INTRODUCTION TO MANIFOLDS — I

Definitions and examples

1. Topologic spaces

♡ **Definition.** A `topological space` M is an abstract point set with explicit indication of which subsets of it are to be considered as open.

Such open-by-definition subsets are to satisfy the following tree axioms:

(1) $\varnothing$ and M are open,
(2) intersection of any `finite number` of open sets is open, and
(3) union of any (infinite) number of open sets is open.

After the notion of an open set is introduced, all the remaining notions of an elementary analysis appear:

- closed sets as complementary to open ones,
- compact subsets as admitting selection of a finite subcovering from any (infinite) open covering,
- connected sets as not representable as unions of disjoint open sets,
- continuous mappings between two topological spaces as mappings yielding open preimages for open subsets of a target space,
- converging sequences of points as the ones which get into any open set containing the limit, after sufficiently many steps,
- *etc.*

Examples.

♢ *Example.* Discrete (finite or infinite) sets with the `discrete` topology: all subsets are declared to be open.

♣ **Problem 1.** Describe all connected discrete spaces, compact discrete spaces.

♢ *Example.* "Normal" spaces like $\mathbb{R}^n$, intervals (open and closed), *etc.*

♣ **Problem 2.** Why a closed interval $[0,1] \subseteq \mathbb{R}^1$ is a topological space?

♢ *Example.* Zarissky topology on $\mathbb{R}$ and $\mathbb{Z}$: open are sets whose complement is finite, and the only other open set is the empty set.

♣ **Problem 3.** Prove that a subset is closed in Zarissky topology, if and only if it is the zero set of a polynomial $\mathbb{R} \to \mathbb{R}$ (resp., $\mathbb{Z} \to \mathbb{Z}$).

♢ *Example.* A set consisting of two elements, a and b, with the following open subsets: $\varnothing, a, \{\, a, b \,\}$.

♣ **Problem 4.** Describe all different types of topological spaces consisting of 3 and 4 points, so that the ones differing only by re-enumeration of points, would be considered as identical. Is there a formula for the number of non-equivalent (in such a sense) spaces for a general n? (I don't know the answer).

♡ **Definition.** A `base` of a topology is a family of open subsets such that any other open set may be represented as the union of subsets constituting the base of the topology.

♢ *Example.* Surgery on "normal" topological spaces: the line with two zeros. Take two copies of the real line, $\mathbb{R}_1$ and $\mathbb{R}_2$, and "glue" them together by all nonzero points. In other words, consider the equivalence relation on $\mathbb{R}_1 \cup \mathbb{R}_2$ as follows, $x_1 \sim x_2 \iff x_1 = x_2 \neq 0$, and look at the quotient space $(\mathbb{R}_1 \cup \mathbb{R}_2)/\sim$. Open (by definition) are the sets such that their full prototypes in $\mathbb{R}_1 \cup \mathbb{R}_2$ are open.

♢ *Example.* In the same way a "fork" can be defined as the result of factorization by the equivalence relation $x_1 \sim x_2 \iff x_1 = x_2 > 0$.

♣ **Problem 5.** What will be wrong with the equivalence relation $x_1 \sim x_2 \iff x_1 = x_2 \geqslant 0$?

♡ **Definition.** A topology is `Hausdorff`, if for any two distinct points x_1, x_2 there exist two `disjoint` open subsets $U_j \ni x_j,\ j = 1, 2$.

Other examples as well as some operations resulting in construction of new topological spaces from old ones, are given below.

Constructions in the category of topological spaces.

- A subset N of a topological space M is the topological space itself, if one declares as open the intersections with N of open subsets in M. Such a topology is called the inherited topology.
- If M, N are topological spaces, then the Cartesian product $M \times N$ is also a topological space.
- Quotient spaces (see above): if there is an equivalence relation $\sim$ on a topological space M, then sometimes the quotient space $M/\sim$ is a topological space also.
- Let M be a `metric space`, that is, the set endowed with a nonnegative symmetric function $\rho\colon M \times M \to \mathbb{R}_+$ satisfying the two axioms,

$$\rho(x, y) = 0 \iff x = y, \tag{1}$$

$$\forall x, y, z \in M \quad \rho(x, y) + \rho(y, z) \geqslant \rho(x, z) \text{ (the triangle inequality)} \tag{2}$$

  The topology is defined by the metric if one chooses open balls $B_r(x) = \{\, y \in M : \rho(x, y) < r \,\}$ as the base of the topology.
- There are lots of different ways to produce new topological spaces. The structure of a topological space is almost the weakest structure within which it makes sense to work in analysis, geometry etc.

♢ *Example.* p-adic numbers. Let p be a prime number and define the p-adic norm for an integer $n = \pm p^d \cdot q,\ (p, q) = 1$, as $\|n\|_p = p^{-d}$. Put $\rho_p(m, n) = \|n - m\|_p$. The metric space $\mathbb{Z}, \rho_p$ is not `complete`, that is, there are fundamental sequences which have no limits, but the standard completion yields some strange object, namely, p-adic integers.

♣ **Problem 6.** Prove that there exists the square root $\sqrt{-1}$ in 5-adics. Find it.

♣ **Problem 7.** Extend this definition to p-adic rationals.

♣ **Problem 8.** Are these spaces Hausdorff?

♢ *Example.* Functional spaces, $C^k(\mathbb{R}^n, \mathbb{R}^m)$, $k = 0, 1, 2, \dots$ The topology is defined via metric, but one has to be careful when dealing with suprema/maxima etc.

One has to be careful when passing to subsets endowed with the inherited topology, however innocent this procedure may look like.

♢ *Example.* Let M be a 2-torus $\mathbb{R}^2/\mathbb{Z}^2$, and $f\colon \mathbb{R} \to M$ be a smooth map,

$$t \mapsto f(t) = (t \mod \mathbb{Z}, \sqrt{2} \mod \mathbb{Z}).$$

This map is bijective and continuous, so one may expect that the topology inherited from this embedding, coincides with the standard topology on $\mathbb{R}$, and it is in fact so if one considers finite segnments of $\mathbb{R}$, but for the entire line the two topologies are different.

♣ **Problem 9.** Prove the latter statement!

♢ *Example.* Some other topological spaces:

- $n \times n$-matrices eventually satisfying some additional requirements (nondegeneracy, zero trace, unitarity etc). These examples are of essential importance since they give rise to the notion of `Lie groups`.
- Riemann surfaces.
- Spaces of loops.
- Spheres, projective spaces, tori etc.
- A lot of other objects: eventually almost any object arising in mathematics admits a certain topological-like structure.

Most of the examples listed above in fact admit some more delicate structures than merely the structure of a topological space. Wait a little!

♡ **Definition.** A `homeomorphism` between two topological spaces M and N is a bijective (=one-to-one) map $f\colon M \to N$ such that both f and f^{-1} are continuous (with respect to the topologies of M and N).

Beware: if, say, M is a topologic space, and N is just a point set, while f is bijective, then N may be endowed with a topology in such a way that f would automatically become a homeomorphism. Don't try to prove trivial statements!

♣ **Problem 10.** Describe formally the `induced topology` on N.

2. Invertible differentiable mappings ≡ diffeomorphisms.

The notion of a topological space does not yet allow for introducing the differentiability concept known from the elementary calculus. So the basic step is to understand the properties of the principal example of the topologic space, namely, $\mathbb{R}^n$ (the Euclidean arithmetic space, or simply the Euclidean space). The coordinate functions on $\mathbb{R}^n$ will be denoted by $x_1, \dots, x_n$. By definition,

$$x_j \colon \mathbb{R}^n \to \mathbb{R}, \qquad (x_1, \dots, x_n) \mapsto x_j$$

are `scalar functions` of many scalar variables.

♣ **Problem 11.** Prove that there is no homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$.

♡ **Definition.** A map $f\colon \mathbb{R}^n \supseteq U \to \mathbb{R}^n$ is a (particular case of) diffeomorphism in an open subset U, if it is differentiable (that is, all the coordinate functions $f_1, \dots, f_n$ of f are smooth), f is the homeomorphism between U and its image $f(U)$, and the inverse map $f^{-1}\colon f(U) \to U$ is also differentiable.

♣ **Problem 12.** Prove that the Jacobian matrix of a diffeomorphism

$$f_*(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is nondegenerate everywhere in U:

$$\forall x \in U \qquad \det f_*(x) \neq 0.$$

In $\mathbb{R}^n$ everything can be explicitly described using coordinate functions.

♣ **Problem 13.** Give the definitions of:

(1) smooth curve,
(2) smooth hypersurface,
(3) tangency between a curve and a hypersurface,
(4) length of a smooth curve,
(5) volume of a domain bounded by a smooth hypersurface,
(6) angle between two curves.

♣ **Problem 14.** Why when giving a definition of a diffeomorphism, it was necessary to restrict oneself by the class of open domains only? Invent a definition of a function smooth in the closed unit ball

$$\overline{B} = \left\{ x \in \mathbb{R}^n \colon \sum_{j=1}^{n} x_j^2 \leqslant 1 \right\}.$$

If U is an open domain, and there is something in it, and if f is a diffeomorphism of U, then something appears in $f(U)$ as well. So some properties may persist (be `invariant`) `under diffeomorphisms`, whilst others may be not.

♣ **Problem 15.** Prove that:

(1) The image of a smooth curve under a diffeomorphism is a smooth curve again.
(2) The same holds true for smooth hypersurfaces.
(3) The length of a smooth curve in general is `not invariant`.
(4) The same holds false for volumes of domains bounded by smooth hypersurfaces.
(5) The tangnecy between curves, or between a curve and a hypersurface persists under diffeomorphisms.
(6) The angles in general are `not preserved`.

♣ **Problem 16.** If two intersecting compact smooth curves in $\mathbb{R}^3$ belong to a smooth surface, then the images also do. Prove this statement. Why it is crazy?

♣ **Problem 17.** The same problem concerning three smooth curves passing through a single point.

3. Definition of a smooth manifold. Examples.

♡ **Definition.** A (smooth) n-`dimensional manifold` M is a Hausdorff (usually) topological space which is locally Euclidean: more precisely, for each point $a \in M$ there is a `local chart`, that is, a map defined in some open neighborhood $U_a \subseteq M$ of a, which takes its values in the Euclidean space $\mathbb{R}^n$ and is a homeomorphism of U_a onto its image:

$$x\colon M \supseteq U_a \to \mathbb{R}^n, \qquad x(\cdot) = (x_1(\cdot), \dots, x_n(\cdot)),\ x_j\colon U_a \to \mathbb{R}.$$

These charts must agree with each other: on any intersection $U_a \cap U_b$ where at least two charts are defined, their `transition functions`

$$y \circ x^{-1}\colon x(U_a \cap U_b) \to \mathbb{R}^n$$

must be a diffeomorphism between $x(U_a \cap U_b)$ and $y(U_a \cap U_b)$.

Such a collection of maps charting all of the manifold, is called an `atlas` of charts.

General principle. *Any property of any object, which is invariant by diffeomorphisms of Euclidean space, can be reformulated for the category of smooth manifolds.*

♣ **Problem 18.** Suggest definitions for:

(1) A smooth curve on a manifold.
(2) A differentiable function on a manifold.
(3) A polynomial function on a manifold*.
(4) Tangency on a manifold
(5) A smooth map between two manifolds.
(6) A diffeomorphism between two manifolds.
(7) A volume-preserving map between two manifolds*.

Why some of the problems are marked by asterisks?

♣ **Problem 19.** Prove that the following are smooth manifolds (each time you should think which natural structure and natural topology are meant).

(1) A $(n-1)$-sphere S^{n-1}.
(2) More generally, a smooth hypersurface defined by one scalar equation $f(x) = 0$ in $\mathbb{R}^n$, provided that the function f has a nonzero gradient at all points of the surface.
(3) The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
(4) The set of all nondegenerate $n \times n$-matrices $GL(n, \mathbb{R})$.
(5) The set of matrices $sl(n, \mathbb{R})$ with the zero trace.
(6) The set $SL(n, \mathbb{R})$ of determinant 1 matrices.

♣ **Problem 20.** Prove that the exponential map

$$\exp\colon \operatorname{Mat}_{n\times n}(\mathbb{R}) \to GL(n,\mathbb{R}), \qquad A \mapsto \exp(A) = E + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots$$

yields a diffeomorphism between $sl(n,\mathbb{R})$ and $SL(n,\mathbb{R})$.

♣ **Problem 21.** Prove that the conjugacy map

$$\operatorname{ad}_C\colon A \mapsto C^{-1}AC, \qquad \det C \neq 0,$$

is a diffeomorphism of all the above manifolds onto themselves.

♣ **Problem 22.** A rotation of the Euclidean space induces a diffeomorphism of the unit sphere. Prove.

♣ **Problem 23.** A shift $x \mapsto x + \omega$, $x, \omega \in \mathbb{R}^n$ induces an automorphism of the torus $\mathbb{R}^n$. Prove.

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