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# Vanishing cycles for formal schemes. II

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## Introduction

Let $k$ be a complete discrete valuation field and $k^\circ$ its ring of integers. In this work we extend the construction of the vanishing cycles functor for formal schemes of locally finite type over $k^\circ$ from [Ber3] to a more broad class of formal schemes that includes, for example, formal completions of the above formal schemes along arbitrary subschemes of their closed fibres. We prove that if $\mathscr{X}$ is a scheme of finite type over a Henselian discrete valuation ring with the completion $k^\circ$ and $\mathscr{Y}$ is a subscheme of the closed fibre $\mathscr{X}_s$, then the vanishing cycles sheaves of the formal completion $\widehat{\mathscr{X}_{/\mathscr{Y}}}$ of $\mathscr{X}$ along $\mathscr{Y}$ are canonically isomorphic to the restrictions of the vanishing cycles sheaves of $\mathscr{X}$ to the subscheme $\mathscr{Y}$. In particular, the restrictions of the vanishing cycles sheaves of $\mathscr{X}$ to $\mathscr{Y}$ depend only on $\widehat{\mathscr{X}_{/\mathscr{Y}}}$, and any morphism $\varphi : \widehat{\mathscr{X}'_{/\mathscr{Y}'}} \to \widehat{\mathscr{X}_{/\mathscr{Y}}}$ induces a homomorphism from the pullback of the restrictions of the vanishing cycles sheaves of $\mathscr{X}$ to $\mathscr{Y}$ to those of $\mathscr{X}'$ to $\mathscr{Y}'$. Furthermore, we prove that, given $\widehat{\mathscr{X}_{/\mathscr{Y}}}$ and $\widehat{\mathscr{X}'_{/\mathscr{Y}'}}$, one can find an ideal of definition of $\widehat{\mathscr{X}'_{/\mathscr{Y}'}}$ such that if two morphisms $\varphi, \psi : \widehat{\mathscr{X}'_{/\mathscr{Y}'}} \to \widehat{\mathscr{X}_{/\mathscr{Y}}}$ coincide modulo this ideal, then the homomorphisms between the vanishing cycles sheaves induced by $\varphi$ and $\psi$ coincide. These facts generalize results from [Ber3], where the case when $\mathscr{Y}$ is open in $\mathscr{X}_s$ was considered, as well as results of G. Laumon, J.-L. Brylinski and the author from [Lau], [Bry] and [Ber5], respectively, where certain cases when $\mathscr{Y}$ is a closed point of $\mathscr{X}_s$ were considered (see Remarks 4.2 and 4.6). Finally, we prove a vanishing theorem which states that the $q$-dimensional étale cohomology groups of certain analytic spaces of dimension $m$ are trivial for $q > m$. This class of analytic spaces includes, for example, the finite étale coverings $\Sigma^{d,n}$ of the Drinfeld half-plane $\Omega^d$ constructed by V. Drinfeld in [Dr]. (The vanishing result for $\Omega^d$ follows from the work of P. Schneider and U. Stuhler [ScSt] where all the cohomology groups of $\Omega^d$ are calculated.)

In §1 we recall a construction of P. Berthelot ([Bert]) that associates with a formal scheme $\mathfrak{X}$ over $k^\circ$ of a certain type (called special in this paper) its generic fibre $\mathfrak{X}_\eta$ (which is a $k$-analytic space), a reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ and, for subscheme $\mathscr{Y} \subset \mathfrak{X}_s$, a canonical isomorphism $(\mathfrak{X}_{/\mathscr{Y}})_\eta \tilde{\to} \pi^{-1}(\mathscr{Y})$. (The analytic domain $\pi^{-1}(\mathscr{Y})$ is called the tube of $\mathscr{Y}$ in $\mathfrak{X}$.) In §2 we construct for a special formal scheme $\mathfrak{X}$ the nearby cycles and vanishing cycles functors $\Theta : \mathfrak{X}_{\eta \text{ét}}^{\sim} \to \mathfrak{X}_{s \text{ét}}^{\sim}$ and $\Psi_\eta : \mathfrak{X}_{\eta \text{ét}}^{\sim} \to \mathfrak{X}_{\bar{s} \text{ét}}^{\sim}$. In §3 we prove the comparison theorem which states that if $\mathscr{X}$ is a scheme of locally finite type over $k^\circ$, $\mathscr{Y}$ is a subscheme of $\mathscr{X}_s$, $\mathscr{F}$ is an abelian constructible sheaf on $\mathscr{X}_\eta$ with torsion orders prime to char$(\widetilde{k})$, and $\widehat{\mathscr{F}_{/\mathscr{Y}}}$ is its pullback on $(\widehat{\mathscr{X}_{/\mathscr{Y}}})_\eta$, then there are canonical isomorphisms $(R\Theta\mathscr{F})\big|_{\mathscr{Y}} \tilde{\to} R\Theta(\widehat{\mathscr{F}_{/\mathscr{Y}}})$ and $(R\Psi_\eta\mathscr{F})\big|_{\overline{\mathscr{Y}}} \tilde{\to} R\Psi_\eta(\widehat{\mathscr{F}_{/\mathscr{Y}}})$. The proof uses the recent stable reduction theorem of A. J. de Jong from [deJ]. The comparison theorem implies that if $\mathscr{Y}$ is a subscheme of the closed fibre of a smooth formal scheme $\mathfrak{X}$ over $k^\circ$ and $\Lambda = \mathbf{Z}/n\mathbf{Z}$, where $n$ is prime to char$(\widetilde{k})$, then $H^q(\overline{\mathscr{Y}}, \Lambda) \tilde{\to} H^q(\overline{\pi^{-1}(\mathscr{Y})}, \Lambda)$, and if, in addition, the closure of $\mathscr{Y}$ in $\mathfrak{X}_s$ is proper then $H_c^q(\overline{\mathscr{Y}}, \Lambda) \tilde{\to} H^q_{\overline{\pi^{-1}(\mathscr{Y})}}(\mathfrak{X}_{\overline{\eta}}, \Lambda)$. This means that the construction of P. Berthelot of $p$-adic cohomology of $\mathscr{Y}$ in terms of certain cohomology of the tube $\pi^{-1}(\mathscr{Y})$ makes sense also for $l$-adic cohomology, $l \neq p = \text{char}(\widetilde{k})$. In §4 we prove the continuity theorem. In §5 we prove a more general version of the Generalized Krasner Lemma 7.3 from [Ber3], introduce a class of analytic spaces called quasi-affine and show that any analytic space that admits a finite étale morphism to a quasi-affine analytic space is quasi-affine. In §6 we prove the vanishing theorem for paracompact quasi-affine analytic spaces. Here the comparison theorem (in the form of [Ber3]) is used to reduce the statement to the affine Lefschetz theorem.

The problem of proving the properties of the vanishing cycles sheaves established in this paper has arisen in P. Deligne's work [Del]. In that work he constructed a certain representation of the group $\text{GL}_2(\mathbf{Q}_p) \times B^*_{2,\mathbf{Q}_p} \times W_{\mathbf{Q}_p}$ in terms of cohomology of the vanishing cycles sheaves of modular curves, where $B_{h,\mathbf{Q}_p}$ is the skew field with center $\mathbf{Q}_p$ and invariant $1/h$. The non-evident fact was that the representation constructed is smooth for the group $B^*_{2,\mathbf{Q}_p}$. In [Car1], a similar representation for a finite extension of $\mathbf{Q}_p$ was constructed in terms of cohomology of the vanishing cycles sheaves of certain Shimura curves (using the properties of the vanishing cycles sheaves of relative curves established in [Bry]). In [Car2], H. Carayol generalized that construction to obtain for every local field $F$ and for every $h \geq 1$ a representation $\mathscr{U}^v_{h,F}$ of the group $\text{GL}_h(F) \times B^*_{h,F} \times W_F$, and conjectured a description of it in terms of the local Langlands and Jacquet-Langlands correspondences. That the representation $\mathscr{U}^v_{h,F}$ is smooth for the group $B^*_{h,F}$ follows from the results of this paper.

Like [Ber3] and [Ber5], this work arose from a suggestion of P. Deligne to apply the étale cohomology theory from [Ber2] to the study of the vanishing

cycles sheaves of schemes. I am very grateful to him for this and to A. J. de Jong who explained me his results from [deJ]. I am also grateful to the referee for useful remarks. I gratefully appreciate the hospitality and support of Harvard University where this work was done.

## 1. Special formal schemes and their generic fibres

Let $R$ be a topological adic Noetherian ring whose Jacobson radical is an ideal of definition (see [EGA1] Ch. 0, §7). We say that a topological $R$-algebra $A$ is *special* if $A$ is an adic ring and, for some ideal of definition $\mathbf{a} \subset A$, the quotient rings $A/\mathbf{a}^n$, $n \geq 1$, are finitely generated over $R$. The following lemma lists properties of special $R$-algebras which follow easily from [Bou], Ch. III and V.

**Lemma 1.1.** *Let $A$ be an $\mathbf{a}$-adic special $R$-algebra. Then*

*(i) $A$ is a Noetherian ring and its Jacobson radical is an ideal of definition;*

*(ii) every ideal $\mathbf{b} \subset A$ is closed, and the quotient ring $B := A/\mathbf{b}$ is an $\mathbf{a}B$-adic special $R$-algebra;*

*(iii) if $A \to B$ is a continuous surjective homomorphism between special $R$-algebras and $\mathbf{b}$ is its kernel, then $A/\mathbf{b}$ is topologically isomorphic to $B$;*

*(iv) if an ideal $\mathbf{b} \subset A$ is open, then the completion $B := \widehat{A}$ of $A$ in the $\mathbf{b}$-topology is a $\mathbf{b}B$-adic special $R$-algebra;*

*(v) if $B$ is a special $R$-algebra, then so is $A\widehat{\otimes}_R B$;*

*(vi) the algebra of restricted power series $B := A\{T_1, \dots, T_n\}$ is an $\mathbf{a}B$-adic special $R$-algebra;*

*(vii) the algebra of formal power series $B := A[[T_1, \dots, T_n]]$ is a $\mathbf{b}$-adic special $R$-algebra, where $\mathbf{b}$ is generated by $\mathbf{a}$ and $T_1, \dots, T_n$.* □

An adic $R$-algebra $A$ is said to be *topologically finitely generated over $R$* if $A$ is topologically $R$-isomorphic to a quotient algebra of the algebra of restricted power series $R\{T_1, \dots, T_n\}$. By [EGA1], Ch. 0, 7.5.3, the latter is equivalent to the fact that $JA$ is an ideal of definition of $A$ and $A/JA$ is finitely generated over $R$, where $J$ is an ideal of definition of $R$. By Lemma 1.1, any adic $R$-algebra topologically finitely generated over $R$ is special.

**Lemma 1.2.** *Let $A$ be an $\mathbf{a}$-adic $R$-algebra. Then the following are equivalent:*

*(a) $A$ is a special $R$-algebra;*

*(b) $A/\mathbf{a}^2$ is finitely generated over $R$;*

*(c) $A$ is topologically $R$-isomorphic to a quotient of the special $R$-algebra $R\{T_1, \dots, T_m\}[[S_1, \dots, S_n]]$.*

We remark that $R\{T_1, \dots, T_m\}[[S_1, \dots, S_n]] = R[[S_1, \dots, S_n]]\{T_1, \dots, T_m\}$.

*Proof* . The implication (c)$\Longrightarrow$(a) follows from Lemma 1.1, and (a)$\Longrightarrow$(b) is trivial. We have to prove (b)$\Longrightarrow$(c). Let $f_1, \dots, f_m$ (resp. $g_1, \dots, g_n$) generate $A/\mathbf{a}$ (resp. $\mathbf{a}/\mathbf{a}^2$) over $R$ (resp. $A/\mathbf{a}$), and let $\varphi : B := R\{T_1, \dots, T_m\}[[S_1, \dots, S_n]] \to A$ be the continuous $R$-homomorphism that takes $T_i$ to $f_i$ and $S_j$ to $g_j$. It suffices to show that $\varphi$ is surjective. By Lemma 1.1, the ideal $\mathbf{b} \subset B$ generated by $J$ and $S_1, \dots, S_n$, where $J$ is an ideal of definition of $R$, is an ideal of definition of $B$, and one has $\varphi(\mathbf{b}) \subset \mathbf{a}$. Therefore, by [Bou], Ch. III, §2, Prop. 14, it suffices to show that all the induced homomorphisms $\mathbf{b}/\mathbf{b}^i \to \mathbf{a}/\mathbf{a}^i$, $i \geq 1$, are surjective. The latter is easily verified by induction. □

A formal scheme $\mathfrak{X}$ over $R$ (i.e., over Spf($R$)) is said to be *special* (resp. of *locally finite type*) if if it is a locally finite union of affine formal schemes of the form Spf($A$), where $A$ is an adic algebra special (resp. topologically finitely generated) over $R$. The category of formal schemes special (resp. of locally finite type) over $R$ will be denoted by $R$-$\mathcal{SF}sch$ (resp. $R$-$\mathcal{F}sch$). This category admits fibre products. (Of course, $R$-$\mathcal{F}sch$ is a full subcategory of $R$-$\mathcal{SF}sch$.) A morphism $\mathfrak{Y} \to \mathfrak{X}$ in $R$-$\mathcal{SF}sch$ is said to be of *locally finite type* if locally it is isomorphic to a morphism of the form Spf($B$) $\to$ Spf($A$), where $B$ is topologically finitely generated over $A$. The latter is equivalent to the fact that, if $\mathcal{J}$ is an ideal of definition of $\mathfrak{X}$, then $\mathcal{J}\mathcal{O}_{\mathfrak{Y}}$ is an ideal of definition of $\mathfrak{Y}$. A quasicompact morphism of locally finite type is said to be of *finite type*. We remark that any open subscheme of a formal scheme special (resp. of locally finite type) over $R$ is of the same type and that, if Spf($A$) $\in R$-$\mathcal{SF}sch$, then $A$ is special (resp. topologically finitely generated) over $R$.

Let $k$ be a non-Archimedean field with a discrete valuation (which is not assumed to be nontrivial), $k^\circ$ the ring of integers of $k$, $k^{\circ\circ}$ the maximal ideal of $k^\circ$, $\widetilde{k} = k^\circ/k^{\circ\circ}$ the residue field of $k$. If $\mathfrak{X} \in k^\circ$-$\mathcal{SF}sch$, then the ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$, where $\mathcal{J}$ is an ideal of definition of $\mathfrak{X}$ that contains $k^{\circ\circ}$, is a scheme of locally finite type over $\widetilde{k}$. It is called the *closed fibre* of $\mathfrak{X}$ and is denoted by $\mathfrak{X}_s$. The scheme $\mathfrak{X}_s$ depends on the choice of the ideal $\mathcal{J}$ but the underlying reduced scheme and the étale topos of $\mathfrak{X}_s$ do not. (For $\mathfrak{X} \in k^\circ$-$\mathcal{F}sch$, one can take $\mathcal{J} = k^{\circ\circ}\mathcal{O}_{\mathfrak{X}}$ and then one gets the closed fibre defined in [Ber3].) We remark that for a subscheme $\mathcal{Y} \subset \mathfrak{X}_s$ the formal completion $\mathfrak{X}_{/\mathcal{Y}}$ of $\mathfrak{X}$ along $\mathcal{Y}$ is a special formal scheme over $k^\circ$. We will define a functor $k^\circ$-$\mathcal{SF}sch \to k$-$\mathcal{A}n$ that associates with a special formal scheme $\mathfrak{X}$ its *generic fibre* $\mathfrak{X}_\eta \in k$-$\mathcal{A}n$, and we will construct a *reduction map* $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$.

If $\mathfrak{X}$ = Spf($A$), where $A = k^\circ\{T_1, \dots, T_m\}[[S_1, \dots, S_n]]$, we set $\mathfrak{X}_\eta = E^m(0;1) \times D^n(0;1)$, where $E^m(0;1)$ and $D^n(0;1)$ are the closed and the open polydiscs of radius 1 with center at zero in $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively. The space $\mathfrak{X}_\eta$ is exhausted by a sequence of affinoid domains $X_1 \subset X_2 \subset \dots$ such that each $X_n$ is a Weierstrass domain in $X_{n+1}$ (i.e., $X$ is a Stein space). The canonical homomorphisms $A \to \mathcal{A}_{X_n}$ are continuous, and the image of $A \otimes_{k^\circ} k$ in each $\mathcal{A}_{X_n}$ is everywhere dense. (If the valuation on $k$ is trivial, then $A \widetilde{\to} \mathcal{O}(\mathfrak{X}_\eta)$.) Any morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, where $\mathfrak{Y}$ is of the same form, induces in the evident way a morphism of $k$-analytic spaces $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$.

Suppose now that $\mathfrak{X}$ = Spf($A$), where $A$ is an arbitrary special $k^\circ$-algebra. We fix a surjective homomorphism $\alpha : A' := k^\circ\{T_1, \dots, T_m\}[[S_1, \dots, S_n]] \to A$. Let $\mathbf{a}$ be the kernel of $\alpha$. We set $\mathfrak{X}'$ = Spf($A'$) and define $\mathfrak{X}_\eta$ as the closed $k$-analytic subspace of $\mathfrak{X}'_\eta$ defined by the subsheaf of ideals $\mathbf{a}\mathcal{O}_{\mathfrak{X}'_\eta}$. In particular, $\mathfrak{X}_\eta$ is identified with the set of continuous multiplicative seminorms on $A$ that extend the valuation on $k^\circ$ and whose values are at most 1, and it is exhausted by a sequence of affinoid domains $X_1 \subset X_2 \subset \dots$ such that each $X_n$ is a Weierstrass domain in $X_{n+1}$. Furthermore, the canonical homomorphisms $A \to \mathcal{A}_{X_n}$ are continuous, and the image of $A \otimes_{k^\circ} k$ in each $\mathcal{A}_{X_n}$ is everywhere dense. It follows that a compact subset $V \subset \mathfrak{X}_\eta$ is an affinoid domain if and only if there exist a $k$-affinoid algebra $\mathcal{A}_V$ and a continuous homomorphism $A \to \mathcal{A}_V^\circ$ such that the image $\mathcal{M}(\mathcal{A}_V)$ in $\mathfrak{X}_\eta$ is contained in $V$ and any continuous homomorphism $A \to \mathcal{B}^\circ$, where $\mathcal{B}$ is an affinoid $k$-algebra, for which the image of $\mathcal{M}(\mathcal{B})$ in $\mathfrak{X}_\eta$ is contained in $V$, is extended in a unique way to a bounded $k$-homomorphism $\mathcal{A}_V \to \mathcal{B}$. This implies easily that the correspondence $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is a functor, and if $\mathfrak{Y}$ is an open affine formal subscheme of $\mathfrak{X}$, then the canonical morphism $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ identifies $\mathfrak{Y}_\eta$ with a closed analytic domain in $\mathfrak{X}_\eta$.

If $\mathfrak{X}$ is arbitrary, we fix a locally finite covering $\{\mathfrak{X}_i\}_{i \in I}$ by open affine subschemes of the form Spf($A$), where $A$ is a special $k^\circ$-algebra. Suppose first that $\mathfrak{X}$ is separated. Then for any pair $i, j \in I$ the intersection $\mathfrak{X}_{ij} = \mathfrak{X}_i \cap \mathfrak{X}_j$ is also of the same form, $\mathfrak{X}_{ij,\eta}$ is a closed analytic domain in $\mathfrak{X}_{i,\eta}$, and the canonical morphism $\mathfrak{X}_{ij,\eta} \to \mathfrak{X}_{i,\eta} \times \mathfrak{X}_{j,\eta}$ is a closed immersion. By [Ber2], 1.3.3, we can glue all $\mathfrak{X}_{i,\eta}$ along $\mathfrak{X}_{ij,\eta}$, and we get a paracompact separated $k$-analytic space $\mathfrak{X}_\eta$. We remark that the correspondence $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is a functor that extends the functor constructed for the affine formal schemes, and if $\mathfrak{Y}$ is an open formal subscheme of $\mathfrak{X}$, then $\mathfrak{Y}_\eta$ is a closed analytic domain in $\mathfrak{X}_\eta$. If $\mathfrak{X}$ is arbitrary, then $\mathfrak{X}_{ij} = \mathfrak{X}_i \cap \mathfrak{X}_j$ are separated formal schemes, and $\mathfrak{X}_{ij,\eta}$ is a closed analytic domain in the $k$-analytic space $\mathfrak{X}_{i,\eta}$. Therefore we can glue all $\mathfrak{X}_{i,\eta}$ along $\mathfrak{X}_{ij,\eta}$ and get a paracompact $k$-analytic space $\mathfrak{X}_\eta$. We remark that the correspondence $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is a functor to the category of paracompact $k$-analytic spaces, and this functor commutes with fibre products. If $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is a morphism of finite type, then the induced morphism $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is compact. We also remark that if $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is finite (resp. flat finite), then so is $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$.

The reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ is constructed as follows. Supposed first that $\mathfrak{X} = \mathrm{Spf}(A)$. Then a point $x \in \mathfrak{X}_\eta$ defines a continuous character $\chi_x : A \to \mathcal{H}(x)$. The latter defines a character $\widetilde{\chi}_x : A_s = A/\mathbf{a} \to \widetilde{\mathcal{H}(x)}$, where $\mathbf{a}$ is an ideal of definition of $A$ that contains $k^{\circ\circ}$. The kernel of $\widetilde{\chi}_x$, which is a prime ideal of $A_s$, is, by definition, the point $\pi(x) \in \mathfrak{X}_s = \mathrm{Spec}(A_s)$. We remark that if $\mathfrak{Y}$ is an open formal subscheme of $\mathfrak{X}$, then the reduction maps for $\mathfrak{X}$ and $\mathfrak{Y}$ are compatible and $\mathfrak{Y}_\eta \xrightarrow{\sim} \pi^{-1}(\mathfrak{Y}_s)$. (This is reduced to $\mathfrak{Y}$ of the form Spf($A_{\{f\}}$), and is easily verified for such $\mathfrak{Y}$.) This remark allows one to extend the construction of the reduction map for arbitrary special formal schemes over $k^\circ$.

**Proposition 1.3.** *Let $\mathfrak{X} \in k^0$-$\mathcal{SF}sch$. Then for any subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, there is a canonical isomorphism $(\mathfrak{X}_{/\mathcal{Y}})_\eta \xrightarrow{\sim} \pi^{-1}(\mathcal{Y})$ .*

*Proof* . We may assume that $\mathcal{Y}$ is closed in $\mathfrak{X}_s$ and $\mathfrak{X} = \mathrm{Spf}(A)$. Let $f_1, \dots, f_n$ be elements of $A$ such that their images in $A_s$ generate the ideal of $\mathcal{Y}$. Then the canonical morphism $\varphi : (\mathfrak{X}_{/\mathcal{Y}})_\eta \to \pi^{-1}(\mathcal{Y}) = \{x \in \mathfrak{X}_\eta \big| |f_i(x)| < 1, 1 \le i \le n\}$ is a homeomorphism. And so it suffices to verify that if $V$ is an affinoid domain in $(\mathfrak{X}_{/\mathcal{Y}})_\eta$, then $\varphi(V)$ is an affinoid domain in $\mathfrak{X}_\eta$ and $\mathcal{A}_{\varphi(V)} \widetilde{\to} \mathcal{A}_V$. This is easily obtained from the description of affinoid domains in $\mathfrak{X}_\eta$. □

## 2. The vanishing cycles functor

A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in $k^\circ$-$\mathcal{SF}sch$ is said to be *étale* if it is of locally finite type and for any ideal of definition $\mathcal{J}$ of $\mathfrak{X}$ the morphism of schemes $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{J}\mathcal{O}_{\mathfrak{Y}}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is étale. The following is a straightforward generalization of Lemmas 2.1-2.2 and Proposition 2.3 from [Ber3].

**Proposition 2.1.** *(i) The correspondence $\mathfrak{Y} \mapsto \mathfrak{Y}_s$ induces an equivalence between the category of formal schemes étale over $\mathfrak{X}$ and the category of schemes étale over $\mathfrak{X}_s$.*

*(ii) If $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is an étale morphism, then $\varphi_\eta(\mathfrak{Y}_\eta) = \pi^{-1}(\varphi_s(\mathfrak{Y}_s))$. In particular, $\varphi_\eta(\mathfrak{Y}_\eta)$ is a closed analytic domain in $\mathfrak{X}_\eta$.*

*(iii) If $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is an étale morphism, then the induced morphism of $k$-analytic spaces $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is quasi-étale (see [Ber3], §3).* □

More generally, let $K$ be a subfield of a separable closure $k^s$ of $k$, and let $\mathfrak{X} \in k^\circ$-$\mathcal{SF}sch$. We denote by $\mathfrak{X}_{s_K}$ and $\mathfrak{X}_{\eta_K}$ the closed and generic fibres of the formal scheme $\mathfrak{X}_K := \mathfrak{X}\widehat{\otimes}_{k^\circ}\widehat{K}^\circ$, i.e., $\mathfrak{X}_{s_K} = \mathfrak{X}_s \otimes \widetilde{K}$ and $\mathfrak{X}_{\eta_K} = \mathfrak{X}_\eta \widehat{\otimes} \widehat{K}$. We also set $\mathfrak{X}_{\bar{s}} = \mathfrak{X}_s \otimes \widetilde{k^s}$ and $\mathfrak{X}_{\bar{\eta}} = \mathfrak{X}_\eta \widehat{\otimes} \widehat{k^s}$. A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}_K$ of formal schemes over $\widehat{K}^\circ$ is said to be *étale* if locally it comes from an étale morphism $\mathfrak{Y}' \to \mathfrak{X}_{k'}$ in $k'^\circ$-$\mathcal{SF}sch$ for some finite extension $k'$ of $k$ in $K$. It follows from Proposition 2.1 that for such $\mathfrak{Y}$ the closed fibre $\mathfrak{Y}_s$, the generic fibre $\mathfrak{Y}_\eta$ and the reduction map $\pi : \mathfrak{Y}_\eta \to \mathfrak{Y}_s$ are well defined, and all the statements of Proposition 2.1 also hold for $\mathfrak{X}_K$ instead of $\mathfrak{X}$. We fix a functor $\mathfrak{Y}_s \mapsto \mathfrak{Y}$ from the category of schemes étale over $\mathfrak{X}_{s_K}$ to the category of formal schemes étale over $\mathfrak{X}_K$ which is inverse to the functor from that more general Proposition 2.1(i). The composition of the latter functor with the functor $\mathfrak{Y} \mapsto \mathfrak{Y}_\eta$ induces a morphism of sites $\nu : \mathfrak{X}_{\eta_K \mathrm{q\acute{e}t}} \to \mathfrak{X}_{s_K \mathrm{\acute{e}t}}$. If $\mu$ denotes the morphism of sites $\mathfrak{X}_{\eta_K \mathrm{q\acute{e}t}} \to \mathfrak{X}_{\eta_K \mathrm{\acute{e}t}}$, we get a left exact functor

$$\Theta^K = \nu_*\mu^* : \mathfrak{X}^\sim_{\eta_K \mathrm{\acute{e}t}} \longrightarrow \mathfrak{X}^\sim_{\eta_K \mathrm{q\acute{e}t}} \longrightarrow \mathfrak{X}^\sim_{s_K \mathrm{\acute{e}t}} .$$

The following is a straightforward generalization of Proposition 4.1 and Corollary 4.2 from [Ber3].

**Proposition 2.2.** *Let $F$ be an étale sheaf on $\mathfrak{X}_{\eta_K}$.*

*(i) If $\mathfrak{Y}_s$ is étale over $\mathfrak{X}_{s_K}$, then $\Theta^K(F)(\mathfrak{Y}_s) = F(\mathfrak{Y}_\eta)$.*

*(ii) If $F$ is an abelian sheaf, then $R^q\Theta^K(F)$ is associated with the presheaf $\mathfrak{Y}_s \mapsto H^q(\mathfrak{Y}_\eta, F)$.*

*(iii) If $F$ is a soft abelian sheaf, then the sheaf $\Theta^K(F)$ is flabby.* □

We denote by $\Theta_K$ the functor $\mathfrak{X}^\sim_{\eta_{\mathrm{\acute{e}t}}} \to \mathfrak{X}^\sim_{s_K \mathrm{\acute{e}t}} : F \mapsto \Theta^K(F_K)$, where $F_K$ is the pullback of $F$ on $\mathfrak{X}_{\eta_K}$. We remark that a sheaf $F$ is soft then, by [Ber3], Lemma 3.2, the sheaf $F_K$ is also soft, and therefore for any $F$ there is a canonical isomorphism $R^q\Theta_K(F)\widetilde{\to}R^q\Theta^K(F_K)$, $q \geq 0$. We are especially interested in the *nearby cycles functor* $\Theta = \Theta_k : \mathfrak{X}^{\sim}_{\eta\,\text{ét}} \to \mathfrak{X}^{\sim}_{s\,\text{ét}}$ and the *vanishing cycles functor* $\Psi_\eta = \Theta_{k^s} : \mathfrak{X}^{\sim}_{\overline{\eta}\,\text{ét}} \to \mathfrak{X}^{\sim}_{\overline{s}\,\text{ét}}$. But to save place (at least in this section) we consider the functor $\Theta_K$ for an arbitrary extension $K$ of $k$ in $k^s$.

**Corollary 2.3.** *(i) For an étale morphism $\mathfrak{Y} \to \mathfrak{X}$ in $k^\circ$-$\mathcal{SF}sch$ and $F \in \mathbf{S}(\mathfrak{X}_\eta)$, one has $(R^q\Theta_K F)\big|_{\mathfrak{Y}_{s_K}} \widetilde{\to} R^q\Theta_K(F\big|_{\mathfrak{Y}_{\eta_K}})$, $q \geq 0$.*

*(ii) For a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in $k^\circ$-$\mathcal{SF}sch$ and $F^\cdot \in D^+(\mathfrak{Y}_\eta)$, one has*

$$R\Theta_K(R\varphi_{\eta_K *}F^\cdot)\widetilde{\to}R\varphi_{s_K *}(R\Theta_K F^\cdot)\ .$$

*In particular, if $F^\cdot \in D^+(\mathfrak{X}_\eta)$, then $R\Gamma(\mathfrak{X}_{s_K}, R\Theta_K F^\cdot)\widetilde{\to}R\Gamma(\mathfrak{X}_{\eta_K}, F^\cdot)$.* □

**Proposition 2.4.** *Let $\mathcal{Y}$ be a closed subset of $\mathfrak{X}_s$ and $F^\cdot \in D^+(\mathfrak{X}_\eta)$. Then there is a canonical isomorphism $R\Gamma_{\mathcal{Y}_{\widetilde{K}}}(\mathfrak{X}_{s_K}, R\Theta_K F^\cdot)\widetilde{\to}R\Gamma_{\pi^{-1}(\mathcal{Y})_K}(\mathfrak{X}_{\eta_K}, F^\cdot)$. In particular, if $\mathfrak{X}$ is of locally finite type over $k^\circ$ and $\mathcal{Y}$ is quasicompact, then $R\Gamma_{\mathcal{Y}_{\widetilde{K}}}(\mathfrak{X}_{s_K}, R\Theta_K F^\cdot)\widetilde{\to}R\Gamma_c(\pi^{-1}(\mathcal{Y})_K, F^\cdot)$.*

*Proof* . It suffices to verify the following two facts for $\mathcal{U} = \pi^{-1}(\mathcal{Y})_K$ and $F \in \mathbf{S}(\mathfrak{X}_{\eta_K})$:

(1) $H^0_{\mathcal{Y}_{\widetilde{K}}}(\mathfrak{X}_{s_K}, \Theta^K(F)) = H^0_{\mathcal{U}}(\mathfrak{X}_{\eta_K}, F)$;

(2) if $F$ is soft, then $H^p_{\mathcal{Y}_{\widetilde{K}}}(\mathfrak{X}_{s_K}, \Theta^K(F)) = 0$ for $p \geq 1$.

(1) One has $H^0_{\mathcal{U}}(\mathfrak{X}_{\eta_K}, F) = \mathrm{Ker}(F(\mathfrak{X}_{\eta_K}) \to F(\mathfrak{X}_{\eta_K}\backslash\mathcal{U}))$. By the definition of $\Theta^K(F)$, one has $F(\mathfrak{X}_{\eta_K}) = \Theta^K(F)(\mathfrak{X}_{\eta_K})$ and $F(\mathfrak{X}_{\eta_K}\backslash\mathcal{U}) = \Theta^K(F)(\mathfrak{X}_{s_{\widetilde{K}}}\backslash\mathcal{Y}_{\widetilde{K}})$, and therefore the group considered coincides with $H^0_{\mathcal{Y}_{\widetilde{K}}}(\mathfrak{X}_{s_K}, \Theta^K(F))$.

(2) If $F$ is soft, then the homomorphism $\Theta^K(F)(\mathfrak{X}_{s_K}) = F(\mathfrak{X}_{\eta_K}) \to$ $\to \Theta^K(F)(\mathfrak{X}_{s_K}\backslash\mathcal{Y}_{\widetilde{K}}) = F(\mathfrak{X}_{\eta_K}\backslash\mathcal{U})$ is surjective and, by Proposition 2.2(iii), the sheaf $\Theta^K(F)$ is flabby. The required fact follows. □

For a $k$-analytic space $X$ (resp. a scheme $\mathcal{Y}$ over $\widetilde{k}$), we set $\overline{X} = X\widehat{\otimes}\widehat{k^s}$ (resp. $\overline{\mathcal{Y}} = \mathcal{Y} \otimes \widetilde{k}^s$).

**Corollary 2.5.** *Let $\mathfrak{X}$ be of locally finite type over $k^\circ$, $\mathcal{Y}$ a quasicompact closed subset of $\mathfrak{X}_s$, and $F^\cdot \in D^+(\mathfrak{X}_\eta)$. Then there is a canonical isomorphism $R\Gamma_{\overline{\mathcal{Y}}}(\mathfrak{X}_{\overline{s}}, R\Psi_\eta F^\cdot)\widetilde{\to}R\Gamma_c(\overline{\pi^{-1}(\mathcal{Y})}, F^\cdot)$. In particular, if all of the irreducible components of $\mathfrak{X}_s$ are proper, then $R\Gamma_c(\mathfrak{X}_{\overline{s}}, R\Psi_\eta F^\cdot)\widetilde{\to}R\Gamma_c(\mathfrak{X}_{\overline{\eta}}, F^\cdot)$.* □

*Remark 2.6.* (i) There is a canonical action of the Galois group $G_\eta := G(k^s/k)$ on $\Psi_\eta(F)$ compatible with the action of $G_s := G(\widetilde{k}^s/\widetilde{k})$ on $\mathfrak{X}_{\overline{s}}$. But for arbitrary special formal schemes over $k^\circ$, this action is not necessarily continuous because the generic fibre $\mathfrak{X}_\eta$ is not necessarily compact even for a quasicompact $\mathfrak{X}$ (see Remark 3.8(iii)). For the same reason, the statement of Proposition 4.6 from [Ber3] is not true for arbitrary special formal schemes. But I don't know if the statement of Theorem 4.9 from [Ber3] is true.

(ii) One can define as follows a version of the above functor $\Theta_K$ which possesses the properties mentioned in (i) (and coincides with $\Theta_K$ for formal schemes of locally finite type). Namely, for an étale abelian sheaf $F$ on $\mathfrak{X}_\eta$ and an étale morphism $f : \mathfrak{U} \to \mathfrak{X}_K$ let $\Theta_{c,K}(F)(\mathfrak{U}_s)$ be the subgroup of all $s \in F(\mathfrak{U}_\eta)$ such that for any open quasicompact subscheme $\mathfrak{V} \subset \mathfrak{U}$ the set $\mathrm{Supp}(s) \cap \mathfrak{V}_\eta$ is compact. Then the correspondence $\mathfrak{U}_s \mapsto \Theta_{K,c}(F)(\mathfrak{U}_s)$ is a sheaf on $\mathfrak{X}_{s_K}$, and one gets a left exact functor $\Theta_{c,K} : \mathbf{S}(\mathfrak{X}_\eta) \to \mathbf{S}(\mathfrak{X}_{s_K})$. From [Ber2], Corollary 5.3.5, it follows that the canonical action of the Galois group $G_\eta$ on $\Psi_{c,\eta}(F) = \Theta_{c,k^s}(F)$ is continuous and $\varinjlim R^q\Theta_{c,k'}(F) \tilde{\to} R^q\Theta_{c,K}(F)$, where the limit is taken over finite extensions $k'$ of $k$ in $K$. Furthermore, the proofs of Proposition 4.6 and Theorem 4.9 from [Ber3] are applicable to the functors $\Theta_c = \Theta_{c,k}$ and $\Psi_{c,\eta}$ and arbitrary special formal schemes, and therefore their statements hold for $\Theta_c$ and $\Psi_{c,\eta}$. We also remark that Proposition 2.4 implies the following fact. Let $\mathfrak{X}$ be of locally finite type, $\mathscr{Y}$ a closed subset of $\mathfrak{X}_s$, $i$ the canonical morphism $\mathfrak{X}_{/\mathscr{Y}} \to \mathfrak{X}$, $F^\cdot \in D^+(\mathfrak{X}_\eta)$, and $F^\cdot_{/\mathscr{Y}}$ the pullback of $F^\cdot$ on $(\mathfrak{X}_{/\mathscr{Y}})_\eta$. Then there is a canonical isomorphism $Ri^!_{s_K}(R\Theta_K F^\cdot) \tilde{\to} R\Theta_{c,K}(F^\cdot_{/\mathscr{Y}})$.

## 3. The comparison theorem

Let $\mathscr{S}$ be the spectrum of a local Henselian ring which is the ring of integers $k^\circ$ of a field $k$ with a discrete valuation (which is not assumed to be nontrivial), and let $\mathscr{X}$ be a scheme of locally finite type over $\mathscr{S}$. For a subscheme $\mathscr{Y} \subset \mathfrak{X}_s$, let $\widehat{\mathscr{X}_{/\mathscr{Y}}}$ denote the formal completion of $\mathscr{X}$ along $\mathscr{Y}$. Since $\widehat{\mathscr{X}_{/\mathscr{Y}}}$ coincides with the formal completion of $\widehat{\mathscr{X}}$ along $\mathscr{Y}$, it follows that this is a special formal scheme over $\widehat{k}^\circ$. Its closed fibre can be identified with $\mathscr{Y}$, and for the generic fibre one has, by Proposition 1.3, a canonical isomorphism $(\widehat{\mathscr{X}_{/\mathscr{Y}}})_\eta \tilde{\to} \pi^{-1}(\mathscr{Y})$, where $\pi$ is the reduction map $\widehat{\mathscr{X}_\eta} \to \mathscr{X}_s$. Furthermore, for a sheaf $\mathscr{F} \in \mathscr{X}^\sim_{\eta_{\text{ét}}}$, let $\widehat{\mathscr{F}}$ and $\widehat{\mathscr{F}_{/\mathscr{Y}}}$ denote the pullbacks of $\mathscr{F}$ on $\widehat{\mathscr{X}_\eta}$ and $(\widehat{\mathscr{X}_{/\mathscr{Y}}})_\eta$, respectively. The nearby cycles and vanishing cycles functors for $\mathscr{X}$ as well as for $\widehat{\mathscr{X}}$ and $(\widehat{\mathscr{X}_{/\mathscr{Y}}})_\eta$ will be denoted in the same way by $\Theta$ and $\Psi_\eta$.

**Theorem 3.1.** *Let $\mathscr{F}$ be an étale abelian constructible sheaf on $\mathscr{X}_\eta$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *Then for any $q \geq 0$ there are canonical isomorphisms*

$$(R^q\Theta\mathscr{F})\big|_{\mathscr{Y}} \tilde{\to} R^q\Theta(\widehat{\mathscr{F}_{/\mathscr{Y}}}) \text{ and } (R^q\Psi_\eta\mathscr{F})\big|_{\overline{\mathscr{Y}}} \tilde{\to} R^q\Psi_\eta(\widehat{\mathscr{F}_{/\mathscr{Y}}}) .$$

*Remark 3.2.* By the comparison theorem 5.1 from [Ber3], if $\mathscr{Y}$ is open in $\mathscr{X}_s$, then the above isomorphisms take place for arbitrary abelian torsion sheaves. If $\mathscr{Y}$ is arbitrary, the assumption on constructibility is necessary even for torsion sheaves with torsion orders prime to $\mathrm{char}(k)$, and the assumption on torsion orders is necessary even for constructible sheaves and the case $\mathrm{char}(k) = 0$ (see Remark 3.8(iii) and (iv)).

*Proof of Theorem 3.1.* We consider only the case when the valuation on $k$ is nontrivial, and at the end of the proof we indicate the changes that should be done in the trivial valuation case. Since the formation of vanishing cycles sheaves of schemes is compatible with any base change ([SGA4$\frac{1}{2}$], Th. finitude, 3.7), these sheaves don't change if we replace the field $k$ by its completion $\widehat{k}$. It follows that the same is true for the nearby cycles sheaves because the Galois groups of $k$ and $\widehat{k}$ are isomorphic. Thus, we may assume that the field $k$ is complete.

Step 1. *The theorem is true if $\mathcal{X}$ is semi-stable over $\mathcal{S}$, $\mathcal{Y}$ is a union of irreducible components of $\mathcal{X}_s$, and $\mathcal{F} = \Lambda_{\mathcal{X}_\eta}$, where $\Lambda = \mathbf{Z}/n\mathbf{Z}$ and $n$ is prime to* $\mathrm{char}(\widetilde{k})$.

Since the statement is local in the étale topology of $\mathcal{X}$, we may assume that all of the irreducible components of $\mathcal{X}_s$ are smooth. Furthermore, for a point $x \in \mathcal{X}_s$ let $\nu(x)$ denote the number of irreducible components that contain $x$. We prove by induction on $\nu(y)$ that the homomorphisms of the theorem are isomorphisms at an étale neighborhood of the point $y \in \mathcal{Y}$. If $\nu(y) = 1$, then $\mathcal{Y}$ contains an open neighborhood of $y$ in $\mathcal{X}_s$, and therefore the required statement follows from the comparison theorem 5.1 from [Ber3]. Assume that $d = \nu(y) - 1 \geq 1$ and that the statement is true for all points $y' \in \mathcal{Y}$ with $\nu(y') \leq d$. Shrinking $\mathcal{X}$, we may assume that there is an étale morphism $\mathcal{X} \to \mathcal{X}' = \mathrm{Spec}(k^\circ[T_1, \dots, T_{d+1}, S_1, \dots, S_t]/(T_1 \cdot \ldots \cdot T_{d+1} - \varpi))$, where $\varpi$ is a uniformizing element of $k^\circ$, such that the irreducible components of $\mathcal{X}_s$ are the preimages of the irreducible components of $\mathcal{X}'_s$. Therefore we may assume that $\mathcal{X} = \mathcal{X}'$. In this case $\mathcal{X}_s$ is a union of $d+1$ irreducible components $X_1, \dots, X_{d+1}$, where each $X_i$ is defined by the equation $T_i = 0$. Consider the canonical projection $\mathcal{X} \to \mathcal{T} := \mathrm{Spec}(k^\circ[S_1, \dots, S_t])$. Let $\mathcal{Y} = X_1 \cup \ldots \cup X_m$, $1 \leq m \leq d+1$, and let $f$ and $g$ denote the induced morphisms of formal schemes $\widehat{\mathcal{X}} \to \widehat{\mathcal{T}}$ and $\mathfrak{Y} := \widehat{\mathcal{X}_{/\mathcal{Y}}} \to \widehat{\mathcal{T}}$, respectively. By the induction, the cohomology sheaves of the complex $\mathcal{G}^\cdot$ defined by the exact triangle

$$\longrightarrow (R\Theta \Lambda_{\mathcal{X}_\eta})\big|_{\mathcal{Y}} \longrightarrow R\Theta(\Lambda_{\mathfrak{Y}_\eta}) \longrightarrow \mathcal{G}^\cdot \longrightarrow$$

$$(\text{resp.} \quad \longrightarrow (R\Psi_\eta \Lambda_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}} \longrightarrow R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta}) \longrightarrow \mathcal{G}^\cdot \longrightarrow \ )$$

are concentrated on $X_1 \cap \ldots \cap X_{d+1}$ (resp. $\overline{X}_1 \cap \ldots \cap \overline{X}_{d+1}$). Since $g$ induces an isomorphism of the intersection with $\mathcal{T}_s$ (resp. $\mathcal{T}_{\overline{s}}$), to prove the statement it suffices to verify that

$$Rg_{s*}((R\Theta \Lambda_{\mathcal{X}_\eta})\big|_{\mathcal{Y}}) \longrightarrow Rg_{s*}(R\Theta \Lambda_{\mathfrak{Y}_\eta}) = R\Theta(Rg_{\eta*} \Lambda_{\mathfrak{Y}_\eta})$$

$$(\text{resp.} \quad Rg_{\overline{s}*}((R\Psi_\eta \Lambda_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}) \longrightarrow Rg_{\overline{s}*}(R\Psi_\eta \Lambda_{\mathfrak{Y}_\eta}) = R\Psi_\eta(Rg_{\eta*} \Lambda_{\mathfrak{Y}_\eta}) \ )$$

is an isomorphism. Since $R\Theta(\Lambda_{\mathscr{X}_\eta})\widetilde{\to}R\Theta(\Lambda_{\widehat{\mathscr{X}_\eta}})$ (resp. $R\Psi_\eta(\Lambda_{\mathscr{X}_\eta})\widetilde{\to}R\Psi_\eta(\Lambda_{\widehat{\mathscr{X}_\eta}})$), it follows that the above homomorphism is an isomorphism for $m = d + 1$. Thus, to prove our statement, it suffices to verify the following two facts for all $p, q \geq 0$.

(1) $R^p f_{s*}(R^q\Theta\Lambda_{\mathscr{X}_\eta}) \widetilde{\to} R^p g_{s*}((R^q\Theta\Lambda_{\mathscr{X}_\eta})\big|_{\mathscr{Y}})$ and $R^p f_{\bar{s}*}(R^q\Psi_\eta\Lambda_{\mathscr{X}_\eta}) \widetilde{\to} R^p g_{\bar{s}*}((R^q\Psi_\eta\Lambda_{\mathscr{X}_\eta})\big|_{\overline{\mathscr{Y}}})$;

(2) $R^q f_{\eta*}(\Lambda_{\widehat{\mathscr{X}_\eta}})\widetilde{\to}R^q g_{\eta*}(\Lambda_{\mathfrak{Y}_\eta})$.

(1) Recall the description of the nearby cycles and vanishing cycles sheaves of $\mathscr{X}^{\cdot}$ due to M. Rapoport and T. Zink ([RapZi], see also [Ill], 3.2). For a subset $J \subset [1, d + 1]$, we set $X_J = \cap_{i\in J} X_i$ and, for $q \geq 0$, we denote by $\alpha^q$ the canonical morphism $X^q \to \mathscr{X}_s^{\cdot}$, where $X^q$ is the disjoint union of all $X_J$ with $\mathrm{card}(J) = q+1$, and by $f^q$ the composition $f_s \circ \alpha^q : X^q \to \mathscr{X}_s^{\cdot} \to \mathscr{T}_s$. (Notice that $X^q$ is a disjoint union of $r_q := \binom{d+1}{q+1}$ affine spaces of dimension $d$ over $\mathscr{T}_s$.) Then $R^q\Theta(\Lambda_{\mathscr{X}_\eta})\widetilde{\to}\alpha_*^{q-1}(\Lambda_{X^{q-1}})(-q)$ for $q \geq 1$, $\Theta(\Lambda_{\mathscr{X}_\eta}) = \Lambda_{\mathscr{X}_s^{\cdot}}$ and this sheaf has an exact resolution (the Čech resolution defined by the covering $X^0 \to \mathscr{X}_s^{\cdot}$)

$$0 \longrightarrow \Lambda_{\mathscr{X}_s^{\cdot}} \longrightarrow \alpha_*^0(\Lambda_{X^0}) \longrightarrow \alpha_*^1(\Lambda_{X^1}) \longrightarrow \ \ldots$$

and for each $q \geq 0$ the sheaf $R^q\Psi_\eta(\Lambda_{\mathscr{X}_\eta})$ has an exact resolution (induced by the same Čech resolution)

$$0 \longrightarrow R^q\Psi_\eta(\Lambda_{\mathscr{X}_\eta}) \longrightarrow \overline{\alpha}_*^q(\Lambda_{\overline{X}^q})(-q) \longrightarrow \overline{\alpha}_*^{q+1}(\Lambda_{\overline{X}^{q+1}})(-q) \longrightarrow \ \ldots$$

We set $Y^q = \mathscr{Y} \times_{\mathscr{X}_s^{\cdot}} X^q$, and denote by $\beta^q$ and $g^q$ the induced morphisms $Y^q \to \mathscr{Y}$ and $Y^q \to \mathscr{T}_s$. Since $Y^q$ is a disjoint union of the same number $r_q$ of affine spaces over $\mathscr{T}_s$, the universal acyclicity of affine spaces implies that $Rf_*^q(\Lambda_{X^q})\widetilde{\to}Rg_*^q(\Lambda_{Y^q}) = \Lambda_{\mathscr{T}_s}^{r_q}$. The latter gives the isomorphism (1) for the functor $\Theta$ and $q \geq 1$. We now remark that a sheaf $\mathscr{F}$ on $\mathscr{X}_s^{\cdot}$ (resp. $\mathscr{X}_{\bar{s}}^{\cdot}$) possesses the property $Rf_{s*}(\mathscr{F})\widetilde{\to}Rg_{s*}(\mathscr{F}\big|_{\mathscr{Y}})$ (resp. $Rf_{\bar{s}*}(\mathscr{F})\widetilde{\to}Rg_{\bar{s}*}(\mathscr{F}\big|_{\overline{\mathscr{Y}}})$) if it admits a resolution $0 \to \mathscr{F} \to \mathscr{G}^{\cdot}$ such that each $\mathscr{G}^q$, $q \geq 0$, possesses the same property. The isomorphism (1) in all other cases is obtained by applying this remark and the universal acyclicity of affine spaces to the above resolutions of the sheaves $\Theta(\Lambda_{\mathscr{X}_\eta}) = \Lambda_{\mathscr{X}_s^{\cdot}}$ and $R^q\Psi_\eta(\Lambda_{\mathscr{X}_\eta})$.

(2) We may assume that $m \leq d$. One has $\widehat{\mathscr{X}_\eta}\widetilde{\to}\{x \in \mathscr{T}_\eta \times \mathbf{A}^d \,\big|\, |w| \leq |T_1(x)|\cdot\ldots\cdot|T_d(x)| \leq 1\}$ and $\mathfrak{Y}_\eta\widetilde{\to}\{x \in \widehat{\mathscr{X}_\eta} \,\big|\, |T_i(x)| < 1 \text{ for some } 1 \leq i \leq m\}$. Therefore our statement follows from the following lemma (where the valuation on $k$ is not necessarily discrete or nontrivial).

**Lemma 3.3.** *Let $S$ be a $k$-analytic space, $0 < \alpha < 1$ a real number, $1 \leq m \leq d$ integers. We set $X = \{x \in S \times \mathbf{A}^d \,\big|\, \alpha \leq |T_i(x)| \leq 1$ for all $1 \leq i \leq d\}$, $Y = \{x \in X \,\big|\, \alpha \leq |T_1(x)|\cdot\ldots\cdot|T_d(x)| \leq 1\}$ and $Z = \{x \in Y \,\big|\, |T_i(x)| < 1$ for some $1 \leq i \leq m\}$. If $f$, $g$ and $h$ denote the canonical projections $X \to S$, $Y \to S$ and $Z \to S$, respectively, then for any $q \geq 0$ there are canonical isomorphisms*

$$R^q f_*(\Lambda_X) \tilde{\to} R^q g_*(\Lambda_Y) \tilde{\to} R^q h_*(\Lambda_Z) \tilde{\to} (\Lambda_S(-q))^{\binom{d}{q}} .$$

*Proof* . The isomorphism between the first and the last sheaves is obtained from the base change theorem 7.7.1 and the Künneth formula 7.7.3 from [Ber2]. The isomorphism $Rf_*(\Lambda_X) \tilde{\to} Rg_*(\Lambda_Y)$ is verified by induction on $d$. If $d = 1$, then $Y = X$. For $d \geq 2$, consider the projections to the first $d-1$ coordinates $\varphi : X \to X' = \{x \in S \times \mathbf{A}^{d-1} \big| \alpha \leq |T_i(x)| \leq 1 \text{ for all } 1 \leq i \leq d-1\}$ and $\psi : Y \to Y' = \{x \in X' \big| \alpha \leq |T_1(x)| \cdot \ldots \cdot |T_{d-1}(x)| \leq 1\}$. For $y' \in Y'$, one has $\varphi^{-1}(y') = \{x \in \mathbf{A}^1_{\mathscr{H}(y')} \big| \alpha \leq |T_d(x)| \leq 1\}$ and $\psi^{-1}(y') = \{x \in \mathbf{A}^1_{\mathscr{H}(y')} \big| \alpha' \leq |T_d(x)| \leq 1\}$, where $\alpha' = \alpha / (|T_1(y')| \cdot \ldots \cdot |T_{d-1}(y')|)$. It follows that $(R\varphi_* \Lambda_X)\big|_{Y'} \tilde{\to} R\psi_*(\Lambda_Y)$. Since the cohomology sheaves of $R\varphi_*(\Lambda_X)$ are $\Lambda_{X'}$ in dimension zero and $\Lambda_{X'}(-1)$ in dimension one, we can apply induction.

To establish the isomorphism $Rg_*(\Lambda_Y) \tilde{\to} Rh_*(\Lambda_Z)$, consider the projections to the first $m$ coordinates $\varphi : Y \to Y' = \{x \in S \times \mathbf{A}^m \big| \alpha \leq |T_1(x)| \cdot \ldots \cdot |T_m(x)| \leq 1\}$ and $\psi : Z \to Z' = \{x \in Y' \big| |T_i(x)| < 1 \text{ for some } 1 \leq i \leq m\}$. We have $Z = \varphi^{-1}(Z')$, and therefore $(R\varphi_* \Lambda_Y)\big|_{Z'} \tilde{\to} R\psi_*(\Lambda_Z)$. Since $R^q \varphi_*(\Lambda_Y)$ is isomorphic to $(\Lambda_{Y'}(-q))^r$, where $r = \binom{d-m}{q}$, the situation is reduced to the case $m = d$.

We set $W = Y \backslash Z = \{x \in Y \big| |T_i(x)| = 1 \text{ for all } 1 \leq i \leq d\}$ and denote by $\Phi$ the $g$-family of supports ([Ber2], §5.1) such that, for an étale morphism $\varphi : U \to S$, $\Phi(\varphi)$ consists of the closed subsets of $Y \times_S U$ that are contained in $W \times_S U$. Then there is an exact triangle

$$\longrightarrow Rg_{\Phi *}(\Lambda_Y) \longrightarrow Rg_*(\Lambda_Y) \longrightarrow Rh_*(\Lambda_Z) \longrightarrow$$

Thus, our problem is to show that $Rg_{\Phi *}(\Lambda_Y) = 0$. We now set $Y' = \{x \in S \times \mathbf{A}^d \big| |T_i(x)| \leq 1 \text{ for all } 1 \leq i \leq d\}$ and $Z' = \{x \in Y' \big| |T_i(x)| < 1 \text{ for some } 1 \leq i \leq d\}$. Since $W = Y' \backslash Z'$ and $Y$ is a neighborhood of $W$ in $Y'$, our problem is to verify that $Rg'_*(\Lambda_{Y'}) \tilde{\to} Rh'_*(\Lambda_{Z'})$, where $g'$ and $h'$ are the canonical projections $Y' \to S$ and $Z' \to S$. But $\Lambda_S \tilde{\to} Rg'_*(\Lambda_{Y'})$, and therefore we have to verify that $\Lambda_S \tilde{\to} Rh'_*(\Lambda_{Z'})$.

Consider the covering of $Z'$ by the open sets $Z_i = \{x \in Z' \big| |T_i(x)| < 1\}$, $1 \leq i \leq d$. For a subset $J \subset [1, d]$, the intersection $Z_J = \cap_{i \in J} Z_i$ is isomorphic to a direct product of $S$ with the $m$-dimensional open unit disc $D^m$ and the $(d-m)$-dimensional closed unit disc $E^{d-m}$, where $m = \text{card}(J)$. From [Ber2], 7.4.2, it follows that $\Lambda_S \tilde{\to} Rp_*(\Lambda_{Z_J})$, where $p$ is the projection $Z_J \to S$. The spectral sequence of the covering now implies that $\Lambda_S \tilde{\to} Rh'_*(\Lambda_{Z'})$. □

Step 2. *The theorem is true in the general case.*

Since the reasoning is completely the same for the nearby cycles and vanishing cycles functors, we consider only the latter ones. We also remark that the validity of the theorem for sheaves is equivalent to its validity for the corresponding complexes of sheaves. (In the following lemma the valuation on $k$ is not necessarily nontrivial.)

**Lemma 3.4.** *Let $k'$ be a finite extension of $k$, $\mathscr{X}'$ a scheme over $\mathscr{S}' = \mathrm{Spec}(k'^{\circ})$, $f : \mathscr{X}' \to \mathscr{X}$ a proper morphism over $\mathscr{S}$, $\mathscr{Y}'$ the preimage of $\mathscr{Y}$ in $\mathscr{X}'_{s'}$, and $\mathscr{F}^{\cdot}$ a bounded complex of constructible sheaves on $\mathscr{X}'_{\eta'}$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$. *If the theorem is true for the triple $(\mathscr{X}', \mathscr{Y}', \mathscr{F}^{\cdot})$ and the functor $\Psi_{\eta'}$, then it is also true for the triple $(\mathscr{X}, \mathscr{Y}, Rf_{\eta *}(\mathscr{F}^{\cdot}))$ and the functor $\Psi_{\eta}$.*

*Proof* . The situation is easily reduced to the case $k' = k$. Let $g$ denote the induced morphism of the formal schemes $\mathfrak{Y}' := \widehat{\mathscr{X}'}_{/\mathscr{Y}'} \to \mathfrak{Y} := \widehat{\mathscr{X}}_{/\mathscr{Y}}$. We have

$$(R\Psi_{\eta}(Rf_{\eta *}\mathscr{F}^{\cdot}))\big|_{\overline{\mathscr{Y}}} \overset{\alpha}{\widetilde{\to}} (Rf_{\bar{s}*}(R\Psi_{\eta}\mathscr{F}^{\cdot}))\big|_{\overline{\mathscr{Y}}} \overset{\beta}{\widetilde{\to}} Rg_{\bar{s}*}((R\Psi_{\eta}\mathscr{F}^{\cdot})\big|_{\overline{\mathscr{Y}}'})$$
$$\overset{\gamma}{\widetilde{\to}} Rg_{\bar{s}*}(R\Psi_{\eta}\widehat{\mathscr{F}}^{\cdot}_{/\overline{\mathscr{Y}}'}) \overset{\delta}{\widetilde{\to}} R\Psi_{\eta}(Rg_{\eta *}\widehat{\mathscr{F}}^{\cdot}_{/\overline{\mathscr{Y}}'}) ,$$

where $\alpha$ is an isomorphism because $f$ is proper, $\beta$ is an isomorphism by the proper base change theorem for schemes, $\gamma$ is an isomorphism by the assumption, and $\delta$ is an isomorphism by Corollary 2.3. The sheaf $\widehat{\mathscr{F}}^{\cdot}_{/\overline{\mathscr{Y}}'}$ is $\mathscr{F}^{\mathrm{an}\cdot}\big|_{\mathfrak{Y}'_{\eta}}$, and we have

$$Rg_{\eta *}(\mathscr{F}^{\mathrm{an}\cdot}\big|_{\mathfrak{Y}'_{\eta}}) \overset{\alpha}{\widetilde{\to}} (Rf^{\mathrm{an}}_{\eta *}\mathscr{F}^{\mathrm{an}\cdot})\big|_{\mathfrak{Y}_{\eta}} \overset{\beta}{\widetilde{\to}} (Rf_{\eta *}\mathscr{F}^{\cdot})^{\mathrm{an}}\big|_{\mathfrak{Y}_{\eta}} = (\widehat{Rf_{\eta *}\mathscr{F}^{\cdot}})_{/\mathscr{Y}} ,$$

where $\alpha$ is an isomorphism by the weak base change theorem [Ber2], 5.3.6, and $\beta$ is an isomorphism by the comparison theorem for cohomology with compact support [Ber2], 7.1.1. The lemma follows. □

We prove the statement by induction on $d = \dim(\mathscr{X}_{\eta})$. Since the statement is local with respect to $\mathscr{X}$, we may assume that $\mathscr{X}$ is affine. If $j$ is an open immersion of $\mathscr{X}$ in a projective $\mathscr{S}$-scheme $\mathscr{X}'$, then replacing $\mathscr{X}$ by $\mathscr{X}'$, $\mathscr{Y}$ by its closure in $\mathscr{X}'_s$ and $\mathscr{F}$ by $Rj_{\eta *}(\mathscr{F})$, we may assume that $\mathscr{X}$ is projective over $\mathscr{S}$.

2.1. *If $\mathscr{F}$ is concentrated on a closed subscheme of dimension $< d$, then the theorem is true for $\mathscr{F}$.* Indeed, this follows from the induction hypothesis and Lemma 3.4 applied to the scheme theoretic closure of the support of $\mathscr{F}$ in $\mathscr{X}$.

2.2. *If $\mathscr{U}$ is an open dense subset of $\mathscr{X}_{\eta}$ and $j$ is the canonical open embedding $\mathscr{U} \hookrightarrow \mathscr{X}_{\eta}$, then the theorem is true for $\mathscr{F}$ if and only if it is true for $j_!(\mathscr{F}\big|_{\mathscr{U}})$.* Indeed, if $i$ denotes the closed immersion $\mathscr{X}_{\eta} \backslash \mathscr{U} \to \mathscr{X}_{\eta}$, then there is an exact sequence $0 \to j_!(\mathscr{F}\big|_{\mathscr{U}}) \to \mathscr{F} \to i_*(i^*\mathscr{F}) \to 0$, and the statement follows from 2.1.

2.3. *To prove the theorem, it suffices to find for each $\mathscr{F}$ an open dense subset $\mathscr{U} \subset \mathscr{X}_{\eta}$ and an embedding of $\mathscr{F}\big|_{\mathscr{U}}$ in a similar constructible sheaf $\mathscr{G}$ on $\mathscr{U}$ such that the theorem is true for the sheaf $j_!(\mathscr{G})$.* Indeed, if this is true, then by 2.2 we can find for each $m \geq 1$ an open dense subset $\mathscr{U} \subset \mathscr{X}_{\eta}$ and an exact sequence of constructible sheaves with torsion orders prime to $\mathrm{char}(\widetilde{k})$, $0 \to \mathscr{F}\big|_{\mathscr{U}} \to \mathscr{G}^0 \to \ldots \to \mathscr{G}^m$, such that the theorem is true for all of the sheaves $j_!(\mathcal{G}^i)$. Then the theorem is true for $j_!(\mathcal{F}|_{\mathcal{U}})$ and, again by 2.2, the theorem is true for $\mathcal{F}$.

2.4. *It suffices to verify the condition from 2.3 for the case when $\mathcal{X}$ is irreducible, reduced and flat over $\mathcal{S}$, and the sheaf $\mathcal{F}$ is constant.* Indeed, we can find an embedding of $\mathcal{F}$ in a finite direct sum of sheaves of the form $f_*(\Lambda_{\mathcal{Z}})$, where $f : \mathcal{Z} \to \mathcal{X}_\eta$ is a finite morphism. We may assume that all such $\mathcal{Z}$ are reduced, and therefore we can replace them by their normalizations and assume that they are irreducible. Furthermore, we can find for each $\mathcal{Z}$ a flat model $\mathcal{X}'$ over $\mathcal{S}$ projective over $\mathcal{X}$. It follows now from Lemma 3.4 that if the condition from 2.3 holds for each $\mathcal{X}'$ and the sheaf $\Lambda_{\mathcal{X}'_\eta}$, then it also holds for $\mathcal{X}$ and $\mathcal{F}$.

2.5. From the stable reduction theorem of de Jong ([deJ], 4.5) it follows that there exist a finite extension $k'$ of $k$, a scheme $\mathcal{X}'$ projective and strictly semi-stable over $\mathcal{S}' = \text{Spec}(k'^0)$, and a proper, dominant and generically finite morphism $f : \mathcal{X}' \to \mathcal{X}$ over $\mathcal{S}$ such that the preimage of $\mathcal{Y}$ in $\mathcal{X}'_s$ is a union of irreducible components of $\mathcal{X}'_s$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ on $\mathcal{X}'_\eta$ (it is also a constant sheaf). By Step 1 and Lemma 3.4, the theorem is true for the complex $Rf_{\eta*}(\mathcal{F}')$. Furthermore, let $\mathcal{U}$ be a nonempty open subset of $\mathcal{X}_\eta$ such that the induced morphism $g : \mathcal{U}' := f_\eta^{-1}(\mathcal{U}) \to \mathcal{U}$ is finite. Then $g_*(\mathcal{F}'|_{\mathcal{U}'}) \tilde{\to} (Rf_{\eta*}\mathcal{F}')|_{\mathcal{U}}$. From 2.2 it follows that the theorem is true for the sheaf $j_!(g_*(\mathcal{F}'|_{\mathcal{U}'}))$, where $j$ is the open immersion $\mathcal{U} \hookrightarrow \mathcal{X}_\eta$. Thus, the condition from 2.3 is satisfied by the subset $\mathcal{U}$ and the sheaf $g_*(\mathcal{F}'|_{\mathcal{U}'})$, and the theorem is proved in the nontrivial valuation case.

If the valuation on $k$ is trivial, then in Step 1 one should consider the case when $\mathcal{X}$ is smooth and $\mathcal{Y}$ is a strict normal crossing divisor in $\mathcal{X}_s = \mathcal{X}$ (see [deJ], 2.4) and use the cohomological purity theorem (instead of the description of the vanishing cycles sheaves), and in Step 2.5 one should use Theorem 3.1 (instead of Theorem 4.5) from [deJ]. □

**Corollary 3.5.** *Let $\mathcal{X}$ be a scheme of locally finite type over $k^\circ$, $\pi : \widehat{\mathcal{X}_\eta} \to \mathcal{X}_s$ the reduction map, and $\mathcal{F}$ a constructible sheaf on $\mathcal{X}_\eta$ with torsion orders prime to* $\text{char}(\widetilde{k})$. *Then for any subscheme $\mathcal{Y} \subset \mathcal{X}_s$ there are canonical isomorphisms*

$$R\Gamma(\mathcal{Y}, R\Theta\mathcal{F}) \tilde{\to} R\Gamma(\pi^{-1}(\mathcal{Y}), \mathcal{F}^{\text{an}}) \ \text{ and}$$

$$R\Gamma(\overline{\mathcal{Y}}, R\Psi_\eta\mathcal{F}) \tilde{\to} R\Gamma(\overline{\pi^{-1}(\mathcal{Y})}, \mathcal{F}^{\text{an}}) \ .$$

*If, in addition, the closure of $\mathcal{Y}$ in $\mathcal{X}_s$ is proper, then there are canonical isomorphisms*

$$R\Gamma_c(\overline{\mathcal{Y}}, R\Psi_\eta\mathcal{F}) \tilde{\to} R\Gamma_{\overline{\pi^{-1}(\mathcal{Y})}}(\mathcal{X}_\eta^{\text{an}}, \mathcal{F}^{\text{an}}) \ \text{ and}$$

$$R\Gamma_c(\mathcal{Y}, R\Theta\mathcal{F}) \tilde{\to} R\Gamma_{\pi^{-1}(\mathcal{Y})}(\mathcal{X}_\eta^{\text{an}}, \mathcal{F}^{\text{an}}) \ .$$

*Proof* . The first two isomorphisms immediately follow from Theorem 3.1 and Corollary 2.3. Assume that the closure $\mathcal{W}$ of $\mathcal{Y}$ in $\mathcal{X}_s$ is proper. Let $\mathcal{Z}$ be the complement of $\mathscr{Y}$ in $\mathscr{W}$, and let $j$ and $i$ denote the canonical morphisms $\mathscr{Y} \to \mathscr{W}$ and $\mathscr{Z} \to \mathscr{W}$. Then there is an exact triangle

$$\longrightarrow \bar{j}_!((R\Psi_\eta \mathscr{F})\big|_{\overline{\mathscr{Y}}}) \longrightarrow (R\Psi_\eta \mathscr{F})\big|_{\overline{\mathscr{W}}} \longrightarrow i_*((R\Psi_\eta \mathscr{F})\big|_{\overline{\mathscr{Z}}}) \longrightarrow$$

Applying the functor $R\Gamma(\overline{\mathscr{W}}, \cdot)$, we get an exact triangle

$$\longrightarrow R\Gamma_c(\overline{\mathscr{Y}}, R\Psi_\eta \mathscr{F}) \longrightarrow R\Gamma(\overline{\pi^{-1}(\mathscr{W})}, \mathscr{F}^{\mathrm{an}}) \longrightarrow R\Gamma(\overline{\pi^{-1}(\mathscr{Z})}, \mathscr{F}^{\mathrm{an}}) \longrightarrow$$

It remains to notice that $\pi^{-1}(\mathscr{W})$ is an open neighborhood of $\pi^{-1}(\mathscr{Y})$ in $\mathscr{X}_\eta^{\mathrm{an}}$ and to apply [Ber2], 5.2.6. □

In the following corollaries the field $k$ is complete.

**Corollary 3.6.** *Let $\mathscr{T}$ be a scheme of locally finite type over $k^\circ$, $\mathfrak{X}$ a special formal scheme over $\widehat{\mathscr{T}}$ which is locally isomorphic to the formal completion of a scheme of finite type over $\mathscr{T}$ along a subscheme of the closed fibre, $F$ an étale sheaf on $\mathfrak{X}_\eta$ locally in the étale topology of $\mathfrak{X}$ isomorphic to the pullback of a constructible sheaf on $\mathscr{T}_\eta$ with torsion orders prime to* char$(\widetilde{k})$. *Then the cohomology sheaves of the complexes $R\Psi_\eta(F)$ and $R\Theta(F)$ are constructible and, for any subscheme $\mathscr{Y} \subset \mathscr{X}_s$, there are canonical isomorphisms*

$$R\Gamma(\overline{\mathscr{Y}}, R\Psi_\eta F) \tilde{\to} R\Gamma(\overline{\pi^{-1}(\mathscr{Y})}, F) \quad \text{and} \quad R\Gamma(\mathscr{Y}, R\Theta F) \tilde{\to} R\Gamma(\pi^{-1}(\mathscr{Y}), F) \ .$$

*If, in addition, the closure of $\mathscr{Y}$ in $\mathfrak{X}_s$ is proper, then there are canonical isomorphisms*

$$R\Gamma_c(\overline{\mathscr{Y}}, R\Psi_\eta F) \tilde{\to} R\Gamma_{\overline{\pi^{-1}(\mathscr{Y})}}(\mathfrak{X}_{\overline{\eta}}, F) \quad \text{and}$$

$$R\Gamma_c(\mathscr{Y}, R\Theta F) \tilde{\to} R\Gamma_{\pi^{-1}(\mathscr{Y})}(\mathfrak{X}_{\overline{\eta}}, F) \ . \qquad \square$$

**Corollary 3.7.** *Let $\mathfrak{X}$ be a smooth formal scheme over $k^\circ$, $\mathscr{Y}$ a subscheme of $\mathfrak{X}_s$, and $\Lambda = \mathbf{Z}/n\mathbf{Z}$, where $n$ is prime to* char$(\widetilde{k})$. *Then for any $q \geq 0$ there is a canonical isomorphism*

$$H^q(\overline{\mathscr{Y}}, \Lambda) \tilde{\to} H^q(\overline{\pi^{-1}(\mathscr{Y})}, \Lambda) \ .$$

*If, in addition, the closure of $\mathscr{Y}$ in $\mathfrak{X}_s$ is proper, then there is a canonical isomorphism*

$$H^q_c(\overline{\mathscr{Y}}, \Lambda) \tilde{\to} H^q_{\overline{\pi^{-1}(\mathscr{Y})}}(\mathfrak{X}_{\overline{\eta}}, \Lambda) \ . \qquad \square$$

*Remark 3.8* (i) In the case when $\mathscr{Y}$ is proper, the first isomorphism of Corollary 3.5 for the vanishing cycles sheaves can be deduced from the comparison theorem 5.1 from [Ber3] in the way indicated by G. Faltings in [Fal]. Namely, in this case the set $\pi^{-1}(\mathscr{Y})$ is open in $\mathscr{X}_\eta^{\mathrm{an}}$. This implies that the dualizing complex of $\pi^{-1}(\mathscr{Y})$ (see [Ber4], §1) is the restriction of the dualizing complex of $\mathscr{X}_\eta^{\mathrm{an}}$, and therefore one can use the local biduality ([SGA4$\frac{1}{2}$], Th. finitude, 4.3), the fact that the vanishing cycles functor commutes with duality ([Ill], 4.2), Proposition

2.4 above, and the comparison statement for the duality functor ([Ber4], 3.3-3.4). If $\mathcal{X}_\eta$ is smooth, as in [Fal], then instead of the latter fact it is enough to use the Poincaré Duality for schemes and analytic spaces.

(ii) Assume that the valuation on $k$ is trivial. Then $k = k^\circ = \widetilde{k}$, $\mathcal{X}_\eta = \mathcal{X} = \mathcal{X}_s$, and therefore $R\Theta(\mathcal{F}) = \mathcal{F}$ and $R\Psi_\eta(\mathcal{F}) = \overline{\mathcal{F}}$. Theorem 3.1 implies that for any subscheme $\mathcal{Y} \subset \mathcal{X}$ one has $R\Theta(\widehat{\mathcal{F}_{/\mathcal{Y}}}) = \mathcal{F}\big|_{\mathcal{Y}}$ and $R\Psi_\eta(\widehat{\mathcal{F}_{/\mathcal{Y}}}) = \overline{\mathcal{F}}\big|_{\overline{\mathcal{Y}}}$. It follows that $H^q(\mathcal{Y}, \mathcal{F}) = H^q(\pi^{-1}(\mathcal{Y}), \mathcal{F}^{\mathrm{an}})$, $H^q(\overline{\mathcal{Y}}, \mathcal{F}) = H^q(\overline{\pi^{-1}(\mathcal{Y})}, \mathcal{F}^{\mathrm{an}})$ and if, in addition, the closure of $\mathcal{Y}$ in $\mathcal{X}$ is proper then $H^q_c(\mathcal{Y}, \mathcal{F}) = H^q_{\pi^{-1}(\mathcal{Y})}(\mathcal{X}^{\mathrm{an}}_\eta, \mathcal{F}^{\mathrm{an}})$ and $H^q_c(\overline{\mathcal{Y}}, \mathcal{F}) = H^q_{\overline{\pi^{-1}(\mathcal{Y})}}(\mathcal{X}^{\mathrm{an}}_{\overline{\eta}}, \mathcal{F}^{\mathrm{an}})$.

(iii) Let $\mathcal{X}$ be the affine line over $\mathcal{S}$, $\mathcal{Y}$ the zero point of the closed fibre, $\{x_i\}_{i\ge1}$ a sequence of closed points of $\mathcal{X}_\eta$ with $|T(x_i)| < 1$ and $|T(x_i)| \to 1$ as $i \to \infty$, and $\mathcal{F} = \oplus_i \Lambda_{x_i}$ the sky-scraper sheaf on $\mathcal{X}_\eta$, where $\Lambda = \mathbf{Z}/n\mathbf{Z}$. Then $(\Psi_\eta \mathcal{F})\big|_{\mathcal{Y}}$ is the direct sum of $\Lambda$'s taken over all points from $A^1(k^{\mathrm{a}}) = k^{\mathrm{a}}$ that lie over the points $x_i$, $i \ge 1$, and $\Psi_\eta(\widehat{\mathcal{F}_{/\mathcal{Y}}})$ is the corresponding direct product. In particular, they don't coincide. By the way, if the fields $k(x_i)$ are separable over $k$ and $[k(x_i) : k] \to \infty$ as $i \to \infty$, then the action of the Galois group of $k$ on $\Psi_\eta(\widehat{\mathcal{F}_{/\mathcal{Y}}})$ is not continuous.

(iv) Assume that $\mathrm{char}(k) = 0$ and $\mathrm{char}(\widetilde{k}) = p > 0$, and let $\mathcal{X} = \mathrm{Spec}(k^\circ[T_1, T_2]/(T_1T_2 - w))$, $\mathcal{Y} = X_1$ (as in Step 1), and $\Lambda = \mathbf{Z}/p\mathbf{Z}$. Then $(R\Psi_\eta \Lambda_{\mathcal{X}_\eta})\big|_{\mathcal{Y}} \to R\Psi_\eta(\Lambda_{\mathfrak{Y}_\eta})$, where $\mathfrak{Y} = \widehat{\mathcal{X}_{/\mathcal{Y}}}$, is not an isomorphism. Indeed, the reasoning from Step 1 shows that it is an isomorphism if and only if for the canonical morphisms $f : \widehat{\mathcal{X}} \to \widehat{\mathcal{S}}$ and $g : \mathfrak{Y} \to \widehat{\mathcal{S}}$ the conditions (1) and (2) hold. But if $X = \widehat{\mathcal{X}_\eta}$ and $Y = \mathfrak{Y}_\eta$, then $H^1(\overline{X}, \Lambda) \to H^1(\overline{Y}, \Lambda)$ is not an isomorphism (i.e., (2) does not hold). To see this, we may use $\mu_p$ instead of $\Lambda$. One has $X = \{x \in \mathbf{A}^1 \big| |w| \le |T(x)| \le 1\}$ and $Y = \{x \in X \big| |T(x)| < 1\}$. The function $\frac{1}{1-T} = 1 + T + T^2 + \dots$ is invertible on $Y$, but its image in $\mathcal{O}(\overline{Y})^*/\mathcal{O}(\overline{Y})^{*p} \subset H^1(\overline{Y}, \mu_p)$ does not come from $\mathcal{O}(\overline{X})^*/\mathcal{O}(\overline{X})^{*p} = H^1(\overline{X}, \mu_p)$.

## 4. The continuity theorem

Let $\mathcal{S}$ be the spectrum of the same local Henselian ring from §3, $\mathcal{T}$ a scheme of finite type over $\mathcal{S}$, and $\mathcal{F}$ an étale abelian constructible sheaf on $\mathcal{T}_\eta$ with torsion orders prime to $\mathrm{char}(\widetilde{k})$. Furthermore, let $\mathcal{X}$ and $\mathcal{X}'$ be schemes of finite type over $\mathcal{T}$, and let $\mathcal{Y} \subset \mathcal{X}_s$ and $\mathcal{Y}' \subset \mathcal{X}'_s$ be subschemes. From Theorem 3.1 it follows that any morphism of formal schemes $\varphi : \widehat{\mathcal{X}'_{/\mathcal{Y}'}} \to \widehat{\mathcal{X}_{/\mathcal{Y}}}$ over $\widehat{\mathcal{T}}$ induces homomorphisms of sheaves on $\mathcal{Y}'$ and $\overline{\mathcal{Y}}'$, respectively,

$$\theta^q(\varphi, \mathcal{F}) : \varphi_s^*((R^q\Theta \mathcal{F}_{\mathcal{X}_\eta})\big|_{\mathcal{Y}}) \longrightarrow (R^q\Theta \mathcal{F}_{\mathcal{X}'_\eta})\big|_{\mathcal{Y}'} ,$$

$$\theta_\eta^q(\varphi, \mathscr{F}) : \varphi_s^*((R^q\Psi_\eta \mathscr{F}_{\mathscr{X}_\eta})\big|_{\overline{\mathscr{Y}}}) \longrightarrow (R^q\Psi_\eta \mathscr{F}_{\mathscr{X}'_\eta})\big|_{\overline{\mathscr{Y}}'} \ .$$

**Theorem 4.1.** *Given $\mathscr{T}$, $\mathscr{F}$, $\widehat{\mathscr{X}_{/\mathscr{Y}}}$ and $\widehat{\mathscr{X}'_{/\mathscr{Y}'}}$ as above, there exists an ideal of definition $\mathscr{J}'$ of $\widehat{\mathscr{X}'_{/\mathscr{Y}'}}$ such that for any pair of $\widehat{\mathscr{T}}$-morphisms $\varphi, \psi : \widehat{\mathscr{X}'_{/\mathscr{Y}'}} \to \widehat{\mathscr{X}_{/\mathscr{Y}}}$ that coincide modulo $\mathscr{J}'$, one has $\theta^q(\varphi, \mathscr{F}) = \theta^q(\psi, \mathscr{F})$ and $\theta_\eta^q(\varphi, \mathscr{F}) = \theta_\eta^q(\psi, \mathscr{F})$.*

*Remark 4.2.* (i) If one considers only open subschemes of the closed fibres then, by Theorem 8.1 from [Ber3], given $\mathscr{T}$, $\mathscr{F}$ and $\widehat{\mathscr{X}_{/\mathscr{Y}}}$, there exists $n \geq 1$ such that the statement of Theorem 4.1 holds for any $\widehat{\mathscr{X}'_{/\mathscr{Y}'}}$ with $\mathscr{J}'$ generated by the $n$-th power of the maximal ideal of $k^\circ$.

(ii) In [Ber5], §7, the statement of Theorem 4.1 was proved in the case when $k^\circ$ is equicharacteristic and $\mathscr{Y}$ and $\mathscr{Y}'$ are closed points. This was done using a formalism of vanishing cycles for non-Archimedean analytic spaces (similar to that from [SGA7], Exp. XIV, for complex analytic spaces) and the same results from [Ber3] used in the proof of Theorem 4.1.

Before proving Theorem 4.1, we will prove a finiteness result which generalizes Corollaries 5.5 and 5.6 from [Ber3] and, in fact, is deduced from them. Let $k$ be a non-Archimedean field (whose valuation is not assumed to be discrete), and let $\mathscr{T}$ be a scheme of finite type over $k$.

**Definition 4.3.** *A $k$-analytic space $X$ over $\mathscr{T}^{\mathrm{an}}$ is said to be* quasi-algebraic over $\mathscr{T}$ *if each point of $X$ has a neighborhood of the form $V_1 \cup \ldots \cup V_n$, where each $V_i$ is isomorphic over $\mathscr{T}^{\mathrm{an}}$ to an affinoid domain in the analytification of a scheme of finite type over $\mathscr{T}$. (If $\mathscr{T} = \mathrm{Spec}(k^\circ)$, the indication to $\mathscr{T}$ will be omited.)*

It is easy to show (see the proof of Corollary 5.6 from [Ber3]) that if $X$ is quasi-algebraic over $\mathscr{T}$, then any $k$-analytic space $Y$ that admits a quasi-étale morphism $Y \to X$ is also quasi-algebraic over $\mathscr{T}$. For example, any analytic domain in a $k$-analytic space smooth over $\mathscr{T}^{\mathrm{an}}$ is quasi-algebraic over $\mathscr{T}$.

**Proposition 4.4.** *Let $X$ be a compact $k$-analytic space quasi-algebraic over $\mathscr{T}$, and let $F$ be an étale sheaf on $X$ which locally in the quasi-étale topology is isomorphic to the pullback of an abelian constructible sheaf on $\mathscr{T}_\eta$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$*. Assume that the residue field $\widetilde{k}$ is separably closed and that for any prime $l$ dividing a torsion order of $F$ one has $s_l(k) := \dim_{\mathbf{F}_l}(|k^*|/|k^*|^l) < \infty$. Then the groups $H^q(X, F)$, $q \geq 0$, are finite.*

*Proof* . The reasoning from the proof of Corollary 5.6 from [Ber3] reduces the situation to the case when $X$ is an analytic domain in the analytification of a scheme of finite type over $\mathscr{T}$ and $F$ is the pullback of a sheaf on $\mathscr{T}_\eta$. In this case Corollary 5.5 from [Ber3] implies that the groups $H^q(\overline{X}, F)$, where $\overline{X} = X \widehat{\otimes} \widehat{k^s}$, are finite. By the Hochschield-Serre spectral sequence, it suffices to show that the groups $H^p(G, H^q(\overline{X}, F))$, where $G = \mathrm{Gal}(k^s/k)$, are finite. For this we may assume that $F$ is $l$-torsion for some prime $l$. Let $Q$ be the minimal closed invariant subgroup of $G$ such that $M := G/Q$ is a pro-$l$-group. Then the indices of all open subgroups of $Q$ are prime to $l$ and $M \widetilde{\to} \mathbf{Z}_l^s$, where $s = s_l(k)$ (see [Ber2], 2.4.4). It follows that $H^p(G, H^q(\overline{X}, F)) = H^p(M, H^q(\overline{X}, F)^Q)$. Thus, our statement follows from the simple fact that the cohomology groups of $M$ with coefficients in a finite discrete $l$-torsion module are finite. □

*Proof of Theorem 4.1.* First of all, all of the sheaves are constructible and equal to zero for $q > 1 + 2\dim(\mathscr{X}_\eta)$ (see [Ber3], Lemma 8.2). In particular, it suffices to find such $\mathscr{J}'$ separately for each $q$. Furthermore, the both vanishing cycles sheaves are epimorphic images of the pullbacks of the nearby cycles sheaves defined for a certain finite extension of $k$ in $k^s$. Therefore it is enough to consider the nearby cycles sheaves and, of course, we may assume that the residue field of $k$ is separable closed. Let us fix a functor $\mathfrak{U}_s \mapsto \mathfrak{U}$ (resp. $\mathfrak{U}'_s \mapsto \mathfrak{U}'$) which is inverse to the functor from Proposition 2.1(i). From Corollary 3.5 it follows that for any étale morphism of finite type $\mathfrak{U}_s \to \mathscr{Y}$ (resp. $\mathfrak{U}'_s \to \mathscr{Y}'$) the groups $H^q(\mathfrak{U}_\eta, \widehat{\mathscr{F}})$ (resp. $H^q(\mathfrak{U}'_\eta, \widehat{\mathscr{F}})$) are finite. Since the space $\mathfrak{U}_\eta$ (resp. $\mathfrak{U}'_\eta$) is quasi-algebraic over $\mathscr{T}$, Proposition 4.4 implies that for any compact analytic domain $V \subset \mathfrak{U}_\eta$ (resp. $V' \subset \mathfrak{U}'_\eta$) the groups $H^q(V, \widehat{\mathscr{F}})$ (resp. $H^q(V', \widehat{\mathscr{F}})$) are finite. Finally, we may assume that the schemes $\mathscr{X}$ and $\mathscr{X}'$ are affine.

As in the proof of Theorem 8.1 from [Ber3], everything is now reduced to the verification of the following fact. Let $\mathfrak{U} = \mathrm{Spf}(A)$ and $\mathfrak{V} = \mathrm{Spf}(B)$ be special affine formal schemes over $\widehat{\mathscr{T}}$ such that their generic fibres are quasi-algebraic over $\mathscr{T}$ and the groups $H^q(\mathfrak{U}_\eta, \widehat{\mathscr{F}})$ and $H^q(\mathfrak{V}_\eta, \widehat{\mathscr{F}})$ are finite. Then there exists an ideal of definition of $\mathfrak{V}$ such that, for any pair of morphisms $\varphi, \psi : \mathfrak{V} \to \mathfrak{U}$ that coincide modulo this ideal, the induced homomorphisms of finite groups $H^q(\mathfrak{U}_\eta, \widehat{\mathscr{F}}) \to H^q(\mathfrak{V}_\eta, \widehat{\mathscr{F}})$ coincide.

Let $\mathbf{a}$ (resp. $\mathbf{b}$) be the maximal ideal of definition of $A$ (resp. $B$). Then for each $0 < r < 1$ the set $U(r)$ (resp. $V(r)$) of the points $x \in \mathfrak{U}_\eta$ (resp. $\mathfrak{V}_\eta$) with $|f(x)| \le r$ for all $f \in \mathbf{a}$ (resp. $\mathbf{b}$) is an affinoid domain, and $\mathfrak{U}_\eta$ (resp. $\mathfrak{V}_\eta$) is exhausted by $U(r)$ (resp. $V(r)$). From [Ber2], Lemma 6.3.12, it follows that there exists $0 < r < 1$ such that the homomorphism $H^q(\mathfrak{U}_\eta, \widehat{\mathscr{F}}) \to H^q(U(r), \widehat{\mathscr{F}})$ (resp. $H^q(\mathfrak{V}_\eta, \widehat{\mathscr{F}}) \to H^q(V(r), \widehat{\mathscr{F}})$) is injective. By Theorem 7.1 from [Ber3], there exists $\varepsilon \in \mathfrak{E}(U(r))$ with the property that, for any pair of morphisms $f, g : Y \to U(r)$ between analytic spaces over $\widehat{\mathscr{T}_\eta}$ with $d(f, g) < \varepsilon$, the induced homomorphisms $H^q(U(r), \widehat{\mathscr{F}}) \to H^q(Y, \widehat{\mathscr{F}})$ coincide. But we can find $n \ge 1$ such that, for the morphisms of formal schemes $\varphi$ and $\psi$ that coincide modulo $\mathbf{b}^n$, one has $d(\varphi', \psi') < \varepsilon$, where $\varphi'$ and $\psi'$ are the induced morphisms $V(r) \to U(r)$. Theorem 4.1 follows. □

Let $\mathscr{G}(\widehat{\mathscr{X}_{/\mathscr{Y}}}/\widehat{\mathscr{T}})$ denote the group of $\widehat{\mathscr{T}}$-automorphisms of $\widehat{\mathscr{X}_{/\mathscr{Y}}}$ and, for an ideal of definition $\mathscr{J}$ of $\widehat{\mathscr{X}_{/\mathscr{Y}}}$, let $\mathscr{G}_{\mathscr{J}}(\widehat{\mathscr{X}_{/\mathscr{Y}}}/\widehat{\mathscr{T}})$ denote its subgroup consisting of the automorphisms that are trivial modulo $\mathscr{J}$. The following corollary is obtained from Theorem 4.1 using Lemma 8.7 from [Ber3].

**Corollary 4.5.** *Given $\mathcal{T}$ and $\widehat{\mathcal{X}_{/\mathcal{Y}}}$ as above and a $\mathbf{Z}_l$-sheaf $\mathcal{F} = (\mathcal{F}_m)_{m\geq 0}$ on $\mathcal{T}_\eta$, where $l$ is prime to* $\mathrm{char}(\widetilde{k})$*, there exists an ideal of definition $\mathcal{J}$ such that the group $\mathcal{G}_{\mathcal{J}}(\widehat{\mathcal{X}_{/\mathcal{Y}}}/\widehat{\mathcal{T}})$ acts trivially on all of the sheaves $(R^q\Psi_\eta\mathcal{F}_{m,\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}$ and $(R^q\Theta\mathcal{F}_{m,\mathcal{X}_\eta})\big|_{\mathcal{Y}}$, $q\geq 0$, $m\geq 0$.* □

*Remark 4.6.* Assume that $\mathcal{T} = \mathcal{S}$ and $\mathcal{Y}$ is a closed point $x$ in $\mathcal{X}_s$. Then $(R^q\Psi_\eta\mathcal{F})_{\overline{x}} = \varinjlim H^q(\mathcal{X}_{(\overline{x})}\otimes_{(k^{\mathrm{nr}})^\circ} K, \mathcal{F})$, where $\mathcal{X}_{(\overline{x})} = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathcal{X},x})$ is the strict Henselization of $\mathcal{X}$ at a geometric point $\overline{x}$ over $x$, and $K$ runs through finite extensions of $k^{\mathrm{nr}}$ in $k^{\mathrm{s}}$. Laumon proved the statement similar to that of Corollary 4.5 for the action of the automorphism group of $\mathcal{X}_{(\overline{x})}$ over $(k^{\mathrm{nr}})^\circ$ on $(R^q\Psi_\eta\mathcal{F})_{\overline{x}}$ under the assumptions that $k^\circ$ is equicharacteristic and the morphism $\mathcal{X}\to\mathcal{S}$ is smooth outside $x$ (see [Lau], p. 34, 6.3.1). Furthermore, assume that $k^\circ$ is of mixed characteristic, the morphism $\mathcal{X}\to\mathcal{S}$ is of relative dimension one, and $\mathcal{X}_\eta$ is smooth. Under these assumptions, Brylinski proved that, for any $K$ as above, one has $H^q(\mathcal{X}_{(\overline{x})}\otimes_{(k^{\mathrm{nr}})^\circ} K, \mathcal{F})\stackrel{\sim}{\to}H^q(\widehat{\mathcal{X}_{(\overline{x})}}\otimes_{(\widehat{k^{\mathrm{nr}}})^\circ}\widehat{K}, \mathcal{F})$, where $\widehat{\mathcal{X}_{(\overline{x})}} = \mathrm{Spec}(\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{X},x})$, and the statement similar to that of Corollary 4.5 holds for the action of the automorphism group of $\widehat{\mathcal{X}_{(\overline{x})}}$ over $\widehat{k^{\mathrm{nr}}}$ on $H^q(\mathcal{X}_{(\overline{x})}\otimes_{(k^{\mathrm{nr}})^\circ} K, \mathcal{F})$ (see [Bry]).

## 5. The Generalized Krasner Lemma and quasi-affine analytic spaces

In this subsection the valuation of the ground non-Archimedean field $k$ is not assumed to be discrete. Recall that, for an element $f$ of a commutative Banach ring $\mathcal{A}$, $\rho(f)$ denotes the spectral radius of $f$, i.e., $\rho(f) = \max_{x\in\mathcal{M}(\mathcal{A})}|f(x)|$ (see [Ber1], §1.3).

**Theorem 5.1.** *Let $\mathcal{A}$ a $k$-affinoid algebra, $p_1,\dots,p_n>0$, $f_1,\dots,f_m$ elements in $\mathcal{A}\{p_1^{-1}T_1,\dots,p_n^{-1}T_n\}$ for which the algebra*

$$\mathcal{B} = \mathcal{A}\{p_1^{-1}T_1,\dots,p_n^{-1}T_n\}/(f_1,\dots,f_m)$$

*is finite étale over $\mathcal{A}$, and $\alpha_j$ the image of $T_j$ in $\mathcal{B}$, $1\leq j\leq n$. Then there exist positive numbers $r_1,\dots,r_m$, $t_1,\dots,t_n$ and series $\Phi_j\in\mathcal{B}\{r_1^{-1}S_1,\dots,r_m^{-1}S_m\}$ with $\rho(\Phi_j)\leq p_j$, $1\leq j\leq n$, such that, for any homomorphism of affinoid algebras $\sigma:\mathcal{A}\to\mathcal{C}$ that defines a homomorphism $\sigma':\mathcal{B}\to\mathcal{D}:=\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}$, and, for any system of elements $g_1,\dots,g_m$ in $\mathcal{C}\{p_1^{-1}T_1,\dots,p_n^{-1}T_n\}$ with $\rho(g_i-\sigma(f_i))\leq r_i$, the system of elements $\{\gamma_j=\sigma'(\Phi_j)(g_1-\sigma(f_1),\dots,g_m-\sigma(f_m))\}_{1\leq j\leq n}$ in $\mathcal{D}$ is a unique one with the properties*

*(1) $\mathcal{C}\{p_1^{-1}T_1,\dots,p_n^{-1}T_n\}/(g_1,\dots,g_m)\stackrel{\sim}{\to}\mathcal{D}:T_j\mapsto\gamma_j$;*

*(2) $\rho(\gamma_j-\sigma'(\alpha_j))<t_j$ for all $1\leq j\leq n$.*

*Proof* . Step 1. Let $X = \mathcal{M}(\mathcal{A})$ and $U = \mathcal{M}(\mathcal{B})$. For $r_1, \dots, r_m > 0$ we set $X_r = \mathcal{M}(\mathcal{A}_r)$ and $U_r = \mathcal{M}(\mathcal{B}_r)$, where $\mathcal{A}_r = \mathcal{A}\{r_1^{-1}S_1, \dots, r_m^{-1}S_m\}$ and $\mathcal{B}_r = \mathcal{B}\{r_1^{-1}S_1, \dots, r_m^{-1}S_m\}$. We also set $Z_r = \mathcal{M}(\mathcal{E}_r)$, where $\mathcal{E}_r = \mathcal{A}_r\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}/(f_1+S_1, \dots, f_m+S_m)$. *We claim that for sufficiently small* $r_1, \dots, r_m$ *the canonical morphism* $Z_r \to X_r$ *is finite étale.* Indeed, the canonical morphism $U \to X_r$ is a composition of the finite morphism $U \to X$ and the closed immersion $X \to X_r$, and therefore $\mathrm{Int}(U/X_r) = U$. On the other hand, it is a composition of the closed immersion $U \to Z_r$ and the morphism $Z_r \to X_r$. From [Ber1], 2.5.8(iii), it follows that the image of $U$ in $Z_r$ is contained in $\mathrm{Int}(Z_r/X_r)$. It follows that we may decrease $r_1, \dots, r_m$ so that $\mathrm{Int}(Z_r/X_r) = Z_r$. Then Corollary 2.5.13(i) from [Ber1] implies that $Z_r \to X_r$ is a finite morphism. Furthermore, let $\Delta_1, \dots, \Delta_m$ be the $(n \times n)$-minors of the matrix $\left(\frac{\partial f_i}{\partial T_j}\right) = \left(\frac{\partial (f_i+S_i)}{\partial T_j}\right)$. By the assumption, the images of these elements in $\mathcal{B}$ generate the unit ideal, i.e., $\sum_{i=1}^m \varphi_i \Delta_i + \sum_{i=1}^m \psi_i f_i = 1$ for some $\varphi_i, \psi_i \in \mathcal{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}$. It follows that $\sum_{i=1}^m \varphi_i \Delta_i + \sum_{i=1}^m \psi_i (f_i + S_i) = 1 + \sum_{i=1}^m \psi_i S_i$. We can decrease $r_1, \dots, r_m$ and assume that $\rho(\psi_i S_i) < 1$ for all $1 \le i \le m$, and therefore the right hand side of the latter equality is invertible $\mathcal{A}_r\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}$. This means that the images of $\Delta_1, \dots, \Delta_m$ in $\mathcal{E}_r$ generate the unit ideal, i.e., the morphism $Z_r \to X_r$ is finite étale.

Step 2. Let $\mathcal{O}_1 = \mathcal{O}_{Z_r}(U)$ and $\mathcal{O}_2 = \mathcal{O}_{U_r}(U)$ be the algebras of functions analytic in a neighborhood of the image of $U$ in $Z_r$ and $U_r$, respectively, and let $I_1 \subset \mathcal{O}_1$ and $I_2 \subset \mathcal{O}_2$ be the ideals generated by the functions $S_1, \dots, S_m$. The both algebras $\mathcal{O}_1$ and $\mathcal{O}_2$ are finite étale over $\mathcal{O}_{X_r}(X)$, and there are canonical isomorphisms $\mathcal{O}_1/I_1 \widetilde{\to} \mathcal{B}$ and $\mathcal{O}_2/I_2 \widetilde{\to} \mathcal{B}$. From [Ber3], Lemma 7.4, it follows that the pairs $(\mathcal{O}_1, I_1)$ and $(\mathcal{O}_2, I_2)$ are Henselian, and therefore the canonical isomorphism $\mathcal{O}_1/I_1 \widetilde{\to} \mathcal{O}_2/I_2$ is induced by a unique isomorphism $\mathcal{O}_1 \widetilde{\to} \mathcal{O}_2$ over $\mathcal{O}_{X_r}(X)$. *We claim that for sufficiently small* $r_1, \dots, r_m$ *the latter comes from an isomorphism* $U_r \widetilde{\to} Z_r$ *over* $X_r$. For this we need the following proposition.

**Proposition 5.2.** *Let* $X$ *and* $Y$ *be* $k$*-affinoid spaces,* $X' \subset X$ *and* $Y' \subset Y$ *Zariski closed subsets, and assume that* $X' \subset \mathrm{Int}(X)$. *Then*

$$\mathrm{Hom}((Y, Y'), (X, X')) \widetilde{\to} \mathrm{Hom}(\mathcal{O}_X(X'), \mathcal{O}_Y(Y')) .$$

*Here the left hand side is the set of morphisms of germs of* $k$*-analytic spaces (see [Ber2], §3.4), and the right hand side is the set of homomorphisms of* $k$*-algebras that induce, for each* $n \ge 1$*, a bounded homomorphism of* $k$*-affinoid algebras* $\mathcal{O}_X(X')/I(X')^n \to \mathcal{O}_Y(Y')/I(Y')^n$*, where* $I(X')$ *and* $I(Y')$ *are the ideals of the functions that vanish on* $X'$ *and* $Y'$*, respectively.*

**Lemma 5.3.** *Let* $X = \mathcal{M}(\mathcal{A})$ *be a* $k$*-affinoid space,* $X' \subset X$ *a Zariski closed subset, and* $I$ *the ideal of elements of* $\mathcal{A}$ *that vanish on* $X'$. *Then*

*(i)* $I(X') = I\mathcal{O}_X(X')$ *and* $\mathcal{A}/I^n \widetilde{\to} \mathcal{O}_X(X')/I(X')^n$ *for all* $n \ge 1$*; in particular, there is an isomorphism of completions* $\widehat{\mathcal{A}} \widetilde{\to} \widehat{\mathcal{O}_X(X')}$*;*

*(ii) the ring* $\mathcal{O}_X(X')$ *is Noetherian and flat over* $\mathcal{A}$*, and the homomorphism* $\mathcal{O}_X(X') \to \widehat{\mathcal{O}_X(X')}$ *is injective.*

*Proof* . First of all we remark that affinoid neighborhoods of $X'$ in $X$ form a fundamental system of neighborhoods. (If $I$ is generated by elements $f_1, \dots, f_n \in \mathcal{A}$, then these are the Weierstrass domains of the form $V = \{x \in X \big| |f_i(x)| \le r_i, 1 \le i \le n\}$ for $r_1, \dots, r_n > 0$.) In particular, $\mathcal{O}_X(X')$ is flat over $\mathcal{A}$. Let $V$ be an affinoid neighborhood of $X'$. Then the closed $k$-analytic subspace of $X$ defined by the ideal $I^n$ is contained in $V$. This implies that the ideal of elements of $\mathcal{A}_V$ that vanish on $X'$ coincides with $I\mathcal{A}_V$ and $\mathcal{A}/I^n \widetilde{\to} \mathcal{A}_V/I^n\mathcal{A}_V$. Furthermore, let $f$ be a non-zero element of $\mathcal{O}_X(X')$. Then $f$ comes from $\mathcal{A}_V$ for some affinoid neighborhood $V$ of $X'$ and its image in $\mathcal{A}_V/I^n\mathcal{A}_V$ is non-zero for some $n \ge 1$. Since the latter coincides with $\mathcal{O}_X(X')/I(X')^n$, it follows that the homomorphism $\mathcal{O}_X(X') \to \widehat{\mathcal{O}_X(X')}$ is injective. This implies that $\mathbf{a} = \mathcal{O}_X(X') \cap \mathbf{a}\widehat{\mathcal{O}_X(X')}$ for any finitely generated ideal $\mathbf{a}$ of $\mathcal{O}_X(X')$, and therefore the latter ring is Noetherian. □

*Proof of Proposition 5.2.* That the map considered is injective is easy. Let $\alpha : \mathcal{O}_X(X') \to \mathcal{O}_Y(Y')$ be a homomorphism with the required property. Let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$, and $I \subset \mathcal{A}$ and $J \subset \mathcal{B}$ the ideals of elements that vanish on $X'$ and $Y'$, respectively. Since $X' \subset \mathrm{Int}(X)$, we can find an admissible surjective epimorphism $\pi : k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathcal{A} : T_i \mapsto f_i$ with $\max_{x \in X'} |f_i(x)| < r_i$, $1 \le i \le n$. We may also assume that the set $\{f_1, \dots, f_n\}$ contains a set of generators of $I$. We have $\max_{y \in Y'} |\alpha(f_i)(y)| < r_i$, and therefore we can shrink $Y$ and assume that each $\alpha(f_i)$ comes from an element $g_i \in \mathcal{B}$ and the spectral radius of $g_i$ in $\mathcal{B}$ is at most $r_i$. Therefore there is a well defined bounded homomorphism $\gamma : k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathcal{B}$ that takes $T_i$ to $g_i$. Furthermore, the ideal $\mathrm{Ker}(\gamma)$ is generated by a finite number of elements $F_1, \dots, F_m$. Since $\alpha(\pi(F_i)) = 0$, we can shrink $Y$ and assume that $\gamma(F_i) = 0$. Therefore $\gamma$ induces a bounded homomorphism $\beta : \mathcal{A} \to \mathcal{B}$. Since the set $\{f_1, \dots, f_n\}$ contains generators of $I$, it follows that $\beta(I) \subset J$, i.e., the morphism $Y \to X$ induced by $\beta$ takes $Y'$ to $X'$. Thus, $\beta$ induces a homomorphism $\alpha' : \mathcal{O}_X(X') \to \mathcal{O}_Y(Y')$, and we have to verify that $\alpha' = \alpha$. By Lemma 5.3, it suffices to verify that, for each $n \ge 1$, the induced homomorphisms $\mathcal{O}_X(X')I(X')^n \to \mathcal{O}_Y(Y')/I(Y')^n$ coincide. But these are bounded homomorphisms between two $k$-affinoid algebras that coincide on a set of $k$-affinoid generators of the first algebra. It follows that they coincide. □

*Remark 5.4.* If the spaces $X$ and $Y$ in Proposition 5.2 are strictly $k$-affinoid, then the boundness assumption is automatically satisfied because any $k$-homomorphism between strictly $k$-affinoid algebras is bounded. But this is not true for arbitrary $k$-affinoid algebras (see [Ber1], 2.1.13).

To apply Proposition 5.2, we have to verify that the homomorphism $\mathcal{O}_1/I_1^n \to \mathcal{O}_2/I_2^n$, $n \ge 1$, induced by the isomorphism $\mathcal{O}_1 \widetilde{\to} \mathcal{O}_2$ is bounded. For this we remark that this is a homomorphism between finite Banach modules over the

$k$-affinoid algebra $\mathcal{O}/I^n$, where $\mathcal{O} = \mathcal{O}_{X_r}(X)$ and the ideal $I$ is generated by $S_1, \dots, S_m$. By [Ber1], 2.1.9, such a homomorphism is always bounded.

Step 3. By the previous step, there is an isomorphism of $\mathcal{A}_r$-affinoid algebras

$$\mathcal{E}_r = \mathcal{A}_r\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}/(f_1{+}S_1, \dots, f_m{+}S_m) \tilde{\to} \mathcal{B}_r = \mathcal{B}\{r_1^{-1}S_1, \dots, r_m^{-1}S_m\}$$

Let $\Phi_j$ be the image of $T_j$ under this isomorphism. (We remark that $\rho(\Phi_j) \le p_j$.) Furthermore, by Corollary 6.3 and Key Lemma 7.3 from [Ber3], we can find positive numbers $t_1, \dots, t_n$ such that, given a cartesian diagram

$$\begin{array}{ccc} Y = \mathcal{M}(\mathcal{C}) & \longrightarrow & X \\ \uparrow & & \uparrow \\ V = \mathcal{M}(\mathcal{D}) & \xrightarrow{\psi} & U \end{array}$$

any morphism $\psi' : V \to U$ with $\rho(\psi'^*\alpha_j - \psi^*\alpha_j) < t_j$ $1 \le j \le n$, that makes the previous diagram (with $\psi'$ instead of $\psi$) cartesian coincides with $\psi$. Finally, since $\Phi_j(0) = \alpha_j$, we can decrease $r_1, \dots, r_m$ and assume that $\rho(\Phi_j - \alpha_j) < t_j$. *We claim that the numbers $r_1, \dots, r_m$, $t_1, \dots, t_n$ and the series $\Phi_1, \dots, \Phi_m$ satisfy the conditions of our theorem.* Indeed, let $\sigma : \mathcal{A} \to \mathcal{C}$ be a homomorphism of affinoid algebras and $g_1, \dots, g_m$ a system of elements of $\mathcal{C}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}$ with $\rho(g_j - \sigma(f_i)) \le r_i$. Then $\sigma$ extends to a well defined homomorphism $\mathcal{A}_r = \mathcal{A}\{r_1^{-1}S_1, \dots, r_m^{-1}S_m\} \to \mathcal{C}$ that takes $S_i$ to $g_i - \sigma(f_i)$. The isomorphism $\mathcal{E}_r \tilde{\to} \mathcal{B}_r$ gives rise to an isomorphism $\mathcal{E}_r \widehat{\otimes}_{\mathcal{A}_r} \mathcal{C} \tilde{\to} \mathcal{B}_r \widehat{\otimes}_{\mathcal{A}_r} \mathcal{C}$. But the left hand side is $\mathcal{C}\{p_1^{-1}T_1, \dots, p_n^{-1}T_1\}/(g_1, \dots, g_m)$, the right hand side $(\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}_r)\widehat{\otimes}_{\mathcal{A}_r}\mathcal{C} = \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C} = \mathcal{D}$, the isomorphism constructed takes $T_j$ to $\Phi_j(S_1, \dots, S_m) \otimes 1 = \sigma'(\Phi_j)(g_1 - \sigma(f_1), \dots, g_m - \sigma(f_m)) = \gamma_j$, and one has $\rho(\gamma_j - \sigma'(\alpha_j)) \le \rho(\Phi_j - \alpha_j) < t_j$. Thus, the conditions (1) and (2) are satisfied. Assume now that $\gamma_1', \dots, \gamma_m'$ is a system of elements with $\mathcal{C}\{p_1^{-1}T_1, \dots, p_n^{-1}T_1\}/(g_1, \dots, g_m) \tilde{\to} \mathcal{D} : T_j \mapsto \gamma_j'$. Then this system gives rise to an automorphism $\chi$ of $V = \mathcal{M}(\mathcal{D})$ over $Y = \mathcal{M}(\mathcal{C})$ that takes $\gamma_j$ to $\gamma_j'$. But if $\rho(\gamma_j' - \sigma'(\alpha_j)) < t_j$, then $\rho(\chi^*\sigma'(\alpha_j) - \sigma'(\alpha_j)) < t_j$, and therefore $\psi\chi$ should coincide with $\psi$. This implies that $\chi = 1_V$, i.e., $\gamma_j' = \gamma_j$ for all $1 \le j \le n$. The theorem is proved. □

Let $\mathcal{X} = \mathrm{Spec}(A)$ be an affine scheme of finite type over $k$. An affinoid domain $V \subset \mathcal{X}^{\mathrm{an}}$ is said to be *Weierstrass* if there exist elements $f_1, \dots, f_n \in A$ and numbers $r_1, \dots, r_n > 0$ such that $V = \{x \in \mathcal{X}^{\mathrm{an}} \big| |f_i(x)| \le r_i, 1 \le i \le n\}$. Furthermore, an affinoid domain $V \subset \mathcal{X}^{\mathrm{an}}$ is said to be *rational* if there exist elements $f_1, \dots, f_n, g \in A$ and numbers $r_1, \dots, r_n > 0$ such that $g$ does not vanish on $V$ and $V = \{x \in \mathcal{X}^{\mathrm{an}} \big| |f_i(x)| \le r_i |g(x)|, 1 \le i \le n\}$. We remark that such $V$ is a Weierstrass domain in $\mathcal{Y}^{\mathrm{an}}$, where $\mathcal{Y} = \mathrm{Spec}(A[\frac{1}{g}])$. For a subset $\Sigma \subset \mathcal{X}^{\mathrm{an}}$, let $A_{(\Sigma)}$ denote the localization of $A$ with respect to the elements that do not vanish on $\Sigma$. From [Ber1], 2.2.10, it follows easily that an affinoid domain $V = \mathcal{M}(\mathcal{A}_V) \subset \mathcal{X}^{\mathrm{an}}$ is Weierstrass (resp. rational) if and only if the image of $A$ (resp. $A_{(V)}$) in $\mathcal{A}_V$ is everywhere dense. In particular, if $V$ is a Weierstrass (resp. rational) domain in $\mathcal{X}^{\mathrm{an}}$, then any Weierstrass (resp. rational) subdomain of $V$ is a Weierstrass (resp. rational) domain in $\mathscr{X}^{\mathrm{an}}$. Let $\mathscr{T}$ be a scheme of finite type over $k$. (For example $\mathscr{T} = \mathrm{Spec}(k)$.)

**Definition 5.5.** *A $k$-analytic space $X$ over $\mathscr{T}^{\mathrm{an}}$ is said to be* quasi-affine over *$\mathscr{T}$ if every compact subset of $X$ is contained in an affinoid domain which is isomorphic over $\mathscr{T}^{\mathrm{an}}$ to a rational domain in the analytification of an affine scheme of finite type over $\mathscr{T}$.*

**Corollary 5.6.** *Any $k$-analytic space over $\mathscr{T}^{\mathrm{an}}$ that admits a finite étale morphism to a $k$-analytic space quasi-affine over $\mathscr{T}$ is quasi-affine over $\mathscr{T}$.*

*Proof* . It suffices to prove that if $\varphi : Y = \mathscr{M}(\mathscr{B}) \to X = \mathscr{M}(\mathscr{A})$ is a finite étale morphism of $k$-affinoid spaces and $X$ is a Weierstrass domain in $\mathscr{X}^{\mathrm{an}}$, where $\mathscr{X} = \mathrm{Spec}(A)$ is an affine scheme of finite type over $k$, then $Y$ is isomorphic to a Weierstrass domain in $\mathscr{Y}^{\mathrm{an}}$ for some affine scheme $\mathscr{Y} = \mathrm{Spec}(B)$ of finite type over $\mathscr{X}$. For this we represent $\mathscr{B}$ is the form $\mathscr{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}/(f_1, \dots, f_m)$. By Theorem 5.1, the latter algebra does not change if we replace the elements $f_1, \dots, f_m$ by sufficiently close elements. Therefore, since $X$ is a Weierstrass domain in $\mathscr{X}^{\mathrm{an}}$, we may assume that $f_1, \dots, f_m$ are polynomials from $A[T_1, \dots, T_n]$. Consider the affine scheme $\mathscr{Y} = \mathrm{Spec}(B)$, where $B = A[T_1, \dots, T_n]/(f_1, \dots, f_m)$. If $g_j$ is the image of $T_j$ in $B$, then $Y$ is isomorphic to the Weierstrass domain $\{y \in \mathscr{Y}^{\mathrm{an}} \big| |g_j(y)| \le p_j, 1 \le j \le m\}$ in $\mathscr{Y}^{\mathrm{an}}$. $\square$

For example, any finite étale covering of the analytification of an affine scheme of finite type over $k$ is quasi-affine. Furthermore, let $F$ be a local non-Archimedean field. Then the Drinfeld half-plane $\Omega^d$ associated with $F$ is quasi-affine. It follows that the finite étale coverings $\Sigma^{d,n}$ of $\Omega^d \widehat{\otimes} \widehat{F}^{\mathrm{nr}}$ constructed by Drinfeld in [Dr] are quasi-affine $\widehat{F}^{\mathrm{nr}}$-analytic spaces.

*Remark 5.7.* If one restricts oneself with strictly $k$-analytic spaces and the case of nontrivial valuation on $k$, then a version of Theorem 5.1 and the statement of Corollary 5.6 follow from results of R. Elkik ([Elk], Lemma 6 and Theorem 7). In this case it is not necessary to assume that the algebra $\mathscr{B}$ is finite over $\mathscr{A}$ (if one uses the usual rigid analytic notion of étaleness which in this case is equivalent to the notion of quasi-étaleness from [Ber3]).

## 6. A vanishing theorem for quasi-affine analytic spaces

In this subsection we assume that the ground non-Archimedean field $k$ is algebraically closed.

**Theorem 6.1.** *Let $\mathscr{T}$ be a scheme of finite type over $k$, $\mathscr{F}$ an abelian constructible sheaf on $\mathscr{T}$ with torsion orders prime to* $\mathrm{char}(\widetilde{k})$*, and $X$ a paracompact*

*$k$-analytic space quasi-affine over $\mathcal{T}$ and of dimension $d$. Then for any $q > d$ one has $H^q(X, \mathcal{F}^{\mathrm{an}}) = 0$.*

*Proof* . First of all, if the valuation on $k$ is trivial, then using the invariance of the étale cohomology groups under algebraically closed extensions of the ground field ([Ber2], 7.6.1), we can increase $k$ and assume that its valuation is nontrivial. Furthermore, we may assume that $X$ is connected. Then $X$ is a union of an increasing sequence of affinoid domains quasi-affine over $\mathcal{T}$. If $V$ is such a domain then, by Proposition 4.4, the groups $H^q(V, \mathcal{F}^{\mathrm{an}})$ are finite and, by [Ber2], Lemma 6.3.12, $H^q(X, \mathcal{F}^{\mathrm{an}})$ is a projective limit of the groups $H^q(V, \mathcal{F}^{\mathrm{an}})$ over all $V$'s. Hence, we may assume that $X = V$, i.e., that $X$ is a rational domain in $\mathcal{X}^{\mathrm{an}}$, where $\mathcal{X} = \mathrm{Spec}(A)$ is an affine scheme of finite type over $k$ and of dimension $d$, and that $\mathcal{X} = \mathcal{T}$. Replacing $\mathcal{X}$ by an open subscheme, we may assume that $X$ is a Weierstrass domain in $\mathcal{X}^{\mathrm{an}}$, i.e., $X = V_r := \{x \in \mathcal{X}^{\mathrm{an}} \big| |f_i(x)| \le r_i, 1 \le i \le n\}$ for some $f_1, \dots, f_n \in A$ and $r_1, \dots, r_n > 0$, and we can complement the set $\{f_1, \dots, f_n\}$ to a system of generators of $A$ over $k$. One has $X = \cap_{r'>r} V_{r'}$, where the intersection is taken over all $r'_i > r_i$ with $r'_i \in |k^*|$, $1 \le i \le n$. By the Continuity Theorem 4.3.5 from [Ber2], one has $H^q(X, \mathcal{F}^{\mathrm{an}}) = \varinjlim H^q(V_{r'}, \mathcal{F}^{\mathrm{an}})$, and therefore we may assume that all $r_i$ are contained in $|k^*|$. In this case we can multiply $f_i$'s by elements of $k$ and assume that $r_i = 1$ for all $1 \le i \le n$.

Let $\mathbf{a}$ be the kernel of the surjective homomorphism $k[T_1, \dots, T_n] \to A : T_j \mapsto f_j$. It is an ideal of $k[T_1, \dots, T_n]$ generated by polynomials $g_1, \dots, g_m$. Multiplying each $g_i$ by an element of $k$, we may assume that $g_i \in k^\circ[T_1, \dots, T_n]$. We set $\mathbf{b} = \{g \in k^\circ[T_1, \dots, T_n] \big| ag \in (g_1, \dots, g_m)$ for some non-zero $a \in k^\circ\}$ and $B = k^\circ[T_1, \dots, T_n]/\mathbf{b}$. Then $\mathcal{Y} = \mathrm{Spec}(B)$ is an affine scheme flat and of finite type over $k^\circ$ with $\mathcal{Y}_\eta = \mathcal{X}$ and $\widehat{\mathcal{Y}_\eta} \tilde{\to} X$. By a result of M. Raynaud and L. Gruson ([RayGr], Corollary 3.4.7), the scheme $\mathcal{Y}$ is finitely presented over $k^\circ$, and therefore from [EGA4], 12.1.1, it follows that the closed fibre $\mathcal{Y}_s$ is of dimension $d$. By [Ber3], Corollaries 4.5(iii) and 5.3, there is a spectral sequence $E_2^{p,q} = H^p(\mathcal{Y}_s, R^q\Psi_\eta \mathcal{F}) \Longrightarrow H^{p+q}(X, \mathcal{F}^{\mathrm{an}})$, and therefore to prove the theorem it suffices to show that $E_2^{p,q} = 0$ for $p + q > d$. For this we use the results from [SGA4], Exp. XIV, on the cohomological dimension of affine schemes. First of all, these results imply that $d(R^q\Psi_\eta \mathcal{F}) \le d - q$, where $d(\mathcal{G})$ denotes the maximum of $\dim(\overline{\{y\}})$ taken over all points $y \in \mathcal{Y}_s$ with $\mathcal{G}_y \ne 0$ (see [SGA7], Exp. I, Theorem 4.2). Furthermore, if $\mathcal{G}$ is an abelian sheaf on $\mathcal{Y}_s$, then $H^p(\mathcal{Y}_s, \mathcal{G}) = 0$ for $p > d(\mathcal{G})$ ([SGA4], Exp. XIV, Theorem 3.1). The both facts imply that $E_2^{p,q} = 0$ for $p + q > d$. The theorem is proved. □

**Corollary 6.2.** *Let $X$ be a paracompact smooth quasi-affine $k$-analytic space of pure dimension $d$, and $\Lambda$ a finite abelian group of order prime to* $\mathrm{char}(\widetilde{k})$*. Then for any $q < d$ one has $H_c^q(X, \Lambda) = 0$.*

*Proof.* The statement follows from Theorem 6.1 and Poincaré Duality ([Ber2], 7.4.3). □

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