation is non-trivial (resp. non-perfect and the valuation is trivial). Then for every point $x \in X$ there exists a finite separable (resp. non necessarily separable) extension k' of k and an open subset $X' \subset X\widehat{\otimes}k'$ such that x has a unique preimage x' in X' and X' is isomorphic to the following k'-analytic curve (depending on the type of x):*

(1) or (4): an open disc with center at zero;

(3) an open annulus with center at zero;

(2) $\mathcal{X}_\eta^{\mathrm{an}} \backslash \coprod_{i=1}^n X_i$, $n \geq 1$, where $\mathcal{X}$ is a connected smooth projective curve over k'°, each X_i is an affinoid subdomain isomorphic to a closed disc with center at zero and all of them are in pairwise different residue classes of $\mathcal{X}_\eta^{\mathrm{an}}$, and x' is the generic point of $\mathcal{X}_s$ in $\mathcal{X}_\eta^{\mathrm{an}}$.

Proof. The proof of Proposition 3.6.1 from [Ber2] consisted of the following three steps. (A) By [Ber2, Lemma 3.6.2], one can shrink X so that there is an open embedding of X in the analytification $\mathcal{X}'^{\mathrm{an}}$ of a smooth affine curve $\mathcal{X}'$ of finite type over k. (B) The claim that there is an open embedding of $\mathcal{X}'$ in a smooth projective curve $\mathcal{X}$ over k, which is of course wrong if the field k is non-perfect. (C) A proof of the required fact under the assumption that there is an open embedding of X in $\mathcal{X}^{\mathrm{an}}$, where $\mathcal{X}$ is a smooth projective curve over k. (The proof of (C) is based on the semi-stable reduction theorem for curves.)

Thus, to correct the proof, it suffices to establish the following fact. *If the valuation on k is nontrivial, then for every point $x \in X$ there exists a finite separable extension k' of k and an open subset $X' \subset X\widehat{\otimes}k'$ such that x has a nonempty preimage in X' and there is an open embedding of X' in $\mathcal{X}^{\mathrm{an}}$, where $\mathcal{X}$ is a smooth projective curve over k'.* Of course, it suffices to consider the case when k is non-perfect of characteristic $p > 0$.

Step 1. We can find an open embedding of X in $\mathcal{X}^{\mathrm{an}}$, where $\mathcal{X}$ is only a regular projective curve. Let $\mathcal{Y}$ be the smooth locus of $\mathcal{X}$. Of course, it suffices to consider the case when $\mathcal{Y}$ does not coincide with $\mathcal{X}$. In this case $\mathcal{Y} = \mathrm{Spec}(A)$ is a smooth affine curve of finite type over k. We may assume that X is relatively compact in $\mathcal{Y}^{\mathrm{an}}$, and we may replace k by a finite separable extension so that the cardinality of the complement $\mathcal{X}\backslash\mathcal{Y}$ does not change after bigger finite separable extensions and that there exists a finite purely inseparable extension k' of k such that the normalization $\mathcal{X}'$ of the curve $\mathcal{X} \otimes k'$ is smooth over k' and all of its points over $\mathcal{X}\backslash\mathcal{Y}$ are k'-rational. Our further construction is of the type used in the proof of (C), and the purpose is to make a surgery of $\mathcal{X}^{\mathrm{an}}$ in a small neighborhood of a point $\mathbf{x} \in \mathcal{X}\backslash\mathcal{Y}$ (after a finite separable extension of k) so that its result is a new regular projective curve whose non-smoothness locus has strictly smaller cardinality than that of the set $\mathcal{X}\backslash\mathcal{Y}$.

Step 2. Let $\mathcal{Y}' = \mathrm{Spec}(A')$ be the preimage of $\mathcal{Y}$ in $\mathcal{X}'$, $\mathbf{x}'$ the preimage of the point $\mathbf{x}$, and X' the preimage of X in $\mathcal{X}'^{\mathrm{an}}$. Of course, $A \otimes k' \overset{\sim}{\to} A'$, $\mathbf{x}' \not\in \mathcal{Y}'$, and X' is relatively compact in $\mathcal{Y}'^{\mathrm{an}}$. Since the point $\mathbf{x}'$ is k'-rational, it has an open neighborhood $\mathcal{V}$ isomorphic to the open unit disc $D(0;1)$ over k'. We may assume that this disc does not intersect with X', and fix an isomorphism $D(0;1) \overset{\sim}{\to} \mathcal{V}$ so that the images of zero is the point $\mathbf{x}'$.

Step 3. By the Riemann-Roch theorem, there exists a rational function f on $\mathcal{X}'$ with the only nontrivial pole at the point $\mathbf{x}'$ and, in particular, $f \in A'$. Since k' is purely inseparable over k, we can replace f by some power of p and assume that in fact $f \in A$. Then f is a rational function on $\mathcal{X}$ with the only nontrivial pole at $\mathbf{x}$. We can shrink the disc $\mathcal{V}$ so that f has no zeros at it.

Step 4. Let a be a big enough positive number such that the preimage of the affinoid subdomain $U = \{x \in \mathcal{X}^{\mathrm{an}} \big| |f(x)| \geq a\}$ of $\mathcal{X}^{\mathrm{an}}$ in $\mathcal{X}'^{\mathrm{an}}$ is contained in $\mathcal{V}$, and so it is a closed disc $E(0;R)_{k'}$. Take a bigger number $b > a$, and consider the affinoid subdomain $W = \{x \in U \big| |f(x)| \leq b\}$ of $\mathcal{X}^{\mathrm{an}}$. The preimage of W in $\mathcal{X}'^{\mathrm{an}}$ is a closed annulus $W' = A(r,R)_{k'}$ with center at zero and $r < R$. If $W = \mathcal{M}(\mathcal{A})$ and $W' = \mathcal{M}(\mathcal{A}')$, one has $\mathcal{A}\widehat{\otimes}k' \overset{\sim}{\to} \mathcal{A}'$.

Step 5. *We claim that, after replacing k by a finite separable extension, one can slightly deform the isomorphism $W' \overset{\sim}{\to} A(r,R)_{k'}$ so that it can be induced by an isomorphism $W \overset{\sim}{\to} A(r,R)$.* Indeed, let $g' \in \mathcal{A}'$ be the pullback of the coordinate function from $A(r,R)_{k'} \subset \mathbf{A}^1_{k'}$. Since the separable closure of k is dense in its algebraic closure, we can replace k by a finite separable extension so that there exists an element $g \in \mathcal{A}$ with $|(g'-g)(x')| < |g'(x')|$ for all points $x' \in W'$. Then the morphism $g : W' \to \mathbf{A}^1_{k'}$ induces an isomorphism $W' \overset{\sim}{\to} A(r,R)_{k'}$ and is induced by the morphism $g : W \to \mathbf{A}^1$. Since the latter gives rise to the isomorphism $W' = W\widehat{\otimes}k' \overset{\sim}{\to} A(r,R)_{k'} = A(r,R)\widehat{\otimes}k'$, it follows that $W \overset{\sim}{\to} A(r,R)$.

Step 6. Let $\mathcal{W}$ be the preimage of the open annulus $B(r,R) = \{x \in A(r,R) \big| r < |T(x)| < R\}$ with respect to the latter isomorphism. One has $\mathcal{W} = \{x \in \mathcal{X}^{\mathrm{an}} \big| a < |f(x)| < b\}$. We now glue the open set $\{x \in \mathcal{X}^{\mathrm{an}} \big| |f(x)| < b\}$ and the open disc $D(0;R)$ via the isomorphism $\mathcal{W} \overset{\sim}{\to} B(r,R)$. The k-analytic space obtained is a proper k-analytic curve and, therefore, it is the analytification $\mathcal{Z}^{\mathrm{an}}$ of a projective regular curve $\mathcal{Z}$. By the construction, X is an open subset of $\mathcal{Z}^{\mathrm{an}}$ and the cardinality of the non-smoothness locus of $\mathcal{Z}$ is strictly less than that of $\mathcal{X}$. Repeating this procedure with other non-smooth points, we get the required fact. ∎

The *genus* of a point $x \in X$ of type (2) is the genus of the smooth projective curve $\mathcal{X}_\eta$ over k' from Proposition 2.2.1. It is well defined, and is equal to the genus of the smooth projective curve over $\widetilde{k}'$ whose field of rational functions is $\widetilde{\mathcal{H}(x')}$ (with the point x' also from Proposition 2.2.1). Since the genus of any point of type (2) in the affine line is zero, it follows that subset of points of type (2) and positive genus in any smooth k-analytic curve is discrete in it.

A smooth k-analytic curve isomorphic to an open disc D or an open annulus B with center at zero, or to $\mathcal{X}_\eta^{\rm an} \backslash \coprod_{i=1}^n X_i$, $n \geq 1$, as above (with $k' = k$), was called in [Ber2] *elementary*. Notice that an open disc $D = D(0; r)$ with $r > 1$ is of the third form since its complement in the projective line $\mathbf{P}^1$ is isomorphic to the closed disc $E(0; r^{-1})$. In particular, if the valuation on k is nontrivial, any open disc is of that form. An open annulus $B = B(0; r, R)$ with $r < 1 < R$ is also of that form since its complement in $\mathbf{P}^1$ is isomorphic to a disjoint union of $E(0; r)$ and $E(0; R^{-1})$.

2.2.2. Remark. The change in the formulation of [Ber2, Proposition 3.6.1] implies that the formulations of Theorem 3.7.2 and Corollary 3.7.3 in [Ber2, §3.7] should be also changed. Recall that [Ber2, Theorem 3.7.2] describes a smooth morphism of good k-analytic spaces $\varphi : Y \to X$ of pure dimension one in an étale neighborhood of a point $y \in Y$. It stated that there is an étale morphism $X' \to X$ and an open subset $Y'' \subset Y' = Y \times_X X'$ such that the point y has a unique preimage y' in Y'' and the induced morphism $Y'' \to X'$ is an elementary fibration of pure dimension one at the point y'. In the case, when the field $\mathcal{H}(\varphi(y))$ is not perfect and its valuation is trivial, the correct statement is that all of the above is true except the requirement on the étaleness of the morphism $X' \to X$. Namely, it is only a composition of a radicial morphism $X' \to X''$ with an étale morphism $X'' \to X$. Here a morphism of good k-analytic spaces $f : X' \to X$ is said to be radicial if it is finite and every point of X has an affinoid neighborhood $V = \mathcal{M}(\mathcal{A})$ such that for its preimage $V' = \mathcal{M}(\mathcal{A}')$ in X' one has $\mathcal{A}' = \mathcal{A}[T_1, \ldots, T_n]/(T_i^{p^{l_i}} - f_i)$ with $f_1, \ldots, f_n \in \mathcal{A}^*$ and $i_1, \ldots, i_n \geq 0$. (Such a morphism is always finite flat, and it induces a homeomorphism between the underlying topological spaces.) In the same case (of the non-perfect field $\mathcal{H}(\varphi(y))$ with the trivial valuation) the correct statement of Corollary 3.7.3 should require that both morphism $X' \to X$ and $Y'' \to Y' = Y \times_X X'$ are compositions of a radicial morphism with an étale one. The above changes do not affect the applications of Theorem 3.7.2 in [Ber2], and here is one more application.

2.2.3. Proposition. *Given smooth k-analytic space of the same dimension n and points $x' \in X'$ and $x'' \in X''$ with $s(x') = s(x'') = n$, there exist étale morphisms $\varphi' : X \to X'$ and $\varphi'' : X \to X''$ that take a point $x \in X$ to x' and x'', respectively.*

Proof. We may assume that $X'' = \mathbf{A}^n$ and x'' is the maximal point of the closed unit polydisc with center at zero. If the valuation on k is trivial, then the image of the point x' under any étale morphism from an open neighborhood of x' to $\mathbf{A}^n$ is the point x''. Assume therefore that the valuation on k is nontrivial. If $n = 1$, then x' is type (2) and so, replacing k by a finite separable extension, we may assume that $X = \mathcal{X}_\eta^{\mathrm{an}} \backslash \coprod_{i=1}^m X_i$, $m \geq 1$, where $\mathcal{X}$ is a connected smooth projective curve over k°, x is the generic point of $\mathcal{X}_s$ in $\mathcal{X}_\eta^{\mathrm{an}}$, and X_i's are as in Proposition 2.2.1(2). Then any étale morphism $\mathcal{Z} \to A^n$ from a (nonempty) open affine subscheme of $\mathcal{X}$ gives rise to an étale morphism from an open neighborhood of the point x to $\mathbf{A}^n$ that takes x to the point x''. Assume now that $n \geq 2$ and the required fact is true for $n-1$. We may then assume that there is a smooth morphism $\varphi : X' \to Y$ of dimension one with $s(y) = n-1$, where $y = \varphi(x')$, such that Y admits an étale morphism to $\mathbf{A}^{n-1}$ that takes y to the maximal point of the closed unit polydisc with center at zero. By [Ber2, Theorem 3.7.2], the situation is reduced to the case when φ is an elementary fibration. It follows easily from the definition of the latter and Proposition 2.2.1 that one can replace Y by an étale neighborhood of the point y and find an étale morphism $\psi : X' \to Y \times \mathbf{A}^1$ over Y that takes x' to the maximal point of the closed unit disc in $\mathbf{A}^1_{\mathcal{H}(y)}$ with center at zero. Then the étale morphism $X' \to \mathbf{A}^n = \mathbf{A}^{n-1} \times \mathbf{A}^1$ induced by the above étale morphism $Y \to \mathbf{A}^{n-1}$ and ψ takes x' to the point x''. ∎

2.3. Étale neighborhoods of a point with with $s(x) < \dim(X)$. In this subsection, k is an arbitrary non-Archimedean field with a nontrivial valuation. Let X is a smooth k-analytic space of dimension n at a point $x \in X$.

2.3.1. Proposition. *Assume that $s(x) < n$. Then*

(i) one can shrink X so that there exists a smooth morphism $\varphi : X \to Y$ to a smooth k-analytic space Y of dimension $n-1$ with $s(y) = s(x)$ for $y = \varphi(x)$;

(ii) given a smooth morphism $\varphi : X \to Y$ as in (i), there exists an étale morphism $Y' \to Y$ and an open subset $X' \subset Y' \times_Y X$ such that the image of X' in X contains the point x and X' is isomorphic over Y' to $Y' \times D$, if $t(y) = t(x)$, and to $Y' \times B$, if $t(y) < t(x)$, where D and B are open disc and annulus with center at zero.

Proof. (i) We may assume that X is an open subset in $\mathcal{X}^{\mathrm{an}}$, where $\mathcal{X} = \mathrm{Spec}(B)$ is a smooth irreducible affine scheme over k. Let $\mathbf{x}$ be the image of x in $\mathcal{X}$. If $\mathbf{x}$ is not the generic point of $\mathcal{X}$ then, by Step 2 of Case (a) from the proof of [Ber7, Theorem 9.1], there exists an isomorphism of an open neighborhood of x onto $D(0; r) \times Y$ that takes x to $\{0\} \times Y$, where Y is a smooth k-analytic space of dimension $n-1$. Thus, we may assume that $\mathbf{x}$ is the generic point of $\mathcal{X}$. Let $s = s(x)$.

The image of the field $k(\mathbf{x})$ is dense in $\mathcal{H}(x)$. It follows that we can find elements $f_1, \ldots, f_s \in$

$k(\mathbf{x})$ with $|f_i(x)| = 1$ such that their images in $\widetilde{\mathcal{H}(x)}$ are algebraically independent over $\widetilde{k}$. It is then clear that $f_1, \dots, f_s$ are algebraically independent over k. We can extend this system of elements to a system $f_1, \dots, f_{n-1}$ of algebraically independent elements of $k(\mathbf{x})$ over k. Shrinking $\mathcal{X}$, we may assume that $f_1, \dots, f_{n-1} \in B$. Consider the morphism f from $\mathcal{X}$ to the $(n-1)$-dimensional affine space, defined by the elements $f_1, \dots, f_{n-1}$. Then $\mathbf{y} = f(\mathbf{x})$ is the generic point of the affine space and, therefore, the morphism f is smooth at the point $\mathbf{x}$. Shrinking $\mathcal{X}$, we may assume that f is smooth. It follows that the morphism of k-analytic spaces $\varphi = f^{\mathrm{an}} : \mathcal{X}^{\mathrm{an}} \to \mathbf{A}^{n-1}$ is also smooth and, by the construction, $s(y) = s$, where $y = \varphi(x)$.

(ii) Since $s(y) = s(x)$, it follows that $s_{\mathcal{H}(y)}(x) = 0$, i.e., the type of x in the fiber of φ at y is not (2). By the local description of smooth analytic curves (Proposition 2.2.1), if x is of type (1) or (4) (resp. (3)), there is an étale morphism $g : Y \to \mathbf{A}^{n-1}$ with $g^{-1}(y) = \{y'\}$ and an open subset $X' \subset X \times_{\mathbf{A}^{n-1}} Y$ such that x has a unique preimage x' in X' and $X'_{y'}$ is isomorphic to $D(0; r)_{\mathcal{H}(y')}$ (resp. $B(0; r, R)_{\mathcal{H}(y')}$). By Proposition 3.7.8 (resp. 3.7.5) from [Ber2], one can shrink Y and X' so that $X' \overset{\sim}{\to} D(0; r') \times Y$ with $0 < r' < r$ (resp. $X' \overset{\sim}{\to} B(0; r', R') \times Y$ with $r < r' < R' < R$). ∎

In the situation of Proposition 2.3.1 we say that the morphism $X' \to X$ is an *Y-split neighborhood of x*. Furthermore, given two Y-split neighborhoods $X' \to X$ and $X'' \to X$ of the point x, we say that the latter *refines* the former if $X'' \to X$ and the corresponding morphism $Y'' \to Y$ go through compatible étale morphisms $X'' \to X'$ and $Y'' \to Y'$.

2.3.2. Corollary. *Given two Y-split neighborhoods $X' \to X$ and $X'' \to X$ of x, there exists an Y-split neighborhood of the same point that refines both of them.*

Proof. It is enough to apply Proposition 2.3.1 to the induced morphism $X' \times_X X'' \to Y' \times_Y Y''$ and a preimage of the point x in $X' \times_X X''$. ∎

2.3.3. Corollary. *Let x be a point of a smooth k-analytic space X with $n = \dim_x(X)$. Then*

(i) $s(x)$ is the minimal number s such that x has a fundamental system of étale neighborhoods isomorphic to a direct product of a smooth k-analytic space of dimension s and a semi-annular space (of dimension $n - s$);

(ii) $t(x)$ is the minimal number t such that x has a fundamental system of étale neighborhoods as in (i) with the semi-annular spaces of dimension $(t, n - s(x) - t)$.

Proof. Proposition 2.3.1 implies that x has a fundamental system of étale neighborhoods isomorphic to a direct product of a smooth k-analytic space of dimension $s(x)$ and a semi-annular space of dimension $(t(x), n - s(x) - t(x))$. Assume that there exists a fundamental system as in (i) with $s < s(x)$, and let f be an element of $\mathcal{O}_{X,x}$ with $|f(x)| = 1$ whose image $\widetilde{f(x)}$ of $f(x)$ in

$\widetilde{\mathcal{H}(x)}$ is transcendent over $\widetilde{k}$. Increasing k, we can find elements $a_1, \dots, a_{n-s+1} \in (k^\circ)^*$ such that their images in $\widetilde{k}$ are pairwise distinct. Shrinking X, we may assume that $f \in \mathcal{O}(X)$ and all of the functions $f - a_i$ are invertible. By the assumption, we may replace X by an étale neighborhood of x isomorphic to $Y \times Z$, where Y is a smooth k-analytic space of dimension s and Z is a semi-annular space of dimension $n-s$. The description of invertible analytic functions on such X implies that there exists $(i_1, \dots, i_{n-s+1}) \in \mathbf{Z}^{n-s+1} \backslash \{0\}$ with $(f-a_1)^{i_1} \cdot \ldots \cdot (f - a_{n-s+1})^{i_{n-s+1}} = gh$, where $g \in \mathcal{O}(Y)^*$ and $h \in \mathcal{O}(X)^*$ is such that $|h(x') - 1| < 1$ for all points $x' \in X$. The image of this equality in $\widetilde{\mathcal{H}(x)}$ gives an equation of algebraic dependence of $\widetilde{f(x)}$ over $\widetilde{\mathcal{H}(y)}$, where y is the image of x in Y. It follows that $\widetilde{\mathcal{H}(x)}$ is algebraic over $\widetilde{\mathcal{H}(y)}$, which is a contradiction. Assume now that there exists a fundamental system as in (ii) with $t < m = t(x)$, and let $f_1, \dots, f_m$ be elements of $\mathcal{O}^*_{X,x}$ whose images in the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$ form its basis. Shrinking X, we may assume that $f_1, \dots, f_m \in \mathcal{O}(X)^*$ and $X = Y \times Z$, where Y is a smooth k-analytic space of dimension $s(x)$ and Z is a semi-annular space of dimension $(t, n - s(x) - t)$. From the description of invertible analytic functions on such X it follows that there exists $(i_1, \dots, i_m) \in \mathbf{Z}^m \backslash \{0\}$ with $f_1^{i_1} \cdot \ldots \cdot f_m^{i_m} = gh$, where $g \in \mathcal{O}(Y)^*$ and $h \in \mathcal{O}(X)^*$ is such that $|h(x') - 1| < 1$ for all points $x' \in X$. Since $\sqrt{|\mathcal{H}(y)^*|} = \sqrt{|k^*|}$, the latter contradicts the property of $f_1, \dots, f_m$ at x. ∎

2.3.4. Corollary. *Let $\mathcal{F}$ be an étale $\mathcal{O}_X$-submodule of $\mathfrak{N}^K_X$ and $\overline{x}$ a geometric point of X such that the stalk $\mathcal{F}_{\overline{x}}$ is a union of free $\mathcal{O}_{X,\overline{x}}$-modules. Then for any morphism $\varphi : Y \to X$ from a smooth k-analytic space Y and any geometric point $\overline{y}$ of Y over $\overline{x}$ the canonical homomorphism $\varphi^*(\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} \to \mathfrak{N}^K_{Y,\overline{y}}$ is injective.*

Proof. If $\varphi^\#(\mathcal{F})$ denotes the image of the homomorphism $\varphi^*(\mathcal{F}) \to \mathfrak{N}^K_Y$, then the required fact is equivalent to the bijectivity of the map $\varphi^*(\mathcal{F})_{\overline{y}} \to \varphi^\#(\mathcal{F})_{\overline{y}}$ and, if the latter is true, then $\varphi^\#(\mathcal{F})_{\overline{y}}$ is a union of free $\mathcal{O}_{Y,\overline{y}}$-module. It follows that, if φ is a composition of two morphisms for which the required fact is true, then it is also true for φ. Since we can shrink X and Y so that φ can be represented as a composition of a closed immersion of Y into $X \times D$ and the canonical projection $X \times D \to X$, where D is an open polydisc, it suffices to consider the following two cases. In both of them we verify that, given elements $f_1, \dots, f_n \in \mathcal{F}(X)$ whose images in the stalk $\mathcal{F}_x$ of a point $x \in X$ are linearly independent over $\mathcal{O}_{X,x}$, their pullbacks in $\varphi^\#(\mathcal{F})_y$, where $y \in \varphi^{-1}(x)$, are linearly independent over $\mathcal{O}_{Y,y}$. Assume the latter is not true. Shrinking X and Y, we may assume that there exist analytic functions $g_1, \dots, g_n \in \mathcal{O}(Y)$ with $\sum_{i=1}^n f_i g_i = 0$.

Case 1: φ is a closed immersion. Since the statement is local with respect to the étale topology, we may assume that $X \overset{\sim}{\to} Y \times D$, where D is the unit open polydisc with center at zero, and the immersion identifies Y with the zero section of the projection $p: X = Y \times D \to Y$. It follows that $\sum_{i=1}^n f_i p^*(g_i) = 0$ and, therefore, all g_i's are zero.

Case 2: φ is smooth. We may assume that φ is of dimension one. To show that $g_1 = \ldots = g_n = 0$, we may replace Y by an étale neighborhood of any point from the preimage $\varphi^{-1}(x)$. We can therefore shrink X and Y in the étale topology and assume that $Y = X \times D$, where D is the unit open disc with center at zero. If σ is a section of φ, then $\sum_{i=1}^n f_i \sigma^*(g_i) = 0$ and, by the assumption on linear independence, $\sigma^*(g_i) = 0$ for all $1 \le i \le n$. Thus, it suffices to show that, given a nonzero analytic function $g \in \mathcal{O}(X \times D)$, there is a section σ with $\sigma^*(g) \neq 0$.

The function g is of the form $\sum_{i=0}^\infty h_i T^i$, where $h_i \in \mathcal{O}(X)$. Every element $\alpha \in k$ with $|\alpha| < 1$ defines a section σ_α that takes z to (z, α) and, if $\sigma_\alpha^*(g) = 0$, one has $\sum_{i=0}^\infty h_i \alpha^i = 0$. Restricting the latter equality to the fiber $\varphi^{-1}(x')$ at an arbitrary point $x' \in X$, we get $h_i(x') = 0$ for all $i \ge 0$. It follows that all h_i's are zero which is a contradiction. ∎

2.4. Basic curves. If $\mathbf{x}$ is a closed point of the closed fiber of a scheme or formal scheme over k° such that the field $\widetilde{k}(\mathbf{x})$ is separable over $\widetilde{k}$, we denote by $k_\mathbf{x}$ the finite unramified extension of k with the residue field $\widetilde{k}(\mathbf{x})$. Assume that k is a closed subfield of $\widehat{k}_0^{\mathrm{a}}$ that contains k_0, where k_0 is a fixed non-Archimedean field whose valuation is nontrivial and discrete.

A *smooth basic curve* is a connected smooth k-analytic space X isomorphic to $X' \widehat{\otimes}_{k'_0} k$ with a finite extension k'_0 of k_0 in k and $X' = \mathcal{X}_\eta^{\mathrm{an}} \backslash \coprod_{i=1}^n X_i$, $n \ge 0$, where $\mathcal{X}$ is a smooth projective curve over $k_0'^\circ$, each X_i is an affinoid subdomain of $\pi^{-1}(\mathbf{x}_i)$ for a closed point $\mathbf{x}_i \in \mathcal{X}_s$, whose field $\widetilde{k}'_0(\mathbf{x}_i)$ is separable over $\widetilde{k}'_0$, and is isomorphic to a closed disc $E(0; r_i) \widehat{\otimes} (k'_0)_{\mathbf{x}_i}$ with $r \in |k_0'^*|$, and the closed points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{X}_s$ are pairwise distinct.

2.4.1. Proposition. *The following properties of a smooth k-analytic curve X are equivalent:*

(a) X is a smooth basic curve;

(b) X is isomorphic to the generic fiber $\mathfrak{X}_\eta$ of a proper marked formal scheme $\mathfrak{X}$ over k°.

Proof. To prove the statement we may assume that $k = k_0$.

(a)$\Longrightarrow$(b) Assume X is isomorphic to $\mathcal{X}_\eta^{\mathrm{an}} \backslash \coprod_{i=1}^n X_i$, $n \ge 0$, as above. *We claim that there is an admissible blow-up $\varphi: \mathcal{X}' \to \mathcal{X}$ whose center is supported in $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and such that the scheme $\mathcal{X}'$ is nondegenerate strictly poly-stable over k° and $X = \pi^{-1}(\mathcal{Y})$, where $\mathcal{Y}$ is the irreducible component of $\mathcal{X}'_s$ for which the induced morphism $\mathcal{Y} \to \mathcal{X}_s$ is surjective.* Indeed, it is enough to construct such a blow-up separately for each point $\mathbf{x} = \mathbf{x}_i$. Since the field $\widetilde{k}(\mathbf{x})$ is separable over $\widetilde{k}$, there is an étale morphism from an open neighborhood of $\mathbf{x}$ to the affine line over k° that takes $\mathbf{x}_i$ to the zero point. It induces an isomorphism of $\pi^{-1}(\mathbf{x}) \subset \mathcal{X}_\eta^{\mathrm{an}}$ with the open disc $D(0;1)\widehat{\otimes} k_\mathbf{x}$ that takes X_i to a closed disc $E(0,r)$ with $r \in |k^*|$. Let $\mathcal{J}$ be the coherent sheaf of ideals which coincides with $\mathcal{O}_{\mathcal{X}}$ outside the point $\mathbf{x}$ and is generated by the coordinate function T of the affine line and an element $a \in k^\circ$ with $|a| = r$. The blow-up $\varphi : \mathcal{X}' \to \mathcal{X}$ with center at $\mathcal{J}$ induces an isomorphism $\mathcal{X}' \backslash \varphi^{-1}(\mathbf{x}) \xrightarrow{\sim} \mathcal{X} \backslash \{\mathbf{x}\}$, and $\varphi^{-1}(\mathbf{x})$ is an irreducible component of $\mathcal{X}'_s$ isomorphic to the projective line over $\widetilde{k}(\mathbf{x})$. There is an étale morphism from an open neighborhood of $\mathbf{x}'$, the zero point of the projective line, to the affine scheme $\mathrm{Spec}(k^\circ[T,u]/(Tu-a))$. The claim now follows from the fact that the preimage of $\varphi^{-1}(\mathbf{x}) \backslash \{\mathbf{x}'\}$ in $\mathcal{X}'^{\mathrm{an}}_\eta$ under the reduction map is identified with the closed disc $E(0;|a|)\widehat{\otimes} k_{\mathbf{x}}$, and the preimage of the infinity point of $\varphi^{-1}(\mathbf{x})$ is identified with the open disc $D(0;|a|)\widehat{\otimes} k_{\mathbf{x}}$. Since $X = \pi^{-1}(\mathcal{Y})$ coincides with $(\widehat{\mathcal{X}}'_{/\mathcal{Y}})_\eta$, we get the required fact.

(b)$\Longrightarrow$(a) Let $\mathfrak{X}$ be the formal completion $\widehat{\mathcal{X}}^{\mathrm{an}}_{/\mathcal{Y}}$ of a nondegenerate strictly poly-stable curve $\mathcal{X}$ over k° along an irreducible component $\mathcal{Y}$ of $\mathcal{X}_s$ proper over $\widetilde{k}$, and let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be the closed points of $\mathcal{Y}$ that lie also in some of the other irreducible components of $\mathcal{X}_s$. The assumption on $\mathcal{X}$ implies that every point $\mathbf{x}_i$ has an open neighborhood in $\mathcal{X}$ that admits an étale morphism to a scheme $\mathrm{Spec}(k^\circ[u,v]/(uv-a_i))$ with $a_i \in k^\circ \backslash \{0\}$.) The point $\mathbf{x}_i$ goes under this étale morphism to the zero point and, therefore, the field $\widetilde{k}(\mathbf{x}_i)$ is separable over $\widetilde{k}$. This easily implies that the open subset $\pi^{-1}(\mathbf{x}_i)$ of $\mathcal{X}^{\mathrm{an}}_\eta$ is isomorphic to the open annulus $B(|a_i|,1)\widehat{\otimes} k_{\mathbf{x}_i}$ over $k_{\mathbf{x}_i}$ with center at zero (see Lemma 3.1.3 for a more general statement). Let X' be the k-analytic space obtained by gluing of X with the open discs $D(0;1)\widehat{\otimes} k_{\mathbf{x}_i}$ along the above annuli, respectively. The space X' is a smooth compact k-analytic curve and, therefore, it is the analytification of a smooth projective curve over k. It is also easy to see that that curve has good reduction, i.e., X' is isomorphic to $\mathcal{X}'^{\mathrm{an}}_\eta$, where $\mathcal{X}'$ is a smooth projective curve over k°. It follows that X is isomorphic to $\mathcal{X}'^{\mathrm{an}}_\eta \backslash \cup_{i=1}^n E(0;|a_i|)\widehat{\otimes} k_{\mathbf{x}_i}$, i.e., it is a basic curve. ∎

A *k-affinoid basic curve* is a connected strictly k-affinoid space X isomorphic to $Y\widehat{\otimes}_{k'_0} k$ with k'_0 a finite extension of k_0 in k and $Y = \mathcal{X}^{\mathrm{an}}_\eta \backslash \coprod_{i=1}^n Y_i$, $n \geq 1$, where $\mathcal{X}$ is a smooth projective curve over k'°_0, each Y_i is an open subset of $\pi^{-1}(\mathbf{x}_i)$ for a closed point $\mathbf{x}_i \in \mathcal{X}_s$, whose field $\widetilde{k}'_0(\mathbf{x}_i)$ is separable over $\widetilde{k}'_0$, and is isomorphic to an open disc $D(0;r_i)\widehat{\otimes}(k'_0)_{\mathbf{x}_i}$ with $r_i \in |k'^*_0|$, and the closed points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{X}_s$ are pairwise distinct. For a k-affinoid basic curve X provided with such an isomorphism, let X' denote the open subset that corresponds to $Y' = \mathcal{X}^{\mathrm{an}}_\eta \backslash \coprod_{i=1}^n Y'_i$, where Y'_i denotes the bigger closed disc $E(0,r_i)\widehat{\otimes}(k'_0)_{\mathbf{x}_i}$. Then X' is a smooth basic curve.

2.4.2. Proposition. *In the above situation, given a k_0-special formal scheme $\mathfrak{Y}$ over k° and a morphism of strictly k-analytic spaces $\varphi : X \to \mathfrak{Y}_\eta$, there exists a finite extension k' of k and a finite open covering $X'\widehat{\otimes} k' = \cup_{i=1}^m X'_i$ such that each X'_i is the generic fiber $\mathfrak{X}^i_\eta$ of a strongly* *marked formal scheme $\mathfrak{X}^i$ over k'° (and, in particular, X'_i is a smooth basic k'-analytic curve) and the induced morphism $X'_i \to \mathfrak{Y}_\eta$ comes from a morphism of formal schemes $\mathfrak{X}^i \to \mathfrak{Y}$ over k°.*

Proof. First of all, we can increase the field k_0 and assume that X comes from a k_0-affinoid basic curve Y. By the proof of Proposition 2.4.1, there is a nondegenerate strictly poly-stable curve $\mathcal{Y}$ projective over k_0° whose closed fiber $\mathcal{Y}_s$ is a union of irreducible components $\mathcal{Z}, \mathcal{Z}_1, \dots, \mathcal{Z}_n$ such that each $\mathcal{Z}_i$ is isomorphic to the projective line over $\widetilde{F}$, $\mathcal{Z}_i \cap \mathcal{Z} = \{\mathbf{x}_i\}$ and $\mathcal{Z}_i \cap \mathcal{Z}_j = \emptyset$ for $i \neq j$, and one has $Y' = (\widehat{\mathcal{Y}}_{/\mathcal{Z}})_\eta$ and $Y = (\widehat{\mathcal{Y}}_{/\mathcal{W}})_\eta$, where $\mathcal{W}$ is an open subset of $\mathcal{Y}_s$ with $\mathcal{Y}_s \backslash \mathcal{W} = \{\mathbf{z}_1, \dots, \mathbf{z}_n\}$ and each $\mathbf{z}_i$ is a closed point of $\mathcal{Z}_i$ different of $\mathbf{x}_i$. Furthermore, as in the proof of Proposition 2.1.1(ii), we can increase the field k_0 and find rational strictly affinoid covering $\{W_i\}_{1\le i\le l}$ of Y such that $\varphi(W_i \widehat{\otimes}_{k_0} k) \subset \mathfrak{Y}^i_\eta$ for some open affine subschemes $\mathfrak{Y}^1, \dots, \mathfrak{Y}^l \subset \mathfrak{Y}$. One can then construct an admissible blow-up $\psi : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ and open subschemes $\mathcal{W}_j \subset \psi^{-1}(\mathcal{W})$, $1 \le j \le l$, with $\mathcal{W} = \cup_{j=1}^l \mathcal{W}_j$ and $W_j = \pi^{-1}(\mathcal{W}_j)$. By the semi-stable reduction theorem for curves, there exists a finite extension k'_0 of k_0 and a morphism $\mathcal{Y}' \to \widetilde{\mathcal{Y}} \otimes_{k_0^\circ} k_0'^\circ$ from a strictly semi-stable projective curve $\mathcal{Y}'$ over $k_0'^\circ$ that gives rise to an isomorphism $\mathcal{Y}'_\eta \xrightarrow{\sim} \mathcal{Y}_\eta \otimes_{k_0} k'_0$. If $\mathcal{Z}'$ denotes the preimage of $\mathcal{Z}$ in $\mathcal{Y}'_s$, we get an isomorphism $(\widehat{\mathcal{Y}}'_{/\mathcal{Z}'})_\eta \xrightarrow{\sim} Y' \otimes_{k_0} k'_0$. If k' is the composite of k and k'_0 in $\widehat{k^{\mathrm{a}}}_0$, one shows as in the Step 4 from the proof of Proposition 2.1.1 that the induced morphism $(\widehat{\mathcal{Y}}'_{/\mathcal{Z}'})_\eta \widehat{\otimes}_{k'_0} k' \to \mathfrak{Y}_\eta$ comes from a morphism of formal schemes $\widehat{\mathcal{Y}}'_{/\mathcal{Z}'} \widehat{\otimes}_{k_0'^\circ} k'^\circ \to \mathfrak{Y}$ over k°. Since $\mathcal{Z}$ is an irreducible component of $\mathcal{Y}_s$, its preimage $\mathcal{Z}'$ is a union of irreducible components $\mathcal{Z}'_1, \dots, \mathcal{Z}'_m$ of $\mathcal{Y}'_s$. It follows that the required fact holds for the strongly marked formal schemes $\mathfrak{X}^i = \widehat{\mathcal{Y}}'_{/\mathcal{Z}'_i} \widehat{\otimes}_{F'^\circ} k'^\circ$ and the smooth basic curves $X'_i = \mathfrak{X}^i_\eta$. ∎

§3. Properties of strictly poly-stable and marked formal schemes

In this section we study properties of a proper marked formal scheme $\mathfrak{X}$ over k° and its generic fiber $\mathfrak{X}_\eta$. For example, the reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ is surjective and the preimage $\pi^{-1}(\mathbf{x})$ of a closed point $\mathbf{x} \in \mathfrak{X}_s$ with $\widetilde{k}(\mathbf{x})$ separable over $\widetilde{k}$ is a semi-annular space. In particular, if the characteristic of k is zero and the field $\widetilde{k}$ is perfect, any closed one-form on $\mathfrak{X}_\eta$, whose restriction to the residue class $\pi^{-1}(\mathbf{x})$ of every closed point $\mathbf{x} \in \mathfrak{X}_s$ is in $\Omega^1_{L^\lambda}(\pi^{-1}(\mathbf{x}))$, has a primitive in the class of functions whose restrictions to every residue class $\pi^{-1}(\mathbf{x})$ are in $L^\lambda(\pi^{-1}(\mathbf{x}))$. The following property is of crucial importance for the construction in §7, which provides a way to connect all of such primitives: the generic point of $\mathfrak{X}$ (i.e., the unique preimage of the generic point of $\mathfrak{X}_s$ in $\mathfrak{X}_\eta$) has a fundamental system of open neighborhoods in $\mathfrak{X}_\eta$ whose intersections with all of the residue classes are nonempty and connected. Furthermore, we consider the stratification of the closed fiber $\mathfrak{X}_s$, constructed in [Ber7], and show that for any morphism of marked formal schemes $\mathfrak{X}' \to \mathfrak{X}$ the image of a stratum of $\mathfrak{X}'_s$ is contained in a stratum of $\mathfrak{X}_s$. We also construct for every stratum closure $\mathcal{Y} \subset \mathfrak{X}_s$ an open neighborhood $\mathfrak{D}_\mathcal{Y}$ of the image of $\pi^{-1}(\mathcal{Y})$ under the diagonal map $\mathfrak{X}_\eta \to \mathfrak{X}_\eta \times \mathfrak{X}_\eta$ and establish a geometric property of the canonical projection to the first coordinate $\mathfrak{D}_\mathcal{Y} \to \pi^{-1}(\mathcal{Y})$.

3.1. Strictly poly-stable formal schemes. Recall (see [Ber7, §1]) that a scheme (resp. formal scheme) over k° is said to be *strictly poly-stable* if it has a locally finite covering by open affine subschemes that admit an étale morphism to an affine scheme of the form $\mathrm{Spec}(A_0) \times \ldots \times \mathrm{Spec}(A_p)$ (resp. $\mathrm{Spf}(A_0) \times \ldots \times \mathrm{Spf}(A_p)$), where each A_i is of the form $k^\circ[T_0, \ldots, T_n]/(T_0 \cdot \ldots \cdot T_n - a)$ (resp. $k^\circ\{T_0, \ldots, T_n\}/(T_0 \cdot \ldots \cdot T_n - a)$) with $a \in k^\circ$ and $n \geq 0$. If all of the elements a are non-zero, the strictly poly-stable scheme (resp. formal scheme) is said to be *nondegenerate* (see [Ber9]). In this case its generic fiber is smooth (resp. rig-smooth). Recall that all irreducible components of the closed fiber of a strictly poly-stable scheme or formal scheme over k° are smooth over $\widetilde{k}$. If a scheme $\mathcal{X}$ is strictly poly-stable over k°, then so is its formal completion along the closed fiber $\widehat{\mathcal{X}}$. If the valuation on k is nontrivial and discrete, any strictly semi-stable scheme or formal scheme over k° is nondegenerate strictly poly-stable. The advantage of strictly poly-stable schemes and formal schemes is in the facts that they are defined over non-Archimedean fields with not necessarily discrete valuation and that their class is preserved under direct products and the ground field extension functor.

For an integral algebra A over a field L, let $\mathfrak{c}(A/L)$ denote the subfield of the elements of A which are algebraic and separable over L. When it is clear what the field L is considered, the field

$\mathfrak{c}(A/L)$ is denoted by $\mathfrak{c}(A)$. For example, by [Ber9, Corollary 8.1.3(ii)], the stalk $\mathfrak{c}_{X,x}$ of the sheaf of constant analytic functions $\mathfrak{c}_X$ on a geometrically reduced strictly k-analytic space X at a point $x \in X$ coincides with $\mathfrak{c}(\mathcal{H}(x)) = \mathfrak{c}(\mathcal{H}(x)/k)$.

3.1.1. Lemma. *Let $\mathfrak{X}$ be a formal scheme locally finitely presented over k°, $\mathcal{Y}$ a smooth irreducible component of $\mathfrak{X}_s$, and σ the generic point of $\mathcal{Y}$ in $\mathfrak{X}_\eta$. Then*

(i) the algebraic closure of k in $\mathcal{H}(\sigma)$ is the finite unramified extension of k with the residue field $\mathfrak{c}(\widetilde{k}(\mathcal{Y}))$ and, in particular, it coincides with $\mathfrak{c}(\mathcal{H}(\sigma))$;

(ii) $\mathfrak{c}(\pi^{-1}(\mathcal{Y})) \stackrel{\sim}{\to} \mathfrak{c}(\mathcal{H}(\sigma))$.

Proof. Let $\mathfrak{Y}$ be a nonempty open subscheme of $\mathfrak{X}$ such that $\mathfrak{Y}_s \subset \mathcal{Y}$ and $\mathfrak{Y}_s$ does not intersect with the other irreducible components of $\mathfrak{X}_s$. Then $\mathfrak{Y}$ is smooth, $\mathfrak{Y}_s$ is irreducible, and σ is its generic point in $\mathfrak{Y}_\eta$. By [Ber7, Corollary 1.7(ii)], one has $\widetilde{\mathcal{H}(\sigma)} = \widetilde{k}(\mathcal{Y})$ and, therefore, the finite unramified extension k' of k with the residue field $\mathfrak{c}(\widetilde{k}(\mathcal{Y}))$ is embedded in $\mathcal{H}(\sigma)$. We claim that each element of k' comes from a constant analytic function from $\pi^{-1}(\mathcal{Y})$. Indeed, consider the induced étale morphism $\varphi : \mathfrak{X}' = \mathfrak{X}\widehat{\otimes}k'^\circ \to \mathfrak{X}$. Since $\mathcal{Y}$ is smooth, there is an irreducible component $\mathcal{Y}'$ of $\mathfrak{X}'_s$ such that φ induces an isomorphism $\mathcal{Y}' \stackrel{\sim}{\to} \mathcal{Y}$, i.e., by [Ber7, Lemma 4.4], it induces an isomorphism $\pi^{-1}(\mathcal{Y}') \stackrel{\sim}{\to} \pi^{-1}(\mathcal{Y})$ and the claim follows. It remains to show that the algebraic closure of k in $\mathcal{H}(\sigma)$ coincides with k'. For this we can replace k by k', $\mathfrak{X}$ by $\mathfrak{X}'$ and $\mathcal{Y}$ by $\mathcal{Y}'$, and so we may assume that $\mathcal{Y}$ is geometrically irreducible. Furthermore, we may shrink $\mathfrak{X}$ and assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine and $\mathfrak{X}_s$ is irreducible. Consider an arbitrary epimorphism $k^\circ\{T_1, \dots, T_n\} \to A$. By [Ber7, Proposition 1.4], one has $\mathcal{A}^\circ = A$ and $|\mathcal{A}|_{\sup} = |k|$, where $\mathcal{A} = A \otimes k$, and, therefore, [BGR, Corollary 6.4.3/6] implies that the spectral norm on $\mathcal{A}$ coincides with the quotient norm with respect to the induced epimorphism $k\{T_1, \dots, T_n\} \to \mathcal{A}$. From [Ber1, 5.2.2 and 5.2.5] it follows that for any non-Archimedean field K over k the norm on the Banach algebra $\mathcal{H}(x)\widehat{\otimes}K$ is multiplicative. Applying this to finite extensions of k, we get the required fact. ∎

From Lemma 3.1.1 it follows that for any connected open neighborhood $\mathcal{U}$ of the point σ in $\pi^{-1}(\mathcal{Y})$, one has $\mathfrak{c}(\pi^{-1}(\mathcal{Y})) \stackrel{\sim}{\to} \mathfrak{c}(\mathcal{U}) \stackrel{\sim}{\to} \mathfrak{c}(\mathcal{H}(\sigma))$.

Recall that a standard formal scheme over k° is a formal scheme of the form $\mathfrak{T}(\mathbf{n}, \mathbf{a}) \times \mathfrak{S}(m)$, where either $\mathbf{n} = (n_0, \dots, n_p) \in \mathbf{Z}^{p+1}$, $n_i \geq 1$, $\mathbf{a} = (a_0, \dots, a_p) \in (k^{\circ\circ})^{p+1}$, $\mathfrak{T}(\mathbf{n}, \mathbf{a}) = \mathrm{Spf}(A_0) \times \dots \times \mathrm{Spf}(A_p)$, $A_i = k^\circ\{T_{i0}, \dots, T_{in_i}\}/(T_{i0} \cdot \ldots \cdot T_{in_i} - a_i)$, $\mathfrak{S}(m) = \mathrm{Spf}(k^\circ\{S_1, \dots, S_m, \frac{1}{S_1}, \dots, \frac{1}{S_m}\})$, or $p = n_0 = 0$, $a_0 = 1$ and $\mathfrak{T}(0, 1) = \mathrm{Spf}(k^\circ)$ (see [Ber7, §1]). Given a strictly poly-stable formal scheme $\mathfrak{X}$ over k°, every point $\mathbf{x} \in \mathfrak{X}_s$ has an open connected affine neighborhood $\mathfrak{X}'$ of $\mathbf{x}$ for which there exists an étale morphism to a standard formal scheme $\varphi : \mathfrak{X}' \to \mathfrak{T} = \mathfrak{T}(\mathbf{n}, \mathbf{a}) \times \mathfrak{S}(m)$ such that the point $\varphi_s(\mathbf{x})$ is contained in the intersection of all irreducible components of $\mathfrak{T}_s$. The tuples $\mathbf{n}$ and $|\mathbf{a}|$ are determined uniquely, up to a permutation of the set $\{0,\dots,p\}$, by the stratum of $\mathfrak{X}_s$ (see §3.3) that contains the point $\mathbf{x}$ (see [Ber7, §4]). The triple $(\mathbf{n},|\mathbf{a}|,m)$ will be called the *type* of $\mathbf{x}$ (or of the stratum).

3.1.2. Lemma. *Given a closed point $\mathbf{x}\in\mathfrak{X}_s$ of type $(\mathbf{n},|\mathbf{a}|,m)$ such that the field $\widetilde{k}(\mathbf{x})$ is separable over $\widetilde{k}$, there is an isomorphism $\pi^{-1}(\mathbf{x})\xrightarrow{\sim}(Y\times D^m)\widehat{\otimes}k_{\mathbf{x}}$, where D^m is an open unit polydisc of dimension m with center at zero, Y is the closed analytic subset of the open unit polydisc $D^{|\mathbf{n}|+p+1}$ defined by the equations $T_{i0}\cdot\ldots\cdot T_{in_i}=a_i$ for $0\le i\le p$, and $|\mathbf{n}|=n_0+n_1+\ldots+n_p$.*

Proof. We may assume that $\mathfrak{X}'=\mathfrak{X}$. There exists a $\widetilde{k}_{\mathbf{x}}$-rational point $\mathbf{x}'\in\mathfrak{X}_s\otimes\widetilde{k}_{\mathbf{x}}$ over $\mathbf{x}$. If $\mathbf{t}'$ is the unique preimage of $\mathbf{t}$ in $\mathfrak{T}_s\otimes\widetilde{k}_{\mathbf{x}}$ then, by [Ber7, Lemma 4.4], the induced étale morphisms $\mathfrak{X}\widehat{\otimes}k^\circ_{\mathbf{x}}\to\mathfrak{T}\widehat{\otimes}k^\circ_{\mathbf{x}}$ and $\mathfrak{X}\widehat{\otimes}k^\circ_{\mathbf{x}}\to\mathfrak{X}$ give rise to isomorphisms $\pi^{-1}(\mathbf{x}')\xrightarrow{\sim}\pi^{-1}(\mathbf{t}')$ and $\pi^{-1}(\mathbf{x}')\xrightarrow{\sim}\pi^{-1}(\mathbf{x})$, and the lemma easily follows. ∎

3.1.3. Corollary. *In the situation of Lemma 3.1.2, assume that $\mathfrak{X}$ is nondegenerate. Then*

(i) $\pi^{-1}(\mathbf{x})$ is a semi-annular $k_{\mathbf{x}}$-analytic space;

(ii) for any $f\in\mathcal{O}(\pi^{-1}(\mathbf{x}))^*$, the real valued function $x\mapsto|f(x)|$ extends by continuity to the closure of $\pi^{-1}(\mathbf{x})$ in $\mathfrak{X}_\eta$.

Proof. Using the notation from the formulation and proof of Lemma 3.1.2, the projection from $\pi^{-1}(\mathbf{x})$ as described in Lemma 3.1.2 to the coordinates $T_{i1},\dots,T_{in_i}$, $0\le i\le p$, gives rise to an isomorphism of $\pi^{-1}(\mathbf{x})$ with $(Z\times D^m)\widehat{\otimes}k_{\mathbf{x}}$, where Z is the open subset of $\mathbf{G}_{\mathrm{m}}^{|\mathbf{n}|}$ defined by the inequalities $|T_{ij}(x)|<1$ and $|(T_{i1}\cdot\ldots\cdot T_{in_i})(x)|>|a_i|$ for $0\le i\le p$ and $1\le j\le n_i$. It is a semi-annular $k_{\mathbf{x}}$-analytic space. To prove (ii), we may assume that $\mathfrak{X}=\mathrm{Spf}(B)$ is affine. From [Ber7, Theorem 5.2] it follows that the multiplicative monoid $B\cap\mathcal{B}^*$, where $\mathcal{B}=B\otimes_{k^\circ}k$, is generated by the subgroup B^* and the coordinate functions T_{ij}, and from [Ber7, Proposition 1.4] it follows that $\mathcal{B}^*=(B\cap\mathcal{B}^*)\cdot k^*$. These facts easily imply that $f=g\cdot h$, where $g\in\mathcal{B}^*$ and $h\in\mathcal{O}(\pi^{-1}(\mathbf{x}))^*$ is such that $|h(x)-1|<1$ for all $x\in\pi^{-1}(\mathbf{x})$, and (ii) follows. ∎

We are going to formulate consequences of Lemmas 3.1.1 and 3.1.2 in the situation we are interested in. Namely, let k_0 be a non-Archimedean field with a nontrivial discrete valuation and a perfect residue field $\widetilde{k}_0$, and k an extension of k_0 which is a closed subfield of $\widehat{k}_0^{\mathrm{a}}$, and let $\mathfrak{X}$ be a marked formal scheme over k° and $\sigma=\sigma_{\mathfrak{X}}$.

3.1.4. Corollary. *(i) $\mathfrak{c}(\mathfrak{X}_\eta)$ is the finite unramified extension of k with the residue field $\widetilde{k}(\mathfrak{X}_s)$, and it coincides with $\mathfrak{c}(\mathcal{H}(\sigma))$;*

(ii) $\mathfrak{X}$ is a marked formal scheme over k'° for any intermediate subfield $k\subset k'\subset\mathfrak{c}(\mathfrak{X}_\eta)$.

Proof. Increasing the field k_0, we may assume that $\mathfrak{X} = \mathfrak{Y}_{\mathcal{Z}}\widehat{\otimes}_{k_0^\circ} k^\circ$, where $\mathfrak{Y}$ is a nondegenerate strictly poly-stable formal scheme over k_0° and $\mathcal{Z}$ is an irreducible component of $\mathfrak{Y}$ for which $\mathcal{Z}' = \mathcal{Z} \otimes_{\widetilde{k}_0} \widetilde{k}$ is an irreducible component of $\mathfrak{Y}'_s$, where $\mathfrak{Y}' = \mathfrak{Y}\widehat{\otimes}_{k_0^\circ} k^\circ$. Since $\mathfrak{X}_\eta = \pi^{-1}(\mathcal{Z}')$ and $\mathfrak{X}_s = \mathcal{Z}'$, Lemma 3.1.1 implies (i). It follows also that $[\mathfrak{c}(\widetilde{k}(\mathcal{Z}')) : \widetilde{k}] = [\mathfrak{c}(\widetilde{k}_0(\mathcal{Z})) : \widetilde{k}_0]$ and, therefore, for any intermediate subfield $k \subset k' \subset \mathfrak{c}(\mathfrak{X}_\eta)$ there is an intermediate subfield $k_0 \subset k'_0 \subset \mathfrak{c}(\pi^{-1}(\mathcal{Z}))$ such that k' is the composite of k and k'_0 and $[k' : k] = [k'_0 : k_0]$. The reasoning from the proof of Lemma 3.1.1 shows that $\mathfrak{Y}_{\mathcal{Z}}$ can be considered as a marked formal scheme over ${k'_0}^\circ$, and (ii) follows. ∎

Furthermore, assume that the characteristic of k_0 is zero. We fix a filtered k-algebra K and a logarithmic character $\lambda : k^* \to K^*$ (see §1.4). For a differential form $\omega \in \Omega^q_{\mathfrak{N}^K}(\mathfrak{X}_\eta)$, $q \geq 0$, and a closed point $\mathbf{x} \in \mathfrak{X}_s$, we denote by $\omega_{\mathbf{x}}$ the restriction of ω to the open subset $\pi^{-1}(\mathbf{x})$. Furthermore, given an integer $n \geq 0$, we denote by $\mathcal{C}^{K,n}_R(\mathfrak{X})$ (resp. $R^{\lambda,n}(\mathfrak{X})$) the set of all functions $f \in \mathfrak{N}^K(\mathfrak{X}_\eta)$ such that $f_{\mathbf{x}} \in \mathcal{C}^{K,n}(\pi^{-1}(\mathbf{x})) = k_{\mathbf{x}} \otimes_k K^n$ (resp. $f_{\mathbf{x}} \in L^{\lambda,n}(\pi^{-1}(\mathbf{x}))$) for all closed points $\mathbf{x} \in \mathfrak{X}_s$. We also denote by $\Omega^q_{R^{\lambda,n}}(\mathfrak{X})$ the set of all $\mathfrak{N}^K$-differential q-forms $\omega \in \Omega^q_{\mathfrak{N}^K}(\mathfrak{X}_\eta)$ such that $\omega_{\mathbf{x}} \in \Omega^q_{L^{\lambda,n}}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$. Notice that $d(R^{\lambda,n}(\mathfrak{X})) \subset \Omega^1_{R^{\lambda,n}}(\mathfrak{X})$.

3.1.5. Corollary. *In the above situation, the following is true:*
(i) $\mathrm{Ker}(R^{\lambda,n}(\mathfrak{X}) \xrightarrow{d} \Omega^1_{R^{\lambda,n}}(\mathfrak{X})) = \mathcal{C}^{K,n}_R(\mathfrak{X})$*;*
(ii) the morphism of complexes $\Omega^\cdot_{R^{\lambda,n}}(\mathfrak{X}) \to \Omega^\cdot_{R^{\lambda,n+1}}(\mathfrak{X})$ *is homotopy equivalent to zero.*

Proof. We increase the field k_0 and use the notation from the proof of Corollary 3.1.3. Assume a closed point $\mathbf{x} \in \mathfrak{Y}'_s$ is of type $(\mathbf{n}, |\mathbf{a}|, m)$ in $\mathfrak{Y}'_s$. Then the projection from $\pi^{-1}(\mathbf{x})$ as described in Lemma 3.1.2 to the coordinates $T_{i1}, \dots, T_{i,n_i}$, $0 \leq i \leq p$, gives rise to an isomorphism of $\pi^{-1}(\mathbf{x})$ with $(Y \times D^m)\widehat{\otimes} k_{\mathbf{x}}$, where Y is the open subset of $\mathbf{G}^{|\mathbf{n}|}_{\mathrm{m}}$ defined by the inequalities $|T_{ij}(x)| < 1$ and $|(T_{i1} \cdot \ldots \cdot T_{in_i})(x)| > |a_i|$ for $0 \leq i \leq p$ and $1 \leq j \leq n_i$. It is a semi-annular $k_{\mathbf{x}}$-analytic space, and the required statements follow from Corollary 1.5.4. ∎

Finally, assume that $\mathrm{Log}^\lambda(a) \in k$ for all $a \in k^*$ with $|a| = 1$. (For example, this assumption is satisfied if the residue field $\widetilde{k}$ is algebraic over a finite field.) Given $n \geq 0$, we denote by $R^{\lambda,n}_0(\mathfrak{X})$ the subspace of functions $f \in R^{\lambda,n}(\mathfrak{X})$ such that $f\big|_{\pi^{-1}(\mathbf{x})} \in L^{\lambda,n}_0(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$, where $L^\lambda_0(\pi^{-1}(\mathbf{x}))$ is the filtered $\mathcal{O}(\pi^{-1}(\mathbf{x}))$-algebra of $L^\lambda(\pi^{-1}(\mathbf{x}))$ generated by $\mathrm{Log}^\lambda(g)$ for $g \in \mathcal{O}(\pi^{-1}(\mathbf{x}))^*$ with $|g(\sigma)| = 1$ (see Corollary 1.5.5). (Notice that the number $|g(\sigma)|$ is well defined by Corollary 3.1.3(ii).) In the similar way we introduce that subspaces $\Omega^q_{R^{\lambda,n}_0}(\mathfrak{X}) \subset \Omega^q_{R^{\lambda,n}}(\mathfrak{X})$ and notice that $d(R^{\lambda,n}_0(\mathfrak{X})) \subset \Omega^1_{R^{\lambda,n}_0}(\mathfrak{X})$. We also denote by $\mathfrak{c}_R(\mathfrak{X})$ the subspace of functions $f \in \mathfrak{n}(\mathfrak{X}_\eta)$ such that $f\big|_{\pi^{-1}(\mathbf{x})} \in \mathfrak{c}(\pi^{-1}(\mathbf{x})) = k_{\mathbf{x}}$.

3.1.6. Corollary. *In the above situation, the following is true:*

(i) $\mathrm{Ker}(R_0^{\lambda,n}(\mathfrak{X}) \xrightarrow{d} \Omega^1_{R_0^{\lambda,n}}(\mathfrak{X})) = \mathfrak{c}_R(\mathfrak{X})$*;*

(ii) the morphism of complexes $\Omega^{\cdot}_{R_0^{\lambda,n}}(\mathfrak{X}) \to \Omega^{\cdot}_{R_0^{\lambda,n+1}}(\mathfrak{X})$ *is homotopy equivalent to zero.* ∎

3.2. Open neighborhoods of the generic point of an irreducible component. In this subsection $\mathfrak{X}$ is a strictly poly-stable formal scheme over k°. For an irreducible component $\mathcal{X}$ of $\mathfrak{X}_s$, let $\mathring{\mathcal{X}}$ denote the maximal subset of $\mathcal{X}$ which does not intersect any of the other irreducible components of $\mathfrak{X}_s$. Notice that $\mathring{\mathcal{X}}$ is open in $\mathfrak{X}_s$ and $\mathfrak{X}$ is smooth at all points of $\mathring{\mathcal{X}}$.

3.2.1. Lemma. *Let* $\mathcal{X}$ *be an irreducible component of* $\mathfrak{X}_s$ *and* σ *the generic point of* $\mathcal{X}$ *in* $\mathfrak{X}_\eta$*. Then every open neighborhood of* σ *in* $\mathfrak{X}_\eta$ *contains a closed subset of the form* $\pi^{-1}(\mathcal{Y})$*, where* $\mathcal{Y}$ *is a nonempty open subset of* $\mathring{\mathcal{X}}$*.*

Proof. Let $\mathfrak{Y} = \mathrm{Spf}(A)$ be a nonempty open affine subscheme of $\mathfrak{X}$ with $\mathfrak{Y}_s \subset \mathring{\mathcal{X}}$. Then σ is the maximal point of the affinoid space $\mathfrak{Y}_\eta = \mathcal{M}(\mathcal{A})$, where $\mathcal{A} = A \otimes k$, i.e., $|f(x)| \le |f(\sigma)|$ for all $x \in \mathfrak{Y}_\eta$ and $f \in \mathcal{A}$ and, in particular, $|f(\sigma)| = |f|_{\sup}$ (see [Ber1, §2.4]). It follows that a fundamental system of open neighborhoods of σ in $\mathfrak{Y}_\eta$ is formed by sets of the form $\mathcal{U} = \{x \in \mathfrak{Y}_\eta \big| |f_i(x)| > r_i$ for $1 \le i \le n\}$, where $f_1, \dots, f_n \in \mathcal{A}$ and $r_i < |f_i|_{\sup}$. Since $|\mathcal{A}|_{\sup} = |k|$ and $\mathcal{A}^\circ = A$ (see [Ber7, Proposition 1.4]), we can multiply each f_i by an element of k and assume that $f_i \in A$ and $|f_i|_{\sup} = 1$. Then $\mathcal{U}$ contains $\pi^{-1}(\mathcal{Y})$, where $\mathcal{Y}$ is the open subset of $\mathfrak{Y}_s$ defined by nonvanishing of the image of $f_1 \cdot \ldots \cdot f_n$ in $\widetilde{A} = A/k^{\circ\circ}A$. ∎

3.2.2. Theorem. *Let* $\mathcal{Y}$ *be a nonempty open subset of* $\mathfrak{X}_s$ *which is contained in only one irreducible component* $\mathcal{X}$ *of* $\mathfrak{X}_s$*, i.e.,* $\mathcal{Y} \subset \mathring{\mathcal{X}}$*. Then there is a fundamental system of open neighborhoods* $\mathcal{U}$ *of* $\pi^{-1}(\mathcal{Y})$ *in* $\mathfrak{X}_\eta$ *such that, for every closed point* $\mathbf{x} \in \mathcal{X}$*, the intersection* $\mathcal{U} \cap \pi^{-1}(\mathbf{x})$ *is nonempty and connected.*

Proof. Step 1. Of course, we may assume that $\mathfrak{X}$ is quasicompact. Here are more remarks of this type.

(1) *If the theorem is true for the pair* $(\mathfrak{X}, \mathcal{Y})$*, then it is true for any pair* $(\mathfrak{X}', \mathcal{Y}')$*, where* $\mathfrak{X}'$ *is an open subscheme of* $\mathfrak{X}$ *and* $\mathcal{Y}' = \mathcal{Y} \cap \mathfrak{X}'_s$. Indeed this follows from the facts that the topology on $\mathfrak{X}'_\eta$ is induced by that on $\mathfrak{X}_\eta$ and that the preimages of any point $\mathbf{x} \in \mathfrak{X}'_s$ in $\mathfrak{X}'_\eta$ and in $\mathfrak{X}_\eta$ coincide.

(2) *If* $\mathcal{Y}'$ *and* $\mathcal{Y}''$ *are open subschemes of* $\mathcal{Y}$ *such that* $\mathcal{Y} = \mathcal{Y}' \cup \mathcal{Y}''$ *and the theorem is true for* $\mathcal{Y}'$*,* $\mathcal{Y}''$ *and* $\mathcal{Y}' \cap \mathcal{Y}''$*, then it is also true for* $\mathcal{Y}$*.* Indeed, let $\mathcal{V}$ be an open neighborhood of $\pi^{-1}(\mathcal{Y})$ in $\mathfrak{X}_\eta$. By the assumption, there exist open neighborhoods $\mathcal{U}'$ of $\pi^{-1}(\mathcal{Y}')$ and $\mathcal{U}''$ of $\pi^{-1}(\mathcal{Y}'')$ in $\mathcal{V}$ and $\mathcal{W}$ of $\pi^{-1}(\mathcal{Y}' \cap \mathcal{Y}'')$ in $\mathcal{U}' \cap \mathcal{U}''$ with the required properties. The union $\mathcal{U} = \mathcal{U}' \cup \mathcal{U}''$ is an open neighborhood of $\pi^{-1}(\mathcal{Y})$ in $\mathcal{V}$. For a closed point $\mathbf{x} \in \mathcal{X}$, $\mathcal{U} \cap \pi^{-1}(\mathbf{x})$ is a union of the nonempty

connected open subsets $\mathcal{U}' \cap \pi^{-1}(\mathbf{x})$ and $\mathcal{U}'' \cap \pi^{-1}(\mathbf{x})$, and the intersection of the latter contains the nonempty subset $\mathcal{W} \cap \pi^{-1}(\mathbf{x})$, i.e., it is nonempty and connected.

(3) *If $\mathfrak{X}'$ and $\mathfrak{X}''$ are open subschemes of $\mathfrak{X}$ such that $\mathfrak{X} = \mathfrak{X}' \cup \mathfrak{X}''$ and the theorem is true for the pairs $(\mathfrak{X}', \mathcal{Y}')$ and $(\mathfrak{X}'', \mathcal{Y}'')$ with $\mathcal{Y}' = \mathcal{Y} \cap \mathfrak{X}'_s$ and $\mathcal{Y}'' = \mathcal{Y} \cap \mathfrak{X}''_s$, then it is also true for the pair* $(\mathfrak{X}, \mathcal{Y})$. Indeed, let $\mathcal{V}$ be an open neighborhood of $\pi^{-1}(\mathcal{Y})$ in $\mathfrak{X}_\eta$. By the assumption, there exist open neighborhoods $\mathcal{U}'$ of $\pi^{-1}(\mathcal{Y}')$ in $\mathfrak{X}'_\eta \cap \mathcal{V}$ and $\mathcal{U}''$ of $\pi^{-1}(\mathcal{Y}')$ in $\mathfrak{X}''_\eta \cap \mathcal{V}$ with the required properties. It suffices to show that the topological interior $\mathcal{U}$ of the set $U = \mathcal{U}' \cup \mathcal{U}''$ in $\mathfrak{X}_\eta$ possesses the required properties. First of all, $\mathcal{U}$ is evidently an open neighborhood of $\pi^{-1}(\mathcal{Y})$ in $\mathcal{V}$. Let $\mathbf{x}$ be a closed point of $\mathcal{X}$, and set $\mathcal{X}' = \mathcal{X} \cap \mathfrak{X}'_s$ and $\mathcal{X}'' = \mathcal{X} \cap \mathfrak{X}''_s$. Since $\pi^{-1}(\mathbf{x})$ is contained in the interior of $\mathfrak{X}_\eta$, it follows that $\mathcal{U} \cap \pi^{-1}(\mathbf{x}) = U \cap \pi^{-1}(\mathbf{x})$. If $\mathbf{x} \notin \mathcal{X}''$, the latter intersection coincides with $\mathcal{U}' \cap \pi^{-1}(\mathbf{x})$. If $\mathbf{x} \in \mathcal{X}' \cap \mathcal{X}''$, it coincides with the union of the sets $\mathcal{U}' \cap \pi^{-1}(\mathbf{x})$ and $\mathcal{U}'' \cap \pi^{-1}(\mathbf{x})$, and so it remains to show that the intersection of both sets is nonempty. But this follows from the remark (1) and the fact that $\mathcal{U}' \cap \mathcal{U}''$ is an open neighborhood of $\pi^{-1}(\mathcal{Y}' \cap \mathcal{Y}'')$ in $\mathfrak{X}'_\eta \cap \mathfrak{X}''_\eta$.

By the previous remarks, the situation is reduced to the case when $\mathfrak{X} = \mathrm{Spf}(A)$ is affine and $\mathcal{Y}$ is a principal open subset of $\mathfrak{X}_s$, i.e., $\mathcal{Y} = \{\mathbf{x} \in \mathfrak{X}_s \big| f(\mathbf{x}) \neq 0\}$ for some $f \in A \backslash k^{\circ\circ} A$. It follows that a fundamental system of open neighborhoods of $\pi^{-1}(\mathcal{Y})$ in $\mathfrak{X}_\eta$ is formed by sets of the form $\mathcal{U}_r = \{x \in \mathfrak{X}_\eta \big| |f(x)| > r\}$ with $0 < r < 1$. Since $\mathcal{Y}$ does not intersect any of the other irreducible components of $\mathfrak{X}_s$, the absolute values of all f_i's at the generic points of those irreducible components in $\mathfrak{X}_\eta$ are strictly less than 1. Thus, it suffices to prove the following fact.

Let $\mathfrak{X} = \mathrm{Spf}(A)$ be an affine strictly poly-stable formal scheme over k°, $\mathcal{X}$ an irreducible component of $\mathfrak{X}_s$, σ the generic point of $\mathcal{X}$ in $\mathfrak{X}_\eta$, f an element of A, and α the maximum of the absolute values of f at the generic points of the other irreducible components of $\mathfrak{X}_s$ in $\mathfrak{X}_\eta$. (If $\mathcal{X} = \mathfrak{X}_s$, one sets $\alpha = 0$.) Assume that $\alpha < |f(\sigma)|$. *Then for every $\alpha \leq r < |f(\sigma)|$ the set $\mathcal{U}_r = \{x \in \mathfrak{X}_\eta \big| |f(x)| > r\}$ possesses the property of the theorem.*

Step 2. Let $\mathbf{x}$ be a closed point of $\mathcal{X}$. We notice that we may always replace the triple $(\mathfrak{X}, \mathcal{X}, \mathbf{x})$ by a triple $(\mathfrak{X}', \mathcal{X}', \mathbf{x}')$ with a morphism $\mathfrak{X}' \to \mathfrak{X}$ of one of the following two forms and where $\mathbf{x}'$ is a point over $\mathbf{x}$ and $\mathcal{X}'$ is the irreducible component of $\mathfrak{X}'_s$ that contains the point $\mathbf{x}'$: (a) $\mathfrak{X}'$ is an étale affine neighborhood of $\mathbf{x}$, and (b) $\mathfrak{X}'$ is an open affine subset of $\mathfrak{X}\widehat{\otimes}_{k^\circ} k'^\circ$, where k' is a finite extension of k. Indeed, this follows from the fact that in both cases the induced map $\pi^{-1}(\mathbf{x}') \to \pi^{-1}(\mathbf{x})$ is open and surjective.

Thus, we may assume that the point $\mathbf{x}$ is $\widetilde{k}$-rational and is contained in the intersection of all irreducible components of $\mathfrak{X}_s$, and that there is an étale morphism $\varphi : \mathfrak{X} \to \mathfrak{T} = \mathfrak{T}(\mathbf{n}, \mathbf{a}) \times \mathfrak{S}(m)$ with $\mathbf{n} = (n_0, \ldots, n_p)$ and $\mathbf{a} = (a_0, \ldots, a_p)$, which induces a bijection between the sets of irreducible components $\mathrm{Irr}(\mathfrak{X}_s) \xrightarrow{\sim} \mathrm{Irr}(\mathfrak{T}_s)$.

Step 3. Assume first that $\mathfrak{X}$ is smooth at $\mathbf{x}$, i.e., $\mathbf{n} = (0)$ and $\mathbf{a} = (1)$. (Notice that it is the only possible case when the valuation on k is trivial.) In [Ber7, §5] we constructed a strong deformation retraction $\Phi : \mathfrak{X}_\eta \times [0, 1] \to \mathfrak{X}_\eta : (x, t) \mapsto x_t$ of $\mathfrak{X}_\eta$ to the skeleton $S(\mathfrak{X})$ which in our case consists of the generic point σ of $\mathcal{X}$ in $\mathfrak{X}_\eta$. Since $\Phi(\pi^{-1}(\mathbf{x}) \times [0, 1[) \subset \pi^{-1}(\mathbf{x})$, it follows that the closure of $\pi^{-1}(\mathbf{x})$ contains the point σ and, therefore, the intersection $\mathcal{U}_r \cap \pi^{-1}(\mathbf{x})$ is nonempty. Furthermore, since $|f(x)| \le |f(x_t)|$ for all $x \in \mathfrak{X}_\eta$ and $t \in [0, 1]$, the set $\mathcal{U}_r$ is preserved under the homotopy Φ. The set $\pi^{-1}(\mathbf{x})$ is isomorphic to the m-dimensional open unit disc D with center at zero, and if the absolute values of all of the coordinate functions on $\pi^{-1}(\mathbf{x}) = D$ at a point $x \in \pi^{-1}(\mathbf{x})$ are equal or less than β then, for every $\beta \le t < 1$, $x_t = 0_t$, where $0_t = \Phi(0, t)$ is the maximal point of the closed disc of radius $(t, \ldots, t)$. If ℓ denotes the homeomorphic embedding $[0, 1] \to \mathfrak{X}_\eta : t \mapsto 0_t$, it follows that $\mathcal{U}_r \cap \ell([0, 1])$ coincides with either $\ell(]\beta, 1[)$ for some $0 \le \beta < 1$, if $|f(0)| \le r$, or $\ell([0, 1[)$, if $|f(0)| > r$. Thus, we may assume that the formal scheme $\mathfrak{X}$ is not smooth at $\mathbf{x}$.

Step 4. By [Ber7, Corollary 7.4], there exists an étale morphism $\mathfrak{S} \to \mathfrak{S}(m)$ and an open subscheme $\mathfrak{X}' \subset \mathfrak{X} \times_{\mathfrak{S}(m)} \mathfrak{S}$ such that $\mathfrak{X}'_s$ contains a point $\mathbf{x}'$ over the point $\mathbf{x}$ and the induced morphism $\mathfrak{X}' \to \mathfrak{S}'$ is geometrically elementary. By Step 2, we may assume that the point $\mathbf{x}'$ is $\widetilde{k}$-rational, and we can replace the triple $(\mathfrak{X}, \mathcal{X}, \mathbf{x})$ by the triple $(\mathfrak{X}', \mathcal{X}', \mathbf{x}')$, where $\mathcal{X}'$ is the irreducible component of $\mathfrak{X}'_s$ that contains $\mathbf{x}'$, and so we may assume that there is an étale morphism $\mathfrak{X} \to \mathfrak{Z} = \mathfrak{T}(\mathbf{n}, \mathbf{a}) \times \mathfrak{S}$ with a smooth formal scheme $\mathfrak{S}$ such that the induced morphism $\psi : \mathfrak{X} \to \mathfrak{S}$ is geometrically elementary. Consider the strong deformation retraction $\Phi_{\mathfrak{S}} : \mathfrak{X}_\eta \times [0, 1] \to \mathfrak{X}_\eta : (x, t) \mapsto x_t$ to the skeleton $S(\mathfrak{X}/\mathfrak{S})$, constructed in [Ber7, §7]. Recall that $S(\mathfrak{X}/\mathfrak{S}) = \cup_{y \in \mathfrak{S}_\eta} S(\mathfrak{X}_y)$, where $\mathfrak{X}_y = \mathfrak{X} \times_{\mathfrak{S}} \mathrm{Spf}(\mathcal{H}(y)^\circ)$. Recall also that there are canonical isomorphisms $\mathfrak{X}_{y,\eta} \xrightarrow{\sim} \mathfrak{X}_{\eta,y}$ and $\mathfrak{X}_{y,s} \xrightarrow{\sim} \mathfrak{X}_{s,\mathbf{y}} \otimes_{\widetilde{k(\mathbf{y})}} \widetilde{\mathcal{H}(y)}$, where $\mathbf{y} = \pi(y)$. The sets $\pi^{-1}(\mathbf{x})$ and $\mathcal{U}_r$ are preserved under $\Phi_{\mathfrak{S}}$ and, therefore, it suffices to show that the set $\mathcal{U}_r \cap S(\mathfrak{X}/\mathfrak{S}) \cap \pi^{-1}(\mathbf{x})$ is nonempty and connected.

Since the morphism ψ is geometrically elementary, the canonical map $S(\mathfrak{X}/\mathfrak{S}) \to S(\mathfrak{Z}/\mathfrak{S})$ identifies the former with its image in the latter. Notice that $S(\mathfrak{Z}/\mathfrak{S}) \xrightarrow{\sim} \Sigma^{\mathbf{n}}_{|\mathbf{a}|} \times \mathfrak{S}$. Since $\pi^{-1}(\mathbf{x}) \xrightarrow{\sim} \pi^{-1}(\mathbf{z})$, where $\mathbf{z}$ is the image of $\mathbf{x}$ in $\mathfrak{Z}_s$, it follows that the latter identifies $S(\mathfrak{X}/\mathfrak{S}) \cap \pi^{-1}(\mathbf{x})$ with $\mathring{\Sigma}^{\mathbf{n}}_{|\mathbf{a}|} \times \pi^{-1}(\mathbf{y})$. Recall that $\pi^{-1}(\mathbf{y}) \xrightarrow{\sim} D$, where D is the m-dimensional open unit polydisc with center at zero.

Furthermore, by [Ber7, Theorem 7.2], the homotopy $\mathfrak{S}_\eta \times [0, 1] \to \mathfrak{S}_\eta$ considered in Step 3 can be lifted to a homotopy $S(\mathfrak{X}/\mathfrak{S}) \times [0,1] \to S(\mathfrak{X}/\mathfrak{S}) : (x,t) \mapsto x_t$. Again, since $|f(x)| \le |f(x_t)|$, the set $\mathcal{U}_r$ is preserved under Φ. Thus, if $\ell : [0,1] \to \mathfrak{S}_\eta$ is the embedding constructed in Step 3 for the point $\mathbf{y}$, $L = S(\mathfrak{X}/\mathfrak{S}) \cap \psi^{-1}(\ell[0,1])) \xrightarrow{\sim} \Sigma^{\mathbf{n}}_{|\mathbf{a}|} \times [0,1]$ and $\mathring{L} = L \cap \pi^{-1}(\mathbf{x}) \xrightarrow{\sim} \mathring{\Sigma}^{\mathbf{n}}_{|\mathbf{a}|} \times [0,1[$, then it suffices to show that the intersection $\mathcal{U}_r \cap \mathring{L} \cap \psi^{-1}(\ell([\beta,1[)$ is nonempty and connected for some $0 \le \beta < 1$.

Finally, the point σ is a vertex of $S(\mathfrak{X}) = L \cap \psi^{-1}(\ell(1)) \xrightarrow{\sim} \Sigma^{\mathbf{n}}_{|\mathbf{a}|}$. It follows that there exists $0 \le \beta < 1$ such that the set $\{\sigma\} \times [\beta,1]$ is contained in $\mathcal{U}_r$ (and, in particular, the set $\mathcal{U}_r \cap \mathring{L} \cap \psi^{-1}(\ell[\beta,1[)$ is nonempty). We can increase β so that $|f(\sigma,t)| > r$ for all $t \in [\beta,1]$, where (σ,t) is identified with the corresponding vertex of $L \cap \psi^{-1}(\ell(t)) \xrightarrow{\sim} \Sigma^{\mathbf{n}}_{|\mathbf{a}|}$. It suffices to show that $\mathcal{U}_r \cap \mathring{L} \cap \psi^{-1}(\ell(t))$ is connected for all $t \in [\beta,1[$. Since for any other vertex τ of $\Sigma^{\mathbf{n}}_{|\mathbf{a}|}$, one has $|f(\tau,t)| \le |f(\tau)| \le \alpha$, it suffices to consider the formal scheme $\mathfrak{X}_{\ell(t)}$ instead of $\mathfrak{X}$, i.e., the situation is reduced to the case when $m = 0$.

Step 5. The étale morphism $\mathfrak{X} \to \mathfrak{Z} = \mathfrak{T}(\mathbf{n},\mathbf{a})$ identifies $S(\mathfrak{X})$ and $S(\mathfrak{X}) \cap \pi^{-1}(\mathbf{x})$ with $\Sigma^{\mathbf{n}}_{|\mathbf{a}|}$ and $\mathring{\Sigma}^{\mathbf{n}}_{|\mathbf{a}|}$, respectively. Recall that $\Sigma^{\mathbf{n}}_{|\mathbf{a}|}$ consists of the points $\mathbf{u} = (u_{ij})_{0 \le i \le p, 0 \le j \le n_i} \in [0,1]^{[\mathbf{n}]}$ with $u_{i0} \cdot \ldots \cdot u_{in_i} = |a_i|$ for all $0 \le i \le p$, and $\mathring{\Sigma}^{\mathbf{n}}_{|\mathbf{a}|}$ consists of the points $\mathbf{u} \in \Sigma^{\mathbf{n}}_{|\mathbf{a}|}$ with $u_{ij} < 1$ for all $0 \le i \le p$ and $0 \le j \le n_i$. Since $\pi^{-1}(\mathbf{x}) \xrightarrow{\sim} \pi^{-1}(\mathbf{z})$, where $\mathbf{z}$ is the image of $\mathbf{x}$ in $\mathfrak{Z}$, the restriction of f to $\pi^{-1}(\mathbf{x})$ can be represented as a power series $\sum_\mu c_\mu T^\mu$, where $\mu = (\mu_0, \ldots, \mu_p)$, $\mu_i = (\mu_{i0}, \ldots, \mu_{in_i}) \in \mathbf{Z}_+^{n_i+1}$, and $c_\mu = 0$ for all μ with $\min\limits_{0 \le j \le n_i} \{\mu_{ij}\} \ge 1$ for some $0 \le i \le p$. If $\mathbf{u} \in \mathring{\Sigma}^{\mathbf{n}}_{|\mathbf{a}|}$, then $|f(\mathbf{u})| = \sup\{|c_\mu| \cdot \mathbf{u}^\mu\}$. First of all, we are going to reduce the situation to the case when $a_i \ne 0$ for all $0 \le i \le p$.

Assume that $a_i = 0$ for $0 \le i \le q$ and $a_i \ne 0$ for $q+1 \le i \le p$. We may assume that the coordinates of the vertex σ are such that $\sigma_{i0} = 0$ and $\sigma_{ij} = 1$ for all $0 \le i \le q$ and $1 \le j \le n_i$. *We claim that for any point* $\mathbf{u} \in \mathring{\Sigma}^{\mathbf{n}}_{|\mathbf{a}|}$ *with* $|f(\mathbf{u})| > r$ *one has* $u_{i0} = 0$ *and* $u_{ij} \ne 0$ *for all* $0 \le i \le q$ *and* $1 \le j \le n_i$. Indeed, assume this is not true, i.e., $u_{ij} = 0$ for some $0 \le i \le q$ and $1 \le j \le n_i$. Let τ be an arbitrary vertex of $\Sigma^{\mathbf{n}}_{|\mathbf{a}|}$ with $\tau_{ij} = 0$ and $\tau_{il} = 1$ for all $0 \le l \le n_i$ with $l \ne j$. Then $|f(\tau)| \ge |f(\mathbf{u})| > r$, and the claim follows.

Furthermore, we set $\mathbf{n}' = (n_0, \ldots, n_q)$ and denote by ψ the projection $\Sigma^{\mathbf{n}}_{|\mathbf{a}|} \to [0,1]^{|\mathbf{n}'|} : \mathbf{u} = (u_{ij})_{0 \le i \le p, 0 \le j \le n_i} \mapsto (u_{ij})_{0 \le i \le q, 1 \le j \le n_i}$. The fiber $\psi^{-1}(\mathbf{v})$ of each point $\mathbf{v} \in [0,1]^{|\mathbf{n}'|}$ is canonically identified with $\Sigma^{\mathbf{m}}_{|\mathbf{b}|}$, where $\mathbf{m} = (n_{q+1}, \ldots, n_p)$ and $\mathbf{b} = (a_{q+1}, \ldots, a_p)$. We can find $0 < \beta < 1$ such that, for every $\mathbf{v} \in [\beta,1]^{|\mathbf{n}'|}$, the absolute value of f at the vertex of $\psi^{-1}(\mathbf{v}) \xrightarrow{\sim} \Sigma^{\mathbf{m}}_{|\mathbf{b}|}$ that corresponds to σ is bigger than r and of the other vertices is smaller than r. By the above claim, if $\mathbf{u}$ is a point from $\mathring{\Sigma}^{\mathbf{n}}_{|\mathbf{a}|}$ with $|f(\mathbf{u})| > r$, then $\psi(\mathbf{u}) \in]0,1[^{|\mathbf{n}|}$, and if, for $t \in [0,1]$, $\mathbf{u}^{(t)}$ denotes the point of $\Sigma^{\mathbf{n}}_{|\mathbf{a}|}$

with the following coordinates: $u^{(t)}_{ij} = u^t_{ij}$ for $0 \le i \le q$, and $u^{(t)}_{ij} = u_{ij}$ for $q+1 \le i \le p$, then $|f(\mathbf{u}^{(t)})| \ge |f(\mathbf{u})| > r$. One also has $\mathbf{u}^{(t)} \in \psi^{-1}([\beta, 1[^{|\mathbf{n}'|})$ for every $0 < t < 1$ sufficiently close to 0. Thus, to prove the theorem, it suffices to show that, for every $\mathbf{v} \in [\beta, 1[^{|\mathbf{n}'|}$, the set of all $\mathbf{u} \in \psi^{-1}(\mathbf{v}) \xrightarrow{\sim} \Sigma^{\mathbf{m}}_{|\mathbf{b}|}$ with $|f(\mathbf{u})| > r$ which lie in $\mathring{\Sigma}^{\mathbf{m}}_{|\mathbf{b}|}$ is connected. (The above reasoning is similar to that from Step 4.)

We get the required reduction, and the theorem now follows from the following simple fact.

3.2.3. Lemma. *Let V be a polytope and $\mathring{V}$ its interior. Assume we are given a continuous function f on V which takes its maximum at only one vertex σ of V and whose restriction to $\mathring{V}$ is the supremum of a family of linear functions. If α is the maximal value of f at all other vertices of V, then for every $\alpha \le r < f(\sigma)$ the set $U = \{x \in \mathring{V} \big| f(x) > r\}$ is connected.*

Proof. Let $f\big|_{\mathring{V}}$ be the supremum of linear functions $\{f_i\}_{i\in I}$, and set $J = \{i \in I \big| f_i(\sigma) > r\}$. Each of the sets $U_i = \{x \in \mathring{V} \big| f_i(x) > r\}$ for $i \in J$ is convex and contains the intersection of an open neighborhood of σ in V with $\mathring{V}$ and, therefore, the union $\cup_{i\in J} U_i$ is connected. *We claim that U coincides with the latter union.* Indeed, assume $x \in U$. Then there exists $i \in I$ with $f_i(x) > r$. Since the function f_i is linear, it takes its maximum at a vertex τ of V. It follows that $f(\tau) \ge f_i(\tau) > r \ge \alpha$ and, therefore, $\tau = \sigma$ and $i \in J$, i.e., $x \in U_i$. ∎

The following is a consequence of the proof of Theorem 3.2.2.

3.2.4. Corollary. *In the situation of Theorem 3.2.2, let $\mathbf{x}$ be a closed point of $\mathcal{X}$ such that the field $\widetilde{k}(\mathbf{x})$ is separable over $\widetilde{k}$. Then*

(i) $\mathfrak{c}(\pi^{-1}(\mathbf{x})) = k_{\mathbf{x}}$;

(ii) if $\mathcal{U}$ is an open neighborhood of σ in $\mathfrak{X}_\eta$ such that the intersection $\mathcal{U} \cap \pi^{-1}(\mathbf{x})$ is connected, then $\mathfrak{c}(\pi^{-1}(\mathbf{x})) \xrightarrow{\sim} \mathfrak{c}(\mathcal{U} \cap \pi^{-1}(\mathbf{x}))$.

Proof. (i) follows from Lemma 3.1.2. To verify (ii), we may assume that the point $\mathbf{x}$ is $\widetilde{k}$-rational. By the proof of Theorem 3.2.2, the intersection $\mathcal{U} \cap \pi^{-1}(\mathbf{x})$ contains a point x with $\mathcal{H}(x) \xrightarrow{\sim} \mathcal{H}(y)$, where y is the maximal point of a closed polydisc. It follows that for any non-Archimedean field K over k the norm on the Banach algebra $\mathcal{H}(x) \widehat{\otimes} K$ is multiplicative and, therefore, the algebraic closure of k in $\mathcal{H}(x)$ coincides with k. This implies (ii). ∎

The following corollary lists consequences of the previous results in the situation we are interested in.

Let k_0 be a non-Archimedean field with a nontrivial discrete valuation and a perfect residue field $\widetilde{k}_0$, k a closed subfield of $\widehat{k^{\mathrm{a}}_0}$ that contains k_0, and K a filtered k-algebra. Furthermore, let $\mathfrak{X}$ be a marked formal scheme over k°, $\mathring{\mathfrak{X}}$ the maximal open subscheme of $\mathfrak{X}$ which is a smooth formal scheme (locally finitely presented) over k°, and $\sigma = \sigma_{\mathfrak{X}}$. For $n \geq 0$ and an open neighborhood $\mathcal{U}$ of σ, let $\mathcal{C}_R^{K,n}(\mathfrak{X},\mathcal{U})$ denote the set of all functions $f \in \mathfrak{N}^K(\mathcal{U})$ such that $f\big|_{\mathcal{U}\cap\pi^{-1}(\mathbf{x})} \in \mathcal{C}^{K,n}(\mathcal{U}\cap\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$. For example, $\mathcal{C}_R^{K,n}(\mathfrak{X},\mathfrak{X}_\eta) = \mathcal{C}_R^{K,n}(\mathfrak{X})$, where the right hand side is introduced at the end of §3.1. We also set $\mathcal{C}_R^{K,n}(\mathfrak{X},\sigma) = \varinjlim \mathcal{C}_R^{K,n}(\mathfrak{X},\mathcal{U})$, where the limit is taken over all open neighborhoods of σ in $\mathfrak{X}_\eta$.

3.2.6. Corollary. *In the above situation, the following is true:*

(i) any open neighborhood of the point σ contains a subset of the form $\pi^{-1}(\mathcal{Y})$, where $\mathcal{Y}$ is a nonempty open subset of $\mathring{\mathfrak{X}}_s$;

(ii) given a nonempty open subset $\mathcal{Y} \subset \mathring{\mathfrak{X}}_s$, a fundamental system of open neighborhoods $\mathcal{U}$ of $\pi^{-1}(\mathcal{Y})$ is formed by those $\mathcal{U}$ with the property that the intersection $\mathcal{U} \cap \pi^{-1}(\mathbf{x})$ is nonempty and connected for all closed points $\mathbf{x} \in \mathfrak{X}_s$;

(iii) for any open neighborhood $\mathcal{U}$ of σ with the property of (ii), one has $\mathcal{C}_R^{K,n}(\mathfrak{X}) \xrightarrow{\sim} \mathcal{C}_R^{K,n}(\mathfrak{X},\mathcal{U})$ and, in particular, $\mathcal{C}_R^{K,n}(\mathfrak{X}) \xrightarrow{\sim} \mathcal{C}_R^{K,n}(\mathfrak{X},\sigma)$. ∎

3.3. A property of strata. Recall (see [Ber7, §2]) that the closed fiber $\mathfrak{X}_s$ of any pluri-nodal formal scheme $\mathfrak{X}$ over k° is provided with a stratification by locally closed irreducible normal subsets so that the closure of any stratum (which is called a stratum closure) is a strata subset (i.e., a union of strata). Furthermore, any étale morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ gives rise to an étale morphism of every stratum of $\mathfrak{X}'_s$ to a stratum of $\mathfrak{X}_s$ and, in particular, the image of a stratum is contained in a stratum. In this subsection we show that, if both formal schemes are nondegenerate strictly poly-stable or marked, the latter property holds without the assumption on étaleness of the morphism.

Recall that, if $\mathfrak{X}$ is strictly poly-stable, then the intersection of any set of irreducible components of $\mathfrak{X}_s$ is smooth over $\widetilde{k}$, and the family of strata coincides with the family of irreducible components of sets of the form $(\cap_{X\in A} X)\backslash(\cup_{Y\notin A} Y)$, where A is a finite set of irreducible components of $\mathfrak{X}_s$. If $\mathfrak{X}$ is marked (and in this case we assume the field k satisfies the assumptions used in the definition of such formal schemes), the above stratification gives rise to a stratification on the closed fiber $\mathfrak{X}_s$. For a stratum closure $\mathcal{X}$ in $\mathfrak{X}_s$, the corresponding stratum will be denoted by $\mathring{\mathcal{X}}$. (This is consistent with the similar notation in the previous subsection.)

3.3.1. Proposition. *Let $\mathfrak{X}$ be a nondegenerate strictly poly-stable formal scheme over k°, and let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism of formal schemes over k°, where $\mathfrak{X}'$ is either strictly poly-stable or marked. Then the preimage of any stratum of $\mathfrak{X}_s$ is a strata subset of $\mathfrak{X}'_s$.*

Proof. Step 1. Let $\mathfrak{T}$ be a standard formal scheme $\mathfrak{T}(\mathbf{n},\mathbf{a}) \times \mathfrak{S}(m)$. In this case there is a bijection between the set $[\mathbf{n}] = [n_0] \times \ldots \times [n_p]$, where $[n] = \{0,\ldots,n\}$, and the set $\mathrm{Irr}(\mathfrak{T}_s)$ of irreducible components of $\mathfrak{T}_s$ that takes an element $\mathbf{j} = (j_0,\ldots,j_p) \in [\mathbf{n}]$ to the irreducible component $\mathcal{Z}_\mathbf{j}$ of $\mathfrak{T}_s$ which is defined by the equations $T_{0j_0} = \ldots = T_{pj_p} = 0$, and each stratum of $\mathfrak{T}_s$ is of the form $(\cap_{\mathbf{j}\in J}\mathcal{Z}_\mathbf{j})\backslash(\cup_{\mathbf{j}\notin J}\mathcal{Z}_\mathbf{j})$, where J is a subset of $[\mathbf{n}]$ of the form $J_0 \times \ldots \times J_p$ with $J_i \subset [n_i]$. We denote it by $\mathring{\mathcal{Z}}_J$ and its closure by $\mathcal{Z}_J$. Furthermore, let M be the multiplicative submonoid of A which is generated by all of the coordinate functions T_{ij} and the subgroup A^*, where $\mathfrak{T} = \mathrm{Spf}(A)$. Notice that $M \supset A \cap \mathcal{A}^*$, where $\mathcal{A} = A \otimes_{k^\circ} k$. If $\mathfrak{T}$ is nondegenerate, then $M = A \cap \mathcal{A}^*$. Given elements $f, g_1, \ldots, g_m \in M$, let $V(f; g_1, \ldots, g_m)$ denote the locally closed subset $\{\mathbf{x} \in \mathfrak{T}_s \big| f(\mathbf{x}) \neq 0 \text{ and } g_i(\mathbf{x}) = 0 \text{ for all } 1 \leq i \leq m\}$. It is easy to see that $V(f; g_1, \ldots, g_m)$ is a strata subset of $\mathfrak{T}_s$ and that each stratum of $\mathfrak{T}_s$ is of the form $V(f; g_1, \ldots, g_m)$. (For example, the above stratum $\mathring{\mathcal{Z}}_J$ is of the required form in which f is the product of the coordinate functions T_{ij} with $j \notin J_i$ and the set $\{g_1, \ldots, g_m\}$ consists of the other coordinate functions.)

Step 2. Let $\psi : \mathfrak{X}'' = \mathrm{Spf}(A'') \to \mathfrak{T}$ be an étale morphism, and assume that $\mathfrak{X}''_s$ is connected and the induced map $\mathrm{Irr}(\mathfrak{X}''_s) \to \mathrm{Irr}(\mathfrak{T}_s)$ is injective. Let also M'' denote the multiplicative submonoid of A'' which is generated by M and the subgroup A''^*. From Step 1 it follows that similar locally closed sets $V(f; g_1, \ldots, g_m)$ are strata subsets of $\mathfrak{X}''_s$ for all $f, g_1, \ldots, g_m \in M''$. Notice that $M'' \supset A'' \cap \mathcal{A}''^*$, where $\mathcal{A}'' = A'' \otimes_{k^\circ} k$, and if $\mathfrak{T}$ is nondegenerate then, by [Ber7, §5], $M'' = A'' \cap \mathcal{A}''^*$. Furthermore, if $\mathfrak{X}' = \mathrm{Spf}(A')$ is the completion of $\mathfrak{X}''$ along an irreducible component, then $A' \cap \mathcal{O}(\mathfrak{X}'_\eta)^*$ is contained in the multiplicative submonoid of A' generated by M'' and the subgroup A'^*. This implies that, given $f, g_1, \ldots, g_m \in A' \cap \mathcal{O}(\mathfrak{X}'_\eta)^*$, the locally closed set $V(f; g_1, \ldots, g_m)$ is a strata subset of $\mathfrak{X}'_s$.

Step 3. The statement of the proposition is local with respect to the étale topologies of $\mathfrak{X}$ and $\mathfrak{X}'$ and, therefore, it suffices to consider the case when $\mathfrak{X} = \mathfrak{T}$ is standard as in Step 1 and $\mathfrak{X}'$ is as in Step 2. In this case the required fact follows from Step 2. ∎

3.3.2. Corollary. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism between marked formal schemes over k°. Then the image of any stratum of $\mathfrak{X}'_s$ is contained in a stratum of $\mathfrak{X}_s$.* ∎

3.4. A tubular neighborhood of the diagonal of a stratum closure. For a formal scheme $\mathfrak{X}$ locally finitely presented over k°, let Δ denote the diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$, and let p_1 and p_2 denote the canonical projections $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$.

3.4.1. Lemma. *Given an étale morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$, the following diagram is cartesian*

$$\begin{array}{ccc} \mathfrak{X}'_\eta & \stackrel{\varphi}{\longrightarrow} & \mathfrak{X}_\eta \\ \uparrow p_1 & & \uparrow p_1 \\ \pi^{-1}(\Delta(\mathfrak{X}'_s)) & \stackrel{(\varphi,\varphi)}{\longrightarrow} & \pi^{-1}(\Delta(\mathfrak{X}_s)) \end{array}$$

Proof. It is clear that the same diagram with $\pi^{-1}(\Gamma_\varphi(\mathfrak{X}'_s))$ instead of $\pi^{-1}(\Delta(\mathfrak{X}'_s))$ is cartesian, where Γ_φ is the graph morphism $\mathfrak{X}' \to \mathfrak{X}' \times \mathfrak{X} : x' \mapsto (x', \varphi(x'))$. But since the étale morphism $(1_{\mathfrak{X}'}, \varphi) : \mathfrak{X}' \times \mathfrak{X}' \to \mathfrak{X}' \times \mathfrak{X}$ induces an isomorphism between closed subschemes $\Delta(\mathfrak{X}'_s) \stackrel{\sim}{\to} \Gamma_\varphi(\mathfrak{X}'_s)$, Lemma 4.4 from [Ber7] implies that $\pi^{-1}(\Delta(\mathfrak{X}'_s)) \stackrel{\sim}{\to} \pi^{-1}(\Gamma_\varphi(\mathfrak{X}'_s))$, and the required statement follows. ∎

We say that a formal scheme $\mathfrak{X}$ is *small* if it is connected affine and admits an étale morphism to a nondegenerate standard formal scheme $\mathfrak{T} = \mathfrak{T}(\mathbf{n}, \mathbf{a}) \times \mathfrak{S}(m)$ such that the induced map between the sets of irreducible components $\mathrm{Irr}(\mathfrak{X}_s) \to \mathrm{Irr}(\mathfrak{T}_s)$ is injective. In this case the latter property holds for any étale morphism to a standard formal scheme, and, if $\mathfrak{X} = \mathrm{Spf}(A)$, the multiplicative monoid $A \cap \mathcal{A}^*$, where $\mathcal{A} = A \otimes_{k^\circ} k$, is generated by the coordinate functions T_{ij} and the subgroup A^* (see [Ber7, §5]). Notice that any open connected affine subscheme of a small formal scheme is also small, and that any nondegenerate strictly poly-stable formal scheme has an open covering by small open subschemes.

Let $\mathfrak{X} = \mathrm{Spf}(A)$ be a small formal scheme, and let $\mathcal{Y}$ be a stratum closure in $\mathfrak{X}_s$. We denote by $M_\mathcal{Y}$ the multiplicative monoid of the elements $f \in A \cap \mathcal{A}^*$ with the property that the image of f in $A/k^{\circ\circ}A$ is not zero at $\mathcal{Y}$, and we set

$$\mathfrak{D}_\mathcal{Y} = \{x \in \pi^{-1}(\Delta(\mathcal{Y})) \big| |(p_1^* f - p_2^* f)(x)| < |p_1^* f(x)| \text{ for all } f \in M_\mathcal{Y}\} \ .$$

3.4.2. Lemma. *In the above situation, the following is true:*

(i) $\mathfrak{D}_\mathcal{Y}$ is open in $\mathfrak{X}_\eta \times \mathfrak{X}_\eta$ and contains $\pi^{-1}(\Delta(\mathring{\mathcal{Y}}))$;

(ii) for every stratum closure $\mathcal{Y}'$ in $\mathfrak{X}_s$ with $\mathcal{Y}' \subset \mathcal{Y}$, one has $\mathfrak{D}_\mathcal{Y} \cap \pi^{-1}(\Delta(\mathcal{Y}')) \subset \mathfrak{D}_{\mathcal{Y}'}$;

(iii) given an étale morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ with small $\mathfrak{X}'$ and a stratum closure $\mathcal{Y}'$ in $\mathfrak{X}'_s$ with $\varphi(\mathring{\mathcal{Y}}') \subset \mathring{\mathcal{Y}}$, the following diagram is cartesian

$$\begin{array}{ccc} \pi^{-1}(\mathcal{Y}') & \longrightarrow & \pi^{-1}(\mathcal{Y}) \\ \uparrow p_1 & & \uparrow p_1 \\ \mathfrak{D}_{\mathcal{Y}'} & \longrightarrow & \mathfrak{D}_\mathcal{Y} \end{array}$$

(iv) if $\mathring{\mathcal{Y}}$ is of type $(\mathbf{n}, |\mathbf{a}|, m)$, then there are morphisms $\alpha : \mathfrak{D}_\mathcal{Y} \to \mathbf{G}_\mathrm{m}^{|\mathbf{n}|}$ and $\beta : \mathfrak{D}_\mathcal{Y} \to \mathbf{A}^m$ such that the image of $(p_1, \alpha) : \mathfrak{D}_\mathcal{Y} \to \pi^{-1}(\mathcal{Y}) \times \mathbf{G}_\mathrm{m}^{|\mathbf{n}|}$ coincide with an open subset Y which is

annular over $\pi^{-1}(\mathcal{Y})$, and the morphism $(p_1, \alpha, \beta) : \mathfrak{D}_{\mathcal{Y}} \to \pi^{-1}(\mathcal{Y}) \times \mathbf{G}_{\mathrm{m}}^{|\mathbf{n}|} \times \mathbf{A}^m$ gives rise to an isomorphism $\mathfrak{D}_{\mathcal{Y}} \stackrel{\sim}{\to} Y \times D^m$.

Proof. (i) Let $\mathfrak{T} = \mathfrak{T}(\mathbf{n}, \mathbf{a}) \times \mathfrak{S}(m)$ be a nondegenerate standard formal scheme with $\mathbf{n} = (n_0, \dots, n_p)$ and $\mathbf{a} = (a_0, \dots, a_p)$ for which there is an étale morphism $\psi : \mathfrak{X} \to \mathfrak{T}$ such that the induced map $\mathrm{Irr}(\mathfrak{X}_s) \to \mathrm{Irr}(\mathfrak{T}_s)$ is injective, and let the image of $\mathring{\mathcal{Y}}$ be contained in $\mathring{\mathcal{Z}}_J$, where is a subset of $[\mathbf{n}]$ of the form $J_0 \times \ldots \times J_p$ with $J_i \subset [n_i]$. (We use here the notation from Step 1 of the proof of Proposition 3.3.1.) We may assume that $J_i = \{0, \dots, n'_i\}$ for all $0 \le i \le p$, where $0 \le n'_i \le n_i$, $n'_i \ge 1$ for $0 \le i \le p'$ and $n'_i = 0$ for $p' + 1 \le i \le p$. Then the type of $\mathring{\mathcal{Y}}$ is $(\mathbf{n}', |\mathbf{a}'|, m')$, where $\mathbf{n}' = (n'_0, \dots, n'_{p'})$, $\mathbf{a}' = (a_0, \dots, a_{p'})$ and $m' = |\mathbf{n}| + m - |\mathbf{n}'|$, and the monoid $M_{\mathcal{Y}}$ is generated by A^* and the coordinate functions T_{ij} for $0 \le i \le p$ and $n'_i + 1 \le j \le n_i$. Since the inequalities in the definition of $\mathfrak{D}_{\mathcal{Y}}$ hold for all elements of A^*, it follows that $\mathfrak{D}_{\mathcal{Y}}$ is defined by a finite number of inequalities, and so it is open in $\mathfrak{X}_\eta \times \mathfrak{X}_\eta$. Since elements of $\mathcal{M}_{\mathcal{Y}}$ do not vanish at any point of $\mathring{\mathcal{Y}}$, it follows that $\mathfrak{D}_{\mathcal{Y}} \supset \pi^{-1}(\Delta(\mathring{\mathcal{Y}}))$, i.e., (i) is true.

(ii) follows from the fact that $M_{\mathcal{Y}'} \subset M_{\mathcal{Y}}$.

(iii) follows from Lemma 3.4.1 and the fact that, if $\mathfrak{X}' = \mathrm{Spf}(A')$, the monoid $M_{\mathcal{Y}'}$ is generated by A'^* and the image of the monoid $\mathcal{M}_{\mathcal{Y}}$ in A'.

(iv) By (iii) we may assume that $\mathfrak{X} = \mathfrak{T}$. In the above situation, the type of $\mathcal{Y}$ is $(\mathbf{n}', \mathbf{a}', m')$, and $\mathfrak{T}_\eta$ is the affinoid subdomain of the analytic torus $\mathbf{G}_{\mathrm{m}}^{|\mathbf{n}|+m}$ defined by the inequalities $|T_{ij}(x)| \le 1$, $|(T_{i1} \cdot \ldots \cdot T_{in_i})(x)| \ge |a_i|$ and $|S_l(x)| = 1$ for $0 \le i \le p$, $1 \le j \le n_i$ and $1 \le l \le m$. The tubular neighborhood $\mathfrak{D}_{\mathcal{Y}}$ is the open subset of $\mathfrak{T}_\eta \times \mathfrak{T}_\eta$ defined by the inequalities $|p_\nu^* T_{ij}(y)| < 1$ for $\nu \in \{1, 2\}$, $0 \le i \le p'$ and $1 \le j \le n'_i$, $|(p_1^* T_{ij} - p_2^* T_{ij})(y)| < |p_1^* T_{ij}(y)|$ for $0 \le i \le p$ and $n'_i + 1 \le j \le n_i$, and $|(p_1^* S_l - p_2^* S_l)(y)| < 1$ for $1 \le l \le m$. If α denotes the morphism $\mathfrak{D}_{\mathcal{Y}} \to \mathbf{G}_{\mathrm{m}}^{|\mathbf{n}'|}$ defined by the functions $\{p_2^* T_{ij}\}_{0 \le i \le p', 1 \le j \le n'_i}$, then the image of $(p_1, \alpha) : \mathfrak{D}_{\mathcal{Y}} \to \pi^{-1}(\mathcal{Y}) \times \mathbf{G}_{\mathrm{m}}^{|\mathbf{n}'|}$ coincides with the open subset Y defined by the inequalities $|V_{ij}(y)| < 1$ for $0 \le i \le p'$ and $1 \le j \le n'_i$ and

$$|(V_{i1} \cdot \ldots \cdot V_{in'_i})(y)| > \frac{|a_i|}{|(T_{i,n'_i+1} \cdot \ldots \cdot T_{i,n_i})(x)|}$$

for $0 \le i \le p'$, where V_{ij} are the pullbacks of the coordinate functions on $\mathbf{G}_{\mathrm{m}}^{|\mathbf{n}'|}$ and x is the image of y in $\pi^{-1}(\mathcal{Y})$. Notice that Y is annular of dimension $|\mathbf{n}'|$ over $\pi^{-1}(\mathcal{Y})$. If now β denotes the morphism $\mathfrak{D}_{\mathcal{Y}} \to \mathbf{A}^{m'}$ defined by the functions

$$\{\{\frac{p_1^* T_{ij}}{p_2^* T_{ij}} - 1\}_{0 \le i \le p, n'_i + 1 \le j \le n_i}, \{\frac{p_1^* S_l}{p_2^* S_l} - 1\}_{1 \le l \le m}\} ,$$

then the morphism $(p_1, \alpha, \beta) : \mathfrak{D}_{\mathcal{Y}} \to \pi^{-1}(\mathcal{Y}) \times \mathbf{G}_{\mathrm{m}}^{|\mathbf{n}'|} \times \mathbf{A}^{m'}$ identifies $\mathfrak{D}_{\mathcal{Y}}$ with the open set $Y \times D^{m'}$, where $D^{m'}$ is the open unit poly-disc in $\mathbf{A}^{m'}$ with center at zero. ∎

If $\mathfrak{X}$ is an arbitrary nondegenerate strictly poly-stable formal scheme over k° and $\mathcal{Y}$ is a stratum closure of $\mathfrak{X}_s$, one defines $\mathfrak{D}_\mathcal{Y}$ as the union $\cup_{i\in I}\mathfrak{D}_{\mathcal{Y}_i}$, where $\{\mathfrak{X}_i\}_{i\in I}$ is a covering of $\mathfrak{X}$ by open small subschemes and $\mathcal{Y}_i = \mathcal{Y}\cap\mathfrak{X}_{i,s}$. By Lemma 3.4.2(iii), $\mathfrak{D}_\mathcal{Y}$ does not depend on the choice of a covering. From Proposition 3.3.1 and its proof it follows easily that, given a morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ of nondegenerate strictly poly-stable formal schemes k°, for any stratum closure $\mathcal{Y}'$ in $\mathfrak{X}'_s$ one has $\varphi(\mathfrak{D}_{\mathcal{Y}'}) \subset \mathfrak{D}_\mathcal{Y}$, where $\mathcal{Y}$ is the stratum closure in $\mathfrak{X}_s$ with $\varphi(\mathring{\mathcal{Y}}') \subset \mathring{\mathcal{Y}}$.

3.5. The same for proper marked formal schemes. Let k_0 be a non-Archimedean field with a nontrivial discrete valuation, k an extension of k_0 which is a closed subfield of $\widehat{k}_0^{\mathrm{a}}$, and let $\mathfrak{X}$ be a proper marked formal scheme over k° which is the formal completion $\widehat{\mathcal{X}}_{/\mathcal{Z}}$ of a nondegenerate strictly poly-stable separated scheme $\mathcal{X}$ over k° along an irreducible component $\mathcal{Z}$ of $\mathcal{X}_s$ proper over $\widetilde{k}$ (see §2.1). Notice that $\mathfrak{X}_s$ coincides with that irreducible component and $\mathfrak{X}_\eta$ is an open subset of the smooth k-analytic space $\mathcal{X}_\eta^{\mathrm{an}}$. Given a stratum closure $\mathcal{Y}$ of $\mathcal{X}_s$ in $\mathfrak{X}_s$, the tubular neighborhood $\mathfrak{D}_\mathcal{Y}$ constructed in the previous subsection for the nondegenerate strictly poly-stable formal scheme $\widehat{\mathcal{X}}$ is contained in $\pi^{-1}(\Delta(\mathcal{Y})) \subset \mathfrak{X}_\eta \times \mathfrak{X}_\eta$. From the construction of $\mathfrak{D}_\mathcal{Y}$ it follows easily that it depends only on the formal scheme $\mathfrak{X}$. If $\mathcal{Y} = \mathfrak{X}_s$, we denote it by $\mathfrak{D}_\mathfrak{X}$. From Proposition 3.3.1 it follows that, given a morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ of proper marked formal schemes over k°, for any stratum closure $\mathcal{Y}'$ in $\mathfrak{X}'_s$ one has $\varphi(\mathfrak{D}_{\mathcal{Y}'}) \subset \mathfrak{D}_\mathcal{Y}$, where $\mathcal{Y}$ is the stratum closure in $\mathfrak{X}_s$ with $\varphi(\mathring{\mathcal{Y}}') \subset \mathring{\mathcal{Y}}$.

We say that an open affine subscheme $\mathfrak{X}' \subset \mathfrak{X}$ is *small* if it is the the formal completion $\widehat{\mathcal{X}}'_{/\mathcal{Y}'}$ of a connected open affine subscheme $\mathcal{X}' \subset \mathcal{X}$ along $\mathcal{Y}' = \mathcal{Y}\cap\mathcal{X}'_s$, where $\mathcal{X}'$ is assumed to admit an étale morphism to a nondegenerate standard scheme $\mathcal{T} = \mathcal{T}(\mathbf{n},\mathbf{a})\times\mathcal{S}(m)$ such that the induced map $\mathrm{Irr}(\mathcal{X}_s) \to \mathrm{Irr}(\mathcal{T}_s)$ is injective. Notice that $\mathfrak{X}'_\eta$ is an open subset of the strictly affinoid subdomain $\widehat{\mathcal{X}}'_\eta$ of the open subset $\mathcal{X}'^{\mathrm{an}}_\eta$ of $\mathcal{X}^{\mathrm{an}}_\eta$. Notice also that since $\mathcal{T}_\eta$ is an open subset of an affine space over k, the sheaf of differential one-forms $\Omega^1_\mathcal{T}$ is free and, therefore, the sheaf of analytic differential one-forms is free over an open neighborhood of $\mathfrak{X}'_\eta$ in $\mathfrak{X}_\eta$. By Lemma 3.4.2(iv), the projection $p_1 : \mathfrak{D}_{\mathcal{Y}'} \to \pi^{-1}(\mathcal{Y}')$ is a discoid morphism, if $\mathcal{Y} = \mathfrak{X}_s$, and a semi-annular morphism, in general.

3.5.1. Proposition. *For every open neighborhood $\mathcal{W}$ of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathfrak{D}_\mathcal{Y}$, there exist open neighborhoods $\pi^{-1}(\mathcal{Y}') \subset \mathcal{U} \subset \mathfrak{X}_\eta$ and $\mathfrak{D}_{\mathcal{Y}'} \subset \mathcal{V} \subset \mathcal{W}$ such that $\Delta(\mathcal{U}) \subset \mathcal{V} \subset p_1^{-1}(\mathcal{U})$ and the projection $p_1 : \mathcal{V} \to \mathcal{U}$ is a discoid morphism, if $\mathcal{Y} = \mathfrak{X}_s$, and a semi-annular morphism, in general.*

Proof. We use the notations from the proof of Lemma 3.4.2. The analytic torus $\mathbf{G}_\mathrm{m}^{|\mathbf{n}|+m}$, considered there, is the analytification $\mathcal{T}_\eta^{\mathrm{an}}$ of $\mathcal{T}_\eta$, $\mathfrak{T}$ is the formal completion $\widehat{\mathcal{T}}$ of $\mathcal{T}$ along its closed fiber $\mathcal{T}_s$, and $\mathring{\mathcal{Y}}'$ is of the type $(\mathbf{n}', |\mathbf{a}'|, m')$. We may assume that $\mathcal{W}$ is an open neighborhood of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathcal{X}'^{\mathrm{an}}_\eta \times \mathcal{X}'^{\mathrm{an}}_\eta$. It what follows, we use the notation $\Pi_{\mathcal{Y}}$ instead $\pi^{-1}(\mathcal{Y})$.

Let $\mathcal{Z}$ denote the stratum closure of $\mathcal{T}_s$ for which $\mathring{\mathcal{Z}}$ contains the image of $\mathring{\mathcal{Y}}'$. For $0 < r < 1$, let $\mathfrak{D}^{\le r}_{\mathcal{Z}}$ (resp. $\widetilde{\mathfrak{D}}^{\le r}_{\mathcal{Z}}$) denote the closed analytic subdomain of $\widehat{\mathcal{T}}_\eta \times \widehat{\mathcal{T}}_\eta$ (resp. $\mathcal{T}^{\mathrm{an}}_\eta \times \mathcal{T}^{\mathrm{an}}_\eta$) defined by the inequalities $|p^*_\nu T_{ij}(y)| \le r$ and $|p^*_\nu(T_{i1} \cdot \ldots \cdot T_{in_i})(y)| \ge \frac{|a_i|}{r}$ for $\nu \in \{1, 2\}$, $0 \le i \le p'$ and $1 \le j \le n'_i$, $|(p^*_1 T_{ij} - p^*_2 T_{ij})(y)| \le r |p^*_1 T_{ij}(y)|$ for $0 \le i \le p$ and $n'_i + 1 \le j \le n_i$, and $|(p^*_1 S_l - p^*_2 S_l)(y)| \le r |p^*_1 S_l(y)|$ for $1 \le l \le m$. We also denote by $\mathfrak{D}^{<r}_{\mathcal{Z}}$ (resp. $\widetilde{\mathfrak{D}}^{<r}_{\mathcal{Z}}$) the open subset of $\widehat{\mathcal{T}}_\eta \times \widehat{\mathcal{T}}_\eta$ (resp. $\mathcal{T}^{\mathrm{an}}_\eta \times \mathcal{T}^{\mathrm{an}}_\eta$) defined by the corresponding strict inequalities. One has $\mathfrak{D}_{\mathcal{Z}} = \cup_{r<1} \mathfrak{D}^{\le r}_{\mathcal{Z}}$.

Furthermore, we denote by $\Pi^{\le r}_{\mathcal{Z}}$ (resp. $\widetilde{\Pi}^{\le r}_{\mathcal{Z}}$) the closed analytic subdomain of $\widehat{\mathcal{T}}_\eta$ (resp. $\mathcal{T}^{\mathrm{an}}_\eta$) defined by the inequalities $|T_{ij}(x)| \le r$ and $|(T_{i1} \cdot \ldots \cdot T_{in_i})(x)| \ge \frac{|a_i|}{r}$ for $0 \le i \le p'$ and $1 \le j \le n'_i$, and we denote by $\Pi^{<r}_{\mathcal{Z}}$ (resp. $\widetilde{\Pi}^{<r}_{\mathcal{Z}}$) the open subset of $\widehat{\mathcal{T}}_\eta$ (resp. $\mathcal{T}^{\mathrm{an}}_\eta$) defined by the corresponding strict inequalities. Notice that if $\mathcal{Z} = \mathcal{T}_s$ (i.e., $\mathcal{Y} = \mathfrak{X}_s$), the system of inequalities is empty and, therefore, $\Pi^{<r}_{\mathcal{Z}} = \Pi^{\le r}_{\mathcal{Z}} = \widehat{\mathcal{T}}_\eta$ and $\widetilde{\Pi}^{<r}_{\mathcal{Z}} = \widetilde{\Pi}^{\le r}_{\mathcal{Z}} = \mathcal{T}^{\mathrm{an}}_\eta$. In the general case $\Pi_{\mathcal{Z}} = \cup_{r<1} \Pi^{<r}_{\mathcal{Z}} = \cup_{r<1} \Pi^{\le r}_{\mathcal{Z}}$, and the union $\cup_{r<1} \widetilde{\Pi}^{\le r}_{\mathcal{Z}}$ is an open subset of $\mathcal{T}^{\mathrm{an}}_\eta$. One also has $\Delta(\Pi^{?r}_{\mathcal{Z}}) \subset \mathfrak{D}^{?r}_{\mathcal{Z}} \subset p^{-1}_\nu(\Pi^{?r}_{\mathcal{Z}})$ (resp. $\Delta(\widetilde{\Pi}^{?r}_{\mathcal{Z}}) \subset \widetilde{\mathfrak{D}}^{?r}_{\mathcal{Z}} \subset p^{-1}_\nu(\widetilde{\Pi}^{?r}_{\mathcal{Z}})$), where $\nu \in \{1, 2\}$ and $? \in \{\le, <\}$.

The formulas for the morphisms α and β from the proof of Lemma 3.4.2(iv) give rise to morphisms $\alpha : \widetilde{\mathfrak{D}}^{\le r}_{\mathcal{Z}} \to \mathbf{G}^{|\mathbf{n}'|}_{\mathrm{m}}$ and $\beta : \widetilde{\mathfrak{D}}^{\le r}_{\mathcal{Z}} :\to \mathbf{A}^{m'}$ such that the image of $(p_1, \alpha) : \widetilde{\mathfrak{D}}^{\le r}_{\mathcal{Z}} \to \widetilde{\Pi}^{\le r}_{\mathcal{Z}} \times \mathbf{G}^{|\mathbf{n}'|}_{\mathrm{m}}$ coincides with the closed analytic subdomain $Y^{\le r}$ defined by the inequalities $|V_{ij}(y)| \le r$ for $0 \le i \le p'$ and $1 \le j \le n'_i$ and

$$|(V_{i1} \cdot \ldots \cdot V_{in'_i})(y)| \ge \frac{|a_i|}{r |(T_{i,n'_i+1} \cdot \ldots \cdot T_{i,n_i})(x)|}$$

for $0 \le i \le p'$, where x is the image of y in $\widetilde{\Pi}^{\le r}_{\mathcal{Z}}$, and the morphism $(p_1, \alpha, \beta) : \widetilde{\mathfrak{D}}^{\le r}_{\mathcal{Z}} \to \widetilde{\Pi}^{\le r}_{\mathcal{Z}} \times \mathbf{G}^{|\mathbf{n}'|}_{\mathrm{m}} \times \mathbf{A}^{m'}$ identifies $\widetilde{\mathfrak{D}}^{\le r}_{\mathcal{Z}}$ with $Y^{\le r} \times D^{\le r}$, where $D^{\le r}$ is the closed polydisc of radius r with center at zero. The same morphism identifies $\widetilde{\mathfrak{D}}^{<r}_{\mathcal{Z}}$ with $Y^{<r} \times D^{<r}$, where $Y^{<r}$ is the open subset of $\widetilde{\Pi}^{<r}_{\mathcal{Z}} \times \mathbf{G}^{|\mathbf{n}'|}_{\mathrm{m}}$ defined by the corresponding strict inequalities and $D^{<r}$ is the open polydisc of radius r with center at zero.

Let $\Pi^{\le r}_{\mathcal{Y}'}$ and $\widetilde{\Pi}^{\le r}_{\mathcal{Y}'}$ (resp. $\Pi^{<r}_{\mathcal{Y}'}$ and $\widetilde{\Pi}^{<r}_{\mathcal{Y}'}$) be the preimages of $\Pi^{\le r}_{\mathcal{Z}}$ and $\widetilde{\Pi}^{\le r}_{\mathcal{Z}}$ (resp. $\Pi^{<r}_{\mathcal{Z}}$ and $\widetilde{\Pi}^{<r}_{\mathcal{Z}}$) in $\mathcal{X}'^{\mathrm{an}}_\eta$, respectively, and let $\mathfrak{D}^{\le r}_{\mathcal{Y}'}$ (resp. $\mathfrak{D}^{<r}_{\mathcal{Y}'}$) be the preimage of $\mathfrak{D}^{\le r}_{\mathcal{Z}}$ (resp. $\mathfrak{D}^{<r}_{\mathcal{Z}}$) in $\mathfrak{D}_{\mathcal{Y}'}$.

Consider the commutative diagram

$$\begin{array}{ccccc} & & \widetilde{\Pi}_{\mathcal{Y}'}^{\le r} & \longrightarrow & \widetilde{\Pi}_{\mathcal{Z}}^{\le r} \\ & \nearrow p_1 & \uparrow & & \uparrow p_1 \\ \mathfrak{D}_{\mathcal{Y}'}^{\le r} & \xrightarrow{\gamma} & \widetilde{\Pi}_{\mathcal{Y}'}^{\le r} \times_{\mathcal{T}_\eta^{\mathrm{an}}} \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r} & \longrightarrow & \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r} \end{array}$$

From Lemma 3.4.2(iii) it follows that the morphism γ identifies $\mathfrak{D}_{\mathcal{Y}'}^{\le r}$ with an affinoid subdomain of $\widetilde{\Pi}_{\mathcal{Y}'}^{\le r} \times_{\mathcal{T}_\eta^{\mathrm{an}}} \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r} \subset {\mathcal{X}'_\eta}^{\mathrm{an}} \times \mathcal{T}_\eta^{\mathrm{an}}$. Recall that γ is the restriction to $\mathfrak{D}_{\mathcal{Y}'}^{\le r}$ of the étale morphism ${\mathcal{X}'_\eta}^{\mathrm{an}} \times {\mathcal{X}'_\eta}^{\mathrm{an}} \to {\mathcal{X}'_\eta}^{\mathrm{an}} \times \mathcal{T}_\eta^{\mathrm{an}}$. If now $\widetilde{\mathfrak{D}}_{\mathcal{Y}'}^{\le r}$ denotes the preimage of ${\mathcal{X}'_\eta}^{\mathrm{an}} \times_{\mathcal{T}_\eta^{\mathrm{an}}} \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r}$ in ${\mathcal{X}'_\eta}^{\mathrm{an}} \times {\mathcal{X}'_\eta}^{\mathrm{an}}$ under the latter étale morphism, we get an étale morphism $\widetilde{\mathfrak{D}}_{\mathcal{Y}'}^{\le r} \to {\mathcal{X}'_\eta}^{\mathrm{an}} \times_{\mathcal{T}_\eta^{\mathrm{an}}} \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r}$ that identifies the affinoid subdomain $\mathfrak{D}_{\mathcal{Y}'}^{\le r}$ of the source with an affinoid subdomain of the target. By [Ber3, Lemma 3.4], it gives rise to an isomorphism $\mathcal{V}^{\le r} \xrightarrow{\sim} \mathcal{W}^{\le r}$, where $\mathcal{V}^{\le r}$ and $\mathcal{W}^{\le r}$ are open neighborhoods of $\mathfrak{D}_{\mathcal{Y}'}^{\le r}$ in $\widetilde{\mathfrak{D}}_{\mathcal{Y}'}^{\le r}$ and of its image in $\widetilde{\Pi}_{\mathcal{Y}'}^{\le r} \times_{\mathcal{T}_\eta^{\mathrm{an}}} \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r}$. We can shrink both open sets so that $\mathcal{V}^{\le r}$ is contained in the open set $\mathcal{W}$ given in the proposition. Furthermore, since the projection $p_1 : \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r} \to \widetilde{\Pi}_{\mathcal{Z}}^{\le r}$ is proper, we may assume that $\mathcal{W}^{\le r}$ is the preimage of an open neighborhood $\Pi_{\mathcal{Y}'}^{\le r} \subset \mathcal{U}^{\le r} \subset \widetilde{\Pi}_{\mathcal{Y}'}^{\le r}$ in ${\mathcal{X}'_\eta}^{\mathrm{an}} \times_{\mathcal{T}_\eta^{\mathrm{an}}} \widetilde{\mathfrak{D}}_{\mathcal{Z}}^{\le r}$. Notice that $\Delta(\mathcal{U}^{\le r}) \subset \mathcal{V}^{\le r}$. We set $\mathcal{U}^{<r} = \mathcal{U}^{\le r} \cap \widetilde{\Pi}_{\mathcal{Y}'}^{<r}$ and denote by $\mathcal{V}^{<r}$ the preimage of $\mathcal{U}^{<r}$ in $\mathfrak{D}_{\mathcal{Y}'}^{<r}$ (with respect to the projection p_1). Since the set $\widetilde{\Pi}_{\mathcal{Y}'}^{<r}$ is open in ${\mathcal{X}'_\eta}^{\mathrm{an}}$, it follows that $\mathcal{U}^{<r}$ is an open neighborhood of $\Pi_{\mathcal{Y}'}^{<r}$ in ${\mathcal{X}'_\eta}^{\mathrm{an}}$, and one has $\Delta(\mathcal{U}^{<r}) \subset \mathcal{V}^{<r} \subset p_1^{-1}(\mathcal{U}^{<r})$.

By the construction, the morphism (p_1, α, β) gives rise to an isomorphism $\mathcal{V}^{<r} \xrightarrow{\sim} Z^{<r} \times D^{<r}$, where $Z^{<r}$ denotes the preimage of $Y^{<r}$ in $\mathcal{U}^{<r} \times \mathbf{G}_{\mathrm{m}}^{|\mathbf{n}'|}$. The set $\mathcal{U} = \cup_{r<1} \mathcal{U}^{<r}$ is an open neighborhood of $\Pi_{\mathcal{Y}'}$ in ${\mathcal{X}'_\eta}^{\mathrm{an}}$, and the set $\mathcal{V} = \cup_{r<1} \mathcal{V}^{<r}$ is an open neighborhood of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathcal{W}$. Since $\Delta(\mathcal{U}^{<r}) \subset \mathcal{V}^{<r} \subset p_1^{-1}(\mathcal{U}^{<r})$ for all $r < 1$, it follows that $\Delta(\mathcal{U}) \subset \mathcal{V} \subset p_1^{-1}(\mathcal{U})$. That the morphism $p_1 : \mathcal{V} \to \mathcal{U}$ is discoid, if $\mathcal{Y} = \mathfrak{X}_s$, and semi-annular, in general, follows from the construction. ∎

An open neighborhood $\mathcal{V}$ of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathfrak{X}_\eta$ is said to be p_1-*discoid* or p_1-*semi-annular* if it possesses the properties of Proposition 3.5.1.

§4. Properties of the sheaves $\Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X$.

In this section we study the quotient sheaf $\Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X$, which measures non-exactness of the de Rham complex of a smooth k-analytic space X (at Ω^1_X). The study is based on the following property of analytic spaces: any two points from the subset X_0 of a connected closed analytic space X can be connected by smooth analytic curves. This property allows one to reduce certain problems to the one-dimensional case. It is used in the proof of the main result in §7 and of the following facts here. Let $\mathcal{O}^1_X$ be the subsheaf of $\mathcal{O}^*_X$ consisting of the functions f with $|f(x)-1|<1$, and $\mathcal{O}^{\mathrm{c}}_X$ the bigger subsheaf consisting of the functions f for which the real valued function $x\mapsto|f(x)|$ is locally constant. We show that $\mathcal{O}^{\mathrm{c}}_X$ is generated by $\mathcal{O}^1_X$ and $\mathfrak{c}^*_X$. The latter implies that, for any local section f of the sheaf $\mathcal{O}^{\mathrm{c}}_X$, the one-form $\frac{df}{f}$ is exact. If $\mathcal{O}^v_X$ denotes the quotient sheaf $\mathcal{O}^*_X/\mathcal{O}^{\mathrm{c}}_X$, we prove that the induced homomorphism $d\mathrm{Log}:\mathcal{O}^v_X\otimes_{\mathbf{Z}}\mathfrak{c}_X\to\Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X$ is injective. We define a subscheaf $\Psi_X\subset\Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X$ and show that its intersection with the image Υ_X of the homomorphism $d\mathrm{Log}$ is zero and that, in the case when $\dim(X)=1$ and $\widetilde{k}$ is algebraic over a finite field, $\Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X=\Upsilon_X\oplus\Psi_X$. Finally, for every point $x\in X$ we define a subspace $\mathcal{V}_{X,x}\subset\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$, which has a direct complement in $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$ with a basis formed by the classes of one-forms $\frac{df_1}{f_1},\dots,\frac{df_t}{f_t}$, where $f_1,\dots,f_t\in\mathcal{O}^*_{X,x}$ are such that $|f_1(x)|,\dots,|f_t(x)|$ is a basis of the abelian group $|\mathcal{H}(x)^*|/|k^*|$.

4.1. Analytic curve connectedness of closed analytic spaces. In this subsection k is an arbitrary non-Archimedean field with a nontrivial valuation.

4.1.1. Theorem. *Let X be a connected closed k-analytic space. Given two points $x,y\in X_0$, there exist morphisms $\varphi_i:Y^i\to X$, $1\le i\le n$, such that $x\in\varphi_i(Y^1)$, $y\in\varphi_n(Y^n)$ and $\varphi_i(Y^i_0)\cap\varphi_{i+1}(Y^{i+1}_0)\ne\emptyset$ for all $1\le i\le n-1$, where each Y^i is an elementary k_i-analytic curve for a finite separable extension k_i of k.*

4.1.2. Remark. Theorem 6.1.1 from J. de Jong's paper [deJ2] states that, under the assumption of discrete valuation on k, the same is true for an arbitrary connected strictly k-analytic space X but with strictly k-affinoid curves Y_i. Besides the assumption on the valuation, Theorem 4.1.1 is different because the curves Y_i in it are required to be closed (in the sense of [Ber1, p. 49] and [Ber2, p. 34]) and, as a consequence, X is required to be closed (otherwise the statement is not true already in dimension one).

Proof. Consider first the case when X is smooth. It suffices to show that for every point of X there exists a flat quasi-finite morphism $X'\to X$ whose image contains the point and in which

X' possesses the required property. (Recall that, by [Ber2, 3.2.3], flat quasi-finite morphisms are open maps.) By the local description of smooth analytic curves (see §2.3), the statement is true if $\dim(X) = 1$. Assume that $\dim(X) \geq 2$ and that the statement is true in the dimension $\dim(X) - 1$. Shrinking X, we may assume that there is a smooth k-analytic space Y of dimension $\dim(X) - 1$ and a smooth morphism $\varphi : X \to Y$ of pure dimension one. By [Ber2, Theorem 3.7.2], there exists an étale morphism $f : Y' \to Y$ and an open subset $X' \subset X \times_Y Y'$ such that x has a unique preimage x' in X' and the induced morphism $\varphi' : X' \to Y'$ is an elementary fibration of pure dimension one at the point x'. The only properties of such an elementary fibration we need are the following: (1) the geometric fibers of φ' are connected, and (2) there is a commutative diagram

$$\begin{array}{ccc} Y' \times B & \stackrel{j}{\longrightarrow} & X' \\ & p_1 \searrow \quad \swarrow \varphi' & \\ & Y' & \end{array}$$

where B is an open annulus with center at zero, j is an open immersion, and p_1 is the canonical projection. Let k' be a finite separable extension of k with $B(k') \neq \emptyset$. After tensoring X', Y' and B with k', we may assume that the projection $p_1 : Y' \times B \to Y'$ has a section. By the induction hypothesis, the required statement is true for Y'. It follows that it is also true for X'.

Consider now the general case. It suffices to show that there is a finite surjective morphism $X' \to X$ such that all connected components of X' possess the required property. We can therefore assume that X is normal and, in particular, that its regularity locus is a dense Zariski open subset. By [Ber4, §5], we can replace k by a finite extension so that we may assume that X is normal and its smoothness locus is a dense Zariski open subset. Let X' be the non-smoothness locus of X. By a result of W. Lütkebohmert [Lüt1, 1.6] (see also [Ber1, 3.3.15]), the complement of any Zariski closed proper subset of X is connected. Thus, by the previous case, it suffices to show that for each point $x \in X'_0$ there is an open neighborhood $\mathcal{U}$ in X and a connected closed analytic subset $Y \subset \mathcal{U}$ of smaller dimension, which contains x but is not contained in X'. Let f be an element of the maximal ideal of the local ring $\mathcal{O}_{X,x}$ which is not contained in the ideal of definition of X', and let $\mathcal{U}$ be a sufficiently small open neighborhood of x in X such that the closed analytic subset $Y = \{y \in \mathcal{U} \big| f(y) = 0\}$ is connected. Then $\mathcal{U}$ and Y satisfy the necessary condition. ∎

4.1.3. Corollary. *Let X be a reduced strictly k-analytic space and $f \in \mathfrak{N}^K(X)$.*

(i) If there exists a surjective flat quasi-finite morphism $\varphi : Y \to X$ from a reduced strictly k-analytic space Y such that $\varphi^*(f) \in \mathcal{C}^K(Y)$, then $f \in \mathcal{C}^K(X)$;

(ii) If X is closed and, for every morphism $\varphi : Y \to X$ from an elementary k'-analytic curve Y with a finite separable extension k' of k, one has $\varphi^*(f) \in \mathcal{C}^{K'}(Y)$, then $\varphi^*(f) \in \mathcal{C}^K(X)$, where $K' = K \otimes_k k'$.

Proof. (i) The assumption implies that $\varphi^*(f) \in \mathrm{Ker}(\mathcal{C}^K(Y) \rightrightarrows \mathcal{C}^K(Z))$, where Z is the reduction of $Y \times_X Y$. The kernel coincides with $\mathcal{C}^K(X) \subset \mathcal{O}^K(X)$ since the correspondence $X' \mapsto \mathcal{O}(X')$ is a sheaf in the flat quasi-finite topology of X (see [Ber2, 4.1.2]).

(ii) By (i), we may replace k by a finite extension and assume that X is connected and contains a k-rational point x. If $f(x) = a \in K$, Theorem 4.1.1 and the assumption imply that $f(y) = a$ for all points $y \in X_0$. ∎

4.1.4. Corollary. *Assume that k is a closed subfield of $\widehat{k}_0^{\mathrm{a}}$ that contains k_0, where k_0 is a fixed non-Archimedean field whose valuation is nontrivial and discrete. Then the conclusions of Theorem 4.1.1 and Corollary 4.1.3(ii) are true with smooth basic curves instead of elementary ones.*

Proof. The statement follows from Propositions 2.1.1 and 2.4.1. ∎

4.2. The sheaves $\mathcal{O}_X^{\mathrm{c}}$, $\mathcal{O}_X^1$ and $\mathcal{O}_X^v$. In this subsection k is again an arbitrary non-Archimedean field with a nontrivial valuation. For a k-analytic space X, let $\mathcal{O}(X)^c$ denote the subgroup of all $f \in \mathcal{O}(X)^*$ for which the restriction of the real valued function $x \mapsto |f(x)|$ to every connected component of X is constant. The correspondence $U \mapsto \mathcal{O}(U)^c$ is a subsheaf of the étale abelian sheaf $\mathcal{O}_X^*$ denoted by $\mathcal{O}_X^{\mathrm{c}}$. Furthermore, we set $\mathcal{O}(X)^1 = \{f \in \mathcal{O}(X)^* \big| |f(x) - 1| < 1$ for all $x \in X\}$. The correspondence $U \mapsto \mathcal{O}(U)^1$ is a subsheaf of the étale abelian sheaf $\mathcal{O}_X^{\mathrm{c}}$ denoted by $\mathcal{O}_X^1$.

4.2.1. Theorem. *Let X be a geometrically reduced closed k-analytic space. Then the étale abelian sheaf $\mathcal{O}_X^{\mathrm{c}}$ is generated by the subsheaves $\mathcal{O}_X^1$ and $\mathfrak{c}_X^*$.*

Proof. The assumption implies that the set of points $x \in X_0$ such that X is smooth at x and the field $\mathcal{H}(x)$ is separable over k is dense ([Ber9, Lemma 8.1.2]). Replacing k by a finite separable extension, we may assume that X contains a k-rational point x, and we may assume that X is connected. Let $f \in \mathcal{O}(X)^c$. Multiplying f by an element of k^*, we may assume that $|f(x) - 1| < 1$. To prove the statement, it suffices to show that $|f(y) - 1| < 1$ for all points $y \in X_0$, and for this we may assume that the field k is algebraically closed and, by Theorem 4.1.1, we may assume that X is an elementary k-analytic curve. If X is an open disc with center at zero, every invertible function on X is of the form ag with $a \in k^*$ and $g \in \mathcal{O}(X)^1$ and, therefore, $f \in \mathcal{O}(X)^1$. If X is an open annulus with center at zero, then any invertible function on X is of the form agT^n with $a \in k^*$, $g \in \mathcal{O}(X)^1$ and $n \in \mathbf{Z}$, and this easily implies that our function f is contained in $\mathcal{O}(X)^1$.

Finally, let $X = \mathcal{X}_\eta^{\rm an} \backslash \coprod_{i=1}^n E_i$, $n \geq 1$, where $\mathcal{X}$ is a smooth projective curve over k°, each E_i is an open subset isomorphic to an open disc with center at zero and all of them are in pairwise different residue classes of $\mathcal{X}_\eta^{\rm an}$. We can always approximate f by a rational function on $\mathcal{X}$, and so we may assume that f itself is a rational function on $\mathcal{X}$. By the previous two cases, $f \in \mathcal{O}(Y)^1$, where Y is the residue class that contains the point x. It follows that $|f(\sigma) - 1| \leq 1$, where σ is the generic point of $\mathcal{X}_s$ in $\mathcal{X}_\eta^{\rm an}$ and, in particular, the reduction of f on $\mathcal{X}_s$ is well defined. By the previous two cases again, the restriction of f to each of the residue classes Z is of the form ag with $a \in k^*$ and $g \in \mathcal{O}(Z)^1$. It follows that the reduction of f on $\mathcal{X}_s$ is a regular function and, therefore, it is constant. Since it is equal to one at the point that corresponds to Y, the required fact follows. ∎

Let $\mathcal{O}_X^v$ denote the quotient sheaf $\mathcal{O}_X^*/\mathcal{O}_X^c$. It is an étale sheaf of torsion free abelian groups, and its restriction to the usual topology of X is a subsheaf of the quotient of the sheaf of positive real valued continuous functions on X by the subsheaf of constant functions. If the characteristic of k is zero, Theorem 4.2.1 implies that, for any $f \in \mathcal{O}(X)^c$, the one-form $\frac{df}{f}$ is a section of the sheaf $d\mathcal{O}_X$ and, therefore, the correspondence $f \mapsto \frac{df}{f}$ gives rise to a homomorphism $d{\rm Log} : \mathcal{O}_X^v \to \Omega_X^{1,{\rm cl}}/d\mathcal{O}_X$. In §§4.3 and 4.4 we'll prove that the induced homomorphism $d{\rm Log} : \mathcal{O}_X^v \otimes_{\mathbf{Z}} \mathfrak{c}_X \to \Omega_X^{1,{\rm cl}}/d\mathcal{O}_X$ is injective if X is of dimension one and of an arbitrary dimension, respectively.

4.2.2. Corollary. *Let X be a geometrically reduced closed k-analytic space X, $\overline{x} : \mathbf{p}_{\mathcal{H}(\overline{x})} \to X$ a geometric point over a point $x \in X$, and $\mathcal{H}_{\overline{x}}$ the algebraic separable closure of $\mathcal{H}(x)$ in $\mathcal{H}(\overline{x})$. Then there is an exact sequence of $G_{\overline{x}/x}$-modules*

$$0 \longrightarrow \widetilde{\mathcal{H}}_{\overline{x}}^*/\widetilde{\mathfrak{c}}_{X,\overline{x}}^* \longrightarrow \mathcal{O}_{X,\overline{x}}^v \longrightarrow \sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|} \longrightarrow 0 \ .$$

Notice that $\mathfrak{c}_{X,\overline{x}}$ coincides with the algebraic separable closure of k in $\mathcal{H}(\overline{x})$ and that $|\mathcal{H}_{\overline{x}}^*| = \sqrt{|\mathcal{H}(x)^*|}$ and $|\mathfrak{c}_{X,\overline{x}}^*| = \sqrt{|k^*|}$ (see [Ber9, Corollary 8.1.3(ii)]).

Proof. By Theorem 4.2.1, there is an exact sequence

$$0 \longrightarrow \mathfrak{c}_{X,\overline{x}}^{\circ *}\mathcal{O}_{X,\overline{x}}^1 \longrightarrow \mathcal{O}_{X,\overline{x}}^c \longrightarrow \sqrt{|k^*|} \longrightarrow 0 \ .$$

Setting $S_{\overline{x}} = \{f \in \mathcal{O}_{X,\overline{x}}^* \big| |f(\overline{x})| = 1\}$, the canonical homomorphism from the above to the exact sequence

$$0 \longrightarrow S_{\overline{x}} \longrightarrow \mathcal{O}_{X,\overline{x}}^* \longrightarrow \sqrt{|\mathcal{H}(x)^*|} \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow S_{\overline{x}}/(\mathfrak{c}_{X,\overline{x}}^{\circ *}\mathcal{O}_{X,\overline{x}}^1) \longrightarrow \mathcal{O}_{X,\overline{x}}^v \longrightarrow \sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|} \longrightarrow 0 \ .$$

Finally, an isomorphism of the first term with $\widetilde{H}^*_{\overline{x}}/\widetilde{\mathfrak{c}}^*_{X,\overline{x}}$ is obtained from the exact sequence

$$0 \longrightarrow \mathcal{O}^1_{X,\overline{x}} \longrightarrow S_{\overline{x}} \longrightarrow \widetilde{H}^*_{\overline{x}} \longrightarrow 0 \ . \qquad \blacksquare$$

4.3. Structure of the sheaves $\Omega^1_X/d\mathcal{O}_X$ for smooth analytic curves. Let k be a non-Archimedean field of characteristic zero with a nontrivial valuation. For a smooth k-analytic space X, we denote by Υ_X the image of the homomorphism $d\mathrm{Log} : \mathcal{O}^v_X \otimes_{\mathbf{Z}} \mathfrak{c}_X \to \Omega^1_X/d\mathcal{O}_X$. Furthermore, for a smooth k-analytic curve X, we denote by $\widetilde{\Psi}(X)$ the subspace of all one-forms $\omega \in \Omega^1(X)$ with the property that every point $x \in X$ has an open neighborhood $\mathcal{U}$ such that $\omega\big|_{\mathcal{U}\backslash\{x\}} \in d\mathcal{O}(\mathcal{U}\backslash\{x\})$. It is easy to see that the correspondence $U \mapsto \widetilde{\Psi}(U)$ is a sheaf in the étale topology of X. We denote it by $\widetilde{\Psi}_X$, and denote by Ψ_X the quotient sheaf $\widetilde{\Psi}_X/d\mathcal{O}_X$.

4.3.1. Theorem. *Let X be a smooth k-analytic curve. Then*

(i) the homomorphism $d\mathrm{Log} : \mathcal{O}^v_X \otimes_{\mathbf{Z}} \mathfrak{c}_X \to \Omega^1_X/d\mathcal{O}_X$ is injective, i.e., $\mathcal{O}^v_X \otimes_{\mathbf{Z}} \mathfrak{c}_X \xrightarrow{\sim} \Upsilon_X$;

(ii) $\Upsilon_X \cap \Psi_X = 0$;

(iii) the support of every element of $\Psi(X)$ is contained in the set of points of type (2) and positive genus;

(iv) the stalk $\Psi_{X,x}$ of Ψ_X at a point $x \in X$ of type (2) and genus g is a vector space over $\mathfrak{c}_{X,x}$ of dimension $2g$;

(v) if the residue field $\widetilde{k}$ is algebraic over a finite field, then $\Omega^1_X/d\mathcal{O}_X = \Upsilon_X \oplus \Psi_X$.

Recall that the first de Rham cohomology group $H^1_{\mathrm{dR}}(\mathcal{X})$ of a geometrically connected smooth projective curve of genus g over an abstract field k of characteristic zero is canonically isomorphic to the quotient of the space of differentials of second kind by the subspace of exact differentials. (A differential of second kind on $\mathcal{X}$ is a rational one-form with the property that its residues at all points of $\mathcal{X}_0$ are zero.) One also has $H^0_{\mathrm{dR}}(\mathcal{X}) = k$ and $H^2_{\mathrm{dR}}(\mathcal{X}) \xrightarrow{\sim} k$. The latter isomorphism is constructed as follows.

First of all, the spectral sequence $E^{p,q}_1 = H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}) \Longrightarrow H^{p+q}_{\mathrm{dR}}(\mathcal{X})$ gives rise to an isomorphism $H^2_{\mathrm{dR}}(\mathcal{X}) \xrightarrow{\sim} H^1(\mathcal{X}, \Omega^1_{\mathcal{X}})$ (and an exact sequence $0 \to H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \to H^1_{\mathrm{dR}}(\mathcal{X}) \to H^1(\mathcal{X}, \mathcal{O}_X) \to 0$). Consider the exact sequence $0 \to \Omega^1_{\mathcal{X}} \to \Omega^1_{k(\mathcal{X})} \to \Omega^1_{k(\mathcal{X})}/\Omega^1_{\mathcal{X}} \to 0$. Since the first cohomology group of the middle term is zero, $H^1(\mathcal{X}, \Omega^1_{\mathcal{X}})$ is canonically isomorphic to the cokernel of the homomorphism from the space of global sections of the middle term to that of global sections of the third term. An element of the latter can be represented as a family $\{\omega_x\}_{x \in \mathcal{X}_0}$ with $\omega_x \in \Omega^1_{k(\mathcal{X})}/\Omega^1_{\mathcal{X},x}$ taken over the set of closed points $\mathcal{X}_0$ such that $\omega_x = 0$ for all but finitely many x.

The correspondence $\{\omega_x\}_{x\in\mathcal{X}_0} \mapsto \sum_{x\in\mathcal{X}_0} \mathrm{Tr}_{k(x)/k}(\mathrm{Res}_x(\omega_x))$ gives rise to an isomorphism between the above cokernel and k.

Furthermore, let $\mathcal{X}'$ be the complement of a subset $\Sigma = \{x_1,\dots,x_n\} \subset \mathcal{X}_0$, $n \geq 1$. The curve $\mathcal{X}'$ is affine and, therefore, the de Rham cohomology groups $H^q_{\mathrm{dR}}(\mathcal{X}')$ coincide with the cohomology groups $H^q(\Omega^\cdot(\mathcal{X}'))$ of the complex $\Omega^\cdot(\mathcal{X}') : 0 \to \mathcal{O}(\mathcal{X}') \to \Omega^1(\mathcal{X}') \to 0$, i.e., $H^0_{\mathrm{dR}}(\mathcal{X}') = k$, $H^2_{\mathrm{dR}}(\mathcal{X}') = 0$ and $H^1_{\mathrm{dR}}(\mathcal{X}') = \Omega^1(\mathcal{X}')/d(\mathcal{O}(\mathcal{X}'))$. The latter is calculated as follows. Consider k as a non-Archimedean field with respect to the trivial valuation, and let $\mathcal{X}^{\mathrm{an}}$ be the corresponding analytification of $\mathcal{X}$, and $H^q_{\mathrm{dR},\Sigma}(\mathcal{X}^{\mathrm{an}})$ the high direct images of the functor of global sections with support in Σ evaluated at the de Rham complex of $\mathcal{X}^{\mathrm{an}}$. By GAGA [Ber1, §3.5], there are canonical isomorphisms $H^q_{\mathrm{dR}}(\mathcal{X}) \stackrel{\sim}{\to} H^q_{\mathrm{dR}}(\mathcal{X}^{\mathrm{an}})$ and $H^q_{\mathrm{dR}}(\mathcal{X}') \stackrel{\sim}{\to} H^q_{\mathrm{dR}}(\mathcal{X}'^{\mathrm{an}})$. If x is a closed point of $\mathcal{X}$, an open neighborhood of x in $\mathcal{X}^{\mathrm{an}}$ is isomorphic to an open neighborhood of zero in the $k(x)$-analytic projective line $\mathbf{P}^1_{k(x)}$. It follows that $H^1_{\mathrm{dR},\{x\}}(\mathcal{X}^{\mathrm{an}}) = 0$ and $H^2_{\mathrm{dR},\{x\}}(\mathcal{X}^{\mathrm{an}}) \stackrel{\sim}{\to} k(x)$ and, therefore, there is an exact sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(\mathcal{X}) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{X}') \longrightarrow H^2_{\mathrm{dR},\Sigma}(\mathcal{X}) = \oplus_{i=1}^n H^2_{\mathrm{dR},\{x_i\}}(\mathcal{X}) \longrightarrow H^2_{\mathrm{dR}}(\mathcal{X}) \longrightarrow 0 \ .$$

For every $1 \leq i \leq n$, the composition of the canonical isomorphism $H^2_{\mathrm{dR},\{x_i\}}(\mathcal{X}) \stackrel{\sim}{\to} k(x_i)$ with the homomorphism $H^1_{\mathrm{dR}}(\mathcal{X}') \to H^2_{\mathrm{dR},\{x_i\}}(\mathcal{X})$ takes a regular one-form on $\mathcal{X}'$ to its residue at the point x_i, and with the homomorphism $H^2_{\mathrm{dR},\{x_i\}}(\mathcal{X}) \to H^2_{\mathrm{dR}}(\mathcal{X})$ corresponds to the trace map $\mathrm{Tr}_{k(x_i)/k} : k(x_i) \to k$. In particular, the dimension of $H^1_{\mathrm{dR}}(\mathcal{X}')$ over k is equal to $2g - 1 + \sum_{i=1}^n [k(x_i) : k]$.

Assume now that k is a non-Archimedean field of characteristic zero (whose valuation is not assumed to be nontrivial). By GAGA (see [Ber1, §§3.4-3.5]), there are canonical isomorphisms $H^q_{\mathrm{dR}}(\mathcal{X}) \stackrel{\sim}{\to} H^q_{\mathrm{dR}}(\mathcal{X}^{\mathrm{an}})$. Assume we are given pairwise disjoint open neighborhoods $D_1,\dots,D_n$ of the points $x_1,\dots,x_n$ in $\mathcal{X}^{\mathrm{an}}$, respectively, such that each D_i is isomorphic to the open unit disc over $\mathcal{H}(x_i)$ with center at zero that corresponds to the point x_i. We fix such isomorphisms and, given $0 < r_1,\dots,r_n < 1$, denote by $D(x_i;r_i)$ and $E(x_i;r_i)$ the preimages of the open and closed discs of radius r_i with center at zero. (Notice that $D(x_i;r_i)$ and $E(x_i;r_i)$ do not depend on the choice of the isomorphisms.) Let X be the open set $\mathcal{X}^{\mathrm{an}} \backslash \coprod_{i=1}^n E(x_i;r_i)$. Since X is a Stein space, the de Rham cohomology groups $H^q_{\mathrm{dR}}(X)$ coincide with the cohomology groups of the complex $\Omega^\cdot(X) : 0 \to \mathcal{O}(X) \xrightarrow{d} \Omega^1(X) \to 0$ and, in particular, $H^1_{\mathrm{dR}}(X) = \Omega^1(X)/d\mathcal{O}(X)$.

4.3.2. Lemma. *There are canonical isomorphisms*

$$H^q_{\mathrm{dR}}(\mathcal{X}') \stackrel{\sim}{\to} H^q_{\mathrm{dR}}(\mathcal{X}'^{\mathrm{an}}) \stackrel{\sim}{\to} H^q_{\mathrm{dR}}(X), \ q \geq 0 \ .$$

Proof. We already mentioned the validity of the first isomorphism in the case when the valuation on k is trivial. In the general case, consider the morphism of long exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H^0_{\mathrm{dR},\Sigma}(\mathcal{X}) & \to & H^0_{\mathrm{dR}}(\mathcal{X}) & \to & H^0_{\mathrm{dR}}(\mathcal{X}') & \to & H^1_{\mathrm{dR},\Sigma}(\mathcal{X}) & \to & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^0_{\mathrm{dR},\Sigma}(\mathcal{X}^{\mathrm{an}}) & \to & H^0_{\mathrm{dR}}(\mathcal{X}^{\mathrm{an}}) & \to & H^0_{\mathrm{dR}}(\mathcal{X}'^{\mathrm{an}}) & \to & H^1_{\mathrm{dR},\Sigma}(\mathcal{X}^{\mathrm{an}}) & \to & \dots \end{array}$$

It follows that the validity of the first isomorphism is equivalent to that of the isomorphism $H^q_{\mathrm{dR},\Sigma}(\mathcal{X}) \stackrel{\sim}{\to} H^q_{\mathrm{dR},\Sigma}(\mathcal{X}^{\mathrm{an}})$. (In particular, the latter isomorphisms hold over fields with trivial valuation.) One has $H^q_{\mathrm{dR},\Sigma}(\mathcal{X}^{\mathrm{an}}) \stackrel{\sim}{\to} \oplus_{i=1}^n H^q_{\mathrm{dR},\{x_i\}}(D(x_i,r_i))$. Let x be one of the points $x_1,\dots,x_n$. Each of the groups in the direct sum is isomorphic to $H^q_{\mathrm{dR},\infty}(\mathbf{P}^1_{\mathcal{H}(x_i)})$. Applying the second long exact sequence for the projective and affine lines and the facts that $H^q_{\mathrm{dR}}(\mathbf{P}^1_{\mathcal{H}(x_i)}) \stackrel{\sim}{\to} \mathcal{H}(x_i)$ for $q=0,2$, $H^1_{\mathrm{dR}}(\mathbf{P}^1_{\mathcal{H}(x_i)})=0$, $H^0_{\mathrm{dR}}(\mathbf{A}^1_{\mathcal{H}(x_i)}) \stackrel{\sim}{\to} \mathcal{H}(x_i)$, and $H^q_{\mathrm{dR}}(\mathbf{A}^1_{\mathcal{H}(x_i)})=0$ for $q=1,2$, we get the required isomorphisms.

The second isomorphisms are verified in the similar way. Namely, consider the long exact sequence

$$0 \to H^0_{\mathrm{dR},S}(\mathcal{X}'^{\mathrm{an}}) \to H^0_{\mathrm{dR}}(\mathcal{X}'^{\mathrm{an}}) \to H^0_{\mathrm{dR}}(X) \to H^1_{\mathrm{dR},S}(\mathcal{X}'^{\mathrm{an}}) \to \dots$$

where $S=\coprod_{i=1}^n (E(x_i,r_i)\backslash\{x_i\})$. It suffices to show that $H^q_{\mathrm{dR},S}(\mathcal{X}'^{\mathrm{an}})=0$ for all $q\geq 0$. For this it suffices to verify that $H^q_{\mathrm{dR},S}(\mathbf{A}^1)=0$ for all $q\geq 0$ with $S=\mathbf{A}^1\backslash D(0;r)$. But this follows from the long exact sequence

$$0 \to H^0_{\mathrm{dR},S}(\mathbf{A}^1) \to H^0_{\mathrm{dR}}(\mathbf{A}^1) \to H^0_{\mathrm{dR}}(D(0;r)) \to H^1_{\mathrm{dR},S}(\mathbf{A}^1) \to \dots$$

and the facts that $H^0_{\mathrm{dR}}(\mathbf{A}^1)=H^0_{\mathrm{dR}}(D(0;r))=k$ and $H^q_{\mathrm{dR}}(\mathbf{A}^1)=H^q_{\mathrm{dR}}(D(0;r))=0$ for $q\geq 1$. ∎

4.3.3. Corollary. *For any non-Archimedean field k' over k, there are canonical isomorphisms $H^q_{\mathrm{dR}}(X)\otimes_k k' \stackrel{\sim}{\to} H^q_{\mathrm{dR}}(X')$, $q\geq 0$, where $X'=X\widehat{\otimes}_k k'$.* ∎

Lemma 4.3.2 implies that there is an exact sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(\mathcal{X}) \longrightarrow H^1_{\mathrm{dR}}(X) \longrightarrow \oplus_{i=1}^n \mathcal{H}(x_i) \longrightarrow k \longrightarrow 0\ ,$$

where the first homomorphism is the canonical one, the second homomorphism takes a one-form $\omega\in\Omega^1(X)$ to the element $\alpha\in\mathcal{H}(x_i)$ for which the restriction of ω to the open annulus $B_i = D_i\backslash E(x_i;r_i)$ is $\alpha\frac{dT}{T}$ up to an exact one-form (where T is a coordinate function on D_i with zero at x_i), and the third homomorphism is induced by the trace maps from $\mathcal{H}(x_i)$ to k. It follows that the first homomorphism identifies $H^1_{\mathrm{dR}}(\mathcal{X})$ with the subspace of the classes of those one-forms $\omega\in\Omega^1(X)$ whose restriction to every annulus B_i is an exact one-form.

4.3.4. Lemma. *In the above situation, assume that the valuation on k is nontrivial, $\mathcal{X} = \mathcal{Y}_\eta$, where $\mathcal{Y}$ is a smooth projective curve over k°, and the open discs $D_1, \dots, D_n$ are the residue classes of closed points of $\mathcal{Y}_s$. Then*

(i) $d\mathcal{O}(X) \xrightarrow{\sim} (d\mathcal{O}_X)(X)$ and $\Omega^1(X)/d\mathcal{O}(X) \xrightarrow{\sim} H^0(X, \Omega^1_X/d\mathcal{O}_X)$;

(ii) the image of $H^1_{\mathrm{dR}}(\mathcal{X})$ in $H^1_{\mathrm{dR}}(X) = \Omega^1(X)/d\mathcal{O}(X)$ coincides with $\Psi(X)$;

(iii) the homomorphism $\mathcal{O}(X)^*/\mathcal{O}(X)^c \otimes_{\mathbf{Z}} k \to \Omega^1(X)/d\mathcal{O}(X) : f \mapsto \frac{df}{f}$ is injective;

(iv) the intersection of the image of the homomorphism (iii) with $\Psi(X)$ is zero.

Proof. Since $H^q(X, \mathfrak{c}_X) = 0$ for $q \geq 1$, one has $H^1(X, d\mathcal{O}_X) \xrightarrow{\sim} H^1(X, \mathcal{O}_X) = 0$, and (i) follows. The remark before the formulation now immediately implies (ii).

To prove (iii) and (iv), we may increase the field k and assume that all of the points $x_1, \dots, x_n$ are k-rational. Notice that the restriction of a function $f \in \mathcal{O}(X)^*$ to each annulus B_j is of the form agT^{l_j}, where $a \in k^*$, $g \in \mathcal{O}(B_j)^1$ and $l_j \in \mathbf{Z}$. If $\mathfrak{Z}$ is the open affine subscheme of $\widehat{\mathcal{Y}}$ with $\mathfrak{Z}_s = \mathcal{Y}_s \backslash \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, where $\mathbf{x}_i = \pi(x_i)$, then the restriction of the real valued function $x \mapsto |f(x)|$ to $\mathfrak{Z}_\eta$ is constant. It follows that $f \in \mathcal{O}(X)^c$ if and only if the integral vector $(l_1, \dots, l_n)$ is zero. Let now $f_1, \dots, f_m$ be invertible analytic functions that generate a free abelian subgroup of $\mathcal{O}(X)^*/\mathcal{O}(X)^c$ of rank m, and let $\omega = \sum_{i=1}^m \lambda_i \frac{df_i}{f_i}$, $\lambda_i \in k$, be such that the restriction of ω to each open annulus B_j, $1 \leq j \leq n$, is an exact one-form. To prove (iii) and (iv), it suffices to show that ω is an exact one-form. The restriction of f_i to B_j is of the form $agT^{l_{ij}}$ as above. The above remark implies that the integral vectors $(l_{i1}, \dots, l_{in})$, $1 \leq i \leq m$, are linearly independent over $\mathbf{Z}$. It follows that they are linearly independent over $\mathbf{Q}$ and, therefore, over k. It remains to notice that the restriction of ω to B_j is $(\sum_{i=1}^m \lambda_i l_{ij}) \frac{dT}{T}$ up to an exact one-form. ∎

Proof of Theorem 4.3.1. The local description of smooth k-analytic curves (see §2.2) and Lemma 4.3.4 straightforwardly imply the statements (i)-(iii). Lemma 4.3.4 implies also the statement (iv) in the case when X is elementary. The general case of (iv) follows from the latter and the simple fact that, given a finite Galois extension k' over k with the Galois group G and a vector space V of dimension n over k' provided with a k-linear action of G such that ${}^\sigma(\alpha v) = {}^\sigma\alpha{}^\sigma v$ for all $\sigma \in G$, $\alpha \in k'$ and $v \in V$, one has $\dim_k(V^G) = n$.

To prove (v), it suffices to show that, in the situation of Lemma 4.3.4, if the points $x_1, \dots, x_n$ are k-rational and the numbers $r_1, \dots, r_n$ are sufficiently close to one, then $H^1_{\mathrm{dR}}(X) = H^1_{\mathrm{dR}}(\mathcal{X}) \oplus \Upsilon(X)$. Notice that in this case the space $H^1_{\mathrm{dR}}(X)$ is of dimension $2g+n-1$ over k. Let $\mathfrak{Z} = \mathrm{Spf}(A)$ be the open affine subscheme of the formal completion $\widehat{\mathcal{Y}}$ with $\mathfrak{Z}_s = \mathcal{Y}_s \backslash \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, where $\mathbf{x}_i = \pi(x_i)$. The assumption on the field k implies that the class of each of the divisors $(\mathbf{x}_1) - (\mathbf{x}_j)$, $2 \leq j \leq n$, is of finite order. This means that there exist elements $f_1, \dots, f_{n-1} \in A^*$ such that the divisor of the rational function on $\mathcal{X}_s$ whose restriction to $\mathfrak{Z}_s$ is the image of f_j in $\widetilde{A} = A/k^{\circ\circ}A$ is a nonzero multiple of $(\mathbf{x}_1) - (\mathbf{x}_j)$. Since the image of $\mathcal{O}(X)$ in $A \otimes_{k^\circ} k$ is dense, we may assume that $f_1, \dots, f_n$ come from analytic functions on X. If now r_i's are sufficiently close to one, those functions are invertible on X. Thus, the subspace of $H^1_{\mathrm{dR}}(X)$, generated by the classes of $\frac{df_1}{f_1}, \dots, \frac{df_{n-1}}{f_{n-1}}$ is of dimension $n-1$, and the required fact follows. ∎

4.3.5. Corollary. *Let $X = \mathcal{X}^{\mathrm{an}}$, where $\mathcal{X}$ is a geometrically connected smooth projective curve over k of genus g, and let $x_1, \dots, x_n$ be the points of X of type (2) and positive genuses $g_1, \dots, g_n$, and b the Betti number of $\overline{X} = X \widehat{\otimes}_k \widehat{k}^{\mathrm{a}}$. Then*

(i) $\dim_k(H^0(X, \Omega^1_X/d\mathcal{O}_X)) = 2g - b$;

(ii) $\dim_k(\Psi(X)) = 2\sum_{i=1}^n g_i = 2(g-b)$;

(iii) if $\widetilde{k}$ is algebraic over a finite field, then $\dim_k(\Upsilon(X)) = b$.

Proof. The spectral sequence $E_2^{p,q} = H^p(X, \Omega^{q,\mathrm{cl}}_X/d\Omega^{q-1}_X) \Longrightarrow H^{p+q}_{\mathrm{dR}}(X)$ gives rise to an exact sequence

$$0 \to H^1(X, \mathfrak{c}_X) \to H^1_{\mathrm{dR}}(X) \to (\Omega^1_X/d\mathcal{O}_X)(X) \to 0 \ .$$

The dimension of the middle term is $2g$. Since $H^1(X, \mathfrak{c}_X) = H^1(\overline{X}, \widehat{k}^{\mathrm{a}}_X)^G$, it follows that the dimension of the first term is b and, therefore, (i) is true. Furthermore, Theorem 4.3.1 implies the first equality in (ii) and the fact that (iii) follows from the second equality in (ii). To prove the latter, we may replace k by a finite extension and assume that $\mathcal{X} = \mathcal{Y}_\eta$, where $\mathcal{Y}$ is a strictly semi-stable projective curve over k° such that all of the double points of $\mathcal{Y}_s$ are $\widetilde{k}$-rational and split. (The latter is equivalent to the fact that, for such a point $\mathbf{y} \in \mathcal{Y}_s$ the preimage $\pi^{-1}(\mathbf{y})$ in $\mathcal{Y}^{\mathrm{an}}_\eta = X$ is isomorphic to an open annulus with center at zero.) The skeleton S of the formal completion $\widehat{\mathcal{Y}}$ of $\mathcal{Y}$ along $\mathcal{Y}_s$ is a finite graph embedded into X whose vertices are the preimages of the generic points of the irreducible components of $\mathcal{Y}_s$ and the edges are the skeletons of the open annuli $\pi^{-1}(\mathbf{y})$ for the double points $\mathbf{y} \in \mathcal{Y}_s$. Let τ be the canonical retraction map $X \to S$ and, for a vertex x of S, let S_x denote the open subset of S which is a union of x and the open edges emanating from x. Then $\tau^{-1}(S_x)$ is an elementary curve, and all of the sets $\tau^{-1}(S_x)$ form an open covering of X. The description of the first de Rham cohomology group of an elementary curve implies that there is an exact sequence

$$0 \to \Psi(X) \to H^0(X, \Omega^1_X/d\mathcal{O}_X) \to \mathrm{Harm}(S, k) \to 0 \ ,$$

where $\mathrm{Harm}(S,k)$ is the space of harmonic cochains on S with values in k (i.e., maps from the set of oriented edges of S to k that satisfy the harmonicity condition at every vertex of S). The second equality in (ii) follows. ∎

4.4. Injectivity of the homomorphism $d\mathrm{Log}: \mathcal{O}_X^v \otimes_{\mathbf{Z}} \mathfrak{c}_X \to \Omega_X^{1,\mathrm{cl}}/d\mathcal{O}_X$.

4.4.1. Theorem. *Let k be a non-Archimedean field of characteristic zero with a nontrivial valuation. Then for any smooth k-analytic space X the homomorphism $\mathcal{O}_X^v \otimes_{\mathbf{Z}} \mathfrak{c}_X \to \Omega_X^{1,\mathrm{cl}}/d\mathcal{O}_X : f \mapsto \frac{df}{f}$ is injective and, therefore, $\mathcal{O}_X^v \otimes_{\mathbf{Z}} \mathfrak{c}_X \xrightarrow{\sim} \Upsilon_X$.*

Let $\mathcal{X}$ be an irreducible reduced scheme of finite type over an abstract field k (of arbitrary characteristic). We say that rational functions $f_1, \dots, f_n \in k(\mathcal{X})^*$ are *multiplicatively independent (modulo constants) at a point* $\mathbf{x} \in \mathcal{X}$ if they are defined and not equal to zero at $\mathbf{x}$ and their images in the quotient group $k(\mathbf{x})^*/L_{\mathbf{x}}^*$ are linearly independent over $\mathbf{Z}$, where $L_{\mathbf{x}}$ is the algebraic closure of k in $k(\mathbf{x})$.

4.4.2. Lemma. *Assume that* $\dim(\mathcal{X}) > 1$. *If rational functions $f_1, \dots, f_n \in k(\mathcal{X})^*$ are multiplicatively independent at the generic point of $\mathcal{X}$, then they are multiplicatively independent at the generic point of an irreducible algebraic curve in $\mathcal{X}$.*

Proof. To prove the statement, we may assume that k is algebraically closed, $\mathcal{X}$ is smooth affine, $f_1, \dots, f_n \in \mathcal{O}(\mathcal{X})^*$, and it suffices to show that the functions $f_1, \dots, f_n$ are independent at the generic point of an irreducible closed subscheme of smaller dimension. Furthermore, we can find an open immersion of $\mathcal{X}$ into a connected projective normal scheme $\overline{\mathcal{X}}$. Let $\mathcal{Z} = \overline{\mathcal{X}} \backslash \mathcal{X}$. By J. de Jong's Theorem 4.1 from [deJ3], there exists a proper, dominant and generically finite morphism $\varphi : \mathcal{X}' \to \overline{\mathcal{X}}$ such that $\mathcal{X}'$ is an irreducible smooth projective scheme and $\varphi^{-1}(\mathcal{Z})$ is a strict normal crossings divisor. Let $D_1, \dots, D_m$ be the irreducible components of the latter. The functions $f_1, \dots, f_n$ are evidently independent at the generic point of $\mathcal{X}'$ and, therefore, their divisors $(f_1), \dots, (f_n)$ in $\mathcal{X}'$, supported in $D_1, \dots, D_m$, are linearly independent over $\mathbf{Z}$. Let $\mathcal{X}'$ be a closed subscheme of a projective space $\mathcal{P}$, and let $\{x_1, \dots, x_l\}$ be k-rational points of $\mathcal{X}'$ such that all irreducible components of the intersections $D_i \cap D_j$ with $i \neq j$ contain at least one of them. By Bertini Theorem (see Theorem 8.18 of Ch. II and Remark 7.9.1 of Ch. III in [Har]), there exists a hyperplane $H \subset \mathcal{P}$ such that the intersection $\mathcal{Y} = H \cap \mathcal{X}'$ is an irreducible smooth suscheme of dimension $\dim(\mathcal{X}) - 1$, each intersection $H \cap D_i$ is a smooth subscheme of dimension $\dim(\mathcal{X}) - 2$, and H does not contain any of the points x_i. It follows that the functions $f_1, \dots, f_n$ are defined and not equal to zero at the generic point $\mathbf{y}$ of $\mathcal{Y}$, and the image $\mathbf{x}$ of $\mathbf{y}$ in $\overline{\mathcal{X}}$ lies in $\mathcal{X}$. By the construction, the canonical map from the divisor subgroup of $\mathcal{X}'$, supported in $D_1, \dots, D_n$, to the divisor group of $\mathcal{Y}$ is injective and, therefore, the images of the functions $f_1, \dots, f_n$ in $k(\mathbf{y})^*/k^*$ are linearly independent over $\mathbf{Z}$. It follows that $f_1, \dots, f_n$ are independent at the point $\mathbf{x}$. ∎

Let now X be a smooth k-analytic space, where k is a non-Archimedean field with a nontrivial valuation. We say that analytic functions $f_1, \dots, f_n \in \mathcal{O}(X)$ are *multiplicatively independent at a point* $x \in X$ if $|f_1(x)| = \dots = |f_n(x)| = 1$ and the images of $f_1(x), \dots, f_n(x)$ in $\widetilde{\mathcal{H}(x)}^*/L_x^*$ are linearly independent over $\mathbf{Z}$, where L_x is the algebraic closure of $\widetilde{k}$ in $\widetilde{\mathcal{H}(x)}$.

4.4.3. Lemma. *If functions $f_1, \dots, f_n \in \mathcal{O}(X)$ are multiplicatively independent at a point $x \in X$ with $s(x) = \dim(X) > 1$, then there is a smooth k'-analytic curve Y for a finite extension k' of k and a morphism $\varphi : Y \to X$ such that the functions $\varphi^*(f_1), \dots, \varphi^*(f_n)$ are multiplicatively independent at a point $y \in Y$.*

Proof. Step 1. *One may assume that $\widehat{\mathcal{X}}_\eta \subset X \subset \mathcal{X}_\eta^{\mathrm{an}}$, where $\mathcal{X}$ is a connected affine smooth scheme over k°, $x \in \widehat{\mathcal{X}}_\eta$ and $\pi(x)$ is the generic point of $\mathcal{X}_s$.* (Notice that, by [Ber7, 1.7], for such a point x one has $\widetilde{k}(\pi(x)) \xrightarrow{\sim} \widetilde{\mathcal{H}(x)}$.) To show this we use the reasoning from the proof of Lemma 2.1.2. Namely, as in Step 1 of that proof, we may assume that X is an open subset in $\mathcal{Y}^{\mathrm{an}}$, where $\mathcal{Y}$ is a smooth irreducible affine scheme over k, and that the point x is contained in a strictly affinoid subdomain W of X such that the image of x under the reduction map $W \to \widetilde{W}$ is the generic point of an irreducible component of $\widetilde{W}$. Furthermore, by [Ber7, Lemma 9.4], there is an open embedding of $\mathcal{Y}$ in $\mathcal{Z}_\eta$, where $\mathcal{Z}$ is an integral scheme proper and flat over k°, and an open subscheme $\mathcal{W} \subset \mathcal{Z}_s$ such that $W = \pi^{-1}(\mathcal{W})$. Finally, by de Jong's result (in the form of [Ber7, Lemma 9.2]), there exists a finite extension k' of k, a poly-stable fibration $\underline{\mathcal{Z}}' = (\mathcal{Z}'_l \to \dots \to \mathcal{Z}'_1 \to \mathcal{Z}'_0 = \mathrm{Spec}(k'^\circ))$, where all morphisms f_i are projective of dimension one, and a dominant morphism $\varphi : \mathcal{Z}'_l \to \mathcal{Z}$ that induces a proper generically étale morphism $\mathcal{Z}'_{l,\eta} \to \mathcal{Z}_\eta$. The latter morphism is evidently étale at every point x' from the preimage of x in $\mathcal{Z}'_{l,\eta}$. By the argument from Step 4 of the proof of Lemma 2.1.2, $\pi(x')$ is the generic point of an irreducible component of $\mathcal{Z}'_{l,s}$. Let now $\mathcal{X}$ be an open connected smooth affine subscheme of $\mathcal{Z}'_l$ with $\pi(x') \in \mathcal{X}_s \subset \varphi^{-1}(\mathcal{W})$. Then $\widehat{\mathcal{X}}_\eta \subset \varphi^{-1}(X) \subset \mathcal{X}_\eta^{\mathrm{an}}$ and $\pi(x')$ is the generic point of $\mathcal{X}_s$.

Step 2. By Lemma 4.4.2, applied to the irreducible smooth scheme $\mathcal{X}_s$ over $\widetilde{k}$ and the images of $f_1(x), \dots, f_n(x)$ in $\widetilde{k}(\mathcal{X}_s) = \widetilde{\mathcal{H}(x)}$, the latter are multiplicatively independent at the generic point $\mathbf{y}$ of a closed subscheme $\mathcal{Y} \subset \mathcal{X}_s$ of dimension $m-1$. Replacing k by a finite extension and $\mathcal{X} = \mathrm{Spec}(A)$ by an open affine neighborhood of $\mathbf{y}$, we may assume that $\mathcal{Y}$ is smooth over $\widetilde{k}$ and even defined by one equation $\widetilde{g} = 0$ for some $\widetilde{g} \in \widetilde{A} = A/k^{\circ\circ}A$. Let g be an element of A whose image in $\widetilde{A}$ is $\widetilde{g}$, and let $\mathcal{Z}$ be the closed subscheme $\mathrm{Spec}(B)$ with $B = A/(g)$. Then the formal completion $\widehat{\mathcal{Z}}$ of $\mathcal{Z}$ along its closed fiber is a smooth formal scheme over k° and $\mathcal{Z}_s = \widehat{\mathcal{Z}}_s \xrightarrow{\sim} \mathcal{Y}$. It follows that there exists a unique point $z \in \widehat{\mathcal{Z}}_\eta$ with $\pi(z) = \mathbf{y}$, and one has $\widetilde{k}(\mathbf{y}) \xrightarrow{\sim} \widetilde{\mathcal{H}(z)}$. In particular, $s(z) = \dim(\mathcal{Z}_\eta^{\mathrm{an}}) = m-1$ and the images of $f_1(z), \dots, f_n(z)$ in $\mathcal{O}^v_{X,z}$ are linearly independent over $\mathbf{Z}$. Since the k-analytic space $\mathcal{Z}_\eta^{\mathrm{an}}$ is smooth at the point z (and in fact at all points from $\widehat{\mathcal{Z}}_\eta$), the lemma is reduced to the case of smaller dimension. ∎

Proof of Theorem 4.4.1. We have to show that if $f_1, \dots, f_n$ are invertible analytic functions on X whose images in the multiplicative group $\mathcal{O}^v_{X,x}$ of a point $x \in X$ are linearly independent over $\mathbf{Z}$, then the images of $\frac{df_1}{f_1}, \dots, \frac{df_n}{f_n}$ in $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$ are linearly independent over $\mathfrak{c}_{X,x}$. Propositions 2.3.1, 1.3.2 and 1.5.1 easily reduce the situation to the case $s(x) = \dim(X)$. We may then assume that $|f_i(x)| = 1$ for all $1 \le i \le n$ and, by Corollary 4.2.2, the assumption means that $f_1, \dots, f_n$ are multiplicatively independent at x. Lemma 4.4.3 now reduces the situation to Theorem 4.3.1. ∎

4.5. A subsheaf $\Psi_X \subset \Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X$ and a subspace $\mathcal{V}_{X,x} \subset \Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$. Let k be a non-Archimedean field of characteristic zero with a non-trivial valuation. For a smooth k-analytic space X, let $\widetilde{\Psi}(X)$ denote the space of all closed one-forms ω whose pullback under any morphism $\varphi : Y \to X$ from an elementary k'-analytic curve over a finite extension k' of k is contained in $\widetilde{\Psi}(Y)$. The correspondence $U \mapsto \widetilde{\Psi}(U)$ is a sheaf $\widetilde{\Psi}_X$ in the étale topology of X, and let Ψ_X denote the quotient sheaf $\widetilde{\Psi}_X/d\mathcal{O}_X$. (Of course, both coincide with the corresponding sheaves defined in the one-dimensional case.)

4.5.1. Theorem. *(i) $\Upsilon_X \cap \Psi_X = 0$;*

(ii) the stalk $\Psi_{X,\overline{x}}$ at a geometric point over a point $x \in X$ with $s(x) > 0$ is of infinite dimension over $\mathfrak{c}_{X,\overline{x}}$;

(iii) if $k_0 \subset k \subset \widehat{k}_0^{\mathrm{a}}$, where k_0 is a subfield whose valuation is discrete, then the stalk $\Psi_{X,x}$ at a point $x \in X$ with $s(x) = \dim(X)$ is the space of the classes of closed one-forms $\omega \in \Omega^1_{X,x}$ for which there exists a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ of x such that ω is defined over the image of φ and $\varphi^*(\omega)_{\mathbf{x}} \in d\mathcal{O}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$.

Notice that the stalks $\Psi_{X,x}$ at points $x \in X$ with $s(x) = 0$ are always zero.

4.5.2. Lemma. *Let $\varphi : Y \to X$ be a smooth morphism, $\overline{y}$ a geometric point of Y over a geometric point $\overline{x}$ of X, and y and x their images in Y and X, respectively. Then*

(i) the canonical homomorphism $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x} \to \Omega^{1,\mathrm{cl}}_{Y,y}/d\mathcal{O}_{Y,y}$ is injective;

(ii) the preimages of $\Upsilon_{Y,y}$ and $\Psi_{Y,y}$ in $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$ are $\Upsilon_{X,x}$ and $\Psi_{X,x}$, respectively;

(iii) if $s(y) = s(x)$, then $\Psi_{X,\overline{x}} \xrightarrow{\sim} \Psi_{Y,\overline{y}}$.

Proof. The statements (i) and (ii) easily follow from the fact that locally in the étale topology any smooth morphism has a section. (This fact is a consequence of [Ber2, Theorem 3.7.2].) To prove (iii), we notice that $\Omega^{1,\mathrm{cl}}_{Y,\overline{y}}/d\mathcal{O}_{Y,\overline{y}}$ is a direct sum of $\Omega^{1,\mathrm{cl}}_{X,\overline{x}}/d\mathcal{O}_{X,\overline{x}}$ and the vector subspace over $\mathfrak{c}_{Y,\overline{y}}$ whose basis is formed by the classes of one-forms $\frac{df_1}{f_1},\dots,\frac{df_t}{f_t}$, where $f_1,\dots,f_t$ are elements of $\mathcal{O}_{Y,y}$ such that $|f_1(y)|,\dots,|f_t(y)|$ form a basis of $\sqrt{|\mathcal{H}(y)^*|}/\sqrt{|\mathcal{H}(x)^*|}$. The intersection of this vector subspace with $\Psi_{Y,\overline{y}}$ is evidently zero, and so (iii) follows from (ii). ∎

Proof of Theorem 4.5.1. (i) It suffices to verify that $\Upsilon_{X,\overline{x}}\cap\Psi_{X,\overline{x}}=0$ for any geometric point $\overline{x}$ over a point $x\in X$. Lemma 4.5.2 reduces the situation to the case $s(x)=\dim(X)$, and Lemma 4.4.3 then reduces the situation to the case $\dim(X)=1$, i.e., to Theorem 4.3.1(ii).

(ii) Assume that $s(x)>0$ and $\Psi_{X,\overline{x}}$ is of finite dimension over $\mathfrak{c}_{X,\overline{x}}$. Shrinking X in the étale topology, we may assume that $\Psi_{X,\overline{x}}$ is generated by the classes of $\omega_1,\dots,\omega_n\in\Omega^{1,\mathrm{cl}}(X)/d\mathcal{O}(X)$ and that there is a function $f\in\mathcal{O}(X)$ with $|f(x)|=1$ for which the image of $f(x)$ in $\widetilde{\mathcal{H}(x)}$ is transcendent over $\widetilde{k}$. The latter implies that the corresponding morphism $f:X\to Y=\mathbf{A}^1$ takes x to the maximal point y of the closed unit disc, which is a point of type (2). It follows from Theorem 4.3.1(iii) that we can replace Y by an étale neighborhood of the point y so that there exist elements $\eta_1,\dots,\eta_{n+1}\in\Omega^1(Y)/d\mathcal{O}(Y)$ whose images in $\Psi_{Y,y}$ are linearly independent over $\mathfrak{c}_{Y,y}$. Shrinking X, we may assume that, for every $1\le i\le n+1$, one has $f^*(\eta_i)=\sum_{j=1}^n\alpha_{ij}\omega_j+\gamma_i$ with $\alpha_{ij}\in\mathfrak{c}(X)$ and $\gamma_i\in\Upsilon(X)$. We now use the fact that locally in the étale topology the morphism f has a section $\psi:Y\to X$. We get equalities $\eta_i=\sum_{j=1}^n\alpha_{ij}\psi^*(\omega_j)+\psi^*(\gamma_i)$ which contradict the assumption on linear independence of the classes of $\eta_1,\dots,\eta_{n+1}$ in $\Psi_{Y,y}$.

(iii) Assume first that the class of a closed one form ω lies in $\Psi(X)$, and let $\varphi:\mathfrak{X}_\eta\to X$ be a marked neighborhood of x. Then the restriction of $\varphi^*(\omega)$ to the residue class $\pi^{-1}(\mathbf{x})$ of every closed point $\mathbf{x}\in\mathfrak{X}_s$ is in $\widetilde{\Psi}(\pi^{-1}(\mathbf{x}))$. Since $\pi^{-1}(\mathbf{x})$ is a semi-annular space, it follows that $\varphi^*(\omega)_{\mathbf{x}}$ is an exact one-form. Conversely, assume that ω is a closed one-form on the generic fiber $\mathfrak{X}_\eta$ of a proper marked formal scheme $\mathfrak{X}$ over k° such that $\varphi^*(\omega)_{\mathbf{x}}\in d\mathcal{O}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x}\in\mathfrak{X}_s$. We have to show that, for any morphism $\varphi:Y\to\mathfrak{X}_\eta$ from an elementary k-analytic curve Y, one has $\varphi^*(\omega)\in\widetilde{\Psi}(Y)$. Since the complement of the generic point (of the comprehensive smooth projective curve with good reduction) in Y is a disjoint union of open discs and annuli (defined over a finite extension of k), it suffices to verify the required fact in the case when Y an open disc $D(0;r)$ or annulus $B(0;r,R)$. In that case, $\widetilde{\Psi}(Y)=\Omega^1(Y)$ and, therefore, we may assume that k is algebraically closed. Furthermore, it suffices to verify the required fact for every smaller open disc $D(0;r')$ with $r'<r$ or annulus $B(0;r',R')$ with $r<r'<R'<R$. We may therefore

assume that Y is $D(0;r)$ or $B(0;r,R)$ as above, but $r, R \in |k^*|$ and the morphism φ is induced by a morphism $\psi : Z \to \mathfrak{X}_\eta$, where Z is the closed disc $E(0;r)$ or annulus $A(0;r,R)$. It suffices to verify the following claim: $S = \{z \in Z \big| \varphi(z) \notin \pi^{-1}(\mathbf{x})$ *for all closed point* $\mathbf{x} \in \mathfrak{X}_s\}$ *is a finite set of points of type (2).* Indeed, assume $\mathfrak{X}$ is the formal completion of a nondegenerate strictly poly-stable formal scheme $\mathfrak{Y}$, and let $\mathfrak{Y}^1, \dots, \mathfrak{Y}^n$ be open affine subscheme of $\mathfrak{Y}$ with $\mathfrak{X}_s \subset \cup_{i=1}^n \mathfrak{Y}_s^i$. The each $Z_i = \varphi^{-1}(\mathfrak{Y}_\eta^i)$ is a strictly affinoid subdomain of Z, and the set S is contained in the union of the boundary points of all Z_i's. The latter union is finite and consists of points of type (2). The claim follows. ∎

For a point x of a smooth k-analytic space X, let $\mathcal{V}_{X,x}$ denote the subspace of $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$ consisting of the classes of closed one-forms $\omega \in \Omega^1_{X,x}$ with the following property: there exists a quasi-étale morphism $\varphi : V \to X$ from a strictly k-affinoid space V such that $x \in \varphi(V)$ and $\varphi^*(\omega)_\mathbf{x} \in d\mathcal{O}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \widetilde{V}$. (For the notion of a quasi-étale morphism see [Ber3, §3] and, if $V = \mathcal{M}(\mathcal{A})$, then $\widetilde{V} = \mathrm{Spec}(\widetilde{\mathcal{A}})$ and π is the reduction map $V \to \widetilde{V}$.) It is clear that $\mathcal{V}_{X,x}$ is functorial with respect to (X,x). Given a geometric point $\overline{x}$ over x, let $\mathcal{V}_{X,\overline{x}}$ denote the inductive limit of all $\mathcal{V}_{Y,y}$ taken over the pairs (φ, α) consisting of an étale morphism $\varphi : Y \to X$ and a morphism $\alpha : \mathbf{p}_{\mathcal{H}(\overline{x})} \to Y$ over $\overline{x}$ whose image is a point $y \in Y$ over x. One evidently has $\mathcal{V}_{X,x} = \mathcal{V}_{X,\overline{x}}^{G_{\overline{x}/x}}$.

4.5.3. Theorem. *(i) The subspace $\mathcal{V}_{X,x}$ has a direct complement in $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$, which is a vector space over $\mathfrak{c}_{X,x}$ with a basis formed by the classes of one-forms $\frac{df_1}{f_1}, \dots, \frac{df_t}{f_t}$, where $f_1, \dots, f_t \in \mathcal{O}^*_{X,x}$ are such that $\{|f_1(x)|, \dots, |f_t(x)|\}$ is a basis of $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$ over $\mathbf{Q}$; in particular, if $t(x) = 0$, then $\mathcal{V}_{X,x} = \Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$;*

(ii) in the situation of Lemma 4.5.2, the preimage of $\mathcal{V}_{Y,y}$ in $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$ is $\mathcal{V}_{X,x}$ and, if in addition $s(y) = s(x)$, then $\mathcal{V}_{X,\overline{x}} \xrightarrow{\sim} \mathcal{V}_{Y,\overline{y}}$;

(iii) $\mathcal{V}_{X,\overline{x}} \cap \Upsilon_{X,\overline{x}}$ is a vector space over $\mathfrak{c}_{X,\overline{x}}$ with the basis $\{\frac{df_i}{f_i}\}_{i \in I}$, where $\{f_i\}_{i \in I} \in \mathcal{O}^*_{X,\overline{x}}$ are such that their images in $\mathcal{O}^v_{X,\overline{x}}$ form a basis of the abelian group $\widetilde{\mathcal{H}}_{\overline{x}}/\widetilde{\mathfrak{c}}_{X,\overline{x}}$ (from Corollary 4.2.2);

(iv) $\Psi_{X,x} \subset \mathcal{V}_{X,x}$.

Proof. Step 1. *(i) is true if $t(x) = 0$.* Indeed, if in the situation of Lemma 4.5.2 $s(y) = s(x)$, then $\Omega^{1,\mathrm{cl}}_{X,\overline{x}}/d\mathcal{O}_{X,\overline{x}} \xrightarrow{\sim} \Omega^{1,\mathrm{cl}}_{Y,\overline{x}}/d\mathcal{O}_{Y,\overline{y}}$. Therefore it suffices to consider the case $s(x) = \dim(X)$. Let ω be a closed one-form on X. By Step 1 from the proof of Lemma 4.4.3, we may replace X by an étale neighborhood of the point x and assume that $\mathfrak{Z} = \widehat{\mathcal{X}}_\eta \subset X \subset \mathcal{X}^{\mathrm{an}}_\eta$ with $\mathcal{X}$ a connected affine smooth scheme over k°, $x \in \widehat{\mathcal{X}}_\eta$ and $\pi(x)$ the generic point of $\mathcal{X}_s$. By [Ber7, Proposition 1.4], one has $\widetilde{\mathfrak{Z}}_\eta \xrightarrow{\sim} \mathfrak{Z}_s$. *We claim that $\omega_\mathbf{x} \in d\mathcal{O}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{Z}_s$.* Indeed, let k' be a finite extension of k such that $\mathfrak{Z}'_s$ with $\mathfrak{Z}' = \mathfrak{Z}\widehat{\otimes}_{k^\circ}k'^\circ$ contains a $\widetilde{k}'$-rational point $\mathbf{x}'$ over $\mathbf{x}$. Then $\pi^{-1}(\mathbf{x}')$ is isomorphic to an open unit polydisc over k' with center at zero and, therefore, the pullback of ω to it is an exact one-form. Since the morphism $\pi^{-1}(\mathbf{x}') \to \pi^{-1}(\mathbf{x})$ is a finite étale covering, the claim follows.

Step 2. *(ii) is true when, in the situation of Lemma 4.5.2, the morphism φ is of dimension one, $s(y) = s(x)$ and $t(y) > t(x)$.* Indeed, let ω be a closed one-form on Y whose class in $\Omega^{1,\mathrm{cl}}_{Y,y}/d\mathcal{O}_{Y,y}$ is in $\mathcal{V}_{Y,y}$, and let $\varphi : V \to Y$ be the corresponding quasi-étale morphism with $y \in \varphi(V)$. Replacing V by a strictly affinoid subdomain whose image also contains the point y, we may assume that V is a strictly affinoid domain in a smooth k-analytic space étale over Y. Replacing Y by the latter, we may assume that V is in fact a strictly affinoid subdomain of Y. Finally, we may replace Y by an étale neighborhood of y and assume that $Y = X \times B$, where $B = B(0; R', R'')$ is an open annulus) with center at zero, and y is a point in the fiber $\varphi^{-1}(x)$ of type (3), i.e., the maximal point of the closed disc radius $R' < r < R''$ with $r \notin \sqrt{|\mathcal{H}(x)^*|}$. (Notice that the fiber is the similar annulus $B_{\mathcal{H}(x)} = B\widehat{\otimes}\mathcal{H}(x)$ over $\mathcal{H}(x)$.)

4.5.4. Lemma. *In the above situation, V contains $U \times A$, where U is a strictly affinoid domain in X with $x \in U$ and $A = A(0; r', r'')$ is a closed annulus in B of radii $r', r'' \in \sqrt{|k^*|}$ with $r' < r < r''$).*

Proof. First of all, we may assume that $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ are strictly k-affinoid with $Y = X \times A(0; R', R'')$ with $R', R'' \sqrt{|k^*|}$ and $R' < r < R''$, and that V is a strictly affinoid rational domain in Y. The latter means that there are (nonzero) elements $g_0, g_1, \dots, g_n \in \mathcal{B}$ without common zeros in Y such that $V = \{y' \in Y \big| |g_i(y')| \le |g_0(y')| \text{ for all } 1 \le i \le n\}$. Each element $g_i \in \mathcal{B}$ has the form $\sum_{j=-\infty}^{\infty} f_{ij}T^j$ with $f_{ij} \in \mathcal{A}$ such that, for every point $x' \in X$ and every $R' \le \alpha \le R''$, $|f_{ij}(x')|\alpha^j \to 0$ as $j \to \pm\infty$. One has $|g_i(y)| = \max\limits_j\{|f_{ij}(x)|r^j\}$. Since $r \notin \sqrt{|\mathcal{H}(x)^*|}$, then for every $0 \le i \le n$ with $g_i(y) \ne 0$ the maximum is achieved at only one value j_i of j. The inclusion $y \in V$ implies that $g_0(y) \ne 0$. Dividing all g_i's by T^{j_0}, we may assume that $j_0 = 0$. If $g_i(y) = 0$ for some $1 \le i \le n$, we can shrink X and Y so that the inequality $|g_i(y')| \le |g_0(y')|$ is satisfied for all points $y' \in Y$. Hence, we may assume that $g_i(y) \ne 0$ for all $1 \le i \le n$. Furthermore, if $j_i \ne 0$ for some $1 \le i \le n$, then $|g_i(y)|$ is *strictly* less than $|g_0(y)|$. We can therefore shrink X and Y so that $|g_i(y')| < |g_0(y')|$ for all $y' \in Y$, and so we may assume that $j_i = 0$ for all $1 \le i \le n$. Notice that the functions $f_{00}, f_{10}, \dots, f_{n0}$ have no common zeros in X. Thus, we get $V = \{y' \in Y \big| \varphi(y') \in U\}$, where U is the strictly affinoid rational domain in X defined by the inequalities $|f_{i0}(x')| \le |f_{00}(x')|$ for all $1 \le i \le n$. ∎

We may assume that $\omega = \varphi^*(\eta) + a\frac{dT}{T}$ with $\eta \in \Omega^{1,\mathrm{cl}}(X)$ and $a \in k$ and that $V = U \times A$ as in Lemma 4.5.4, and our purpose is to show that $a = 0$ and the class of η in $\Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$ lies in $\mathcal{V}_{X,x}$. Replacing k by a finite extension, we may assume that $r', r'' \in |k^*|$. The reduction $\widetilde{A}$ is a union of two affine lines over $\widetilde{k}$ intersecting in the zero point $\mathbf{0}$ and $\pi^{-1}(\mathbf{0}) = B(0; r', r'')$. Let $\mathbf{y}$ be a closed point of $\widetilde{V}$ whose image in $\widetilde{A}$ is $\mathbf{0}$. If $\mathbf{x}$ is the image of $\mathbf{y}$ in $\widetilde{U}$, then $\pi^{-1}(\mathbf{y}) \xrightarrow{\sim} \pi^{-1}(\mathbf{x}) \times B(0; r', r'')$. By the assumption, $\omega_{\mathbf{y}} \in d\mathcal{O}(\pi^{-1}(\mathbf{y}))$, it follows that $a = 0$ and $\eta_{\mathbf{x}} \in d\mathcal{O}(\pi^{-1}(\mathbf{x}))$. Since such a point $\mathbf{y}$ exists for every closed point $\mathbf{x} \in \widetilde{U}$, the required fact follows.

Step 3. *(i) is true.* Indeed, this follows from the Steps 1 and 2 and the fact that, in the situation of Step 2, $\Omega^{1,\mathrm{cl}}_{Y,\overline{y}}/d\mathcal{O}_{Y,\overline{y}}$ is a direct sum of $\Omega^{1,\mathrm{cl}}_{X,\overline{x}}/d\mathcal{O}_{X,\overline{x}}$ and a one-dimension vector subspace over $\mathfrak{c}_{Y,\overline{y}}$ generated by the class of $\frac{df}{f}$ for an element $f \in \mathcal{O}^*_{Y,y}$ with $|f(y)| \notin \sqrt{|\mathcal{H}(x)^*|}$.

Step 4. *(ii), (iii) and (iv) are true.* For (ii) it suffices to consider the case when φ is of dimension one. In this case the claim follows from (i) and the fact mentioned in the previous Step 3. The statement (iii) follows from Step 1, Corollary 4.2.2 and Theorem 4.4.1, and (iv) is already evident. ∎

4.5.5. Remark. By Theorem 4.3.1(v), if $\widetilde{k}$ is algebraic over a finite field, then for any smooth k-analytic curve X one has $\Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X = \Upsilon_X \oplus \Psi_X$. It would be interesting to know if the same equality holds in all dimensions.

§5. Isocrystals

In this section we study various objects related to a wide germ of a strictly k-affinoid space, i.e., a germ of an analytic space (X,Y) in which Y is a strictly affinoid domain in the interior of a separated analytic space X, and consider a related notion of a wide germ of a formal scheme $(X,\mathfrak{Y})$ (which gives rise to a wide germ of a strictly k-affinoid germ $(X,\mathfrak{Y}_\eta)$). First of all, we show that the correspondence $(X,Y)\mapsto B=\mathcal{O}(X,Y)$ gives rise to an anti-equivalence between the category of such germs and that of dagger algebras. We define categories of sheaves on a germ and, if X is smooth, relate $\mathcal{D}_{(X,Y)}$-modules to isocrystals over B. In particular, we show that the correspondence $\mathcal{F}\mapsto\mathcal{F}(X,Y)$ gives rise to an equivalence between the category of $\mathcal{O}_{(X,Y)}$-coherent $\mathcal{D}_{(X,Y)}$-modules and that of isocrystals finitely generated over B, which preserves de Rham cohomology groups. We then construct an increasing sequence of isocrystals $E^0_B=B\subset E^1_B\subset\dots$ with the property that the canonical homomorphisms $H^1_{\mathrm{dR}}(E^i_B)\to H^1_{\mathrm{dR}}(E^{i+1}_B)$ are zero. The construction in fact depends on certain choices, but the object constructed is unique up to a non-canonical isomorphism. Under the assumption that the isocrystal $E_B=\cup_{i=1}^\infty E^i_B$ can be provided with the structure of a filtered B-algebra that satisfies the Leibniz rule, we relate it to a classical object, the shuffle algebra, which appeared in the work of K.-T. Chen on iterated integrals (see [Chen]). Finally, if the germ (X,Y) is a lifting of a similar germ (X',Y') defined over a closed subfield $k'\subset k$ such that the valuation on k' is discrete and k is isomorphic to a closed subfield $\widehat{k'^{\mathrm{a}}}$, we show that all of the isocrystals E^i_B are unipotent and $E^i_{B'}\otimes_{B'}B\overset{\sim}{\to}E^i_B$, where $B'=\mathcal{O}(X',Y')$. This is deduced from a result of O. Gabber (Lemma 5.5.1) and a finiteness result of E. Grosse-Klönne [GK2]. (In its turn, the latter is an extension of a result of P. Berthelot and Z. Mebkhout.)

5.1. Wide germs of analytic spaces and of formal schemes. Let k be an arbitrary non-Archimedean field with a nontrivial valuation. Recall ([Ber2, §3.4]) that the category of germs of a k-analytic space is the localization of the category of pairs (X,S) (consisting of a k-analytic space X and a subset $S\subset X$) with respect to the system of morphisms $\varphi:(Y,T)\to(X,S)$ such that φ induces an isomorphism of Y with an open neighborhood of S in X. This system admits a calculus of right fractions, and so the set of morphisms $\mathrm{Hom}((Y,T),(X,S))$ is the inductive limit of the sets of morphisms $\varphi:\mathcal{V}\to X$ with $\varphi(T)\subset S$, where $\mathcal{V}$ runs through open neighborhoods of T in Y. We say that a germs (X,S) is *strictly k-affinoid* if S is strictly affinoid domain in X. We say that a germ (X,S) is *wide* if X is separated and S is contained in the interior $\mathrm{Int}(X)$ of X. In this subsection we study the category of wide strictly k-affinoid germs.

For a germ (X,S) we denote by $\mathcal{O}(X,S)$ the algebra of functions analytic in a neighborhood of S in X, i.e., $\mathcal{O}(X,S) = \varinjlim \mathcal{O}(\mathcal{U})$, where $\mathcal{U}$ runs through open neighborhoods of S in X. For example, if S is a point $x \in X$, then $\mathcal{O}(X,x)$ is the local ring $\mathcal{O}_{X,x}$. If $X = \mathbf{A}^n$ is the n-dimensional affine space and $S = E$ is the closed unit polydisc with center at zero, then $\mathcal{O}(\mathbf{A}^n, E)$ is the algebra of overconvergent power series

$$k\{T_1, \ldots, T_n\}^\dagger = \cup_{\rho>1} k\{\rho^{-1}T_1, \ldots, \rho^{-1}T_n\} \ .$$

Recall (see [MW], [GK1]) that a k-dagger algebra is a k-algebra A isomorphic to a quotient of $k\{T_1, \ldots, T_n\}^\dagger$. For a k-dagger algebra A, the equivalence class of the norm on A induced from the Gauss norm of $k\{T_1, \ldots, T_n\}^\dagger$ does not depend on the representation of A as a quotient, and all ideals of A are closed and finitely generated (see [GK1, §1]). In particular, A is Noetherian, and there is a well defined completion $\widehat{A}$, which is a strictly k-affinoid algebra. Furthermore, the canonical homomorphism $A \to \widehat{A}$ is injective, and gives rise to a bijection between the sets of maximal ideals. Finally, any homomorphism of k-dagger algebras $A \to B$ is bounded with respect to the norms on the algebras, and so it extends in a canonical way to a homomorphism of strictly k-affinoid algebras $\widehat{A} \to \widehat{B}$.

5.1.1. Lemma. *The correspondence $(X,Y) \mapsto \mathcal{O}(X,Y)$ gives rise to an anti-equivalence between the category of wide strictly k-affinoid germs and the category of k-dagger algebras.*

Proof. First of all, we have to verify that, for a wide strictly k-affinoid germ (X,Y), $\mathcal{O}(X,Y)$ is a k-dagger algebra. By a result of M. Temkin ([Tem, Theorem 5.1]), there exists a bigger strictly affinoid domain $V \subset X$ such that $Y \subset \mathrm{Int}(V)$ and Y is a Weierstrass domain in V. We may assume that $X = V = \mathcal{M}(\mathcal{A})$. In this case it follows from [Ber1, 2.5.2] that there exists an admissible epimorphism $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{A} : T_i \mapsto f_i$ such that $r_i \in \sqrt{|k^*|}$ and $\max\limits_{x \in Y} |f_i(x)| = 1 < r_i$. Since Y is Weierstrass in X, there exist elements $g_1, \ldots, g_m \in \mathcal{A}$ such that $Y = \{x \in X \big| |g_i(x)| \le 1$ for all $1 \le i \le m\}$. Of course, we may assume that $\max\limits_{x \in X} |g_i(x)| > 1$ for all $1 \le i \le m$ and, therefore, we may replace the system of elements $\{f_1, \ldots, f_n\}$ by $\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$. Thus, we may assume that $Y = \{x \in X \big| |f_i(x)| \le 1$ for all $1 \le i \le n\}$ and there is an admissible epimorphism $k\{T_1, \ldots, T_n\} \to \mathcal{B} : T_i \mapsto f_i$, where $Y = \mathcal{M}(\mathcal{B})$. It follows that there is an epimorphism $k^\circ\{T_1, \ldots, T_n\}^\dagger \to \mathcal{O}(X,Y)$, i.e., $\mathcal{O}(X,Y)$ is a k-dagger algebra.

Furthermore, we have to verify that, for a similar germ (X', Y'), the canonical map

$$\mathrm{Hom}((X',Y'),(X,Y)) \stackrel{\sim}{\to} \mathrm{Hom}(\mathcal{O}(X,Y), \mathcal{O}(X',Y'))$$

is bijective. That it is injective is easy. Suppose we are given a homomorphism $\mathcal{O}(X,Y) \to \mathcal{O}(X',Y')$. It extends in a canonical way to a bounded homomorphism of strictly k-affinoid algebras

$\mathcal{B} \to \mathcal{B}'$, where $Y' = \mathcal{M}(\mathcal{B}')$ and $\mathcal{B}$ (as well as some other objects below) are from the previous paragraph. Let f'_i be the image of f_i in $\mathcal{O}(X', Y')$. One has $\max\limits_{x' \in Y'} |f'_i(x)| \leq 1$. We can shrink X' and assume that $X' = \mathcal{M}(\mathcal{A}')$ is k-affinoid and that each f'_i comes from $\mathcal{A}'$ and is of spectral radius at most r_i. Then there is a well defined bounded homomorphism of k-affinoid algebras $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{A}' : T_i \mapsto f'_i$. We can shrink X' and assume that this homomorphism takes the finitely generated kernel of the epimorphism $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{A} : T_i \mapsto f_i$ to zero. This means that there is a bounded homomorphism $\mathcal{A} \to \mathcal{A}'$ which induces a morphism of germs $(X', Y') \to (X, Y)$. The latter evidently gives rise to the homomorphism $\mathcal{O}(X, Y) \to \mathcal{O}(X', Y')$ we started from.

Finally, let B be a k-dagger algebra, and consider a surjective homomorphism $k\{T_1, \ldots, T_n\}^\dagger \to B$. Its kernel is generated by a finite number of elements that come from $k\{r_1^{-1}T_1, \ldots, r_n^{-1}\}$ for some $r_i \in \sqrt{|k^*|}$ with $r_i > 1$, and let $\mathcal{A}$ be the quotient of the latter by the ideal generated by those elements. Then for $X = \mathcal{M}(\mathcal{A})$, the set $Y = \{x \in X \big| |f_i(x)| \leq 1 \text{ for all } 1 \leq i \leq n\}$, where f_i is the image of T_i in $\mathcal{A}$, is a Weierstrass domain in X which is contained in $\mathrm{Int}(X)$, and one has $\mathcal{O}(X, Y) = B$. ∎

We now introduce a related category of *germs of a formal scheme over* k°. Its objects are the triples $(X, \mathfrak{Y}, \alpha)$ consisting of a k-analytic space X, a formal scheme $\mathfrak{Y}$ locally finitely presented over k°, and an isomorphism α between the generic fiber $\mathfrak{Y}_\eta$ and a strictly analytic subdomain $Y \subset X$, and morphisms $(X', \mathfrak{Y}', \alpha') \to (X, \mathfrak{Y}, \alpha)$ are the pairs consisting of a morphism of strictly k-analytic spaces $(X', Y') \to (X, Y)$ (with $Y' = \alpha'(\mathfrak{Y}'_\eta)$) and a morphism of formal schemes $\mathfrak{Y}' \to \mathfrak{Y}$, which are compatible in the evident sense. For brevity, we identify $\mathfrak{Y}_\eta$ with its image in X and denote the corresponding germ by $(X, \mathfrak{Y})$. There are evident functors of generic fiber $(X, \mathfrak{Y}) \mapsto (X, \mathfrak{Y}_\eta)$ and of closed fiber $(X, \mathfrak{Y}) \mapsto \mathfrak{Y}_s$. We say that $(X, \mathfrak{Y})$ is a *germ of a smooth formal scheme* if $\mathfrak{Y}$ is smooth over k°. A germ $(X, \mathfrak{Y})$ is said to be *wide* if X and $\mathfrak{Y}$ are separated and $\mathfrak{Y}_\eta$ is contained in the interior $\mathrm{Int}(X)$ of X. For example, if $\mathfrak{X}$ is a separated formal scheme locally finitely presented over k° and $\mathfrak{Y}$ is a open subscheme smooth over k° and such that $\mathfrak{Y}_s$ is contained in an irreducible component of $\mathfrak{X}_s$ which is proper over $\widetilde{k}$, then $(\mathfrak{X}_\eta, \mathfrak{Y})$ is a wide germ of a smooth formal scheme. (This germ is wide, by a result of M. Temkin [Tem, Theorem 4.1].)

A morphism $(\varphi, \psi) : (X', \mathfrak{Y}') \to (X, \mathfrak{Y})$ is said to be *étale* if the morphisms $\varphi : (X', \mathfrak{Y}'_\eta) \to (X, \mathfrak{Y}_\eta)$ and $\psi : \mathfrak{Y}' \to \mathfrak{Y}$ are étale in the sense of [Ber2, §3.4] and [Ber3, §2], respectively. Notice that any germ $(X', \mathfrak{Y}')$ étale over a wide germ $(X, \mathfrak{Y})$ is also wide.

5.1.2. Lemma. *Given a wide germ* $(X, \mathfrak{Y})$, *the correspondence* $(X', \mathfrak{Y}') \mapsto \mathfrak{Y}'_s$ *gives rise to* *an equivalence between the category of germs étale over $(X, \mathfrak{Y})$ and that of schemes étale over $\mathfrak{Y}_s$.*

Recall that, by [Ber3, Lemma 2.1], the latter category is equivalent to the category of formal schemes étale over $\mathfrak{Y}$.

Proof. The functor is clearly fully faithful. Therefore to show that it is essentially surjective, it suffices to verify this property locally on $\mathfrak{Y}$. This is done using the reasoning from the proof of [Ber3, Proposition 2.3] as follows. By the local description of étale morphisms, we may assume that we are given an étale morphism $\psi : \mathfrak{Y}' = \mathrm{Spf}(A'_{\{f\}}) \to \mathfrak{Y} = \mathrm{Spf}(A)$, where $A' = A[T]/(P) = A\{P\}/(P)$, P is a monic polynomial over A and f is an element of A' such that the image of the derivative P' in $\widetilde{A}'_{\widetilde{f}} = (\widetilde{A}[T]/(\widetilde{P}))_{\widetilde{f}}$ is invertible. Since $\mathfrak{Y}_\eta$ lies in the interior of X, we can shrink X and assume that $X = \mathcal{M}(\mathcal{B})$ is strictly k-affinoid and $\mathfrak{Y}_\eta$ is a Weierstrass domain in it. Then the image of $\mathcal{B}$ in $\mathcal{A} = A \otimes_{k^\circ} k$ is dense. We may therefore assume that the coefficients of the polynomial P are contained in $\mathcal{B}$ and the element f is contained in $\mathcal{B}' = \mathcal{B}[T]/(P)$. Consider the finite flat morphism $\varphi : X' = \mathcal{M}(\mathcal{B}') \to X$. One has $\mathfrak{Y}'_\eta = \{x' \in \varphi^{-1}(\mathfrak{Y}_\eta) \big| |f(x')| = 1\}$. We claim that the morphism φ is étale at all points of $\mathfrak{Y}'_\eta$. Indeed, if g denote the image of the derivative P' in $\mathcal{B}'$, then φ is étale at a point $x' \in X'$ if and only if $g(x') \neq 0$. Since $|g(x')| = 1$ for all points $x' \in \mathfrak{Y}'_\eta$, the claim follows. ∎

5.1.3. Corollary. *Let k' be a closed subfield of k such that the residue field $\widetilde{k}$ is algebraic over $\widetilde{k}'$. Then any wide germ of a smooth formal scheme $(X, \mathfrak{Y})$ is, locally on $\mathfrak{Y}$, induced by a wide germ of a smooth formal scheme over k''° for some finite extension k'' of k' in k.*

Proof. By the definition of a smooth formal scheme over k°, one can shrink $\mathfrak{Y}$ so that there is an étale morphism from $\mathfrak{Y}$ to the formal affine scheme over k°. the generic fiber of the latter is the closed unit polydisc with center at zero in the affine space. Both are defined over k'° and, therefore, the statement follows from Lemma 5.1.2. ∎

5.1.4. Remarks. (i) In [GK1], E. Große-Klönne introduced a category of dagger spaces which are glued from affinoid dagger spaces, the maximal spectra of k-dagger algebras. The functor of Lemma 5.1.1 gives rise to an equivalence between the category of wide germs of a strictly k-affinoid space and the category of affinoid dagger spaces. One can show that this functor can be extended to a fully faithful functor from the category of wide germs (X, Y), in which Y is a strictly analytic subdomain of X, to the category of separated dagger spaces, and one can describe reasonable subcategories of the above which are equivalent under that functor (cf. [Ber2, §1.6]). Similarly, there is a fully faithful functor from the category of wide germs of formal schemes over k° to the category of weak formal schemes over k°, introduced by D. Meredith [Mer] in the case when the valuation on k is discrete.

(iii) If the valuation on k is discrete, the results of M. Temkin mentioned above were proven earlier by W. Lütkebohmert in [Lüt2].

(iv) It is very likely the statement of Lemma 5.1.2 is true without the assumption that the germ $(X, \mathfrak{Y})$ is wide.

5.2. $\mathcal{D}$-modules on smooth strictly k-affinoid germs and isocrystals. The category of sheaves of sets on a germ (X, S) is defined as the inductive limit of the categories of sheaves of sets on open neighborhoods of S in X (see [SGA4, Exp. VI]). Namely, a sheaf of sets on (X, S) is a pair $(\mathcal{U}, F)$ consisting of an open neighborhood $\mathcal{U}$ of S in X and a sheaf of sets F on $\mathcal{U}$. A representative of a morphism $(\mathcal{V}, G) \to (\mathcal{U}, F)$ is a pair $(\mathcal{W}, \alpha)$ consisting of an open neighborhood $\mathcal{W}$ of S in $\mathcal{U} \cap \mathcal{V}$ and a morphism of sheaves $\alpha : G\big|_{\mathcal{W}} \to F\big|_{\mathcal{W}}$. Two representatives $(\mathcal{W}, \alpha)$ and $(\mathcal{W}', \alpha')$ of a morphism $(\mathcal{V}, G) \to (\mathcal{U}, F)$ are said to be equivalent if the restrictions of α and α' to some open neighborhood of S in $\mathcal{W} \cap \mathcal{W}'$ coincide. A morphism $(\mathcal{V}, G) \to (\mathcal{U}, F)$ is an equivalence class of its representatives. The pullback of a sheaf of sets on $\mathcal{U}$ to S gives rise to a functor from the category of sheaves of sets on (X, S) to that on the topological space S (the underlying topological space of (X, S)).

In the similar way one defines categories of sheaves on (X, S) with additional algebraic structures. In particular, one has the abelian category of abelian sheaves on (X, S). This category has injectives, and the values of the high direct images of the left exact functor $F \mapsto F(X, S) = \varinjlim_{\mathcal{U}} F(\mathcal{U})$ are the groups $H^q((X, S), F) = \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, F)$. Notice that if X is Hausdorff and S has a fundamental system of paracompact neighborhoods in X (e.g., S is compact), then the latter inductive limit coincide with $H^q(S, F)$, the cohomology group of the space S with coefficients in the pullback of F on S.

For example, the *structural sheaf* $\mathcal{O}_{(X,S)}$ on (X, S) is the sheaf represented by the pair $(X, \mathcal{O}_X)$ (it is isomorphic to the sheaf represented by any pair $(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$). The category of $\mathcal{O}_{(X,S)}$*-modules* (resp. *coherent* $\mathcal{O}_{(X,S)}$*-modules*) is the inductive limit of the categories of $\mathcal{O}_{\mathcal{U}}$-modules (resp. coherent $\mathcal{O}_{\mathcal{U}}$-modules) taken over open neighborhoods $\mathcal{U}$ of S in X. Notice that the correspondence $(\mathcal{U}, \mathcal{F}) \mapsto \mathcal{F}(X, Y)$ gives rise to a functor from the category of $\mathcal{O}_{(X,S)}$-modules to that of $\mathcal{O}(X, S)$-modules.

5.2.1. Lemma. *Let (X, Y) be a wide strictly k-affinoid germ (X, Y) with $B = \mathcal{O}(X, Y)$.*

Then

(i) the above functor induces an equivalence between the category of coherent (resp. and locally free) $\mathcal{O}_{(X,Y)}$-modules and that of finitely generated (resp. and projective) B-modules;

(ii) for any coherent $\mathcal{O}_{(X,Y)}$-module $\mathcal{F}$ and any $q \geq 1$, one has $H^q((X,Y),\mathcal{F}) = 0$;

(iii) if $\mathcal{F}$ is a locally free $\mathcal{O}_{(X,Y)}$-module of finite rank, then for any $\mathcal{O}_{(X,Y)}$-module $\mathcal{G}$ the canonical map $\mathcal{F}(X,Y) \otimes_B \mathcal{G}(X,Y) \to (\mathcal{F} \otimes_{\mathcal{O}_{(X,Y)}} \mathcal{G})(X,Y)$ is a bijection.

Proof. (i) As in the proof of Lemma 5.1.1, we may assume that $X = \mathcal{M}(\mathcal{A})$ is strictly k-affinoid, there is a surjective morphism $k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\} \to \mathcal{A}$ with $r_i > 1$, and $Y = \{x \in X \big| |f_i(x)| \leq 1 \text{ for all } 1 \leq i \leq n\}$, where f_i is the image of T_i in $\mathcal{A}$. Moreover, we may assume the coherent (resp. and locally free) $\mathcal{O}_{(X,Y)}$-module considered comes from a coherent (resp. and locally free) $\mathcal{O}_X$-module $\mathcal{F}$. Then $\mathcal{F}(X)$ is a finitely generated (resp. and projective) $\mathcal{A}$-module. In particular, there is a surjective $\mathcal{A}$-homomorphism $\mathcal{A}^m \to \mathcal{F}(X)$. Given an affinoid neighborhood V of Y in X, the latter induces a surjective homomorphism $\mathcal{A}_V^m \to \mathcal{F}_V(V)$, where $\mathcal{F}_V$ is the corresponding coherent $\mathcal{O}_V$-module. Since $B = \varinjlim_V \mathcal{A}_V$ and $\mathcal{F}(X,Y) = \varinjlim_V \mathcal{F}_V(V)$, there is a surjective homomorphism $B^m \to \mathcal{F}(X,Y)$. This implies that $\mathcal{F}(X,Y)$ is a finitely generated (resp. and projective) B-module, i.e., the functor is well defined. That it is an equivalence of categories is established in the same way.

(ii) The inductive limit defining the cohomology group considered coincides with the inductive limit $\varinjlim_V H^q(V,\mathcal{F}_V)$ taken over affinoid neighborhoods V of Y in X. Since $H^q(V,\mathcal{F}_V) = 0$, the required fact follows.

(iii) The statement is trivial if $\mathcal{F}$ is a free $\mathcal{O}_{(X,Y)}$-module. In the general case, by (i), we may assume that there is a coherent $\mathcal{O}_{(X,Y)}$-module $\mathcal{H}$ such that $\mathcal{F} \oplus \mathcal{H}$ is a free $\mathcal{O}_{(X,Y)}$-module of finite rank, and the required fact easily follows. ∎

For any germ of a formal scheme $(X,\mathfrak{Y})$ over k°, one can define as follows a *nearby cycles functor* θ from the category of sheaves on its generic fiber $(X,\mathfrak{Y}_\eta)$ to the category of sheaves in the Zariski topology of its closed fiber $\mathfrak{Y}_s$ (see [Ber8, §4]). Given a sheaf F on $(X,\mathfrak{Y}_\eta)$, one sets $\theta(F)(\mathfrak{Z}_s) = F(X,\mathfrak{Z}_\eta)$ for any open subset $\mathfrak{Z}_s \subset \mathfrak{Y}_s$, where $\mathfrak{Z}$ is the open subscheme of $\mathfrak{Y}$ with the underlying space $\mathfrak{Z}_s$ and $F(X,\mathfrak{Z}_\eta)$ is the set of global sections of the pullback of F to $(X,\mathfrak{Z}_\eta)$. For example, it follows from Lemma 3.1.1 that the restriction of the sheaf $\mathfrak{c}_{\mathfrak{Y}} = \theta(\mathfrak{c}_{(X,\mathfrak{Y}_\eta)})$ to any connected component $\mathfrak{Z}_s$ of $\mathfrak{Y}_s$ is the constant sheaf associated to the field $\mathfrak{c}(\mathfrak{Z}_\eta)$ which is the finite unramified extension of k with the residue field $\widetilde{k}(\mathfrak{Z}_s)$. We denote by $\mathcal{O}_{(X,\mathfrak{Y})}$ the sheaf $\theta(\mathcal{O}_{(X,\mathfrak{Y}_\eta)})$. It is a coherent sheaf of rings.

5.2.2. Corollary. *Let $(X, \mathfrak{Y})$ be a wide germ of a formal scheme over k°. Then*

(i) the functor θ gives rise to an equivalence between the category of coherent $\mathcal{O}_{(X,\mathfrak{Y}_\eta)}$-modules and that of coherent $\mathcal{O}_{(X,\mathfrak{Y})}$-modules;

(ii) for any coherent $\mathcal{O}_{(X,\mathfrak{Y}_\eta)}$-module $\mathcal{F}$, one has $H^q(\mathfrak{Y}_s, \theta(\mathcal{F})) \xrightarrow{\sim} H^q((X, \mathfrak{Y}_\eta), \mathcal{F})$, $q \geq 0$. ∎

Furthermore, we say that a germ (X, S) is *smooth* if X is a separated k-analytic space smooth in a neighborhood of the set S. Such a germ is automatically wide. For example, if $(X, \mathfrak{Y})$ is a wide germ of a smooth formal scheme, then the germ $(X, \mathfrak{Y}_\eta)$ is smooth.

Assume that the characteristic of k is zero, and let (X, Y) be a smooth germ. As above, we define the category of $\mathcal{D}_{(X,Y)}$-modules and those of $\mathcal{O}_X$-coherent and of unipotent $\mathcal{D}_{(X,Y)}$-modules on (X, Y) as the inductive limit of the corresponding categories taken over open neighborhoods of Y in X. These three categories are abelian. The *de Rham cohomology groups $H^i_{\mathrm{dR}}((X, Y), \mathcal{F})$ of a $\mathcal{D}_{(X,Y)}$-module $\mathcal{F}$* are the hypercohomology groups of the complex $\mathcal{F} \otimes_{\mathcal{O}_{(X,Y)}} \Omega^\cdot_{(X,Y)}$ with respect to the functor of global sections on (X, Y). If $\mathcal{F} = \mathcal{O}_X$, they are denoted by $H^i_{\mathrm{dR}}(X, Y)$.

Let (X, Y) be a smooth strictly k-affinoid germ and $B = \mathcal{O}(X, Y)$. An *isocrystal* (resp. a *finite isocrystal*) *over B* is an B-module (resp. a finitely generated B-module) M provided with an integrable connection $\nabla : M \to M \otimes_B \Omega^1_B$. It is known that any finitely generated B-module that can be provided with a connection is projective and, in particular, any finite isocrystal is a projective B-module. A *unipotent isocrystal over B* is a finite isocrystal M over B which is a successive extension of the trivial isocrystal B. Notice that such M is always a free B-module. A *trivial isocrystal over B* is a finite isocrystal isomorphic to the isocrystal B^n, $n \geq 0$. The *de Rham cohomology groups $H^i_{\mathrm{dR}}(M)$ of an isocrystal M over B* are the cohomology groups of the complex

$$M \otimes_B \Omega^\cdot_B : M \xrightarrow{\nabla} M \otimes_B \Omega^1_B \xrightarrow{\nabla} M \otimes_B \Omega^2_B \xrightarrow{\nabla} \dots \ .$$

5.2.3. Lemma. *Let (X, Y) be a smooth strictly k-affinoid germ. Then*

(i) the correspondence $\mathcal{F} \mapsto \mathcal{F}(X, Y)$ gives rise to a functor from the category of $\mathcal{D}_{(X,Y)}$-modules to that of isocrystals over B;

(ii) the above functor induces an equivalence between the category of $\mathcal{O}_{(X,Y)}$-coherent (resp. unipotent) $\mathcal{D}_{(X,Y)}$-modules and that of finite (resp. unipotent) isocrystals over B;

(iii) for any $\mathcal{O}_X$-coherent $\mathcal{D}_{(X,Y)}$-module $\mathcal{F}$ one has $H^i_{\mathrm{dR}}((X, Y), \mathcal{F}) \xrightarrow{\sim} H^i_{\mathrm{dR}}(\mathcal{F}(X, Y))$.

Proof. (i) Since the $\mathcal{O}_{(X,Y)}$-module $\Omega^1_{(X,Y)}$ is locally free of finite rank, the statement follows from Lemma 5.2.1(iii).

(ii) That the functor is fully faithful follows from Lemma 5.2.1. To prove that it is essentially surjective, it suffices to show that, given a locally free $\mathcal{O}_X$-module of finite rank $\mathcal{F}$, any connection

$\nabla : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y) \otimes_B \Omega^1_B$ comes from a connection on $\mathcal{F}\big|_{\mathcal{U}}$ for some open neighborhood $\mathcal{U}$ of Y in X. But this is easily seen since the connection is determined by its values on a finite number of elements of $\mathcal{F}(X,Y)$.

(iii) follows from Lemma 5.2.1(ii). ∎

Let $(X,\mathfrak{Y})$ be a wide germ of a smooth formal scheme over k°. A $\mathcal{D}_{(X,\mathfrak{Y})}$-*module* is a sheaf of $\mathcal{O}_{(X,\mathfrak{Y})}$-modules $\mathcal{F}$ provided with an integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{(X,\mathfrak{Y})}} \Omega^1_{(X,\mathfrak{Y})}$, where $\Omega^q_{(X,\mathfrak{Y})} = \theta(\Omega^q_{(X,\mathfrak{Y}_\eta)})$. The de Rham cohomology groups $H^q_{\mathrm{dR}}(\mathfrak{Y}_s,\mathcal{F})$ of such a module are defined in the same way as above. If $\mathcal{F}$ is the trivial $\mathcal{D}_{(X,\mathfrak{Y})}$-module $\mathcal{O}_{(X,\mathfrak{Y})}$ (and the valuation on k is discrete), its de Rham cohomology groups coincide with the Monsky-Washnitzer cohomology groups $H^q_{\mathrm{MW}}(\mathfrak{Y}_s/k)$ of $\mathfrak{Y}_s$ over k ([MW]).

5.2.4. Corollary. *(i) The functor θ gives rise to a functor from the category of $\mathcal{D}_{(X,\mathfrak{Y}_\eta)}$-modules to that of $\mathcal{D}_{(X,\mathfrak{Y})}$-modules;*

(ii) the functor θ induces an equivalence between the category of $\mathcal{O}_{(X,\mathfrak{Y}_\eta)}$-coherent (resp. unipotent) $\mathcal{D}_{(X,\mathfrak{Y}_\eta)}$-modules and that of $\mathcal{O}_{(X,\mathfrak{Y})}$-coherent (resp. unipotent) $\mathcal{D}_{(X,\mathfrak{Y})}$-modules;

(iii) $H^q_{\mathrm{dR}}(\mathfrak{Y}_s,\theta(\mathcal{F})) \xrightarrow{\sim} H^q_{\mathrm{dR}}((X,\mathfrak{Y}_\eta),\mathcal{F})$ for every $\mathcal{O}_{(X,\mathfrak{Y}_\eta)}$-coherent $\mathcal{D}_{(X,\mathfrak{Y}_\eta)}$-module $\mathcal{F}$ and every $q \geq 0$. ∎

5.2.5. Remarks. (i) Germs of analytic spaces are particular cases of pro-analytic spaces from [Ber4]. The above definition of the category of sheaves on a germ is a simple variant of a more general definition in [Ber4, §2].

(ii) Our terminology is slightly different from the standard one in the sense that finite isocrystals are usually called isocrystals (see also Remark 6.2.2).

(iii) In the theory of (finite) isocrystals an additional property of overconvergence plays an important role. All of the finite isocrystals considered here are unipotent and known to be overconvergent (see [LeCh]). We do not invoke this property explicitly, but consider a closely related property in §7.1.

5.3. A construction of isocrystals. Let k be a field of characteristic zero, and B a commutative k-algebra which is provided with a projective B-module of finite rank Ω^1_B and a k-linear derivation $d : B \to \Omega^1_B$ with $k = \mathrm{Ker}(d)$ and such that locally in the Zariski topology of B the B-module Ω^1_B is freely generated by exact differentials. The latter assumption implies that, setting $\Omega^q_B = \wedge^q \Omega^1_B$, one can define a de Rham complex $\Omega^{\cdot}_B : 0 \to B \xrightarrow{d} \Omega^1_B \xrightarrow{d} \Omega^2_B \xrightarrow{d} \dots$ whose cohomology groups, denoted by $H^q_{\mathrm{dR}}(B)$, are vector spaces over k. Similarly, any B-module M

with an integrable connection $\nabla : M \to M \otimes_B \Omega^1_B$ defines a de Rham complex $M \otimes_B \Omega^{\cdot}_B$ whose cohomology groups, denoted by $H^q_{\mathrm{dR}}(M)$, are vector spaces over k. Such a B-module M will be called an isocrystal and, if it is finitely generated over B, it will be called a finite isocrystal. Trivial and unipotent isocrystals are defined in the same way as in §1.3.

Given an isocrystal M over B and a k-vector subspace $V \subset H^1_{\mathrm{dR}}(M)$, we construct as follows an isocrystal M_V. As a B-module, M_V is the direct sum $M \oplus (V \otimes_k B)$. For an element $v \in V$, we denote by $\overline{v}$ the corresponding element $v \otimes 1$ of M_V. Furthermore, we fix a k-linear section to the subspace of closed one-forms $s : V \to (M \otimes_B \Omega^1_B)^{\mathrm{cl}}$, and extend the connection $\nabla : M \to M \otimes_B \Omega^1_B$ to a connection $\nabla : M_V \to M_V \otimes_B \Omega^1_B$ by $\nabla(\overline{v}) = s(v)$ for all $v \in V$. This provides M_V with the structure of an isocrystal over B. Notice that different choices of the section s give rise to non-canonically isomorphic isocrystal structures on M_V. Indeed, a second section $s' : V \to (M \otimes_B \Omega^1_B)^{\mathrm{cl}}$ defines a connection ∇' on M_V with $\nabla'(\overline{v}) = s'(v)$ and, therefore, $s'(v) - s(v) \in \nabla(M)$. Let μ be a k-linear map $V \to M$ for which $s'(v) - s(v) = \nabla(\mu(v))$ for all $v \in V$. Then the B-linear map $\varphi : M_V \to M_V$ identical on M and defined by $\varphi(\overline{v}) = \mu(v) + \overline{v}$ for all $v \in V$ gives rise to an isomorphism of isocrystals $(M_V, \nabla') \xrightarrow{\sim} (M_V, \nabla)$.

5.3.1. Lemma. *(i) There is a canonical isomorphism of isocrystals* $M_V/M \xrightarrow{\sim} V \otimes_k B$*;*

(ii) $H^0_{\mathrm{dR}}(M) \xrightarrow{\sim} H^0_{\mathrm{dR}}(M_V)$*;*

(iii) $\mathrm{Ker}(H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(M_V)) = V$.

Proof. (i) trivially follows from the construction.

Let $e_1, \dots, e_n$ be elements of V linearly independent over k, and assume that for a closed one-form $\omega \in M \otimes_B \Omega^1_B$ one has $\omega = \nabla(m + \sum_{i=1}^n f_i \overline{e}_i)$ with $m \in M$ and $f_i \in B$. It follows that $\sum_{i=1}^n \overline{e}_i df_i = \omega - \nabla(m) - \sum_{i=1}^n f_i s(e_i) \in M \otimes_M \Omega^1_B$ and, therefore, $df_i = 0$, i.e., $f_i \in k$. This implies that the class of ω is contained in V, i.e., (iii) is true, and, if $\omega = 0$, we get (ii). ∎

Notice that the exact sequence of isocrystals $0 \to M \to M_V \to M_V/M \to 0$ gives rise to an isomorphism $H^0_{\mathrm{dR}}(M_V/M) \xrightarrow{\sim} V \subset H^1_{\mathrm{dR}}(M)$.

Let $M \subset M'$ be isocrystals over B such that $H^0_{\mathrm{dR}}(M'/M) \otimes_k B \xrightarrow{\sim} M'/M$, and let $V = \mathrm{Ker}(H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(M'))$. Then any homomorphism of isocrystals $\varphi : M \to N$ with $V \subset \mathrm{Ker}(H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(N))$ can be extended to a homomorphism of isocrystals $\varphi' : M' \to N$. Indeed, as a B-module, M' can be identified with the direct sum $M \oplus (U \otimes_k B)$, where $U = H^0_{\mathrm{dR}}(M'/M)$. The assumption implies that $\varphi(\nabla(u \otimes 1)) \in \nabla(N)$ for all $u \in U$. If β is a k-linear map $U \to N$ with $\varphi(\nabla(u \otimes 1)) = \nabla(\beta(u))$ for all $u \in U$, then an extension $\varphi' : M' \to N$ can be defined by $\varphi'(u \otimes 1) = \beta(u)$ for $u \in U$.

5.3.2. Lemma. *Assume that B is Noetherian and any finite isocrystal over B is a projective B-module. Given isocrystals $M \subset M'$ as above with $H^0_{\mathrm{dR}}(M) \xrightarrow{\sim} H^0_{\mathrm{dR}}(M')$, any homomorphism of isocrystals $M' \to N$ injective on M is injective on M'.*

Proof. We can replace M' by the preimage of $U' \otimes_k B$ for some finitely dimensional subspace $U' \subset U = H^0_{\mathrm{dR}}(M)$, and so we may assume that $\dim_k(U) < \infty$. Suppose $P = \mathrm{Ker}(M' \to N) \neq 0$. It suffices to show that $H^0_{\mathrm{dR}}(P) \neq 0$ because this would contradict the assumption. For this we consider the exact sequence of isocrystals $0 \to P \to M'/M = U \otimes_k B \to Q \to 0$. By the assumption, Q is a projective B-module. It follows that the canonical map $H^0_{\mathrm{dR}}(Q) \otimes_k B \to Q$ is injective (see the proof of Lemma 1.3.1). Since B is Noetherian, it follows that $\dim_k(H^0_{\mathrm{dR}}(Q)) < \dim_k(U)$ which implies that $H^0_{\mathrm{dR}}(P) \neq 0$. ∎

We now construct as follows an increasing sequence of unipotent isocrystals $E^0_B \subset E^1_B \subset E^2_B \subset \ldots$: $E^0_B = B$ and, for $i \geq 0$, $E^{i+1}_B = (E^i_B)_V$ for $V = H^1_{\mathrm{dR}}(E^i_B)$. The inductive limit $E_B = \varinjlim E^i_B$ is a filtered isocrystal over B. For an isocrystal M over B, we denote the isocrystals $M \otimes_B E_B$ and $M \otimes_B E^i_B$ by M_E and M_{E^i}, respectively.

Let M be a unipotent isocrystal of rank m over B, and fix a filtration $M^0 = 0 \subset M^1 \subset \ldots \subset M^n = M$ such that each quotient M^i/M^{i-1} is a trivial isocrystal. Consider the filtration on the isocrystal M_E defined by the sub-isocrystals

$$\widetilde{M}_i = M^n_{E^i} + M^{n-1}_{E^{i+1}} + \ldots + M^1_{E^{i+n-1}} \ .$$

5.3.3. Lemma. *(i) $(\widetilde{M}_i \otimes_B \Omega^1_B)^{\mathrm{cl}} \subset \nabla(\widetilde{M}_{i+1})$;*
(ii) $\widetilde{M}^\nabla_0$ is a vector space over k of dimension m;
(iii) $M^\nabla_E = \widetilde{M}^\nabla_0$.

Proof. If $n = 1$, the claim follows from Lemma 5.3.1. Assume that $n \geq 2$ and the claim is true for $n-1$. We set $N = M^{n-1}$ and notice that there are exact sequences of isocrystals $0 \to \widetilde{N}_{i+1} \to \widetilde{M}_i \to (M/N)_{E^i} \to 0$ and $0 \to \widetilde{N}_{i+1} \to (\widetilde{M}_i + M_{E^{i+1}}) \to (M/N)_{E^{i+1}} \to 0$, where $\widetilde{N}_{i+1}$ is considered for the induced filtration of N of length $n-1$.

First of all, let $\omega \in (\widetilde{M}_i \otimes_B \Omega^1_B)^{\mathrm{cl}}$. By the validity of the claim for the trivial isocrystal M/N, there is an element $f \in M_{E^{i+1}}$ such that $\omega - \nabla(f) \in (\widetilde{N}_{i+1} \otimes_B \Omega^1_B)^{\mathrm{cl}}$. By the induction hypotheses, there exists an element $g \in \widetilde{N}_{i+2}$ with $\omega - \nabla(f) = \nabla(g)$, and so $\omega = \nabla(f+g)$ with $f + g \in M_{E^{i+1}} + \widetilde{N}_{i+2} = \widetilde{M}_{i+1}$, i.e., (i) is true.

Furthermore, the first of the above exact sequences gives rise to an exact sequence of vector spaces $0 \to \widetilde{N}^\nabla_{i+1} \to \widetilde{M}^\nabla_i \to (M/N)^\nabla_{E^i} = (M/N)^\nabla$. To prove (ii) and (iii), it suffices to show that the second homomorphism is surjective. Let f be an element of M whose image in M/N is contained in $(M/N)^\nabla$. Then $\nabla(f) \in (N \otimes_B \Omega^1_B)^{\mathrm{cl}}$ and, by the induction hypothesis, there exists an element $g \in \widetilde{N}_1$ with $\nabla(g) = \nabla(f)$. The element $f - g$ lies in $\widetilde{M}_0$ and its image in $(M/N)^\nabla$ coincides with that of f. ∎

Assume that the ring B is Noetherian and any finite isocrystal over B is a projective B-module (as in Lemma 5.3.2). Assume also that the isocrystal E_B can be provided with a structure of a commutative filtered B-algebra which satisfies the Leibniz rule (i.e., $\nabla(fg) = \nabla(f)g + f\nabla(g)$). (In §6, it will be shown that in the case of interest for us these assumptions are satisfied.) Then for any isocrystal M over B there is a canonical homomorphism of isocrystals $M_E^\nabla \otimes_k E \to M_E$, which is injective if M is finite over B.

5.3.4. Corollary. *Under the above assumptions, the following properties of a finite isocrystal M of rank m over B are equivalent:*

(a) M is unipotent;

(b) the vector space M_E^∇ is of dimension m and $M_E^\nabla \otimes_k E \xrightarrow{\sim} M_E$;

(c) there exists an embedding of isocrystals $M \hookrightarrow (E)^l$ with $l \geq 1$.

Furthermore, if M is unipotent, its level is equal to the minimal n for which there exists an embedding of isocrystals $M \hookrightarrow (E^{n-1})^l$ with $l \geq 1$.

The *level* of a unipotent isocrystal M is the minimal n for which there exists a filtration $M^0 = 0 \subset M^1 \subset \ldots \subset M^n = M$ such that each quotient M^i/M^{i-1} is a trivial isocrystal.

Proof. (c)$\Longrightarrow$(a) Since M is finitely generated over B, its image is contained in $(E^{n-1})^l$ for some $n \geq 1$. The proof of Lemma 5.3.2 implies that M is unipotent of level at most n.

The implication (a)$\Longrightarrow$(b) follows easily from Lemma 5.3.3 (and its proof), and the implication (b)$\Longrightarrow$(c) is trivial.

Assume that M is unipotent of level n, and let $M^0 = 0 \subset M^1 \subset \ldots \subset M^n = M$ be a filtration such that each quotient M^i/M^{i-1} is a trivial isocrystal. By Lemma 5.3.3, for the corresponding filtration of M_E one has $M_E^\nabla = \widetilde{M}_0^\nabla$. *We claim that there is a basis of M_E^∇ such that the isomorphism of isocrystals $M_E \xrightarrow{\sim} (E)^m$, induced by it, takes every $\widetilde{M}_i$ into $(E^{i+n-1})^m$.* Indeed, if $n = 1$, the claim is trivial. Assume that $n \geq 2$ and the claim is true for $n-1$. Then we can find a basis $h_1, \ldots, h_l$ of N_E^∇ with the required property for $N = M^{n-1}$. The proof of Lemma 5.3.3 shows that, if $f_{l+1}, \ldots, f_m$ are elements of M whose images in M/N form a basis of $(M/N)^\nabla$, then there exist elements $g_{l+1}, \ldots, g_m \in \widetilde{N}_1$ such that the elements $h_1, \ldots, h_l, h_{l+1} =$

$f_{l+1} - g_{l+1}, \dots, h_m = f_m - g_m$ form a basis of M_E^∇. Since $\widetilde{M}_i = M_{E^i} + \widetilde{N}_{i+1}$, it follows that the constructed basis possesses the required property. ∎

5.4. The filtered isocrystals E_B and the shuffle algebras. We continue to work here in the general setting of the previous subsection.

Recall that the shuffle algebra $\mathrm{Sh}(V)$ of a vector space V over a field k is a commutative $\mathbf{Z}_+$-graded algebra which, as a vector space, is the tensor algebra $\oplus_{n=0}^\infty V^{\otimes n}$ of V with the following multiplication

$$(v_1 \otimes \dots \otimes v_p)\cdot(v_{p+1} \otimes \dots \otimes v_{p+q}) = \sum_\sigma v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(p+q)} \ ,$$

where the sum is taken over all (p,q)-shuffles, i.e., those of the permutations σ of $\{1, \dots, p+q\}$ for which $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$.

Assume we are in the situation of the beginning of §5.3. For a vector subspace $V \subset H^1_{\mathrm{dR}}(B)$, we denote by $\mathrm{Sh}_B(V)$ the $\mathbf{Z}_+$-graded B-algebra $\mathrm{Sh}(V) \otimes_k B$. (If $V = H^1_{\mathrm{dR}}(B)$, it will be denoted by Sh_B.) Fixing a k-linear section $s : V \to \Omega_B^{1,\mathrm{cl}}$, the k-linear map

$$\mathrm{Sh}(V) \to \mathrm{Sh}(V) \otimes_k \Omega^1_B : v_1 \otimes \dots \otimes v_p \mapsto (v_1 \otimes \dots \otimes v_{p-1}) \otimes s(v_p)$$

extends to a k-linear connection on the B-module $\mathrm{Sh}_B(V)$

$$\nabla : \mathrm{Sh}_B(V) \to \mathrm{Sh}_B(V) \otimes_B \Omega^1_B = \mathrm{Sh}(V) \otimes_k \Omega^1_B \ .$$

Notice that this connection is not integrable (unless the rank of Ω^1_B is at most one). The connection ∇ preserves each B-submodule $\mathrm{Sh}^n_B(V) = \oplus_{p=0}^n (V^{\otimes p} \otimes_k B)$, and it gives rise to an integrable connection on each quotient $\mathrm{Gr}^n(\mathrm{Sh}_B(V)) = \mathrm{Sh}^n_B(V)/\mathrm{Sh}^{n-1}_B(V)$, which does not depend on the choice of the section s. One has $\mathrm{Gr}^n(\mathrm{Sh}_B(V))^\nabla = V^{\otimes n}$ and $\mathrm{Gr}^n(\mathrm{Sh}_B(V))$ is isomorphic to the free B-module with an integrable connection $\mathrm{Gr}^n(\mathrm{Sh}_B(V))^\nabla \otimes_k B$.

5.4.1. Lemma. *The connection ∇ satisfies the Leibniz rule, i.e., $\nabla(ab) = a\nabla(b) + b\nabla(a)$, and its kernel coincides with k.*

Proof. To verify the Leibniz rule, it suffices to consider the case when $a = v_1 \otimes \dots \otimes v_p$ and $b = v_{p+1} \otimes \dots \otimes v_{p+q}$. One has

$$\nabla(a\cdot b) = \sum_\sigma (v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(p+q-1)}) \otimes s(v_{\sigma^{-1}(p+q)}) \ ,$$

where the sum is taken over all (p,q)-shuffles σ of the set $\{1,\dots,p+q\}$. On the other hand, one has

$$a{\cdot}\nabla(b)=\sum_\tau(v_{\tau^{-1}(1)}\otimes\ldots\otimes v_{\tau^{-1}(p+q-1)})\otimes s(v_{p+q})$$

and

$$b{\cdot}\nabla(a)=\sum_\tau(v_{\tau^{-1}(1)}\otimes\ldots\otimes v_{\tau^{-1}(p-1)}\otimes v_{\tau^{-1}(p+1)}\otimes\ldots\otimes v_{\tau^{-1}(p+q)})\otimes s(v_p)\ ,$$

where the first sum is taken over $(p,q-1)$-shuffles τ of the set $\{1,\dots,p+q-1\}$, and the second one is taken over $(p-1,q)$-shuffles τ of the set $\{1,\dots,p-1,p+1,\dots,p+q\}$. It remains to notice that, for every (p,q)-shuffle σ of the set $\{1,\dots,p+q\}$, one has either $\sigma^{-1}(p+q)=p+q$ or $\sigma^{-1}(p+q)=p$, and so there is a one-to-one correspondence between the shuffles σ of each of the latter types and the shuffles τ from each of the above sums, respectively. Assume now that there exists an element $a\in\mathrm{Sh}^n_B(V)\backslash\mathrm{Sh}^{n-1}_B(V)$ with $da=0$ and $n\geq1$. If $\{v_1,\dots,v_m\}$ is a basis of the k-vector space V, then a basis of the k-vector space $\mathrm{Sh}_B(V)$ is formed by the elements $v_{i_1,\dots,i_p}=v_{i_1}\otimes\ldots\otimes v_{i_p}$, and $a=\sum v_{i_1,\dots,i_p}g_{i_1,\dots,i_p}$ with $g_{i_1,\dots,i_p}\in B$. The description of $\mathrm{Sh}^n_B(V)/\mathrm{Sh}^{n-1}_B(V)$ implies that $g_{i_1,\dots,i_n}\in k$ for all $(i_1,\dots,i_n)$. Furthermore, given $(i_1,\dots,i_{n-1})$, the coefficient of da at $v_{i_1,\dots,i_{n-1}}$ is equal to $dg_{i_1,\dots,i_{n-1}}+\sum_{j=1}^m g_{i_1,\dots,i_{n-1},j}s(v_j)$. It follows that $g_{i_1,\dots,i_n}=0$ for all $(i_1,\dots,i_n)$ which is a contradiction. ∎

Let us call a *D_B-algebra* any B-algebra which is an isocrystal over B whose connection satisfies the Leibniz rule. Lemma 5.4.1 implies that $\mathrm{Gr}^{\cdot}(\mathrm{Sh}_B(V))$ is a graded D_B-algebra isomorphic to $\mathrm{Gr}^{\cdot}(\mathrm{Sh}_B(V))^\nabla\otimes_k B$, and there is a canonical isomorphism of graded k-algebras $\mathrm{Sh}(V)\xrightarrow{\sim}\mathrm{Gr}^{\cdot}(\mathrm{Sh}_B(V))^\nabla$.

Assume now that the filtered isocrystal E_B, constructed in §5.3, is provided with a structure of a commutative filtered D_B-algebra. Recall that each quotient $\mathrm{Gr}^n(E_B)=E^n_B/E^{n-1}_B$ is canonically isomorphic to the trivial isocrystal $H^1_{\mathrm{dR}}(E^{n-1}_B)\otimes_k B$. In particular, there are canonical isomorphisms of graded D_B-algebras $\mathrm{Gr}^{\cdot}(E_B)^\nabla\otimes_k B\xrightarrow{\sim}\mathrm{Gr}^{\cdot}(E_B)$ and of k-vector spaces $\mathrm{Gr}^n(E_B)^\nabla\xrightarrow{\sim}H^1_{\mathrm{dR}}(E^{n-1}_B)$ and $H^1_{\mathrm{dR}}(\mathrm{Gr}^n(E_B))\xrightarrow{\sim}H^1_{\mathrm{dR}}(E^{n-1}_B)\otimes_k H^1_{\mathrm{dR}}(B)$. Since the canonical map $H^1_{\mathrm{dR}}(E^{n-1}_B)\to H^1_{\mathrm{dR}}(E^n_B)$ is zero, the latter isomorphism and the exact sequence $0\to E^{n-1}_B\to E^n_B\to\mathrm{Gr}^n(E_B)\to0$ give rise to an injective k-linear map $\mathrm{Gr}^n(E_B)^\nabla\hookrightarrow H^1_{\mathrm{dR}}(B)^{\otimes n}$. In this way we get an injective k-linear map

$$\psi:\mathrm{Gr}^{\cdot}(E_B)^\nabla\to\mathrm{Sh}(H^1_{\mathrm{dR}}(B))\ .$$

5.4.2. Proposition. *The map ψ is a homomorphism of k-algebras.*

Proof. Let $a\in E^p_B$ and $b\in E^q_B$ be such that $\nabla(a)\in E^{p-1}_B\otimes_B\Omega^1_B$ and $\nabla(b)\in E^{q-1}_B\otimes_B\Omega^1_B$. We have to show that $\psi(a{\cdot}b)=\psi(a){\cdot}\psi(b)$. If $p=0$ or $q=0$, the equality is trivial, and so assume that $p, q \geq 1$ and that the equality is true for all of the pairs (p', q') with either $p' < p$ and $q' \leq q$ or $p' \leq p$ and $q' < q$. Notice that the element $\psi(a)$ is constructed as follows. First of all, consider the image of $\nabla(a)$ in $(E_B^{p-1}/E_B^{p-2} \otimes_B \Omega_B^1)^{\mathrm{cl}} = (E_B^{p-1}/E_B^{p-2})^\nabla \otimes_k \Omega_B^{1,\mathrm{cl}}$ (with $E_B^{-1} = 0$). If the image of the latter in $H^1_{\mathrm{dR}}(B)^{\otimes(p-1)} \otimes_k \Omega_B^{1,\mathrm{cl}}$ under the map induced by $\psi : (E_B^{p-1}/E_B^{p-2})^\nabla \to H^1_{\mathrm{dR}}(B)^{\otimes(p-1)}$ is $\sum_{i=1}^m u_1^{(i)} \otimes \ldots \otimes u_{p-1}^{(i)} \otimes (s(u_p^{(i)}) + df_i)$, where $f_i \in B$, then $\psi(a) = \sum_{i=1}^m u_1^{(i)} \otimes \ldots \otimes u_p^{(i)}$. Similarly, if the image of $\nabla(b)$ in $H^1_{\mathrm{dR}}(B)^{\otimes(q-1)} \otimes_k \Omega_B^{1,\mathrm{cl}}$ is $\sum_{j=1}^n v_1^{(j)} \otimes \ldots \otimes v_{q-1}^{(j)} \otimes (s(v_q^{(j)}) + dg_j)$, where $g_j \in B$, then $\psi(a) = \sum_{j=1}^n v_1^{(j)} \otimes \ldots \otimes v_q^{(j)}$. To find $\psi(a{\cdot}b)$ we apply the same procedure to the element $\nabla(a{\cdot}b) = a{\cdot}\nabla(b) + b{\cdot}\nabla(a)$. By induction, one has

$$\psi(a{\cdot}b) = (\psi(a){\cdot}\sum_{j=1}^n v_1^{(j)} \otimes \ldots \otimes v_{q-1}^{(j)}) \otimes v_q^{(j)} + (\psi(b){\cdot}\sum_{i=1}^m u_1^{(i)} \otimes \ldots \otimes u_{p-1}^{(i)}) \otimes u_p^{(i)} \ .$$

Thus, the required equality $\psi(a{\cdot}b) = \psi(a){\cdot}\psi(b)$ is a consequence of the formula which tells that the element $(u_1 \otimes \ldots \otimes u_p){\cdot}(v_1 \otimes \ldots \otimes v_q)$ is equal to

$$((u_1 \otimes \ldots \otimes u_p){\cdot}(v_1 \otimes \ldots \otimes v_{q-1})) \otimes v_q + ((v_1 \otimes \ldots \otimes v_q){\cdot}(u_1 \otimes \ldots \otimes u_{p-1})) \otimes u_p \ .$$

This formula is verified in the same way as the Leibniz rule in the proof of Lemma 5.4.1. ∎

Recall (see [Rad, Theorem 3.1.1]) that the shuffle algebra $\mathrm{Sh}(V)$ of a vector space V over a field of characteristic zero is isomorphic to the ring of polynomials over the field with a set of variables consisting of homogeneous elements.

5.4.3. Corollary. *(i) If B has no zero divisors, then so is the graded B-algebra $\mathrm{Gr}^{\cdot}(E_B)$, and B is algebraically closed in it;*

(ii) $\mathrm{Gr}^{\cdot}(E_B)^* = B^*$ and, in particular, $(E_B)^* = B^*$. ∎

Finally, assume that the B-module Ω_B^1 is of rank one. Then the connection ∇ on Sh_B is integrable and, in particular, Sh_B is an isocrystal over B. One has $H^0_{\mathrm{dR}}(\mathrm{Sh}_B^n) = k$ and $H^q_{\mathrm{dR}}(\mathrm{Sh}_B^n) = 0$ for all $n \geq 0$ and $q \geq 2$.

5.4.4. Lemma. *If the B-module Ω_B^1 is of rank one, then*

(i) the canonical homomorphism $H^1_{\mathrm{dR}}(\mathrm{Sh}_B^{n-1}) \to H^1_{\mathrm{dR}}(\mathrm{Sh}_B^n)$ is zero;

(ii) $H^1_{\mathrm{dR}}(\mathrm{Sh}_B^n) \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathrm{Gr}^n(\mathrm{Sh}_B)) = H^1_{\mathrm{dR}}(B)^{\otimes(n+1)}$;

(iii) the map ψ of the Proposition 5.4.2 is an isomorphism and, in particular, there is a canonical isomorphism of graded D_B-algebras $\mathrm{Gr}^{\cdot}(E_B) \xrightarrow{\sim} \mathrm{Gr}^{\cdot}(\mathrm{Sh}_B)$.

Proof. (i) The statement is trivially true for $n = 0$ (with $\mathrm{Sh}_B^{-1} = 0$). Assume that $n \geq 1$ and that the statement is true for $n - 1$. Each element of $\mathrm{Sh}_B^{n-1} \otimes_B \Omega_B^1 = \mathrm{Sh}^{n-1}(H^1_{\mathrm{dR}}(B)) \otimes_k \Omega_B^1$

is a finite sum of elements of the form $v_1 \otimes \ldots \otimes v_{n-1} \otimes \omega$ with $v_i \in H^1_{\mathrm{dR}}(B)$ and $\omega \in \Omega^1_B$. Let $\omega = s(v) + df$ for some $v \in H^1_{\mathrm{dR}}(B)$ and $f \in B$. One has $v_1 \otimes \ldots \otimes v_{n-1} \otimes s(v) = \nabla(v_1 \otimes \ldots \otimes v_{n-1} \otimes v)$ and $v_1 \otimes \ldots \otimes v_{n-1} \otimes df = \nabla(fv_1 \otimes \ldots \otimes v_{n-1}) - v_1 \otimes \ldots \otimes v_{n-2} \otimes (fs(v_{n-1}))$. It remains to notice that the element $v_1 \otimes \ldots \otimes v_{n-2} \otimes (fs(v_{n-1}))$ is contained in $\mathrm{Sh}^{n-2}_B \otimes_B \Omega^1_B$ and, by induction, it is contained in $\nabla(\mathrm{Sh}^{n-1}_B)$.

(ii) The required isomorphism is obtained from the exact sequence $0 \to \mathrm{Sh}^{n-1}_B \to \mathrm{Sh}^n_B \to \mathrm{Gr}^n(\mathrm{Sh}_B) \to 0$ using (i) and the fact that $H^2_{\mathrm{dR}}(\mathrm{Sh}^{n-1}_B) = 0$.

(iii) As in (ii), the required isomorphism is obtained from the exact sequence $0 \to E^{n-1}_B \to E^n_B \to \mathrm{Gr}^n(E_B) \to 0$ and the facts that the canonical map $H^1_{\mathrm{dR}}(E^{n-1}_B) \to H^1_{\mathrm{dR}}(E^n_B)$ is zero and $H^2(E^{n-1}) = 0$. ∎

5.4.5. Remark. In §6.4 it will be shown that, in the situation considered there, the isomorphism of Lemma 5.4.4(iii) is induced by a (non-canonical) isomorphism of filtered D_B-algebras $E_B \xrightarrow{\sim} \mathrm{Sh}_B$.

5.5. Unipotent isocrystals $E^i(X, \mathfrak{Z})$.

5.5.1. Lemma *(O. Gabber). Let k be a non-Archimedean field with a nontrivial valuation. Given an inductive system $\{F^\cdot_n\}_{n\geq 1}$ of complexes of Banach spaces over k with $\dim_k H^q(\varinjlim F^\cdot_n) < \infty$ for some $q \in \mathbf{Z}$, for any Banach space W over k the canonical linear map*

$$H^q(\varinjlim F^\cdot_n) \otimes_k W \to H^q(\varinjlim (F^\cdot_n \widehat{\otimes}_k W))$$

is an isomorphism.

Proof. We set $Z_n = \mathrm{Ker}(F^q_n \to F^{q+1}_n)$ and $B_n = \mathrm{Coim}(F^{q-1}_n \to F^q_n)$ (i.e., the quotient of F^{q-1}_n by $\mathrm{Ker}(F^{q-1}_n \to F^q_n)$). These are Banach spaces over k, and there is a canonical injective bounded linear operator $B_n \to Z_n$. Since the functor $V \mapsto V \widehat{\otimes}_k W$ is exact in the sense that it takes a short exact sequence of Banach spaces $0 \to V' \to V \to V'' \to 0$ with $V' \xrightarrow{\sim} \mathrm{Ker}(V \to V'')$ to a similar short exact sequence (see [Gru]), it suffices to show that the canonical linear map

$$(\varinjlim \mathrm{Coker}(B_n \to Z_n)) \otimes_k W \to \mathrm{Coker}(\varinjlim (B_n \widehat{\otimes}_k W) \to \varinjlim (Z_n \widehat{\otimes}_k W))$$

is an isomorphism provided we know that the inductive limit on the left hand side is of finite dimension over k.

Given $n \leq m$, let $D_{n,m}$ denote the fiber product $B_m \times_{Z_m} Z_n$, i.e., the closed subspace of the direct product $B_m \times Z_n$ that consists of the pairs (b, z) such that the images of b and z in Z_m

coincide. Given $n \geq 1$, the Banach spaces $D_{n,m}$ form an inductive system that maps to Z_n. Assume n is big enough so that there is a finite dimensional subspace $H \subset Z_n$ which maps bijectively onto $\varinjlim \mathrm{Coker}(B_n \to Z_n)$. Then the canonical injective map from the inductive limit of Banach space $\varinjlim_m (H \times D_{n,m})$ to the Banach space Z_n is surjective. *We claim that there exists $m \geq n$ such that the injective map $H \times D_{n,m} \to Z_n$ is surjective.* Indeed, if none of those maps are surjective then, by the Banach theorem (see [Bou2, Ch. I, §3, n° 3]), their images are countable unions of nowhere dense subsets, and so Z_n itself is a countable union of nowhere dense subsets which contradicts the Baire theorem (see [Bou1, Ch. 9, §5, n° 3]).

Thus, for sufficiently big $m \geq n$ the map $H \times D_{n,m} \to Z_n$ is bijective. The same Banach theorem, implies that it is an isomorphism and, in particular, there is an isomorphism

$$(H \otimes_k W) \times (D_{n,m} \widehat{\otimes}_k W) \xrightarrow{\sim} Z_n \widehat{\otimes}_k W \ .$$

This immediately gives the bijectivity of the considered linear map. ∎

Let k be a non-Archimedean field of characteristic zero with a nontrivial discrete valuation, X a smooth k-analytic space, and Y a compact strictly analytic subdomain of X. By a theorem of E. Grosse-Klönne ([GK2, Theorem A]), the de Rham cohomology groups $H^q_{\mathrm{dR}}(X, Y)$ are of finite dimension over k (see Remark 5.5.5(i)). Furthermore, for a non-Archimedean field k' over k, we set $X' = X \widehat{\otimes}_k K'$ and $Y' = Y \widehat{\otimes}_k K'$. Notice that, if k' is finite over k, then one has $H^q_{\mathrm{dR}}(X, Y) \otimes_k k' \xrightarrow{\sim} H^q_{\mathrm{dR}}(X', Y')$ for all $q \geq 0$.

5.5.2. Corollary. *The canonical map $H^q_{\mathrm{dR}}(X, Y) \otimes_k k' \to H^q_{\mathrm{dR}}(X', Y')$, $q \geq 0$, is always an isomorphism.*

Proof. Let $V_1 \supset V_2 \supset \dots$ be a decreasing sequence of compact strictly analytic domains in X with $Y = \cap_{n=1}^{\infty} V_n$ and $V_{n+1} \subset \mathrm{Int}(V_n)$, and let $F_n^{\cdot} : 0 \to F_n^0 \to F_n^1 \to \dots$ be the de Rham complex of the strictly k-analytic space V_n. Then F_n^q are Banach spaces over k, the complexes form an inductive system $F_1^{\cdot} \to F_2^{\cdot} \to \dots$ and, if $F^{\cdot} : 0 \to F^0 \to F^1 \to \dots$ is the corresponding inductive limit, one has $H^q_{\mathrm{dR}}(X, Y) = H^q(F^{\cdot})$. Furthermore, for $V'_n = V_n \widehat{\otimes}_k k'$ one has $Y' = \cap_{n=1}^{\infty} V'_n$ and $V'_{n+1} \subset \mathrm{Int}(V'_n)$ for all $n \geq 1$. The de Rham complex of V'_n is $F_n^{\cdot} \widehat{\otimes}_k k'$ and, if $F'^{\cdot}$ is the corresponding inductive limit, one has $H^q_{\mathrm{dR}}(X', Y') = H^q(F'^{\cdot})$. Since k' is a Banach space over k, the required fact follows from Lemma 5.5.1. ∎

Let now k be a non-Archimedean field of characteristic zero, which is an extension of a non-Archimedean field k' with a nontrivial discrete valuation and a closed subfield of $\widehat{k'^{\mathrm{a}}}$.

5.5.3. Corollary. *Let X be a smooth k-analytic space. Then* *(i) for any compact strictly analytic subdomain $Y \subset X$, the de Rham cohomology groups $H^q_{\mathrm{dR}}(X,Y)$ are of finite dimension over k;*

(ii) if $X = X' \otimes_{k'} k$ and φ denotes the canonical morphism $X \to X'$, then the canonical morphism of étale abelian sheaves on X'

$$\varphi^{-1}(\Omega^{1,\mathrm{cl}}_{X'}/d\mathcal{O}_{X'}) \otimes_{k''} k \longrightarrow \Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X$$

is an isomorphism, where $k'' = k \cap k'^{\mathrm{a}}$.

Proof. (i) By Lemma 2.1.3, we can find systems of open subsets $X_i \subset X$ and of strictly affinoid subdomains $Y_i \subset X_i$ such that $Y = \cup Y_i$ and, for every i, X_i and Y_i come from a smooth k_i-analytic space X'_i and a strictly affinoid subdomain $Y'_i \subset X'_i$, respectively, where k_i is a finite extension of k' in k. By Corollary 5.5.2, the de Rham cohomology groups $H^q_{\mathrm{dR}}(X_i,Y_i)$ are of finite dimension over k and, therefore, the same is true for $H^q_{\mathrm{dR}}(X,Y)$.

(ii) The statement is an easy consequence of the same Lemma 2.1.3 and Corollary 5.5.2. ∎

Let Y be a strictly affinoid subdomain of X. Then $B = \mathcal{O}(X,Y)$ is a k-dagger algebra, and, by Corollary 5.5.3(i), the de Rham cohomology groups $H^q_{\mathrm{dR}}(B)$ are of finite dimension over k. The construction of §5.3 can be applied to the algebra B, and so it provides a filtered isocrystal E_B over B. *Assume, in addition, that $X = X' \otimes_{k'} k$ and $Y = Y' \otimes_{k'} k$, where X' and $Y' \subset X'$ are smooth k'-analytic space and a strictly affinoid subdomain, respectively.* Then $B' = \mathcal{O}(X',Y')$ is a k'-dagger algebra, and the construction of §5.3 provides a filtered isocrystal $E_{B'}$ over B'.

5.5.4. Corollary. *In the above situation, the following is true:*

(i) the isocrystals E^i_B and $E^i_{B'}$ are unipotent;

(ii) there is compatible system of isomorphisms of isocrystals $E^i_{B'} \otimes_{B'} B \xrightarrow{\sim} E^i_B$;

(iii) the isomorphism from (ii) gives rise to an isomorphism of de Rham cohomology groups $H^q_{\mathrm{dR}}(E^i_{B'}) \otimes_{k'} k \xrightarrow{\sim} H^q_{\mathrm{dR}}(E^i_B)$ for all $q \geq 0$.

Proof. The statements are true, by Corollary 5.5.2, for $i = 0$ and are deduced, by induction, from the exact sequence of isocrystals $0 \to E^i_B \to E^{i+1}_B \to H^1_{\mathrm{dR}}(E^i_B) \otimes_{\mathfrak{c}(B)} B \to 0$. ∎

5.5.5. Corollary. *In the above situation, the following is true:*

(i) every unipotent isocrystal M over B is isomorphic to $M' \otimes_{B'} B$ for an unipotent isocrystal M' over B';

(ii) $\mathrm{Hom}(M'_1, M'_2) \otimes_{k'} k \xrightarrow{\sim} \mathrm{Hom}(M'_1 \otimes_{B'} B, M'_2 \otimes_{B'} B)$ for every pair of unipotent isocrystals M'_1 and M'_2 over B'. ∎

The above construction will be applied in the following situations. Let $(X, \mathfrak{Z})$ be a wide germ of a smooth *affine* formal scheme over k°. Then $(X, \mathfrak{Z}_\eta)$ is a smooth strictly k-affinoid germ. The filtered isocrystal E_B, provided by the previous construction for $B = \mathcal{O}(X, \mathfrak{Z}_\eta)$, will be denoted by $E(X, \mathfrak{Z})$. Notice that the isocrystals $E^i(X, \mathfrak{Z})$ are unipotent. In §6 we consider germs $(X, \mathfrak{Z})$ which are liftings of a similar germ $(X', \mathfrak{Z}')$ over k'° as in Lemma 5.5.4. By that lemma, there is compatible system of isomorphisms of isocrystals $E^i(X', \mathfrak{Z}') \otimes_{B'} B \xrightarrow{\sim} E^i(X, \mathfrak{Z})$. At the end of §6 and in §7, we consider germs of the form $(\mathfrak{X}_\eta, \mathfrak{Z})$, where $\mathfrak{X}$ is a proper marked formal scheme over k° and $\mathfrak{Z}$ is an open affine subscheme of the smooth locus $\mathring{\mathfrak{X}}$ of $\mathfrak{X}$. Notice that such a germ is of the previous type. Finally, after Theorem 1.6.1 is proved we consider again (in §8.1) arbitrary germs of a smooth affine formal scheme over k°.

5.5.6. Remark. In the paper E. Grosse-Klönne as well as in those of P. Berthelot and Z. Mebkhout only the case $\mathrm{char}(\widetilde{k}) > 0$ is considered since the case $\mathrm{char}(\widetilde{k}) = 0$ is much easier and is believed to be known earlier.

§6. F-isocrystals

Beginning with this section the ground field k is assumed to be a closed subfield of $\mathbf{C}_p$. We apply constructions of the previous section to a wide germ of a smooth affine formal scheme $(X, \mathfrak{Z})$ which is a lifting of a similar germ defined over a finite extension of $\mathbf{Q}_p$. First of all, we consider the notions of a Frobenius lifting on the associated germ $(X, \mathfrak{Z}_\eta)$ and of a Frobenius structure on isocrystals over $B = \mathcal{O}(X, \mathfrak{Z}_\eta)$ (F-isocrystals). We provide the unipotent isocrystals $E^i(X, \mathfrak{Z}) = E^i_B$ with such a structure and, using a result of B. Chiarellotto [Chi], show that they possess the following nice property: any morphism of F-isocrystals $E^i(X, \mathfrak{Z}) \to M$ for which the induced map $H^1_{\mathrm{dR}}(E^i(X, \mathfrak{Z})) \to H^1_{\mathrm{dR}}(M)$ is zero can be extended in a unique way to a morphism of F-isocrystals $E^{i+1}(\mathfrak{X}, \mathfrak{Z}) \to M$. This property implies, for example, that the unipotent F-isocrystals $E^i(X, \mathfrak{Z})$ are unique up to a unique isomorphism, and is used to provide $E(X, \mathfrak{Z}) = \cup E^i(X, \mathfrak{Z})$ with a unique structure of a filtered B-algebra which satisfies the Leibniz rule and commutes with the Frobenius structure. One more application is as follows. Let $R^\lambda(\mathfrak{X}, \mathfrak{Z}_\eta)$ denote the filtered F-isocrystal for which $R^{\lambda,i}(\mathfrak{X}, \mathfrak{Z}_\eta)$ consists of all naive analytic functions f defined in an open neighborhood $\mathcal{U}$ of $\mathfrak{Z}_\eta$ such that $f\big|_{\mathcal{U}\cap\pi^{-1}(\mathbf{x})} \in L^{\lambda,i}(\mathcal{U} \cap \pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$. Then, if $\dim(\mathfrak{X}_\eta) = 1$, the canonical map $H^1_{\mathrm{dR}}(R^{\lambda,i}(\mathfrak{X}, \mathfrak{Z}_\eta)) \to H^1_{\mathrm{dR}}(R^{\lambda,i+1}(\mathfrak{X}, \mathfrak{Z}_\eta))$ is zero and, therefore, the canonical map $B \to R^{\lambda,0}(\mathfrak{X}, \mathfrak{Z}_\eta)$ extends in a unique way to a homomorphism of filtered B-algebras and F-isocrystals $E(X, \mathfrak{Z}) \to R^\lambda(\mathfrak{X}, \mathfrak{Z}_\eta)$. We show that, given a closed one-form with coefficients in $E^i(X, \mathfrak{Z})$ embedded in $R^{\lambda,i}(\mathfrak{X}, \mathfrak{Z}_\eta)$, its primitive in $E^{i+1}(X, \mathfrak{Z})$ embedded in $R^{\lambda,i+1}(\mathfrak{X}, \mathfrak{Z}_\eta)$ is precisely the one given by R. Coleman's construction. We also show that the induced homomorphism $E^K(X, \mathfrak{Z}) = E(X, \mathfrak{Z}) \otimes_k K \to R^\lambda(\mathfrak{X}, \mathfrak{Z}_\eta)$ is injective.

6.1. Frobenius liftings. Assume first that k is finite over $\mathbf{Q}_p$, and let $(X, \mathfrak{Z})$ be a wide germ of a smooth affine formal scheme over k°. We set $B = \mathcal{O}(X, \mathfrak{Z}_\eta)$ and $A^\dagger = A \cap B$, where $\mathfrak{Z} = \mathrm{Spf}(A)$. Furthermore, let $\widetilde{\phi}$ be the Frobenius endomorphism of $\mathfrak{Z}_s$ which corresponds to the endomorphism $\widetilde{\phi}^* : \widetilde{A} \to \widetilde{A}$ that takes α to α^q, where $\widetilde{A} = A/k^{\circ\circ}A$ and q is the number of elements in $\widetilde{k}$. It is well known (see, for example, [Col2]) that there is a lifting $\phi^* : A^\dagger \to A^\dagger$ of $\widetilde{\phi}^*$. It gives rise to a homomorphism $\phi^* : B \to B$, which induces the Frobenius automorphism of $\mathfrak{c}(B)$ over k. By Lemma 5.1.1(ii), ϕ^* is induced by a unique morphism of germs $\phi : (X, \mathfrak{Z}_\eta) \to (X, \mathfrak{Z}_\eta)$ which is called a *Frobenius lifting on* $(X, \mathfrak{Z}_\eta)$ *of degree* $[\widetilde{k} : \mathbf{F}_p]$. The latter is defined by a morphism $\phi : \mathcal{U} \to X$, where $\mathcal{U}$ is an open neighborhood of $\mathfrak{Z}_\eta$ in X. Notice that, given an open neighborhood $\mathcal{V}$ of $\mathfrak{Z}_\eta$ in X and $m \geq 1$, we can always shrink $\mathcal{U}$ so that $\phi^i(\mathcal{U}) \subset \mathcal{V}$ for $i \leq m$.

A Frobenius lifting ϕ on $(X, \mathfrak{Z}_\eta)$ induces a k-linear endomorphism ϕ^* on $H^i_{\mathrm{dR}}(X, \mathfrak{Z}_\eta) = H^i_{\mathrm{dR}}(B)$

which is also semi-linear with respect to the Frobenius automorphism of $\mathfrak{c}(B)$ over k. Recall that a Weil number of weight n (with respect to ϕ) is an algebraic number such that the absolute value of all of its conjugates in $\mathbf{C}$ are equal to $q^{\frac{n}{2}}$. By a result of B. Chiarellotto from [Chi], the eigenvalues of ϕ^* on $H^i_{\mathrm{dR}}(B)$ are Weil numbers of weights in the interval $[i, 2i]$.

6.1.1. Lemma. *Let $\mathfrak{X}$ be a special formal scheme over k°, which is the formal completion of a separated formal scheme of finite type over k° along an irreducible component of its closed fiber which is proper over $\widetilde{k}$, and let $\mathfrak{Z}$ is an open affine subscheme of the smooth locus $\mathring{\mathfrak{X}}$ of $\mathfrak{Z}$. Given a Frobenius lifting $\phi : (\mathfrak{X}_\eta, \mathfrak{Z}_\eta) \to (\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$ and an integer $n \geq 1$, there exists an open neighborhood $\mathcal{U}$ of $\mathfrak{Z}_\eta$ in $\mathfrak{X}_\eta$ at which all ϕ^i, $1 \leq i \leq n$, are defined and such that $\phi^i(\mathcal{U} \cap \pi^{-1}(\mathbf{x})) \subset \pi^{-1}(\mathbf{x})$ for all $1 \leq i \leq n$ and all closed points $\mathbf{x} \in \mathfrak{X}_s$.*

Proof. Let $\mathfrak{Z} = \mathrm{Spf}(A)$, $k^\circ\{T_1, \ldots, T_m\}^\dagger \to A : T_j \mapsto f_j$ an epimorphism, and $\mathcal{V}$ an open neighborhood of $\mathfrak{Z}_\eta$ at which all ϕ^i, $1 \leq i \leq n$, are defined. One has $|(\phi^{*^i} f_j - f_j^{q^i})(x)| < 1$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ and $x \in \mathfrak{Z}_\eta$, and we can find an open neighborhood $\mathcal{U}$ of $\mathfrak{Z}_\eta$ in $\mathcal{V}$ such that the above inequalities hold for all points $x \in \mathcal{U}$. It follows that $|(\phi^{*^i} f - f^{q^i})(x)| < 1$ for all $f \in A$, $1 \leq i \leq n$ and $x \in \mathcal{U}$. In particular, given $f \in A$ and $x \in \mathcal{U}$, $|f(x)| < 1$ if and only if $|f(\phi^i(x))| < 1$ for all $1 \leq i \leq n$, i.e., $\pi(\phi^i(x)) = \pi(x)$ for all $1 \leq i \leq n$ and $x \in \mathcal{U}$. The set $\mathcal{U}$ possesses the required property. ∎

Assume now that k is an arbitrary closed subfield of $\mathbf{C}_p$, and let $(X, \mathfrak{Z})$ be a wide germ of a smooth affine formal scheme over k° which is a lifting of a similar germ over a finite extension of $\mathbf{Q}_p$ in k. A *Frobenius lifting on* $(X, \mathfrak{Z}_\eta)$ is a morphism of germs $\phi : (X, \mathfrak{Z}_\eta) \to (X, \mathfrak{Z}_\eta)$ which is induced by a Frobenius lifting ϕ' on a germ $(X', \mathfrak{Z}')$ over a finite extension k' of $\mathbf{Q}_p$ in k whose lift to k is $(X, \mathfrak{Z})$. The *degree* $\deg(\phi)$ *of* ϕ is that of ϕ'. If $n = \deg(\phi)$, then $|(\phi^* f - f^{p^n})(x)| < |f|_{\sup}$ for all $f \in B = \mathcal{O}(X, \mathfrak{Z}_\eta)$ and $x \in \mathfrak{Z}_\eta$ (where $|f|_{\sup}$ is the supremum norm of f on $\mathfrak{Z}_\eta$).

The Frobenius lifting ϕ extends in a unique way to a Frobenius lifting on $(X, \mathfrak{Y}_\eta)$ for every nonempty open affine subscheme $\mathfrak{Y} \subset \mathfrak{Z}$ of the form $\mathfrak{Y}' \widehat{\otimes}_{k'^\circ} k^\circ$, where $\mathfrak{Y}'$ is a nonempty open affine subscheme of $\mathfrak{Z}'$. Notice that the intersection of all $\mathfrak{Y}_\eta$ is the generic point σ of $\mathfrak{Z}$. The Frobenius lifting ϕ gives rise to a morphism of germs $\phi : (X, \sigma) \to (X, \sigma)$, and the latter is defined by a homomorphism of local rings $\phi^* : \mathcal{O}_{X,\sigma} \to \mathcal{O}_{X,\sigma}$. A *Frobenius lifting at the point* σ is a morphism of germs $\phi : (X, \sigma) \to (X, \sigma)$ which is induced by a Frobenius lifting on $(X, \mathfrak{Y}_\eta)$ for a nonempty open affine subscheme $\mathfrak{Y} \subset \mathfrak{Z}$.

6.2. A Frobenius structure on the isocrystals $E^i(X, \mathfrak{Z})$. Let ϕ be a Frobenius lifting on $(X, \mathfrak{Z}_\eta)$ which comes from a triple $(k', X', \mathfrak{Z}')$ as at the end of the previous subsection. A *Frobenius* *structure on an isocrystal M over $B = \mathcal{O}(X, \mathfrak{Z}_\eta)$* is a ϕ^*-semi-linear homomorphism of B-modules $F : M \to M$ which commutes with ∇. The homomorphism F induces a k-linear endomorphism of the de Rham cohomology groups $H^i_{\mathrm{dR}}(M)$. An isocrystal provided with a Frobenius structure is called an *F-isocrystal* (see Remark 6.2.2). An *unipotent* (resp. *trivial*) *F-isocrystal over B* is a finite F-isocrystal which is unipotent (resp. trivial) as an isocrystal. Let $E^0(X, \mathfrak{Z}) = B \subset E^1(X, \mathfrak{Z}) \subset E^2(X, \mathfrak{Z}) \subset \dots$ be the increasing sequence of unipotent isocrystals over B, constructed in §5.5.

6.2.1. Lemma. *(i) The isocrystals $E^i(X, \mathfrak{Z})$ can be provided with a compatible system of Frobenius structures;*

(ii) the eigenvalues of F on $H^1_{\mathrm{dR}}(E^i(X, \mathfrak{Z}))$ are Weil numbers of weights in $[i+1, 2(i+1)]$.

Proof. (i) It suffices to show that, given an F-isocrystal M and a $\mathfrak{c}(B)$-vectors subspace $V \subset H^1_{\mathrm{dR}}(M)$ invariant under the action F, one can extend the Frobenius structure from M to M_V. Indeed, let $s : V \to (M \otimes_B \Omega^1_B)^{\mathrm{cl}}$ be a $\mathfrak{c}(B)$-linear section that defines M_V. Then $F(s(v)) - s(F(v)) \in \nabla(M)$ for all $v \in V$, and an extension of F can be defined by $F(v \otimes 1) = F(v) \otimes 1 + \mu(v)$, where $\mu : V \to M$ is a $\mathfrak{c}(B)$-linear map with $F(s(v)) - s(F(v)) = \nabla(\mu(v))$ for all $v \in V$.

(ii) The statement is deduced, by induction, from the result of B. Chiarellotto, cited above, and the fact that $H^1_{\mathrm{dR}}(E^i) \subset H^1_{\mathrm{dR}}(E^i/E^{i-1}) = H^1_{\mathrm{dR}}(E^{i-1}) \otimes_{\mathfrak{c}(B)} H^1_{\mathrm{dR}}(B)$, where $E^i = E^i(X, \mathfrak{Z})$. ∎

6.2.2. Remark. In the usual definition of a (finite) F-isocrystal one requires that the induced map $F : \phi^* M = M \otimes_{B, \phi^*} B \to M$ is bijective. All finite F-isocrystals we consider satisfy that condition.

6.3. A uniqueness property of certain F-isocrystals. Let $M \subset M'$ be F-isocrystals over B such that the quotient F-isocrystal M'/M is trivial, and let $V = \mathrm{Ker}(H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(M'))$. Assume that the eigenvalues of F on $H^0_{\mathrm{dR}}(M'/M)$ are not roots of unity (and, in particular, the same is true for the eigenvalues of F on V).

6.3.1. Proposition. *Let N be an F-isocrystal with $H^0_{\mathrm{dR}}(N) = \varprojlim H^0_{\mathrm{dR}}(N)/W_\nu$, where $\{W_\nu\}$ is a filtered family of F-invariant $\mathfrak{c}(B)$-vector subspaces such that the action of F on each quotient is of finite order. Then any homomorphism of F-isocrystals $\varphi : M \to N$ with $V \subset \mathrm{Ker}(H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(N))$ can be extended in a unique way to a homomorphism of F-isocrystals $\varphi' : M' \to N$.*

Proof. We set $K = \mathfrak{c}(B)$ and denote by g the Frobenius automorphism of K over k. As a B-module, M' can be identified with the direct sum $M \oplus (U \otimes_K B)$, where $U = H^0_{\mathrm{dR}}(M'/M)$. One has $\nabla(u \otimes 1) \in (M \otimes_B \Omega^1_B)^{\mathrm{cl}}$ and $F(u \otimes 1) = F(u) \otimes 1 + \alpha(u)$ for all $u \in U$, where $\alpha : U \to M$ is a g-semi-linear map. By the assumption, there is a K-linear map $\beta : U \to N$ such that $\varphi(\nabla(u\otimes 1)) = \nabla(\beta(u))$ for all $u \in U$. It follows that

$$\nabla(F(\beta(u)) = F(\varphi(\nabla(u \otimes 1))) = \nabla(\varphi(F(u \otimes 1)))$$
$$= \nabla(\varphi(F(u) \otimes 1 + \alpha(u))) = \nabla(\beta(F(u)) + \varphi(\alpha(u))) \ ,$$

and so the formula $(F \circ \beta)(u) = (\beta \circ F)(u) + (\varphi \circ \alpha)(u) + A(u)$ defines a g-semi-linear map $A : U \to H^0_{\mathrm{dR}}(N)$. If an extension $\varphi' : M' \to N$ exists, it must provide a g-semi-linear map $B : U \to H^0_{\mathrm{dR}}(N)$ with $B(u) = \varphi'(u \otimes 1) - \beta(u)$ for all $u \in U$, and the necessary and sufficient condition for its existence is the validity of the equality $\varphi'(F(u \otimes 1)) = F(\varphi'(u \otimes 1))$ for all $u \in U$, which is equivalent to the equality $B \circ F - F \circ B = A$. Thus, if we set $W = H^0_{\mathrm{dR}}(N)$ and denote by $\mathrm{Hom}_K(U, W)$ and $\mathrm{Hom}_g(U, W)$ the K-vector spaces of K-linear and of g-semi-linear maps $U \to W$, respectively, the required statement follows from the following lemma. (The first case of the lemma will be used later.)

6.3.2. Lemma. *Let K be a finite extension of k, g an automorphism of K over k, W a K-vector space provided with a g-semi-linear operator F such that $W = \varprojlim W/W_\nu$, where $\{W_\nu\}$ is a filtered family of F-invariant K-vector subspaces, and the action of F on each quotient W/W_ν is of finite order. Then*

(i) given a polynomial $P(T) \in k[T]$ with no roots-of-unity roots, the k-linear operator $P(F) : W \to W$ is invertible;

(ii) given a finitely dimensional K-vector space U provided with a g-semi-linear operator G such that its eigenvalues considered as a k-linear operator are not roots of unity, the k-linear operator $\mathrm{Hom}_K(U, W) \to \mathrm{Hom}_g(U, W) : B \mapsto F \circ B - B \circ G$ is bijective.

Proof. In both cases, it suffices to prove the statements for each quotient W/W_ν instead of W, and so we may assume that the action of F on W is of finite order. Furthermore, the space W is a union $\cup_{i\in I} W_i$ of a filtered family of finitely dimensional K-vector subspaces W_i invariant under the action of F. Since U is of finite dimension over K, it suffices to prove the statements for each of the subspaces W_i instead of W. Thus, we may assume that $\dim_K(W) < \infty$ and, therefore, it suffices to prove that the k-linear operators considered are injective. Let m be a positive integer with $F^m = 1_W$ (and, in particular, $g^m = 1$).

(i) By the assumption, there exist $Q(T), R(T) \in k[T]$ with $Q(T)P(T) + R(T)(T^m - 1) = 1$. If we substitute F instead of T, we get $Q(F) \circ P(F) = 1_W$, and the required fact follows.

(ii) For every $n \geq 1$, one has $F^nB - BG^n = F^{n-1}(FB - BG) + (F^{n-1}B - BG^{n-1})G$. Thus, if $FB - BG = 0$, it follows by induction that $F^nB - BG^n = 0$ for all $n \geq 1$. Since $F^m = 1_W$, we get $B(1_W - G^m) = 0$. By the assumption, the operator $1_W - G^m$ is invertible and, therefore, $B = 0$. ∎

We now return to the situation of §6.2, and let us fix a Frobenius lifting ϕ on $(X, \mathfrak{Z}_\eta)$.

6.3.3. Corollary. *Given two Frobenius structure F and F' on $E(X, \mathfrak{Z})$, there exists a unique isomorphism of filtered F-isocrystals $(E(X, \mathfrak{Z}), F) \xrightarrow{\sim} (E(X, \mathfrak{Z}), F')$.* ∎

Furthermore, let $\mathfrak{Y}$ be a nonempty open affine subscheme of $\mathfrak{Z}$ as at the end of §6.1. We extend ϕ to a Frobenius lifting on $(X, \mathfrak{Y}_\eta)$ and provide the isocrystals $E^i(X, \mathfrak{Y})$ with a Frobenius structure with respect to that Frobenius lifting. Let j denote the canonical morphism of germs $(X, \mathfrak{Y}_\eta) \to (X, \mathfrak{Z}_\eta)$.

6.3.4. Corollary. *There is a unique system of j-homomorphisms of isocrystals $E^i(X, \mathfrak{Z}) \to E^i(X, \mathfrak{Y})$ that commute with the Frobenius structures.* ∎

6.4. Structure of a commutative filtered D_B-algebra on $E(X, \mathfrak{Z})$.

6.4.1. Proposition. *The F-isocrystal $E(X, \mathfrak{Z})$ can be provided with a unique structure of a commutative filtered D_B-algebra which is compatible with the action of F.*

To prove the proposition, we need the following lemma in which $E^n = E^n(X, \mathfrak{Z})$.

6.4.2. Lemma. *Let $N' \subset N$ be F-isocrystals with N satisfying the assumptions of Proposition 6.3.1 and such that the map $H^1_{\mathrm{dR}}(N') \to H^1_{\mathrm{dR}}(N)$ is zero. Then any pair of homomorphisms of F-isocrystals $E^m \otimes_B E^{n-1} \to N'$ and $E^{m-1} \otimes_B E^n \to N'$, which coincide on $E^{m-1} \otimes_B E^{n-1}$, can be uniquely extended to a homomorphism of F-isocrystals $E^m \otimes_B E^n \to N$.*

Proof. We set $M' = E^m \otimes_B E^n$ and denote by M the sub-F-isocrystal of M' which is the sum of $E^m \otimes_B E^{n-1}$ and $E^{m-1} \otimes_B E^n$ in M'. The quotient F-isocrystal M'/M is isomorphic to the tensor product $E^m/E^{m-1} \otimes_B E^n/E^{n-1}$ and, therefore, it is trivial and, by Lemma 6.2.1(ii), the eigenvalues of F on $H^0_{\mathrm{dR}}(M'/M)$ are not roots of unity. Since the induced homomorphism of F-isocrystals $M \to N$ goes through a homomorphism $M \to N'$, the assumption implies that the map $H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(N)$ is zero, and the required fact follows from Proposition 6.3.1. ∎

Proof of Proposition 6.4.1. We set $E^i = E^i(X, \mathfrak{Z})$. Since the canonical maps $H^1_{\mathrm{dR}}(E^i) \to H^1_{\mathrm{dR}}(E^{i+1})$ are zero, we can apply Lemma 6.4.2 and construct, by induction, a unique system of compatible homomorphisms of F-isocrystals $E^m \otimes_B E^n \to E^{m+n} : f \otimes g \mapsto fg$. The uniqueness implies that this operation is commutative. That it is associative is proved in the similar way by induction. Namely, to show that the following diagram is commutative

$$\begin{array}{ccc} E^m \otimes_B E^n \otimes_B E^l & \longrightarrow & E^{m+n} \otimes_B E^l \\ \downarrow & & \downarrow \\ E^m \otimes_B E^{n+l} & \longrightarrow & E^{m+n+l} \end{array}$$

one assumes that the corresponding diagrams for the triples $(m, n, l-1)$, $(m, n-1, l)$ and $(m-1, n, l)$ are commutative, and applies Proposition 6.3.1 to $M' = E^m \otimes_B E^n \otimes_B E^l$ and the sub-F-isocrystal M which is the sum of $E^m \otimes_B E^n \otimes_B E^{l-1}$, $E^m \otimes_B E^{n-1} \otimes_B E^l$ and $E^{m-1} \otimes_B E^n \otimes_B E^l$ in M'. (Notice that the quotient M'/M is canonically isomorphic to the tensor product $E^m/E^{m-1} \otimes_B E^n/E^{n-1} \otimes_B E^l/E^{l-1}$.) Thus, E is provided with a unique structure of a commutative filtered B-algebra on $E = E(X, \mathfrak{Z})$ which satisfies the Leibniz rule and is compatible with the action of Frobenius. ∎

6.4.3. Corollary. *If* $\dim(X) = 1$, *there is an isomorphism of filtered* D*-algebras* $E(X, \mathfrak{Z}) \xrightarrow{\sim} \mathrm{Sh}_B$ *which is compatible with the isomorphism of Lemma 5.4.4(iii).*

Proof. Let ϕ be a Frobenius lifting on $(X, \mathfrak{Z}_\eta)$. *We claim that the isocrystals* Sh_B^n *admit a system of compatible Frobenius structures, which is unique up to a unique automorphism.* Indeed, the claim is trivially true for $n = 0$, and so assume $n \geq 1$ and that Sh_B^{n-1} is already provided with a Frobenius structure. If $\{v_1, \dots, v_m\}$ is a basis of the k-vector space $H^1_{\mathrm{dR}}(B)$, then to construct F, it suffices to define the images of the elements $v_{i_1,\dots,i_n} = v_{i_1} \otimes \dots \otimes v_{i_n}$. Since $\nabla(v_{i_1,\dots,i_n}) = v_{i_1,\dots,i_{n-1}} \otimes s(v_{i_n}) \in \mathrm{Sh}_B^{n-1}$, an element $F(\nabla(v_{i_1,\dots,i_n}))$ of $\mathrm{Sh}_B^{n-1} \otimes_B \Omega^1_B$ is defined and, by Lemma 5.4.4(i), it is equal to $\nabla(w)$ for some $w \in \mathrm{Sh}_B^n$, and we define $F(v_{i_1,\dots,i_n}) = w$. The ϕ^*-semi-linear map F extended to Sh_B^n commutes with ∇ and, therefore, it is a required one. The uniqueness of F follows from Proposition 6.3.1 and the fact that the eigenvalues of F on $H^1_{\mathrm{dR}}(\mathrm{Sh}_B^n) = H^1_{\mathrm{dR}}(B)^{\otimes(n+1)}$ are Weil numbers of weights at least $n+1$.

Thus, the identity map $E^0(X, \mathfrak{Z}) = B \to \mathrm{Sh}_B^0 = B$ extends in a unique way to an injective homomorphism of filtered F-isocrystals and B-algebras $E(X, \mathfrak{Z}) \xrightarrow{\sim} \mathrm{Sh}_B$. Assume that for some $n \geq 1$ the map $E^{n-1}(X, \mathfrak{Z}) \to \mathrm{Sh}_B^{n-1}$ is an isomorphism. Then it induces an isomorphism between their first de Rham cohomology groups. Since $\mathrm{Gr}^n(E(X, \mathfrak{Z})) = H^1_{\mathrm{dR}}(E^{n-1}(X, \mathfrak{Z})) \otimes_k B$ and $\mathrm{Gr}^n(\mathrm{Sh}_B) = H^1_{\mathrm{dR}}(\mathrm{Sh}_B^{n-1}) \otimes_k B$, it follows that $E^n(X, \mathfrak{Z}) \to \mathrm{Sh}_B^n$. That it is compatible with the isomorphism of Lemma 4.4.4(iii) is easy. ∎

6.5. Filtered F-isocrystals $E^K(X, \mathfrak{Z})$ and $\mathcal{F}^\lambda(\mathfrak{X}, \mathfrak{Z})$. Let K be a commutative filtered k-algebra. We denote by $E^K(X, \mathfrak{Z})$ the filtered F-isocrystal $E(X, \mathfrak{Z}) \otimes_k K$. It is a commutative filtered B-algebra with $\mathrm{Gr}^{\cdot}(E^K(X,\mathfrak{Z})) = \mathrm{Gr}^{\cdot}(E(X,\mathfrak{Z})) \otimes_k \mathrm{Gr}^{\cdot}(K)$ and, given a homomorphism of filtered k-algebras $K \to K'$, there is the evident isomorphism of filtered B-algebras and of isocrystals $E^K(X,\mathfrak{Z}) \otimes_K K' \stackrel{\sim}{\to} E^{K'}(X,\mathfrak{Z})$. Notice that $H^0_{\mathrm{dR}}(E^{K,i}(X,\mathfrak{Z})) = \mathfrak{c}(B) \otimes_k K^i = \mathcal{C}^{K,i}(X,\mathfrak{Z}_\eta)$ for all $i \geq 0$. If the algebra K possesses the property $\dim_k(K^i) < \infty$ for all $i \geq 0$, then each $E^{K,i}(X,\mathfrak{Z})$ is a unipotent F-isocrystal isomorphic to the direct sum $\oplus_{j=0}^i (E^j(X,\mathfrak{Z}) \otimes_k K^{i-j}/K^{i-j-1})$ and, for $i \geq 1$, and the quotient $E^{K,i}(X,\mathfrak{Z})/E^{K,i-1}(X,\mathfrak{Z})$ is isomorphic to the trivial F-isocrystal $\oplus_{j=0}^i (E^j(X,\mathfrak{Z})/E^{j-1}(X,\mathfrak{Z}) \otimes K^{i-j}/K^{i-j-1})$ with $K^{-1} = 0$ and $E^{-1}(X,\mathfrak{Z}) = 0$. If K is arbitrary, it is a union of filtered subalgebras with the above property (e.g., of finitely generated k-subalgebras), and so each $E^{K,i}(X,\mathfrak{Z})$ is a union of unipotent sub-F-isocrystals.

Let now $\mathfrak{X}$ be a proper marked formal scheme over k°, $\mathfrak{Z}$ an open affine subscheme of $\mathring{\mathfrak{X}}$, and λ an element of K^1. Given an open neighborhood $\mathcal{U}$ of $\sigma = \sigma_{\mathfrak{X}}$, let $R^{\lambda,i}(\mathfrak{X},\mathcal{U})$ denote the set of all functions $f \in \mathfrak{N}^{K,i}(\mathcal{U})$ such that $f_{\mathbf{x}} \in L^{\lambda,i}(\mathcal{U} \cap \pi^{-1}(\mathbf{x}))$ for every closed point $\mathbf{x} \in \mathfrak{X}_s$, and let $R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)$ denote the inductive limit of $R^{\lambda,i}(\mathfrak{X},\mathcal{U})$ taken over all open neighborhoods of $\mathfrak{Z}_\eta$ in $\mathfrak{X}_\eta$. We also denote by $R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta)$ the similar inductive limit of $L^{\lambda,i}(\mathcal{U} \cap \pi^{-1}(\mathbf{x}))$. Lemma 6.1.1 implies that both $R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)$ and $R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta)$ are F-isocrystals over B. Notice that there is a canonical projection $R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta) \to R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta)$ and, if $\mathbf{x} \in \mathfrak{Z}_s$, then $R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta) = L^{\lambda,i}(\pi^{-1}(\mathbf{x})) = \mathcal{O}(\pi^{-1}(\mathbf{x})) \otimes_k K^i$.

Assume in addition that $\dim(\mathfrak{X}_\eta) = 1$. In this case $\mathfrak{X}_\eta$ is isomorphic to a smooth basic curve $\mathcal{X}^{\mathrm{an}}_\eta \backslash \cup_{i=1}^m X_i^{r_i}$, $m \geq 0$, where $\mathcal{X}$ is a connected projective smooth curve over k°, $X_i^{r_i}$ are affinoid subdomains lying in pairwise different residue classes of closed points $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathcal{X}_s = \mathfrak{X}_s$ and, if an isomorphism of each residue class $\pi^{-1}(\mathbf{x}_i)$ with an open disc of radius one with center at zero over $k_{\mathbf{x}_i}$ is fixed, then each $X_i^{r_i}$ is isomorphic to a closed subdisc of radius $r_i \in |k^*|$, $r_i < 1$, with center at zero. If $\mathfrak{X}_s \backslash \mathfrak{Z}_s = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, $n \geq \max(m,1)$, then a fundamental system of open neighborhoods of $\mathfrak{Z}_\eta$ in $\mathfrak{X}_\eta$ is formed by sets of the form $\mathcal{X}^{\mathrm{an}}_\eta \backslash \cup_{i=1}^n X_i^{t_i}$ with $X_i^{t_i} \subset \pi^{-1}(\mathbf{x}_i)$ having the same meaning and $r_i \leq t_i < 1$. Since each open subset $\pi^{-1}(\mathbf{x}_i) \backslash X_i^{t_i}$ is isomorphic to an open annulus over $k_{\mathbf{x}_i}$, it follows the homomorphisms $H^1_{\mathrm{dR}}(R^{\lambda,n}(\mathfrak{X},\mathfrak{Z}_\eta)) \to H^1_{\mathrm{dR}}(R^{\lambda,n+1}(\mathfrak{X},\mathfrak{Z}_\eta))$ are zero. One also has $H^0_{\mathrm{dR}}(R^{\lambda,n}(\mathfrak{X},\mathfrak{Z}_\eta)) = \mathcal{C}^{K,n}_R(\mathfrak{X},\mathfrak{Z}_\eta)$. Thus, the assumptions of Proposition 6.3.1 are satisfied and, therefore, the canonical injective homomorphism $E^0(\mathfrak{X}_\eta,\mathfrak{Z}) = \mathcal{O}(\mathfrak{X}_\eta,\mathfrak{Z}_\eta) \to R^{\lambda,0}(\mathfrak{X},\mathfrak{Z}_\eta)$ extends in a unique way to a compatible system of injective homomorphisms of F-isocrystals $E^i(\mathfrak{X}_\eta,\mathfrak{Z}) \to R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)$, $i \geq 0$, which, by Lemma 6.4.2, takes products to products. The latter extends by linearity to a homomorphism of filtered F-isocrystals $E^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z}) \to R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)$, $i \geq 0$, which also takes products to products. In the same way one constructs, for every closed point $\mathbf{x} \in \mathfrak{X}_s$, a homomorphism of F-isocrystals $E^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z}) \to R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta)$ which coincides in fact with the composition of the latter homomorphism with the canonical projection $R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)\to R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta)$.

6.5.1. Lemma. *In the above situation, the homomorphisms $E^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z})\to R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta)$ are injective for all closed points $\mathbf{x}\in\mathfrak{X}_s$ and, in particular, the homomorphism $E^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z})\to R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)$ is injective.*

Proof. The statement is evidently true for $i=0$, and so assume that $i\ge 1$ and that it is true for $i-1$. Let us write E^i and $E^{K,i}$ instead of $E^i(\mathfrak{X}_\eta,\mathfrak{Z})$ and $E^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z})$, respectively, and let $\widetilde{E}$ denote the B-submodule of $E^{K,i}$ generated by $E^{K,i-1}$ and $E^0\otimes_k K^i$. It is a sub-F-isocrystal of $E^{K,i}$. Notice that the quotient $E^{K,i}/\widetilde{E}$ is isomorphic to the F-isocrystal $\oplus_{j=1}^i(E^j/E^{j-1}\otimes_k K^{i-j}/K^{i-j-1})$, and one has $H^0_{\mathrm{dR}}(\widetilde{E})\xrightarrow{\sim}H^0_{\mathrm{dR}}(E^{K,i})$. Thus, to prove the statement it suffices to verify the injectivity of the induced homomorphism $\widetilde{E}\to R^{\lambda,i}_{\mathbf{x}}(\mathfrak{X},\mathfrak{Z}_\eta)$. By the induction hypothesis, the latter follows from the trivial fact that, for an open annulus with center at zero Y, the intersection of $\mathcal{O}(Y)\otimes_k K^i$ and $L^{\lambda,i-1}(Y)$ in $L^{\lambda,i}(Y)$ coincides with $\mathcal{O}(Y)\otimes_k K^{i-1}$. ∎

We denote by $\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})$ the image of $E^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z})$ in $R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)$ and by $\mathcal{F}^{\lambda,i}_{\mathfrak{X},\sigma}$ the union of the images of $\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})$ in $\mathfrak{N}^{K,i}_{\mathfrak{X}_\eta,\sigma}$ taken over all nonempty open affine subschemes $\mathfrak{Z}$ of $\mathring{\mathfrak{X}}$ (see Corollary 6.3.4). We now describe a primitive of a one-form $\omega\in\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})\otimes_B\Omega^1_B$ in $\mathcal{F}^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z})$. Recall that, by Lemma 6.2.1(ii), there exists a polynomial $P(T)\in k[T]$, whose roots are Weil numbers of weights ≥ 1, with $P(\phi^*)\omega\in d(\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z}))$.

6.5.2. Lemma. *(i) There exists a function $f\in R^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$, unique up to an element of $\mathcal{C}^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z}_\eta)$, such that $df=\omega$ and, for every polynomial $P(T)\in k[T]$ with no roots-of-unity roots and $P(\phi^*)\omega\in d(\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z}))$, one has $P(\phi^*)f\in\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})$;*

(ii) the function f is contained in $\mathcal{F}^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z})$.

Proof. (i) We know that a function $f'\in R^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$ with $df'=\omega$ exists. Let $P(T)\in k[T]$ be a polynomial with no roots-of-unity roots such that $P(\phi^*)\omega=dg$ for some $g\in\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})$. The function $h=P(\phi^*)f'-g$ lies in $R^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$ and, since $dh=0$, it follows that $h\in\mathcal{C}^{K,i+1}_R(\mathfrak{X},\mathfrak{Z}_\eta)$. By Lemma 6.3.2(i), the map $P(\phi^*):\mathcal{C}^{K,i+1}_R(\mathfrak{X},\mathfrak{Z}_\eta)\to\mathcal{C}^{K,i+1}_R(\mathfrak{X},\mathfrak{Z}_\eta)$ is a bijection. We can therefore find an element $\alpha\in\mathcal{C}^{K,i+1}_R(\mathfrak{X},\mathfrak{Z}_\eta)$ with $P(\phi^*)f'-g=P(\phi^*)\alpha$. The function $f=f'-\alpha$ is contained in $R^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$ and satisfies the required properties. The bijectivity of the above map implies that f is uniquely defined by the element g and the polynomial P.

We claim that the function f does not depend on the choice of g and P up to an element of $\mathcal{C}^{\lambda,i}(\mathfrak{X}_\eta,\mathfrak{Z}_\eta)$. Indeed, let g' be another element of $\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})$ with $P(\phi^*)\omega=dg'$, and let f' be the function from $R^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$ with $df'=\omega$ and $P(\phi^*)f'=g'$. Since $d(g'-g)=0$, it follows that $g'-g\in\mathcal{C}^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z}_\eta)$. On the other hand, $f'-f\in\mathcal{C}^{K,i+1}_R(\mathfrak{X},\mathfrak{Z}_\eta)$ and $P(\phi^*)(f'-f)=g'-g$. Since the actions of $P(\phi^*)$ on $\mathcal{C}_R^{K,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$ and $\mathcal{C}^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z}_\eta)$ are bijective, it follows that $f'-f\in\mathcal{C}^{K,i}(\mathfrak{X}_\eta,\mathfrak{Z}_\eta)$. Furthermore, let $Q(T)$ be the monic polynomial of minimal degree with $Q(\phi^*)\omega\in d(\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z}))$. It is clear that $Q(T)$ divides $P(T)$, and so let $P(T)=Q(T)Q'(T)$, and let $Q(\phi^*)\omega=dg'$ for some $g'\in\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})$. We have $P(\phi^*)\omega=d(Q'(\phi^*)g')$. If f' is the function from $R^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$ with $df'=\omega$ and $Q(\phi^*)f'=g'$, then $P(\phi^*)f'=Q'(\phi^*)g'$, and the independence on P follows from that on g.

(ii) We know that a function $f'\in\mathcal{F}^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z})$ with $df'=\omega$ exists and that $f'\in R^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$. If $P(\phi^*)\omega=dg$ with P and g from (i), then $P(\phi^*)f'-g\in\mathcal{F}^{\lambda,i+1}(\mathfrak{X},\mathfrak{Z})$ and $d(P(\phi^*)f'-g)=0$, i.e., $P(\phi^*)(f'-f)\in\mathcal{C}^{K,i+1}(\mathfrak{X},\mathfrak{Z}_\eta)$. Since the action of $P(\phi^*)$ on the latter space is bijective, the required fact follows. ∎

6.5.3. Remark. The construction of Lemma 6.5.2 is due to R. Coleman. His methods will be also used to generalize it to higher dimensions and to show, in particular, that the sub-isocrystals $\mathcal{F}^{\lambda,i}(\mathfrak{X},\mathfrak{Z})\subset R^{\lambda,i}(\mathfrak{X},\mathfrak{Z}_\eta)$ do not depend on the choice of the Frobenius lifting ϕ.

§7. Construction of the sheaves $\mathcal{S}_X^\lambda$

The construction of the sheaves $\mathcal{S}_X^{\lambda,n}$ is carried out by the double induction on $m = \dim(X)$ and the number n. Assume that the sheaves $\mathcal{S}_X^{\lambda,i}$ with all required properties exist for all pairs (i, X) with $0 \le i \le n$, or $i = n+1$ and $\dim(X) \le m$. In order to construct the sheaves $\mathcal{S}_X^{\lambda,n+1}$ for X of dimension $m+1$, we have to construct a primitive of a closed one-form ω with coefficients in $\mathcal{S}_X^{\lambda,n}$ in an open neighborhood of every point of X. For this we make additional induction hypotheses that describe the form of a function $f \in \mathcal{S}^{\lambda,i}(X)$ in an étale neighborhood of a point $x \in X$. If $s(x) < \dim(X)$ then, given a smooth morphism $\varphi : X \to Y$ to a smooth k-analytic space Y of dimension $\dim(X)-1$ such that $s(y) = s(x)$ for $y = \varphi(y)$, there exists an Y-split étale neighborhood $\psi : X' \to X$ of x over an étale morphism $Y' \to Y$ such that $\psi^*(f)$ is contained in the $\mathcal{O}(X')$-module generated by functions of the form $g\mathrm{Log}^\lambda(h)^j$ with $g \in \mathcal{S}^{\lambda,i-j}(Y')$ and $h \in \mathcal{O}(X')^*$, and a primitive of ω at such a point x is easily found among functions of a similar form. If $s(x) = \dim(X)$, there exists a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ such that $\varphi^*(f)\big|_{\pi^{-1}(\mathbf{x})} \in L^{\lambda,i}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$, and, for some open affine subscheme $\mathfrak{Z} \subset \mathring{\mathfrak{X}}$, the restriction $\varphi^*(f)\big|_{(\mathfrak{X}_\eta,\mathfrak{Z}_\eta)}$ is contained in the image $\mathcal{G}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z})$ of an injective morphism of F-isocrystals $E^{K,i}(\mathfrak{X}_\eta, \mathfrak{Z}) \to \mathcal{S}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$ provided by the construction of §6. In fact, there is one more technical assumption as well as the hypothesis that, if $\dim(\mathfrak{X}_\eta) = 1$, $\mathcal{G}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z})$ coincides with the image of $E^{K,i}(\mathfrak{X}_\eta, \mathfrak{Z})$ under the homomorphism constructed at the end of §6. The induction hypotheses reduce the problem of constructing a primitive of ω at x to that of constructing a primitive of a closed one-form which is defined on $\mathfrak{X}_\eta$ and possesses certain properties. The latter is done in a way which is a generalization of R. Coleman's construction; the miraculous fact is that the primitive does not depend on all of the objects used in its construction (marked neighborhoods, Frobenius liftings and so on). The proof is based on the geometrical properties of analytic spaces established in §2 and §3 and in [Ber7] and [Ber9].

7.1. Induction hypotheses. We construct the $\mathcal{D}_X$-submodules $\mathcal{S}_X^{\lambda,n}$ of $\mathfrak{N}_X^{K,n}$ by the double induction on $n \ge 0$ and $m = \dim(X) \ge 0$ (and then define the filtered $\mathcal{D}_X$-subalgebra $\mathcal{S}_X^\lambda$ of $\mathfrak{N}_X^K$ as the inductive limit of all $\mathcal{S}_X^{\lambda,n}$). Notice that it is enough to consider only pure dimensional smooth k-analytic spaces, and we assume that all the spaces considered are such ones.

First of all, if $n = 0$ then $\mathcal{S}_X^{\lambda,0} = \mathcal{O}_X^{K,0}$, and if $m = 0$ then $\mathcal{S}_X^{\lambda,n} = \mathcal{O}_X^{K,n}$. Given $m, n \ge 0$, assume that the $\mathcal{D}_X$-modules $\mathcal{S}_X^{\lambda,i}$, with the induction hypotheses we are going to specify, are already constructed for all pairs (i, X) with $0 \le i \le n$, or $i = n+1$ and $\dim(X) \le m$, and we will construct the $\mathcal{D}_X$-modules $\mathcal{S}_X^{\lambda,n+1}$ with $\dim(X) = m+1$. The induction hypotheses include the properties of Theorems 1.6.1 (with the same condition on (i, X') in the property (f)); the following property

(g) $\mathcal{S}_X^{\lambda,i} \cdot \mathcal{S}_X^{\lambda,j} \subset \mathcal{S}_X^{\lambda,i+j}$ for all (i, j, X) with $i + j \leq n$, or $i + j = n + 1$ and $\dim(X) \leq m$;

the hypothesis on the uniqueness of the sheaves $\mathcal{S}_X^{\lambda,i}$ with the above properties; the properties (i)-(iii) of Theorem 1.6.2, and additional hypotheses (IH1)-(IH3). Before formulating them we make a preliminary observation.

Let $\Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i},X}$ denote the subsheaf of closed one-forms $\mathrm{Ker}(\Omega^1_{\mathcal{S}^{\lambda,i},X} \xrightarrow{d} \Omega^2_{\mathcal{S}^{\lambda,i},X})$, where $0 \leq i \leq n$ or $i = n + 1$ and $\dim(X) \leq m$. Notice that if $0 \leq i \leq n - 1$ or $i = n$ and $\dim(X) \leq m$ then, by the property (c), $\Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i},X} \subset d\mathcal{S}_X^{\lambda,i+1}$. Let $\mathcal{P}_X^{\lambda,i+1}$ denote the preimage of $\Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i},X}$ in $\mathcal{S}_X^{\lambda,i+1}$. By the property (d), $\mathcal{S}_X^{\lambda,i+1}$ is generated over $\mathcal{O}_X$ by local sections of $\mathcal{P}_X^{\lambda,i+1}$. We have an exact sequence

$$0 \longrightarrow \mathcal{C}_X^{K,i+1} \longrightarrow \mathcal{P}_X^{\lambda,i+1} \xrightarrow{d} \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i},X} \longrightarrow 0\ .$$

It follows that, if $H^1(X, \mathfrak{c}_X) = 0$, then every closed one-form $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(X)$ has a primitive in $\mathcal{P}^{\lambda,i+1}(X)$ which is defined uniquely up to an element of $\mathcal{C}^{K,i+1}(X)$. Similarly, if Y is a strictly affinoid domain in X with $H^1(Y, \mathfrak{c}_Y) = 0$, then every closed one-form $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(X, Y)$ on the germ (X, Y) has a primitive in $\mathcal{P}^{\lambda,i+1}(X, Y)$.

Suppose now we are given a wide germ of a smooth affine formal scheme $(X, \mathfrak{Z})$ such that, if $i = n$, then $\dim(X) \leq m + 1$, and let $B = \mathcal{O}(X, \mathfrak{Z}_\eta)$. Since $\mathfrak{Z}$ is smooth over k°, all of the connected component of $\mathfrak{Z}_\eta \widehat{\otimes}_k k'$ are contractible for any non-Archimedean field k' over k and, therefore, $H^q(\mathfrak{Z}_\eta, \mathfrak{c}_{\mathfrak{Z}_\eta}) = 0$ for all $q \geq 1$ (see [Ber9, §8]). By the above remark, if $0 \leq i \leq n - 1$, or $i = n$ and $\dim(X) \leq m$, every closed one-form $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(X, \mathfrak{Z}_\eta)$ has a primitive in $\mathcal{P}^{\lambda,i+1}(X, \mathfrak{Z}_\eta) \subset \mathcal{S}^{\lambda,i+1}(X, \mathfrak{Z}_\eta)$. Thus, if the germ $(X, \mathfrak{Z})$ is a lifting of a similar germ over a finite extension of $\mathbf{Q}_p$ in k, then by the construction of §6.3, given a Frobenius lifting ϕ on $(X, \mathfrak{Z}_\eta)$, there is a unique system of injective homomorphisms of F-isocrystals $E^i(X, \mathfrak{Z}) \to \mathcal{S}^{\lambda,i}(X, \mathfrak{Z}_\eta)$ for $0 \leq i \leq n$, or $i = n+1$ and $\dim(X) \leq m$. Lemma 6.4.2 implies that these homomorphisms take the product $f \cdot g \in E^{i+j}(X, \mathfrak{Z})$ of two elements $f \in E^i(X, \mathfrak{Z})$ and $g \in E^j(X, \mathfrak{Z})$ to the product of the corresponding functions in $\mathcal{S}^{\lambda,i+j}(X, \mathfrak{Z}_\eta)$ for $i + j \leq n$, or $i + j = n + 1$ and $\dim(X) \leq m$. The above homomorphisms give rise to a homomorphism from the F-isocrystal $E^{K,i}(X, \mathfrak{Z})$, introduced in §6.5, to $\mathcal{S}^{\lambda,i}(X, \mathfrak{Z}_\eta)$.

7.1.1. Lemma. *The homomorphism of F-isocrystals $E^{K,i}(X, \mathfrak{Z}) \to \mathcal{S}^{\lambda,i}(X, \mathfrak{Z}_\eta)$ is injective.*

Proof. The statement is proved in the same way as Lemma 6.5.1. Namely, it is evidently true for $i = 0$. Assume it is true for $i - 1$ with $i \geq 1$, and let us write E^i and $E^{K,i}$ instead of $E^i(X, \mathfrak{Z})$ and $E^{K,i}(X, \mathfrak{Z})$, respectively. Let $\widetilde{E}$ denote the B-submodule of $E^{K,i}$ generated by $E^{K,i-1}$ and

$E^0 \otimes_k K^i$. It is a sub-F-isocrystal of $E^{K,i}$. Notice that the quotient $E^{K,i}/\widetilde{E}$ is isomorphic to the F-isocrystal $\oplus_{j=1}^i (E^j/E^{j-1} \otimes_k K^{i-j}/K^{i-j-1})$, and one has $H^0_{\mathrm{dR}}(\widetilde{E}) \xrightarrow{\sim} H^0_{\mathrm{dR}}(E^{K,i})$. Thus, to prove the statement it suffices to verify the injectivity of the induced homomorphism $\widetilde{E} \to \mathcal{S}^{\lambda,i}(X, \mathfrak{Z}_\eta)$. The latter follows from the injectivity of the homomorphism $E^{K,i-1} \to \mathcal{S}^{\lambda,i-1}(X, \mathfrak{Z}_\eta)$ and the trivial fact that the intersection of the image of $E^0 \otimes K^i$ in $\mathfrak{N}^{K,i}(X, \mathfrak{Z}_\eta)$ with $\mathfrak{N}^{K,i-1}(X, \mathfrak{Z}_\eta)$ coincides with $E^0 \otimes K^{i-1}$. ∎

Let $\mathcal{G}^{\lambda,i}(X, \mathfrak{Z})$ denote the image of $E^{K,i}(X, \mathfrak{Z})$ in $\mathcal{S}^{\lambda,i}(X, \mathfrak{Z}_\eta)$. It does not depend on the choice of the Frobenius lifting and depends functorially on $(X, \mathfrak{Z})$ since $\mathcal{G}^{\lambda,i}(X, \mathfrak{Z})$ is generated over B by the primitives of all closed one-forms from $\mathcal{G}^{\lambda,i-1}(X, \mathfrak{Z}) \otimes_B \Omega^1_B$ which, by the property (b), are defined uniquely up to an element of $\mathcal{C}^{K,i}(X, \mathfrak{Z}_\eta) = \mathfrak{c}(B) \otimes_k K^i$. Notice that $\mathcal{G}^{\lambda,i}(X, \mathfrak{Z}) \cdot \mathcal{G}^{\lambda,j}(X, \mathfrak{Z}) \subset \mathcal{G}^{\lambda,i+j}(X, \mathfrak{Z})$ for $i + j \le n$, or $i + j = n + 1$ and $\dim(X) \le m$.

Till the end of this section we consider only germs of the form $(\mathfrak{X}_\eta, \mathfrak{Z})$, where $\mathfrak{X}$ is a proper marked formal scheme over k° and $\mathfrak{Z}$ is a nonempty open affine subscheme of $\mathring{\mathfrak{X}}$. (Recall that if $i = n$, we assume that $\dim(\mathfrak{X}_\eta) \le m + 1$.) Let $\mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma}$ denote the union of $\mathcal{G}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z})$ in $\mathcal{S}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma} \subset \mathfrak{N}^{K,i}_{\mathfrak{X}_\eta,\sigma}$ taken over all nonempty open affine subschemes $\mathfrak{Z}$ of $\mathring{\mathfrak{X}}$ (see Corollary 6.3.4).

Recall that in the case $\dim(\mathfrak{X}_\eta) = 1$ (and an arbitrary $i \ge 0$) we considered a similar injective homomorphism of F-isocrystals $E^{K,i}(\mathfrak{X}_\eta, \mathfrak{Z}) \to R^{\lambda,i}(\mathfrak{X}, \mathfrak{Z}_\eta) \subset \mathfrak{N}^{K,i}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$, denoted by $\mathcal{F}^{\lambda,i}(\mathfrak{X}, \mathfrak{Z})$ its image and by $\mathcal{F}^{\lambda,i}_{\mathfrak{X},\sigma}$ the corresponding union in $\mathfrak{N}^{K,i}_{\mathfrak{X}_\eta,\sigma}$ (see §6.5). Here is the first of the additional induction hypotheses.

(IH1) If $\dim(\mathfrak{X}_\eta) = 1$, then $\mathcal{G}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z}) = \mathcal{F}^{\lambda,i}(\mathfrak{X}, \mathfrak{Z})$.

The property (IH1) implies that the sub-F-isocrystals $\mathcal{F}^{\lambda,i}(\mathfrak{X}, \mathfrak{Z}) \subset \mathfrak{N}^{K,i}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$, which are defined in §6.5 for all $i \ge 0$, do not depend on the choice of the Frobenius lifting and are functorial on $(\mathfrak{X}, \mathfrak{Z})$ at least for $i \le n$.

The last two of the additional induction hypotheses describe the local form of a function $f \in \mathcal{S}^{\lambda,i}(X)$ in an étale neighborhood of a point $x \in X$ in the cases $s(x) < \dim(X)$ and $s(x) = \dim(X)$, respectively.

(IH2) (split case) If $s(x) < \dim(X)$ then, given a smooth morphism $\varphi : X \to Y$ to a smooth k-analytic space Y of dimension $\dim(X) - 1$ with $s(y) = s(x)$ for $y = \varphi(x)$, there exists an Y-split étale neighborhood $\psi : X' \to X$ of x over an étale morphism $Y' \to Y$ such that $\psi^*(f) \in \sum_{j=0}^i p_1^{\#}(\mathcal{S}^{\lambda,j}(Y')) \cdot L^{\lambda,i-j}(X')$.

(IH3) (marked case) If $s(x) = \dim(X)$, there is a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ of x with the following properties:

(IH3.1) $\varphi^*(f) \in R^{\lambda,i}(\mathfrak{X})$ (see the end of §3.1);

(IH3.2) $\varphi^*(f)_\sigma \in \mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma}$;

(IH3.3) for every stratum closure $\mathcal{Y} \subset \mathfrak{X}_s$, there exists a small open affine subscheme $\mathfrak{Y} \subset \mathfrak{X}$ (see §3.5) with $\mathfrak{Y}_s \cap \mathring{\mathcal{Y}} \neq \emptyset$ and the following property: if $\mathcal{Y} = \mathfrak{X}_s$ (resp. $\mathcal{Y}$ is arbitrary), there is a p_1-discoid (resp. p_1-semi-annular) open neighborhood $\mathcal{V}$ of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathfrak{D}_{\mathcal{Y}}$, where $\mathcal{Y}' = \mathcal{Y} \cap \mathfrak{Y}_s$, such that for $\mathcal{U} = p_1(\mathcal{V})$ one has $p_2^*(\varphi^*(f)) \in p_1^{\#}(\mathcal{S}^{\lambda,i}(\mathcal{U}))$ (resp. $p_2^*(\varphi^*(f)) \in \sum_{j=0}^{i} p_1^{\#}(\mathcal{S}^{\lambda,j}(\mathcal{U}))\cdot L^{\lambda,i-j}(\mathcal{V})$), where p_1 and p_2 are the canonical projections $\mathcal{V} \to \mathcal{U}$ and $\mathcal{V} \to \mathfrak{X}_\eta$, respectively.

In the property (IH3), if the proper marked formal scheme $\mathfrak{X}$ is defined over k'°, then $R^{\lambda,i}(\mathfrak{X})$ and $\mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma}$ are considered for the k'-algebra $K \otimes_k k'$ and the element $\lambda \otimes 1$.

The properties (IH2) and (IH3.2) can be used to describe the stalk $\mathcal{S}^{\lambda,i}_{X,\overline{x}}$ of the sheaf $\mathcal{S}^{\lambda,i}_X$ at a geometric point $\overline{x}$ of X over a point $x \in X$. Namely, in the situation (IH2), if $\overline{y}$ is a geometric point of Y over the point $y \in Y$ which is under the geometric point $\overline{x}$, let $\varphi^{\#}(\mathcal{S}^{\lambda,i}_{Y,\overline{y}})$ denote the $\mathcal{O}_{X,\overline{x}}$-subalgebra of $\mathcal{S}^{\lambda,i}_{X,\overline{x}}$ generated by the functions $\varphi^*(g)$ with $g \in \mathcal{S}^{\lambda,i}_{Y,\overline{y}}$. In the situation (IH3), if $\overline{\sigma}$ is a geometric point of $\mathfrak{X}_\eta$ over the point σ and the geometric point $\overline{x}$, let $\mathcal{G}^{\lambda,i}_{\mathfrak{X},\overline{\sigma}}$ denote the inductive limit of $\mathcal{G}^{\lambda,i}_{\mathfrak{X}',\sigma'}$ taken over commutative triangles

$$\begin{array}{ccc} \mathfrak{X}'_\eta & \longrightarrow & \mathfrak{X}_\eta \\ & \nwarrow \quad \nearrow & \\ & \overline{\sigma} & \end{array}$$

where the upper arrow is a marked neighborhood of the point σ that comes from a morphism of formal schemes $\mathfrak{X}' \to \mathfrak{X}$, and the image of the left arrow is the generic point σ' of $\mathfrak{X}'$. The statements (i) and (ii) of the following lemma are easy consequences of (IH2) and (IH3.2), respectively. Notice also that, by Lemma 1.4.2, the sum in the second part of (i) is a direct one.

7.1.2. Lemma. *(i) In the situation of (IH2), if $t(y) = t(x)$ then $\mathcal{S}^{\lambda,i}_{X,\overline{x}} = \varphi^{\#}(\mathcal{S}^{\lambda,i}_{Y,\overline{y}})$, and if $t(y) < t(x)$ then $\mathcal{S}^{\lambda,i}_{X,\overline{x}} = \sum_{j=0}^{i} \varphi^{\#}(\mathcal{S}^{\lambda,j}_{Y,\overline{y}})\mathrm{Log}^{\lambda}(f)^{i-j}$, where f is a function from $\mathcal{O}^*_{X,x}$ with $|f(x)| \notin \sqrt{|\mathcal{H}(y)^*|}$;*

(ii) in the situation of (IH3), one has $\mathcal{S}^{\lambda,i}_{X,\overline{x}} = \mathcal{G}^{\lambda,i}_{\mathfrak{X},\overline{\sigma}}$. ∎

7.2. Split one-forms. Let X be a smooth k-analytic space. If $i = n$, we assume that $\dim(X) \leq m+1$. We say that X is *split* if it is provided with an isomorphism with $Y \times D$ or $Y \times B$, where Y is a smooth k-analytic space of dimension $\dim(X) - 1$ whose sheaf of analytic one-forms Ω^1_Y is free over $\mathcal{O}_Y$, and D and B are open disc and annulus, respectively. For a split space X, let p_1 be the projection $X \to Y$. We say that a closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,i}}(X)$ is *split* if it is contained in $\Omega^1_M(X)$ with $M = \sum_{j=0}^{i} p_1^{\#}(\mathcal{S}^{\lambda,j}(Y)) \cdot L^{\lambda,i-j}(X)$ and admits a primitive f_ω in $N =$

$\sum_{j=0}^{i+1} p_1^\#(\mathcal{S}^{\lambda,j}(Y)) \cdot L^{\lambda,i+1-j}(X)$. Notice that if X is isomorphic to $Y \times D$ then $M = p_1^\#(\mathcal{S}^{\lambda,i}(Y))$ and $N = p_1^\#(\mathcal{S}^{\lambda,i+1}(Y))$.

7.2.1. Lemma. *Let X be a smooth k-analytic space such that* $\dim(X) \le m+1$ *if $i = n$, and let $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(X)$. Then every point $x \in X$ with $s(x) < \dim(X)$ has an étale neighborhood $\varphi : X' \to X$ such that X' and $\varphi^*(\omega)$ are split.*

Proof. First of all, we can shrink X and assume that $\omega = \sum_{\nu=1}^{l} f_\nu \omega_\nu$ with $f_\nu \in \mathcal{S}^{\lambda,i}(X)$ and $\omega_\nu \in \Omega^1(X)$. Furthermore, by Proposition 2.3.1(i), we can shrink X and find smooth morphism $\psi : X \to Y$ to a smooth k-analytic space Y of dimension $\dim(X) - 1$ with $s(y) = s(x)$ for $y = \varphi(x)$. By the property (IH2), we can find, for every $1 \le \nu \le l$, an Y-split étale neighborhood $\varphi_\nu : X_\nu \to X$ of x over an étale morphism $Y_\nu \to Y$ such that $\varphi_\nu^*(f_\nu) \in \sum_{j=0}^{i} p_1^\#(\mathcal{S}^{\lambda,i}(Y_\nu)) \cdot L^{\lambda,i-j}(X_\nu)$. Corollary 2.3.2 implies that there exists an Y-split neighborhood $\varphi : X' \to X$ over $Y' \to Y$ that refines all of the neighborhoods $X_\nu \to X$. It follows that $\varphi^*(\omega) \in \Omega^1_M(X')$, where $M = \sum_{j=0}^{i} p_1^\#(\mathcal{S}^{\lambda,j}(Y')) \cdot L^{\lambda,i-j}(X')$. Of course, we may shrink Y' and assume that the sheaf $\Omega^1_{Y'}$ is free over $\mathcal{O}_{Y'}$. Finally, from Propositions 1.3.2 and 1.5.1 it follows that $\varphi^*(\omega) = dg + p_1^*(\eta)$ with $g \in N = \sum_{j=0}^{i+1} p_1^\#(\mathcal{S}^{\lambda,j}(Y')) \cdot L^{\lambda,i+1-j}(X')$ and a closed one-form $\eta \in \Omega^1_{\mathcal{S}^{\lambda,i}}(Y')$. By the induction hypotheses, we can shrink Y' and assume that $\eta = dh$ for some $h \in \mathcal{S}^{\lambda,i+1}(Y')$, and we get $\varphi^*(\omega) = df$ for $f = g + p_1^*(h) \in N$, i.e., the one-form $\varphi^*(\omega)$ is split. ∎

7.2.2. Lemma. *Let X be a split smooth k-analytic space (with $p_1 : X \to Y$) such that* $\dim(X) \le m+1$ *if $i = n$, and let ω be a split closed one-form from $\Omega^1_{\mathcal{S}^{\lambda,i}}(X)$. Then*

(i) the primitive $f_\omega \in N = \sum_{j=0}^{i+1} p_1^\#(\mathcal{S}^{\lambda,j}(Y)) \cdot L^{\lambda,i+1-j}(X)$ is unique up to an element of $\mathcal{C}^{K,i+1}(X)$;

(ii) if $i \le n-1$, or $i = n$ and $\dim(X) \le m$, *then $f_\omega \in \mathcal{S}^{\lambda,i+1}(X)$;*

(iii) given a split smooth k-analytic space X' with $\dim(X') \le m+1$ *if $i = n$ and a morphism $\varphi : X' \to X$ such that the one-form $\varphi^*(\omega)$ is also split, one has $f_{\varphi^*(\omega)} - \varphi^*(f_\omega) \in \mathcal{C}^{K,i+1}(X')$.*

Proof. (i) and (ii). If $i \le n-1$ or $i = n$ and $\dim(X) \le m$ (as in (ii)) then, by the induction hypotheses, one has $N \subset \mathcal{S}^{\lambda,i+1}(X)$. This implies (i) (in the case considered) and (ii). Assume therefore that $i = n$ and $\dim(X) = m+1$. If $m = 0$, X is an open disc or annulus and $N = L^{\lambda,i+1}(X)$, and the statement is trivial. Assume that $m \ge 1$, i.e., $\dim(X) \ge 2$, and that there is another primitive $f'_\omega \in N$. By Corollary 4.1.3(ii), it suffices to show that for any morphism $\psi : Z \to X$ from an elementary k'-analytic curve Z, where k' is a finite extension of k, one has $\psi^*(f_\omega - f'_\omega) \in \mathcal{C}^{K',n+1}(Z)$, where $K' = K \otimes_k k'$. Since both $\psi^*(f_\omega)$ and $\psi^*(f'_\omega)$ are functions from $\mathcal{S}^{\lambda,n+1}(Z)$, the induction hypotheses imply (i).

(iii) Let $g = f_{\varphi^*(\omega)} - \varphi^*(f_\omega)$. If $i \le n-1$, then $g \in \mathcal{S}^{\lambda,i+1}(X')$, and the statement follows from the induction hypotheses. Assume therefore that $i = n$. If $m = 0$, then each of X and X' is either zero-dimensional, or an open disc or annulus, and in all of the cases one has $g \in L^{\lambda,n+1}(X')$, and the statement follows. If $m \ge 1$, then as above it suffices to show that for any morphism $\psi : Z \to X$ from an elementary k'-analytic curve Z, where k' is a finite extension of k, one has $\psi^*(g) \in \mathcal{C}^{K',n+1}(Z)$, where $K' = K \otimes_k k'$. But this again follows from the induction hypotheses since $\psi^*(g) \in \mathcal{S}^{\lambda,n+1}(Z)$. ∎

7.3. Marked and weakly marked one-forms. Let $\mathfrak{X}$ be a proper marked formal scheme over k°. If $i = n$, we assume that $\dim(\mathfrak{X}_\eta) \le m+1$. We say that a closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,i}}(\mathfrak{X}_\eta)$ is *marked* if it satisfies the following properties:

(MF1) $\omega \in \Omega^1_{R^{\lambda,i}}(\mathfrak{X})$ (see the end of §3.1);

(MF2) there exists a Frobenius lifting ϕ at the point $\sigma = \sigma_{\mathfrak{X}}$ (see the end of §6.1) and a polynomial $P(T) \in k[T]$ with no roots-of-unity roots such that $P(\phi^*)\omega_\sigma \in d\mathcal{S}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma}$;

(MF3) for every stratum closure $\mathcal{Y} \subset \mathfrak{X}_s$, there exists a small open affine subscheme $\mathfrak{Y} \subset \mathfrak{X}$ (see §3.5) with $\mathfrak{Y}_s \cap \mathring{\mathcal{Y}} \ne \emptyset$ and the following property: if $\mathcal{Y} = \mathfrak{X}_s$ (resp. $\mathcal{Y}$ is arbitrary), there is a p_1-discoid (resp. p_1-semi-annular) open neighborhood $\mathcal{V}$ of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathfrak{D}_{\mathcal{Y}}$ with $\mathcal{Y}' = \mathcal{Y} \cap \mathfrak{Y}_s$ and $p_2^*(\omega) \in \Omega^1_M(\mathcal{V})$, where $M = p_1^\#(\mathcal{S}^{\lambda,i}(\mathcal{U}))$ (resp. $M = \sum_{j=0}^i p_1^\#(\mathcal{S}^{\lambda,j}(\mathcal{U}))\cdot L^{\lambda,i-j}(\mathcal{V})$) and $\mathcal{U} = p_1(\mathcal{V})$.

7.3.1. Lemma. *Let X be a smooth k-analytic space such that $\dim(X) \le m+1$ if $i = n$, and let $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(X)$. Then every point $x \in X$ with $s(x) = \dim(X)$ has a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ such that the closed one-form $\varphi^*(\omega)$ is marked and in fact $\varphi^*(\omega)_\sigma \in \mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma} \otimes_{\mathcal{O}_{\mathfrak{X}_\eta,\sigma}} \Omega^1_{\mathfrak{X}_\eta,\sigma}$.*

Proof. We can shrink X so that $\eta = \sum_{j=1}^l f_j \eta_j$ with $f_j \in \mathcal{S}^{\lambda,i}(X)$ and $\eta_j \in \Omega^1(X)$. By Propositions 2.1.1, Corollary 3.3.2 and the property (IH3), we can find a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ of x such that all $\varphi^*(f_j)$ possess the properties (IH3.1)-(IH3.3) on $\mathfrak{X}_\eta$. We claim that $\varphi^*(\omega)$ is marked. Indeed, the properties (IH3.1) and (IH3.3) imply (MF1) and (MF3). The property (IH3.2) implies that $\varphi^*(\omega)_\sigma \in \mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma} \otimes_{\mathcal{O}_{\mathfrak{X}_\eta,\sigma}} \Omega^1_{\mathfrak{X}_\eta,\sigma}$. In its turn, the latter implies that for any Frobenius lifting ϕ at σ there exists a polynomial $P(T) \in k[T]$ whose roots are Weil numbers with weights in $[1, 2(i+1)]$ and such that $P(\phi^*)\varphi^*(\omega)_\sigma \in d\mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma} \subset d\mathcal{S}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma}$, i.e., (MF2) is true. ∎

Given a smooth k-analytic space X such that if $i = n$ then $\dim(X) \le m+1$, we introduce the following $\mathcal{D}_X$-submodule of $\mathfrak{N}^{K,i+1}_X$:

$$\widetilde{\mathcal{S}}^{\lambda,i}_X = \sum_{j=0}^{i} \mathcal{S}^{\lambda,j}_X \cdot \mathcal{L}^{\lambda,i+1-j}_X \ .$$

Notice that if $i \le n-1$ or $i=n$ and $\dim(X)\le m$, then $\widetilde{\mathcal{S}}^{\lambda,i}_X \subset \mathcal{S}^{\lambda,i+1}_X$. In any case, given a morphism $\varphi : X' \to X$ as in the property (e) with $\dim(X')\le m+1$ if $i=n$, then $\varphi^{\#}(\widetilde{\mathcal{S}}^{\lambda,i}_X) \subset \widetilde{\mathcal{S}}^{\lambda',i}_{X'}$.

7.3.2. Lemma. $\mathrm{Ker}(\widetilde{\mathcal{S}}^{\lambda,i}_X \xrightarrow{d} \Omega^1_{\widetilde{\mathcal{S}}^{\lambda,i},X}) = \mathcal{C}^{K,i+1}_X$.

Proof. If $i\le n-1$ or $i=n$ and $\dim(X)\le m$, the statement follows from the induction hypotheses since $\widetilde{\mathcal{S}}^{\lambda,i}_X \subset \mathcal{S}^{\lambda,i+1}_X$. Assume therefore that $i=n$ and $\dim(X)=m+1$.

Case $m=0$, i.e., $\dim(X)=1$. Let $f\in \widetilde{\mathcal{S}}^{\lambda,n}(X)$ be such that $df=0$. It suffices to show that every point $x\in X$ has an étale neighborhood $\varphi : X'\to X$ such that $\varphi^*(f)\in \mathcal{C}^{\lambda,n+1}(X')$. Replacing X by an étale neighborhood of x, we may assume that $f=\sum_{j=0}^n g_j h_j$, where $g_j\in \mathcal{S}^{\lambda,j}(X)$ and $h_j \in L^{\lambda,n+1-j}(X)$. If x is not of the type (2), the property (IH2) implies that there is a finite extension k' of k and an open subset $X'\subset X\otimes k'$ such that the induced morphism $\varphi: X'\to X$ is an étale neighborhood of the point x and, if x of type (1) or (4) (resp. (3)), X' is isomorphic to an open disc (resp. open annulus) with center at zero over k' and $\varphi^*(g_j)\in L^{\lambda,j}(X')$ for all $0\le j\le n$. We get $\varphi^*(f)\in L^{\lambda,n+1}(X')$, and the required fact follows. Assume now that the point x is of type (2). By Propositions 2.2.1 and 2.4.1 and the properties (IH1) and (IH3), there exists a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ of x such that $\varphi^*(g_j)\in \mathcal{G}^{\lambda,j}_{\mathfrak{X},\sigma} \subset \mathcal{F}^{\lambda,j}_{\mathfrak{X},\sigma}$ for all $0\le j\le n$. We get $\varphi^*(f)\in \mathcal{F}^{\lambda,n+1}_{\mathfrak{X},\sigma}$, and Lemma 6.5.1 implies the required fact.

Case $m\ge 1$, i.e., $\dim(X)\ge 2$. By Corollary 4.1.3(ii), it suffices to show that for any morphism $\varphi: Y\to X$ from an elementary k'-analytic curve Y one has $\varphi^*(f)\in \mathcal{C}^{K',n+1}(Y)$, where k' is a finite extension of k and $K'=K\otimes_k k'$. Since $\varphi^*(f)\in \widetilde{\mathcal{S}}^{\lambda,n}(Y)\subset \mathcal{S}^{\lambda,n+1}(Y)$, the induction hypothesis implies the required fact. ∎

In the situation of the beginning of this subsection, we say that a closed one-form $\omega\in \Omega^1_{\mathcal{S}^{\lambda,i}}(\mathfrak{X}_\eta)$ is *weakly marked* if it satisfies the properties (MF1), (MF3) and the following weaker form of the property (MF2):

($\widetilde{\mathrm{MF2}}$) there exists a Frobenius lifting ϕ at σ and a polynomial $P(T)\in k[T]$ with no roots-of-unity roots such that $P(\phi^*)\omega_\sigma \in d\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma}$.

The usefulness of weakly marked one-forms will be seen in Proposition 7.4.3. Notice that if the property (MF2) (resp. ($\widetilde{MF}2$)) holds for a Frobenius lifting ϕ, then it holds for any Frobenius lifting ϕ' with $\phi'=\phi^l$ or $\phi={\phi'}^l$, where l is a positive integer.

7.3.3. Proposition. *Let $\omega\in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(\mathfrak{X}_\eta)$ be a marked (resp. weakly marked) one-form. If the property (MF2) (resp. ($\widetilde{MF}2$)) holds for a polynomial P and a Frobenius lifting ϕ, then it holds for the same polynomial P and any other Frobenius lifting ϕ' of the same degree as ϕ.*

Let $\omega\in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(\mathfrak{X}_\eta)$ be a closed one-form that possesses the property (MF3) for the maximal stratum closure $\mathfrak{X}_s$. Suppose we are given a closed subfield $k' \subset \mathbf{C}_p$, a proper marked formal scheme $\mathfrak{X}'$ over k'° with $\dim(\mathfrak{X}'_\eta) \le m+1$ if $i = n$, an embedding of fields $k \hookrightarrow k'$, and a morphism of formal schemes $\varphi : \mathfrak{X}' \to \mathfrak{X}$ over the embedding $k^\circ \hookrightarrow k'^\circ$. Suppose also we are given a filtered k'-algebra K' and a homomorphism of filtered algebras $K \to K'$ over the embedding $k \hookrightarrow k'$ that takes λ to an element $\lambda' \in K'$.

7.3.4. Lemma. *Assume that φ is* dominant, *i.e., $\varphi(\sigma') = \sigma$, where σ and σ' are the generic points of $\mathfrak{X}$ and $\mathfrak{X}'$, respectively. Then for every pair of Frobenius liftings ϕ and ϕ' of the same degree at σ and σ', respectively, and every $j \ge 0$ one has*

$$\phi'^{*j}(\varphi^*(\omega)_{\sigma'}) - \varphi^*(\phi^{*j}(\omega))_{\sigma'} \in d\mathcal{S}^{\lambda',i}_{\mathfrak{X}'_\eta,\sigma'} .$$

Proof. Let $\mathfrak{Y}$ be a small open affine subscheme of $\mathfrak{X}$ from the property (MF3) (for the maximal stratum closure $\mathfrak{X}_s$). Furthermore, let $\mathfrak{Z}$ and $\mathfrak{Z}'$ be open affine subschemes of $\mathring{\mathfrak{X}} \cap \mathfrak{Y}$ and $\mathring{\mathfrak{X}}'$ with $\varphi(\mathfrak{Z}') \subset \mathfrak{Z}$ and such that ϕ and ϕ' are Frobenius liftings of the germs $(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$ and $(\mathfrak{X}'_\eta, \mathfrak{Z}'_\eta)$, and let $\mathcal{U}$ and $\mathcal{U}'$ be open neighborhoods of $\mathfrak{Z}_\eta$ and $\mathfrak{Z}'_\eta$ at which the morphisms ϕ^j and ϕ'^j are defined, respectively, and the sheaf $\Omega^1_{\mathcal{U}}$ is free over $\mathcal{O}_{\mathcal{U}}$. Consider the morphism of analytic spaces

$$\psi_j = (\varphi \circ \phi'^j, \phi^j \circ \varphi) : \mathcal{U}' \to \mathfrak{X}_\eta \times \mathfrak{X}_\eta .$$

Since the morphisms between closed fibers $\mathfrak{Z}'_s \to \mathfrak{Z}_s$ induced by $\varphi \circ \phi'^j$ and $\phi^j \circ \varphi$ coincide, it follows that $\psi_j(\mathfrak{Z}'_\eta) \subset \mathfrak{D}_{\mathfrak{Z}} \subset \mathfrak{D}_{\mathfrak{X}}$ and, therefore, $\psi_j(\mathfrak{Z}'_\eta)$ is contained in an open neighborhood $\mathcal{V}$ of $\mathfrak{D}_{\mathfrak{Z}}$ in $\mathfrak{D}_{\mathfrak{X}}$. By Proposition 3.5.1 and the property (MF3), we can shrink $\mathcal{V}$ and $\mathcal{U}$ and assume that $\mathcal{V}$ is a p_1-discoid open neighborhood of $\mathfrak{D}_{\mathfrak{Z}}$ in $\mathfrak{D}_{\mathfrak{X}}$, $\mathcal{U} = p_1(\mathcal{V})$ and $p_2^*(\omega) \in \Omega^1_{p_1^\#(\mathcal{S}^{\lambda,i}(\mathcal{U}))}(\mathcal{V})$. The one-form considered coincides with $\psi_j^*(p_1^*(\omega) - p_2^*(\omega))$. Hence it suffices to show that $p_1^*(\omega) - p_2^*(\omega) \in d\mathcal{S}^{\lambda,i}(\mathcal{V})$. But the restriction of the latter one-form to the diagonal $\Delta(\mathcal{U})$ is zero, and therefore the required fact follows from Proposition 1.3.2 applied to the surjective discoid morphism $p_1 : \mathcal{V} \to \mathcal{U}$ and the $D_{\mathcal{U}}$-module $\mathcal{S}^{\lambda,i}(\mathcal{U})$. ∎

Proof of Proposition 7.3.3. We apply Lemma 7.3.4 to the identity morphism $\mathfrak{X} \to \mathfrak{X}$ and the Frobenius liftings ϕ and ϕ'. If $P(\phi^*)\omega_\sigma \in d\mathcal{S}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma}$ (resp. $d\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma}$), it follows that $P(\phi'^*)\omega_\sigma \in d\mathcal{S}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma}$ (resp. $d\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_\eta,\sigma}$). ∎

7.4. Construction of a primitive of a weakly marked one-form. Let $\mathfrak{X}$ be a proper marked formal scheme over k°. If $i = n$, we assume that $\dim(\mathfrak{X}_\eta) \le m+1$. Furthermore, let $\omega \in \Omega^1_{\mathcal{S}^{\lambda,i}}(\mathfrak{X}_\eta)$ be a closed one-form that possesses the properties (MF1) and (MF2) (resp. ($\widetilde{\text{MF2}}$)), and fix a Frobenius lifting ϕ at σ for which (MF2) (resp. ($\widetilde{\text{MF2}}$)) holds.

7.4.1. Proposition. *(i) There exists a function $f_\omega \in R^{\lambda,i+1}(\mathfrak{X})$, unique up to an element of $\mathcal{C}^{K,i}(\mathfrak{X}_\eta)$ (resp. $\mathcal{C}^{K,i+1}(\mathfrak{X}_\eta)$), such that $df_\omega = \omega$ and, for every polynomial $P(T) \in k[T]$ with no roots-of-unity roots and $P(\phi^*)\omega_\sigma \in d\mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $d\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$), one has $P(\phi^*)(f_\omega)_\sigma \in \mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$);*

(ii) if ω also satisfies (MF3), f_ω does not depend on the choice of the Frobenius lifting ϕ.

Proof. By Corollary 3.1.5, a function $f' \in R^{\lambda,i+1}(\mathfrak{X})$ with $df' = \omega$ exists. Let $P(T) \in k[T]$ be a polynomial with no roots-of-unity roots such that $P(\phi^*)\omega_\sigma = dg$ for some $g \in \mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$). Then for the function $h = P(\phi^*)f'_\sigma - g$ one has $dh = 0$, and Lemma 7.3.2 implies that $h \in \mathcal{C}^{K,i+1}_R(\mathfrak{X},\sigma)$. By Lemma 6.3.2(i), the map $P(\phi^*) : \mathcal{C}^{K,i+1}_R(\mathfrak{X}) \to \mathcal{C}^{K,i+1}_R(\mathfrak{X},\sigma) = \mathcal{C}^{K,i+1}_R(\mathfrak{X})$ is a bijection. We can therefore find an element $\alpha \in \mathcal{C}^{K,i+1}_R(\mathfrak{X})$ with $P(\phi^*)f'_\sigma - g = P(\phi^*)\alpha_\sigma$. The function $f_\omega = f' - \alpha$ is contained in $R^{\lambda,i+1}(\mathfrak{X})$ and satisfies the required properties. The bijectivity of the above map implies that f_ω is uniquely defined by the element g and the polynomial P.

We claim that, up to an element of $\mathcal{C}^{\lambda,i}(\mathfrak{X}_\eta)$ (resp. $\mathcal{C}^{\lambda,i+1}(\mathfrak{X}_\eta)$), the function f_ω does not depend on the choice of g and P and, if ω satisfies (MF3), of the Frobenius lifting ϕ.

Let g' be another element of $\mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$) with $P(\phi^*)\omega_\sigma = dg'$. Then $d(g'-g) = 0$. The induction hypothesis (resp. Lemma 7.3.2) implies that $g' - g \in \mathcal{C}^{K,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}}$). By Lemma 3.1.1, the element $g' - g$ is the restriction of a unique element from $\mathcal{C}^{K,i}(\mathfrak{X}_\eta)$ (resp. $\mathcal{C}^{K,i+1}(\mathfrak{X}_\eta)$). If f' is the function from $R^{\lambda,i+1}(\mathfrak{X})$ with $df' = \omega$ and $P(\phi^*)f'_\sigma = g'$, then $f' - f \in \mathcal{C}^{K,i+1}_R(\mathfrak{X})$ and $P(\phi^*)(f'_\sigma - f_\sigma) = g' - g \in \mathcal{C}^{K,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}}$), where $f = f_\omega$. Since the actions of $P(\phi^*)$ on $\mathcal{C}^{K,i+1}_R(\mathfrak{X}_\eta)$ and $\mathcal{C}^{K,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}}$) are bijective, it follows that $f' - f \in \mathcal{C}^{K,i}(\mathfrak{X}_\eta)$ (resp. $\mathcal{C}^{K,i+1}(\mathfrak{X}_\eta)$).

Furthermore, let $Q(T) \in k[T]$ be the monic polynomial of minimal degree with $Q(\phi^*)\omega_\sigma \in d\mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $d\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$). It is clear that $Q(T)$ divides $P(T)$, and so let $P(T) = Q(T)Q'(T)$. Let $Q(\phi^*)\omega_\sigma = dg'$ for some $g' \in \mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$). We have $P(\phi^*)\omega_\sigma = d(Q'(\phi^*)g')$. If f' is the function from $R^{\lambda,i+1}(\mathfrak{X}_\eta)$ with $df' = \omega$ and $Q(\phi^*)f'_\sigma = g'$, then $P(\phi^*)f'_\sigma = Q'(\phi^*)g'$, and the required fact follows from the previous paragraph.

Finally, let ϕ' be another Frobenius lifting at σ, and let f'_ω be the corresponding primitive. Assume first that $\phi' = \phi^l$ for some integer $l \geq 1$. If $P(T) = \prod_j (T - \alpha_j)$ with $\alpha_j \in k^{\mathrm{a}}$, the polynomial $Q(T) = \prod_j (T - \alpha_j^l)$ has coefficients in k, and one has $Q(T^l) = P(T)P'(T)$ for some $P'(T) \in k[T]$. It follows that $Q({\phi'}^*)(f_\omega)_\sigma = g'$ with $g' = P'(\phi^*)g \in \mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$), and the required fact follows from the independence of f'_ω on the choice of g'. It remains to consider the case when ϕ and ϕ' are of the same degree. In this case Lemma 7.3.4 implies that $P({\phi'}^*)\omega_\sigma = dg'$

for some $g' \in \mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$ (resp. $\widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$), and again the required fact follows from the independence of f'_ω on the choice of the element g'. ∎

7.4.2. Lemma. *Let $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(\mathfrak{X}_\eta)$ be a marked (resp. weakly marked) one-form. Then*

(i) if $i \le n-1$ or $i = n$ and $\dim(\mathfrak{X}_\eta) \le m$, then $f_\omega \in \mathcal{S}^{\lambda,i+1}(\mathfrak{X}_\eta)$

(ii) if $\omega = dh$ for some $h \in \widetilde{\mathcal{S}}^{\lambda,i}(\mathfrak{X}_\eta)$, then $f_\omega - h \in \mathcal{C}^{K,i+1}(\mathfrak{X}_\eta)$;

(iii) given a dominant morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ as in Lemma 7.3.4, one has $f_{\varphi^*(\omega)} - \varphi^*(f_\omega) \in \mathcal{C}^{K',i}(\mathfrak{X}'_\eta)$ (resp. $\mathcal{C}^{K',i+1}(\mathfrak{X}'_\eta)$);

(iv) if $\dim(\mathfrak{X}_\eta) = 1$, the construction of f_ω is compatible with that of Lemma 6.5.2, i.e., if $\omega_\sigma \in \mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma} \otimes_{\mathcal{O}_{\mathfrak{X}_{\eta,\sigma}}} \Omega^1_{\mathfrak{X}_{\eta,\sigma}}$, then $(f_\omega)_\sigma \in \mathcal{F}^{\lambda,i+1}_{\mathfrak{X},\sigma}$.

Notice that in (iii) the one-form $\varphi^*(\omega)$ is marked (resp. weakly marked), by Lemma 7.3.4.

Proof. (i) We know that a function $f' \in \mathcal{S}^{\lambda,i+1}(\mathfrak{X}_\eta)$ with $df' = \omega$ exists. Let $\mathbf{x}$ be a closed point of $\mathfrak{X}_s$. Since $\omega_{\mathbf{x}} \in \Omega^1_{L^{\lambda,i}}(\pi^{-1}(\mathbf{x}))$ and $L^{\lambda,i}(\pi^{-1}(\mathbf{x})) \subset \mathcal{S}^{\lambda,i}(\pi^{-1}(\mathbf{x}))$, it follows that $f'_{\mathbf{x}} \in L^{\lambda,i+1}(\pi^{-1}(\mathbf{x}))$, i.e., $f' \in R^{\lambda,i+1}(\mathfrak{X})$. One has $P(\phi^*)f'_\sigma - g \in \mathcal{S}^{\lambda,i+1}_{\mathfrak{X}_{\eta,\sigma}}$ and $d(P(\phi^*)f'_\sigma - g) = 0$. It follows that $P(\phi^*)f'_\sigma - g \in \mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}} = \mathcal{C}^{K,i+1}(\mathfrak{X}_\eta)$. Since the map $P(\phi^*) : \mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}} \to \mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}}$ is a bijection, there exists an element $\alpha \in \mathcal{C}^{K,i+1}(\mathfrak{X}_\eta)$ with $P(\phi^*)f'_\sigma - g = P(\phi^*)\alpha_\sigma$. The function $f' - \alpha$ is contained $\mathcal{S}^{\lambda,i+1}(\mathfrak{X}_\eta) \cap R^{\lambda,i+1}(\mathfrak{X})$, and Proposition 7.4.1 implies that $f_\omega = f' - \alpha$.

(ii) One has $d(P(\phi^*)h_\sigma - g) = 0$ and $P(\phi^*)h_\sigma - g \in \widetilde{\mathcal{S}}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$. Lemma 7.3.2 implies that $P(\phi^*)h_\sigma - g \in \mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}} = \mathcal{C}^{K,i+1}(\mathfrak{X}_\eta)$. Since the map $P(\phi^*) : \mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}} \to \mathcal{C}^{K,i+1}_{\mathfrak{X}_{\eta,\sigma}}$ is a bijection, to prove the required fact it suffices to show that $h \in R^{\lambda,i+1}(\mathfrak{X})$. Let $\mathbf{x}$ be a closed point of $\mathfrak{X}_s$. One has $(f_\omega)_{\mathbf{x}} - h_{\mathbf{x}} \in \widetilde{\mathcal{S}}^{\lambda,i}(\pi^{-1}(\mathbf{x}))$ and $d((f_\omega)_{\mathbf{x}} - h_{\mathbf{x}}) = 0$ and, by Lemma 7.3.2 again, we get $(f_\omega)_{\mathbf{x}} - h_{\mathbf{x}} \in \mathcal{C}^{K,i+1}(\pi^{-1}(\mathbf{x}))$, i.e., $h \in R^{\lambda,i+1}(\mathfrak{X})$.

(iii) and (iv) trivially follow from Proposition 7.4.1 and Lemma 6.5.2, respectively. ∎

We now want to extend the conclusion of Lemma 7.4.2(iii) to not necessarily dominant morphisms $\mathfrak{X}' \to \mathfrak{X}$. In the following proposition k' is a closed subfield of $\mathbf{C}_p$, $\mathfrak{X}'$ is a proper marked formal scheme over k'° with $\dim(\mathfrak{X}'_\eta) \le m+1$ if $i = n$, and K' is a filtered k'-algebra.

7.4.3. Proposition. *Given a marked one-form $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,i}}(\mathfrak{X}_\eta)$, there exists an open neighborhood $\mathcal{U}$ of σ such that, for any triple $(k', \mathfrak{X}', K')$ as above, any embedding of fields $k \hookrightarrow k'$, any morphism of formal schemes $\varphi : \mathfrak{X}' \to \mathfrak{X}$ over the embedding $k^\circ \hookrightarrow k'^\circ$ with $\varphi(\sigma') \in \mathcal{U}$, and any homomorphism of filtered algebras $K \to K'$ over the embedding $k \hookrightarrow k'$, the one-form $\varphi^*(\omega)$ is weakly marked and $f_{\varphi^*(\omega)} - \varphi^*(f_\omega) \in \mathcal{C}^{K',i+1}(\mathfrak{X}'_\eta)$. If $\dim(\mathfrak{X}_\eta) = 1$, one can take $\mathcal{U} = \mathfrak{X}_\eta$.*

For needs of the following subsection we denote by $\mathcal{U}_\omega$ the maximal open neighborhood of σ that possesses the property of Proposition 7.4.3. (It will be shown in fact that $\mathcal{U}_\omega$ is always $\mathfrak{X}_\eta$.)

Proof. First of all, it follows from Lemma 7.4.2(iii) that the construction of the primitives is preserved under the base change with respect to an embedding of fields $k \hookrightarrow k'$ and under the change with respect to a homomorphism of filtered algebras $K \to K'$, and so it suffices to consider only morphisms over the same field k and with the same filtered k-algebra K.

Let us fix a Frobenius lifting ϕ at the point σ and a polynomial $P(T) \in k[T]$ without roots-of-unity roots for which $P(\phi^*)\omega_\sigma \in d\mathcal{S}^{\lambda,i}_{\mathfrak{X}_{\eta,\sigma}}$. Furthermore, let $\mathcal{U}$ be an open neighborhood of σ such that the morphisms ϕ^j, $0 \le j \le \deg(P)$, are defined on $\mathcal{U}$, $P(\phi^*)\omega \in d\mathcal{S}^{\lambda,i}(\mathcal{U})$, and the sheaf of $\mathcal{O}_\mathcal{U}$-modules $\Omega^1_\mathcal{U}$ is free. *We claim that the set $\mathcal{U}$ possesses the required property.*

Indeed, let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism with $\varphi(\sigma') \in \mathcal{U}$. Let $\mathring{\mathcal{Y}}$ be the stratum of $\mathfrak{X}_s$ that contains the point $\pi(\varphi(\sigma'))$ and $\mathcal{Y}$ its closure in $\mathfrak{X}_s$. Since $\mathfrak{X}'_s$ is proper, its image is closed in $\mathfrak{X}_s$ and, therefore, one has $\varphi(\mathfrak{X}'_\eta) \subset \pi^{-1}(\mathcal{Y})$. Let $\mathfrak{Y}$ be a small open affine subscheme of $\mathfrak{X}$ with $\mathfrak{Y}_s \cap \mathring{\mathcal{Y}} \ne \emptyset$, and let $\mathcal{V}$ be a p_1-semi-annular open neighborhood of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathfrak{D}_\mathcal{Y}$, where $\mathcal{Y}' = \mathcal{Y} \cap \mathfrak{Y}_s$, such that $\mathcal{W} = p_1(\mathcal{V}) \subset \mathcal{U}$ and $p_2^*(\omega) \in \Omega^1_M(\mathcal{V})$, where $M = \sum_{j=0}^i p_1^\#(\mathcal{S}^{\lambda,j}(\mathcal{W}))L^{\lambda,i-j}(\mathcal{V})$ (see the property (IH3.3)).

Furthermore, let $\mathfrak{Z}'$ be a non-empty open affine subscheme of $\mathring{\mathfrak{X}}'$ with $\varphi(\mathfrak{Z}'_s) \subset \mathfrak{Y}_s \cap \mathring{\mathcal{Y}}$, ϕ' a Frobenius lifting on the germ $(\mathfrak{X}'_\eta, \mathfrak{Z}'_\eta)$, and $\mathcal{U}'$ an open neighborhood of $\mathfrak{Z}'_\eta$ such that the morphisms ${\phi'}^j$, $0 \le j \le \deg(P)$, are defined on $\mathcal{U}'$ and $\varphi(\mathcal{U}') \subset \mathcal{U}$. Given $0 \le j \le \deg(P)$, consider the morphism of analytic spaces

$$\psi_j = (\varphi \circ {\phi'}^j, \phi^j \circ \varphi) : \mathcal{U}' \to \mathfrak{X}_\eta \times \mathfrak{X}_\eta \ .$$

Since $\varphi \circ {\phi'}^j$ and $\phi^j \circ \varphi$ induce the same morphism $\mathfrak{Z}'_s \to (\mathring{\mathcal{Y}} \cap \mathfrak{Y}_s) \times (\mathring{\mathcal{Y}} \cap \mathfrak{Y}_s)$, it follows that $\psi_j(\mathfrak{Z}'_\eta) \subset \pi^{-1}(\Delta(\mathring{\mathcal{Y}} \cap \mathfrak{Y}_s)) \subset \mathfrak{D}_{\mathcal{Y}'}$. We can therefore shrink $\mathcal{U}'$ and assume that $\psi_j(\mathcal{U}') \subset \mathcal{V}$ for all $0 \le j \le \deg(P)$. We have

$${\phi'}^{*j}(\varphi^*(\omega)) - \varphi^*(\phi^{*j}(\omega)) = \psi_j^*(p_1^*(\omega) - p_2^*(\omega)) \ .$$

Thus, to prove our claim it suffices to show that $p_1^*(\omega) - p_2^*(\omega) \in dN$, where

$$N = \sum_{j=0}^i p_1^\#(\mathcal{S}^{\lambda,j}(\mathcal{W}))L^{\lambda,i-j+1}(\mathcal{V}) \ .$$

But the restriction of the latter one-form to the diagonal $\Delta(\mathcal{W})$ is zero, and so the required fact follows from Corollary 1.5.6.

Assume now that $\dim(\mathfrak{X}_\eta) = 1$. If φ is not dominant, the image of $\mathfrak{X}'_s$ in $\mathfrak{X}_s$ is a closed point $\mathbf{x}$, and one has $\varphi(\mathfrak{X}'_\eta) \subset \pi^{-1}(\mathbf{x})$. Since $(f_\omega)_{\mathbf{x}} \in L^{\lambda,i+1}(\pi^{-1}(\mathbf{x}))$, then $\varphi^*(f_\omega) \in L^{\lambda,i+1}(\mathfrak{X}'_\eta) \subset \widetilde{\mathcal{S}}^{\lambda,i}(\mathfrak{X}'_\eta)$, and the required fact follows from Lemma 7.4.2(ii) because $\varphi^*(\omega) = d(\varphi^*(f_\omega))$. ∎

7.5. Construction of the $\mathcal{D}_X$-modules $\mathcal{S}_X^{\lambda,n+1}$. We say that a smooth k-analytic space X is *atomic* if it is *split* or *marked*, i.e., it is provided with an isomorphism with a space of the form $Y \times D$ or $Y \times B$, where Y is a smooth k'-analytic space of dimension $\dim(X) - 1$ whose sheaf Ω^1_Y is free over $\mathcal{O}_Y$, and D and B are open disc and annulus with center at zero, or with the generic fiber $\mathfrak{X}_\eta$ of a proper marked formal scheme over k'°, respectively, where k' is a finite extension of k (see the end of §1.1). Let X be an atomic k-analytic space of dimension at most $m+1$. We say that a closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$ is *atomic* if it is split or marked in each of the corresponding cases. Given such a form ω, we denote by f_ω its primitive constructed in §7.2 and Proposition 7.4.1, respectively. The primitive f_ω is defined uniquely up to an element of $\mathcal{C}^{K,n+1}(X)$. We also denote by $\mathcal{U}_\omega$ the open subset of X introduced after the formulation of Proposition 7.4.3 in the case when X is marked and of dimension at least 2, and set $\mathcal{U}_\omega = X$ in all other cases.

7.5.1. Lemma. *Let $\varphi : X' \to X$ be a morphism between atomic k-analytic spaces of dimension at most $m+1$, and let $\omega \in \Omega^1_{\mathcal{S}^n}(X)$ be an atomic closed one-form on X such that its pullback $\varphi^*(\omega)$ is also atomic. Then $(f_{\varphi^*(\omega)} - \varphi^*(f_\omega))\big|_{\mathcal{V}} \in \mathcal{C}^{K,n+1}(\mathcal{V})$, where $\mathcal{V} = \mathcal{U}_{\varphi^*(\omega)} \cap \varphi^{-1}(\mathcal{U}_\omega)$.*

Proof. If both spaces are split, the statement follows from Lemma 7.2.2(iii). In the general case, by Corollary 4.1.4, it suffices to show that the function $g = \psi^*(f_{\varphi^*(\omega)} - \varphi^*(f_\omega))$ is contained in $\mathcal{C}^{K,n+1}(Z)$ for any morphism $\psi : Z \to \mathcal{V}$ from a smooth basic k'-analytic curve Z, where k' is a finite extension of k.

Assume that both spaces X and X' are marked. Increasing the field k, we may assume that X and X' are the generic fibers of proper marked formal schemes $\mathfrak{X}$ and $\mathfrak{X}'$ over k°, respectively, and $Z = \mathcal{X}_\eta^{\mathrm{an}} \backslash \cup_{\mu=1}^{l} E(0; r_\mu)$, where $\mathcal{X}$ is a smooth projective curve over k°, $\mathbf{x}_1, \ldots, \mathbf{x}_l$ are pairwise distinct $\widetilde{k}$-rational points of $\mathcal{X}_s$ and $0 < r_\mu < 1$ with $r_\mu \in |k^*|$. Given numbers $r_\mu < r'_\mu < 1$ with $r'_\mu \in \sqrt{|k^*|}$, let W denote the strictly affinoid domain $\mathcal{X}_\eta^{\mathrm{an}} \backslash \cup_{\mu=1}^{l} D(0, r'_\mu)$ and W' the open set $\mathcal{X}_\eta^{\mathrm{an}} \backslash \cup_{\mu=1}^{l} E(0, r'_\mu)$. It suffices to show that $g\big|_{W'} \in \mathcal{C}^{K,n+1}(W')$. For this we may increase the field k and assume that $r'_\mu \in |k^*|$ for all $1 \le \mu \le l$, i.e., that W is a k-affinoid basic curve. By Proposition 2.4.2 applied to the induced morphism $W \to (\mathfrak{X} \times \mathfrak{X}')_\eta$, we can increase the field k and find a finite open covering $W' = \cup_\nu W'_\nu$ such that each W'_ν is the generic fiber of a proper marked formal scheme $\mathfrak{Y}_\nu$ over k° and the induced morphisms $W_\nu \to \mathfrak{X}_\eta$ and $W'_\nu \to \mathfrak{X}'_\eta$ come from morphisms of formal schemes $\mathfrak{Y}_\nu \to \mathfrak{X}$ and $\mathfrak{Y}_\nu \to \mathfrak{X}'$, respectively. Since the latter two morphisms satisfy the assumption of Proposition 7.4.3, it follows that the functions $\psi^*(f_{\varphi^*(\omega)}) - f_{(\varphi\psi)^*(\omega)}$ and $(\varphi\psi)^*(f_\omega) - f_{(\varphi\psi)^*(\omega)}$ are contained in $\mathcal{C}^{K,n+1}(Z)$, i.e., $g \in \mathcal{C}^{K,n+1}(Z)$.

If one of the spaces is split and another one is marked, then the reasoning from the previous paragraph reduces the situation to the case when the morphism from Z to the marked space is induced by a morphism of the corresponding proper marked formal schemes and so, by Proposition 7.4.3, the pullback of the primitive on the marked space coincide with the primitive of the pullback of the one-form on Z (up to an element of $\mathcal{C}^{K,n+1}(Z)$). The similar fact is true for the morphism of Z to the split space, by the induction hypotheses. ∎

7.5.2. Lemma. *Let X be a smooth k-analytic space of dimension at most $m+1$ with $H^1(X, \mathfrak{c}_X) = 0$. Then every closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$ has a primitive $f \in \mathfrak{N}^{K,n+1}(X)$ which is defined uniquely up to an element of $\mathcal{C}^{K,n+1}(X)$ by the following property: given a morphism $\varphi : V \to X$ from an atomic k-analytic space of dimension at most $m+1$ such that the one-form $\varphi^*(\omega)$ is atomic, one has $\varphi^*(f) - f_{\varphi^*(\omega)} \in \mathcal{C}^{K,n+1}(V)$.*

Proof. Step 1. Given a closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$, we say that a morphism $\varphi : V \to X$ is *ω-atomic* if (1) V is an atomic k-analytic space; (2) the one-form $\varphi^*(\omega)$ is atomic; (3) φ is étale, if V is split, or a marked neighborhood of the image of the generic point of V, otherwise. Furthermore, we say that a family of morphisms $\{V_i \xrightarrow{\varphi_i} X\}_{i\in I}$ is an *ω-atomic covering of X* if all of the morphisms φ_i are ω-atomic and $X = \cup_{i\in I}\varphi_i(\mathcal{V}_i)$, where $\mathcal{V}_i$ denotes the étaleness locus of the morphism $\mathcal{U}_{\varphi_i^*(\omega)} \to X$ (i.e., $\{\mathcal{V}_i \xrightarrow{\varphi_i} X\}_{i\in I}$ is an étale covering of X). By the induction hypotheses and Lemmas 7.2.1 and 7.3.1, an ω-atomic covering of X always exists. Given a pair $i, j \in I$, consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{V}_i & \xrightarrow{\varphi_i} & X \\ \uparrow\psi_j & & \uparrow\varphi_j \\ \mathcal{V}_{ij} & \xrightarrow{\psi_i} & \mathcal{V}_j \end{array}$$

We claim that $\psi_i^*(f_{\varphi_j^*(\omega)}) - \psi_j^*(f_{\varphi_i^*(\omega)}) \in \mathcal{C}^{K,n+1}(\mathcal{V}_{ij})$. Indeed, if η is the one-form $\psi_i^*(\varphi_j^*(\omega)) = \psi_j^*(\varphi_i^*(\omega))$, it suffices to verify that, for any η-atomic morphism $\alpha : W \to \mathcal{V}_{ij}$, the restrictions of the functions $f_{\alpha^*(\eta)} - \alpha^*(\psi_i^*(f_{\varphi_j^*(\omega)}))$ and $f_{\alpha^*(\eta)} - \alpha^*(\psi_j^*(f_{\varphi_i^*(\omega)}))$ to the open subset $\mathcal{U}_{\alpha^*(\eta)}$ are contained in $\mathcal{C}^{K,n+1}(\mathcal{U}_{\alpha^*(\eta)})$, but this follows from Lemma 7.5.1.

Thus, the one-form ω defines a one-cocycle of the étale covering $\{\mathcal{V}_i \xrightarrow{\varphi_i} X\}_{i\in I}$ with coefficients in the étale sheaf $\mathcal{C}_X^{K,n+1}$. Since $H^1(X, \mathfrak{c}_X) = 0$, this one-cocycle is a co-boundary and, therefore, there is a primitive $f \in \mathfrak{N}^{K,n+1}(X)$ of ω, unique up to an element of $\mathcal{C}^{K,n+1}(X)$, such that

$(\varphi_i^*(f) - f_{\varphi_i^*(\omega)})\big|_{\mathcal{V}_i} \in \mathcal{C}^{K,n+1}(\mathcal{V}_i)$ for all $i \in I$. From the construction it follows that, for any open subset $X' \subset X$ with $H^1(X', \mathfrak{c}_{X'}) = 0$, the restriction of f to X' coincides, up to an element of $\mathcal{C}^{K,n+1}(X')$, with the primitive of the form $\omega\big|_{X'}$ constructed on X'. Of course, if X is of dimension at most m, the induction hypotheses imply that $f \in \mathcal{S}^{\lambda,n+1}(X)$.

Step 2. *If, in the above situation, the space X and the one-form ω are atomic, then $f - f_\omega \in \mathcal{C}^{K,n+1}(X)$.* Indeed, it remains to verify the claim only in the case when X is marked of dimension $m+1 \geq 2$. Let X be the generic form of a proper marked formal scheme $\mathfrak{X}$ over k°. We know that $(f - f_\omega)\big|_{\mathcal{U}} \in \mathcal{C}^{K,n+1}(\mathcal{U})$ for an open neighborhood $\mathcal{U}$ of the generic point of $\mathfrak{X}$. Corollary 3.2.6 then implies that it suffices to verify that $(f - f_\omega)_{\mathbf{x}} \in \mathcal{C}^{K,n+1}(\pi^{-1}(\mathbf{x}))$ for every closed point $\mathbf{x} \in \mathfrak{X}_s$. Since ω is marked, one has $\omega_{\mathbf{x}} \in \Omega^1_{L^{\lambda,n}}(\pi^{-1}(\mathbf{x}))$ and $(f_\omega)_{\mathbf{x}} \in L^{\lambda,n+1}(\pi^{-1}(\mathbf{x}))$. Increasing the field k, we may assume that the point $\mathbf{x}$ is $\widetilde{k}$-rational. It follows from the description of Lemma 3.1.2 that the space $\pi^{-1}(\mathbf{x})$ can be covered by open subsets $\mathcal{U}$ isomorphic to a product of open annuli and discs. Of course, such a subset $\mathcal{U}$ is split and the restriction of the form ω to it is split. The claim now follows from the fact that it is known to be true for split spaces.

The statement of the lemma now easily follows from the construction of f, Step 2 and Lemma 7.5.1. ∎

7.5.3. Corollary. *Given an atomic k-analytic space X of dimension at most $m+1$ and an atomic closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$, the one-form ω is atomic with respect to any other structure of an atomic space on X, and its primitive f_ω does not depend, up to an element of $\mathcal{C}^{K,n+1}(X)$, on the choice of such a structure.* ∎

7.5.4. Corollary. *Let $\varphi : X' \to X$ be a morphism between smooth k-analytic spaces of dimension at most $m+1$ with $H^1(X, \mathfrak{c}_X) = H^1(X', \mathfrak{c}_{X'}) = 0$. Given a closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$, one has $f_{\varphi^*(\omega)} - \varphi^*(f_\omega) \in \mathcal{C}^{K,n+1}(X')$.* ∎

For a smooth k-analytic space X of dimension at most $m+1$ with $H^1(X, \mathfrak{c}_X) = 0$, let $\mathcal{P}^{\lambda,n+1}(X)$ denote the k-vector subspace of $\mathfrak{N}^{K,n+1}(X)$ generated by the primitives f_ω for all closed one-forms $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$. It follows that there is an exact sequence of vector spaces

$$0 \longrightarrow \mathcal{C}^{K,n+1}(X) \longrightarrow \mathcal{P}^{\lambda,n+1}(X) \xrightarrow{d} \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,n}}(X) \longrightarrow 0$$

and, given a morphism $\varphi : X' \to X$ as in Corollary 7.5.4, one has $\varphi^*(f) \in \mathcal{P}^{\lambda,n+1}(X')$ for all functions $f \in \mathcal{P}^{\lambda,n+1}(X)$. Furthermore, given an open covering $X = \cup_{i \in I} \mathcal{U}_i$ by open subsets with $H^1(\mathcal{U}_i, \mathfrak{c}_X) = 0$, if a function $f \in \mathfrak{N}^{K,n+1}(X)$ is such that $f\big|_{\mathcal{U}_i} \in \mathcal{P}^{\lambda,n+1}(\mathcal{U}_i)$ for all $i \in I$, then

$f \in \mathcal{P}^{\lambda,n+1}(\mathrm{X})$. Indeed, since the one-forms $(df)\big|_{\mathcal{U}_i} \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,n}}(\mathcal{U}_i)$ are compatible on intersections, there exists a one-form $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,n}}(X)$ with $df = \omega$ and, therefore, $(f - f_\omega)\big|_{\mathcal{U}_i} \in \mathcal{C}^{K,n+1}(\mathcal{U}_i)$ for all $i \in I$, and Lemma 7.5.2 easily implies that $f - f_\omega \in \mathcal{C}^{K,n+1}(X)$, i.e., $f \in \mathcal{P}^{\lambda,n+1}(X)$.

We now extend the definition of $\mathcal{P}^{\lambda,n+1}(X)$ to an arbitrary smooth k-analytic space of dimension $m+1$ using the main result of [Ber9], which implies that a basis of topology of X is formed by the open subsets $\mathcal{U}$ with $H^1(\mathcal{U}, \mathfrak{c}_{\mathcal{U}}) = 0$. Namely, we let $\mathcal{P}^{\lambda,n+1}(X)$ denote the vector space of all functions $f \in \mathfrak{N}^{K,n+1}(X)$ such that $f\big|_{\mathcal{U}} \in \mathcal{P}^{\lambda,n+1}(\mathcal{U})$ for all open subsets $\mathcal{U}$ with $H^1(\mathcal{U}, \mathfrak{c}_{\mathcal{U}}) = 0$. It follows that, given a morphism $\varphi : X' \to X$ of smooth k-analytic spaces of dimension at most $m+1$, one has $\varphi^*(f) \in \mathcal{P}^{\lambda,n+1}(X')$ for all $f \in \mathcal{P}^{\lambda,n+1}(X)$. Lemma 7.5.2 implies that the correspondence $X' \mapsto \mathcal{P}^{\lambda,n+1}(X')$ is a sheaf in the étale topology of X, and so there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{C}^{K,n+1}_X \longrightarrow \mathcal{P}^{\lambda,n+1}_X \xrightarrow{d} \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,n},X} \longrightarrow 0 \ .$$

Of course, if $\dim(X) \le m$, the sheaf $\mathcal{P}^{\lambda,n+1}_X$ coincides with that considered in §7.1.

Finally, we define $\mathcal{S}^{\lambda,n+1}_X$ as the étale subsheaf of $\mathcal{O}_X$-modules in $\mathfrak{N}^{K,n+1}_X$ generated by $\mathcal{P}^{\lambda,n+1}_X$. Since $d(\mathcal{P}^{\lambda,n+1}_X) \subset \Omega^1_{\mathcal{S}^{\lambda,n},X}$, it follows that $d(\mathcal{S}^{\lambda,n+1}_X) \subset \Omega^1_{\mathcal{S}^{\lambda,n+1},X}$, i.e., $\mathcal{S}^{\lambda,n+1}_X$ is a $\mathcal{D}_X$-submodule of $\mathfrak{N}^{K,n+1}_X$.

7.6. End of the proof. It remains to verify that the $\mathcal{D}_X$-modules $\mathcal{S}^{\lambda,n+1}_X$ satisfy all of the induction hypotheses and the properties (i)-(iii) of Theorem 1.6.2, and that they are unique. First of all, the validity of the properties (a), (c), (d), (e) and (f) from Theorem 1.6.1 and of the property (IH2) trivially follow from the construction.

The property (b). We have to prove that, if $df = 0$ for a function $f \in \mathcal{S}^{\lambda,n+1}(X)$, then $f \in \mathcal{C}^{K,n+1}(X)$. Corollary 4.1.3(ii) reduces the situation to the case when X is a smooth k-analytic curve. Since the statement is local in the étale topology of X, we may assume that X is atomic and $f = \sum_{i=1}^l f_i g_i$, where $f_i \in \mathcal{O}(X)$ and all $g_i \in \mathcal{P}^{\lambda,n+1}(X)$ are such that the one-forms $dg_i \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$ are atomic. If they are in fact split and, in particular, X is isomorphic to D or B, it follows that $f \in L^{\lambda,n+1}(X)$, and the required fact is trivial. Assume therefore, that X and all dg_i are marked. In this case we may even assume that $dg_{i,\sigma} \in \mathcal{G}^{\lambda,n}_{\mathfrak{X},\sigma} \otimes_{\mathcal{O}_{\mathfrak{X}_{\eta,\sigma}}} \Omega^1_{\mathfrak{X}_{\eta,\sigma}}$ for all $1 \le i \le l$, where $\mathfrak{X}$ is a proper marked formal scheme with $\mathfrak{X}_\eta = X$ and σ is its generic point. Lemma 7.4.2(iv) implies that $g_{i,\sigma} \in \mathcal{F}^{\lambda,n+1}_{\mathfrak{X},\sigma}$ for all $1 \le i \le l$ and, therefore, $f_\sigma \in \mathcal{F}^{\lambda,n+1}_{\mathfrak{X},\sigma}$. The latter implies that $f_\sigma \in \mathcal{C}^{K,n+1}_{\mathfrak{X}_\eta,\sigma}$. Since $f_{\mathbf{x}} \in \mathcal{C}^{K,n+1}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$ it follows that $f \in \mathcal{C}^{K,n+1}(\mathfrak{X}_\eta)$.

To verify the remaining properties and, in particular, the property (g) formulated in §7.1, we make a preliminary observation similar to that from §7.1.

Suppose we are given a proper marked formal scheme $\mathfrak{X}$ over k° with $\dim(\mathfrak{X}_\eta) \le m+1$ and a nonempty open affine subscheme $\mathfrak{Z} \subset \mathfrak{X}^\circ$. Since $H^1(\mathfrak{Z}_\eta, \mathfrak{c}_{\mathfrak{Z}_\eta}) = 0$, every closed one-form $\omega \in \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,n}}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$ has a primitive in $\mathcal{S}^{\lambda,n+1}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$ and, therefore, given a Frobenius lifting ϕ on $(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$, the injective homomorphisms of F-isocrystals $E^i(\mathfrak{X}_\eta, \mathfrak{Z}) \to \mathcal{S}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$ considered in §7.1 for $0 \le i \le n$ extends to a similar homomorphism for $i = n+1$. The properties of the sheaves $\mathcal{S}^{\lambda,n+1}_X$ already known imply that the induced homomorphism $E^{K,n+1}(\mathfrak{X}_\eta, \mathfrak{Z}) \to \mathcal{S}^{\lambda,n+1}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$ is injective and that its image $\mathcal{G}^{\lambda,n+1}(\mathfrak{X}, \mathfrak{Z})$ does not depend on the choice of the Frobenius lifting. Let $\mathcal{G}^{\lambda,n+1}_{\mathfrak{X},\sigma}$ be the union of the latter in $\mathcal{S}^{\lambda,n+1}_{\mathfrak{X}_\eta,\sigma}$ taken over all $\mathfrak{Z}$'s. Given $0 \le i,j \le n+1$ with $i+j = n+1$, Lemma 6.4.2, applied to the multiplication homomorphisms from $E^i(\mathfrak{X}_\eta, \mathfrak{Z}) \otimes_B E^{j-1}(\mathfrak{X}_\eta, \mathfrak{Z})$ and $E^{i-1}(\mathfrak{X}_\eta, \mathfrak{Z}) \otimes_B E^j(\mathfrak{X}_\eta, \mathfrak{Z})$ to $E^{i+j-1}(\mathfrak{X}_\eta, \mathfrak{Z})$, implies that $\mathcal{G}^{\lambda,i}(\mathfrak{X}, \mathfrak{Z}) \cdot \mathcal{G}^{\lambda,j}(\mathfrak{X}, \mathfrak{Z}) \subset \mathcal{G}^{\lambda,n+1}(\mathfrak{X}, \mathfrak{Z})$ and, therefore, $\mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma} \cdot \mathcal{G}^{\lambda,j}_{\mathfrak{X},\sigma} \subset \mathcal{G}^{\lambda,n+1}_{\mathfrak{X},\sigma}$.

The property (g). It is enough to show that, given $f \in \mathcal{P}^{\lambda,i}(X)$ and $g \in \mathcal{P}^{\lambda,j}(X)$, one has $f \cdot g \in \mathcal{P}^{\lambda,n+1}(X)$ in the case when $\dim(X) = m+1$, $1 \le i,j \le n$ and $i+j = n+1$. Since this fact is local in the étale topology, we may assume that the space X and the one-forms $df \in \Omega^1_{\mathcal{S}^{\lambda,i-1}}(X)$ and $dg \in \Omega^1_{\mathcal{S}^{\lambda,j-1}}(X)$ are atomic. If they are in fact split, then the required fact easily follows from the induction hypotheses. Assume they are marked. Using the induction hypothesis (IH3) and increasing the field k, we may assume in addition that X is the generic fiber $\mathfrak{X}_\eta$ of a proper marked formal scheme $\mathfrak{X}$ over k°, $f_\sigma \in \mathcal{G}^{\lambda,i}_{\mathfrak{X},\sigma}$ and $g_\sigma \in \mathcal{G}^{\lambda,j}_{\mathfrak{X},\sigma}$, where σ is the generic point of $\mathfrak{X}$. It follows that $f \cdot g \in R^{\lambda,n+1}(\mathfrak{X})$ and $(f \cdot g)_\sigma \in \mathcal{G}^{\lambda,n+1}_{\mathfrak{X},\sigma}$. On the other hand, the closed one-form $\omega = f \cdot dg + g \cdot df \in \Omega^1_{\mathcal{S}^{\lambda,n}}(\mathfrak{X}_\eta)$ possesses the property (MF1), and since $\omega_\sigma \in \mathcal{G}^{\lambda,n}_{\mathfrak{X},\sigma} \otimes_{\mathcal{O}_{\mathfrak{X}_{\eta,x}}} \Omega^1_{\mathfrak{X}_\eta,\sigma}$, it also possesses the property (MF2), i.e., given a Frobenius lifting ϕ at σ, there exists a polynomial $P(T) \in k[T]$ with no roots-of-unity roots and $P(\phi^*)\omega_\sigma \in d\mathcal{G}^{\lambda,n}_{\mathfrak{X},\sigma}$. Since $d(f \cdot g) = \omega$, it follows that $d(P(\phi^*)(f \cdot g)_\sigma) \in d\mathcal{G}^{\lambda,n}_{\mathfrak{X},\sigma}$, and since the kernel of the differential on $\mathcal{G}^{\lambda,n+1}_{\mathfrak{X},\sigma}$ coincides with $\mathcal{C}^{K,n+1}_{\mathfrak{X}_\eta,\sigma} = \mathcal{C}^{K,n+1}(\mathfrak{X}_\eta)$, it follows that $P(\phi^*)(f \cdot g)_\sigma \in \mathcal{G}^{\lambda,n}_{\mathfrak{X},\sigma} + \mathcal{C}^{K,n+1}_{\mathfrak{X}_\eta,\sigma} \subset \widetilde{\mathcal{S}}^{\lambda,n}_{\mathfrak{X}_\eta,\sigma}$. Proposition 7.4.1(i) now implies that the function $f \cdot g$ is precisely the primitive of ω given by the proposition, i.e., $f \cdot g \in \mathcal{P}^{\lambda,n+1}(\mathfrak{X}_\eta)$.

The property (IH1). It suffices to verify that $\mathcal{G}^{\lambda,n+1}(\mathfrak{X}_\eta, \mathfrak{Z}) \subset R^{\lambda,n+1}(\mathfrak{X}, \mathfrak{Z}_\eta)$ and, for the latter, it suffices to show that any element $f \in \mathcal{G}^{\lambda,n+1}(\mathfrak{X}_\eta, \mathfrak{Z})$ with $df \in \mathcal{G}^{\lambda,n}(\mathfrak{X}_\eta, \mathfrak{Z}) \otimes_B \Omega^1_B$ is contained in $R^{\lambda,n+1}(\mathfrak{X}, \mathfrak{Z}_\eta)$. By the induction hypothesis, $\mathcal{G}^{\lambda,n}(\mathfrak{X}_\eta, \mathfrak{Z}) \subset R^{\lambda,n}(\mathfrak{X}, \mathfrak{Z}_\eta)$. Let $\mathcal{U}$ be an open neighborhood of $\mathfrak{Z}_\eta$ in $\mathfrak{X}_\eta$ with $f \in \mathcal{S}^{\lambda,n+1}(\mathcal{U})$ and such that, for every closed point $\mathbf{x} \in \mathfrak{X}_s$, the intersection $\pi^{-1}(\mathbf{x}) \cap \mathcal{U}$ is either an open disc or annulus and the restriction of df to it is contained in $\Omega^1_{L^{\lambda,n}}(\pi^{-1}(\mathbf{x}) \cap \mathcal{U})$. The latter has a primitive in $L^{\lambda,n+1}(\pi^{-1}(\mathbf{x}) \cap \mathcal{U}) \subset \mathcal{S}^{\lambda,n+1}(\pi^{-1}(\mathbf{x}) \cap \mathcal{U})$, and the required fact follows.

The property (IH3). It is enough to verify the properties (IH3.1)-(IH3.3) for a function $f \in \mathcal{P}^{\lambda,n+1}(X)$ which is a primitive of a closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$. First of all, Lemma 7.3.1 reduces the situation to the case when X is the generic fiber $\mathfrak{X}_\eta$ of a proper marked formal scheme $\mathfrak{X}$ over k°, the one-form ω is marked and $\omega_\sigma \in \mathcal{G}^{\lambda,n}_{\mathfrak{X},\sigma} \otimes_{\mathcal{O}_{\mathfrak{X}_{\eta,\sigma}}} \Omega^1_{\mathfrak{X}_{\eta,\sigma}}$. The construction of f from Proposition 7.4.1 immediately implies that $f \in R^{\lambda,n+1}(\mathfrak{X})$, i.e., (IH3.1) is true. Since $f_\sigma \in \mathcal{S}^{\lambda,n+1}_{\mathfrak{X}_{\eta,\sigma}}$ and $\omega_\sigma = dg$ for some $g \in \mathcal{G}^{\lambda,n+1}_{\mathfrak{X},\sigma}$, it follows that $f_\sigma - g \in \mathcal{C}^{K,n+1}_{\mathfrak{X}_{\eta,\sigma}} = \mathcal{C}^{K,n+1}(\mathfrak{X}_\eta)$ and, therefore, $f_\sigma \in \mathcal{G}^{\lambda,n+1}_{\mathfrak{X},\sigma}$, i.e., (IH3.2) is true. Finally, given a stratum closure $\mathcal{Y} \subset \mathfrak{X}_s$, the property (MF3) of ω tells that there exists a small open affine subscheme $\mathfrak{Y} \subset \mathfrak{X}$ with $\mathfrak{Y} \cap \mathring{\mathcal{Y}} \neq \emptyset$ and the following property: if $\mathcal{Y} = \mathfrak{X}$ (resp. $\mathcal{Y}$ is arbitrary), there is a p_1-discoid (resp. p_1-semi-annular) open neighborhood $\mathcal{V}$ of $\mathfrak{D}_{\mathcal{Y}'}$ in $\mathfrak{D}_{\mathcal{Y}}$ with $\mathcal{Y}' = \mathcal{Y} \cap \mathfrak{Y}_s$ such that $p_2^*(\omega) \in \Omega^1_M(\mathcal{V})$, where $M = p_1^\#(\mathcal{S}^{\lambda,n}(\mathcal{U}))$ (resp. $M = \sum_{j=0}^n p_1^\#(\mathcal{S}^{\lambda,j}(\mathcal{U}))\cdot L^{\lambda,n-j}(\mathcal{V})$) and $\mathcal{U} = p_1(\mathcal{V})$. Shrinking $\mathcal{U}$, we may assume that the $\mathcal{O}_{\mathcal{U}}$-module $\Omega^1_{\mathcal{U}}$ is free and, therefore, we can apply Proposition 1.3.2 (resp. Corollary 1.5.6). It follows that $p_2^*(\omega) = dg$ for some $g \in N$, where $N = p_1^\#(\mathcal{S}^{\lambda,n+1}(\mathcal{U}))$ (resp. $N = \sum_{j=0}^{n+1} p_1^\#(\mathcal{S}^{\lambda,j}(\mathcal{U}))\cdot L^{\lambda,n+1-j}(\mathcal{V})$). The restriction of g to the diagonal $\Delta(\mathcal{U})$ lies in $\mathcal{S}^{\lambda,n+1}(\Delta(\mathcal{U}))$, and the restriction of $p_2^*(f)$ coincides with that of $p_1^*(f)$ and, therefore, also lies in $\mathcal{S}^{\lambda,n+1}(\Delta(\mathcal{U}))$. By the property (a) already verified, we may assume that both restrictions are equal. *We then claim that* $p_2^*(f) = g$. Indeed, it suffices to verify the equality at the fiber $p_1^{-1}(x)$ of every point $x \in \mathcal{U}_0$. The restriction of g to it lies in $L^{\lambda,n+1}(p_1^{-1}(x))$ and, by the property (f), the restriction of $p_2^*(f)$ lies in $\mathcal{S}^{\lambda,n+1}(p_1^{-1}(x))$. The claim now follows from the property (b).

The properties (ii) and (iii) of Theorem 1.6.2 easily follow from the already established properties of Theorem 1.6.1 and the construction of the sheaves $\mathcal{S}^{\lambda,n+1}_X$, respectively.

The property (i). By Theorem 4.1.1, we may assume that either X is isomorphic to an open disc or annulus and $f \in L^{\lambda,n+1}(X)$, or $X = \mathfrak{X}_\eta$, where $\mathfrak{X}$ is a proper marked formal scheme over k° with $\dim(\mathfrak{X}_\eta) = 1$, and $f \in R^{\lambda,n+1}(\mathfrak{X})$ and $f_\sigma \in \mathcal{F}^{\lambda,n+1}_{\mathfrak{X},\sigma}$. In the first case, the required fact follows from Lemma 1.4.2. In the second case, the same lemma implies that if $\mathcal{U} \cap \pi^{-1}(\mathbf{x}) \neq \emptyset$, then $f\big|_{\pi^{-1}(\mathbf{x})} = 0$, and the required fact follows from Lemma 6.5.1.

Uniqueness of the $\mathcal{D}_X$-modules $\mathcal{S}^{\lambda,n+1}_X$. Assume one can construct another $\mathcal{D}_X$-submodule $\widehat{\mathcal{S}}^{\lambda,n+1}_X \subset \mathfrak{N}^{K,n+1}_X$ which together with the $\mathcal{D}_X$-submodules $\mathcal{S}^{\lambda,i}_X$, $0 \le i \le n$, possesses the properties of Theorem 1.6.1. It follows easily that these sheaves also satisfy the property (ii) of Theorem 1.6.2. It suffices to verify that, given a closed one-form $\omega \in \Omega^1_{\mathcal{S}^{\lambda,n}}(X)$, its local primitives in $\widehat{\mathcal{S}}^{\lambda,n+1}_X$ are local sections of $\mathcal{P}^{\lambda,n+1}_X$. For this we may assume that X and ω are atomic, and let f and $\widehat{f}$ be primitives of ω in $\mathcal{S}^{\lambda,n+1}(X)$ and $\widehat{\mathcal{S}}^{\lambda,n+1}(X)$, respectively. If X and ω are in fact split, the required fact $f - \widehat{f} \in \mathcal{C}^{K,n+1}(X)$ follows from the construction of the primitive of ω in §7.2 and the property (e). Assume therefore that they are marked, and let $X = \mathfrak{X}_\eta$, where $\mathfrak{X}$ is a proper marked scheme over k°. By (MF1), one has $\omega \in \Omega^1_{R^{\lambda,n}}(\mathfrak{X})$ and, therefore, the property (e) and the Logarithmic Poincaré Lemma imply that $\widehat{f} \in R^{\lambda,n+1}(\mathfrak{X})$. Furthermore, by (MF2), there exists a Frobenius lifting ϕ at σ and a polynomial $P(T) \in k[T]$ without roots-of-unity roots such that $P(\phi^*)\omega_\sigma = dg$ with $g \in \mathcal{S}^{\lambda,n}_{\mathfrak{X}_{\eta,\sigma}}$. It follows that $d(P(\phi^*)\widehat{f}_\sigma - g) = 0$. But from the property (f) it follows that $P(\phi^*)\widehat{f}_\sigma - g \in \widehat{\mathcal{S}}^{\lambda,n+1}_{\mathfrak{X}_{\eta,\sigma}}$, and the property (b) implies that $P(\phi^*)\widehat{f}_\sigma - g \in \mathcal{C}^{K,n+1}(\mathfrak{X}_\eta)$, i.e., $P(\phi^*)\widehat{f}_\sigma \in \mathcal{S}^{\lambda,n}_{\mathfrak{X}_{\eta,\sigma}}$. It follows now from Proposition 7.4.1(i) that $f - \widehat{f} \in \mathcal{C}^{K,n+1}(X)$.

§8. Properties of the sheaves $\mathcal{S}^\lambda_X$

In this section we refine information on the sheaves $\mathcal{S}^\lambda_X$ using the fact that they exist. First of all, we show that any connected wide germ with good reduction (X,Y) can be provided with a unique filtered $D_{(X,Y)}$-subalgebra $\mathcal{E}^\lambda(X,Y) \subset \mathcal{S}^\lambda(X,Y)$ with the properties (a)-(d) and (f) of Coleman's algebras mentioned in the introduction, and we relate it to the isocrystal $E^\lambda(X,\mathfrak{Y})$ considered in §5.5 and §7.1. We also show that any proper marked formal scheme $\mathfrak{X}$ over k° can be provided with a unique filtered $\mathcal{O}(\mathfrak{X}_\eta)$-subalgebra $\mathcal{E}^\lambda(\mathfrak{X}) \subset \mathcal{S}^\lambda(\mathfrak{X}_\eta)$ which is a filtered $D_{\mathfrak{X}_\eta}$-algebra that possesses similar properties and some sort of continuity (see Theorem 8.2.1(1)). Given a function $f \in \mathcal{S}^\lambda(X)$ on a smooth k-analytic space X, every point $x \in X$ with $s(x) = \dim(X)$ has a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ with $\varphi^*(f)$ in the image of the canonical injective homomorphism $\mathcal{E}^\lambda(\mathfrak{X}) \otimes_k K \to \mathcal{S}^\lambda(\mathfrak{X}_\eta)$. The latter is used to provide each stalk of $\mathcal{S}^\lambda_X$ at a geometric point $\overline{x}$ over a point $x \in X$ with a $G_{\overline{x}/x}$-invariant filtered $D_{\mathcal{O}_{X,\overline{x}}}$-subalgebra $\mathcal{E}^\lambda_{X,\overline{x}}$ which is functorial with respect to $(k, X, \overline{x}, K, \lambda)$ and such that the homomorphism $k_{\mathrm{Log}} \to K : \mathrm{Log}(p) \mapsto \lambda$ gives rise to an isomorphism $\mathcal{E}_{X,\overline{x}} \xrightarrow{\sim} \mathcal{E}^\lambda_{X,\overline{x}}$ and, if $f_1, \dots, f_t$ are elements of $\mathcal{O}^*_{X,x}$ for which $|f_1(x)|, \dots, |f_t(x)|$ for a basis of the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$, then there is a $G_{\overline{x}/x}$-equivariant isomorphism of filtered $D_{\mathcal{O}_{X,\overline{x}}}$-algebras $\mathcal{E}^\lambda_{X,\overline{x}}[T_1, \dots, T_t] \otimes_k K \xrightarrow{\sim} \mathcal{S}^\lambda_{X,\overline{x}} : T_i \mapsto \mathrm{Log}^\lambda(f_i)$. We prove that the subspace $\mathcal{V}_{X,x} \subset \Omega^{1,\mathrm{cl}}_{X,x}/d\mathcal{O}_{X,x}$, introduced in §4.5, coincides with the space of the classes of closed one-forms $\Omega^1_{X,x}$ that admit a primitive in $\mathcal{E}^\lambda_{X,x} = (\mathcal{E}^\lambda_{X,\overline{x}})^{G_{\overline{x}/x}}$. If $\dim(X) = 1$ and x is a point of type (2), we construct a $G_{\overline{x}/x}$-equivariant isomorphism between $\mathcal{E}_{X,\overline{x}}$ and the shuffle algebra $\mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}}$. In §8.4, we prove a uniqueness result which implies the following generalization of the property of the logarithmic function mentioned at the beginning of the introduction. Assume that X is connected, the $\mathcal{O}_X$-module Ω^1_X is free over a nonempty Zariski open subset and $H^1(X, \mathfrak{c}_X) = 0$. Assume also we are given closed analytic one-forms $\{\omega_i\}_{i\in I}$ which are linearly independent over k modulo exact one-forms and, for a point $x \in X(k)$, let f_i be a primitive of ω_i in $\mathcal{O}_{X,x}$. Then the functions $\{f_i\}_{i\in I}$ are algebraically independent over the image of $\mathcal{O}(X)$ in $\mathcal{O}_{X,x}$. Finally, in §8.5, we consider the filtered $\mathcal{D}_X$-subalgebra $\mathfrak{s}_X = \mathcal{S}_X \cap \mathfrak{n}_X$ and prove that $\mathfrak{s}_X \cap \mathcal{L}_X = \mathcal{O}_X$ and that the subsheaf $\Psi_X \subset \Omega^{1,\mathrm{cl}}_X/d\mathcal{O}_X$, introduced in §4.5, coincides with the sheaf of the classes of closed one-forms that admit local primitives which are sections of $\mathfrak{s}_X$.

8.1. Filtered $D_{(X,Y)}$-algebras $\mathcal{E}^\lambda(X,Y)$ for germs with good reduction. We say that a k-germ (X,Y) (or a k-analytic space Y) has *good reduction* if Y is isomorphic to the generic fiber $\mathfrak{Y}_\eta$ of a smooth formal scheme $\mathfrak{Y}$ over k°. Notice that any wide germ with good reduction is smooth. A *$D_{(X,Y)}$-module* on a smooth k-germ (X,Y) is an $\mathcal{O}(X,Y)$-submodule $M \subset \mathfrak{N}^K(X,Y)$

such that $dM \subset \Omega^1_M(X,Y)$, where $\Omega^q_M(X,Y)$, $q \geq 0$, denotes the image of $M \otimes_{\mathcal{O}(X,Y)} \Omega^q(X,Y)$ in $\Omega^q_{\mathfrak{N}^K}(X,Y)$. A $D_{(X,Y)}$-*algebra* is a $D_{(X,Y)}$-module which is also a subalgebra of $\mathfrak{N}^K(X,Y)$.

8.1.1. Theorem. *Every connected wide k-germ with good reduction (X,Y) can be provided with a unique filtered $\mathcal{O}(X,Y)$-subalgebra $\mathcal{E}^\lambda(X,Y) \subset \mathcal{S}^\lambda(X,Y)$ such that the following is true:*

(a) $\mathcal{E}^\lambda(X,Y)$ is a filtered $D_{(X,Y)}$-algebra;

(b) $\mathcal{E}^{\lambda,0}(X,Y) = \mathcal{O}(X,Y)$;

(c) $\mathrm{Ker}(\mathcal{E}^\lambda(X,Y) \stackrel{d}{\to} \Omega^1_{\mathcal{E}^\lambda}(X,Y)) = \mathfrak{c}(X,Y)$ $(= \mathfrak{c}(Y))$;

(d) every closed one-form $\omega \in \Omega^1_{\mathcal{E}^{\lambda,i}}(X,Y)$ has a primitive $f_\omega \in \mathcal{E}^{\lambda,i+1}(X,Y)$;

(e) $\mathcal{E}^{\lambda,i+1}(X,Y)$ is generated over $\mathcal{O}(X,Y)$ by the above primitives f_ω;

(f) there exists a point $x \in Y_{st}$ such that $f_x \in \mathcal{O}_{X,x}$ for all $f \in \mathcal{E}^\lambda(X,Y)$.

Furthermore, the $D_{(X,Y)}$-algebra $\mathcal{E}^\lambda(X,Y)$ possesses the following properties:

(1) $f_x \in \mathcal{O}_{X,x}$ for all points $x \in Y_{st}$ and all $f \in \mathcal{E}^\lambda(X,Y)$;

(2) the canonical homomorphism $\mathcal{E}^\lambda(X,Y) \otimes_k K \to \mathcal{S}^\lambda(X,Y)$ is injective;

(3) $\mathcal{E}^\lambda(X,Y)$ is functorial with respect to $(k,(X,Y),K,\lambda)$;

(4) the homomorphism $k_{\mathrm{Log}} \to K$ that takes $\mathrm{Log}(p)$ to λ gives rise to an isomorphism $\mathcal{E}(X,Y) \stackrel{\sim}{\to} \mathcal{E}^\lambda(X,Y)$.

8.1.2. Corollary. *Assume that $(X,Y) = (X,\mathfrak{Y}_\eta)$, where $(X,\mathfrak{Y})$ is a wide germ of a smooth affine formal scheme over k°. Then*

(i) if, in addition, $(X,\mathfrak{Y})$ is induced by a similar germ over a finite extension of $\mathbf{Q}_p$ in k, then $\mathcal{E}^\lambda(X,Y)$ is the image of the injective homomorphism of F-isocrystals $E(X,\mathfrak{Y}) \to \mathcal{S}^\lambda(X,\mathfrak{Y}_\eta)$ (from §7.1) and, in particular, that image does not depend on the choice of the Frobenius lifting;

(ii) $\mathcal{E}^\lambda(X,Y)$ is isomorphic to the filtered isocrystal $E(X,\mathfrak{Y})$ (from §5.5) and, in particular, $E(X,\mathfrak{Y})$ can be provided with the structure of a filtered algebra which satisfies the Leibniz rule;

(iii) given a morphism $(X',\mathfrak{Y}') \to (X,\mathfrak{Y})$ of wide germs of smooth affine formal schemes over an embedding $k \hookrightarrow k'$ such that the induced morphism $\mathfrak{Y}'_s \to \mathfrak{Y}_s \otimes_{\widetilde{k}} \widetilde{k}'$ is dominant and quasifinite, for every $i \geq 0$ the induced homomorphism $\mathcal{E}^i(X,Y) \otimes_{\mathcal{O}(X,Y)} \mathcal{O}(X',Y') \to \mathcal{E}^i(X',Y')$ (with $Y' = \mathfrak{Y}'_\eta$) is injective and its cokernel is a free $\mathcal{O}(X',Y')$-module.

Proof of Theorem 8.1.1 and Corollary 8.1.2. First of all, we notice that if the $\mathcal{O}(X,Y)$-subalgebra $\mathcal{E}^\lambda(X,Y)$ with the properties (a)-(f) exists, it is unique. Indeed, for this it is enough to verify that, given a closed one-form ω as in (d), one has $f - f_\omega \in \mathfrak{c}(Y)$ for any primitive f of ω in $\mathcal{S}^{\lambda,i+1}(X,Y)$ with the property (f). We know that $f - f_\omega \in \mathfrak{c}(Y) \otimes_k K^{i+1}$ and, by (f), $(f-f_\omega)_x \in \mathfrak{c}_{X,x}$. Since the intersection of $\mathfrak{c}(Y) \otimes_k K^{i+1}$ and $\mathfrak{c}_{X,x}$ in $\mathfrak{c}_{X,x} \otimes_k K^{i+1}$ coincides with

$\mathfrak{c}(Y)$, the required fact follows.

Suppose first we are in the situation of Corollary 8.1.2(i). In this case we define $\mathcal{E}^\lambda(X,Y)$ as the image of the injective homomorphism of F-isocrystals $E(X,\mathfrak{Y}) \to \mathcal{S}^\lambda(X,\mathfrak{Y}_\eta)$ from §7.1. It obviously possesses the properties (a)-(e), and so, to verify the property (f), it suffices to verify the property (1). Assume that, for some $i \geq 0$, $f_x \in \mathcal{O}_{X,x}$ for all points $x \in (\mathfrak{Y}_\eta)_{st}$ and all $f \in \mathcal{E}^{\lambda,i}(X,\mathfrak{Y})$, and let ω be a closed one-form in $\mathcal{E}^{\lambda,i}(X,\mathfrak{Y}) \otimes_B \Omega^1_B$, where $B = \mathcal{O}(X,\mathfrak{Y}_\eta)$. It has a primitive f in $\mathcal{E}^{\lambda,i+1}(X,\mathfrak{Y})$, and it suffices to verify that $f_{\mathbf{x}} \in \mathcal{O}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{Y}_s$. Recall that, by the construction, there is a monic polynomial $P(T) \in k[T]$ with no roots-of-unity roots such that $P(\phi^*)f = g \in \mathcal{E}^{\lambda,i}(X,\mathfrak{Y})$, where ϕ is a fixed Frobenius lifting on $(X,\mathfrak{Y}_\eta)$. By the induction hypothesis, for every closed point $\mathbf{x} \in \mathfrak{Y}_s$ one has $g_{\mathbf{x}} \in \mathcal{O}(\pi^{-1}(\mathbf{x}))$ and $\omega_{\mathbf{x}} \in \Omega^{1,\mathrm{cl}}(\pi^{-1}(\mathbf{x}))$. Since $\pi^{-1}(\mathbf{x})$ is isomorphic to the open unit polydisc with center at zero over $k_{\mathbf{x}}$, there is a primitive $h_{\mathbf{x}}$ of $\omega_{\mathbf{x}}$ in $\mathcal{O}(\pi^{-1}(\mathbf{x}))$. It follows that $f_{\mathbf{x}} - h_{\mathbf{x}} = \alpha \in k_{\mathbf{x}} \otimes_k K^{i+1}$ and, therefore, $P(1)\alpha = g_{\mathbf{x}} - P(\phi^*)h_{\mathbf{x}} \in \mathcal{O}(\pi^{-1}(\mathbf{x}))$. The latter implies that $\alpha \in k_{\mathbf{x}}$ and, therefore, $f_{\mathbf{x}} \in \mathcal{O}(\pi^{-1}(\mathbf{x}))$.

Suppose now $(X,Y) = (X,\mathfrak{Y}_\eta)$, where $(X,\mathfrak{Y})$ is a wide germ of a smooth formal scheme over k°, and assume that, for some $i \geq 0$, the $D_{(X,Y)}$-module $\mathcal{E}^{\lambda,i}(X,Y)$ is already constructed and, besides the properties (a)-(f), it possesses the following property: for any open affine subscheme $\mathfrak{Z} \subset \mathfrak{Y}$ with $(X,\mathfrak{Z})$ satisfying the assumption of Corollary 8.1.2(i), the restriction of any functions from $\mathcal{E}^{\lambda,i}(X,Y)$ is contained in $\mathcal{E}^{\lambda,i}(X,\mathfrak{Z}_\eta)$. To construct $\mathcal{E}^{\lambda,i+1}(X,Y)$, it suffices to find, for any closed one-form ω as in (d), a primitive $f \in \mathcal{S}^{\lambda,i+1}(X,Y)$ of ω with the property (1). We know that the restriction of ω to any $(X,\mathfrak{Z}_\eta)$ as above has a primitive in $\mathcal{E}^{\lambda,i+1}(X,\mathfrak{Z}_\eta)$ which is unique up to an element of $\mathfrak{c}(\mathfrak{Z}_\eta) = \mathfrak{c}(Y)$. All such primitives give rise to a one-cocycle on $\mathfrak{Y}_s$ with coefficients in $\mathfrak{c}(Y)$. Since the latter is always a co-boundary, there is a primitive of ω in $\mathcal{S}^{\lambda,i+1}(X,Y)$ whose restriction to every $(X,\mathfrak{Z}_\eta)$ as above is contained in $\mathcal{E}^{\lambda,i+1}(X,\mathfrak{Z})$.

Thus, the $\mathcal{D}_{(X,Y)}$-algebra $\mathcal{E}^\lambda(X,Y)$ is constructed, and it is easy to see from the construction that it possesses all of the properties of Theorem 8.1.1 and the properties (i)-(ii) of Corollary 8.1.2. To prove the property (iii), we may assume that $\mathfrak{c}(Y) = k$ and $\mathfrak{c}(Y') = k'$. The proof is easily done, by induction on i, using the fact that the quotient of $\mathcal{E}^{i+1}(X,Y)$ by $\mathcal{E}^i(X,Y)$ is the trivial isocrystal $H^1_{\mathrm{dR}}(\mathcal{E}^i(X,Y)) \otimes_k \mathcal{O}(X,Y)$, the similar fact for (X',Y') and the following claim: *for every $i \geq 0$, the canonical homomorphism $H^1_{\mathrm{dR}}(\mathcal{E}^i(X,Y)) \otimes_k k' \to H^1_{\mathrm{dR}}(\mathcal{E}^i(X',Y'))$ is injective.*

By Lemma 5.5.4(iii), to prove the claim it suffices to consider the case $k' = k$. Furthermore, since there are canonical embeddings $H^1_{\mathrm{dR}}(\mathcal{E}^i(X,Y)) \hookrightarrow H^1_{\mathrm{dR}}(X,Y)^{\otimes(i+1)}$ and $H^1_{\mathrm{dR}}(\mathcal{E}^i(X',Y')) \hookrightarrow H^1_{\mathrm{dR}}(X',Y')^{\otimes(i+1)}$, it suffices to prove the claim for $i = 0$. If $X' = X$ and $\mathfrak{Y}'$ is an open subscheme of $\mathfrak{Y}$, the injectivity of the homomorphism $H^1_{\mathrm{dR}}(X,Y) \to H^1_{\mathrm{dR}}(X',Y')$ follows from [Bert, Corollary 5.7]. In the general case, we can shrink both schemes $\mathfrak{Y}$ and $\mathfrak{Y}'$ so that the induced homomorphism $\mathfrak{Y}'_s \to \mathfrak{Y}_s$ is finite and flat, and then the required injectivity follows from [Bert, Proposition 3.6]. ∎

Let now $(X,\mathfrak{Y})$ be a wide germ of a smooth formal scheme over k°. We denote by $\widetilde{\mathcal{E}}_{(X,\mathfrak{Y})}$ the sheaf on $\mathfrak{Y}_s$ associated to the presheaf $\mathfrak{Z}_s \mapsto \mathcal{E}(X,\mathfrak{Z}_\eta)$. This sheaf is a filtered $\mathcal{D}_{(X,\mathfrak{Y})}$-subalgebra of $\theta(\mathcal{S}_{(X,\mathfrak{Y})})$. Notice that $(\widetilde{\mathcal{E}}_{(X,\mathfrak{Y})})^\nabla = \mathfrak{c}_{\mathfrak{Y}}$ $(= \theta(\mathfrak{c}_{(X,\mathfrak{Y}_\eta)}))$. For a $\mathcal{D}_{(X,\mathfrak{Y})}$-module $\mathcal{F}$, let $\mathcal{F}_{\mathcal{E}}$ and $\mathcal{F}_{\mathcal{E}^i}$ denote the $\mathcal{D}_{(X,\mathfrak{Y})}$-modules $\mathcal{F} \otimes_{\mathcal{O}_{(X,\mathfrak{Y})}} \widetilde{\mathcal{E}}_{(X,\mathfrak{Y})}$ and $\mathcal{F} \otimes_{\mathcal{O}_{(X,\mathfrak{Y})}} \widetilde{\mathcal{E}}^i_{(X,\mathfrak{Y})}$, respectively.

8.1.3. Proposition. *Let $\mathcal{F}$ be a unipotent $\mathcal{D}_{(X,\mathfrak{Y})}$-module $\mathcal{F}$ of rank m and of level n. Then*

(i) the $\mathfrak{c}_{\mathfrak{Y}}$-module $\mathcal{F}^\nabla_{\mathcal{E}}$ is free of rank m and $\mathcal{F}^\nabla_{\mathcal{E}} \otimes_{\mathfrak{c}_{\mathfrak{Y}}} \widetilde{\mathcal{E}}_{(X,\mathfrak{Y})} \xrightarrow{\sim} \mathcal{F}_{\mathcal{E}}$;

(ii) $\mathcal{F}^\nabla_{\mathcal{E}} = (\mathcal{F}^n + \mathcal{F}^{n-1}_{\mathcal{E}^1} + \ldots + \mathcal{F}^1_{\mathcal{E}^{n-1}})^\nabla$, where $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \ldots \subset \mathcal{F}^n = \mathcal{F}$ is a filtration such that each quotient $\mathcal{F}^i/\mathcal{F}^{i-1}$ is a trivial $\mathcal{D}_{(X,\mathfrak{Y})}$-module;

(iii) there is an embedding of $\mathcal{D}_{(X,\mathfrak{Y})}$-modules $\mathcal{F} \hookrightarrow (\widetilde{\mathcal{E}}^{n-1}_{(X,\mathfrak{Y})})^l$ with $l \geq 1$.

Proof. We may assume that $\mathfrak{Y}$ is connected. By Lemma 5.3.3, the properties (i) and (ii) are true for the restrictions of $\mathcal{F}^\nabla_{\mathcal{E}}$ to any open affine subscheme $\mathfrak{Z} \subset \mathfrak{Y}$. This immediately implies (ii) and the second half of (i). It follows also that the sheaf $\mathcal{F}^\nabla_{\mathcal{E}}$ gives rise to a one-cocycle in the Zariski topology of $\mathfrak{Y}_s$ with coefficients in the constant sheaf of groups $\mathrm{GL}_m(\mathfrak{c}(\mathfrak{Y}_\eta))$. Since $\mathfrak{Y}_s$ is irreducible, that cocycle is a coboundary, i.e., the sheaf $\mathcal{F}^\nabla_{\mathcal{E}}$ is free of rank m over $\mathfrak{c}_{\mathfrak{Y}}$, and (i) is true. Finally, an embedding $\mathcal{F} \hookrightarrow (\widetilde{\mathcal{E}}^{n-1}_{(X,\mathfrak{Y})})^l$ is constructed in the same way as the corresponding embedding in the proof of Corollary 5.3.4. Namely, for $i \geq 0$ we set $\widetilde{\mathcal{F}}_i = \mathcal{F}^n_{\mathcal{E}^i} + \mathcal{F}^{n-1}_{\mathcal{E}^{i+1}} + \ldots + \mathcal{F}^1_{\mathcal{E}^{i+n-1}}$ (and so (2) means that $\mathcal{F}^\nabla_{\mathcal{E}} = \widetilde{\mathcal{F}}^\nabla_0$), and *we claim that there is a basis $h_1,\ldots,h_m$ of $\mathcal{F}^\nabla_{\mathcal{E}}$ over $\mathfrak{c}(\mathfrak{Y}_\eta)$ such that the induced isomorphism $\mathcal{F}_{\mathcal{E}} \xrightarrow{\sim} (\widetilde{\mathcal{E}}_{(X,\mathfrak{Y})})^m$ takes each $\widetilde{\mathcal{F}}_i$ into $(\widetilde{\mathcal{E}}^{i+n-1}_{(X,\mathfrak{Y})})^m$.* Indeed, if $n=1$, the claim is trivial. Assume that $n \geq 2$ and the claim is true for $n-1$. We set $\mathcal{G} = \mathcal{F}^{n-1}$ and consider the canonical epimorphism of vector spaces $\mathcal{F}^\nabla_{\mathcal{E}} \to (\mathcal{F}/\mathcal{G})^\nabla_{\mathcal{E}} = (\mathcal{F}/\mathcal{G})^\nabla$. Its kernel is the space $\mathcal{G}^\nabla_{\mathcal{E}}$, and we can find a basis $h_1,\ldots,h_l$ of it with the required property (for $\mathcal{G}$). Furthermore, we take elements $h_{l+1},\ldots,h_m \in \mathcal{F}^\nabla_{\mathcal{E}} = \widetilde{\mathcal{F}}^\nabla_0$ whose images in $(\mathcal{F}/\mathcal{G})^\nabla$ form a basis of that space. Then the elements $h_1,\ldots,h_m$ form a basis of $\mathcal{F}^\nabla_{\mathcal{E}}$, which possesses the required property, since the restriction of the induced isomorphism $\mathcal{F}_{\mathcal{E}} \xrightarrow{\sim} (\widetilde{\mathcal{E}}_{(X,\mathfrak{Y})})^m$ to every open affine subscheme possesses that property (by the proof of Corollary 5.3.4). ∎

8.1.4. Lemma. *Let $\mathcal{F}$ be a $\mathcal{D}_{(X,\mathfrak{Y})}$-module with the property that every point of $\mathfrak{Y}_s$ has an open neighborhood such that the restriction of $\mathcal{F}$ to it is unipotent. Then $\mathcal{F}$ is unipotent.*

Proof. It suffices to show that $\mathcal{F}$ contains a nonzero trivial $\mathcal{D}_{(X,\mathfrak{Y})}$-submodule.

Let $\mathfrak{Z}$ be an open subscheme of $\mathfrak{Y}$ such that $(X,\mathfrak{Z})$ is induced by a similar germ over a finite extension of $\mathbf{Q}_p$ and the restriction of $\mathcal{F}$ to $\mathfrak{Z}_s$ is unipotent. *Then for any open subscheme* $\mathfrak{Z}' \subset \mathfrak{Z}$ *the canonical map* $\mathcal{F}^\nabla(\mathfrak{Z}_s) \to \mathcal{F}^\nabla(\mathfrak{Z}'_s)$ *is an isomorphism.* Indeed, it suffices to show that $H^1_{\mathrm{dR},W}(\mathfrak{Z}_s, \mathcal{F}) = 0$, where $W = \mathfrak{Z}_s \backslash \mathfrak{Z}'_s$ and, since the restriction of $\mathcal{F}$ to $\mathfrak{Z}_s$ is unipotent, it suffices to consider the case $\mathcal{F} = \mathcal{O}_{(X,\mathfrak{Y})}$. By Corollary 5.5.2, we may assume that k is a finite extension of $\mathbf{Q}_p$ and, in that case, the required fact is a consequence of [Bert, Corollary 5.7].

Let now $\{\mathfrak{Z}^i\}_{i\in I}$ be a covering of $\mathfrak{Y}$ by sufficiently small open subschemes which satisfy the assumptions of the previous paragraph. By the above claim, one has $\mathcal{F}^\nabla(\mathfrak{Z}^i_s) \overset{\sim}{\to} \mathcal{F}^\nabla(\mathfrak{Z}^i_s \cap \mathfrak{Z}^j_s)$ for every pair $i, j \in I$. This easily implies that $\mathcal{F}^\nabla(\mathfrak{Y}_s) \overset{\sim}{\to} \mathcal{F}^\nabla(\mathfrak{Z}^i_s)$ for every $i \in I$. In particular, $\mathcal{F}$ contains a nonzero trivial $\mathcal{D}_{(X,\mathfrak{Y})}$-submodule. ∎

8.1.5. Remark. Assume that the valuation on k is discrete, and let $\mathfrak{X}$ be a formal scheme of finite type over k° and $\mathfrak{Y}$ an open smooth subscheme of $\mathfrak{X}$ such that the Zariski closure $\overline{\mathfrak{Y}}_s$ of $\mathfrak{Y}_s$ in $\mathfrak{X}_s$ is proper over $\widetilde{k}$. The triple $T = (\mathfrak{Y}, \overline{\mathfrak{Y}}_s, \mathfrak{X})$ is what A. Besser calls a rigid triple (see [Bes, 2.8]). In [Bes, §4], he defined a ring $A_{\mathrm{Col}}(T)$ of so called abstract Coleman functions and constructed an embedding of $A_{\mathrm{Col}}(T)$ into the ring of naive analytic functions $\mathfrak{n}(\mathfrak{Y}_\eta)$. Notice that the generic fiber $\mathfrak{Y}_\eta$ is a closed analytic domain in $\mathfrak{X}_\eta$, and it does not coincide with $\mathfrak{X}_\eta$ if $\mathfrak{Y}$ is not proper (e.g., if $\mathfrak{Y}$ is affine). Thus, the naive analytic functions, constructed by A. Besser, are defined only on the closed subset $\mathfrak{Y}_\eta$ whereas the functions constructed here are defined on an open neighborhood of $\mathfrak{Y}_\eta$ in $\mathfrak{X}_\eta$. This is one of the main differences between the two constructions. On the other hand, it is very likely that the image of $A_{\mathrm{Col}}(T)$ in $\mathfrak{n}(\mathfrak{Y}_\eta)$ coincides with that of the restriction map $\widetilde{\mathcal{E}}(\mathfrak{X}_\eta, \mathfrak{Y}) \to \mathfrak{n}(\mathfrak{Y}_\eta) : f \mapsto f\big|_{\mathfrak{Y}_\eta}$ (and so both rings are isomorphic) and, in particular, if $\mathfrak{X}$ is proper smooth over k° and $\mathfrak{Y} = \mathfrak{X}$ the image of $A_{\mathrm{Col}}(T)$ in $\mathfrak{n}(\mathfrak{X}_\eta)$ coincides with $\widetilde{\mathcal{E}}(\mathfrak{X}_\eta, \mathfrak{X})$.

8.2. Filtered $D_{\mathfrak{X}_\eta}$-algebras $\mathcal{E}^\lambda(\mathfrak{X})$ for proper marked formal schemes.

8.2.1. Theorem. *Every proper marked formal scheme $\mathfrak{X}$ over k° can be provided with a unique filtered $\mathcal{O}(\mathfrak{X}_\eta)$-subalgebra $\mathcal{E}^\lambda(\mathfrak{X}) \subset \mathcal{S}^\lambda(\mathfrak{X}_\eta)$ such that the following is true:*

(a) $\mathcal{E}^\lambda(\mathfrak{X})$ is a filtered $D_{\mathfrak{X}_\eta}$-algebra;

(b) $\mathcal{E}^{\lambda,0}(\mathfrak{X}) = \mathcal{O}(\mathfrak{X}_\eta)$;

(c) $\mathrm{Ker}(\mathcal{E}^\lambda(\mathfrak{X}) \xrightarrow{d} \Omega^1_{\mathcal{E}^\lambda}(\mathfrak{X})) = \mathfrak{c}(\mathfrak{X}_\eta)$*;*

(d) every closed one form $\omega \in \Omega^1_{\mathcal{E}^{\lambda,i}}(\mathfrak{X})$ has a primitive f_ω in $\mathcal{E}^{\lambda,i+1}(\mathfrak{X})$;

(e) $\mathcal{E}^{\lambda,i+1}(\mathfrak{X})$ is generated over $\mathcal{O}(\mathfrak{X}_\eta)$ by the above primitives f_ω;

(f) there exists a point $x \in (\mathring{\mathfrak{X}}_\eta)_{st}$ with $f_x \in \mathcal{O}_{X,x}$ for all $f \in \mathcal{E}^\lambda(\mathfrak{X})$.

Furthermore, the filtered $D_{\mathfrak{X}_\eta}$-algebra $\mathcal{E}^\lambda(\mathfrak{X})$ possesses the following properties:

(1) $\mathcal{E}^{\lambda,i}(\mathfrak{X}) \subset R_0^{\lambda,i}(\mathfrak{X})$ (see the end of §3.1);

(2) for every $f \in \mathcal{E}^\lambda(\mathfrak{X})$, one has $f\big|_{(\mathfrak{X}_\eta, \mathring{\mathfrak{X}}_\eta)} \in \mathcal{E}^\lambda(\mathfrak{X}_\eta, \mathring{\mathfrak{X}}_\eta)$, and, if $\dim(\mathfrak{X}_\eta) = 1$ or $\mathring{\mathfrak{X}} = \mathfrak{X}$, there is an isomorphism $\mathcal{E}^\lambda(\mathfrak{X}) \otimes_{\mathcal{O}(\mathfrak{X}_\eta)} \mathcal{O}(\mathfrak{X}_\eta, \mathring{\mathfrak{X}}_\eta) \xrightarrow{\sim} \mathcal{E}^\lambda(\mathfrak{X}_\eta, \mathring{\mathfrak{X}}_\eta)$;

(3) $\mathcal{E}^\lambda(\mathfrak{X})$ is functorial with respect to $(k, \mathfrak{X}, K, \lambda)$;

(4) the homomorphism $k_{\mathrm{Log}} \to K : \mathrm{Log}(p) \mapsto \lambda$ gives rise to an isomorphism $\mathcal{E}(\mathfrak{X}) \xrightarrow{\sim} \mathcal{E}^\lambda(\mathfrak{X})$;

(5) the canonical homomorphism of filtered $\mathcal{O}(\mathfrak{X}_\eta)$-algebras $\mathcal{E}^\lambda(\mathfrak{X}) \otimes_k K \to \mathcal{S}^\lambda(\mathfrak{X}_\eta)$ is injective;

(6) given a function $f \in \mathcal{S}^{\lambda,i}(X)$ on a smooth k-analytic space X, every point $x \in X$ with $s(x) = \dim(X)$ has a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ such that $\varphi^*(f) \in (\mathcal{E}^\lambda(\mathfrak{X}) \otimes_k K)^i$.

Proof. The uniqueness of the algebras $\mathcal{E}^\lambda(\mathfrak{X})$ is verified in the same way as the correspondent fact in the proof of Theorem 8.1.1, and their construction is done by induction as follows. Assume that $D_{\mathcal{X}_\eta}$-module $\mathcal{E}^{\lambda,i}(\mathfrak{X})$ with all of the properties is already constructed for some $i \geq 0$. To construct $\mathcal{E}^{\lambda,i+1}(\mathfrak{X})$, it suffices to show that every closed one-form $\omega \in \Omega^1_{\mathcal{E}^{\lambda,i}}(\mathfrak{X})$ has a primitive $f_\omega \in \mathcal{S}^{\lambda,i+1}(\mathfrak{X}_\eta) \cap R_0^{\lambda,i+1}(\mathfrak{X})$. First of all, by [Ber9, Corollary 8.3.3], $H^1(\mathfrak{X}_\eta, \mathfrak{c}_{\mathfrak{X}_\eta}) = 0$ and, therefore, there exists a primitive f_ω of ω in $\mathcal{S}^{\lambda,i+1}(\mathfrak{X}_\eta)$. Let $\mathfrak{Z}$ be an open affine subscheme of $\mathring{\mathfrak{X}}$. By the property (2), $\omega\big|_{(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)} \in \mathcal{E}^{\lambda,i} \otimes_B \Omega^1_B$, where $B = \mathcal{O}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$, and, therefore, there exists a primitive f'_ω of ω in $\mathcal{E}^{\lambda,i+1}(\mathfrak{X}_\eta, \mathfrak{Z})$. We get $f_\omega\big|_{(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)} - f'_\omega \in \mathfrak{c}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta) \otimes_k K$. Since $\mathfrak{c}(\mathfrak{X}_\eta, \mathfrak{Z}_\eta) = \mathfrak{c}(\mathfrak{X}_\eta)$, we may assume that $f_\omega\big|_{(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)} \in \mathcal{E}^{\lambda,i+1}(\mathfrak{X}_\eta, \mathfrak{Z})$, and *we claim that such f_ω possesses the necessary property, i.e., $f_\omega \in R_0^{\lambda,i+1}(\mathfrak{X})$.* Indeed, by the property (1), one has $\omega \in \Omega^1_{R_0^{\lambda,i}}(\mathfrak{X})$, and Corollary 3.1.6 implies that there exists a primitive f of ω in $R_0^{\lambda,i+1}(\mathfrak{X})$. It follows that $f_\omega - f = \alpha \in \mathfrak{c}_R(\mathfrak{X}) \otimes_k K$, and we have to show that in fact $\alpha \in \mathfrak{c}_R(\mathfrak{X})$. Let ϕ be a Frobenius lifting on $(\mathfrak{X}_\eta, \mathfrak{Z}_\eta)$. By the construction of f_ω, there exists a monic polynomial $P(T) \in k[T]$ with no roots-of-unity roots such that $P(\phi^*) f_\omega = g \in \mathcal{E}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z})$. It follows that $g - P(\phi^*) f = P(\phi^*)\alpha$, and the latter is the restriction of a unique element $\beta \in \mathfrak{c}_R(\mathfrak{X}) \otimes_k K$. Since $P(\phi^*)$ induces a bijection on $\mathfrak{c}_R(\mathfrak{X}) \otimes_k K$, it suffices to verify that $\beta \in \mathfrak{c}_R(\mathfrak{X})$.

Let $\mathcal{U}$ be an open neighborhood of $\mathfrak{Z}_\eta$ in $\mathfrak{X}_\eta$ such that all ϕ^j for $j \leq \deg(P)$ are defined on $\mathcal{U}$. For a closed point $\mathbf{x} \in \mathfrak{X}_s$, we set $\mathcal{U}_\mathbf{x} = \pi^{-1}(\mathbf{x}) \cap \mathcal{U}$ and denote by $M_\mathbf{x}$ be the subgroup of all $h \in \mathcal{O}(\mathcal{U}_\mathbf{x})^*$ for which the real valued function $x \mapsto |h(x)|$ extends by continuity to the generic point σ of $\mathfrak{X}$ and such that $|h(\sigma)| = 1$. Furthermore, we denote by $L_0^\lambda(\mathcal{U}_\mathbf{x})$ the filtered $D_{\mathcal{U}_\mathbf{x}}$-subalgebra of $L^\lambda(\mathcal{U}_\mathbf{x})$ generated over $\mathcal{O}(\mathcal{U}_\mathbf{x})$ by $\mathrm{Log}^\lambda(h)$ for $h \in M_\mathbf{x}$, and we denote by $R_0^{\lambda,i}(\mathfrak{X}, \mathcal{U})$ the subspace of all $h \in R^{\lambda,i}(\mathfrak{X}, \mathcal{U})$ with $h_\mathbf{x} \in L_0^\lambda(\mathcal{U}_\mathbf{x})$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$. For example, $P(\phi^*) f \in R_0^{\lambda,i+1}(\mathfrak{X}, \mathcal{U})$.

By the property (5), we can find a dominant morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ from a proper marked formal scheme $\mathfrak{X}'$ over k'° with is a finite extension k' of k such that $\varphi(\mathfrak{X}'_\eta) \subset \mathcal{U}$ and $\varphi^*(g) \in \mathcal{E}^{\lambda,i}(\mathfrak{X}') \otimes_k K$.

Since $g \in \mathcal{E}^{\lambda,i}(\mathfrak{X}_\eta, \mathfrak{Z})$, it follows that in fact $\varphi^*(g) \in \mathcal{E}^{\lambda,i}(\mathfrak{X}')$, and since $\varphi^*(h) \in R_0^{\lambda,i}(\mathfrak{X}')$ for all $h \in R_0^{\lambda,i}(\mathfrak{X},\mathcal{U})$, it follows that $\varphi^*(\beta) = \varphi^*(g) - \varphi^*(P(\phi^*)f) \in R_0^{\lambda,i}(\mathfrak{X}')$. This implies that $\beta \in \mathfrak{c}_R(\mathfrak{X})$. That the $D_{\mathfrak{X}_\eta}$-modules $\mathcal{E}^{\lambda,i+1}(\mathfrak{X})$ possess all of the other properties easily follows from the construction. ∎

8.2.2. Remark. If $K = k$ and $\dim(\mathfrak{X}_\eta) = 1$, i.e., $\mathfrak{X}_\eta$ is a smooth basic curve, the filtered algebra $\mathcal{E}^\lambda(\mathfrak{X})$ is precisely the algebra $A(\mathfrak{X}_\eta)$ introduced by R. Coleman in [Col1] and [CoSh] and mentioned in the introduction to this paper.

8.3. A filtered $D_{\mathcal{O}_{X,x}}$-subalgebra $\mathcal{E}^\lambda_{X,x} \subset \mathcal{S}^\lambda_{X,x}$ and the space $\mathcal{V}_{X,x}$.

8.3.1. Theorem. *There is a unique way to provide each stalk $\mathcal{S}^\lambda_{X,\overline{x}}$ at a geometric point $\overline{x}$ of a smooth k-analytic space X over a point $x \in X$ with a $G_{\overline{x}/x}$-invariant filtered $D_{\mathcal{O}_{X,\overline{x}}}$-subalgebra $\mathcal{E}^\lambda_{X,\overline{x}}$ so that the following is true (with $\mathcal{E}^\lambda_{X,x} = (\mathcal{E}^\lambda_{X,\overline{x}})^{G_{\overline{x}/x}}$):*

(a) if $s(x) = \dim(X)$, then $\mathcal{E}^\lambda_{X,x}$ consists of the elements $f \in \mathcal{S}_{X,x}$ such that there exists a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ of x with $\varphi^*(f) \in \mathcal{E}^\lambda(\mathfrak{X})$;

(b) $\mathcal{E}^\lambda_{X,\overline{x}}$ is functorial with respect to $(k, X, \overline{x}, K, \lambda)$;

(c) for any smooth morphism $\varphi : Y \to X$ and any geometric point $\overline{y}$ of Y over $\overline{x}$ and a point $y \in Y$ with $s(y) = s(x)$, there is a $G_{\overline{y}/y}$-equivariant isomorphism of filtered $D_{\mathcal{O}_{Y,\overline{y}}}$-algebras $\mathcal{E}^\lambda_{X,\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}} \overset{\sim}{\to} \mathcal{E}^\lambda_{Y,\overline{y}}$.

Furthermore, the filtered $D_{\mathcal{O}_{X,\overline{x}}}$-algebra $\mathcal{E}^\lambda_{X,\overline{x}}$ possesses the following properties:

(1) $\mathcal{E}^\lambda_{X,\overline{x}}$ is a free $\mathcal{O}_{X,\overline{x}}$-module of at most countable rank;

(2) if $f_1, \dots, f_t$ are elements of $\mathcal{O}^*_{X,x}$ such that $|f_1(x)|, \dots, |f_t(x)|$ form a basis of the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$, then there is an isomorphism of filtered $D_{\mathcal{O}_{X,x}}$-algebras

$$\mathcal{E}^\lambda_{X,x}[T_1, \dots, T_t] \otimes_k K \overset{\sim}{\to} \mathcal{S}^\lambda_{X,x} : T_i \mapsto \mathrm{Log}^\lambda(f_i) \ ;$$

(3) the homomorphism $k_{\mathrm{Log}} \to K : \mathrm{Log}(p) \mapsto \lambda$ gives rise to an isomorphism $\mathcal{E}_{X,x} \overset{\sim}{\to} \mathcal{E}^\lambda_{X,x}$;

(4) $\mathcal{E}_{X,x}$ consists of the elements $f \in \mathcal{S}_{X,x}$ such that there exists a strictly affinoid domain V in an open neighborhood of x, at which f is defined, with $x \in V$ and $f\big|_V \in \mathfrak{n}(V)$.

Proof. If the $D_{\mathcal{O}_{X,\overline{x}}}$-algebras $\mathcal{E}^\lambda_{X,\overline{x}}$ exist, the uniqueness is trivial. Their construction is done in several steps.

Step 1. Assume first that $s(x) = \dim(X)$. We define $\mathcal{E}^\lambda_{X,x}$ (and together with it $\mathcal{E}^\lambda_{X,\overline{x}}$) by the property from (a), and *we claim that $\mathcal{E}^\lambda_{X,\overline{x}}$ is a free $\mathcal{O}_{X,\overline{x}}$-module of at most countable rank.* Indeed, since k is a closed subfield of $\mathbf{C}_p$, $\mathcal{H}(x)$ contains a countable dense subfield. It follows that set of finite extensions of $\mathcal{H}(x)$ in $\mathcal{H}(\overline{x})$ is countable and, therefore, there is a marked neighborhood $\varphi : \mathfrak{X}_{1,\eta} \to X$ of the point x and a countable sequence of morphisms of proper marked formal schemes $\ldots \stackrel{\psi_2}{\to} \mathfrak{X}_2 \stackrel{\psi_1}{\to} \mathfrak{X}_1$ such that each induced morphism $\mathfrak{X}_{n,\eta} \to X$ is also a marked neighborhood of x and $\mathcal{E}^\lambda_{X,\overline{x}} = \varinjlim \mathcal{E}^\lambda(\mathfrak{X}_n)$. If $\mathfrak{Z}_n \subset \mathring{\mathfrak{X}}_n$ are open affine subschemes with $\psi_n(\mathfrak{Z}_{n+1}) \subset \mathfrak{Z}_n$, then $\mathcal{E}^\lambda_{X,\overline{x}} = \varinjlim \mathcal{E}^\lambda(\mathfrak{X}_{n,\eta}, \mathfrak{Z}_{n,\eta})$, and the claim follows from Corollary 8.1.2(iii).

Step 2. Assume now that $s(x) < \dim(X)$. We take a smooth morphism $\varphi : X' \to Y$ from an open neighborhood X' of x such that $s(y) = s(x) = \dim(Y)$, where $y = \varphi(x)$, and a geometric point $\overline{y}$ of Y over y and under $\overline{x}$, and define $\mathcal{E}^\lambda_{X,\overline{x}}$ (and together with it $\mathcal{E}^\lambda_{X,x} = (\mathcal{E}^\lambda_{X,\overline{x}})^{G_{\overline{x}/x}}$) as the image of the injective homomorphism $\mathcal{E}^\lambda_{Y,\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \to \mathcal{S}^\lambda_{X,\overline{x}}$. (The injectivity follows from Step 1 and Corollary 2.3.4.) Although we do not yet know that the above image does not depend on the choice of φ and $\overline{y}$, and we do know that the property (2) holds. To prove the theorem, it suffices to verify the property (4). Let $\overline{\mathcal{E}}^\lambda_{X,x}$ denote the set of all elements of $\mathcal{S}^\lambda_{X,x}$ with the property (4).

Step 3. *The inclusion* $\mathcal{E}^\lambda_{X,x} \subset \overline{\mathcal{E}}^\lambda_{X,x}$ *is true.* Indeed, to prove this, we may assume that $s(x) = \dim(X)$. Let $f \in \mathcal{E}^\lambda_{X,x}$. Shrinking X, we may assume that $f \in \mathcal{S}^\lambda(X)$, and let us take a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ of x with $\varphi^*(f) \in \mathcal{E}^\lambda(\mathfrak{X})$, and a nonempty open affine subscheme $\mathfrak{Z} \subset \mathring{\mathfrak{X}}$ such that φ is étale at all points of the strictly affinoid domain $\mathfrak{Z}_\eta$. By the construction, $\varphi^*(f)\big|_{\mathfrak{Z}_\eta} \in \mathfrak{n}(\mathfrak{Z}_\eta)$ and, by Raynaud's theorem (see [BoLü2, Corollary 5.11]), $\varphi(\mathfrak{Z}_\eta)$ is a finite union of strictly affinoid subdomains of X. If V is one of them that contains the point x, then $f\big|_V \in \mathfrak{n}(V)$, i.e., $f \in \overline{\mathcal{E}}^\lambda_{X,x}$.

Step 4. *If* $t(x) = 0$, *then* $\overline{\mathcal{E}}^\lambda_{X,x} \subset \mathcal{E}^\lambda_{X,x}$. Indeed, since $\mathcal{E}^\lambda_{X,x}[\mathrm{Log}(p)] \stackrel{\sim}{\to} \mathcal{S}^\lambda_{X,x}$, every element $f \in \overline{\mathcal{E}}^\lambda_{X,x}$ is of the form $\sum_{i=1}^n g_i \mathrm{Log}(p)^i$ with $g_i \in \mathcal{E}^\lambda_{X,x}$. Shrinking X, we may assume that $g_0.g_1, \ldots, g_n \in \mathcal{S}^\lambda(X)$. Assume that $n \geq 1$ and $g_n \neq 0$. Then every strictly affinoid domain V with $x \in V$ contains a point $z \in V_{st}$ with $g_n(x) \neq 0$, and so, for such a point z, $f(z)$ is a polynomial in $\mathrm{Log}(p)$ over $\mathcal{O}_{Z,z}$ of positive degree. This contradicts the assumption on f.

Step 5. The previous two steps reduce the theorem to the verification of the following fact. *Given a smooth morphism* $\varphi : Y \to X$ *of dimension one, if the required statement is true for a point* $x \in X$, *then it is also true for any point* $y \in \varphi^{-1}(x)$ *with* $t(y) > t(x)$. Indeed, by Proposition 2.3.1(ii), we may assume that $Y = X \times B$, where $B = B(0; R', R'')$ is an open annulus of radii $R' < R''$ with center at zero, and y is the point of the fiber $\varphi^{-1}(x)$ which is the maximal point of the closed disc of radius $R' < r < R''$ with $r \notin \sqrt{|\mathcal{H}(x)^*|}$.

One has $\mathcal{E}^\lambda_{X,x}[\mathrm{Log}(T)][\mathrm{Log}(p)] \stackrel{\sim}{\to} \mathcal{S}^\lambda_{X,x}$ and, in particular, every element $f \in \overline{\mathcal{E}}^\lambda_{Y,y}$ is of the form $\sum_{i,j=0}^n g_{ij} \mathrm{Log}(T)^i \mathrm{Log}(p)^j$ with $g_{ij} \in \mathcal{E}^\lambda_{X,x}$. Shrinking X, we may assume that $g_{ij} \in \mathcal{S}^\lambda(X)$ and, therefore, $f \in \mathcal{S}^\lambda(Y)$. Let d be the maximal value of $i+j$ with $g_{ij} \neq 0$. We are going to show that, if $d > 0$, then every strictly affinoid domain V with $y \in V$ contains a point $y' \in V_0$ with $f_{y'} \notin \mathcal{O}_{Y,y'}$. By Lemma 4.5.4, we may assume that $V = U \times A$ (as in the formulation of the lemma). Shrinking U, we may assume that $(g_{ij})\big|_U \in \mathfrak{n}(U)$ for all $i, j \geq 0$. The maximal possible degree of $\mathrm{Log}(p)$ in $f_{y'}$ is d, and the coefficient is equal to $\alpha = \sum_{i=0}^{j} g_{d-i,i}(x')v(T(y'))^i$, where j is the maximal $i \leq d$ with $g_{d-i,i} \neq 0$, $x' = \varphi(y')$, $v(T(y'))$ is the rational number with $\mathrm{Log}(T)_{y'} - v(T(y'))\mathrm{Log}(p) \in \mathcal{O}_{Y,y'}$. (Notice that $v(T(y'))$ is the value of the real logarithm with basis $|p|$ at $|T(y')|$.) Let x' be a point in U_0 with $g_{d-j,j}(x') \neq 0$. If $j = 0$ then, for every point $y' \in V_0$ over x', $f_{y'}$ is a polynomial over $\mathcal{O}_{Y,y'}$ of degree d, and so assume that $j > 0$. Since $V = U \times A$, we can find a point $y' \in V_0$ over x' for which the denominator of the rational number $v(T(y'))$ is divisible by a big power of p so that

$$|v(T(y'))| > \max_{0 \leq i \leq j-1} \left(\frac{|g_{d-i.i}(x')|}{|g_{d-j,j}(x')|} \right)^{\frac{1}{j-i}} .$$

We get $|\alpha(y')| = |g_{d-j,j}(x')||v(T(y'))|^j \neq 0$, and the required fact follows. ∎

8.3.2. Corollary. *The statements (iv)-(vi) of Theorem 1.6.2 are true.*

Proof. The statement (vi) and (iv) follow from Theorem 8.3.1(1) and (2), respectively. In the situation of the second part of (v), Lemma 5.5.4 implies that $\varphi^*(\mathcal{S}_X^{\lambda,i}) \xrightarrow{\sim} \varphi^\#(\mathcal{S}_X^{\lambda,i}) \xrightarrow{\sim} \mathcal{S}_{X'}^{\lambda',i}$, and so to verify the first part we may assume that $k' = k$. In that case, the required fact follows from Corollary 2.3.4. ∎

8.3.3. Corollary. *Let X be a smooth k-analytic space. Then*

(i) the following properties of a point $x \in X$ are equivalent:

(a) $\mathcal{S}_{X,x}^\lambda = \mathcal{O}_{X,x}^K$;

(b) $\mathcal{L}_{X,x}^\lambda = \mathcal{O}_{X,x}^K$;

(c) $s(x) = t(x) = 0$, i.e., $\widetilde{\mathcal{H}(x)}$ is algebraic over $\widetilde{k}$ and the group $|\mathcal{H}(x)^*|/|k^*|$ is torsion;

(ii) the following properties of a geometric point $\overline{x}$ of X over a point $x \in X$ are equivalent:

(a) $\mathcal{S}_{X,\overline{x}}^\lambda = \mathcal{L}_{X,\overline{x}}^\lambda$;

(b) $\mathcal{E}_{X,\overline{x}}^\lambda = \mathcal{O}_{X,\overline{x}}$;

(c) $s(x) = 0$, i.e., $\widetilde{\mathcal{H}(x)}$ is algebraic over $\widetilde{k}$. ∎

8.3.4. Corollary. *If elements $\{f_i\}_{i \in I} \in \mathcal{O}_{X,\overline{x}}^*$ are such that their images in $\mathcal{O}_{X,\overline{x}}^v$ form a basis of the abelian group $\widetilde{\mathcal{H}}_{\overline{x}}/\widetilde{\mathfrak{c}}_{X,\overline{x}}$ (from Corollary 4.2.2), then there is an isomorphism of filtered $D_{\mathcal{O}_{X,\overline{x}}}$-algebras*

$$\mathcal{O}_{X,\overline{x}}[T_i]_{i \in I} \xrightarrow{\sim} \mathcal{E}_{X,\overline{x}}^\lambda \cap \mathcal{L}_{X,\overline{x}}^\lambda : T_i \mapsto \mathrm{Log}^\lambda(f_i) .$$ ∎

8.3.5. Corollary. *The class of a closed one-form $\omega \in \Omega^1_{X,x}$ lies in $\mathcal{V}_{X,x}$ if and only if it admits a primitive $f_\omega \in \mathcal{E}_{X,x}$.*

Proof. For a geometric point $\overline{x}$ over x, let $\mathcal{V}'_{X,\overline{x}}$ denote the space of the classes of those closed one-forms ω from $\Omega^1_{X,\overline{x}}$ that admit a primitive f_ω in $\mathcal{E}_{X,\overline{x}}$. It suffices to verify that $\mathcal{V}_{X,\overline{x}} = \mathcal{V}'_{X,\overline{x}}$. If $\varphi : Y \to X$ is a smooth morphism and y is a point over x with $s(y) = s(x) = \dim(X)$, then, by Theorem 4.5.3(ii), $\mathcal{V}_{X,\overline{x}} \xrightarrow{\sim} \mathcal{V}_{X,\overline{y}}$ for any geometric point $\overline{y}$ over y and $\overline{x}$ and, by Theorems 4.5.3(i) and 8.2.1(d), $\mathcal{V}_{X,\overline{x}} = \Omega^{1,\mathrm{cl}}_{X,\overline{x}}/d\mathcal{O}_{X,\overline{x}} = \mathcal{V}'_{X,\overline{x}}$. This gives the inclusion $\mathcal{V} \subset \mathcal{V}'$ and, to prove the converse inclusion, it suffices to verify the following claim. *Given a smooth morphism $\varphi : Y \to X$ of dimension one, if the required fact is true for a point $x \in X$, it is also true for any point $y \in \varphi^{-1}(x)$ with $t(y) > t(x)$.* For this, we may assume that $Y = X \times B$, where $B = B(0; R', R'')$ and y is the point of the fiber $\varphi^{-1}(x)$ which is the maximal point of a closed disc of radius $R' < r < R''$ with $r \notin \sqrt{|\mathcal{H}(x)^*|}$. Given a closed one-form $\omega \in \Omega^1_{Y,\overline{y}}$ whose class is in $\mathcal{V}'_{Y,\overline{y}}$, one has $\omega = \eta + \alpha\frac{dT}{T} + dg$, where η is a closed one-form in $\Omega^1_{Y,\overline{y}}$ with class in $\mathcal{V}_{Y,\overline{y}}$, $\alpha \in \mathfrak{c}_{Y,\overline{y}}$ and $g \in \mathcal{O}_{Y,\overline{y}}$. We can shrink X and Y (in the étale topology) so that $g \in \mathcal{O}(Y)$ and the primitives f_η and f_ω are defined on all Y and $f_\omega = f_\eta + \alpha\mathrm{Log}(T) + g$. We are going to show that, if $\alpha \neq 0$, every strictly affinoid domain V with $y \in V$ contains a point $y' \in V_{st}$ with $(f_\omega)_{y'} \notin \mathcal{O}_{Y,y'}$. Indeed, shrinking V, we may assume that $(f_\eta)\big|_V \in \mathfrak{n}(V)$ and, by Lemma 4.5.4, we may assume that $V = U \times A$ as in its formulation. For a point $y' \in V_{st}$, $(f_\omega)_{y'}$ is a polynomial over $\mathcal{O}_{Y,y'}$ in $\mathrm{Log}(p)$ of degree at most one, and the coefficient at $\mathrm{Log}(p)$ is equal to $\alpha v(T(y'))$. We can of course find a point $y' \in V_{st}$ with $v(T(y')) \neq 0$, and the required fact follows. ∎

Recall that the construction of §5.4 applied to the $\mathfrak{c}_{X,\overline{x}}$-algebra $\mathcal{O}_{X,\overline{x}}$, the differential $d : \mathcal{O}_{X,\overline{x}} \to \Omega^1_{X,\overline{x}}$ and the vector subspace $\mathcal{V}_{X,\overline{x}} \subset \Omega^{1,\mathrm{cl}}_{X,\overline{x}}/d\mathcal{O}_{X,\overline{x}}$, provides a filtered $D_{\mathcal{O}_{X,\overline{x}}}$-algebra $\mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}}(\mathcal{V}_{X,\overline{x}}) = \mathcal{O}_{X,\overline{x}} \otimes_{\mathfrak{c}_{X,\overline{x}}} \mathrm{Sh}(\mathcal{V}_{X,\overline{x}})$. If $\varphi : X' \to Y$ is a smooth morphism from an open neighborhood X' of x to a smooth k-analytic space Y with $s(y) = \dim(Y) = s(x)$, where $y = \varphi(x)$, and $\overline{y}$ is a geometric point of Y under $\overline{x}$ and over y, then there is a $G_{\overline{x}/x}$-equivariant isomorphism $\mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}}(\mathcal{V}_{X,\overline{x}}) \xrightarrow{\sim} \mathcal{O}_{X,\overline{x}} \otimes_{\mathfrak{c}_{Y,\overline{y}}} \mathrm{Sh}_{\mathcal{O}_{Y,\overline{y}}}$. Thus, the $G_{\overline{y}/y}$-equivariant injective homomorphism of graded $D_{\mathcal{O}_{Y,\overline{y}}}$-algebras $\mathrm{Gr}^{\cdot}(\mathcal{E}_{Y,\overline{y}}) \hookrightarrow \mathrm{Gr}^{\cdot}(\mathrm{Sh}_{\mathcal{O}_{Y,\overline{y}}})$, constructed in §5.4, gives rise to a $G_{\overline{x}/x}$-equivariant injective homomorphism of graded $D_{\mathcal{O}_{X,\overline{x}}}$-algebras

$$\mathrm{Gr}^{\cdot}(\mathcal{E}_{X,\overline{x}}) \hookrightarrow \mathrm{Gr}^{\cdot}(\mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}}(\mathcal{V}_{X,\overline{x}})) \ .$$

Notice that the right hand side is isomorphic, as an $\mathcal{O}_{X,\overline{x}}$-algebra, to $\mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}}(\mathcal{V}_{X,\overline{x}})$, and recall that, by a result of R. Radford (see [Rad, Theorem 3.1.1]), the shuffle algebra of a vector space over a field of characteristic zero is isomorphic to the ring of polynomials over the field with a set of variables consisting of homogeneous elements.

8.3.6. Corollary. *(i) The graded $\mathcal{O}_{X,\overline{x}}$-algebra* $\mathrm{Gr}^{\cdot}(\mathcal{E}_{X,\overline{x}})$ *has no zero divisors and $\mathcal{O}_{X,\overline{x}}$ is algebraically closed in it and, in particular, the same is true for the filtered $\mathcal{O}_{X,\overline{x}}$-algebra $\mathcal{E}_{X,\overline{x}}$;*

(ii) $\mathrm{Gr}^{\cdot}(\mathcal{E}_{X,\overline{x}})^* = \mathcal{O}^*_{X,\overline{x}}$ *and, in particular, $\mathcal{E}^*_{X,\overline{x}} = \mathcal{O}^*_{X,\overline{x}}$.* ∎

From Theorem 8.3.1 and Corollary 8.3.5 it follows that if the k-algebra K (resp. $\mathrm{Gr}^{\cdot}(K)$) has no zero divisors and k is algebraically closed in it, then the sheaf of $\mathcal{O}_X$-algebras $\mathcal{S}^\lambda_X$ (resp. $\mathrm{Gr}^{\cdot}(\mathcal{S}^\lambda_X)$) has no zero divisors and $\mathcal{O}_X$ is algebraically closed in it. It follows also that the $\mathcal{O}_X$-modules $\mathcal{S}^\lambda_X/\mathcal{S}^{\lambda,n}_X$ are torsion free.

8.3.7. Theorem. *Let X be a smooth k-analytic curve, and $\overline{x}$ a geometric point of X over a point x of type (2). Then there is a (non-canonical) $G_{\overline{x}/x}$-equivariant isomorphism of filtered $D_{\mathcal{O}_{X,\overline{x}}}$-algebras $\mathcal{E}_{X,\overline{x}} \stackrel{\sim}{\to} \mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}}$.*

Proof. There is a countable sequence of étale morphism $\dots \stackrel{\varphi_2}{\to} X_2 \stackrel{\varphi_1}{\to} X_1 \stackrel{\varphi}{\to} X$ with a set of compatible morphisms $\mathbf{p}_{\mathcal{H}(\overline{x})} \to X_n$ over $\overline{x}$, which form a fundamental system for calculating the stalk $F_{\overline{x}}$ of an étale sheaf F at $\overline{x}$, i.e., $F_{\overline{x}} = \varinjlim F(X_n)$, (see §1.1). We may assume that each X_n is an elementary and basic curve over a finite Galois extension k_n over k, and is a Galois covering of X whose Galois group G_n coincides with that of $\mathcal{H}(x_n)$ over $\mathcal{H}(x)$, where x_n is the generic point of X_n as well as the image of $\overline{x}$ in X_n. We may also assume that the above system is induced by an equivariant system of morphisms $\dots \stackrel{\varphi_2}{\to} \mathfrak{X}_2 \stackrel{\varphi_1}{\to} \mathfrak{X}_1 \stackrel{\varphi}{\to} X$, where each $\mathfrak{X}_n$ is a proper marked formal scheme over k°_n with $\mathfrak{X}_{n,\eta} = X_n$ and the generic point x_n. We set $V_n = \Omega^1(X_n)/d\mathcal{O}(X_n)$ and $V = \Omega^1_{X,\overline{x}}/d\mathcal{O}_{X,\overline{x}}$. Notice that $\cup_{n=1}^\infty k_n = \mathfrak{c}_{X,\overline{x}}$ is the algebraic closure of k in $\mathcal{H}(\overline{x})$, $V = \cup_{n=1}^\infty V_n$, and $G_{\overline{x}/x} = \varprojlim G_n$. We now choose the following objects:

(1) a compatible system of k_n-linear G_n-equivariant sections $s_n : V_n \to \Omega^1(X_n)$ of the canonical epimorphisms $\Omega^1(X_n) \to V_n$ (we denote by Ω_n the image of s_n);

(2) a compatible system of k_n-linear G_n-equivariant sections $t_n : \Omega_n \to \mathcal{P}^1(X_n) \cap \mathcal{E}^1(\mathfrak{X}_n)$ of the epimorphisms $d : \mathcal{P}^1(X_n) \to \Omega^1(X_n)$ (over Ω_n);

(3) a basis $\{v_i\}_{i\geq 1}$ of V over $\mathfrak{c}_{X,\overline{x}}$ such that, for every $n \geq 1$, the first l_n vectors $v_1, \dots, v_{l_n}$ form a basis of V_n over k_n.

Notice that the systems of sections $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ define $\mathfrak{c}_{X,\overline{x}}$-linear $G_{\overline{x}/x}$-equivariant sections $s : V \to \Omega^1_{X,\overline{x}}$ and $t : \Omega = \cup_{n=1}^\infty \Omega_n \to \mathcal{P}^1_{X,\overline{x}}$. We are now going to construct a $G_{\overline{x}/x}$-equivariant embedding of $\mathcal{D}_{X,\overline{x}}$-algebras $\mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}} \to \mathcal{E}_{X,\overline{x}}$ for which the operator $\nabla : \mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}} \to \mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}} \otimes_{\mathfrak{c}_{X,\overline{x}}} \Omega^1_{X,\overline{x}}$, that takes $v_{i_1} \otimes \dots \otimes v_{i_m}$ to $v_{i_1} \otimes \dots \otimes v_{i_{m-1}} s(v_{i_m})$, corresponds to the differential on $\mathcal{E}_{X,\overline{x}}$. Since $\mathrm{Sh}(V)=\cup_{n=1}^{\infty}\mathrm{Sh}(V_n)$, for this it suffices to construct a compatible system of G_n-equivariant embeddings of k_n-algebras $\mathrm{Sh}_{\mathcal{O}(X_n)}=\mathcal{O}(X_n)\otimes_{k_n}\mathrm{Sh}(V_n)\to\mathcal{E}(\mathfrak{X}_n)$ with the same properties. We now recall results of D. Radford [Rad] on the structure of the shuffle algebra $\mathrm{Sh}(V)$.

First of all, let S denote the free semigroup generated by symbols $x_1,x_2,\dots$. Each element $x=x_{i_1}\cdot\ldots\cdot x_{i_m}\in S$ defines an element $v_x=v_{i_1}\otimes\ldots\otimes v_{i_m}$ of $\mathrm{Sh}(V)$, and one sets $|x|=m$. There is a subset $P\subset S$ (whose elements are called primes) such that the elements v_x for $x\in P$ form a polynomial basis of $\mathrm{Sh}(V)$. Moreover, if $P^m=\{x\in P\big||x|=m\}$ and $P^{[m]}=\cup_{i=1}^m P^m$, the elements v_x for $x\in P^{[m]}$ form a polynomial basis of the subalgebra of $\mathrm{Sh}(V)$ generated by the subspace $\mathrm{Sh}^m(V)=\oplus_{i=1}^m V^{\otimes i}$. By the definition of primes, all elements x_i are contained in P^1, and if S_n is the semigroup generated by $x_1,\dots,x_n$, then the set of primes in S_n coincides with the intersections $P\cap S_n$.

Thus, to construct the required homomorphism $\alpha:\mathrm{Sh}_{\mathcal{O}_{X,\overline{x}}}\to\mathcal{E}_{X,\overline{x}}$, it suffices to define the images of the elements v_x for $x\in P$. First of all, for $i\geq 1$ we set $\alpha(v_i)=t(s(v_i))$ and, in particular, $d\alpha(v_i)=s(v_i)$. This defines a compatible system of k_n-linear G_n-equivariant maps $\mathrm{Sh}^1(V_n)\to\mathcal{E}^1(\mathfrak{X}_n)$ and, therefore, a $\mathfrak{c}_{X,\overline{x}}$-linear $G_{\overline{x}/x}$-equivariant map $\alpha_1:\mathrm{Sh}^1(V)\to\mathcal{E}^1_{X,\overline{x}}$. Assume that $m\geq 2$ and that we have already constructed a compatible system of k_n-linear G_n-equivariant maps $\alpha_{m-1}:\mathrm{Sh}^{m-1}(V_n)\to\mathcal{E}^{m-1}(\mathfrak{X}_n)$, $n\geq 1$, which take products (when they are defined) to products and for which the operators $\nabla:\mathrm{Sh}^{m-1}(V_n)\to\mathrm{Sh}^{m-2}(V_n)\otimes_{k_n}\Omega^1(X_n)$ correspond to the usual differentials. The correspondence $v_{i_1}\otimes\ldots\otimes v_{i_{m-1}}s(v_{i_m})\mapsto v_{i_1}\otimes\ldots\otimes v_{i_m}$ gives rise to a k_n-linear G_n-equivariant isomorphism between the cokernel of the induced homomorphism $\mathrm{Sh}^{m-1}_{\mathcal{O}(X_n)}\to\mathrm{Sh}^{m-1}_{\mathcal{O}(X_n)}\otimes_{\mathcal{O}(X_n)}\Omega^1(X_n)$ and the tensor product $V_n^{\otimes m}$ (see Lemma 5.4.4). It also defines a k_n-linear G_n-equivariant section $s_n^m:V^{\otimes m}\to\mathrm{Sh}^{m-1}_{\mathcal{O}(X_n)}\otimes_{\mathcal{O}(X_n)}\Omega^1(X_n)$. As in (3) above, we denote by Ω_n^m the image of the latter, and choose a compatible system of k_n-linear G_n-invariant sections $t_n^m:\Omega_n^m\to\mathcal{P}^m(X_n)\cap\mathcal{E}^m(\mathfrak{X}_n)$ of the epimorphism $d:\mathcal{P}^m(X_n)\to\Omega^1_{\mathcal{S}^{m-1}}(X_n)$ (over Ω_n^m). Finally, for a prime $x=x_{i_1}\cdot\ldots\cdot x_{i_m}\in P^m\cap S_n$, we set $\alpha_n(v_x)=t_n^m(v_{i_1}\otimes\ldots\otimes v_{i_{m-1}}s(v_{i_m}))$. In this way we get a compatible system of k_n-linear G_n-equivariant maps $\alpha_m:\mathrm{Sh}^m(V_n)\to\mathcal{E}^m(\mathfrak{X}_n)$ which extend the maps α_{m-1} and give rise to the required isomorphism. ∎

8.3.8. Corollary. *In the situation of Theorem 8.3.7, there is an isomorphism of filtered $D_{\mathcal{O}_{X,x}}$-algebras $\mathrm{Sh}^{G_{\overline{x}/x}}_{\mathcal{O}_{X,\overline{x}}}\otimes_k K\xrightarrow{\sim}\mathcal{S}^{\lambda}_{X,x}$.* ∎

Notice that the algebra on the left hand side in the above isomorphism is bigger than the algebra $\mathrm{Sh}_{\mathcal{O}_{X,x}}=\mathcal{O}_{X,x}\otimes_{\mathfrak{c}_{X,x}}\otimes\mathrm{Sh}(\Omega_{X,x}/d\mathcal{O}_{X,x})$. The following corollary is proved in the same way as Theorem 4.5.1(ii). ∎

8.3.9. Corollary. *Let X be a smooth k-analytic space. For $n \ge 1$, let $\mathfrak{S}^n_X$ denote the subalgebra of $\mathcal{S}_X$ generated by $\mathcal{S}^n_X$ and K, and set $\mathfrak{S}^0_X = \mathcal{L}_X$. Then for every geometric point $\overline{x}$ of X over a point x with $s(x) > 0$ and every $n \ge 0$, $\mathcal{S}^{n+1}_{X,\overline{x}}$ is not contained in any finitely generated $\mathfrak{S}^n_{X,\overline{x}}$-subalgebra of $\mathcal{S}_{X,\overline{x}}$.* ∎

8.4. More uniqueness properties.

8.4.1. Theorem. *Let X be a connected smooth k-analytic space such that the $\mathcal{O}_X$-module Ω^1_X is free over a nonempty Zariski open subset of X. Let $\{f_i\}_{i\in I}$ be a system of functions from $\mathcal{P}^{\lambda,1}(X)$ such that the classes of the analytic one-forms df_i in $\Omega^{1,\mathrm{cl}}(X)/d\mathcal{O}(X)$ are linearly independent over $\mathfrak{c}(X)$. Then the following homomorphism of filtered $\mathcal{D}_X$-algebras is injective:*

$$\mathcal{O}(X)[T_i]_{i\in I} \otimes_k K \to \mathcal{S}^\lambda(X) : T_i \mapsto f_i \ .$$

8.4.2. Proposition. *Let X be a smooth k-analytic space, and X' a dense Zariski open subset of X. If a function $f \in \mathcal{S}^\lambda(X)$ is such that $f\big|_{X'} \in \mathcal{S}^{\lambda,n}(X')$, then $f \in \mathcal{S}^{\lambda,n}(X)$.*

Proof. The statement is local in the étale topology of X, and we prove it by induction on m, the dimension of the proper Zariski closed subset $Y = X \backslash X'$. If $m = 0$, we may assume that Y is a point $x \in X_0$. In this case $\mathcal{S}^\lambda_{X,x} = \mathcal{O}_{X,x} \otimes_k K$, and the statement immediately follows. Assume that $1 \le m \le \dim(X) - 1$ and that the statement is true for smaller dimensions. We may assume that Y is connected, and we provide it with the structure of a reduced k-analytic space. The non-smoothness locus Z of Y is a Zariski closed subset of X of smaller dimension, and so if the statement is true for the pair $(X\backslash Z, X'\backslash Z)$, it is also true for the pair (X, X'). This reduces the situation to the case when Y is smooth. It suffices to show that, given a function $f \in \mathcal{S}^{\lambda,n'}(X)$ with $n' > n$ and $f\big|_{X'} \in \mathcal{S}^{\lambda,n}(X')$, every point $x \in Y$ has an étale neighborhood $\varphi : U \to X$ with connected U such that the preimage $V = \varphi^{-1}(Y)$ is connected and $\varphi^*(f)\big|_{U'} \in \mathcal{S}^{\lambda,n}(U')$ for a Zariski open subset $U' \subset U$ which is strictly bigger than $\varphi^{-1}(X') = U \backslash V$.

First of all, replacing X by an étale neighborhood of the point x, we may assume that $X \xrightarrow{\sim} Y \times D$, where D is the open unit polydisc with center at zero of dimension $q = \dim(X) - m$. By the property (IH2) from the proof of Theorem 1.7.1, we may then assume that $f \in p_1^\#(\mathcal{S}^{\lambda,n'}(Y))$, i.e., $f = \sum_{i=1}^l g_i f_i$ with $g_i \in \mathcal{O}(X)$ and $f_i \in \mathcal{S}^{\lambda,n'}(Y)$. Assume that for some $p \le l$ the functions $g_1, \dots, g_p$ form a maximal subset of linearly independent functions over the fraction field of $\mathcal{O}(Y)$. If $p < l$, we can replace Y by a Zariski open subset so that all of the functions $g_{p+1}, \dots, g_l$ are linear combinations of $g_1, \dots, g_p$ over $\mathcal{O}(Y)$. This reduces the situation to the case when the functions $g_1, \dots, g_l$ are linearly independent over the fraction field of $\mathcal{O}(Y)$.

8.4.3. Lemma. *Let $g_1, \ldots, g_l$ be analytic functions on $X = Y \times D$ linearly independent over the fraction field of $\mathcal{O}(Y)$. Then there exist nonzero k-rational points $\alpha_1, \ldots, \alpha_l \in D$ such that the determinant of the matrix $(g_i(\alpha_j))_{1 \le i,j \le l}$ is not equal to zero.*

Proof. If $l = 1$, the statement means that, given a nonzero analytic function $g \in \mathcal{O}(X)$, there exists a k-rational point $\alpha \in D$ with $g(\alpha) \neq 0$, and the required fact is easily verified by induction on $q = \dim(D)$. Assume the statement is true for $l - 1$ with $l \ge 2$. Then there exist nonzero k-rational points $\alpha_1, \ldots, \alpha_{l-1} \in D$ such that the determinant of the matrix $(g_i(\alpha_j))_{1 \le i,j \le l-1}$ is not equal to zero. To find the required value of α_l, consider the expansion of the determinant of the original matrix in terms of the elements of the last column. It follows that this determinant is a nonzero linear combination of the elements $g_i(\alpha_l)$ with coefficients in $\mathcal{O}(Y)$ and, therefore, the required value of α_l exists. ∎

Replacing Y by the Zariski open subset where the above determinant does not vanish, we may assume that it is invertible on Y. Each point α_j defines a section $\sigma_j : Y \to X \backslash Y$ of the canonical projection $X \backslash Y \to Y$. Since $f\big|_{X \backslash Y} \in \mathcal{S}^{\lambda,n}(X \backslash Y)$, it follows that the function $\sigma_j^*(f) = \sum_{i=1}^{l} g_i(\alpha_j) f_i$ is contained in $\mathcal{S}^{\lambda,n}(Y)$. The fact that the determinant of the matrix $(g_i(\alpha_j))_{1 \le i,j \le l}$ is invertible on Y implies that all of the functions f_i are also contained in $\mathcal{S}^{\lambda,n}(Y)$ and, therefore, $f \in \mathcal{S}^{\lambda,n}(X)$. ∎

Proof of Theorem 8.4.1. We may assume that the system considered is finite, i.e., we are given $\{f_1, \ldots, f_m\} \subset \mathcal{P}^{\lambda,1}(X)$. For $\nu = (\nu_1, \ldots, \nu_m) \in \mathbf{Z}_+^m$, we set $|\nu| = \nu_1 + \ldots + \nu_m$, and we provide $\mathbf{Z}_+^m$ with an ordering possessing the property that if $|\nu| < |\nu'|$ then $\nu < \nu'$. For a polynomial $P = \sum_{\nu \in \mathbf{Z}_+^m} g_\nu T^\nu \in \mathcal{O}^K[T_1, \ldots, T_m]$, we denote by $\nu(P)$ the maximal ν with $g_\nu \neq 0$. Furthermore, let $\{\gamma_j\}_{j \in J}$ be a basis of K over k and, for a function $g = \sum_{j \in J} g_j \gamma_j \in \mathcal{O}^K(X)$, let $\ell(g)$ denote the number of $j \in J$ with $g_j \neq 0$. We have to show that if a polynomial P as above is not zero then $P(f) \neq 0$. We prove the latter fact by double induction on (μ, l) with $\mu = \nu(P)$ and $l = \ell(g_\mu)$. Of course, the fact is true if $\mu = (0, \ldots, 0)$, and so assume that $\mu \neq (0, \ldots, 0)$ and $l \ge 1$, and that the fact is true for all (μ', l') with either $\mu' < \mu$, or $\mu' = \mu$ and $l' < l$.

If X' is a nonempty Zariski closed subset of X then, by Lemma 1.1.1, X' is connected and $\mathfrak{c}(X) \stackrel{\sim}{\to} \mathfrak{c}(X')$. Proposition 8.4.2 also implies that the restrictions of the analytic one-forms $df_1, \ldots, df_m$ to X' are linearly independent in $\Omega^{1,\mathrm{cl}}(X')/d\mathcal{O}(X')$. Thus, we can replace X by a nonempty Zariski open subset at which g_{μ,j_0} is invertible and the $\mathcal{O}_X$-module Ω_X^1 is free. After that we can replace P by $P/g_{\mu,j_0}$, and so we may assume that $g_{\mu,j_0} = 1$.

Let $\eta_1, \ldots, \eta_q \in \Omega^1(X)$ be a basis of Ω_X^1 over $\mathcal{O}_X$. Then $df_i = \sum_{p=1}^{q} f_i^{(p)} \eta_p$ and $dg_\nu =$

$\sum_{p=1}^{q} g_\nu^{(p)} \eta_p$, where $f_i^{(p)} \in \mathcal{O}(X)$ and $g_\nu^{(p)} \in \mathcal{O}^K(X)$ with $\ell(g_\mu^{(p)}) < l$ for all $1 \le p \le q$. The coefficient of $dP(f)$ at η_p is equal to

$$\sum_{\nu \in \mathbf{Z}_+^m} \sum_{i=1}^{m} \nu_i g_\nu f_i^{(p)} f^{\nu - e_i} + \sum_{\nu \in \mathbf{Z}_+^m} g_\nu^{(p)} f^\nu \ ,$$

where $f^{\nu-e_i} = f_1^{\nu_1} \cdot \ldots \cdot f_i^{\nu_i - 1} \cdot \ldots \cdot f_m^{\nu_m}$. Thus, if $P(f) = 0$, all of the above coefficients are equal to zero. Since $\ell(g_\mu^{(p)}) < l$, the induction hypothesis implies that the coefficient of the above expression at every f^ν is zero.

If $|\nu| = |\mu|$, the coefficient at f^ν is $g_\nu^{(p)}$ for all $1 \le p \le q$ and, therefore, $g_\nu = \alpha_\nu \in \mathcal{C}(X)$. Furthermore, for every $1 \le i \le m$ with $\mu_i \ge 1$ the coefficient at $f^{\mu - e_i}$ is equal to

$$g_{\mu-e_i}^{(p)} + \mu_i \alpha_\mu f_i^{(p)} + \sum_{\substack{j=1 \\ j \ne i}}^{m} (\mu_j + 1) \alpha_{\mu - e_i + e_j} f_j^{(p)} \ ,$$

where $\mu - e_i + e_j = (\mu_1, \ldots, \mu_i - 1, \ldots, \mu_j + 1, \ldots, \mu_m)$. The summation by p gives the equality

$$dg_{\mu-e_i} + \mu_i \alpha_\mu df_i + \sum_{\substack{j=1 \\ j \ne i}}^{m} (\mu_j + 1) \alpha_{\mu - e_i + e_j} df_j = 0 \ .$$

Decomposing this equality in the basis $\{\gamma_j\}_{j \in J}$ and using the fact that $\alpha_{\mu, j_0} \ne 0$, we get a nontrivial linear relation between the classes of $df_1, \ldots, df_m$ in $\Omega^{1,\mathrm{cl}}(X)/d\mathcal{O}(X)$ over $\mathfrak{c}(X)$, which is a contradiction. ∎

8.4.4. Corollary. *In the situation of Theorem 8.4.1, let $\{f_i\}_{i \in I}$ be a maximal system of invertible analytic functions on X such that the classes of the one-forms $\frac{df_i}{f_i}$ in $\Omega^{1,\mathrm{cl}}(X)/d\mathcal{O}(X)$ are linearly independent over $\mathfrak{c}(X)$. Then the following homomorphism of filtered D_X-algebras is bijective:*

$$\mathcal{O}(X)[T_i]_{i \in I} \otimes_k K \to L^\lambda(X) : T_i \mapsto \mathrm{Log}^\lambda(f_i) \ . \qquad \blacksquare$$

8.4.5. Corollary. *In the situation of Theorem 8.4.1, let $\{\omega_i\}_{i \in I} \subset \Omega^{1,\mathrm{cl}}(X)$ be such that the classes of ω_i's in $\Omega^{1,\mathrm{cl}}(X)/d\mathcal{O}(X)$ are linearly independent over $\mathfrak{c}(X)$, and assume that either they are contained in the $\mathfrak{c}(X)$-vector subspace generated by the classes of the one-forms $\frac{df}{f}$ for $f \in \mathcal{O}(X)^*$, or $H^1(X, \mathfrak{c}_X) = 0$. Given a point $x \in X(k)$, let f_i be a primitive of ω_i in $\mathcal{O}_{X,x}$. Then the analytic functions $\{f_i\}_{i \in I}$ are algebraically independent over the image of $\mathcal{O}(X)$ in $\mathcal{O}_{X,x}$.*

Notice that the existence of a k-rational point implies that $\mathfrak{c}(X) = k$.

Proof. In both cases, for every $i \in I$, there exists a primitive g_i of ω_i in $\mathcal{P}^{\lambda,i}(X)$ with $g_{i,x} = f_i$, and the required fact follows from Theorems 8.4.1 and 1.6.2(i). ∎

The following is a consequence of Proposition 8.4.2.

8.4.6. Corollary. *Let X be a smooth k-analytic space. Then*

(i) for any étale morphism $Y \to X$ with connected Y and the property that the $\mathcal{O}_Y$-module Ω^1_Y is free over a nonempty Zariski open subset of Y, the map $\mathcal{P}^{\lambda,n+1}(Y)/\mathcal{S}^{\lambda,n}(Y) \otimes_{\mathfrak{c}(Y)} \mathcal{O}(Y) \to \mathcal{S}^{\lambda,n+1}(Y)/\mathcal{S}^{\lambda,n}(Y)$ is injective;

(ii) there is an isomorphism of $\mathcal{D}_X$-modules $\mathcal{P}^{\lambda,n+1}_X/\mathcal{S}^{\lambda,n}_X \otimes_{\mathfrak{c}_X} \mathcal{O}_X \xrightarrow{\sim} \mathcal{S}^{\lambda,n+1}_X/\mathcal{S}^{\lambda,n}_X$.

Proof. By Proposition 8.4.2, local sections of the sheaf $\mathcal{S}^{\lambda,n+1}_X/\mathcal{S}^{\lambda,n}_X$ satisfy the assumption of Lemma 1.3.1, and so (i) follows from that lemma and the fact that the canonical maps $\mathcal{P}^{\lambda,n+1}(Y)/\mathcal{S}^{\lambda,n}(Y) \to (\mathcal{P}^{\lambda,n+1}_X/\mathcal{S}^{\lambda,n}_X)(Y)$ and $\mathcal{S}^{\lambda,n+1}(Y)/\mathcal{S}^{\lambda,n}(Y) \to (\mathcal{S}^{\lambda,n+1}_X/\mathcal{S}^{\lambda,n}_X)(Y)$ are injective. That the homomorphism in (ii) is injective follows from the same lemma, and its surjectivity follows from the property (d) of Theorem 1.6.1. ∎

8.4.7. Remark. It would be interesting to know if the statement of Proposition 8.4.2 is true with the weaker assumption $f \in \mathfrak{N}^K(X)$ instead of $f \in \mathcal{S}^\lambda(X)$. This is true for trivial reason in the case when $X \backslash X'$ is of dimension zero. It would be also interesting to know if the correspondence $U \mapsto \mathcal{S}^\lambda(U)$ is a sheaf in the flat quasifinite topology of X (see [Ber2, §4.1]). Again, this is true for trivial reason for X of dimension one.

8.5. A filtered $\mathcal{D}_X$-subalgebra $\mathfrak{s}_X \subset \mathcal{S}_X$ and the sheaf Ψ_X. For $i \geq 0$, let $\mathfrak{s}^i_X$ denote the $\mathcal{D}_X$-module which is the intersection of $\mathcal{S}^i_X$ and $\mathfrak{n}_X$ in $\mathfrak{N}_X$, and set $\mathfrak{s}_X = \varinjlim \mathfrak{s}^i_X$. Notice that, since the sheaves $\mathcal{S}^{\lambda,i}_X$ are functorial with respect to (K,λ), it follows that $\mathfrak{s}^i_X \subset \mathcal{S}^{\lambda,i}_X$ for every (K,λ). Notice also that, by Theorem 8.3.1(4), one has $\mathfrak{s}_{X,x} \subset \mathcal{E}_{X,x}$ for all points $x \in X$.

8.5.1. Theorem. *(i) The class of a closed one-form $\omega \in \Omega^1_{X,x}$ is contained in $\Psi_{X,x}$ if and only if it admits a primitive $f_\omega \in \mathfrak{s}_{X,x}$;*

(ii) $\mathfrak{s}_X \cap \mathcal{L}_X = \mathcal{O}_X$.

Proof. (i) Assume first that $s(x) = \dim(X)$. If the class of ω is in $\Psi_{X,x}$ then, by Theorem 4.5.1(iii), there exists a marked neighborhood $\varphi : \mathfrak{X}_\eta \to X$ of x such that $\varphi^*(\omega)_{\mathbf{x}} \in d\mathcal{O}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$. We can find a monic polynomial $P(T) \in k[T]$ with no roots-of-unity roots such that $P(\phi^*)(\varphi^*(\omega)) = dg$ for some Frobenius lifting ϕ around the generic point σ of $\mathfrak{X}$ and a function $g \in \mathcal{O}_{\mathfrak{X}_\eta,\sigma}$. Then there exists a primitive f of $\varphi^*(\omega)$ in $\mathcal{S}^1(\mathfrak{X}_\eta)$ with $P(\phi^*)f = g$. on the other hand, by Theorem 8.2.1, there exists a primitive f' of $\varphi^*(\omega)$ in $\mathcal{E}^1(\mathfrak{X})$ and, therefore,

$f - f' = \alpha + \beta\mathrm{Log}(p)$ with $\alpha, \beta \in \mathfrak{c}(\mathfrak{X})$. It follows that $g - P(\phi^*)f' = P(\phi^*)\alpha + P(\phi^*)\beta\mathrm{Log}(p)$. The left hand side is the restriction of an element of $\mathfrak{c}(\mathfrak{X})$ and, therefore, $P(\phi^*)\beta = 0$. Since $P(\phi^*)$ induces a bijection on $\mathfrak{c}(\mathfrak{X})$, it follows that $\beta = 0$, i.e., $f_{\mathbf{x}} \in \mathcal{O}(\pi^{-1}(\mathbf{x}))$ for all closed points $\mathbf{x} \in \mathfrak{X}_s$. The latter means that $f \in \mathfrak{s}(\mathfrak{X}_\eta)$ and, therefore, ω has a primitive in $\mathfrak{s}_{X,x}$. Conversely, assume ω has a primitive f in $\mathfrak{s}_{X,x}$, and let $\varphi : \mathfrak{X}_\eta \to X$ be a marked neighborhood of the point x whose image is contained in an open neighborhood of x over which f is defined. Then for every closed point $\mathbf{x} \in \mathfrak{X}_s$, one has $\varphi^*(f)_{\mathbf{x}} \in \mathfrak{s}(\pi^{-1}(\mathbf{x})) \cap L^1(\pi^{-1}(\mathbf{x})) = \mathcal{O}(\pi^{-1}(\mathbf{x}))$ and, therefore, $\varphi^*(\omega)_{\mathbf{x}} \in d\mathcal{O}(\pi^{-1}(\mathbf{x}))$. Theorem 4.5.1(iii) implies that the class of ω lies in $\Psi_{X,x}$.

If the point x is arbitrary, we may shrink X and assume that there exists a smooth morphism $\varphi : X \to Y$ that takes x to a point y with $s(x) = s(y) = \dim(Y)$. By Lemma 4.5.2(iii), $\Psi_{X,\overline{x}} \xrightarrow{\sim} \Psi_{Y,\overline{y}}$ for any pair of compatible geometric points $\overline{x}$ and $\overline{y}$ over x and y, respectively, and the first case implies that, if the class of ω is in $\Psi_{X,x}$, then it has a primitive in $\mathfrak{s}_{X,x}$. On the other hand, assume that ω has a primitive in $\mathfrak{s}_{X,x}$. Since the latter is contained in $\mathcal{E}_{X,x}$, then the class of ω lies in $\mathcal{V}_{X,x}$ and, by Theorem 4.5.2(ii), we can shrink X and Y in the étale topology so that the class of ω coincides with that of $\varphi^*(\eta)$ for some $\eta \in \Omega^{1,\mathrm{cl}}_{X,x}$ with class in $\mathcal{V}_{Y,y}$. It follows that $\varphi^*(\eta)$ has a primitive in $\mathfrak{s}_{X,x}$. Since any smooth morphism has a local section in the étale topology, it follows that η has a primitive in $\mathfrak{s}_{Y,y}$. By the previous case, the class of η lies in $\Psi_{Y,y}$ and, therefore, that of ω lies in $\Psi_{X,x}$.

(ii) By Corollary 8.3.4, it suffices to prove that, given functions $f_1, \dots, f_n \in \mathcal{O}^*_{X,x}$ with $|f_i(x)| = 1$ such that their images in $\mathcal{O}^v_{X,x}$ are linearly independent, for any polynomial of positive degree $P \in \mathcal{O}_{X,x}[T_1, \dots, T_n]$ the function $P(\mathrm{Log}(f_1), \dots, \mathrm{Log}(f_n))$ does not lie in $\mathfrak{s}_{X,x}$. Shrinking X, we may assume that it is connected, $f_1, \dots, f_n \in \mathcal{O}(X)^*$ and and all of the coefficients of the polynomial P are defined over all X, and we are going to show that $P(\mathrm{Log}(f_1), \dots, \mathrm{Log}(f_n)) \notin \mathfrak{s}(X)$.

Step 1. *The required fact is true if* $s(x) = \dim(X) = 1$. Indeed, shrinking X in the étale topology, we may assume that $X = \mathcal{X}^{\mathrm{an}}_\eta \backslash \coprod_{i=1}^m X_i$, where $\mathcal{X}$ is a smooth projective curve over k° and each X_i is an affinoid subdomain of $\pi^{-1}(\mathbf{x}_i)$ isomorphic to a closed disc with center at zero, and $\mathbf{x}_1, \dots, \mathbf{x}_m$ are pairwise distinct $\widetilde{k}$-rational points of $\mathcal{X}_s$. As in the proof of Theorem 4.3.1, we can shrink X (increasing each X_i in $\pi^{-1}(\mathbf{x}_i)$) so that there exist functions $g_1, \dots, g_{m-1} \in \mathcal{O}(X)^*$ such that the reduction of every g_i is a well defined rational function on $\mathcal{X}_s$ whose divisor is a nonzero integral multiple of $(\mathbf{x}_i) - (\mathbf{x}_m)$. It follows (see §4.3) that a nonzero integral power of every function f_j is contained in the subgroup generated by $g_1, \dots, g_{m-1}$ and $\mathcal{O}(X)^1$ and, therefore, $P(\mathrm{Log}(f_1), \dots, \mathrm{Log}(f_n)) = Q(\mathrm{Log}(g_1), \dots, \mathrm{Log}(g_{m-1}))$ for some polynomial Q of positive degree over $\mathcal{O}(X)$. We may therefore assume that $\{f_1, \dots, f_n\} = \{g_1, \dots, g_{m-1}\}$. Assume now that $P(T_1, \dots, T_n) = \sum_{i=1}^{d} P_i(T_2, \dots, T_n) T_1^i$, where $d \geq 1$ and $P_d \neq 0$. By the construction, the restrictions of the functions $\mathrm{Log}(f_2), \dots, \mathrm{Log}(f_n)$ to the open annulus $B = \pi^{-1}(\mathbf{x}_1) \backslash X_1$ are analytic and, by Corollary 8.4.5, $P_d(\mathrm{Log}(f_2), \dots, \mathrm{Log}(f_n)) \neq 0$. Thus, the restriction of $P(\mathrm{Log}(f_1), \dots, \mathrm{Log}(f_n))$ to B is a polynomial in $\mathrm{Log}(f_1)$ of positive degree over $\mathcal{O}(B)$. It is clear that such a functions is not contained in $\mathfrak{s}(B)$.

Step 2. *The required fact is true if* $s(x) = \dim(X)$. Since all of the nonzero coefficients of P are not equal to zero at x, we may shrink X and assume that they are even invertible. By Lemma 4.4.3, there exists a morphism $\varphi : Y \to X$ from smooth k-analytic curve Y such that, for some point $y \in Y$ of type (2), the images of $\varphi^*(f_1), \dots, \varphi^*(f_n)$ in $\mathcal{O}_{Y,y}^{v}$ are linearly independent. Since $\varphi^*(P)$ is a polynomial of positive degree over $\mathcal{O}(Y)$, this reduces the situation to Step 1.

Step 3. *The required fact is true in the general case.* Indeed, shrinking X, we may assume that there exists a smooth morphism $\varphi : X \to Y$ that takes the point x to a point $y \in Y$ with $s(x) = s(y) = \dim(Y)$. Let x' be an arbitrary point of the fiber $X_y = \varphi^{-1}(y)$ with $s(x') = \dim(X_y)$. Then $s(x') = \dim(X)$. Let $\overline{y}$ be a geometric point of Y over y. If $\overline{x}$ and $\overline{x}'$ are geometric points of X over $\overline{y}$ and the points x and x', then the canonical homomorphisms from $\widetilde{\mathcal{H}}_{\overline{y}}^{*}/\widetilde{\mathfrak{c}}_{Y,\overline{y}}^{*}$ to $\widetilde{\mathcal{H}}_{\overline{x}}^{*}/\widetilde{\mathfrak{c}}_{X,\overline{x}}^{*}$ and $\widetilde{\mathcal{H}}_{\overline{x}'}^{*}/\widetilde{\mathfrak{c}}_{X,\overline{x}'}^{*}$ are bijective and injective, respectively. It follows that the validity of the required fact for x' implies that for the point x. ∎

If X is a proper smooth k-analytic space with good reduction, then, by Theorem 8.1.1(1), the $\mathcal{O}(X)$-algebra $\mathcal{E}^{\lambda}(X) = \mathcal{E}^{\lambda}(X, X)$ is contained in $\mathfrak{s}(X)$. But the $\mathcal{O}(X)$-algebra $\mathfrak{s}(X)$ is much larger than $\mathcal{E}(X)$.

8.5.2. Lemma. *For every* $i \geq 1$, *the quotient space* $\mathfrak{s}^i(\mathbf{P}^1)/\mathfrak{s}^{i-1}(\mathbf{P}^1)$ *is of infinite dimension over* k.

Notice that $\mathcal{E}(\mathbf{P}^1) = \mathcal{O}(\mathbf{P}^1) = k$.

Proof. Let $\mathcal{X}$ be an elliptic curve over k with good reduction all of whose points of order two are k-rational, ω a nonzero invariant one-form on $\mathcal{X}$, and f a primitive of ω in $\mathcal{E}^1(\mathcal{X}^{\mathrm{an}})$ with $f(0) = 0$. Then f has zero of first order at all points of finite order in $\mathcal{X}^{\mathrm{an}}(k)$. If σ denotes the automorphism $x \mapsto -x$ on $\mathcal{X}$, then the quotient of $\mathcal{X}$ by $\{1, \sigma\}$ is the projective line. We denote by φ the corresponding homomorphism $\mathcal{X}^{\mathrm{an}} \to \mathbf{P}^1$, and we may assume that $\varphi(0) = \infty$. Since ${}^{\sigma}\omega = -\omega$, then ${}^{\sigma}f = -f$. Furthermore, if g is a rational function on $\mathcal{X}$ whose divisor is $(P_1) + (P_2) + (P_3) - 3(0)$, where P_i are the points of order two, then ${}^{\sigma}g = -g$ and, therefore, for the function $h = \frac{f}{g} \in \mathfrak{s}^1(\mathcal{X}^{\mathrm{an}})$ one has ${}^\sigma h = h$, i.e., $h \in \mathfrak{s}^1(\mathbf{P}^1)$. Finally, let α be an element of k^* with $|\alpha| \neq 1$ and, for $n \in \mathbf{Z}$, let ψ_n denote the automorphism $\mathbf{P}^1 \to \mathbf{P}^1 : z \mapsto \alpha^n z$ and set $h_n = \psi_n^*(h)$. *We claim that, for every $i \geq 1$, the images of the functions $\{h_n^i\}_{n \in \mathbf{Z}}$ in $\mathfrak{s}^i(\mathbf{P}^1)/\mathfrak{s}^{i-1}(\mathbf{P}^1)$ are linearly independent over k.* Indeed, since the function f is analytic everywhere except the generic point x of $\mathcal{X}^{\mathrm{an}}$, the function h is analytic everywhere except the point $y = \varphi(x)$ of type (2). It follows that each function h_n is analytic everywhere except the point $y_n = \psi_n^{-1}(y)$. By Corollary 8.3.6(i), the image of h_n^i in $\mathfrak{s}^i_{\mathbf{P}^1, y_n}/\mathfrak{s}^{i-1}_{\mathbf{P}^1, y_n}$ is not zero. Since all of the points $\{y_n\}_{n \in \mathbf{Z}}$ are pairwise distinct, the claim follows. ∎

§9. Integration and parallel transport along a path

For a k-analytic space X, we set $\overline{X} = X\widehat{\otimes}\widehat{k}^{\mathrm{a}}$. At the beginning of this section we construct, for every smooth k-analytic space X with $H_1(\overline{X}, \mathbf{Q}) \xrightarrow{\sim} H_1(X, \mathbf{Q})$, every closed one-form $\omega \in \Omega^1_{\mathcal{S}^\lambda,n}(X)$ and every path $\gamma : [0,1] \to X$ with ends in $X(k)$, an integral $\int_\gamma \omega \in K^{n+1}$. This integral possesses all of the natural properties and, in particular, it only depends on the homotopy class of γ. That this dependence is nontrivial is shown in §9.2. Furthermore, for a $\mathcal{D}_X$-module $\mathcal{F}$, let $\mathcal{F}_{\mathcal{S}^\lambda}$ denote the $\mathcal{D}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{S}^\lambda_X$. We prove that a locally unipotent $\mathcal{D}_X$-module $\mathcal{F}$ is trivial over an open neighborhood of every point $x \in X_{st}$, and its $\mathcal{C}^K_X$-module of horizontal sections $\mathcal{F}^\nabla_{\mathcal{S}^\lambda}$ is locally free in the usual topology of X. It is not difficult to derive from this that for any path $\gamma : [0,1] \to \overline{X}$ with ends $x, y \in \overline{X}_{st}$ there is an associated isomorphism of $(k^{\mathrm{a}} \otimes_k K)$-modules $\mathrm{T}^{\mathcal{F}}_\gamma : \mathcal{F}^\nabla_x \otimes_k K \xrightarrow{\sim} \mathcal{F}^\nabla_y \otimes_k K$ which possesses all of the properties of the classical parallel transport. It is more difficult to prove that the parallel transport is uniquely determined by those properties. We also show that an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module $\mathcal{F}$ is locally quasi-unipotent (i.e., locally unipotent in the étale topology) if and only if the $\mathcal{C}^K_X$-module $\mathcal{F}^\nabla_{\mathcal{S}^\lambda}$ is locally free in the étale topology of X, and we construct a similar parallel transport $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}} : \mathcal{F}^\nabla_{\overline{x}} \otimes_k K \xrightarrow{\sim} \mathcal{F}^\nabla_{\overline{y}} \otimes_k K$ along an étale path $\overline{\gamma}$ from $\overline{x}$ to $\overline{y}$. In particular, every locally quasi-unipotent $\mathcal{D}_X$-module $\mathcal{F}$ gives rise to a semi-linear representation of the étale fundamental group $\pi_1^{\text{ét}}(X, \overline{x})$ of X in the free $(k^{\mathrm{a}} \otimes_k K)$-module $\mathcal{F}^\nabla_{\overline{x}}$ of rank equal to the rank of $\mathcal{F}$. (Semi-linearity here is considered with respect to the action of $\pi_1^{\text{ét}}(X, \overline{x})$ on k^{a} through its homomorphism to the Galois group of k^{a} over k.)

9.1. Integration of closed one-forms along a path.

9.1.1. Theorem. *Given a closed subfield $k \subset \mathbf{C}_p$, a filtered k-algebra K and an element $\lambda \in K^1$, there is a unique way to construct, for every smooth k-analytic space X with $H_1(\overline{X}, \mathbf{Q}) \xrightarrow{\sim} H_1(X, \mathbf{Q})$, every closed one-form $\omega \in \Omega^1_{\mathcal{S}^\lambda,n}(X)$ and every path $\gamma : [0,1] \to X$ with ends in $X(k)$, an integral $\int_\gamma \omega \in K^{n+1}$ such that the following is true:*

(a) if $\omega = df$ with $f \in \mathcal{S}^{\lambda,n+1}(X)$, then $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$;

(b) $\int_\gamma \omega$ depends only on the homotopy class of γ;

(c) given a second path $\gamma' : [0,1] \to X$ with ends in $X(k)$ and $\gamma'(0) = \gamma(1)$, one has $\int_{\gamma'\circ\gamma} \omega = \int_\gamma \omega + \int_{\gamma'} \omega$.

Furthermore, the integral possesses the following properties:

(1) $\int_\gamma \omega$ depends linearly on ω;

(2) $\int_\gamma \omega$ is functorial with respect to $(k, X, \gamma, K, \lambda)$;

(3) if $\gamma([0,1]) \subset Y$ and $\omega\big|_{(X,Y)} \in \Omega^1_{\mathcal{E}^\lambda}(X,Y)$, where Y is an analytic domain with good reduction, then $\int_\gamma \omega \in k$.

Here $H_1(X, \mathbf{Q})$ is the singular homology group of X with coefficients in $\mathbf{Q}$. The condition $H_1(\overline{X}, \mathbf{Q}) \stackrel{\sim}{\to} H_1(X, \mathbf{Q})$ is equivalent to the isomorphism of singular cohomology groups $H^1(X, \mathbf{Q}) \stackrel{\sim}{\to} H^1(\overline{X}, \mathbf{Q})$ (which coincide with the étale cohomology groups with coefficients in the constant sheaf $\mathbf{Q}_X$). The full formulation of (2) is as follows. Given a closed subfield $k' \subset \mathbf{C}_p$, a filtered k'-algebra K', a smooth k'-analytic space X' with $H_1(\overline{X}', \mathbf{Q}) \stackrel{\sim}{\to} H_1(X', \mathbf{Q})$, a morphism $\varphi : X' \to X$ over an isometric embedding $k \hookrightarrow k'$, a homomorphism of filtered algebras $K \to K'$ over the embedding $k \hookrightarrow k'$ that takes λ to an element $\lambda' \in {K'}^1$, and a path $\gamma' : [0,1] \to X'$ with ends in $X'(k')$, one has $\int_{\gamma'} \varphi^*(\omega) = \int_{\varphi\circ\gamma'} \omega$.

Proof. Consider first the case when k is algebraically closed, i.e., $k = \mathbf{C}_p$. In this case the condition on X is evidently satisfied, $\mathfrak{c}_X$ is the constant sheaf k_X associated to k and, therefore, $\mathcal{C}^n_X = K^n_X$. The pullback of the exact sequence of abelian groups $0 \to K^{n+1}_X \to \mathcal{P}^{\lambda,n+1}_X \to \Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,n},X} \to 0$ with respect to γ gives rise to an exact sequence on $[0,1]$

$$0 \longrightarrow K^{n+1}_{[0,1]} \longrightarrow \gamma^*(\mathcal{P}^{\lambda,n+1}_X) \stackrel{d}{\longrightarrow} \gamma^*(\Omega^{1,\mathrm{cl}}_{\mathcal{S}^{\lambda,n},X}) \longrightarrow 0 \ .$$

Since the unit interval $[0,1]$ is contractible, there is a section g of the sheaf $\gamma^*(\mathcal{P}^{\lambda,n+1}_X)$, defined uniquely up to an element of K^{n+1}, with $dg = \gamma^*(\omega)$. and we set $\int_\gamma \omega = g(1) - g(0)$. (Notice that the stalks of $\gamma^*(\mathcal{P}^{\lambda,n+1}_X)$ at 0 and 1 coincide with $\mathcal{P}^{\lambda,n+1}_{X,\gamma(0)} \subset \mathcal{O}_{X,\gamma(0)} \otimes_k K^{n+1}$ and $\mathcal{P}^{\lambda,n+1}_{X,\gamma(1)} \subset \mathcal{O}_{X,\gamma(1)} \otimes_k K^{n+1}$, respectively, and therefore, $g(0)$ and $g(1)$ are elements of K^{n+1}.) If $\omega = df$, one can take $g = \gamma^*(f)$, and we get $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$, i.e., (a) is true. Assume we are given a path $\gamma' : [0,1] \to X$ with the same ends as γ and which is homotopy equivalent to γ. This implies that there is a continuous map $\Phi : [0,1]^2 \to X$ with $\Phi(t,0) = \gamma(t)$, and $\Phi(t,1) = \gamma'(t)$ for all $t \in [0,1]$. The same reasoning gives a section g of the sheaf $\Phi^*(\mathcal{P}^{\lambda,n+1}_X)$ with $dg = \Phi^*(\omega)$, and this easily implies the equality $\int_{\gamma'} \omega = \int_\gamma \omega$, i.e., (b) is true. The validity of (c) and (1) is trivial, that of (2) easily follows from the property (f) of Theorem 1.6.1, and that of (3) follows from Theorem 8.1.1. To prove uniqueness of the integral, we take open subsets $\mathcal{U}_1, \dots, \mathcal{U}_m \subset X$ such that ω has a primitive f_i at each $\mathcal{U}_i$ and $\gamma([\frac{i-1}{m}, \frac{i}{m}]) \subset \mathcal{U}_i$ for all $1 \le i \le m$. Furthermore, given $1 \le i \le m-1$, let x_i be a k-rational point of $\mathcal{U}_i \cap \mathcal{U}_{i+1}$ which is contained in the same connected component of the intersection as the point $\gamma(\frac{i}{m})$, and set $x_0 = \gamma(0)$ and $x_m = \gamma(1)$. By the property (a), the integral of ω along any path $[0,1] \to \mathcal{U}_i$ with ends x_{i-1} and x_i is equal to $f_i(x_i) - f_i(x_{i-1})$ and, by the properties (b) and (c), we get the equality

$$\int_\gamma \omega = \sum_{i=1}^{m} (f_i(x_i) - f_i(x_{i-1})) \ ,$$

which implies the uniqueness.

In the general case we need the following fact.

9.1.2. Lemma. *Let k be a non-Archimedean field whose residue field is at most countable, X a k-analytic space, and α the canonical map $\overline{X} = X\widehat{\otimes}\widehat{k}^{\mathrm{a}} \to X$. Then for every path $\gamma : [0,1] \to X$ and every point $x' \in \overline{X}$ with $\alpha(x') = \gamma(0)$ there exists a path $\gamma' : [0,1] \to \overline{X}$ with $\alpha \circ \gamma' = \gamma$ and $\gamma'(0) = x'$.*

Proof (cf. [Ber1, Lemma 3.2.5]). We may assume that the space X is compact, and let us consider the cartesian diagram

$$\begin{array}{ccc} [0,1] & \xrightarrow{\gamma} & X \\ \uparrow \alpha' & & \uparrow \alpha \\ \Sigma & \xrightarrow{\tau} & \overline{X} \end{array}$$

Notice that there is a homeomorphism $\overline{X} \xrightarrow{\sim} \varprojlim X\widehat{\otimes}k'$, where k' runs through finite extensions of k in k^{a}. It follows that the map $\alpha' : \Sigma \to [0,1]$ is open, proper and surjective. Since the connected component of a point in a compact space coincides with the intersection of its open-closed neighborhoods, it follows that the map $\alpha' : \Sigma' \to [0,1]$ is surjective, where Σ' is the connected component of the point $(0, x')$ in Σ. The assumption on the residue field of k and [Ber1, 3.2.9] imply that the compact space Σ' has a countable basis of open sets. From [En, 4.2.8] it follows that Σ' is metrizable. Since connected metrizable compact spaces are arcwise connected ([En, 6.3.11]), it follows that Σ' is arcwise connected. This implies that there exists a homeomorphic embedding $\psi : [0,1] \to \Sigma'$ with $\psi(0) = (0, x')$ and $\alpha'(\psi(1)) = 1$. The composition $\beta = \alpha' \circ \psi$ is a homeomorphism of $[0,1]$ with itself that fixed its ends, and the path $\gamma' = \tau \circ \psi \circ \beta^{-1} : [0,1] \to \overline{X}$ possesses the required properties. ∎

By Lemma 9.1.2, there exists a path $\gamma' : [0,1] \to \overline{X}$ with $\alpha \circ \gamma' = \gamma$. Since the points $\gamma(0)$ and $\gamma(1)$ are in $X(k)$, they have unique preimages x' and y' in $\overline{X}$, respectively, and it follows that $\gamma'(0) = x'$ and $\gamma'(1) = y'$. We set $\int_\gamma \omega = \int_{\gamma'} \overline{\omega} \in K^{n+1} \otimes_k \mathbf{C}_p$. First of all, the latter does not depend on the choice of γ'. Indeed, if γ'' is another lifting of γ, then the class of ${\gamma'}^{-1} \circ \gamma''$ in $H_1(|\overline{X}|, \mathbf{Q})$ is zero (since its image in $H_1(|X|, \mathbf{Q})$ is zero) and, therefore, $\int_{\gamma''} \overline{\omega} = \int_{\overline{\gamma}} \overline{\omega}$. For the same reason, given an element σ of the Galois group of k^{a} over k, the class of ${\gamma'}^{-1} \circ {}^{\sigma}\gamma'$ in $H_1(|\overline{X}|, \mathbf{Q})$ is zero. This implies that $\int_\gamma \omega \in K^{n+1}$. If $\tau : [0,1] \to X$ is a path homotopy equivalent to γ, then the class of $\gamma^{-1}\circ\tau$ in $H_1(|X|,\mathbf{Q})$ is zero and, therefore, the class of ${\gamma'}^{-1}\circ\tau'$ in $H_1(|\overline{X}|,\mathbf{Q})$ is zero, i.e., $\int_\tau\omega=\int_\gamma\omega$. All of the other properties of the integral and its uniqueness easily follow from the case of an algebraically closed field k. ∎

9.1.3. Remarks. (i) If the field k is not algebraically closed, the assumption $H_1(\overline{X},\mathbf{Q})\xrightarrow{\sim} H_1(X,\mathbf{Q})$ is really necessary for the existence of the integral along a path in X. Indeed, let X be a twisted Tate elliptic curve. Then X is contractible, but $\overline{X}$ is homotopy equivalent to a circle (and, in particular, the assumption is not satisfied). Let ω be a nonzero analytic one-form from $\Omega^1(X)$. The existence of the integral $\int_\gamma\omega$ along paths γ in X would imply that ω has a primitive on X. But this is impossible since $\overline{X}$ is isomorphic to the quotient $\mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$ for some $q\in(k^{\mathrm{a}})^*$ with $|q|<1$, and the primitive of the pullback of ω on $\mathbf{G}_{\mathrm{m}}$ is, up to some constants, the logarithm $\mathrm{Log}^\lambda(T)$, which cannot come from a function on X if $\mathrm{Log}^\lambda(q)\neq 0$.

(ii) The integral of a closed analytic one-form on a smooth algebraic variety $\mathcal{X}$, constructed by Yu. Zarhin ([Zar]) and P. Colmez ([Colm]), depends only on two points and not on a path that connects them. The reason is that the integral is required to be functorial with respect to morphisms in the category of algebraic varieties, and the latter category is too coarse to distinguish a nontrivial homotopy type of the analytification $\mathcal{X}^{\mathrm{an}}$ of $\mathcal{X}$. If $\mathcal{X}$ is proper and has good reduction, $\mathcal{X}^{\mathrm{an}}$ is contractible, and the integral of a closed analytic one-form along a path (from Theorem 9.1.1) depends only on its ends and coincides with the integral constructed by R. Coleman in [Col2] as well as with those mentioned above.

9.2. Nontrivial dependence on the homotopy class of a path. In this subsection $K=k_{\mathrm{Log}}$ and $\lambda=\mathrm{Log}(p)$. Let $\mathcal{X}$ be a geometrically connected separated smooth scheme over k and $\overline{\mathcal{X}}=\mathcal{X}\otimes_k\widehat{k}^{\mathrm{a}}$. The construction of the previous subsection gives rise to a $\widehat{k}^{\mathrm{a}}$-linear pairing $\Omega^{1,\mathrm{cl}}(\overline{\mathcal{X}})/d\mathcal{O}(\overline{\mathcal{X}})\times H_1(\overline{\mathcal{X}}^{\mathrm{an}},\widehat{k}^{\mathrm{a}})\to(\widehat{k}^{\mathrm{a}})^1_{\mathrm{Log}}:(\omega,\gamma)\mapsto\int_\gamma\omega$. This pairing is equivariant with respect to the action of the Galois group $G=\mathrm{Gal}(k^{\mathrm{a}}/k)$. From [Ber9, Corollary 8.3.4] it follows that there is a canonical isomorphism of finite dimensional k-vector spaces $H^1(\mathcal{X}^{\mathrm{an}},\mathfrak{c}_{\mathcal{X}^{\mathrm{an}}})\xrightarrow{\sim} H^1(\overline{\mathcal{X}}^{\mathrm{an}},\widehat{k}^{\mathrm{a}})^G$. Thus, if we set $H_1(\mathcal{X}^{\mathrm{an}},\mathfrak{c}_{\mathcal{X}^{\mathrm{an}}})=H_1(\overline{\mathcal{X}}^{\mathrm{an}},\widehat{k}^{\mathrm{a}})^G$, we get a k-linear pairing

$$\Omega^{1,\mathrm{cl}}(\mathcal{X})/d\mathcal{O}(\mathcal{X})\times H_1(\mathcal{X}^{\mathrm{an}},\mathfrak{c}_{\mathcal{X}^{\mathrm{an}}})\to k^1_{\mathrm{Log}}:(\omega,\gamma)\mapsto\int_\gamma\omega\ .$$

Our first purpose is to extend this pairing to the de Rham cohomology group $H^1_{\mathrm{dR}}(\mathcal{X})$, which contains $\Omega^{1,\mathrm{cl}}(\mathcal{X})/d\mathcal{O}(\mathcal{X})$ and coincides with it if $\mathcal{X}$ is affine. Recall that, by R. Kiehl's theorem [Kie], there are canonical isomorphisms of de Rham cohomology groups $H^n_{\mathrm{dR}}(\mathcal{X})\xrightarrow{\sim} H^n_{\mathrm{dR}}(\mathcal{X}^{\mathrm{an}})$.

9.2.1. Lemma. *Let $\mathcal{X}$ be an irreducible scheme over a non-Archimedean field k, and let $F^\cdot$ be a complex of sheaves of abelian groups on $\mathcal{X}^{\mathrm{an}}$. Assume that the restriction of the sheaf* $\mathrm{Ker}(F^0 \to F^1)$ *to the Zariski topology of $\mathcal{X}$ is constant. Then the correspondence $\mathcal{Y} \mapsto H^1(\mathcal{Y}^{\mathrm{an}}, F^\cdot)$ is a sheaf on $\mathcal{X}$.*

Proof. Let $\mathcal{U} = \{\mathcal{Y}_i\}_{i \in I}$ be an open covering of $\mathcal{Y}$, and consider the spectral sequence $E_2^{p,q} = \check{H}^p(\mathcal{U}^{\mathrm{an}}, \mathcal{H}^q) \Longrightarrow H^{p+q}(\mathcal{Y}^{\mathrm{an}}, \mathcal{F}^\cdot)$, where $\mathcal{H}^q$ denotes the presheaf $\mathcal{Y} \mapsto H^q(\mathcal{Y}^{\mathrm{an}}, F^\cdot)$. The assumption implies that $E_2^{p,0} = 0$ for all $p \geq 1$ and, therefore, $H^1(\mathcal{Y}^{\mathrm{an}}, F^\cdot) \xrightarrow{\sim} E_2^{0,1}$. The required fact follows. ∎

By Lemma 1.1.1, the assumption of Lemma 9.2.1 is satisfied for the complex $0 \to \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}} \to 0$. It follows that the correspondence $\mathcal{Y} \mapsto H^1(\mathcal{Y}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{Y}^{\mathrm{an}}})$ is a sheaf on $\mathcal{X}$. For the same reason, the assumption of Lemma 9.2.1 is satisfied for the complex $\Omega^\cdot_{\mathcal{X}^{\mathrm{an}}}$ and, therefore, the correspondence $\mathcal{Y} \mapsto H^1_{\mathrm{dR}}(\mathcal{Y}^{\mathrm{an}}) = H^1_{\mathrm{dR}}(\mathcal{Y})$ is a sheaf on $\mathcal{Y}$. Thus, the above k-linear pairings for open affine subschemes of $\mathcal{X}$ give rise to a k-linear pairing

$$H^1_{\mathrm{dR}}(\mathcal{X}) \times H_1(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) \to k^1_{\mathrm{Log}} : (\omega, \gamma) \mapsto \int_\gamma \omega \ .$$

Recall that there is an exact sequence $0 \to H^1(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) \to H^1_{\mathrm{dR}}(\mathcal{X}) \to H^0(\mathcal{X}^{\mathrm{an}}, \Omega^{1,\mathrm{cl}}_{\mathcal{X}^{\mathrm{an}}}/d\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}) \to H^2(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}})$ (see §1.3).

9.2.2. Lemma. *The induced pairing $H^1(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) \times H_1(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) \to k^1_{\mathrm{Log}}$ takes values in k and is nondegenerate.*

Proof. We may assume that and k is algebraically closed and $\mathcal{X}$ is affine. Then $H^1_{\mathrm{dR}}(\mathcal{X}) = \Omega^{1,\mathrm{cl}}(\mathcal{X})/d\mathcal{O}(\mathcal{X})$. Let ω be a closed one-form in $\Omega^1(\mathcal{X})$ whose image in $H^1_{\mathrm{dR}}(\mathcal{X}) = H^1_{\mathrm{dR}}(\mathcal{X}^{\mathrm{an}})$ is in $H^1(\mathcal{X}^{\mathrm{an}}, k)$. Then the restriction of ω to an open neighborhood of every point of $\mathcal{X}^{\mathrm{an}}$ is an exact one-form, and so $\int_\gamma \omega \in k$ for all $\gamma \in H_1(\mathcal{X}^{\mathrm{an}}, k)$. Assume that $\int_\gamma \omega = 0$ for all $\gamma \in H_1(\mathcal{X}^{\mathrm{an}}, k)$. Then any local primitive of ω admits an analytic continuation to the whole space $\mathcal{X}^{\mathrm{an}}$ and, therefore, ω is an exact one-form. Since the dimensions of both spaces over k are equal, the pairing considered is nondegenerate. ∎

9.2.3. Theorem. *Assume that $\mathcal{X}$ is proper and that it is either an abelian variety or can be defined over a finite extension of $\mathbf{Q}_p$, and let $\gamma \in H_1(\mathcal{X}^{\mathrm{an}}, \mathfrak{c}_{\mathcal{X}^{\mathrm{an}}})$. If $\int_\gamma \omega \in k$ for all $\omega \in \Omega^{1,\mathrm{cl}}(\mathcal{X})$, then $\gamma = 0$.*

Proof. We may assume that the field k is algebraically closed, and we make the following simple remark concerning the above statement in a more general setting. Let $\varphi : Y \to X$ be a morphism between separated smooth k-analytic spaces. (A) If the map $H_1(Y, k) \to H_1(X, k)$ is injective, then the validity of the statement for X implies that for Y. (B) If the map $H_1(Y,k) \to H_1(X,k)$ is bijective and the map $\Omega^{1,\mathrm{cl}}(X) \to \Omega^{1,\mathrm{cl}}(Y)/d\mathcal{O}(Y)$ is surjective, then the validity of the statement for Y implies that for X.

Step 1. *The theorem is true if $\mathcal{X}$ is an abelian variety.* Indeed, by Raynaud's uniformization of abelian varieties (in the form of [BoLü1]), we may replace $\mathcal{X}$ by an isogenous abelian variety so that there is an exact sequence of proper smooth k-analytic groups $0 \to Y \to \mathcal{X}^{\mathrm{an}} \to Z \to 0$, where Z has good reduction and Y is an analytic torus, i.e., $Y \xrightarrow{\sim} \mathbf{G}_{\mathrm{m}}^n/\Gamma$, where Γ is a lattice of maximal rank in $(k^*)^n$. By [Ber1, §6.5], $H_1(Y,k) \xrightarrow{\sim} H_1(\mathcal{X}^{\mathrm{an}},k)$, and so the remark (B) reduces the verification of the statement to Y instead of $\mathcal{X}$. Let $\gamma_1,\dots,\gamma_n$ be a basis of Γ over $\mathbf{Z}$, and let $\gamma_i = (q_{i1},\dots,q_{in}) \in (k^*)^n$. For an element $\gamma \in H_1(Y,k)$ one has $\gamma = \sum_{i=1}^n \lambda_i\gamma_i$ with $\lambda_i \in k$. If ω_i is the invariant one-form on Y which corresponds to the one-form $\frac{dT_i}{T_i}$ on $\mathbf{G}_{\mathrm{m}}^n$, then $\int_\gamma \omega_i = \sum_{j=1}^n \lambda_j \mathrm{Log}(q_{ij})$. It follows that, if $q_{ij} = u_{ij}p^{\alpha_{ij}}$ with $|u_{ij}| = 1$ and $\alpha_{ij} \in \mathbf{Q}$, then the coefficient of $\int_\gamma \omega_i$ at $\mathrm{Log}(p)$ is equal to $\sum_{j=1}^n \lambda_j\alpha_{ij}$. Thus, if $\int_\gamma \omega_i \in k$ for all $1 \le i \le n$, then $\sum_{j=1}^n \lambda_j\alpha_{ij} = 0$ for all $1 \le i \le n$. Since Γ is a lattice in $(k^*)^n$, it follows that the system of row vectors $\{(\alpha_{i1},\dots,\alpha_{in})\}_{1\le i\le n}$, is linearly independent. Hence $\lambda_i = 0$ for all $1 \le i \le n$, i.e., $\gamma = 0$.

Step 2. *The theorem is true if $\mathcal{X}$ can be defined over a finite extension of $\mathbf{Q}_p$.* By the remark (A), it suffices to verify that, for the canonical morphism $\mathcal{X} \to \mathcal{A}$ from $\mathcal{X}$ to its Albanese variety $\mathcal{A}$, the induced map $H_1(\mathcal{X}^{\mathrm{an}},k) \to H_1(\mathcal{A}^{\mathrm{an}},k)$ is bijective. Notice that the required fact is equivalent to the bijectivity of the map $H^1(|\mathcal{A}^{\mathrm{an}}|,\mathbf{Q}_l) \to H^1(|\mathcal{X}^{\mathrm{an}}|,\mathbf{Q}_l)$ for some prime number l, and that there is a well known isomorphism of l-adic étale cohomology groups $H^1(\mathcal{A},\mathbf{Q}_l) \xrightarrow{\sim} H^1(\mathcal{X},\mathbf{Q}_l)$. We now use the assumption that $\mathcal{X}$ is defined over a finite extension k' of $\mathbf{Q}_p$ in k, i.e., $\mathcal{X} = \mathcal{Y} \otimes_{k'} k$ and, therefore, $\mathcal{A} = \mathcal{B} \otimes_{k'} k$, where $\mathcal{Y}$ is a proper smooth scheme over k' and $\mathcal{B}$ is its Albanese variety. We may assume that k' is big enough so that $H^1(|\mathcal{Y}^{\mathrm{an}}|,\mathbf{Q}_l) \xrightarrow{\sim} H^1(|\mathcal{X}^{\mathrm{an}}|,\mathbf{Q}_l)$ and $H^1(|\mathcal{B}^{\mathrm{an}}|,\mathbf{Q}_l) \xrightarrow{\sim} H^1(|\mathcal{A}^{\mathrm{an}}|,\mathbf{Q}_l)$ (see [Ber7, Theorem 10.1]), i.e., it suffices to check the bijectivity of the map $H^1(|\mathcal{B}^{\mathrm{an}}|,\mathbf{Q}_l) \to H^1(|\mathcal{Y}^{\mathrm{an}}|,\mathbf{Q}_l)$. By [Ber8, Corollary 1.2], there are canonical isomorphisms $H^1(|\mathcal{Y}^{\mathrm{an}}|,\mathbf{Q}_l) \xrightarrow{\sim} H^1(\mathcal{X},\mathbf{Q}_l)^G$ and $H^1(|\mathcal{B}^{\mathrm{an}}|,\mathbf{Q}_l) \xrightarrow{\sim} H^1(\mathcal{A},\mathbf{Q}_l)^G$, where G is the Galois group of k'^{a} over k', and the required fact follows from the isomorphism $H^1(\mathcal{A},\mathbf{Q}_l) \xrightarrow{\sim} H^1(\mathcal{X},\mathbf{Q}_l)$. ∎

9.2.4. Corollary. *In the situation of Theorem 9.2.3, the following is true:*

(i) the integration gives rise to a surjective homomorphism $H^1_{\mathrm{dR}}(\mathcal{X}) \to H^1(\mathcal{X}^{\mathrm{an}},\mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) \otimes_k k^1_{\mathrm{Log}}$;

(ii) the intersection of $H^1(\mathcal{X}^{\mathrm{an}},\mathfrak{c}_{\mathcal{X}^{\mathrm{an}}})$ and $\Omega^{1,\mathrm{cl}}(\mathcal{X})/d\mathcal{O}(\mathcal{X})$ in $H^1_{\mathrm{dR}}(\mathcal{X})$ is zero;

(iii) the canonical homomorphisms $H^1(\mathcal{X}^{\mathrm{an}},\mathfrak{c}_{\mathcal{X}^{\mathrm{an}}}) \to H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ and $\Omega^{1,\mathrm{cl}}(\mathcal{X})/d\mathcal{O}(\mathcal{X}) \to H^0(\mathcal{X}^{\mathrm{an}},\Omega^{1,\mathrm{cl}}_{\mathcal{X}^{\mathrm{an}}}/d\mathcal{O}_{\mathcal{X}^{\mathrm{an}}})$ are injective.

Proof. The statement (i) follows from Theorem 9.2.3, (ii) follows from its proof, and (iii) follows from (ii). ∎

9.2.5. Remarks. The assumption on $\mathcal{X}$ in Theorem 9.2.1 is certainly superfluous since the isomorphism $H_1(\mathcal{X}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} H_1(\mathcal{A}^{\mathrm{an}}, \mathbf{Q})$ must always take place.

9.3. Locally unipotent and quasi-unipotent $\mathcal{D}_X$-modules. A $\mathcal{D}_X$-module $\mathcal{F}$ on a smooth k-analytic space X is said to be *unipotent* (resp. *quasi-unipotent*) *at a point* $x \in X$ if x has an open neighborhood $U \subset X$ (resp. an étale neighborhood $U \to X$) for which $\mathcal{F}\big|_U$ is unipotent. A $\mathcal{D}_X$-module $\mathcal{F}$ is said to be *locally unipotent* (resp. *locally quasi-unipotent*) if it is unipotent (resp. quasi-unipotent) at all points of X.

Furthermore, the *level* of a unipotent $\mathcal{D}_X$-module $\mathcal{F}$ on X is the minimal n for which there is a filtration of $\mathcal{D}_X$-submodules $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \ldots \subset \mathcal{F}^n = \mathcal{F}$ such that each quotient $\mathcal{F}^i/\mathcal{F}^{i-1}$ is a trivial $\mathcal{D}_X$-module. If a $\mathcal{D}_X$-module $\mathcal{F}$ is unipotent (resp. quasi-unipotent) at a point $x \in X$, its *level at* x is the minimal number n, which is the level of the unipotent $\mathcal{D}_U$-module $\mathcal{F}\big|_U$ for some U from the previous paragraph (see Remark 9.3.6(i)).

9.3.1. Lemma. *Assume that a $\mathcal{D}_X$-module $\mathcal{F}$ is unipotent at a point $x \in X$. Then its level at x is the same whether it is considered as unipotent or quasi-unipotent at x.*

Proof. Lemma 5.2.1 reduces the lemma to the verification of the following fact. Given a finite étale morphism of connected smooth strictly k-affinoid germs $\varphi : (X', Y') \to (X, Y)$ and a unipotent isocrystal M over $B = \mathcal{O}(X, Y)$, the level of M is equal to the level of the unipotent isocrystal $M' = M \otimes_B B'$ over $B' = \mathcal{O}(X', Y')$. To show this, we may assume that φ is Galois with Galois group G and $k = \mathfrak{c}(B)$, and set $k' = \mathfrak{c}(B')$.

The canonical homomorphisms $M^\nabla \otimes_k B \to M$ and ${M'}^\nabla \otimes_{k'} B' \to B'$ are injective and their images coincide with the maximal trivial sub-isocrystals of M and M', respectively. Since the B'-submodule ${M'}^\nabla \otimes_{k'} B'$ is invariant under the action of G, it coincides with $N \otimes_B B'$ for a B-submodule $N \subset M$ (see [SGA1, Exp. VIII, §1]). Replacing M by N, we may assume that M' is a trivial isocrystal, and our purpose is to show that M is also trivial. We prove the latter by induction on the rank n of M. Since M is unipotent, the required fact is evident for $n = 1$. Assume that $n \geq 2$ and the statement is true for $n - 1$. Consider an exact sequence of isocrystals $0 \to B \to M \to P \to 0$. Since M' is trivial, $P' = P \otimes_B B'$ is also trivial and, by induction, P is trivial and $\dim_k(P^\nabla) = n - 1$. The exact sequence $0 \to k' \to {M'}^\nabla \to {P'}^\nabla \to 0$ and the fact that $H^1(G, k') = 0$ imply that there is an exact sequence $0 \to k \to M^\nabla \to P^\nabla \to 0$. It follows that $\dim_k(M^\nabla) = n$, i.e., M is trivial. ∎

9.3.2. Corollary *(cf. Remark 9.5.4(i).) Any $\mathcal{D}_X$-module $\mathcal{F}$ unipotent at a point $x \in X_{st}$ is trivial over an open neighborhood of x.*

Proof. Corollaries 2.3.3 and 1.3.3 imply that the level of $\mathcal{F}$ at x is zero if we consider $\mathcal{F}$ as a $\mathcal{D}_X$-module quasi-unipotent at x and, therefore, the claim follows from Lemma 9.3.1. ∎

9.3.3. Theorem. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module, $x \in X$ and $n \geq 1$. Then the following are equivalent:*

(a) $\mathcal{F}$ is quasi-unipotent at x of level at most n;

(b) the point x has an étale neighborhood $U \to X$ such that, for some $m \geq 1$, there is an embedding of $\mathcal{D}_U$-modules $\mathcal{F}\big|_U \hookrightarrow (\mathcal{S}_U^{\lambda,n-1})^m$.

For a $\mathcal{D}_X$-module $\mathcal{F}$, let $\mathcal{F}_{\mathcal{S}^\lambda}$ denote the $\mathcal{D}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{S}_X^\lambda$. It is a $\mathcal{D}_{\mathcal{S}^\lambda}$-*module*, i.e., a sheaf of $\mathcal{D}$-modules over the $\mathcal{D}_X$-algebra $\mathcal{S}_X^\lambda$, and its sheaf of horizontal sections $\mathcal{F}_{\mathcal{S}^\lambda}^\nabla$ is a sheaf of modules over $\mathcal{C}_X^K$. Moreover, it is a filtered $\mathcal{D}_{\mathcal{S}^\lambda}$-module with respect to the filtration defined by the $\mathcal{D}_X$-submodules $\mathcal{F}_{\mathcal{S}^{\lambda,i}} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{S}^{\lambda,i}$, $i \geq 0$. Assume that $\mathcal{F}$ is unipotent of rank m over $\mathcal{O}_X$, and fix a filtration $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \ldots \subset \mathcal{F}^n = \mathcal{F}$ such that each quotient $\mathcal{F}^i/\mathcal{F}^{i-1}$ is a trivial $\mathcal{D}_X$-module. Then there is a different structure of a filtered $\mathcal{D}_{\mathcal{S}^\lambda}$-module on $\mathcal{F}_{\mathcal{S}^\lambda}$ defined by the $\mathcal{D}_X$-submodules

$$\widetilde{\mathcal{F}}_i = \mathcal{F}^n_{\mathcal{S}^{\lambda,i}} + \mathcal{F}^{n-1}_{\mathcal{S}^{\lambda,i+1}} + \ldots + \mathcal{F}^1_{\mathcal{S}^{\lambda,i+n-1}} \ .$$

9.3.4. Lemma. *In the above situation, every point of X has an open neighborhood U such that the following is true:*

(i) $(\widetilde{\mathcal{F}}_i \otimes_{\mathcal{O}_X} \Omega_X^1)^{\mathrm{cl}}(U) \subset \nabla(\widetilde{\mathcal{F}}_{i+1}(U))$;

(ii) each $\mathfrak{c}_U$-module $\widetilde{\mathcal{F}}_i^\nabla\big|_U$ is free;

(iii) the $\mathcal{C}_U^K$-module $\mathcal{F}_{\mathcal{S}^\lambda}^\nabla\big|_U$ is free with free generators $h_1, \ldots, h_m \in \widetilde{\mathcal{F}}_0^\nabla(U)$;

(iv) the elements $h_1, \ldots, h_m$ are free generators of the $\mathcal{S}_U^\lambda$-module $\mathcal{F}_{\mathcal{S}^\lambda}\big|_U$;

(v) the isomorphism of $\mathcal{D}_{\mathcal{S}^\lambda}$-modules $\mathcal{F}_{\mathcal{S}^\lambda}^\nabla\big|_U \otimes_{\mathcal{C}_U^K} \mathcal{S}_U^\lambda \xrightarrow{\sim} (\mathcal{S}_U^\lambda)^m$, defined by the elements $h_1, \ldots, h_m$, takes each $\widetilde{\mathcal{F}}_i\big|_U$ into $(\mathcal{S}_U^{\lambda,i+n-1})^m$.

Proof. Shrinking X, we may assume that X is connected, the $\mathcal{O}_X$-module Ω_X^1 is free, every surjection of $\mathcal{O}_X$-modules $\mathcal{F}^i \to \mathcal{F}^i/\mathcal{F}^{i-1}$ has a section, and $H^1(X, \mathfrak{c}_X) = 0$. *We claim that all of the statements are true for $U = X$.* If $n = 1$, the claim follows from Theorem 1.6.1. Assume that $n \geq 2$ and the claim is true for $n-1$. We set $\mathcal{G} = \mathcal{F}^{n-1}$ and notice that $\widetilde{\mathcal{F}}_i = \mathcal{F}_{\mathcal{S}^{\lambda,i}} + \widetilde{\mathcal{G}}_{i+1}$, where the second summand is considered for the induced filtration of $\mathcal{G}$ of length $n-1$.

First of all, given $\omega \in (\widetilde{\mathcal{F}}_i \otimes_{\mathcal{O}_X} \Omega_X^1)^{\mathrm{cl}}(X)$, the above assumptions imply that there exists an element $f \in \mathcal{F}_{\mathcal{S}^{\lambda,i+1}}(X)$ with $\omega - \nabla(f) \in (\widetilde{\mathcal{G}}_{i+1} \otimes_{\mathcal{O}_X} \Omega_X^1)^{\mathrm{cl}}(X)$. By the induction hypotheses, there exists an element $g \in \widetilde{\mathcal{G}}_{i+2}(X)$ with $\omega - \nabla(f) = \nabla(g)$, and so $\omega = \nabla(f+g)$ with $f+g \in \mathcal{F}_{\mathcal{S}^{\lambda},i+1}(X) + \widetilde{\mathcal{G}}_{i+2}(X) \subset \widetilde{\mathcal{F}}_{i+1}(X)$, and (i) follows.

Furthermore, consider the exact sequence of $\mathcal{D}_X$-modules $0 \to \widetilde{\mathcal{G}}_{i+1} \to \widetilde{\mathcal{F}}_i \to (\mathcal{F}/\mathcal{G})_{\mathcal{S}^{\lambda},i} \to 0$. It gives rise to an exact sequence $0 \to \widetilde{\mathcal{G}}_{i+1}^{\nabla} \to \widetilde{\mathcal{F}}_i^{\nabla} \to (\mathcal{F}/\mathcal{G})_{\mathcal{S}^{\lambda},i}^{\nabla} = (\mathcal{F}/\mathcal{G})^{\nabla} \otimes_k K^i$. To prove (ii) , it suffices to show that the second homomorphism is surjective and has a section. For this we take elements $f_1, \ldots, f_l \in \mathcal{F}(X)$ whose images in $\mathcal{F}(X)/\mathcal{G}(X)$ generate $(\mathcal{F}/\mathcal{G})^{\nabla}$ over $\mathfrak{c}(X)$. By the induction hypotheses, there exist elements $g_1, \ldots, g_l \in \widetilde{\mathcal{G}}_1(X)$ with $\nabla(g_j) = \nabla(f_j)$ for all $1 \le j \le l$. We get elements $f_1 - g_1, \ldots, f_l - g_l \in \widetilde{\mathcal{F}}_0^{\nabla}(X)$ which are linearly independent over $\mathfrak{c}(X)$ modulo $\widetilde{\mathcal{G}}_1^{\nabla}(X)$ and generate $\widetilde{\mathcal{F}}_0^{\nabla}$ modulo $\widetilde{\mathcal{G}}_1^{\nabla}$. If V denotes the subspace of $\widetilde{\mathcal{F}}_0^{\nabla}(X)$ generated by $f_1 - g_1, \ldots, f_l - g_l$ over $\mathfrak{c}(X)$, then the subspace $V \otimes_k K^i$ of $\widetilde{\mathcal{F}}_i^{\nabla}(X)$ maps isomorphically onto $(\mathcal{F}/\mathcal{G})^{\nabla}(X) \otimes_k K^i$, and the required fact follows.

If now $h_1, \ldots, h_{m-l}$ are elements of $\widetilde{\mathcal{G}}_0^{\nabla}(X)$ with the properties (iii)-(v) for $\mathcal{G}$, then the elements $h_1, \ldots, h_{m-l}, h_{m-l+1} = f_1 - g_1, \ldots, h_m = f_l - g_l$ possess the same properties for $\mathcal{F}$. ∎

Proof of Theorem 9.3.3. The implication (a)$\Longrightarrow$(b) follows from Lemma 9.3.4.

(b)$\Longrightarrow$(a) It suffices to show that every point of X has an étale neighborhood such that the restriction of $\mathcal{F}$ to it has an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-submodule $\mathcal{G}$ such that its image in $(\mathcal{S}_X^{\lambda,n-1})^m$ is contained in $(\mathcal{S}_X^{\lambda,n-2})^m$ and the quotient $\mathcal{F}/\mathcal{G}$ is a trivial $\mathcal{D}_X$-module. Shrinking X, we may assume that $\mathcal{F}$ is free over $\mathcal{O}_X$. Consider the induced morphism of $\mathcal{D}_X$-modules $\mathcal{F} \to \mathcal{H} = (\mathcal{S}_X^{\lambda,n-1}/\mathcal{S}_X^{\lambda,n-2})^m$. By Corollary 8.4.6(ii), there is an isomorphism of $\mathcal{D}_X$-modules $\mathcal{H}^{\nabla} \otimes_{\mathfrak{c}_X} \mathcal{O}_X \stackrel{\sim}{\to} \mathcal{H}$. We can therefore shrink X in the étale topology so that the images of generators of $\mathcal{F}$ over $\mathcal{O}_X$ are contained in $\sum_{j=1}^{l} h_j \mathcal{O}(X)$ for some $h_j \in \mathcal{H}^{\nabla}(X)$. Let Y be an affinoid neighborhood of a point $x \in X$, and let M be the submodule of $\mathcal{H}(X, Y)$ generated by the elements h_j over $B = \mathcal{O}(X, Y)$. Then M is a trivial isocrystal over B, and so the image of the finite isocrystal $\mathcal{F}(X, Y)$ in M is also trivial. This easily implies the required fact. ∎

9.3.5. Corollary. *Let $\mathcal{F}$ be a locally quasi-unipotent $\mathcal{D}_X$-module. Then*

(i) the étale $\mathcal{C}_X^K$-module $\mathcal{F}_{\mathcal{S}^{\lambda}}^{\nabla}$ is locally free;

(ii) there is an isomorphism of $\mathcal{D}_{\mathcal{S}^{\lambda}}$-modules $\mathcal{F}_{\mathcal{S}^{\lambda}}^{\nabla} \otimes_{\mathcal{C}_X^K} \mathcal{S}_X^{\lambda} \stackrel{\sim}{\to} \mathcal{F}_{\mathcal{S}^{\lambda}}$. ∎

9.3.6. Remark. (i) Let X be a Tate elliptic curve, and ω a nonzero invariant one-form on X. Then the $\mathcal{D}_X$-module $\mathcal{F} = \mathcal{O}_X e$, defined by $\nabla(e) = \omega e$, is not unipotent, but is locally unipotent of level one at all points of X.

(ii) Lemma 9.3.4 implies that, if a $\mathcal{D}_X$-module $\mathcal{F}$ is unipotent at a point $x \in X$ of level at most n, then x has an open neighborhood U for which there is an embedding of $\mathcal{D}_U$-modules

$\mathcal{F} \hookrightarrow (\mathcal{S}_U^{\lambda,n-1})^m$ with $m \geq 1$. It would be interesting to know if the converse implication is also true.

9.4. Parallel transport along a path. Let k^{a} be an algebraic closure of k, and X a smooth k-analytic space. For an étale $\mathcal{O}_X$-module $\mathcal{F}$, we denote by $\overline{\mathcal{F}}$ the pullback of $\mathcal{F}$ on $\overline{X} = X\widehat{\otimes}\widehat{k}^{\mathrm{a}}$ and, for a point $x \in \overline{X}$, we denote by $\mathcal{F}_x$ the inductive limit of the stalks $\mathcal{F}_{x'}$ taken over finite extensions k' of k in k^{a}, where x' is the image of x in $X\widehat{\otimes}k'$. Notice that $\overline{\mathcal{F}}_x = \mathcal{F}_x \otimes_{k^{\mathrm{a}}} \widehat{k}^{\mathrm{a}}$. Recall that, by Corollary 9.3.2, any locally unipotent $\mathcal{D}_X$-module $\mathcal{F}$ is a trivial $\mathcal{D}_X$-module in an open neighborhood of every point from X_{st} and, in particular, the stalk $\mathcal{F}_x^{\nabla}$ at a point $x \in \overline{X}_{st}$ is a k^{a}-vector space whose dimension is equal to the rank of $\mathcal{F}$ at x. Let $X(k^{\mathrm{a}})$ be the inductive limit of the sets $(X\widehat{\otimes}k')(k')$ taken over finite extensions k' of k in k^{a}. Notice that it is identified with the projective limit of the sets $(X\widehat{\otimes}k')_0$ and, in particular, with a dense subset of $\overline{X}$. One has $X(k^{\mathrm{a}}) \subset \overline{X}(\widehat{k}^{\mathrm{a}}) \subset \overline{X}_{st}$.

9.4.1. Theorem. *Given a closed subfield $k \subset \mathbf{C}_p$, a filtered k-algebra K and an element $\lambda \in K^1$, there is a unique way to construct, for every smooth k-analytic space X, every locally unipotent $\mathcal{D}_X$-module $\mathcal{F}$, every path $\gamma : [0,1] \to \overline{X}$ with ends $x, y \in \overline{X}_{st}$, an isomorphism of $k^{\mathrm{a}} \otimes_k K$-modules (the* parallel transport*)*

$$\mathrm{T}_\gamma^{\mathcal{F}} = \mathrm{T}_\gamma^{\mathcal{F},\lambda} : \mathcal{F}_x^{\nabla} \otimes_k K \xrightarrow{\sim} \mathcal{F}_y^{\nabla} \otimes_k K$$

such that the following is true:

(a) $\mathrm{T}_\gamma^{\mathcal{F}}$ depends only on the homotopy type of γ;

(b) given a second path $\tau : [0,1] \to \overline{X}$ with ends $y, z \in \overline{X}_{st}$, one has $\mathrm{T}_{\tau\circ\gamma}^{\mathcal{F}} = \mathrm{T}_\tau^{\mathcal{F}} \circ \mathrm{T}_\gamma^{\mathcal{F}}$;

(c) $\mathrm{T}_\gamma^{\mathcal{F}}$ is functorial with respect to $\mathcal{F}$;

(d) $\mathrm{T}_\gamma^{\mathcal{F}}$ commutes with tensor products;

(e) $\mathrm{T}_\gamma^{\mathcal{F}}$ is functorial with respect to X;

(f) if $\mathcal{F}$ is the unipotent $\mathcal{D}_X$-module $\mathcal{O}_X e_1 \oplus \mathcal{O}_X e_2$ on $X = \mathbf{G}_{\mathrm{m}}$ with $\nabla(e_1) = 0$ and $\nabla(e_2) = \frac{dT}{T} e_1$, $\gamma(0) = 1$ and $\gamma(1) = a \in k^*$, then $\mathrm{T}_\gamma^{\mathcal{F}}(e_2 - \log(T)e_1) = (e_2 - \log(\frac{T}{a})e_1) - \mathrm{Log}^\lambda(a)e_1$.

Furthermore, the parallel transport possesses the following properties:

(1) $\mathrm{T}_\gamma^{\mathcal{F}}$ commutes with the Hom-functor;

(2) if $\mathcal{F}$ is unipotent of level n, then $\mathrm{T}_\gamma^{\mathcal{F}}(\mathcal{F}_x^{\nabla}) \subset \mathcal{F}_y^{\nabla} \otimes_k K^{n-1}$;

(3) if $\mathcal{F}$ is unipotent and $\gamma([0,1]) \subset Y$, where Y an analytic subdomain of $\overline{X}$ with good reduction, then $\mathrm{T}_\gamma^{\mathcal{F}}(\mathcal{F}_x^{\nabla}) \subset \mathcal{F}_y^{\nabla}$;

(4) $\mathrm{T}_\gamma^{\mathcal{F}}$ is functorial with respect to $(k, X, \gamma, K, \lambda)$;

(5) the system of parallel transports $\mathrm{T}^{\mathcal{F}}_\gamma$ is uniquely determined by the properties (a)-(f) restricted to the paths with ends in $X(k^{\mathrm{a}})$.

9.4.2. Remarks. (i) In the situation of (f), the stalk $\mathcal{F}^\nabla_a$ at a point $a \in k^*$ is a two dimensional vector space generated by the elements e_1 and $e_2 - \log(\frac{T}{a})e_1$. The restriction of $e_2 - \mathrm{Log}^\lambda(T)e_1$ to an open neighborhood of the point a is an element of $\mathcal{F}^\nabla_a \otimes_k K^1$ (and in fact the K-module $\mathcal{F}^\nabla_a \otimes_k K$ is generated by e_1 and that element), and the property (f) simply means that the parallel transport $\mathrm{T}^{\mathcal{F}}_\gamma$ takes $e_2 - \mathrm{Log}^\lambda(T)e_1$ to itself.

(ii) The complete formulation of (4) (and of its particular case (e)) is as follows. Given a similar tuple $(k', X', \gamma', K', \lambda')$, compatible isometric embeddings $k \hookrightarrow k'$ and $k^{\mathrm{a}} \hookrightarrow k'^{\mathrm{a}}$, a morphism $\varphi : X' \to X$ and a homomorphism of filtered algebras $K \to K' : \lambda \mapsto \lambda'$ over the embedding $k \hookrightarrow k'$, the following diagram (in which $\mathcal{F}' = \varphi^*\mathcal{F}$ and x', y' are the ends of γ') is commutative

$$\begin{array}{ccc} \mathcal{F}^\nabla_{\varphi(x')} \otimes_k K & \xrightarrow{\mathrm{T}^{\mathcal{F},\lambda}_{\varphi\circ\gamma'}} & \mathcal{F}^\nabla_{\varphi(y')} \otimes_k K \\ \downarrow & & \downarrow \\ \mathcal{F}'^\nabla_{x'} \otimes_{k'} K' & \xrightarrow{\mathrm{T}^{\mathcal{F}',\lambda'}_{\gamma'}} & \mathcal{F}'^\nabla_{y'} \otimes_{k'} K' \end{array}$$

Proof. *Construction.* The inverse image of the étale sheaf $\mathfrak{c}_X$ with respect to the morphism $\overline{X} \to X$ is the constant sheaf $k^{\mathrm{a}}_{\overline{X}}$. From Lemma 9.3.4 it follows that the inverse image of the étale sheaf $\mathcal{F}^\nabla_{\mathcal{S}^\lambda}$ is an étale sheaf of k^{a}-vector spaces whose restriction to the usual topology of $\overline{X}$ is locally constant. It follows that the inverse image of the latter with respect to any path γ is a constant sheaf on $[0,1]$. Thus, if the ends x, y of γ lie in $\overline{X}_{st}$, we get an isomorphism of $k^{\mathrm{a}} \otimes_k K$-modules $\mathrm{T}^{\mathcal{F}}_\gamma : \mathcal{F}^\nabla_x \otimes_k K \xrightarrow{\sim} \mathcal{F}^\nabla_y \otimes_k K$. The validity of the properties (a)-(f),(1) and (4) is easily verified. To verify (2), we again use Lemma 9.3.4. Fixing a filtration $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \ldots \subset \mathcal{F}^n = \mathcal{F}$ such that each quotient $\mathcal{F}^i/\mathcal{F}^{i-1}$ is a trivial $\mathcal{D}_X$-module, it follows that the sheaf $\widetilde{\mathcal{F}}^\nabla_0$ of horizontal sections of $\widetilde{\mathcal{F}}_0 = \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{S}^{\lambda,0}_X + \ldots + \mathcal{F}^1 \otimes_{\mathcal{O}_X} \mathcal{S}^{\lambda,n-1}_X$ is an étale sheaf of $\mathfrak{c}_X$-vector spaces whose restriction to the usual topology of X is locally free, and its stalk at every point $x \in X_{st}$ contains the subspace $\mathcal{F}^\nabla_x$. The above reasoning implies the property (2). The property (3) is an easy consequence of Proposition 8.1.3. The statement on the uniqueness in (5) follows from the fact that the restriction of $\mathcal{F}$ to an open neighborhood $\mathcal{U}$ of every point $x \in X_{st}$ is a trivial $\mathcal{D}_{\mathcal{U}}$-module, the property (e) applied to the canonical morphism $\mathcal{U} \to \mathcal{M}(k)$, and the property (b).

Uniqueness. Let $\widetilde{\mathrm{T}}^{\mathcal{F}}_\gamma$ be a system of isomorphisms possessing the properties (a)-(f). We have to show that it coincides with $\mathrm{T}^{\mathcal{F}}_\gamma$.

First of all, by Theorem 4.1.1 and the property (e), it suffices to prove the uniqueness on the class of smooth basic curves. Furthermore, the last reasoning from the construction part shows that both parallel transports extend uniquely to all paths with ends in $\overline{X}(\widehat{k}^{\mathrm{a}})$. Thus, applying Corollary 5.5.5, we see that it suffices to prove that both parallel transports coincide on the class of smooth basic curves under the assumptions that k is algebraically closed and the weaker form of the property (e), in which morphisms considered are defined over a finite extension of $\mathbf{Q}_p$, is true.

Let X be a smooth basic curve. Since X is simply connected, $\mathcal{F}^{\nabla}_{\mathcal{S}^\lambda}(X)$ is a free K-module of rank equal to the rank m of $\mathcal{F}$ over $\mathcal{O}_X$, and $\mathcal{F}^{\nabla}_{\mathcal{S}^\lambda}$ is the constant sheaf associated to it. Furthermore, the isomorphisms $\mathrm{T}^{\mathcal{F}}_\gamma$ and $\widetilde{\mathrm{T}}^{\mathcal{F}}_\gamma$ depend only on the ends $x=\gamma(0)$ and $y=\gamma(1)$ of γ, and so they will be denoted by $\mathrm{T}^{\mathcal{F}}_{x,y}$ and $\widetilde{\mathrm{T}}^{\mathcal{F}}_{x,y}$ respectively. It suffices to show that each isomorphism $\widetilde{\mathrm{T}}^{\mathcal{F}}_{x,y}$ is induced by the canonical isomorphisms of $\mathcal{F}^{\nabla}_{\mathcal{S}^\lambda}(X)$ with $\mathcal{F}^{\nabla}_{\mathcal{S}^\lambda,x}=\mathcal{F}^{\nabla}_x\otimes_k K$ and $\mathcal{F}^{\nabla}_{\mathcal{S}^\lambda,y}=\mathcal{F}^{\nabla}_y\otimes_k K$.

If the $\mathcal{D}_X$-module $\mathcal{F}$ is trivial, the required fact follows from the property (e) applied to the canonical morphism $X\to\mathcal{M}(k)$ and, in particular, it is true if X is an open disc. Assume now that X is an open annulus in $\mathbf{A}^1$ with center at zero. In this case the category of unipotent $\mathcal{D}_X$-modules is described as follows (see [Crew, Proposition 6.7]). Given a nilpotent linear operator N on a finite dimensional k-vector space V, let $\mathcal{F}_V$ denote the $\mathcal{O}_X$-module $V\otimes_k\mathcal{O}_X$ provided with the connection $\nabla_N=N\otimes\frac{dT}{T}$. Then the correspondence $(V,N)\mapsto(\mathcal{F}_V,\nabla_N)$ gives rise to an equivalence of the category of such pairs (V,N) with that of unipotent $\mathcal{D}_X$-modules. Notice that $(\mathcal{F}_{V_1},\nabla_{N_1})\otimes_{\mathcal{O}_X}(\mathcal{F}_{V_2},\nabla_{N_2})=(\mathcal{F}_{V_1\otimes V_2},\nabla_N)$, where $N=N_1\otimes 1+1\otimes N_2$. Let (V,N) be a pair that corresponds to $\mathcal{F}$. By the description of nilpotent matrices and the property (c), we may assume that there is a basis $v_1,\dots,v_m$ of $V_m=V$ such that $N(e_i)=e_{i-1}$ for all $1\le i\le m$, where $v_0=0$. If $m=2$, the required fact follows from the property (f) (as explained in Remark 9.4.2(i)). If $m\ge 3$, we use the surjective homomorphism $V_{m-1}\otimes V_2\to V_m$ that takes the element $v_i\otimes v_1$ to v_{i-1} and the element $v_i\otimes v_2$ to iv_{i+1}, where $1\le i\le m-1$. This homomorphism gives rise to a surjective homomorphism of unipotent $\mathcal{D}_X$-modules $\mathcal{F}_{V_{m-1}}\otimes_{\mathcal{O}_X}\mathcal{F}_{V_2}\to\mathcal{F}_{V_m}$, and the properties (c) and (d) imply the required fact. We notice the following consequence of the above consideration. For a unipotent $\mathcal{D}_X$-module $\mathcal{F}$, let $F_{L^{\lambda,n}}$ denote the D_X-module $\mathcal{F}(X)\otimes_{\mathcal{O}(X)}L^{\lambda,n}(X)$. If $\mathcal{F}$ is of level n, then there is an isomorphism of D_X-modules $F^{\nabla}_{L^{\lambda,n-1}}\otimes_k\mathcal{O}(X)\xrightarrow{\sim}F_{L^{\lambda,n-1}}$. In particular, $\mathcal{F}^{\nabla}_{\mathcal{S}^{\lambda,n-1}}(X)=F^{\nabla}_{L^{\lambda,n-1}}$ and $(\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^1_X)(X)\subset\nabla(F_{L^{\lambda,n-1}})$.

Thus, it remains to consider the case when X is an arbitrary smooth basic curve, i.e., it is isomorphic to the generic fiber $\mathfrak{X}_\eta$ of a proper marked formal scheme $\mathfrak{X}$ over k°. By the previous two cases and the property (e), it suffices to verify the required fact for an étale neighborhood of the generic point of $\mathfrak{X}$. By the property (c) and Theorem 9.3.3, we may assume that for an open affine subscheme $\mathfrak{Z}\subset\mathring{\mathfrak{X}}$, the isocrystal associated to $\mathcal{F}\big|_{(X,\mathfrak{Z}_\eta)}$ is $E^n(X,\mathfrak{Z})$ for some $n\ge 0$ and even that $E^n(X,\mathfrak{Z})=\mathcal{F}(X)\otimes_{\mathcal{O}(X)}\mathcal{O}(X,\mathfrak{Z})$.

The required fact is true for $n = 0$, and so assume that $n \geq 1$ and that the fact is true for $n-1$. Let $\mathcal{G}$ be the unipotent $\mathcal{D}_X$-submodule of $\mathcal{F}$ with $\mathcal{G}\big|_{(X,\mathfrak{Z}_\eta)} = E^{n-1}(X,\mathfrak{Z})$, and let $\mathcal{H}$ denote the trivial $\mathcal{D}_X$-module $\mathcal{F}/\mathcal{G}$. Given an element $e \in \mathcal{H}^\nabla(X)$, let f be its preimage in $\mathcal{F}(X)$. Then $\nabla(f) = \omega \in (\mathcal{G} \otimes_{\mathcal{O}_X} \Omega^1_X)(X)$. For a point $x \in X(k)$, the space $\mathcal{F}^\nabla_x$ is generated by $\mathcal{G}^\nabla_x$ and elements of the form $f - g_x$, where f is as above and g_x is a primitive of $\omega = \nabla(f)$ in $\mathcal{G}_x$. Given a point $x_0 \in X(k)$ and a primitive g_{x_0} of ω in $\mathcal{G}_{x_0} \otimes_k K$, consider the function $x \mapsto g_x$ that takes a point $x \in X(k)$ to the primitive g_x of ω in $\mathcal{G}_x \otimes_k K$ such that $\widetilde{\mathrm{T}}^{\mathcal{F}}_{x_0,x}(f - g_{x_0}) = f - g_x$. For a closed point $\mathbf{x} \in \mathfrak{X}_s$, the preimage $\pi^{-1}(\mathbf{x})$ is either an open disc or annulus. It follows that there exists an element $g_{\mathbf{x}} \in \mathcal{G}(\pi^{-1}(\mathbf{x})) \otimes_{\mathcal{O}(\pi^{-1}(\mathbf{x}))} L^\lambda(\pi^{-1}(\mathbf{x}))$ such that, for every point $x \in \pi^{-1}(\mathbf{x})$, the image of $g_{\mathbf{x}}$ in $\mathcal{G}_x \otimes_k K$ coincides with g_x. Let $R^\lambda(\mathfrak{X})$ denote the space of systems of functions $\alpha_{\mathbf{x}} \in L^\lambda(\pi^{-1}(\mathbf{x}))$ taken over all closed points $\mathbf{x} \in \mathfrak{X}_s$. (Notice that all $R^{\lambda,i}(\mathfrak{X})$, defined at the end of §3.1, are contained in $R^\lambda(\mathfrak{X})$.) Then the above system of elements $g_{\mathbf{x}}$ can be considered as an element g of $\mathcal{G}(X) \otimes_{\mathcal{O}(X)} R^\lambda(\mathfrak{X})$ which is a primitive of ω.

We claim that g is defined by the element $e \in \mathcal{H}^\nabla(X)$ uniquely up to an element of $\mathcal{G}^\nabla_{\mathcal{S}^\lambda}(X)$. Indeed, if g'_{x_0} is another primitive of ω in $\mathcal{G}_{x_0} \otimes_k K$ that gives rise to a primitive g' of ω in $\mathcal{G}(X) \otimes_{\mathcal{O}(X)} R^\lambda(\mathfrak{X})$, then $g'_{x_0} - g_{x_0} \in \mathcal{G}^\nabla_{x_0} \otimes_k K$. By the induction hypothesis, one has $\widetilde{\mathrm{T}}^{\mathcal{F}}_{x_0,x}(g'_{x_0} - g_{x_0}) = \mathrm{T}^{\mathcal{F}}_{x_0,x}(g'_{x_0} - g_{x_0})$, and so the function $x \mapsto \mathrm{T}^{\mathcal{F}}_{x_0,x}(g'_{x_0} - g_{x_0})$ is an element of $\mathcal{G}^\nabla_{\mathcal{S}^\lambda}(X)$. Furthermore, if x_1 is another point of $X(k)$, then $\widetilde{\mathrm{T}}^{\mathcal{F}}_{x_0,x}(f - g_{x_0}) = \widetilde{\mathrm{T}}^{\mathcal{F}}_{x_1,x}(f - g_{x_1})$. Finally, if f' is another preimage of e in $\mathcal{F}(X)$, then $f' = f + h$ for some $h \in \mathcal{G}(X)$, $g'_{x_0} = g_{x_0} + h$ is a primitive of $\nabla(f') = \omega + \nabla(h)$ in $\mathcal{G}_{x_0} \otimes_k K$ and, therefore, $\widetilde{\mathrm{T}}^{\mathcal{F}}_{x_0,x}(f' - g'_{x_0}) = \widetilde{\mathrm{T}}^{\mathcal{F}}_{x_0,x}(f - g_{x_0})$.

Let now h be a primitive of ω in $\mathcal{G}_{\mathcal{S}^\lambda}(X)$. Notice that h is also contained in $\mathcal{G}(X) \otimes_{\mathcal{O}(X)} R^\lambda(\mathfrak{X})$ as the element g, and $g_{\mathbf{x}} - h_{\mathbf{x}} \in K$ for every closed point $\mathbf{x} \in \mathfrak{X}_s$. Thus, to prove the required fact it suffices to show that $g - h \in \mathcal{G}^\nabla_{\mathcal{S}^\lambda}(X)$, and to verify the latter, it suffices to show that $(g-h)_\sigma \in \mathcal{G}_\sigma \otimes_{\mathcal{O}_{X,\sigma}} \mathcal{S}^\lambda_{X,\sigma}$, where σ is the generic point of $\mathfrak{X}$. For this we consider a Frobenius lifting ϕ on $(X, \mathfrak{Z}_\eta)$ and a compatible system of Frobenius structures on the isocrystals $E^i(X,\mathfrak{Z})$, $i \geq 0$. Replacing ϕ by its power, we may assume that there exists a closed point $\mathbf{x}_0 \in \mathfrak{Z}_s$ stable under ϕ and, therefore, there exists a k-rational point $x_0 \in \pi^{-1}(\mathbf{x}_0)$ also stable under ϕ. By Lemma 6.2.1, we can find a monic polynomial $P(T) \in k[T]$ with no roots-of-unity roots such that $P(\phi^*)\omega \in \nabla(\mathcal{G}(\mathcal{U}))$ for some open neighborhood $\mathcal{U}$ of $\mathfrak{Z}_\eta$ in $\mathfrak{X}_\eta$. We may assume that $\mathcal{U}$ is connected and all of the morphisms ϕ^i with $0 \leq i \leq \deg(P)$ are defined at it. The above uniqueness claim implies that $P(\phi^*)g \in \mathcal{G}_{\mathcal{S}^\lambda}(\mathcal{U})$. The equality here is considered in the tensor product $\mathcal{G}(\mathcal{U}) \otimes_{\mathcal{O}(X)} R^\lambda(\mathfrak{X},\mathcal{U})$, where $R^\lambda(\mathfrak{X},\mathcal{U})$ is the space of all systems of functions $\alpha_{\mathbf{x}} \in L(\mathcal{U} \cap \pi^{-1}(\mathbf{x}))$ taken over all closed points $\mathbf{x} \in \mathfrak{X}_s$ (cf. §6.5). On the other hand, one also has $P(\phi^*)h \in \mathcal{G}_{\mathcal{S}^\lambda}(\mathcal{U})$. It follows that

$P(\phi^*)(g-h) \in \mathcal{G}_{\mathcal{S}^\lambda}(\mathcal{U})$, and Lemma 6.3.2 implies the required fact. ∎

Assume that X is connected, and let $\gamma : [0,1] \to X$ be a path with ends $x, y \in X$ such that $\mathfrak{c}(X) \stackrel{\sim}{\to} \mathfrak{c}_{X,x}$ and $\mathfrak{c}(X) \stackrel{\sim}{\to} \mathfrak{c}_{X,x}$. Then the points x and y have unique preimages x' and y' in each connected component $\overline{X}'$ of $\overline{X}$. By Lemma 9.1.2, the path γ lifts to a path $\gamma' : [0,1] \to \overline{X}'$ with the ends x' and y'. Thus, if $\pi_1(\overline{X}, x') \stackrel{\sim}{\to} \pi_1(X, x)$ and $x, y \in X_{st}$, then the parallel transport $\mathrm{T}^{\mathcal{F}}_{\gamma'} : \mathcal{F}_{x'} \otimes_k K \stackrel{\sim}{\to} \mathcal{F}_{y'} \otimes_k K$ does not depend on the lifting γ' of γ and, by functoriality, it gives rise to an isomorphism of K-modules $\mathrm{T}^{\mathcal{F}}_{\gamma} : \mathcal{F}_x \otimes_k K \stackrel{\sim}{\to} \mathcal{F}_y \otimes_k K$.

9.4.3. Corollary. *Given a closed subfield $k \subset \mathbf{C}_p$, a filtered k-algebra K and an element $\lambda \in K^1$, there is a unique way to construct, for every connected smooth k-analytic space X with $\pi_1(\overline{X}, z') \stackrel{\sim}{\to} \pi_1(X, z)$, every locally unipotent $\mathcal{D}_X$-module $\mathcal{F}$, every path $\gamma : [0,1] \to X$ with ends $x, y \in X_{st}$ such that $\mathfrak{c}(X) \stackrel{\sim}{\to} \mathfrak{c}_{X,x}$ and $\mathfrak{c}(X) \stackrel{\sim}{\to} \mathfrak{c}_{X,x}$, an isomorphism of K-modules*

$$\mathrm{T}^{\mathcal{F}}_{\gamma} = \mathrm{T}^{\mathcal{F},\lambda}_{\gamma} : \mathcal{F}^{\nabla}_x \otimes_k K \stackrel{\sim}{\to} \mathcal{F}^{\nabla}_y \otimes_k K$$

for which the properties (a)-(f) hold. Moreover, the properties (1)-(5) (appropriately modified) also hold. ∎

Recall that the *fundamental groupoid* of a topological space X is a category $\Pi_1(X)$ whose objects are points of X and sets of morphisms $\Pi_X(x, y)$ are the sets of homotopy classes of paths from x to y. For a smooth k-analytic space X, let $\Pi_1(X)_{st}$ denote the full subcategory of $\Pi_1(X)$ whose objects are points from X_{st}, and let $\Pi_1(X)_{st,e}$ denote the full subcategory of $\Pi_1(X)_{st}$ whose objects are the points x with $\mathfrak{c}(X') \stackrel{\sim}{\to} \mathfrak{c}_{X,x}$, where X' is the connected component of X that contains x. Let also L-$\mathcal{M}od$ denote the category of L-modules of a commutative algebra L. The properties (a) and (b) of the parallel transport mean that the correspondences and $x \mapsto \mathcal{F}^{\nabla}_x \otimes_k K$ and $\gamma \mapsto \mathrm{T}^{\mathcal{F}}_{\gamma}$ give rise to a functor $\mathrm{T}^{\mathcal{F}} = \mathrm{T}^{\mathcal{F},\lambda} : \Pi_1(\overline{X})_{st} \to (k^{\mathrm{a}} \otimes_k K)$-$\mathcal{M}od$ (resp. $\Pi_1(X)_{st,e} \to K$-$\mathcal{M}od$), which is uniquely defined by them and the properties (d)-(f). In particular, for every point $x \in \overline{X}_{st}$ (resp. $X_{st,e}$), there is an associated representation of the fundamental group $\pi_1(\overline{X}, x)$ (resp. $\pi_1(X, x)$) in $\mathcal{F}^{\nabla}_x \otimes_k K$. In order to extend the construction to the class of locally quasi-unipotent $\mathcal{D}_X$-modules, we recall a different interpretation of the fundamental groupoid, which is possible due to the fact that smooth k-analytic spaces are locally simply connected.

Let $\mathrm{Cov}(X)$ denote the category of topological covering spaces over X, i.e., continuous maps $\varphi : Y \to X$ with the property that each point of X has an open neighborhood $\mathcal{U} \subset X$ for which $\varphi^{-1}(\mathcal{U})$ is a disjoint union of spaces such that each of them maps homeomorphically onto $\mathcal{U}$. (Notice that such Y always carries a canonical structure of a smooth k-analytic space.) Every point $x \in X$

defines a functor $F_x : \mathrm{Cov}(X) \to \mathcal{E}ns$ to the category of sets that takes $\varphi : Y \to X$ to $\varphi^{-1}(x)$. Then there is a canonical bijection between the set $\Pi_X(x,y)$ of homotopy classes of paths from x to y and the set of isomorphisms of functors $F_x \xrightarrow{\sim} F_y$ and, in particular, the fundamental group $\pi_1(X,x)$ is canonically isomorphic to the automorphism group of the functor F_x.

9.4.4. Remark. In [Vol], V. Vologodsky constructed a parallel transport $\mathrm{T}^{\mathcal{F}}_{x,y} : \mathcal{F}_x \otimes_k k_{\mathrm{Log}} \xrightarrow{\sim} \mathcal{F}_y \otimes_k k_{\mathrm{Log}}$ on a smooth geometrically connected scheme $\mathcal{X}$ over k with discrete valuation, where $\mathcal{F}$ is an unipotent $\mathcal{D}_{\mathcal{X}}$-module and $x, y \in \mathcal{X}(k)$. The relation between it and the parallel transport on $\mathcal{X}^{\mathrm{an}}$ from Corollary 9.4.3 is similar to that between integrals mentioned in Remark 9.1.3(ii). In particular, they do coincide if $\mathcal{X}$ is proper and has good reduction. (Notice that $\mathcal{X}^{\mathrm{an}}$ and $\overline{\mathcal{X}}^{\mathrm{an}}$ are simply connected for such $\mathcal{X}$.)

9.5. Parallel transport along an étale path. Let $\mathrm{Cov}^{\text{ét}}(X)$ denote the category of étale covering spaces over X, i.e., étale morphisms $\varphi : Y \to X$ with the property that each point of X has an open neighborhood $\mathcal{U} \subset X$ for which $\varphi^{-1}(\mathcal{U})$ is a disjoint union of spaces such that the induced morphism from each of them to $\mathcal{U}$ is finite étale (such φ is called an étale covering map). Every geometric point $\overline{x} : \mathbf{p}_{\mathcal{H}(\overline{x})} \to X$ defines a functor $F_{\overline{x}} : \mathrm{Cov}^{\text{ét}}(X) \to \mathcal{E}ns$ that takes $\varphi : Y \to X$ to the set of all morphisms $\mathbf{p}_{\mathcal{H}(\overline{x})} \to Y$ over $\overline{x}$ (i.e., it takes $\varphi : Y \to X$ to the stalk at $\overline{x}$ of the sheaf representable by Y). Given two geometric points $\overline{x}$ and $\overline{y}$ of X, the *homotopy class of an étale path from $\overline{x}$ to $\overline{y}$* is an isomorphism of functors $\overline{\gamma} : F_{\overline{x}} \xrightarrow{\sim} F_{\overline{y}}$. For brevity, we call it an *étale path from $\overline{x}$ to $\overline{y}$* and denote $\overline{\gamma} : \overline{x} \mapsto \overline{y}$. By de Jong's results from [deJ1], if X is connected, an étale path from $\overline{x}$ to $\overline{y}$ exists for every pair of geometric points $\overline{x}$ and $\overline{y}$.

The *étale fundamental groupoid of* X is the category $\Pi_1^{\text{ét}}(X)$ whose objects are geometric points $\overline{x}$ of X and sets of morphisms $\Pi_X^{\text{ét}}(\overline{x},\overline{y})$ are the sets of homotopy classes of étale paths $\overline{\gamma} : \overline{x} \mapsto \overline{y}$. For example, the automorphism group of a geometric point $\overline{x}$ in $\Pi_1^{\text{ét}}(X)$ is the étale fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$ introduced in [deJ1]. Given an étale covering map $Y \to X$ and geometric points $\overline{x}' : \mathbf{p}_{\mathcal{H}(\overline{x})} \to Y$ and $\overline{y}' : \mathbf{p}_{\mathcal{H}(\overline{y})} \to Y$ over $\overline{x}$ and $\overline{y}$, respectively, let $H_Y(\overline{x}',\overline{y}')$ denote the set of all étale paths $\overline{x} \mapsto \overline{y}$ that take $\overline{x}'$ to $\overline{y}'$ under the induced maps $F_{\overline{x}}(Y) \to F_{\overline{y}}(Y)$. The set $\Pi_X^{\text{ét}}(\overline{x},\overline{x}')$ is provided with the weakest topology with respect to which all of the subsets $H_Y(\overline{x}',\overline{y}')$ are open. In particular, the étale fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$ is provided with a topology. Notice that the composition maps between the spaces of morphisms are continuous. Notice also that for every morphism $X' \to X$ in the category of analytic spaces there is an induced functor $\Pi_1^{\text{ét}}(X') \to \Pi_1^{\text{ét}}(X)$ with continuous maps between the spaces of morphisms. In particular, the canonical morphism $X \to \mathbf{p}_k = \mathcal{M}(k)$ induces, for every pair of geometric points $\overline{x}$ and $\overline{y}$ of X, a continuous map

$\Pi_X^{\text{ét}}(\overline{x},\overline{y}) \to \Pi_{\mathbf{p}_k}^{\text{ét}}(\overline{x},\overline{y}) : \overline{\gamma} \mapsto \sigma_{\overline{\gamma}}$. From [deJ1, 2.12] it follows that the latter map is always open and, if X is geometrically connected, it is surjective.

If the valuation on k is nontrivial and the k-analytic space X is smooth, the canonical functor $\Pi_1^{\text{ét}}(X) \to \Pi_1(X)$, which is evidently surjective on the families of objects, is also surjective and continuous on the sets of morphisms (where the sets $\Pi_X(x,y)$ of morphisms in $\Pi_1(X)$ are provided with the discrete topology). Indeed, this follows from the fact that such a space X is locally simply connected. In particular, the map $\pi_1^{\text{ét}}(X,\overline{x}) \to \pi_1(X,x)$ is surjective.

9.5.1. Lemma. *Let X be a geometrically reduced k-analytic space, and K a k-algebra. Then any étale sheaf of $\mathcal{C}_X^K$-modules L, which is locally free of finite rank over $\mathcal{C}_X^K$, is representable by an étale covering space over X.*

Proof. Consider first the case $K = k$, i.e., $\mathcal{C}_X^K = \mathfrak{c}_X$, and $L = \mathfrak{c}_X$. The sheaf $\mathfrak{c}_X$ is the pullback with respect to the canonical morphism $X \to \mathbf{p}_k$ of the sheaf $\mathfrak{c}_{\mathbf{p}_k}$ (which coincides with the structural sheaf of $\mathbf{p}_k$). The latter is representable by the étale covering space $W_k = \coprod V_{\mathbf{m}}$ over $\mathbf{p}_k$, where the disjoint union is taken over all maximal ideals $\mathbf{m} \subset k[T]$ whose residue field is separable over k and $V_{\mathbf{m}} = \mathcal{M}(k[T]/\mathbf{m})$. From [Ber2, Corollary 4.1.4(ii)] it follows that the sheaf $\mathfrak{c}_X$ is representable by the étale covering space $W_X = X \times_{\mathbf{p}_k} W_k$ over X.

In the general case, we fix a basis $\{e_i\}_{i\in I}$ of the k-vector space K with $e_0 = 1$ for a fixed element $0 \in I$. Let A be the set of pairs (J,f) consisting of a finite subset $J \subset I$ and a map $f : j \mapsto \mathbf{m}_{f(j)}$ from J to the set of maximal ideals of $k[T]$ such that, if $J \neq \{0\}$, then $\mathbf{m}_{f(j)} \neq (T)$ for all $j \in J$. For every $(J,f) \in A$, we fix a total ordering on J (i.e., represent J as $\{j_1,\dots,j_n\}$) and set $W_{(J,f)} = V_{\mathbf{m}_{f(j_1)}} \times_{\mathbf{p}_k} \times \dots \times_{\mathbf{p}_k} V_{\mathbf{m}_{f(j_n)}}$. Then $W_k^K = \coprod_{(J,f)\in A} W_{(J,f)}$ is an étale covering space over $\mathbf{p}_k$, and it represents the étale sheaf $\mathcal{C}_{\mathbf{p}_k}^K = \mathfrak{c}_{\mathbf{p}_k} \otimes_k K$. It follows that the étale covering space $W_X^K = X \times_{\mathbf{p}_k} W_k^K$ over X represents the étale sheaf $\mathcal{C}_X^K = \mathfrak{c}_X \otimes_k K$. If the étale $\mathcal{C}_X^K$-module L is free of rank n, then it is representable by the fiber product over X of n copies of W_X^K, which is an étale covering space over X. If L is arbitrary, there is an étale covering $\{U_j \to X\}_{j\in J}$ of X such that each $L\big|_{U_j}$ is free over $\mathcal{C}_{U_j}^K$, i.e., it is representable by an étale covering space over U_j, and the claim follows from [deJ1, Lemma 2.3]. ∎

In the situation of Lemma 9.5.1, for any geometric point $\overline{x}$ of X, there is a canonical isomorphism $\mathfrak{c}_{\mathbf{p}_k,\overline{x}} \xrightarrow{\sim} \mathfrak{c}_{X,\overline{x}}$ and so, for any étale path $\overline{\gamma} : \overline{x} \mapsto \overline{y}$, the induced isomorphism $\mathfrak{c}_{X,\overline{x}} \xrightarrow{\sim} \mathfrak{c}_{X,\overline{y}}$ is compatible with the isomorphism $\sigma_{\overline{\gamma}} : \mathfrak{c}_{\mathbf{p}_k,\overline{x}} \xrightarrow{\sim} \mathfrak{c}_{\mathbf{p}_k,\overline{y}}$. Because of that the former isomorphism is also denoted by $\sigma_{\overline{\gamma}}$. More generally, any étale path $\overline{\gamma} : \overline{x} \mapsto \overline{y}$ gives rise to a bijection $\mathrm{T}_{\overline{\gamma}}^L : L_{\overline{x}} \xrightarrow{\sim} L_{\overline{y}}$, which is compatible with all algebraic structures on L defined in a functorial way and, in particular, $\mathrm{T}^L_{\overline{\gamma}}$ is an isomorphism of K-modules which is $\sigma_{\overline{\gamma}}$-semi-linear in the sense that it is compatible with the isomorphism $\sigma_{\overline{\gamma}} : \mathfrak{c}_{X,\overline{x}} \stackrel{\sim}{\to} \mathfrak{c}_{X,\overline{y}}$. Furthermore, it commutes with tensor products and the internal Hom-functor, and is functorial on L and commutes with any base change in the category of analytic spaces. Notice also that if the field k is algebraically closed and L is locally free in the usual topology of X, then $\mathrm{T}^L_{\overline{\gamma}}$ coincides with the parallel transport $\mathrm{T}^L_\gamma : L_x \stackrel{\sim}{\to} L_y$, where γ is the homotopy class of a usual path with ends $x, y \in X$ that lies under $\overline{\gamma}$.

Let now k be a closed subfield of $\mathbf{C}_p$, K a filtered k-algebra, λ an element K^1, X a smooth k-analytic space, and $\mathcal{F}$ a locally quasi-unipotent $\mathcal{D}_X$-module. We apply the above construction to the sheaf of horizontal sections $\mathcal{F}^\nabla_{\mathcal{S}^\lambda}$. It follows that each étale path $\overline{\gamma} : \overline{x} \mapsto \overline{y}$ defines a $\sigma_{\overline{\gamma}}$-semi-linear isomorphism of K-modules $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}} = \mathrm{T}^{\mathcal{F},\lambda}_{\overline{\gamma}} : \mathcal{F}^\nabla_{\mathcal{S}^\lambda,\overline{x}} \stackrel{\sim}{\to} \mathcal{F}^\nabla_{\mathcal{S}^\lambda,\overline{y}}$. Recall that, by Corollary 9.3.2, if $\mathcal{F}$ is locally unipotent then, for every geometric point $\overline{x}$ of X over a point $x \in X_{st}$, there is a canonical isomorphism $\mathcal{F}^\nabla_x \otimes_{\mathfrak{c}_{X,x}} \mathfrak{c}_{X,\overline{x}} \stackrel{\sim}{\to} \mathcal{F}^\nabla_{\overline{x}}$. If now $\Pi^{\text{ét}}_1(X)_{st}$ denotes the full subcategory of $\Pi^{\text{ét}}_1(X)$ whose family of objects consists of the geometric points of X over points in X_{st}, we get a functor $\mathrm{T}^{\mathcal{F}} = \mathrm{T}^{\mathcal{F},\lambda} : \Pi^{\text{ét}}_1(X)_{st} \to K\text{-}\mathcal{M}od$ which takes a geometric point $\overline{x}$ to $\mathcal{F}^\nabla_{\overline{x}} \otimes_k K$ and an étale path $\overline{\gamma} : \overline{x} \mapsto \overline{y}$ to $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}}$. The following theorem lists properties of this functor.

9.5.2. Theorem. *(a) The K-linear isomorphism $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}} : \mathcal{F}^\nabla_{\overline{x}} \otimes_k K \stackrel{\sim}{\to} \mathcal{F}^\nabla_{\overline{y}} \otimes_k K$ is $\sigma_{\overline{\gamma}}$-semi-linear;*

(b) the map $\Pi^{\text{ét}}_X(\overline{x}, \overline{y}) \times (\mathcal{F}^\nabla_{\overline{x}} \otimes_k K) \to \mathcal{F}^\nabla_{\overline{y}} \otimes_k K$ is continuous;

(c) $\mathrm{T}^{\mathcal{F}}$ is functorial with respect to $\mathcal{F}$;

(d) $\mathrm{T}^{\mathcal{F}}$ commutes with tensor products and the internal Hom-functor;

(e) $\mathrm{T}^{\mathcal{F}}$ is functorial with respect to $(k, X, \overline{\gamma}, K, \lambda)$;

(f) in the situation of Corollary 9.4.3, $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}}$ is the $\sigma_{\overline{\gamma}}$-semi-linear extension of the parallel transport $\mathrm{T}^{\mathcal{F}}_\gamma : \mathcal{F}^\nabla_x \otimes_k K \stackrel{\sim}{\to} \mathcal{F}^\nabla_y \otimes_k K$.

Proof. The only property, which is not so evident, is (f) in the case when the field k is not necessarily algebraically closed (i.e., strictly smaller than $\mathbf{C}_p$). To verify it, we may assume that $\mathfrak{c}(X) = k$ (and therefore, $\mathfrak{c}_{X,x} = \mathfrak{c}_{X,y} = k$). Let x' and y' be the preimages in $\overline{X}$ of the points x and y, and let $\overline{x}' : \mathbf{p}_{\mathcal{H}(\overline{x})} \to \overline{X}$ and $\overline{y}' : \mathbf{p}_{\mathcal{H}(\overline{x})} \to \overline{X}$ be geometric points of $\overline{X}$ over the geometric points $\overline{x}$ and $\overline{y}$, respectively. By the assumption, there is a unique homotopy class of a path γ' from x' to y' in $\overline{X}$ over the path γ and, by the construction of Theorem 9.4.1, one has $\mathrm{T}^{\mathcal{F}}_\gamma(f) = \mathrm{T}^{\overline{\mathcal{F}}}_{\gamma'}(f)$ for all $f \in \mathcal{F}^\nabla_x$. Since the map $\Pi^{\text{ét}}_{\overline{X}}(\overline{x}', \overline{y}') \to \Pi_{\overline{X}}(x', y')$ is surjective, we can find an étale path $\overline{\gamma}' : \overline{x}' \mapsto \overline{y}'$ over the path γ' and, by the case of an algebraic closed field, one has $\mathrm{T}^{\overline{\mathcal{F}}}_{\overline{\gamma}'}(f) = \mathrm{T}^{\overline{\mathcal{F}}}_{\gamma'}(f)$ for all $f \in \mathcal{F}^\nabla_x$. Thus, to verify the property (f), we can replace the étale path $\overline{\gamma} : \overline{x} \mapsto \overline{y}$ by the étale path ${\overline{\gamma}'}^{-1} \circ \overline{\gamma}$ (where $\overline{\gamma}'$ is considered an an element of $\Pi^{\text{ét}}_X(\overline{x}, \overline{y})$), and so we may assume that $\overline{\gamma}$ is an element of the étale fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$ whose image in the topological fundamental group $\pi_1(X,x)$ is trivial, and we have to show that $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}}(f)=f$ for all $f\in\mathcal{F}^\nabla_x$. Since the dimension of $\mathcal{F}^\nabla_x$ over k is finite, there is an open subgroup $H\subset\pi_1^{\text{ét}}(X,\overline{x})$ such that $\mathrm{T}^{\mathcal{F}}_{\tau}(f)=f$ for all $\tau\in H$ and $f\in\mathcal{F}^\nabla_x$. We may shrink H and assume that it is contained in $\mathrm{Ker}(\pi_1^{\text{ét}}(X,\overline{x})\to\pi_1(X,x))$. Furthermore, the canonical morphism $\mathbf{p}_{\mathcal{H}(x)}\to X$ gives rise to a homomorphism of étale fundamental groups $\pi_1^{\text{ét}}(\mathbf{p}_{\mathcal{H}(x)},\overline{x})\to\pi_1^{\text{ét}}(X,\overline{x})$. It is clear that elements from the image of the latter act trivially on $\mathcal{F}^\nabla_{\overline{x}}\otimes_k K$. Since the canonical homomorphism from $\pi_1^{\text{ét}}(\mathbf{p}_{\mathcal{H}(x)},\overline{x})$ to the Galois group $\mathrm{Gal}(k^{\mathrm{a}}/k)$ of the algebraic closure k^{a} of k in $\mathcal{H}(\overline{x})$ over k is surjective, we can multiply $\overline{\gamma}$ by an element of $\pi_1^{\text{ét}}(\mathbf{p}_{\mathcal{H}(x)},\overline{x})$ so that we may assume the image of $\overline{\gamma}$ in $\mathrm{Gal}(k^{\mathrm{a}}/k)$ is trivial. We now use the fact that the image of $\pi_1^{\text{ét}}(\overline{X},\overline{x}')$ in $\pi_1^{\text{ét}}(X,\overline{x})$ is dense in $\mathrm{Ker}(\pi_1^{\text{ét}}(X,\overline{x})\to\mathrm{Gal}(k^{\mathrm{a}}/k))$ (see [deJ1, Remark 2.15]). It follows that $\overline{\gamma}=\overline{\gamma}'\tau$ for some $\overline{\gamma}'\in\pi_1^{\text{ét}}(\overline{X},\overline{x}')$ and $\tau\in H$ and, therefore, $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}}(f)=f$ for all $f\in\mathcal{F}^\nabla_x$. ∎

9.5.3. Example. Let $\mathcal{F}=\mathcal{O}_Xe$ be the $\mathcal{D}_X$-module on the affine line $X=\mathbf{A}^1$ defined by $\nabla(e)=-dTe$. A horizontal section of $\mathcal{F}$ at zero is given by $\exp(T)e$, where $\exp(T)$ is the usual exponential function convergent on the open disc with center at zero and of radius $|p|^{\frac{1}{p-1}}$. The logarithmic map $\varphi:Y=D(1;1)\to X=\mathbf{A}^1$ that takes z to $\log(z)$ is an étale covering map (see §9.5), and since $\nabla(\varphi^*(e))=-\frac{dz}{z}\varphi^*(e)$ it follows that $\varphi^*(\mathcal{F})$ is a trivial $\mathcal{D}_Y$-module. In particular, $\mathcal{F}$ is a locally quasi-unipotent $\mathcal{D}_X$-module of level one. Assume now that the field k is algebraically closed, i.e., $k=\mathbf{C}_p$. For any geometric point $\overline{a}$ over a point $a\in X(k)=k$, $\mathcal{F}^\nabla_{\overline{a}}=\mathcal{F}^\nabla_a$ is a one-dimensional vector space over k generated by the function $\exp(T-a)$. Given an étale path $\overline{\gamma}:\overline{0}\mapsto\overline{a}$, the isomorphism $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}}:\mathcal{F}^\nabla_0\xrightarrow{\sim}\mathcal{F}^\nabla_a$ depends only on the image of $1\in\varphi^{-1}(0)$ in $\varphi^{-1}(a)$ under $\overline{\gamma}$. If $\alpha=\overline{\gamma}(1)$, then $\log(\alpha)=a$, and one has $\mathrm{T}^{\mathcal{F}}_{\overline{\gamma}}(\exp(T))=\alpha\exp(T-a)$.

9.5.4. Remarks. (i) The statement of Corollary 9.3.2 is not true if $\mathcal{F}$ is only assumed to be quasi-unipotent at a point $x\in X_{st}$. Indeed, assume that k is algebraically closed, and let x be a point of the affine line $\mathbf{A}^1$ of type (4) which corresponds to a family of embedded discs in k of radii $>|p|^{\frac{1}{p-1}}$ with empty intersection. Then for $\mathcal{F}$ from the previous Example 9.5.3 one has $\mathcal{F}^\nabla_x=0$.

(ii) In a recent preprint [DeWe], C. Deninger and A. Werner constructed a parallel transport along an étale path for a certain class of vector bundles on a smooth projective curve. It is of different nature than the parallel transport from Theorem 9.5.2. For example, it is not related to an integrable connection on such a vector bundle, and it is continuous with respect to the nontrivial p-adic topology on the fibers of the bundle in comparison with the continuity of Theorem 9.5.2(c) with respect to the discrete topology. Of course, it would be interesting to find a relation between the two parallel transports.

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