NON-ARCHIMEDEAN ANALYTIC SPACES

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Contents

§1. Introduction

1.1. Fields complete with respect to a valuation.

1.1.1. Definition. A (real) *valuation* on a field k is a function $|\ | : k \to \mathbf{R}_+$ with the following properties:

(1) $|a| = 0$ if and only if $a = 0$;

(2) (multiplicativity) $|ab| = |a| \cdot |b|$;

(3) (triangle axiom) $|a + b| \le |a| + |b|$.

Any valuation $|\ | : k \to \mathbf{R}_+$ defines a metric on k with respect to which the distance between two elements $a, b \in k$ equals $|a - b|$. The completion of k with respect to this metric is a field $\widehat{k}$ which contains k, and it is provided with a valuation $|\ | : \widehat{k} \to \mathbf{R}_+$ that extends that on k. We are mostly interested in complete fields.

1.1.2. Examples. (i) The fields of real and complex numbers $\mathbf{R}$ and $\mathbf{C}$ are complete with respect to the usual Archimedean valuation $|\ |_\infty$. More generally, given a number $1 < \varepsilon \le 1$, $\mathbf{R}$ and $\mathbf{C}$ are complete with respect to the valuation $|\ |_\infty^\varepsilon$.

(ii) The field of formal Laurent power series $k((T))$ over a field k is complete with respect to the following valuation: given $1 < \varepsilon < 1$, $|\sum_{i=-\infty}^{\infty} a_i T^i| = \varepsilon^n$, where n is the minimal integer with $a_n \ne 0$.

(iii) Given a prime number p and a number $0 < \varepsilon < 1$, the field of rational numbers $\mathbf{Q}$ is provided with the following (*p-adic*) valuation $|\ |_{p,\varepsilon}$: $|\frac{a}{b} p^n|_{p,\varepsilon} = \varepsilon^n$, where a and b are nonzero integers prime to p and $n \in \mathbf{Z}$. The completion $\mathbf{Q}_p$ of $\mathbf{Q}$ with respect to $|\ |_{p,\varepsilon}$ does not depend on ε, and is called the *field of p-adic numbers.*

(iv) Any field k is complete with respect to the *trivial valuation* $|\ |_0$ defined as follows: $|0|_0 = 0$ and $|a|_0 = 1$ for $a \ne 0$.

Notice that in the examples (ii)-(iv) the valuation satisfies the stronger (*non-Archimedean*) form of the triangle axiom: $|a+b| \le \max\{|a|,|b|\}$.

1.1.3. Fact (Ostrowski Theorem). Any valuation on the field of rational numbers $\mathbf{Q}$ is either $|\ |_\infty^\varepsilon$ for $0 < \varepsilon \le 1$, or $|\ |_{p,\varepsilon}$ for a prime p and $0 < \varepsilon < 1$, or $|\ |_0$.

1.1.4. Fact. Any field k complete with respect to a valuation is either Archimedean (i.e., it is $\mathbf{R}$ or $\mathbf{C}$ provided with $|\ |_\infty^\varepsilon$ for $0 < \varepsilon \le 1$), or non-Archimedean (i.e., its valuation is non-Archimedean).

1.1.5. Fact (Kürschák, Ostrowski). Let k be a field complete with respect to a non-Archimedean valuation. Then the valuation on k extends in a unique way to any algebraic extension of k, and the completion $\widehat{k^{\rm a}}$ of an algebraic closure $k^{\rm a}$ of k is algebraically closed. If the valuation on k is nontrivial, then the separable closure $k^{\rm s}$ of k in $k^{\rm a}$ is dense in $\widehat{k^{\rm a}}$.

1.1.6. Notation. For a non-Archimedean field k, one sets $k^\circ = \{a \in k \big| |a| \le 1\}$, the *ring of integers* of k. It is a local ring with the maximal ideal $k^{\circ\circ} = \{a \in k \big| |a| < 1\}$. One sets $\widetilde{k} = k^\circ / k^{\circ\circ}$, the *residue field* of k. One also denotes by $|k^*|$ the subgroup of $\mathbf{R}_+^*$ which is the image of k^* with respect to the valuation on k.

1.1.7. Exercise (Gauss Lemma). Let k be a non-Archimedean field, i.e., a field complete with respect to a non-Archimedean valuation $|\ |$. Given a number $r > 0$, we define as follows a real valued function $|\ |_r$ on the field of rational functions $k(T)$: $|\sum_{i=0}^n a_i T^i|_r = \max\limits_{0 \le i \le n} \{|a_i| r^i\}$, and $|\frac{f}{g}|_r = \frac{|f|_r}{|g|_r}$ for $f, g \in k[T] \backslash \{0\}$. Then

(i) the function $|\ |_r$ is a well defined non-Archimedean valuation;

(ii) $k(T)$ is complete with respect to $|\ |_r$ if and only if the valuation on k is trivial and $r \ge 1$;

(iii) Describe the residue field $\widetilde{K}$ and the group $|K^*|$ for the completion K of $k(T)$ with respect to the valuation $|\ |_r$.

1.2. Commutative Banach rings.

1.2.1. Definition. (i) A *Banach norm* on a commutative ring with unity $\mathcal{A}$ is a function $||\ || : \mathcal{A} \to \mathbf{R}_+$ with the following properties:

(1) $||f|| = 0$ if and only if $f = 0$;

(2) $||fg|| \le ||f|| \cdot ||g||$;

(3) $||f+g|| \le ||f|| + ||g||$.

(ii) A *commutative Banach ring* is a commutative ring with unity $\mathcal{A}$ provided with a Banach norm and complete with respect to it (i.e., each Cauchy sequence has a limit).

1.2.2. Examples (Exercise). (i) Every field k complete with respect to a valuation is commutative Banach ring.

(ii) The ring of integers $\mathbf{Z}$ provided with the usual Archimedean norm $|\ |_\infty$ is a commutative Banach ring.

(iii) The ring of integers k° of a finite extension k of $\mathbf{Q}$ provided with the norm $||\ || = \max\{|\ |_{\sigma,\infty}\}$, where σ runs through all embeddings $k \hookrightarrow \mathbf{C}$, is a commutative Banach ring.

(iv) The field of complex numbers $\mathbf{C}$ provided with the Banach norm $||\ || = \max\{|\ |_\infty, |\ |_0\}$ is a commutative Banach ring.

(v) Given a non-Archimedean field k and numbers $r_1, \ldots, r_n > 0$, let $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ (for brevity $k\{r^{-1}T\}$) be the set of all formal power series $f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu$ with the property that $|a_\nu| r^\nu \to 0$ as $|\nu| = \nu_1 + \ldots + \nu_n \to \infty$. Then $k\{r^{-1}T\}$ is a commutative Banach ring with respect to the Banach norm $||f|| = \max\{|a_\nu| r^\nu\}$.

1.2.3. Definition. The *spectrum* $\mathcal{M}(\mathcal{A})$ of a commutative Banach ring $\mathcal{A}$ is the set of all nonzero bounded multiplicative seminorms on $\mathcal{A}$, i.e., functions $|\ | : \mathcal{A} \to \mathbf{R}_+$ with the following properties:

(1) $|fg| = |f| \cdot |g|$;

(2) $|f + g| \le |f| + |g|$;

(3) $|f| \ne 0$ for some $f \in \mathcal{A}$;

(4) there exists $C > 0$ such that $|f| \le C||f||$ for all $f \in \mathcal{A}$.

1.2.4. Exercise. If $|\ |$ possesses the above properties, then $|1| = 1$ and (4) is true with $C = 1$.

Let x be a point of $\mathcal{M}(\mathcal{A})$, and $|\ |_x$ the corresponding seminorm. Then the kernel $\mathrm{Ker}(|\ |_x)$ of $|\ |_x$ is a prime ideal of $\mathcal{A}$. It follows that the quotient $\mathcal{A}/\mathrm{Ker}(|\ |_x)$ has no zero divisors, the seminorm $|\ |_x$ defines a Banach norm on it and extends to a multiplicative Banach norm on its fraction field. The completion of the latter is denoted by $\mathcal{H}(x)$. It is a field complete with respect to a valuation $|\ |$, and we get a bounded character $\mathcal{A} \to \mathcal{H}(x) : f \mapsto f(x)$ such that $|f|_x = |f(x)|$.

1.2.5. Definition. The spectrum $\mathcal{M}(\mathcal{A})$ is provided with the weakest topology with respect to which all functions $\mathcal{M}(\mathcal{A}) \to \mathbf{R}_+$ of the form $x \mapsto |f(x)|$ for some $f \in \mathcal{A}$ are continuous.

1.2.6. Fact. The spectrum $\mathcal{M}(\mathcal{A})$ of a nontrivial (i.e., $\mathcal{A} \ne \{0\}$) commutative Banach ring $\mathcal{A}$ is a nonempty compact topological space.

1.2.7. Exercise. (i) Describe the spectrum of the commutative Banach rings from Example 1.2.2 (in (v) only for the case when the valuation on k is trivial and $n = 1$).

(ii) An element f of a commutative Banach ring $\mathcal{A}$ is invertible if and only if $f(x) \neq 0$ for all $x \in \mathcal{M}(\mathcal{A})$.

(iii) $\mathcal{M}(\mathcal{A}\widehat{\otimes}_k\widehat{k^{\rm a}})/\mathrm{Gal}(k^{\rm a}/k) \xrightarrow{\sim} \mathcal{M}(\mathcal{A})$.

1.2.8. Definition. The *spectral radius* of an element $f \in \mathcal{A}$ is the number

$$\rho(f) = \inf_n \sqrt[n]{||f^n||} \ .$$

1.2.9. Fact. (i) $\rho(f) = \lim\limits_{n\to\infty} \sqrt[n]{||f^n||}$.

(ii) $\rho(f) = \max\limits_{x\in\mathcal{M}(\mathcal{A})} |f(x)|$.

1.2.10. Exercise. The function $f \mapsto \rho(f)$ is a power multiplicative seminorm, i.e., $\rho(1) = 1$, $\rho(f+g) \leq \rho(f) + \rho(g)$, $\rho(fg) \leq \rho(f)\cdot\rho(g)$, and $\rho(f^n) = \rho(f)^n$, which is bounded with respect to the Banach norm, i.e., $\rho(f) \leq ||f||$.

1.3. Analytic spaces over a commutative Banach ring. Let k be a commutative Banach ring.

1.3.1. Definition. The *n-dimensional affine space over k* is the space $\mathbf{A}^n = \mathbf{A}^n_k$ of multiplicative seminorms on the ring of polynomials $\mathcal{A} = k[T_1, \ldots, T_n]$ whose restriction to k is bounded with respect to the Banach norm on k.

As in §1.2, given a point $x \in \mathbf{A}^n$, the corresponding seminorm is denoted by $|\ |_x$, and there is a corresponding bounded character $\mathcal{A} \to \mathcal{H}(x)$ to a field $\mathcal{H}(x)$ complete with respect to a valuation $|\ |$ so that $|f|_x = |f(x)|$ for all $f \in \mathcal{A}$.

1.3.2. Definition. The space $\mathbf{A}^n$ is provided with the weakest topology with respect to which all functions $\mathbf{A}^n \to \mathbf{R}_+$ of the form $x \mapsto |f(x)|$ for $f \in \mathcal{A}$ are continuous.

1.3.3. Exercise. (i) The affine space $\mathbf{A}^n$ is a locally compact topological space.

(ii) There is a canonical continuous map $\mathbf{A}^n \to \mathcal{M}(k)$ whose fiber at a point $x \in \mathcal{M}(k)$ is the affine space $\mathbf{A}^n_{\mathcal{H}(x)}$ over $\mathcal{H}(x)$.

1.3.4. Definition. An *analytic function* on an open subset $\mathcal{U} \subset \mathbf{A}^n$ is a map $f : \mathcal{U} \to \coprod\limits_{x\in\mathcal{U}} \mathcal{H}(x)$ with $f(x) \in \mathcal{H}(x)$ which is a *local limit of rational functions*, i.e., such that every point $x \in \mathcal{U}$ has an open neighborhood $\mathcal{U}'$ in $\mathcal{U}$ with the following property: for every $\varepsilon > 0$, there exist $g, h \in \mathcal{A}$ with $h(x') \neq 0$ and $\left|f(x') - \frac{g(x')}{h(x')}\right| < \varepsilon$ for all $x' \in \mathcal{U}'$.

1.3.5. Exercise. (i) The correspondence that takes an open subset $\mathcal{U} \subset \mathbf{A}^n$ to the set of analytic functions $\mathcal{O}(\mathcal{U})$ on $\mathcal{U}$ is a sheaf of local rings $\mathcal{O}_{\mathbf{A}^n}$.

(ii) If $k = \mathbf{C}$, then $\mathbf{A}^n$ is the vector space $\mathbf{C}^n$ and the sheaf $\mathcal{O}_{\mathbf{A}^n}$ is the sheaf of complex analytic functions on $\mathbf{C}^n$.

(iii) Describe the affine space $\mathbf{A}^n_{\mathbf{R}}$ over the field of real numbers $\mathbf{R}$.

(iv) Describe the sheaf of analytic functions on the zero dimensional affine space $\mathbf{A}^0_{\mathbf{Z}}$ or, more generally, $\mathbf{A}^0_{k^\circ}$ for a finite extension k of $\mathbf{Q}$.

(v) Let $\mathcal{U}$ be the complement in $\mathbf{A}^0_{k^\circ}$ (from (iv)) of the point x_0 that corresponds to the trivial valuation on k°, and let j denote the open embedding $\mathcal{U} \hookrightarrow \mathbf{A}^0_{k^\circ}$. Show that the stalk $(j_*\mathcal{O}_{\mathcal{U}})_{x_0}$ of the sheaf $j_*\mathcal{O}_{\mathcal{U}}$ at the point x_0 coincides with the adele ring of k.

(vi) Describe $\mathbf{A}^1_k$ over a field k provided with the trivial valuation.

1.3.6. Exercise. (i) Let k be an algebraically closed non-Archimedean field. There is a canonical embedding $k \hookrightarrow \mathbf{A}^1 : a \mapsto p_a$ defined by $|f(p_a)| = |f(a)|$ for $f \in \mathcal{A}$. These are points of *type (1)*. Furthermore, let $E = E(a; r)$ be the closed disc in k with center $a \in k$ and radius $r > 0$. Then the function that takes a polynomial $f = \sum_{i=1}^n \alpha_i (T-a)^i$ to $\max\limits_{0 \le i \le n} |\alpha_i| r^i$ is a multiplicative norm on $k[T]$, and so it gives rise to a point p_E. (Show that this point depends only on the disc, and not on its center.) If $r \in |k^*|$, the point p_E is said to be of *type (2)*. If $r \notin |k^*|$, the point p_E is said to be of *type (2)*. Finally, let $\mathcal{E}$ be a family of nested closed discs in k (i.e., any two discs from $\mathcal{E}$ have a nonempty intersection and, in particular, one of them lies in another one). Show that the function $f \mapsto \inf\limits_{E \in \mathcal{E}} |f(p_E)|$ is a multiplicative seminorm on $k[T]$. Let $\sigma = \bigcap\limits_{E \in \mathcal{E}} E$. Show that, if σ is nonempty, it is either a point $a \in k$ and $p_{\mathcal{E}} = p_a$, or it is a closed disc E and $p_{\mathcal{E}} = p_E$. A field k in which the intersection of any family of nested discs is nonempty are said to be *spherically complete*. Thus, if k is not spherically complete and, therefore, there exist $\mathcal{E}$ with $\sigma \ne \emptyset$, we get a new point which is said to be of *type (4)*. Show that any point of the affine line $\mathbf{A}^1$ is of one of the above types (1)-(4).

(ii) If a point $x \in \mathbf{A}^1$ is of type (1), then the stalk $\mathcal{O}_{\mathbf{A}^1,x}$ of $\mathcal{O}_{\mathbf{A}^1}$ at x is the local ring of convergent power series at a. Otherwise, $\mathcal{O}_{\mathbf{A}^1,x}$ is a field whose completion is the field $\mathcal{H}(x)$. If x is of type (2), then the residue field $\widetilde{\mathcal{H}(x)}$ is the field of rational functions in one variable over $\widetilde{k}$ and $|\mathcal{H}(x)^*| = |k^*|$. If x is of type (3), then $\widetilde{\mathcal{H}(x)} = \widetilde{k}$ and $|\mathcal{H}(x)^*|$ is the subgroup of $\mathbf{R}^*_+$ generated by $|k^*|$ and the number r. If x is f type (4), then $\widetilde{\mathcal{H}(x)} = \widetilde{k}$ and $|\mathcal{H}(x)^*| = |k^*|$ (i.e., $\mathcal{H}(x)$ is a so called *immediate extension of* k).

The following definition of an analytic space over k gives reasonable objects at least for fields complete with respect to a nontrivial valuation and the ring of integers $\mathbf{Z}$. But it does not give all possible analytic spaces over non-Archimedean fields.

1.3.7. Definition. (i) A *local model* of an analytic space over k is a locally ringed space $(X, \mathcal{O}_X)$ defined by an open subset $\mathcal{U} \subset \mathbf{A}^n$ and a finite set of analytic functions $f_1, \dots, f_n \in \mathcal{O}(\mathcal{U})$ so that $X = \{x \in \mathcal{U} \big| f_i(x) = 0 \text{ for all } 1 \le i \le n\}$ and $\mathcal{O}_X$ is the restriction of the sheaf $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ to X, where $\mathcal{J}$ is the subsheaf of ideals generated by the functions $f_1, \dots, f_n$.

(ii) An *analytic space over* k is a locally ring space locally isomorphic to a local model.

§2. Affinoid algebras and affinoid spaces

2.1. Affinoid algebras. Let k be a non-Archimedean field. If X is a Banach space over k and Y is a closed subspace of X, the quotient X/Y is provided with the following Banach norm (the *quotient norm*): $||\overline{x}|| = \inf_{x \in \overline{x}} ||x||$.

2.1.1. Definition. A bounded k-linear map between Banach space $f : X \to Y$ is said to be *admissible* if the canonical bijective map $X/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is an isomorphism of Banach spaces.

Here $X/\mathrm{Ker}(f)$ is provided with the quotient norm, and $\mathrm{Im}(f)$ is provided with the norm induced from Y. (In particular, if f is admissible, $\mathrm{Im}(f)$ is closed in Y.) Recall that, by the Banach openness theorem, if the valuation on k is nontrivial, every surjective bounded k-linear map between Banach spaces over k is always admissible.

2.1.2. Definition. A *k-affinoid algebra* is a commutative Banach k-algebra $\mathcal{A}$ for which there exists an admissible epimorphism $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathcal{A}$. If such an epimorphism can be found with $r_1 = \ldots = r_n = 1$, $\mathcal{A}$ is said to be *strictly k-affinoid*.

2.1.3. Fact. (i) Any k-affinoid algebra $\mathcal{A}$ is Noetherian, and all of its ideals are closed.

(ii) The k-affinoid algebra $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is strictly k-affinoid if and only if $r_i \in \sqrt{|k^*|} = \{\alpha \in \mathbf{R}_+^* \big| \alpha^n \in |k^*| \text{ for some } n \ge 1\}$.

2.1.4. Fact. (i) If an element $f \in \mathcal{A}$ is not nilpotent, then there exists a constant $C > 0$ such that $||f^n|| \le C\rho(f)^n$ for all $n \ge 1$.

(ii) If $\mathcal{A}$ is reduced (i.e., it has no nonzero nilpotent elements), then the Banach norm on $\mathcal{A}$ is equivalent to the spectral norm, i.e., there exists a constant $C > 0$ such that $||f|| \le C\rho(f)$ for all $f \in \mathcal{A}$.

2.1.5. Exercise. (i) Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a bounded homomorphism between k-affinoid algebras. Given elements $f_1, \dots, f_n \in \mathcal{B}$ and positive numbers $r_1, \dots, r_n$ with $r_i \ge \rho(f_i)$ for all $1 \le i \le n$, there exists a unique bounded homomorphism $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$ extending φ and sending T_i to f_i, $1 \le i \le n$.

(ii) A k-affinoid algebra $\mathcal{A}$ is strictly k-affinoid if and only if $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$ for all $f \in \mathcal{A}$.

Let $\mathcal{A}$ be a k-affinoid algebra.

2.1.6. Definition. (i) $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ is the commutative Banach algebra of formal power series $f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu$ over $\mathcal{A}$ with $||a_\nu|| r^\nu \to 0$ as $|\nu| \to \infty$, provided with the norm $||f|| = \max_\nu \{||a_\nu|| r^\nu\}$.

(ii) An $\mathcal{A}$-*affinoid algebra* is a commutative Banach $\mathcal{A}$-algebra for which there exists an admissible epimorphism $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$. (Exercise: show that such $\mathcal{B}$ is k-affinoid.)

2.1.7. Definition. A Banach $\mathcal{A}$-module M is said to be *finite* if there exists an admissible epimorphism $A^n \to M$. The category of finite Banach (resp. finite) $\mathcal{A}$-modules is denoted by $\mathrm{Mod}_b^h(\mathcal{A})$ (resp. $\mathrm{Mod}^h(\mathcal{A})$).

2.1.8. Fact. (i) The forgetful functor $\mathrm{Mod}_b^h(\mathcal{A}) \to \mathrm{Mod}^h(\mathcal{A})$ is an equivalence of categories.

(ii) Any $\mathcal{A}$-linear map between finite Banach $\mathcal{A}$-modules is admissible.

(iii) Given $M, N \in \mathrm{Mod}_b^h(\mathcal{A})$ and an $\mathcal{A}$-affinoid algebra $\mathcal{B}$, one has $M \otimes_{\mathcal{A}} N \xrightarrow{\sim} M \widehat{\otimes}_{\mathcal{A}} N \in \mathrm{Mod}_b^h(\mathcal{A})$ and $M \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} M \widehat{\otimes}_{\mathcal{A}} \mathcal{B} \in \mathrm{Mod}_b^h(\mathcal{B})$.

2.1.9. Exercise. A commutative Banach $\mathcal{A}$-algebra $\mathcal{B}$ is said to be *finite* if it is a finite Banach $\mathcal{A}$ module.

(i) Any finite Banach $\mathcal{A}$-algebra is $\mathcal{A}$-affinoid.

(ii) The forgetful functor from the category of finite Banach $\mathcal{A}$-algebras to that of finite $\mathcal{A}$-algebras is an equivalence of categories.

2.1.10. Fact. Suppose that $\mathcal{A}$ is strictly k-affinoid.

(i) Every maximal ideal of $\mathcal{A}$ has finite codimension and, in particular, there is a canonical injective map $\mathrm{Max}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$, where $\mathrm{Max}(\mathcal{A})$ is the set of maximal ideals of $\mathcal{A}$.

(ii) If the valuation on k is nontrivial, the image of $\mathrm{Max}(\mathcal{A})$ is dense in $\mathcal{M}(\mathcal{A})$.

(iii) Every homomorphism $\mathcal{A} \to \mathcal{B}$ to a strictly k-affinoid algebra $\mathcal{B}$ is bounded.

2.1.11. Definition. The category of *k-affinoid spaces* k-$\mathcal{A}ff$ is the category dual to that of k-affinoid algebras with bounded homomorphisms between them.

The k-affinoid space corresponding to a k-affinoid algebra $\mathcal{A}$ is mentioned by its spectrum $X = \mathcal{M}(\mathcal{A})$.

2.1.12. Definition. The *dimension* $\dim(X)$ *of a k-affinoid space* $X = \mathcal{M}(\mathcal{A})$ is the Krull dimension of the algebra $\mathcal{A}\widehat{\otimes}k'$ for some non-Archimedean field k' over k such that $\mathcal{A}\widehat{\otimes}k'$ is strictly k'-affinoid.

2.1.13. Fact. (i) The dimension $\dim(X)$ does not depend on the choice of the field k'.

(ii) For any finite affinoid covering $\{X_i\}_{i\in I}$ of X, one has $\dim(X) = \max_i \dim(X_i)$.

(iii) For any point $x \in X$, one has $\mathrm{cd}_l(\mathcal{H}(x)) \le \mathrm{cd}_l(k) + \dim(X)$.

Here l is a prime integer and $\mathrm{cd}_l(k)$ is the l-cohomological dimension of k, i.e., the minimal integer n (or ∞) such that $H^i(G_k, A) = 0$ for all $i > n$ and all l-torsion discrete G_k-modules A, where G_k is the Galois group of k.

2.2. Affinoid domains. Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space.

2.2.1. Definition. An *affinoid domain* in X is a closed subset $V \subset X$ for which there is a morphism of k-affinoid spaces $\varphi : \mathcal{M}(\mathcal{A}_V) \to X$ with $\mathrm{Im}(\varphi) = V$ and such that, for any morphism of k-affinoid spaces $\psi : Y \to X$ with $\mathrm{Im}(\psi) \subset V$, there exists a unique morphism $Y \to \mathcal{M}(A_V)$ whose composition with φ is ψ. If, in addition, $\mathcal{A}_V$ is a strictly k-affinoid algebra, V is said to be a *strictly affinoid domain.*

2.2.2. Exercise. (i) Given tuples $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_n)$ of elements of $\mathcal{A}$ and $p = (p_1, \dots, p_m)$ and $q = (q_1, \dots, q_n)$ of positive numbers, show that the set $V = X(p^{-1}f, qg^{-1}) = \{x \in X \big| |f_i(x)| \le p_i$, and $|g_j(x)| \ge q_j$ for all $1 \le i \le m$, and $1 \le j \le n\}$ is an affinoid domain with respect to the homomorphism

$$\mathcal{A} \to \mathcal{A}_V = \mathcal{A}\{p_1^{-1}T_1, \dots, p_m^{-1}T_m, q_1S_1, \dots, q_nS_n\}/(T_i - f_i, g_jS_j - 1) \ .$$

Such an affinoid domain is said to be a *Laurent domain.* If $n = 0$, it is said to be a *Weierstrass domain.*

(ii) Given elements $g, f_1, \dots, f_n \in \mathcal{A}$ without common zeros in X and positive numbers $p = (p_1, \dots, p_m)$, show that the set $V = X(p^{-1}\frac{f}{g}) = \{x \in X \big| |f_i(x)| \le p_i|g_i(x)|$ for all $1 \le i \le n\}$ is an affinoid domain with respect to the homomorphism

$$\mathcal{A} \to \mathcal{A}_V = \mathcal{A}\{p^{-1}\frac{f}{g}\} = \mathcal{A}\{p_1^{-1}T_1, \dots, p_m^{-1}T_n\}/(gT_i - f_i) \ .$$

Such an affinoid domain is said to be a *rational domain.*

(iii) Show that the intersection of two affinoid (resp. Weierstrass, resp. Laurent, resp. rational) domains is an affinoid domain of the same type. In particular, every Laurent domain is a rational domain.

(iv) Every point of X has a fundamental system of neighborhoods consisting of Laurent domains.

2.2.3. Fact. Let V be an affinoid domain in X. Then

(i) $\mathcal{M}(\mathcal{A}_V) \stackrel{\sim}{\to} V$;

(ii) $\mathcal{A}_V$ is a flat $\mathcal{A}$-algebra;

(iii) for any point $x \in V$, one has $\mathcal{H}(x) \stackrel{\sim}{\to} \mathcal{H}_V(x)$;

(iv) V is a Weierstrass (resp. rational) domain if and only if the image of $\mathcal{A}$ (resp. $\mathcal{A}_{(V)}$) is dense in $\mathcal{A}_V$, where $\mathcal{A}_{(V)}$ is the localization of $\mathcal{A}$ with respect to the elements that do not vanish at any point of V.

2.2.4. Exercise. If U is a Weierstrass resp. rational) domain in V and V is a Weierstrass resp. rational) domain in X, then U is a Weierstrass resp. rational) domain in X.

2.2.5. Fact. (Gerritzen-Grauert theorem) Every affinoid domain is a finite union of rational domains.

Let $\{V_i\}_{i \in I}$ be a finite affinoid covering of X.

2.2.6. Fact. (i) (Tate's acyclicity theorem) For any finite Banach $\mathcal{A}$-module M, the Čech complex

$$0 \to M \to \prod_i M_{V_i} \to \prod_{i,j} M_{V_i \cap V_j} \to \dots$$

is exact and admissible.

(ii) (Kiehl's theorem). Suppose we are given, for each $i \in I$, a finite $\mathcal{A}_{V_i}$-module M_i and, for each pair $i, j \in I$, an isomorphism of finite $\mathcal{A}_{V_i \cap V_j}$-modules $\alpha_{ij} : M_i \otimes_{\mathcal{A}_{V_i}} \mathcal{A}_{V_i \cap V_j} \stackrel{\sim}{\to} M_j \otimes_{\mathcal{A}_{V_j}} \mathcal{A}_{V_i \cap V_j}$ such that $\alpha_{il}\big|_W = \alpha_{jl}\big|_W \circ \alpha_{ij}\big|_W$, $W = V_i \cap V_j \cap V_l$, for all $i, j, l \in I$. Then there exists a finite $\mathcal{A}$-module M that gives rise to the $\mathcal{A}_{V_i}$-modules M_i and the isomorphisms α_{ij}.

2.2.7. Definition. A morphism of k-affinoid spaces $\varphi : Y \to X$ is said to be an *affinoid domain embedding* if it induces an isomorphism of Y with an affinoid domain in X.

The category of k-affinoid spaces with affinoid domain embeddings as morphisms is denoted by k-$\mathcal{A}ff^{ad}$.

2.3. The relative interior, the Shilov boundary, and the reduction map. Let $\varphi : Y = \mathcal{M}(\mathcal{B}) \to X = \mathcal{M}(\mathcal{A})$ be a morphism of k-affinoid spaces.

2.3.1. Definition. The *relative interior* of the morphism φ is the subset $\mathrm{Int}(Y/X) \subset Y$ that consists of the points $y \in Y$ with the following property: there exists an admissible epimorphism $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B} : T_i \mapsto f_i$ such that $|f_i(y)| < r_i$ for all $1 \le i \le n$. The *relative boundary* of φ is the set $\partial(Y/X) = Y \backslash \mathrm{Int}(Y/X)$. If $X = \mathcal{M}(k)$, one denotes them by $\mathrm{Int}(Y)$ and $\partial(Y)$, respectively.

2.3.2. Fact. (i) $\mathrm{Int}(Y/X)$ is open and $\partial(Y/X)$ is closed in Y.

(ii) For morphisms $Z \stackrel{\psi}{\to} Y \stackrel{\varphi}{\to} X$, one has $\mathrm{Int}(Z/X) = \mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X))$.

(iii) If $\{X_i\}_{i \in I}$ is a finite affinoid covering of X and $Y_i = \varphi^{-1}(X_i)$. Then

$$\mathrm{Int}(Y/X) = \{y \in Y \big| y \in \mathrm{Int}(Y_i/X_i) \text{ for all } i \in I \text{ with } y \in Y_i\} \ .$$

(iv) $\mathrm{Int}(Y/X) = Y$ if and only if φ is a finite morphism (i.e., $\mathcal{B}$ is a finite Banach $\mathcal{A}$-algebra).

(v) If Y is an affinoid domain in X, then $\mathrm{Int}(Y/X)$ coincides with the topological interior of Y in X.

2.3.3. Definition. A closed subset Γ of the spectrum of a commutative Banach algebra $\mathcal{A}$ is called a *boundary* if every element of $\mathcal{A}$ has its maximum in Γ. If there exists a unique minimal boundary, it is said to be the *Shilov boundary* of $\mathcal{A}$, and it is denoted by $\Gamma(\mathcal{A})$.

2.3.4. Fact. Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space.

(i) The Shilov boundary $\Gamma(\mathcal{A})$ exists and finite.

(ii) If $x_0 \in \Gamma(\mathcal{A})$, then for any open neighborhood $\mathcal{U}$ of x, there exist $f \in \mathcal{A}$ and $\varepsilon > 0$ such that $|f(x_0)| = \rho(f)$ and $\{x \in X \big| |f(x)| > \rho(f) - \varepsilon\} \subset \mathcal{U}$.

(iii) For an affinoid domain V in X, one has $\Gamma(\mathcal{A}) \cap V \subset \Gamma(\mathcal{A}_V) \subset \partial(V/X) \cup (\Gamma(\mathcal{A}) \cap V)$.

For a (non-Archimedean) commutative Banach algebra $\mathcal{A}$, the set $\mathcal{A}^\circ = \{f \in \big| \rho(f) \le 1\}$ is a ring and $\mathcal{A}^{\circ\circ} = \{f \in \big| \rho(f) < 1\}$ is an ideal in it. The residue ring $\mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$ is denoted by $\widetilde{\mathcal{A}}$. Every bounded homomorphism of commutative Banach algebras $\varphi : \mathcal{A} \to \mathcal{B}$ induces homomorphisms $\varphi^\circ : \mathcal{A}^\circ \to \mathcal{B}^\circ$ and $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$. For example, for any point $x \in \mathcal{M}(\mathcal{A})$, there is a homomorphism $\widetilde{\chi}_x : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$. Since $\widetilde{\mathcal{H}(x)}$ is a field, $\mathrm{Ker}(\widetilde{\chi}_x)$ is a prime ideal ideal of $\widetilde{\mathcal{A}}$.

2.3.5. Definition. The *reduction map* is the map

$$\pi : \mathcal{M}(\mathcal{A}) \to \mathrm{Spec}(\widetilde{\mathcal{A}}) : x \mapsto \mathrm{Ker}(\widetilde{\chi}_x) \ .$$

2.3.6. Exercise. (i) If $\mathcal{A}$ is Noetherian, the preimage of an open (resp. closed) subset of $\mathrm{Spec}(\widetilde{\mathcal{A}})$ with respect to π is a closed (resp. open) subset of $\mathcal{M}(\mathcal{A})$.

(ii) The preimage of a minimal prime ideal of $\widetilde{\mathcal{A}}$ is nonempty and closed.

2.3.7. Fact. Let $\mathcal{A}$ be a strictly k-affinoid algebra and set $X = \mathcal{M}(\mathcal{A})$ and $\widetilde{X} = \mathrm{Spec}(\widetilde{\mathcal{A}})$.

(i) The reduction map $\pi : X \to \widetilde{X}$ is surjective;

(ii) The preimage $\pi^{-1}(\widetilde{x})$ of the generic point $\widetilde{x}$ of an irreducible component of $\widetilde{X}$ consists of one point $x \in X$, and one has $\widetilde{k}(\widetilde{x}) \widetilde{\to} \widetilde{\mathcal{H}(x)}$, where $\widetilde{k}(\widetilde{x})$ is the fraction field of $\widetilde{\mathcal{A}}/\mathrm{Ker}(\widetilde{\chi}_x)$.

(iii) The Shilov boundary $\Gamma(\mathcal{A})$ coincides with the set of points x from (ii).

(iv) For a morphism of strictly k-affinoid spaces $\varphi : Y = \mathcal{M}(\mathcal{B}) \to X$, one has $\mathrm{Int}(Y/X) = \{y \in Y \big| \widetilde{\chi}_y(\widetilde{\mathcal{B}}) \text{ is integral over } \widetilde{\chi}_y(\widetilde{\mathcal{A}})\}$.

(v) The homomorphism $\mathcal{A} \to \mathcal{B}$ (from (iv) is finite if and only if the homomorphism $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is finite.

2.3.8. Exercise. In the situation of fact 2.3.7, one has $\mathrm{Int}(X) = \pi^{-1}(\mathrm{Max}(\widetilde{\mathcal{A}}))$, where $\mathrm{Max}(\widetilde{\mathcal{A}})$ is the set of maximal ideals of $\widetilde{\mathcal{A}}$.

§3. Analytic spaces.

3.1. The category of k-analytic spaces.

3.1.1. Definition. A family τ of subsets of a topological space X is said to be a *quasinet* if, for every point $x \in X$, there exist $V_1, \dots, V_n \in \tau$ such that $x \in V_1 \cap \dots \cap V_n$ and $V_1 \cup \dots \cup V_n$ is a neighborhood of x.

3.1.2. Exercise. Let τ be a quasinet on a topological space X.

(i) A subset $\mathcal{U} \subset X$ is open if and only if for every $V \in \tau$ the intersection $\mathcal{U} \cap V$ is open in V.

(ii) Suppose that τ consists of compact subsets. Then X is Hausdorff if and only if for any pair $U, V \in \tau$ the intersection $U \cap V$ is compact.

3.1.3. Definition. A family τ of subsets of a topological space X is said to be a *net* if it is a quasinet and, for every pair $U, V \in \tau$, $\tau\big|_{U \cap V}$ is a quasinet on $U \cap V$.

We consider a net τ as a category and denote by $\mathcal{T}$ the canonical functor $\tau \to \mathcal{T}op$ to the category of topological spaces $\mathcal{T}op$. We also denote by $\mathcal{T}^a$ the forgetful functor $k\text{-}\mathcal{A}ff^{ad} \to \mathcal{T}op$ that takes a k-affinoid space to the underlying topological space.

3.1.4. Definition. A *k-analytic space* is a triple (X, A, τ), where X is a locally Hausdorff topological space, τ is a net of compact subsets on X, and A is a *k-affinoid atlas on X with the net τ*, i.e., a pair consisting of a functor $A : \tau \to k\text{-}\mathcal{A}ff^{ad}$ and an isomorphism of functors $\mathcal{T}^a \circ A \widetilde{\to} \mathcal{T}$.

In other words, a k-affinoid atlas $\mathcal{A}$ on X with the net τ is a map which assigns to each $V \in \tau$ a k-affinoid space $\mathcal{M}(\mathcal{A}_V)$ and a homeomorphism $\mathcal{M}(\mathcal{A}_V) \xrightarrow{\sim} V$ and, to each pair $U, V \in \tau$ with $U \subset V$ an affinoid domain embedding $\mathcal{M}(\mathcal{A}_U) \to \mathcal{M}(\mathcal{A}_V)$. These data satisfy natural compatibility conditions (formulate them). To define morphisms between k-analytic spaces, one needs a preliminary work.

3.1.5. Fact. (i) If W is an affinoid domain in some $U \in \tau$, then it is an affinoid domain in any $V \in \tau$ that contains W.

(ii) The family $\overline{\tau}$ of all W's from (i) is a net on X, and there is a unique (up to a unique isomorphism) k-affinoid atlas $\overline{\mathcal{A}}$ with the net $\overline{\tau}$ that extends $\mathcal{A}$.

3.1.6. Definition. A *strong morphism* of k-analytic spaces $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is a pair consisting of a continuous map $\varphi : X \to X'$, such that for each $V \in \tau$ there exists $V' \in \tau'$ with $\varphi(V) \subset V'$, and of a system of compatible morphisms of k-affinoid spaces $\varphi_{V/V'} : (V, \mathcal{A}_V) \to (V', \mathcal{A}'_{V'})$ for all pairs $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subset V'$.

3.1.7. Fact. (i) Any strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ extends in a unique way to a strong morphism $\overline{\varphi} : (X, \overline{\mathcal{A}}, \overline{\tau}) \to (X', \overline{\mathcal{A}}', \overline{\tau}')$.

(ii) for any pair of strong morphisms $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ and $\psi : (X', \mathcal{A}', \tau') \to (X'', \mathcal{A}'', \tau'')$, there is a well defined composition morphism $\psi \circ \varphi : (X, \mathcal{A}, \tau) \to (X'', \mathcal{A}'', \tau'')$ so that one gets a category k-$\widetilde{\mathcal{A}n}$.

3.1.8. Definition. A strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if it induces a homeomorphism $X \xrightarrow{\sim} X'$ and, for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subset V'$, $\varphi_{V/V'}$ is an affinoid domain embedding.

3.1.9. Fact. The system of quasi-isomorphisms in k-$\widetilde{\mathcal{A}n}$ admits calculus of right fractions, i.e., it possesses the following properties:

(1) all identity morphisms are quasi-isomorphisms;

(2) the composition of two quasi-isomorphisms is a quasi-isomorphism;

(3) any diagram $(X, \mathcal{A}, \tau) \xrightarrow{\varphi} (X', \mathcal{A}', \tau') \xleftarrow{g} (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}')$ with quasi-isomorphism g can be complemented to a commutative square with quasi-isomorphism f

$$\begin{array}{ccc} (X, \mathcal{A}, \tau) & \xrightarrow{\varphi} & (X', \mathcal{A}', \tau') \\ \uparrow f & & \uparrow g \\ (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}') \end{array}$$

(4) given two strong morphisms $\varphi, \psi : (X, \mathcal{A}, \tau) \rightrightarrows (X', \mathcal{A}', \tau')$ and a quasi-isomorphism $g : (X', \mathcal{A}', \tau') \to (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}')$ with $g\varphi = g\psi$, there exists a quasi-isomorphism $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X, \mathcal{A}, \tau)$ with $\varphi f = \psi f$. (In this case, one in fact has $\varphi = \psi$.)

3.1.10. Definition. The *category of k-analytic spaces k-$\mathcal{A}n$* is the category of fractions of k-$\widetilde{\mathcal{A}n}$ with respect to the system of quasi-isomorphisms.

3.1.11. Exercise. (i) For a net σ on X, one writes $\sigma \prec \tau$ if $\sigma \subset \overline{\tau}$. Let $\mathcal{A}_\sigma$ denote the restriction of the k-affinoid atlas $\overline{\mathcal{A}}$ to σ. Show that the system of nets σ with $\sigma \prec \tau$ is filtered and, for any k-analytic space $(X', \mathcal{A}', \tau')$, one has

$$\mathrm{Hom}((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau')) = \varinjlim_{\sigma \prec \tau} \mathrm{Hom}_{\widetilde{\mathcal{A}n}}((X, \mathcal{A}_\sigma, \sigma), (X', \mathcal{A}', \tau')) \ .$$

(ii) The functor k-$\mathcal{A}ff \to k$-$\mathcal{A}n : X = \mathcal{M}(\mathcal{A}) \mapsto (X, \mathcal{A}, \{X\})$ is fully faithful.

One can introduce a similar category *st-k-$\mathcal{A}n$* of *strictly k-analytic spaces* using only strictly k-affinoid spaces and strictly affinoid domains.

3.1.12. Fact (Temkin). The canonical functor *st-k-$\mathcal{A}n$* $\to$ k-$\mathcal{A}n$ is fully faithful.

3.2. Analytic domains and G-topology. Let $(X, \mathcal{A}, \tau)$ be a k-analytic space.

3.2.1. Definition. A subset $Y \subset X$ is said to be an *analytic domain* if, for every point $y \in Y$, there exist $V_1, \ldots, V_n \in \overline{\tau}\big|_Y$ such that $y \in V_1 \cap \ldots \cap V_n$ and the set $V_1 \cup \ldots \cup V_n$ is a neighborhood of y in Y.

For example, an open subset of X and a finite union of sets from $\overline{\tau}$ are analytic domains.

3.2.2. Exercise. (i) The property to be an analytic domain does not depend on the choice of the net τ.

(ii) The restriction $\overline{\mathcal{A}}\big|_Y$ of $\overline{\mathcal{A}}$ to $\overline{\tau}\big|_Y$ is a k-affinoid atlas on Y, and the k-analytic space $(Y, \overline{\mathcal{A}}\big|_Y, \overline{\tau}\big|_Y)$ does not depend on the choice of τ up to a canonical isomorphism.

(iii) The canonical morphism $Y \to X$ possesses the following property: any morphism $\varphi : Z \to X$ with $\varphi(Z) \subset Y$ goes through a unique morphism $Z \to Y$.

(iv) The intersection of two analytic domains is an analytic domain, and the preimage of an analytic domain with respect to a morphism of k-analytic space is an analytic domain.

(v) If $\{X_i\}_{i \in I}$ is a family of analytic domains in X which forms a quasinet, then for any k-analytic space X' the following sequence of maps is exact

$$\mathrm{Hom}(X, X') \to \prod_i \mathrm{Hom}(X_i, X') \rightrightarrows \prod_{i,j} \mathrm{Hom}(X_i \cap X_j, X') \ .$$

3.2.3. Definition. An *affinoid domain* in $(X, \mathcal{A}, \tau)$ is an analytic domain isomorphic to a k-affinoid space.

3.2.4. Fact. (i) A subset $Y \subset X$ is an affinoid domain if and only there is a finite covering $\{V_i\}_{i\in I}$ of V by sets from $\overline{\tau}$ with the following properties:

(1) for every pair $i, j \in I$, $V_i \cap V_j \in \overline{\tau}$ and $\mathcal{A}_{V_i}\widehat{\otimes}\mathcal{A}_{V_j} \to \mathcal{A}_{V_i\cap V_j}$ is an admissible epimorphism;

(2) the Banach k-algebra $\mathcal{A}_V = \mathrm{Ker}(\prod_i \mathcal{A}_{V_i} \to \prod_{i,j} \mathcal{A}_{V_i\cap V_j})$ is k-affinoid and $V \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_V)$.

(ii) The family $\widehat{\tau}$ of all affinoid domains is a net, and there exists a unique (up to a canonical isomorphism) k-affinoid atlas $\widehat{\mathcal{A}}$ with the net $\widehat{\tau}$ that extends the atlas $\mathcal{A}$.

The k-affinoid atlas $\widehat{\mathcal{A}}$ is said to be the *maximal k-affinoid atlas* on X. In practice, one does not make a difference between $(X, \mathcal{A}, \tau)$ and the k-analytic spaces isomorphic to it, it is simply denoted by X and is assumed to be endowed by the maximal k-affinoid atlas. For a point $x \in X$, one sets $\mathcal{H}(x) = \varinjlim \mathcal{H}_V(x)$, where the inductive limit is taken over all affinoid domains that contain the point x. (Notice that this inductive system is filtered, and all transition homomorphisms in it are isomorphisms.)

Let X be a k-analytic space. The family of analytic domains in X considered as a category (with inclusions as morphisms) gives rise to a Grothendieck topology generated by the pretopology for which the set of coverings of an analytic domain $Y \subset X$ is formed by families $\{Y_i\}_{i\in I}$ of analytic domains in Y which are quasinets on Y. This Grothendieck topology is called the G-*topology on* X, and the corresponding site is denoted by X_{G}. The G-topology is a natural framework for working with coherent sheaves.

3.2.5. Exercise. Let $\mathbf{A}^n$ be the n-dimensional affine space over k introduced in the Definition 1.3.1, and let τ be the family of the compact subsets $E(0; r)$, $r = (r_1, \dots, r_n) \in (\mathbf{R}_+^*)^n$, where $E(a; r)$ is the closed polydisc of polyradius r with center at $a = (a_1, \dots, a_n) \in k^n$, i.e., $E(a; r) = \{x \in \mathbf{A}^n \big| |(T_i - a_i)(x)| \le r_i \text{ for all } 1 \le i \le n\}$.

(i) The family τ is a net on $\mathbf{A}^n$, and the functor $\tau \to k\text{-}\mathcal{A}ff : E(0; r) \mapsto \mathcal{M}(k\{r^{-1}T\})$ defines a k-analytic space structure on $\mathbf{A}^n$.

(ii) If $X = \mathcal{M}(\mathcal{A})$ is a k-affinoid space, then $\mathrm{Hom}(X, \mathbf{A}^1) \xrightarrow{\sim} \mathcal{A}$;

(iii) the functor on k-$\mathcal{A}n$ that takes a k-analytic space to $\mathrm{Hom}(X, \mathbf{A}^1)$ is a sheaf of rings in the G-topology of X. It is called the *structural sheaf* of X and denoted by $\mathcal{O}_{X_{\mathrm{G}}}$. The restriction of the latter to the usual topology of X is denoted by $\mathcal{O}_X$. Show that the pair $(X, \mathcal{O}_X)$ is a locally ringed space.

3.2.6. Exercise. Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space. Show that, for any finite $\mathcal{A}$-module

M, the correspondence $V \mapsto M \otimes_{\mathcal{A}} \mathcal{A}_V$ gives rise to a sheaf of $\mathcal{O}_{X_{\mathrm{G}}}$-module. It is denoted by $\mathcal{O}_{X_{\mathrm{G}}}(M)$.

3.2.7. Definition. A $\mathcal{O}_{X_{\mathrm{G}}}$-module F is said to be *coherent* if there exists a quasinet τ of affinoid domains such that, for every $V \in \tau$, $F\big|_V$ is isomorphic to $\mathcal{O}_{V_{\mathrm{G}}}(M)$ for some finite $\mathcal{A}_V$-module M.

3.2.8. Exercise. If $X = \mathcal{M}(\mathcal{A})$ is k-affinoid, the correspondence $M \mapsto \mathcal{O}_{X_{\mathrm{G}}}(M)$ gives rise to an equivalence between the category of finite $\mathcal{A}$-modules and that of coherent $\mathcal{O}_{X_{\mathrm{G}}}$-modules.

3.2.9. Definition. (i) A k-analytic space X is said to be *good* if every point of X has an affinoid neighborhood.

(ii) X is said to be *without boundary* (or *closed*) if every point of X lies in the interior $\mathrm{Int}(V)$ of an affinoid domain V (see Definitions 2.3.1 and 3.3.5).

3.2.10. Fact. Suppose that the valuation on k is nontrivial.

(i) The functor $X \mapsto (X, \mathcal{O}_X)$ from the full subcategory of good k-analytic spaces to that of locally ringed spaces is fully faithful.

(ii) The functor of (i) gives rise to an equivalence between the full subcategory of k-analytic spaces without boundary and the category of analytic spaces over k introduced in the Definition 1.3.7.

3.2.11. Definition. An $\mathcal{O}_X$-module $\mathcal{F}$ on a good k-analytic space X is said to be *coherent* if every point of X has an open neighborhood $\mathcal{U}$ such that $F\big|_{\mathcal{U}}$ is isomorphic to the cokernel of a homomorphism of locally free $\mathcal{O}_{\mathcal{U}}$-modules of finite rank. The category of coherent $\mathcal{O}_X$-modules is denoted by $\mathrm{Coh}(X)$.

3.2.12. Fact. Let X be a good k-analytic space.

(i) the functor of restriction to the usual topology gives rise to an equivalence of categories $\mathrm{Coh}(X_{\mathrm{G}}) \xrightarrow{\sim} \mathrm{Coh}(X) : F \mapsto \mathcal{F}$;

(ii) a coherent $\mathcal{O}_{X_{\mathrm{G}}}$-module F is locally free (in X_{G}) if and only if the coherent $\mathcal{O}_X$-module $\mathcal{F}$ is a locally free.

Let $\{X_i\}_{i \in I}$ be a family of k-analytic spaces, and suppose that, for each pair $i, j \in I$, we are given an analytic domain $X_{ij} \subset X_i$ and an isomorphism of k-analytic spaces $\nu_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$ so that $X_{ii} = X_i$, $\nu_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$, and $\nu_{ik} = \nu_{jk} \circ \nu_{ij}$ on $X_{ij} \cap X_{ik}$.

3.2.13. Exercise. Suppose that either (a) all X_{ij} are open in X_i, or (b) for any $i \in I$, all

X_{ij} are closed in X_i and the number of $j \in I$ with $X_{ij} \neq \emptyset$ is finite. Then there exists a k-analytic space X and a family of morphisms $\mu_i : X_i \to X$, $i \in I$, such that:

(1) μ_i is an isomorphism of X_i with an analytic domain in X;

(2) $\{\mu_i(X_i)\}_{i\in I}$ is a covering of X in X_{G};

(3) $\mu_i(X_{ij}) = \mu_i(X_i) \cap \mu_j(X_j)$;

(4) $\mu_i = \mu_j \circ \nu_{ij}$ on X_{ij}.

Moreover, in the case (a), all $\mu_i(X_i)$ are open in X. In the case (b), all $\mu_i(X_i)$ are closed in X and, if all X_i are Hausdorff (resp. paracompact), then X is Hausdorff (resp. paracompact). The k-analytic space X is unique up to a unique isomorphism.

3.2.14. Fact. The category k-$\mathcal{A}n$ admits fiber products.

3.2.15. Exercise. (i) For any non-Archimedean field k' over k, the functor k-$\mathcal{A}ff \to k'$-$\mathcal{A}ff : \mathcal{M}(\mathcal{A}) \mapsto \mathcal{M}(\mathcal{A}\widehat{\otimes}_k k')$ extends to a functor k-$\mathcal{A}n \to k'$-$\mathcal{A}n : X \mapsto X\widehat{\otimes}_k k'$ (the *ground field extension functor*).

(ii) For a point $x \in X$ and an affinoid domain $x \in V \subset X$, let x' denote the point of $V\widehat{\otimes}\mathcal{H}(x)$ that corresponds to the character $\mathcal{A}_V\widehat{\otimes}\mathcal{H}(x) : f \otimes \lambda \mapsto \lambda f(x)$. Show that the point x', considered as a point of $X\widehat{\otimes}\mathcal{H}(x)$, does not depend on the choice of V.

3.2.16. Definition. The *fiber of a morphism* $\varphi : Y \to X$ *at a point* $x \in X$ is the $\mathcal{H}(x)$-analytic space

$$Y_x = (Y\widehat{\otimes}\mathcal{H}(x))\widehat{\otimes}_{X\widehat{\otimes}\mathcal{H}(x)}\mathcal{M}(\mathcal{H}(x)) \ ,$$

where the morphism $\mathcal{M}(\mathcal{H}(x)) \to X\widehat{\otimes}\mathcal{H}(x)$ corresponds to the point x' (from Exercise 3.2.15(ii)).

3.2.17. Exercise. Show that there is a canonical homeomorphism $|Y_x| \stackrel{\sim}{\to} \varphi^{-1}(x)$.

3.3. Classes of morphisms of analytic spaces.

Finite morphisms and closed immersions

3.3.1. Definition. (i) A morphism of k-affinoid spaces $\varphi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$ is said to be *finite* (resp. a *closed immersion*) if the canonical homomorphism $\mathcal{A} \to \mathcal{B}$ makes $\mathcal{B}$ a finite Banach $\mathcal{A}$-algebra (resp. is surjective and admissible).

(ii) A morphism of k-analytic spaces $\varphi : Y \to X$ is said to be *finite* (resp. a *closed immersion*) if there exists a family of affinoid domains $\{V_i\}_{i\in I}$ which is a covering in X_{G} such that all $\varphi^{-1}(V_i) \to V_i$ are finite morphisms (resp. closed immersions) of k-affinoid spaces.

3.3.2. Exercise. (i) If a morphism of k-analytic spaces $\varphi : Y \to X$ is finite (resp. a closed immersion), then for any affinoid domain $V \subset X$ the induced morphism $\varphi^{-1}(V) \to V$ is a finite morphism (resp. a closed immersion) of k-affinoid spaces.

(ii) If φ is finite, the induced map between the underlying topological spaces $|Y| \to |X|$ is compact (i.e., the preimage of a compact is a compact) and has finite fibers, and $\varphi_{G_*}(\mathcal{O}_{Y_G})$ is a coherent $\mathcal{O}_{X_G}$-module.

(iii) If φ is a closed immersion, it induced a homeomorphism between $|Y|$ and its image in in $|X|$, and the homomorphism $\mathcal{O}_{X_G} \to \varphi_{G_*}(\mathcal{O}_{Y_G})$ is surjective.

(iv) The classes of finite morphisms and closed immersions are preserved under composition, any base change, and any ground field extension functor.

3.3.3. Definition. A *closed analytic subspace* of a k-analytic space X is an isomorphism class of closed immersions $Y \to X$. The underlying closed subset of such a subspace is said to be a *Zariski closed subset* of X. The complements of Zariski closed subsets are said to be *Zariski open subsets* of X.

Separated morphisms

3.3.3. Definition. A morphism of k-analytic spaces $\varphi : Y \to X$ is said to be *separated* if the diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ is a closed immersion. If the canonical morphism $X \to \mathcal{M}(k)$ is separated, X is said to be *separated*.

3.3.4. Exercise. (i) If X is separated, the underlying topological space $|X|$ is Hausdorff. If X is good, the converse implication is also true. Give an example of a (non-good) non-separated k-analytic space X with Hausdorff space $|X|$.

(ii) If a morphism $\varphi : Y \to X$ is separated, the map $|Y| \to |X|$ is Hausdorff (i.e., the image of $|Y|$ in $|Y| \times_{|X|} |Y|$ is closed). If both X and Y are good, the converse implication is also true.

(iii) The class of separated morphisms is preserved under composition, any base change and any ground field extension functor.

Proper morphisms

3.3.5. Definition. The *relative interior* of a morphism $\varphi : Y \to X$ is the set $\mathrm{Int}(Y/X)$ consisting of the points $y \in Y$ such that, for any affinoid domain $\varphi(x) \in U \subset X$, there exists an affinoid neighborhood V of y in $\varphi^{-1}(U)$ with $y \in \mathrm{Int}(V/U)$. The *relative boundary* of φ is the set $\partial(Y/X) = Y \backslash \mathrm{Int}(Y/X)$. The morphism φ is said to be *without boundary* (or *closed*) if $\partial(Y/X) = \emptyset$.

For example, any finite morphism is without boundary.

3.3.6. Fact. (i) If Y is an analytic domain in X, then $\mathrm{Int}(Y/X)$ coincides with the topological interior of Y in X.

(ii) If $\{X_i\}_{i\in I}$ is a quasinet of analytic domains in X and $Y_i = \varphi^{-1}(X_i)$, then $\mathrm{Int}(Y/X) = \{y \in Y \big| y \in \mathrm{Int}(Y_i/X_i) \text{ for all } i \in I \text{ with } y \in Y_i\}$.

(iii) The class of morphisms without boundary is preserved under composition, any base change and any ground field extension functor.

(iv) (Temkin) If $\varphi : Y \to X$ is a separated morphism without boundary and X is k-affinoid, then for any affinoid domain $U \subset Y$ there exists a bigger affinoid domain $V \subset Y$ such that $U \subset \mathrm{Int}(V/X)$ and U is a Weierstrass domain in V.

3.3.7. Definition. A morphism $\varphi : Y \to X$ is said to be *proper* if it is compact (i.e., proper as a map of topological spaces) and has no boundary.

For example, any finite morphism is proper.

3.3.8. Fact (Temkin). The class of proper morphisms is preserved under composition, any base change and any ground field extension functor.

Étale and quasi-étale morphisms

Let $\varphi : Y \to X$ be a morphism of k-analytic spaces.

3.3.9. Definition. (i) φ is said to be *finite étale* if it is finite and, for any affinoid domain $U = \mathcal{M}(\mathcal{A}) \subset X$ with $\varphi^{-1}(U) = \mathcal{M}(\mathcal{B})$, $\mathcal{B}$ is a finite étale $\mathcal{A}$-algebra (in the sense of algebraic geometry).

(ii) φ is said to be *étale* if, for every point $y \in Y$, there exist open neighborhoods $y \in \mathcal{V} \subset Y$ and $\varphi(y) \in \mathcal{U} \subset X$ such that φ induces a finite étale morphism $\mathcal{V} \to \mathcal{U}$.

3.3.10. Fact. (i) Every étale morphism is an open map.

(ii) The class of étale morphisms is preserved under composition, any base change and any ground field extension functor.

3.3.11. Definition. A *germ of a k-analytic space* (or simply a *k-germ*) is a pair (X, S) consisting of a k-analytic space X and a subset $S \subset X$. (If $S = \{x\}$, it is denoted by (X, x).) The k-germs form a category k-$\mathcal{G}erms$ with respect to the following sets of morphisms

$$\mathrm{Hom}((Y,T),(X,S)) = \varinjlim \mathrm{Hom}'(\mathcal{V}, X) \ ,$$

where the inductive limit is taken over all open neighborhoods $\mathcal{V}$ of T in Y, and $\mathrm{Hom}'(\mathcal{V}, X)$ is the set of morphisms of k-analytic spaces $\varphi : \mathcal{V} \to X$ with $\varphi(T) \subset S$ (such morphisms are called *representatives* of a morphism $(Y, T) \to (X, S)$.)

3.3.12. Exercise. (i) Let $k\text{-}\widetilde{\mathcal{G}erms}$ denote the category of k-germs in which morphisms from (Y, T) to (X, S) are the morphisms $\varphi : Y \to X$ with $\varphi(T) \subset S$. Then $k\text{-}\mathcal{G}erms$ is the category of fractions of $k\text{-}\widetilde{\mathcal{G}erms}$ with respect to the family of morphisms $\varphi : (Y, T) \to (X, S)$ such that φ induces an isomorphism of Y with an open neighborhood of S in X.

(ii) There is a fully faithful functor $k\text{-}\mathcal{A}n \to k\text{-}\mathcal{G}erms : X \mapsto (X, |X|)$.

(iii) The category $k\text{-}\mathcal{G}erms$ admits fiber products and a ground field extension functor.

3.3.13. Fact. For a point $x \in X$, let $\mathrm{Fét}(X, x)$ denote the category of morphisms $(Y, T) \to (X, x)$ with an étale representative $\varphi : \mathcal{V} \to X$ such that the induced morphism $\mathcal{V} \to \varphi(\mathcal{V})$ is finite. Let also $\mathrm{Fét}(\mathcal{H}(x))$ denote the category of schemes finite and étale over $\mathrm{Spec}(\mathcal{H}(x))$. Then there is an equivalence of categories $\mathrm{Fét}(X, x) \widetilde{\to} \mathrm{Fét}(\mathcal{H}(x))$.

3.3.14. Exercise. Let $\varphi : Y \to X$ be a morphism of k-analytic spaces, and suppose that for a point $y \in Y$ with $x = \varphi(y)$ the maximal purely inseparable extension of $\mathcal{H}(x)$ in $\mathcal{H}(y)$ is dense in $\mathcal{H}(y)$. Then there is an equivalence of categories $\mathrm{Fét}(X, x) \widetilde{\to} \mathrm{Fét}(Y, y)$.

3.3.15. Definition. A morphism $\varphi : Y \to X$ is said to be *quasi-étale* if for every point $y \in Y$ there exist affinoid domains $V_1, \dots, V_n \subset Y$ such that $V_1 \cup \dots \cup V_n$ is a neighborhood of y and each V_i can be identified with an affinoid domain in a k-analytic space étale over X.

3.3.16. Fact. The class of quasi-étale morphisms is preserved under composition, any base change and any ground field extension functor.

Smooth morphisms (see also §4.5)

For a k-analytic space X, one sets $\mathbf{A}^n_X = X \times \mathbf{A}^n$ (the *n-dimensional affine space over X*).

3.3.15. Definition. A morphism $\varphi : Y \to X$ is said to be *smooth at a point $y \in Y$* if there exists an open neighborhood $y \in \mathcal{V} \subset X$ such that the induced morphism $\mathcal{V} \to X$ can be represented as a composition of an étale morphism $\mathcal{V} \to \mathbf{A}^n_X$ with the canonical projection $\mathbf{A}^n_X \to X$. φ is said to be *smooth* if it is smooth at all points of Y. If the canonical morphism $X \to \mathcal{M}(k)$ is smooth, X is said to be *smooth*.

3.3.16. Exercise. The class of smooth morphisms is preserved under composition, any base change and any ground field extension functor.

3.3.17. Definition. A strictly k-analytic space X is said to be *rig-smooth* if, for any connected strictly affinoid domain V the sheaf of differentials Ω_V is a locally free $\mathcal{O}_V$-module of rank $\dim(\mathcal{A}_V)$, the Krull dimension of $\mathcal{A}_V$.

3.3.18. Fact. A strictly k-analytic space X is smooth if and only if it is rig-smooth and has no boundary.

3.4. Topological properties of analytic spaces. Let X be a k-analytic space.

3.4.1. Fact. (i) Every point of X has a fundamental system of open neighborhoods which are locally compact, arcwise connected and countable at infinity.

(ii) If X is paracompact, the topological dimension of X is at most $\dim(X)$. If, in addition, X is strictly k-analytic, both numbers are equal.

3.4.2. Exercise. (i) For $r = (r_1, \ldots, r_n) \in \mathbf{R}_+^n$, let p_r denote the point of the affine space $\mathbf{A}^n$ that corresponds to the following multiplicative seminorm on $k[T] = k[T_1, \ldots, T_n]$: if $f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu$, then $|f(p_r)| = \max_\nu |a_\nu| r^\nu$. Show that the map $\mathbf{R}_+^n \to \mathbf{A}^n : r \mapsto p_r$ is continuous and induces a homeomorphism of $\mathbf{R}_+^n$ with a closed subset of $\mathbf{A}^n$.

(ii) Let $\Phi : \mathbf{A}^n \times [0,1] \to \mathbf{A}^n$ be the map that takes a pair (x,t) to the point x_t that corresponds to the following multiplicative seminorm on $k[T]$: $|f(x_t)| = \max_i |\partial_i f(x)| t^i$, where ∂_i for $i = (i_1, \ldots, i_n) \in \mathbf{Z}_+^n$ denotes the operator $\frac{1}{i!} T^i \frac{\partial^i}{dT^i} : k[T] \to k[T]$, i.e.,

$$\partial_i(\sum_\nu a_\nu T^\nu) = \sum_{\nu=i}^{\infty} \binom{\nu}{i} a_\nu T^\nu \ .$$

Show that Φ is a strong deformation retraction of $\mathbf{A}^n$ to the image of $\mathbf{R}_+^n$ in it.

(iii) Show that any nonempty open subset $\mathcal{U} \subset \mathbf{A}^n$, whose complement is a union of Zariski closed subsets, contains the image of $\mathbf{R}_+^n$ and is preserved under the retraction Φ, i.e., Φ induces a strong deformation of $\mathcal{U}$ to the image of $\mathbf{R}_+^n$. (A subset of $\mathbf{A}^n$ is Zariski closed if it is the set of zeros of a family of entire analytic functions.)

Suppose that the valuation on k is nontrivial.

3.4.3. Definition. A k-analytic space is said to be *locally embeddable in a smooth space* if each point has an open neighborhood isomorphic to a strictly analytic domain in a smooth k-analytic space. (Such a space is automatically strictly k-analytic.)

3.4.4. Fact. Any k-analytic space locally embeddable in a smooth space is locally contractible.

Here is a stronger form of Fact 3.4.4.

3.4.5. Fact. Let X be a k-analytic space locally embeddable in a smooth space. Then every point $x \in X$ has a fundamental system of open neighborhoods $\mathcal{U}$ which possess the following properties:

(a) there is a contraction Φ of $\mathcal{U}$ to a point $x_0 \in \mathcal{U}$;

(b) there is an increasing sequence of compact strictly analytic domains $X_1 \subset X_2 \subset \dots$ which are preserved under Φ and such that $\mathcal{U} = \bigcup_{i=1}^{\infty} X_i$;

(c) given a non-Archimedean field K over k, $\mathcal{U}\widehat{\otimes}K$ has a finite number of connected components, and Φ lifts to a contraction of each of the connected components to a point over x_0;

(d) there is a finite separable extension k' of k such that, if K from (c) contains k', then the map $\mathcal{U}\widehat{\otimes}K \to \mathcal{U}\widehat{\otimes}k'$ induces a bijection between the sets of connected components.

§4. Analytic spaces associated to algebraic varieties and formal schemes

4.1. GAGA. Let $\mathcal{X}$ be a scheme of locally finite type over k, and let $\Phi_{\mathcal{X}}$ be the functor from the category k-$\mathcal{A}n$ to the category of sets that takes a k-analytic space Y to the set of morphisms of locally ringed space $\mathrm{Hom}((Y, \mathcal{O}_X), (\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$.

4.1.1. Fact. The functor $\Phi_{\mathcal{X}}$ is representable by a K-analytic space without boundary $\mathcal{X}^{\mathrm{an}}$ and a morphism $\pi : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$. They possess the following properties:

(1) For any non-Archimedean field K over k, there is a bijection $\mathcal{X}^{\mathrm{an}}(K) \stackrel{\sim}{\to} \mathcal{X}(K)$.

(2) The map π is surjective and induces a bijection $\mathcal{X}_0^{\mathrm{an}} \stackrel{\sim}{\to} \mathcal{X}_0$.

(3) For any point $x \in \mathcal{X}^{\mathrm{an}}$, the local homomorphism $\pi_x : \mathcal{O}_{\mathcal{X},\pi(x)} \to \mathcal{O}_{\mathcal{X}^{\mathrm{an}},x}$ is faithfully flat.

(4) If $x \in \mathcal{X}_0^{\mathrm{an}}$, then π_x induces an isomorphism of completions $\widehat{\mathcal{O}}_{\mathcal{X},\pi(x)} \stackrel{\sim}{\to} \widehat{\mathcal{O}}_{\mathcal{X}^{\mathrm{an}},x}$.

Here $\mathcal{X}_0$ denotes the set of closed points of $\mathcal{X}$ and, for a k-analytic space X, one sets $X_0 = \{x \in X \big| [\mathcal{H}(x) : k] < \infty\}$.

4.1.2. Exercise. (i) The k-analytic space $\mathbf{A}^n$ is the analytification of the scheme theoretic n-dimentional affine space A^n over k (i.e., $\mathrm{Spec}(k[T_1, \dots, T_n])$.

(ii) Let $\mathbf{P}^n$ be the *n-dimensional projective space*, i.e., the k-analytic space associated to the scheme theoretic projective space P^n over k. The canonical morphism $\mathrm{A}^{n+1}\backslash\{0\} \to \mathrm{P}^n$ gives rise to a morphism $\mathbf{A}^{n+1}\backslash\{0\} \to \mathbf{P}^n$. Show that the images of two points $x, y \in \mathbf{A}^{n+1}$ coincide in $\mathbf{P}^n$ if and only if there exists $\lambda > 0$ such that, for every homogeneous polynomial $f \in k[T_0, \dots, T_n]$ of degree $d \geq 0$, one has $|f(y)| = \lambda^d |f(x)|$.

(iii) Show that the embedding $\mathbf{R}_+^{n+1} \to \mathbf{A}^{n+1}$ and the strong deformation retraction $\Phi : \mathbf{A}^{n+1} \times [0,1] \to \mathbf{A}^{n+1}$ from Exercise 3.4.2 give rise to an embedding $\mathrm{P}^n(\mathbf{R}_+) \to \mathbf{P}^n(\mathbf{R}_+)$ and a strong deformation retraction $\Phi : \mathbf{P}^n \times [0,1] \to \mathbf{P}^n$ to the image of $\mathrm{P}^n(\mathbf{R}_+)$, where $\mathrm{P}^n(\mathbf{R}_+)$ is the quotient of $\mathbf{R}_+^{n+1}\backslash\{0,\dots,0\}$ by the canonical action of the group $\mathbf{R}_+^*$.

4.1.3. Fact. Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes of locally finite type over k, and let $\varphi^{\mathrm{an}} : \mathcal{Y}^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ be the corresponding morphism of k-analytic spaces.

(i) φ is (1) étale, (2) smooth, (3) separated, (4) an open immersion, and (5) an isomorphism if and only if φ^{an} possesses the same property.

(ii) Suppose that φ is of finite type. Then φ is (1) a closed immersion, (2) finite, and (3) proper if and only if φ^{an} possesses the same property.

4.1.4. Fact. (i) $\mathcal{X}$ is separated $\Longleftrightarrow$ $|\mathcal{X}^{\mathrm{an}}|$ is Hausdorff;

(ii) $\mathcal{X}$ is proper $\Longleftrightarrow$ $|\mathcal{X}^{\mathrm{an}}|$ is compact;

(iii) $\mathcal{X}$ is connected $\Longleftrightarrow$ $|\mathcal{X}^{\mathrm{an}}|$ is arcwise connected;

(iv) the dimension of $\mathcal{X}$ is equal to the topological dimension of $|\mathcal{X}^{\mathrm{an}}|$.

For an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, let $\mathcal{F}^{\mathrm{an}}$ denote the $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}$-module

$$\pi^*\mathcal{F} = \pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}^{\mathrm{an}}} \ .$$

(Show that the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is exact, faithful, and takes coherent sheaves to coherent sheaves.)

4.1.5. Fact. Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a proper morphism of schemes of locally finite type over k. Then for any coherent $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{F}$, there is a canonical isomorphism $(R^n\varphi_*\mathcal{F})^{\mathrm{an}} \xrightarrow{\sim} R^n\varphi_*^{\mathrm{an}}\mathcal{F}^{\mathrm{an}}$.

4.1.6. Fact. Let $\mathcal{X}$ be a proper k-scheme. Then the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ gives rise to an equivalence of categories $\mathrm{Coh}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{X}^{\mathrm{an}})$.

4.1.7. Exercise. (i) The functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ is fully faithful on the category of proper k-schemes.

(ii) Let $\mathcal{X}$ be a proper k-scheme. Then the functor $\mathcal{Y} \mapsto \mathcal{Y}^{\mathrm{an}}$ induces an equivalence between the category of finite (resp. finite étale) schemes over $\mathcal{X}$ and the category of finite (resp. finite étale) k-analytic spaces over $\mathcal{X}^{\mathrm{an}}$.

(iii) Every reduced proper k-analytic space X of dimension one is algebraic, i.e., there exists a projective algebraic curve $\mathcal{Y}$ over k such that $X \xrightarrow{\sim} \mathcal{Y}^{\mathrm{an}}$.

Suppose now that the valuation on k is trivial. For a k-analytic space X, let X_t denote the set of points $x \in X$ for which the canonical valuation on $\mathcal{H}(x)$ is trivial.

4.1.8. Exercise. (i) The set X_t is closed in X, and $X_0 \subset X_t$.

(ii) Every morphism $\varphi : Y \to X$ induces a continuous map $Y_t \to X_t$.

4.1.9. Fact Let $\mathcal{X}$ be a scheme of locally finite type over k.

(i) $\mathcal{X}_t^{\mathrm{an}} \stackrel{\sim}{\to} \mathcal{X}$ and $\overline{\mathcal{X}_0^{\mathrm{an}}} = \mathcal{X}_t^{\mathrm{an}}$.

(ii) If $x \in \mathcal{X}_t^{\mathrm{an}}$, then $\widehat{\mathcal{O}}_{\mathcal{X},\pi(x)} \stackrel{\sim}{\to} \widehat{\mathcal{O}}_{\mathcal{X}^{\mathrm{an}},x}$.

(iii) The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ gives rise to an equivalence of categories $\mathrm{Coh}(\mathcal{X}) \stackrel{\sim}{\to} \mathrm{Coh}(\mathcal{X}^{\mathrm{an}})$.

(iv) The functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ is fully faithful.

4.2. Generic fibers of formal schemes locally finitely presented over k°. If the valuation on k is nontrivial, we fix a nonzero element $a \in k^{\circ\circ}$. If the valuation on k is trivial (then $\widetilde{k} = k^\circ = k$ and $k^{\circ\circ} = 0$), we set $a = 0$.

4.2.1. Definition. The *ring of restricted power series over k° in n-variables* is the ring $k^\circ\{T\} = k\{T_1, \dots, T_n\}$ of the formal power series $f = \sum_\nu a_\nu T^\nu$ over k° such that, for any $m \geq 0$, the number of ν's with $a^m \not| a_\nu$ is finite, i.e., $k^\circ\{T\} = \varprojlim k^\circ/(a^m)[T]$ (or $k^\circ\{T\} = k\{T\} \cap k^\circ[[T]]$).

4.2.2. Fact (Artin-Rees Lemma). For any finitely generated ideal $\mathbf{a} \subset k^\circ\{T\}$, there exists $m_0 \geq 0$ such that $\mathbf{a} \cap a^m k^\circ\{T\} \subset a^{m-m_0}\mathbf{a}$ for all $m \geq m_0$.

4.2.3. Definition. A *topologically finitely presented ring over* k° is a ring of the form $k^\circ\{T\}/\mathbf{a}$ for a finitely generated ideal $\mathbf{a} \subset k^\circ\{T\}$.

4.2.4. Exercise. Let A be a topologically finitely presented ring over k°.

(i) A is separated and complete in the a-adic topology.

(ii) $\widetilde{A} = A/k^{\circ\circ}A$ is a finitely generated $\widetilde{k}$-algebra. In particular, any open subset of the formal scheme $\mathrm{Spf}(A)$ is a finite union of open affine subschemes of the form $\mathrm{Spf}(A_{\{f\}})$, $f \in A$.

(iii) $\mathcal{A} = A \otimes_{k^\circ} k$ is a strictly k-affinoid algebra, and the image of A in $\mathcal{A}$ lies in $\mathcal{A}^\circ$.

(iv) Let $\mathfrak{X}$ be the affine formal scheme $\mathrm{Spf}(A)$, $\mathfrak{X}_s$ the affine $\widetilde{k}$-scheme $\mathrm{Spec}(\widetilde{A})$, and $\mathfrak{X}_\eta$ the strictly k-affinoid space $\mathcal{M}(\mathcal{A})$. Each point $x \in \mathfrak{X}_\eta$ gives rise to a character $\widetilde{\chi}_x : \widetilde{A} \to \widetilde{\mathcal{H}(x)}$ whose kernel is a prime ideal of $\widetilde{A}$. It this way one gets a *reduction map* $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$. Show that the preimage of a closed subset of $\mathfrak{X}_s$ is an open subset of $\mathfrak{X}_\eta$.

(v) The preimage $\pi^{-1}(\mathcal{Y})$ of an open subset $\mathcal{Y} \subset \mathfrak{X}_s$ is a compact strictly analytic domain in $\mathfrak{X}_\eta$. If $\mathcal{Y}$ is an open affine subscheme of $\mathfrak{X}_s$ and $\mathfrak{Y}$ is the open affine formal subscheme of $\mathfrak{X}$ with the underlying space $\mathcal{Y}$ (i.e., $\mathfrak{Y}_s = \mathcal{Y}$), then $\mathfrak{Y}_\eta \stackrel{\sim}{\to} \pi^{-1}(\mathcal{Y})$.

4.2.5 Definition. (i) A formal scheme *locally finitely presented over* k° is a formal scheme $\mathcal{X}$ over $\mathrm{Spf}(k^\circ)$ which is a locally finite union of open affine formal subschemes of the form $\mathrm{Spf}(A)$, where A is topologically finitely presented over k°. The category of such formal schemes is denoted by k°-$\mathcal{F}sch$.

(ii) For $\mathfrak{X} \in k^\circ$-$\mathcal{F}sch$, the locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/k^{\circ\circ}\mathcal{O}_{\mathfrak{X}})$ is a scheme of locally finite type over $\widetilde{k}$, it is called the *closed* (or *special*) *fiber of* $\mathfrak{X}$ and denoted by $\mathfrak{X}_s$.

(iii) For $\mathfrak{X} \in k^\circ$-$\mathcal{F}sch$ and $n \geq 1$, the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/a^n\mathcal{O}_{\mathfrak{X}})$ is denoted by $\mathfrak{X}_n$.

We are going to construct a functor k°-$\mathcal{F}sch \to k$-$\mathcal{A}n$ that associates to a formal scheme $\mathfrak{X} \in k^\circ$-$\mathcal{F}sch$ its *generic fiber* $\mathfrak{X}_\eta \in k$-$\mathcal{A}n$, and we will construct a *reduction map* $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$. Both constructions extend those of Exercise 4.2.4.

We fix a locally finite covering $\{\mathfrak{X}_i\}_{i\in I}$ by open affine subschemes of the form $\mathrm{Spf}(A)$, where A is topologically finitely presented over k°. Suppose first that $\mathfrak{X}$ is separated. Then for any pair $i, j \in I$ the intersection $\mathfrak{X}_{ij} = \mathfrak{X}_i \cap \mathfrak{X}_j$ is an open affine suscheme of both $\mathfrak{X}_i$ and $\mathfrak{X}_j$ and, therefore, $\mathfrak{X}_{ij,\eta}$ is a strictly affinoid domain in $\mathfrak{X}_{i,\eta}$ and $\mathfrak{X}_{j,\eta}$ and the canonical morphism $\mathfrak{X}_{ij,\eta} \to \mathfrak{X}_{i,\eta} \times \mathfrak{X}_{j,\eta}$ is a closed immersion. By Exercise 3.2.13, we can glue all $\mathfrak{X}_{i,\eta}$ along $\mathfrak{X}_{ij,\eta}$, and we get a paracompact separated strictly k-analytic space $\mathfrak{X}_\eta$. If $\mathfrak{Y}$ is an open formal subscheme of $\mathfrak{X}$, then $\mathfrak{Y}_\eta$ is a closed strictly analytic domain of $\mathfrak{X}_\eta$.

Suppose now that $\mathfrak{X}$ is arbitrary. Then each $\mathfrak{X}_{ij} = \mathfrak{X}_i \cap \mathfrak{X}_j$ is a separated formal scheme, and $\mathfrak{X}_{ij,\eta}$ is a compact strictly analytic domain in both $\mathfrak{X}_{i,\eta}$ and $\mathfrak{X}_{j,\eta}$. We can therefore glue all $\mathfrak{X}_{i,\eta}$ along $\mathfrak{X}_{ij,\eta}$, and we get a paracompact strictly k-analytic space $\mathfrak{X}_\eta$.

4.2.6. Exercise. (i) The reduction maps $\mathfrak{X}_{i,\eta} \to \mathfrak{X}_{i,s}$ induce a reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$.

(ii) If $\mathcal{Y}$ is a closed subset of $\mathfrak{X}_s$, then $\pi^{-1}(\mathcal{Y})$ is an open subset of $\mathfrak{X}_\eta$.

(iii) If $\mathcal{Y}$ is an open subset of $\mathfrak{X}_s$, then $\pi^{-1}(\mathcal{Y})$ is a closed strictly analytic domain in $\mathfrak{X}_\eta$. If $\mathfrak{Y}$ is the open formal subscheme of $\mathfrak{X}$ with the underlying space $\mathcal{Y}$, then $\mathfrak{Y}_\eta \widetilde{\to} \pi^{-1}(\mathcal{Y})$.

(iv) The correspondence $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is a functor that commutes with fiber products and extensions of the ground field.

(v) If a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in k°-$\mathcal{F}sch$ is finite, then the morphisms $\varphi_s : \mathfrak{Y}_s \to \mathfrak{X}_s$ and $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ are finite.

4.2.7. Definition. A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in k°-$\mathcal{F}sch$ is said to be *étale* if for all $n \geq 1$ the induced morphisms of schemes $\varphi_n : \mathfrak{Y}_n \to \mathfrak{X}_n$ are étale.

4.2.8. Fact. (i) The correspondence $\mathfrak{Y} \mapsto \mathfrak{Y}_s$ induces an equivalence between the category of formal schemes étale over $\mathfrak{X}$ and the category of schemes étale over $\mathfrak{X}_s$.

(ii) For an étale morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, one has $\varphi_\eta(\mathfrak{Y}_\eta) = \pi^{-1}(\varphi_s(\mathfrak{Y}_s))$.

(iii) For an étale morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, the morphism of k-analytic spaces $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is quasi-étale.

4.2.9. Fact (Temkin). If a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in k°-$\mathcal{F}sch$ is proper (i.e., the morphism of schemes $\varphi_s : \mathfrak{Y}_s \to \mathfrak{X}_s$ is proper), then the morphism of k-analytic space $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is proper.

Let now $\mathcal{X}$ be a scheme which admits a locally finite covering by open affine subschemes of the form $\mathrm{Spec}(A)$, where A is a finitely presented k°-algebra. Then the formal completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along the subscheme $(\mathcal{X}_s, \mathcal{O}_\mathcal{X}/a\mathcal{O}_\mathcal{X})$ is a formal scheme from k°-$\mathcal{F}sch$, and it has the generic fiber $\widehat{\mathcal{X}}_\eta$. On the other hand, there is a k-analytic space $\mathcal{X}_\eta^{\mathrm{an}} = (\mathcal{X}_\eta)^{\mathrm{an}}$. From the construction of both spaces it follows that there is a canonical morphism of k-analytic spaces $\widehat{\mathcal{X}}_\eta \to \mathcal{X}_\eta^{\mathrm{an}}$.

4.2.10. Fact. (i) If $\mathcal{X}$ is affine (resp. separated and quasicompact), the above morphism identifies $\widehat{\mathcal{X}}_\eta$ with a strictly affinoid (resp. a compact strictly analytic) domain in $\mathcal{X}_\eta^{\mathrm{an}}$.

(ii) If $\mathcal{X}$ is proper over k°, then $\widehat{\mathcal{X}}_\eta \stackrel{\sim}{\to} \mathcal{X}_\eta^{\mathrm{an}}$.

(iii) A proper morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ induces an isomorphism $\widehat{\mathcal{Y}}_\eta \stackrel{\sim}{\to} \mathcal{Y}_\eta^{\mathrm{an}} \times_{\mathcal{X}_\eta^{\mathrm{an}}} \widehat{\mathcal{X}}_\eta$.

4.3. Generic fibers of special formal schemes over k°. *In this subsection, the valuation on the ground field k is assumed to be discrete* (but not necessarily nontrivial).

4.3.1. Definition. A *special k°-algebra* is an adic ring A such that, for some ideal of definition $\mathbf{a} \subset A$, the quotient rings $A/\mathbf{a}^n$ are finitely generated over k° for all $n \geq 1$.

For example, the k°-algebra $k^\circ\{T_1, \dots, T_m\}[[S_1, \dots, S_n]] = k^\circ[[S_1, \dots, S_n]]\{T_1, \dots, T_m\}$ is special.

4.3.2. Fact. Let A be an $\mathbf{a}$-adic special k°-algebra.

(i) A is a Noetherian ring, and its Jacobson radical is an ideal of definition.

(ii) every ideal $\mathbf{b} \subset A$ is closed, and the quotient ring $B = A/\mathbf{b}$ is an $\mathbf{a}B$-adic special k°-algebra.

(iii) if $A \to B$ is a continuous surjective homomorphism to a special k°-algebra B and $\mathbf{b}$ is its kernel, then $A/\mathbf{b}$ is topologically isomorphic to B.

(iv) if an ideal $\mathbf{a} \subset A$ is open, then the completion $B = \widehat{A}$ of A in the $\mathbf{b}$-adic topology is a $\mathbf{a}B$-adic special k°-algebra.

(v) if B is a special k°-algebra, then so is $A \widehat{\otimes}_{k^\circ} B$;

(vi) $B = A\{T_1, \dots, T_m\}$ is an $\mathbf{a}B$-adic special k°-algebra.

(vii) $B = A[[S_1, \dots, S_n]]$ is a $\mathbf{b}$-adic special k°-algebra, where $\mathbf{b}$ is the ideal of B generated by $\mathbf{a}$ and $T_1, \dots, T_n$.

4.3.3. Fact. The following properties of an $\mathbf{a}$-adic k°-algebra A are equivalent:

(a) A is a special k°-algebra;

(b) $A/\mathbf{a}^2$ is finitely generated over k°;

(c) A is topologically isomorphic over k° to a quotient of $k^\circ\{T_1, \dots, T_m\}[[S_1, \dots, S_n]]$.

4.3.4. Definition. (i) A formal scheme $\mathfrak{X}$ over k° is said to be *special* if it is a locally finite of open affine formal subschemes of the form $\mathrm{Spf}(A)$, where A is a special k°-algebra. The category of such formal schemes is denoted by k°-$\mathcal{SF}sch$.

(ii) For $\mathfrak{X} \in k^\circ$-$\mathcal{SF}sch$, the locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}\mathcal{O}_{\mathfrak{X}})$, where $\mathcal{J}$ is an ideal of definition of $\mathfrak{X}$ that contains $k^{\circ\circ}$, is scheme of locally finite type over $\widetilde{k}$, it is called the *closed* (or *special*) *fiber of* $\mathfrak{X}$ and is denoted by $\mathfrak{X}_s$.

Notice that the scheme $\mathfrak{X}_s$ depends on the choice of the ideal of definition $\mathcal{J}$, but the underlying reduced scheme (and in particular the étale site of $\mathfrak{X}_s$) does not.

4.3.5. Exercise. (i) k°-$\mathcal{F}sch$ is a full subcategory of k°-$\mathcal{SF}sch$.

(ii) If $\mathfrak{X} \in k^\circ$-$\mathcal{SF}sch$ and $\mathcal{Y}$ is a subscheme $\mathfrak{X}_s$, the formal completion $\mathfrak{X}_{/\mathcal{Y}}$ of $\mathfrak{X}$ along $\mathcal{Y}$ is a special formal scheme over k°.

We are going to construct a functor k°-$\mathcal{SF}sch \to k$-$\mathcal{A}n$ that associates to a formal scheme $\mathfrak{X} \in \mathrm{Ob}(k^\circ$-$\mathcal{SF}sch)$ its *generic fiber* $\mathfrak{X}_\eta \in k$-$\mathcal{A}n$, and we will construct a *reduction map* $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$. Both constructions extend those from §4.2.

4.3.6. Exercise. Let A be a special k°-algebra and $\mathfrak{X} = \mathrm{Spf}(A)$, and fix an admissible epimorphism $\alpha : A' = k^\circ\{T_1, \dots, T_m\}[[S_1, \dots, S_n]] \to A$. If $\mathbf{a}$ is the kernel of α, we define $\mathfrak{X}_\eta$ as the closed k-analytic subspace of $Y = E^m \times D^n$ defined by the subsheaf of ideals $\mathbf{a}\mathcal{O}_Y$, where E^m and D^n are the closed and the open polydiscs of radius 1 with center at zero in $\mathbf{A}^m$ and $\mathbf{A}^n$, respectively.

(i) $\mathfrak{X}_\eta$ can be identified with the set of multiplicative seminorms on A that extend the valuation on k° and whose values at all elements of A are at most 1 and at elements of an ideal of definition of A are strictly less than 1.

(ii) There is an increasing sequence of affinoid domains $V_1 \subset V_2 \subset \dots$ in $\mathfrak{X}_\eta$ such that $\mathfrak{X}_\eta = \bigcup_{n=1}^\infty V_n$, each V_n is a Weierstrass domain in V_{n+1}, the image of an ideal of definition with respect to the canonical homomorphism $A \to \mathcal{A}_{V_n}^\circ$ lie in $\mathcal{A}_{V_n}^{\circ\circ}$, and the image of $A \otimes_{k^\circ} k$ in $\mathcal{A}_{V_n}$ is dense.

(iii) A compact subset $V \subset \mathfrak{X}_\eta$ is an affinoid domain if and only if there exist a k-affinoid algebra $\mathcal{A}_V$ and a homomorphism $A \to \mathcal{A}_V^\circ$ that take an ideal of definition to $\mathcal{A}_V^{\circ\circ}$ such that the image of $\mathcal{M}(\mathcal{A}_V)$ in $\mathfrak{X}_\eta$ is V and any homomorphism $A \to \mathcal{B}^\circ$ that take an ideal of definition to $\mathcal{B}^{\circ\circ}$, where $\mathcal{B}$ is a k-affinoid algebra, for which the image of $\mathcal{M}(\mathcal{B})$ in $\mathfrak{X}_\eta$ lies in V, goes through a unique bounded homomorphism $\mathcal{A}_V \to \mathcal{B}$.

(iv) The correspondence $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is a functor.

(v) The construction of Exercise 4.2.4 gives rise to a *reduction map* $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$ such that the preimage of a closed subset of $\mathfrak{X}_s$ is an open subset of $\mathfrak{X}_\eta$.

(vi) The preimage $\pi^{-1}(\mathcal{Y})$ of an open subset $\mathcal{Y} \subset \mathfrak{X}_s$ is a closed analytic domain in $\mathfrak{X}_\eta$.

(vii) For any affine subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, there is a canonical isomorphism $(\mathfrak{X}_{/\mathcal{Y}})_\eta \stackrel{\sim}{\to} \pi^{-1}(\mathcal{Y})$.

If $\mathfrak{X}$ is arbitrary, we do the same construction as in §2. Namely, we fix a locally finite covering $\{\mathfrak{X}_i\}_{i\in I}$ by open affine subschemes of the form $\mathrm{Spf}(A)$, where A is a special k°-algebra. Suppose first that $\mathfrak{X}$ is separated. Then for any pair $i, j \in I$ the intersection $\mathfrak{X}_{ij} = \mathfrak{X}_i \cap \mathfrak{X}_j$ is an open affine suscheme of both $\mathfrak{X}_i$ and $\mathfrak{X}_j$ and, therefore, $\mathfrak{X}_{ij,\eta}$ is a closed analytic domain in $\mathfrak{X}_{i,\eta}$ and $\mathfrak{X}_{j,\eta}$ and the canonical morphism $\mathfrak{X}_{ij,\eta} \to \mathfrak{X}_{i,\eta} \times \mathfrak{X}_{j,\eta}$ is a closed immersion. By Exercise 3.2.13, we can glue all $\mathfrak{X}_{i,\eta}$ along $\mathfrak{X}_{ij,\eta}$, and we get a paracompact separated k-analytic space $\mathfrak{X}_\eta$. If $\mathfrak{Y}$ is an open formal subscheme of $\mathfrak{X}$, then $\mathfrak{Y}_\eta$ is a closed analytic domain of $\mathfrak{X}_\eta$.

Suppose now that $\mathfrak{X}$ is arbitrary. Then each $\mathfrak{X}_{ij} = \mathfrak{X}_i \cap \mathfrak{X}_j$ is a separated formal scheme, and $\mathfrak{X}_{ij,\eta}$ is a closed analytic domain in both $\mathfrak{X}_{i,\eta}$ and $\mathfrak{X}_{j,\eta}$. We can therefore glue all $\mathfrak{X}_{i,\eta}$ along $\mathfrak{X}_{ij,\eta}$, and we get a paracompact k-analytic space $\mathfrak{X}_\eta$.

4.3.7. Exercise. (i) The reduction maps $\mathfrak{X}_{i,\eta} \to \mathfrak{X}_{i,s}$ induce a reduction map $\pi : \mathfrak{X}_\eta \to \mathfrak{X}_s$.

(ii) If $\mathcal{Y}$ is a closed subset of $\mathfrak{X}_s$, then $\pi^{-1}(\mathcal{Y})$ is an open subset of $\mathfrak{X}_\eta$.

(iii) If $\mathcal{Y}$ is an open subset of $\mathfrak{X}_s$, then $\pi^{-1}(\mathcal{Y})$ is a closed analytic domain in $\mathfrak{X}_\eta$.

(iv) For any subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, there is a canonical isomorphism $(\mathfrak{X}_{/\mathcal{Y}})_\eta \stackrel{\sim}{\to} \pi^{-1}(\mathcal{Y})$.

(v) The correspondence $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ is a functor that commutes with fiber products and extensions of the ground field.

4.3.8. Definition. (i) A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in k°-$\mathcal{SF}sch$ is said to be of *locally finite type* if locally it is isomorphic to a morphism of the form $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$, where B is topologically finitely generated over A (i.e., it is a quotient of $A\{T_1, \dots, T_n\}$).

(ii) A morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in k°-$\mathcal{SF}sch$ is said to be *étale* if it is of locally finite type and, for any ideal of definition $\mathcal{J}$ of $\mathfrak{X}$, the morphism of schemes $(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}/\mathcal{J}\mathcal{O}_\mathfrak{Y}) \to (\mathfrak{X}, , \mathcal{O}_\mathfrak{X}/\mathcal{J}\mathcal{O}_\mathfrak{X})$ is étale.

4.3.9. Fact. (i) The correspondence $\mathfrak{Y} \mapsto \mathfrak{Y}_s$ induces an equivalence between the category of formal schemes étale over $\mathfrak{X}$ and the category of schemes étale over $\mathfrak{X}_s$.

(ii) For an étale morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, one has $\varphi_\eta(\mathfrak{Y}_\eta) = \pi^{-1}(\varphi_s(\mathfrak{Y}_s))$.

(ii) For an étale morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$, the morphism of k-analytic spaces $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is quasi-étale.

Let now $\mathcal{X}$ be a scheme which admits a locally finite covering by open affine subschemes of the form Spec(A), where A is a finitely presented k°-algebra. For a subscheme $\mathcal{Y} \subset \mathfrak{X}_s$, let $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ denote the formal completion of $\mathcal{X}$ along $\mathcal{Y}$. Since $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ coincides with the formal completion of $\widehat{\mathcal{X}}$ along $\mathcal{Y}$, it follows that this is a special formal scheme over k°. Its closed fibre can be identified with $\mathcal{Y}$, and one has $(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta \widetilde{\to} \pi^{-1}(\mathcal{Y})$, where π is the reduction map $\widehat{\mathcal{X}}_\eta \to \mathcal{X}_s$.

4.4. The local structure of a smooth analytic curve.

4.4.1. Exercise. Let x be a point of the affine line $\mathbf{A}^1$.

(i) The types of all points from the preimage of x in $\mathbf{A}^1 \widehat{\otimes} \widehat{k}^{\mathrm{a}}$ (which were introduced in Exercise 1.3.6) are the same. This defines the type of x.

(ii) If x is of type (1) or (4), then $\widetilde{\mathcal{H}(x)}$ is algebraic over $\widetilde{k}$ and $\sqrt{|\mathcal{H}(x)^*|} = \sqrt{|k^*|}$.

(iii) If x is of type (2), then $\widetilde{\mathcal{H}(x)}$ is finitely generated of transcendence degree one over $\widetilde{k}$ and the group $|\mathcal{H}(x)^*|/|k^*|$ is finite.

(iv) If x is of type (3), then $\widetilde{\mathcal{H}(x)}$ is finite over $\widetilde{k}$ and the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|k^*|}$ is of dimension one.

4.4.2. Exercise. Let X be a smooth k-analytic curve. For every point $x \in X$, there is an étale morphism from an open neighborhood of x to the affine line.

(i) The type of the image of x in $\mathbf{A}^1$ does not depend on the choice of the étale morphism. It is said to be the *type of* x.

(ii) If $x \in X_0$ (i.e., $[\mathcal{H}(x) : k] < \infty$), it is of type (1) and its local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring. In all other cases, $\mathcal{O}_{X,x}$ is a field.

4.4.3. Fact. Assume that the field k is perfect or its valuation is non-trivial (resp. non-perfect and the valuation is trivial). Then for every point x of a smooth k-analytic curve X there exists a finite separable (resp. finite) extension k' of k and an open subset $X' \subset X \widehat{\otimes} k'$ such that x has a unique preimage x' in X' and X' is isomorphic to the following k'-analytic curve (depending on the type of x):

(1) or (4): an open disc with center at zero;

(3) an open annulus with center at zero;

(2) $\mathcal{X}_\eta^{\rm an}\backslash\coprod_{i=1}^n E_i$, $n\ge 1$, where $\mathcal{X}$ is a connected smooth projective curve over k'°, each E_i is an affinoid domain isomorphic to a closed disc with center at zero, all of E_i's are in pairwise different residue classes of $\mathcal{X}_s$, and x' is the preimage of the generic point of $\mathcal{X}_s$ in $\mathcal{X}_\eta^{\rm an}$.

4.4.4. Definition. A triple (X,Y,x) consisting of a smooth k-analytic curve X, an open subset $Y\subset X$ and a point $x\in Y$ is said to be *elementary* if

(a) if x is of type (1) or (4) (resp. (3)), $X={\bf P}^1$, Y is an open disc with center at zero (resp. $x=p_{E(0;r)}$ and Y is an open annulus $\{y\in{\bf A}^1\big|r'<|T(y)|<r''\}$ with $r'<r<r''$);

(c) if x is of type (2), $Y=\mathcal{X}_\eta^{\rm an}\backslash\coprod_{i=1}^n E_i$, $n\ge 1$, where $\mathcal{X}$ is a connected smooth projective curve over k°, each E_i is an affinoid domain isomorphic to a closed disc with center at zero, all of E_i's are in pairwise different residue classes of $\mathcal{X}_s$, and x is the preimage of the generic point of $\mathcal{X}_s$ in $\mathcal{X}_\eta^{\rm an}$.

4.5. The local structure of a smooth morphisms.

4.5.1. Definition. A morphism $\varphi:Y\to X$ is said to be an *elementary fibration of dimension one at a point* $y\in Y$ if it can be included to a commutative diagram

$$\begin{array}{ccccc} Y & \hookrightarrow & Z & \longleftarrow & V=\coprod_{i=1}^m (X\times E_i) \\ & \varphi\searrow & \downarrow\psi & \swarrow pr & \\ & & X & & \end{array}$$

such that

(a) $\psi:Z\to X$ is a smooth proper morphism whose geometric fibres are irreducible curves of genus $g\ge 0$;

(b) Y is an open subset of Z, and $V=Z\backslash Y$;

(c) V is an analytic domain in Z isomorphic to a disjoint union $\coprod_{i=1}^m(X\times E_i)$, $m\ge 1$, where E_i are closed discs in ${\bf A}^1$ with center at zero, and pr is the canonical projection;

(d) there exists an analytic domain $V\subset V'$ such that the isomorphism $V\stackrel{\sim}{\to}\coprod_{i=1}^m(X\times E_i)$ from (c) extends to an isomorphism $V'\stackrel{\sim}{\to}\coprod_{i=1}^m(X\times E_i')$, where E_i' is a closed disc in ${\bf A}^1$ which contains E_i and has a bigger radius;

(e) the triple (Z_x,Y_x,y), where $x=\varphi(y)$, is elementary.

4.5.2. Definition. A morphism of good k-analytic spaces $\varphi:Y\to X$ is said to be an *almost étale neighborhood of a point* $x\in X$ if x has a unique preimage y in Y and, if ${\rm char}(k)=0$, φ is étale, and, if $p={\rm char}(k)>0$, there exist affinoid neighborhoods V of y and U of $\varphi(y)$ with $\varphi(V)\subset U$

such that the induced morphism $V \to U$ is a composition of finite morphisms of k-affinoid spaces $V \stackrel{\chi}{\to} U' = \mathcal{M}(\mathcal{A}') \stackrel{\psi}{\to} U = \mathcal{M}(\mathcal{A})$ such that χ is étale and $\mathcal{A}' = \mathcal{A}[T_1, \ldots, T_n]/(T_i^{p^{m_i}} - f_i)$ for some $f_1, \ldots, f_n \in \mathcal{A}$ and $m_1, \ldots, m_n \geq 0$.

Notice that, if the field $\mathcal{H}(x)$ is perfect with trivial valuation, such a morphism φ is étale at y.

4.5.2. Fact. Given a smooth morphism of pure dimension one between good k-analytic spaces $\varphi : Y \to X$ and a point $y \in Y$, suppose that the valuation on $\mathcal{H}(\varphi(y))$ is nontrivial (resp. trivial). Then there exists an étale (resp. almost étale) neighborhood $f : X' \to X$ of the point $\varphi(y)$ and an open subset $Y'' \subset Y' = Y \times_X X'$

$$
\begin{array}{ccc}
Y & \stackrel{\varphi}{\longrightarrow} & X \\
\uparrow f' & & \uparrow f \\
Y' & \stackrel{\varphi'}{\longrightarrow} & X' \\
\uparrow & \varphi'' \nearrow & \\
Y'' & &
\end{array}
$$

such that y has a unique preimage y' in Y'' and $\varphi'' : Y'' \to X'$ is an elementary fibration of dimension one at the point y'.

4.5.3. Exercise. A smooth morphism is an open map.

§5. Étale cohomology of analytic spaces

5.1. Étale topology on an analytic space. Let X be a k-analytic space, and let $\text{Ét}(X)$ denote the category of étale morphisms $U \to X$.

5.1.1. Definition. The *étale topology* on X is the Grothendieck topology on the category $\text{Ét}(X)$ generated by the pretopology for which the set of coverings of $(U \to X) \in \text{Ét}(X)$ is formed by the families $\{U_i \stackrel{f_i}{\to} U\}_{i \in I}$ such that $U = \cup_{i \in I} f_i(U_i)$. The corresponding site (the *étale site* of X) is denoted by $X_{\text{ét}}$, the category of sheaves of sets on $X_{\text{ét}}$ (the *étale topos* of X) is denoted by $X_{\widetilde{\text{ét}}}$, and the category of abelian étale sheaves is denoted by $\mathbf{S}(X)$. The cohomology groups of an abelian étale sheaf F are denoted by $H^q(X, F)$.

There is a canonical morphism of sites $\pi : X_{\text{ét}} \to |X|$. For an étale sheaf F on X and a point $x \in X$, one sets $F_x = (\pi_* F)_x$ (the stalk of the restriction of F to the usual topology of X at x). For a section of F over X, f_x denotes the image of f in F_x.

5.1.2. Definition. (i) A *geometric point* of X is a morphism of the form $\overline{x} : \mathbf{p}_{\mathcal{H}(\overline{x})} \to X$, where $\mathbf{p}_{\mathcal{H}(\overline{x})}$ is the spectrum of an algebraically closed non-Archimedean field $\mathcal{H}(\overline{x})$ over k.

(ii) The *stalk* $F_{\overline{x}}$ of an étale sheaf F at a geometric point $\overline{x}$ is the pullback of F with respect to the morphism $\overline{x}$, i.e., the inductive limit of $F(U)$ taken over all pairs (φ, α) consisting of an étale morphism $\varphi : U \to X$ and a morphism $\alpha : \mathbf{p}_{\mathcal{H}(\overline{x})} \to U$ over $\overline{x}$.

Let $G_{\overline{x}/x}$ denote the Galois group of the separable closure of $\mathcal{H}(x)$ in $\mathcal{H}(\overline{x})$ over $\mathcal{H}(x)$. For any étale sheaf F on X, there is a canonical discrete action of the group $G_{\overline{x}/x}$ on the stalk $F_{\overline{x}}$.

5.1.3. Fact. (i) For any étale sheaf F on X, one has $F_x = F_{\overline{x}}^{G_{\overline{x}/x}}$.

(ii) For any abelian étale sheaf F on X and any $n \geq 0$, one has $(R^n\pi_*F)_x = H^n(G_{\overline{x}/x}.F_{\overline{x}})$.

5.1.4. Exercise. Show that $\mathrm{cd}_l(X) \leq \mathrm{cd}_l(k) + 2\dim(X)$, where l is a prime integer and $\mathrm{cd}_l(X)$ is the l-cohomological dimension of X, i.e., the minimal $n \geq 0$ such that $H^i(X, F) = 0$ for all $i > n$ and all l-torsion abelian étale sheaves F. (Use the spectral sequence of the morphism of sites $\pi : X_{\text{ét}} \to |X|$ and Facts 2.1.13(iii) and 3.4.1(ii).)

5.1.5. Definition. (i) The *support of a section* $f \in F(X)$ of an abelian étale sheaf F is the set $\mathrm{Supp}(f) = \{x \in X \big| f_x \neq 0\}$. (Show that it is a closed subset of X.)

(ii) For a Hausdorff k-analytic space X, there is a left exact functor $\Gamma_c : \mathbf{S}(X) \to \mathcal{A}b$ defined as follows: $\Gamma_c(F) = \{f \in F(X) \big| \mathrm{Supp}(f) \text{ is compact}\}$. One set $H^n_c(X, F) = R^n\Gamma_c(F)$ (the *cohomology groups with compact support*).

(iii) Given a Hausdorff morphism $\varphi : Y \to X$, there is a left exact functor $\varphi_! : \mathbf{S}(Y) \to \mathbf{S}(X)$ defined as follows: for an étale morphism $U \to X$, $(\varphi_!F)(U) = \{f \in F(Y \times_X U) \big|$ the map $\mathrm{Supp}(f) \to U$ is compact$\}$.

5.1.6. Fact (Weak base change theorem for cohomology with compact support). Given a Hausdorff morphism $\varphi : Y \to X$, an abelian étale sheaf F on Y, and a geometric point $\overline{x}$ on X, one has $(R^n\varphi_!F)_{\overline{x}} \stackrel{\sim}{\to} H^n_c(Y_{\overline{x}}, F_{\overline{x}})$, where $Y_{\overline{x}} = Y_x \widehat{\otimes}_{\mathcal{H}(x)} \mathcal{H}(\overline{x})$ and $F_{\overline{x}}$ is the pullback of F on $Y_{\overline{x}}$.

5.1.7. Exercise. If fibers of φ are of dimension at most d, then for any étale abelian torsion sheaf F on Y, one has $R^n\varphi_!F = 0$ for all $n > 2d$.

Let $\mathrm{Q\acute{e}t}(X)$ denote the category of quasi-étale morphisms $U \to X$.

5.1.8. Definition. The *quasi-étale topology* on X is the Grothendieck topology on the category $\mathrm{Q\acute{e}t}(X)$ generated by the pretopology for which the set of coverings of $(U \to X) \in \mathrm{Q\acute{e}t}(X)$ is formed by the families $\{U_i \stackrel{f_i}{\to} U\}_{i \in I}$ such that each point of U has a neighborhood of the form $f_{i_1}(V_1) \cup \ldots \cup f_{i_n}(V_n)$ for some affinoid domains $V_1 \subset U_{i_1}, \ldots, V_n \subset U_{i_n}$. The corresponding site (the *quasi-étale site* of X) is denoted by $X_{\text{qét}}$, and the category of sheaves of sets on $X_{\text{qét}}$ (the *quasi-étale topos* of X) is denoted by $X_{\widetilde{\text{qét}}}$.

There is a commutative diagram of morphisms of sites:

$$\begin{array}{ccc} X_{\mathrm{G}} & \longrightarrow & |X| \\ \uparrow & & \uparrow \pi \\ X_{\mathrm{q\acute{e}t}} & \xrightarrow{\mu} & X_{\mathrm{\acute{e}t}} \end{array}$$

5.1.9. Fact. (i) For any étale sheaf F on X, one has $F \xrightarrow{\sim} \mu_* \mu^* F$ and, in particular, the functor $\mu^* : X_{\widetilde{\mathrm{\acute{e}t}}} \to X_{\widetilde{q\mathrm{\acute{e}t}}}$ is fully faithful.

(ii) For any abelian étale sheaf on X, one has $H^n(X, F) \xrightarrow{\sim} H^n(X_{\mathrm{q\acute{e}t}}, \mu^* F)$.

(iii) If $\varphi : Y \to X$ is a compact morphism, then for any abelian étale sheaf F on Y, one has $\mu^*(R^n \varphi_* F) \xrightarrow{\sim} R^n \varphi_*(\mu^* F)$.

5.2. Basic facts on étale cohomology.

5.2.1. Fact (Comparison theorem for cohomology with compact support). Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a compactifiable morphism between schemes of locally finite type over k. Then for any abelian torsion sheaf $\mathcal{F}$ on $\mathcal{Y}$, one has

$$(R^n \varphi_! \mathcal{F})^{\mathrm{an}} \xrightarrow{\sim} R^n \varphi_!^{\mathrm{an}} \mathcal{F}^{\mathrm{an}} \ .$$

In particular, if $\mathcal{Y}$ is compactifiable, then $H_c^n(\mathcal{Y}, \mathcal{F}) \xrightarrow{\sim} H_c^n(\mathcal{Y}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$.

5.2.2. Fact (Base change theorem for cohomology with compact support). Given a morphism of k-analytic spaces $\varphi : Y \to X$, a non-Archimedean field k' over k, and a morphism of k'-analytic spaces $\varphi' : Y' \to X'$ such that the following diagram is cartesian

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow f' & & \uparrow f \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

(i.e., $Y' \xrightarrow{\sim} (Y \widehat{\otimes} k') \times_{X \widehat{\otimes} k'} X'$), for any abelian torsion sheaf F on Y with torsion orders prime to $\mathrm{char}(\widetilde{k})$, one has

$$f^*(R^n \varphi_! F) \xrightarrow{\sim} R^n \varphi'_!(f'^* F) \ .$$

5.2.3. Fact (Poincaré Duality). (i) One can assign to every smooth morphism $\varphi : Y \to X$ of pure dimension d a trace mapping

$$\mathrm{Tr}_\varphi : R^{2d} \varphi_!(\mu_{n,Y}^d) \to (\mathbf{Z}/n\mathbf{Z})_X, \ (n, \mathrm{char}(k)) = 1 \ ,$$

These mappings possess certain natural properties and are uniquely determined by them.

(ii) Suppose that n is prime to $\mathrm{char}(\widetilde{k})$. Then Tr_φ induces for every $G^{\cdot} \in D^-(Y, \mathbf{Z}/n\mathbf{Z})$ and $F^{\cdot} \in D^+(X, \mathbf{Z}/n\mathbf{Z})$ an isomorphism

$$R\varphi_*(\underline{\mathcal{H}om}(G^{\cdot}, \varphi^* F^{\cdot}(d)[2d])) \xrightarrow{\sim} \underline{\mathcal{H}om}(R\varphi_! G^{\cdot}, F^{\cdot}) \ .$$

5.2.4. Exercise. (i) Let $\varphi : Y \to X$ be a separated smooth morphism of pure dimension d. Then for any $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$, $(n, \mathrm{char}(\widetilde{k})) = 1$, there is a canonical isomorphism

$$R\varphi_*(\varphi^* F) \xrightarrow{\sim} \underline{\mathcal{H}om}(R\varphi_!(\mathbf{Z}/n\mathbf{Z})_Y, F(-d)[-2d]) \ .$$

(ii) Let φ be the canonical projection $X \times D \to X$, where D is an open disc in $\mathbf{A}^1$. Then for any torsion sheaf F on X with torsion orders prime to $\mathrm{char}(\widetilde{k})$, one has

$$F \xrightarrow{\sim} \varphi_* \varphi^* F \text{ and } R^q \varphi_*(\varphi^* F) = 0 \text{ for } q \geq 1 \ .$$

(iii) Suppose that k is algebraically closed, and let X be a separated smooth k-analytic space of pure dimension d. Then for any $F \in \mathbf{S}(X, \mathbf{Z}/n\mathbf{Z})$ and $q \geq 0$, there is a canonical isomorphism

$$\mathrm{Ext}^q(F, \mu_{n,Y}^d) \xrightarrow{\sim} H_c^{2d-q}(X, F)^\vee \ .$$

In particular, if F is finite locally constant, then one has

$$H^q(X, F^\vee(d)) \xrightarrow{\sim} H_c^{2d-q}(X, F)^\vee \ .$$

5.2.5. Fact (Smooth base change theorem). Given a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow f' & & \uparrow f \\ Y' & \xrightarrow{\varphi'} & X' \end{array}$$

with smooth morphism f, for any $(\mathbf{Z}/n\mathbf{Z})_Y$-module F with n prime to $\mathrm{char}(\widetilde{k})$, one has

$$f^*(R^q \varphi_* F) \xrightarrow{\sim} R^q \varphi'_*(f'^* F) \ .$$

5.2.6. Fact (Comparison theorem). Given a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ of finite type between schemes of locally finite type over k and a constructible sheaf $\mathcal{F}$ on $\mathcal{Y}$ with torsion orders prime to $\mathrm{char}(k)$, one has

$$(R^q \varphi_* F)^{\mathrm{an}} \xrightarrow{\sim} R^q \varphi_*^{\mathrm{an}} \mathcal{F}^{\mathrm{an}} \ .$$

5.2.7. Exercise. Let $\mathcal{X}$ be a scheme of finite type over k.

(i) For any constructible sheaf $\mathcal{F}$ on $\mathcal{X}$ with torsion orders prime to $\mathrm{char}(k)$, one has

$$H^q(\mathcal{X}, \mathcal{F}) \stackrel{\sim}{\to} H^q(\mathcal{X}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) \ .$$

(ii) If $(n, \mathrm{char}(k)) = 1$, then for any $\mathcal{F}^{\cdot} \in D_c^-(\mathcal{X}, \mathbf{Z}/n\mathbf{Z})$ and $\mathcal{G}^{\cdot} \in D_c^+(\mathcal{X}, \mathbf{Z}/n\mathbf{Z})$, one has

$$(\underline{\mathcal{H}om}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}))^{\mathrm{an}} \stackrel{\sim}{\to} \underline{\mathcal{H}om}(\mathcal{F}^{\mathrm{an}\cdot}, \mathcal{G}^{\mathrm{an}\cdot}) \ .$$

(iii) The functor $D_c^b(\mathcal{X}, \mathbf{Z}/n\mathbf{Z}) \to D_c^b(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}/n\mathbf{Z})$ is fully faithful.

5.2.8. Fact (Finiteness theorem). Suppose that k is algebraically closed, and let X be a compact k-analytic space with the property that every point of X has a neighborhood of the form $V_1 \cup \ldots \cup V_n$, where each V_i is isomorphic to an affinoid domain in some $\mathcal{X}^{\mathrm{an}}$. Furthermore, let F be a torsion sheaf with torsion orders prime to $\mathrm{char}(\widetilde{k})$ and such that either F is finite locally constant, or X is isomorphic to an analytic domain in some $\mathcal{X}^{\mathrm{an}}$ and F is a pullback of a constructible sheaf on $\mathcal{X}$. Then the groups $H^q(X, F)$ are finite.

5.3. The action of automorphisms on étale cohomology groups.

5.3.1. Definition. A (non-Archimedean) *analytic space* is a pair (k, X) consisting of a non-Archimedean field k and a k-analytic space X. A *morphism of analytic spaces* $(k', X') \to (k, X)$ is a pair consisting of an isometric embedding $k \hookrightarrow k'$ and a morphism of k'-analytic spaces $X' \to X\widehat{\otimes}k'$.

Let X be an analytic space. (For brevity, one writes X instead of (k, X).) One denotes by $\mathcal{G}(X)$ the automorphism group of X (automorphisms in the category of analytic spaces).

5.3.2. Definition. Let ε be the following data:

(1) $\{U_i\}_{i \in I}$ a finite family of compact analytic domains in X;

(2) for every $i \in I$, $\{f_{ij}\}_{j \in J_i}$ a finite family of analytic functions on U_i, and $\{t_{ij}\}_{j \in J_i}$ a finite family of positive numbers.

One sets $\mathcal{G}_\varepsilon(X) = \{\sigma \in \mathcal{G}(X) \big| \sigma(U_i) = U_i, \ \sup_{x \in U_i} |(\sigma^* f_{ij} - f_{ij})(x)| \leq t_{ij} \text{ for all } i \in I \text{ and } j \in J_i\}$. It is a subgroup of $\mathcal{G}(X)$.

The family of subgroups $\mathcal{G}_\varepsilon(X)$ defines a topology on $\mathcal{G}(X)$.

5.3.3. Definition. An action of a topological group G on an analytic space X is said to be *continuous* if the homomorphism $G \to \mathcal{G}(X)$ is continuous.

5.3.4. Examples. (i) If X is k-analytic, then the action of the Galois group of k on $\overline{X} = X\widehat{\otimes}\widehat{k^{\mathrm{a}}}$ is continuous.

(ii) If a k-analytic group G acts on k-analytic space X, then the action of the group of k-points $G(k)$ on X and $\overline{X}$ is continuous.

(iii) If a topological group acts continuously on a formal scheme $\mathfrak{X}$ locally finitely presented over k°, then it acts continuously on the generic fiber $\mathfrak{X}_\eta$.

(iv) The same fact (as in (iii)) is true for special formal schemes over k° (when the valuation on k is discrete).

5.3.5. Fact. Let $U \to X$ be a quasi-étale morphism with compact U. Then there exist open subgroups $\mathcal{G}_U \subset \mathcal{G}(X)$ and $\mathcal{G}' \subset \mathcal{G}(U)$ and a unique continuous homomorphism $\mathcal{G}_U \to \mathcal{G}'$ compatible with the morphism $U \to X$.

Let a topological group G act continuously on an analytic space X. It follows from Fact 5.3.5, that for any quasi-étale morphism $U \to X$ with compact U there exists an open subgroup $G_U \subset G$ and a canonical extension of the action of G_U on X to a continuous action of G_U on U.

5.3.6. Definition. (i) An action of G on an étale sheaf F on X *compatible with the action of G on X* is a system of isomorphisms $\tau(g) : F \xrightarrow{\sim} g^*F$, $g \in G$, with $\tau(gh) = h^*(\tau(g)) \circ \tau(h)$.

(ii) An action of G on F is said to be *discrete* (or F is a *discrete G-sheaf*) if, for any quasi-étale morphism $U \to X$ with compact U, the action of G_U on $F(U)$ is discrete.

(iii) For a ring Λ, the category of discrete Λ-G-module is denoted by $\mathbf{S}(X(G), \Lambda)$.

Let $\varphi : Y \to X$ be a G-equivariant Hausdorff morphism. By Fact 5.3.5, the functor $\varphi_! = \Gamma_c : \mathbf{S}(Y) \to \mathbf{S}(X)$ induces a left exact functor $\widetilde{\varphi}_! : \mathbf{S}(Y(G), \Lambda) \to \mathbf{S}(X(G), \Lambda)$.

5.3.7. Fact. For any $F \in \mathbf{S}(Y(G), \Lambda)$, one has $R^q\widetilde{\varphi}_!F \xrightarrow{\sim} R^q\varphi_!F$.

5.3.8. Exercise. (i) The canonical action of G on $R^q\varphi_!F$ is discrete.

(ii) if X is Hausdorff, then the canonical action of G on the groups $H^q_c(\overline{X}, \Lambda)$ is discrete.

5.3.9. Fact. (i) (Verdier Duality). There exists an exact functor $R\varphi^! : D^+(X(G), \Lambda) \to D^+(X(G), \Lambda)$ such that, for every pair $E^\cdot \in D^-(Y(G), \Lambda)$ and $F^\cdot \in D^+(X(G), \Lambda)$, there is a functorial isomorphism

$$R\varphi_*(\underline{\mathcal{H}om}(E^\cdot, R\varphi^! F^\cdot)) \xrightarrow{\sim} \underline{\mathcal{H}om}(R\varphi_! E^\cdot, F^\cdot) \ .$$

(ii) (Poincaré Duality). Assume that $\varphi : Y \to X$ is smooth of pure dimension d and $n\Lambda = 0$ for some n prime to char$(\widetilde{k})$. Then for every $F^\cdot \in D^+(X(G), \Lambda)$ there is a functorial isomorphism

$$\varphi^* F^\cdot(d)[2d] \xrightarrow{\sim} R\varphi^! F^\cdot \ .$$

5.4. Vanishing cycles for formal schemes. Let k be a non-Archimedean field (resp. with discrete valuation), and let $\mathfrak{X} \in k^\circ\text{-}\mathcal{F}sch$ (resp. $\mathfrak{X} \in k^\circ\text{-}\mathcal{SF}sch$). We fix a functor $\mathfrak{Y}_s \mapsto \mathfrak{Y}$ from the category of schemes étale over $\mathfrak{X}_s$ to the category of category of formal schemes étale over $\mathfrak{X}$ which is inverse to the functor from Fact 4.2.8(i) (resp. 4.3.9(i)). By Fact 4.2.8 (resp. 4.3.9) (ii) and (iii), the composition of the functor $\mathfrak{Y}_s \mapsto \mathfrak{Y}$ with the functor $\mathfrak{Y} \mapsto \mathfrak{Y}_\eta$ gives rise to a morphism of sites $\nu : \mathfrak{X}_{\eta_{\text{qét}}} \to \mathfrak{X}_{s\text{ét}}$. We get a left exact functor

$$\Theta = \nu_* \mu^* : \mathfrak{X}^\sim_{\eta_{\text{ét}}} \longrightarrow \mathfrak{X}^\sim_{\eta_{\text{qét}}} \longrightarrow \mathfrak{X}^\sim_{s\text{ét}} .$$

The similar functor for a bigger field (resp. finite extension) K is denoted by Θ_K. We set $\mathfrak{X}_{\overline{s}} = \mathfrak{X}_s \otimes \widetilde{k}^{\mathrm{s}}$ and $\mathfrak{X}_{\overline{\eta}} = \mathfrak{X}_\eta \widehat{\otimes} \widehat{k}^{\mathrm{a}}$.

5.4.1. Definition. The *vanishing cycles functor* is the functor $\Psi_\eta : \mathfrak{X}^\sim_{\eta_{\text{ét}}} \to \mathfrak{X}^\sim_{\overline{s}\text{ét}}$ which is defined as follows.

(i) If $\mathfrak{X} \in k^\circ\text{-}\mathcal{F}sch$, then $\Psi_\eta(F) = \Theta_{\widehat{k^{\mathrm{a}}}}(\overline{F})$, where $\overline{F}$ is the pullback of F on $\mathfrak{X}_{\overline{\eta}}$.

(ii) If $\mathfrak{X} \in k^\circ\text{-}\mathcal{SF}sch$, then $\Psi_\eta(F) = \varinjlim \overline{\Theta_K(F_K)}$, where the limit is taken over all finite extensions K of k in k^{s} and $\overline{\Theta_K(F_K)}$ is the lift of $\Theta_K(F_K)$ to $\mathfrak{X}_{\overline{s}}$.

If $\mathfrak{X} \in k^\circ\text{-}\mathcal{F}sch$, the definitions (i) and (ii) are compatible.

Let now $\mathcal{X}$ be a scheme finitely presented over a local Henselian ring, which is the ring of integers of a field with valuation whose completion is k, and let Ψ_η be the vanishing cycles functor for the scheme $\mathcal{X}$. For an étale sheaf $\mathcal{F}$ on $\mathcal{X}_\eta$, let $\widehat{\mathcal{F}}$ denote its pullback on the k-analytic space $\widehat{\mathcal{X}}_\eta$.

5.4.2. Fact (Comparison theorem for an arbitrary k). Let $\mathcal{F}$ be an abelian torsion sheaf on $\mathcal{X}_\eta$. Then there are canonical isomorphisms

$$R^q\Psi_\eta(\mathcal{F}) \stackrel{\sim}{\to} R^q\Psi_\eta(\widehat{\mathcal{F}}) .$$

Till the end of this subsection the valuation on k is assumed to be discrete.

Let $\mathcal{Y}$ be a subscheme of $\mathcal{X}_s$, and let $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ be the formal completion of $\mathcal{X}$ along $\mathcal{Y}$. It is special formal scheme over k°. For an étale sheaf $\mathcal{F}$ on $\mathcal{X}_\eta$ on $\mathcal{X}_\eta$, let $\widehat{\mathcal{F}}_{/\mathcal{Y}}$ denote the pullback of $\mathcal{F}$ on the k-analytic space $(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta$.

5.4.3. Fact (Comparison theorem for a discretely valued k). Let $\mathcal{F}$ be an étale abelian constructible sheaf on $\mathcal{X}_\eta$ with torsion orders prime to char$(\widetilde{k})$. Then there are canonical isomorphisms

$$(R^q\Psi_\eta\mathcal{F})\big|_{\overline{\mathcal{Y}}} \stackrel{\sim}{\to} R^q\Psi_\eta(\widehat{\mathcal{F}}_{/\mathcal{Y}}) .$$

The sheaves on the left hand side are the restrictions of the vanishing cycles sheaves of $\mathfrak{X}$ to the subscheme $\overline{\mathcal{Y}}$ of $\mathcal{X}_{\overline{s}}$.

Fact 5.4.3 implies that, given a second scheme $\mathcal{X}'$ of finite type over the same local Henselian ring, a subscheme $\mathcal{Y}' \subset \mathcal{X}'_s$, and an integer n prime to $\mathrm{char}(\widetilde{k})$, any morphism of formal schemes $\varphi : \widehat{\mathcal{X}'}_{/\mathcal{Y}'} \to \widehat{\mathcal{X}}_{/\mathcal{Y}}$ induces a homomorphism $\theta_n(\varphi)$ from the pullback of $(R^q\Psi_\eta(\mathbf{Z}/n\mathbf{Z})_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}$ to $(R^q\Psi_\eta(\mathbf{Z}/n\mathbf{Z})_{\mathcal{X}'_\eta})\big|_{\overline{\mathcal{Y}}'}$. In particular, given a prime l different from $\mathrm{char}(\widetilde{k})$, the automorphism group of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ acts on the l-adic sheaves $(R^q\Psi_\eta(\mathbf{Q}_l)_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}$.

5.4.4. Fact (Continuity theorem). (i) Given $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, $\widehat{\mathcal{X}'}_{/\mathcal{Y}'}$ and n as above, there exists an ideal of definition $\mathcal{J}'$ of $\widehat{\mathcal{X}'}_{/\mathcal{Y}'}$ such that for any pair of morphisms $\varphi, \psi : \widehat{\mathcal{X}'}_{/\mathcal{Y}'} \to \widehat{\mathcal{X}}_{/\mathcal{Y}}$, which coincide modulo $\mathcal{J}'$, one has $\theta_n(\varphi) = \theta_n(\psi)$.

(ii) Given $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ and l as above, there exists an ideal of definition $\mathcal{J}$ of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ such that any automorphism of $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, trivial modulo $\mathcal{J}$, acts trivially on the sheaves $(R^q\Psi_\eta(\mathbf{Q}_l)_{\mathcal{X}_\eta})\big|_{\overline{\mathcal{Y}}}$.