INTRODUCTION TO MANIFOLDS — I

Supplementary problems

Matrix manifolds

♣ **Problem 1.** Let $M = \mathrm{Mat}_{n\times n} \simeq \mathbb{R}^{n^2}$ be the set of all square matrices. Prove that the map

$$\mathrm{Ad}_C\colon A \mapsto C^{-1}AC, \qquad \det C \neq 0,$$

defines a diffeomorphism of the manifold M onto itself.

♣ **Problem 2.** The same question about the manifold $N \subset M$ of matrices of determinant 1.

♣ **Problem 3.** The same question but for the map

$$M \ni A \mapsto B^{-1}AC, \qquad \det B, \det C \neq 0. \tag{1}$$

♣ **Problem 4.** Prove that for any two matrices of the same rank there exists a diffeomorphism $M \to M$ of the form (1) taking one into the other.

♣ **Problem 5.** Prove that

$$\det(E + \varepsilon B) = 1 + \varepsilon \operatorname{tr} B + O(\varepsilon^2).$$

Using the previous problem, deduce the formula for the first order term in the expansion $\det(A + \varepsilon B)$, when $\det A \neq 0$.

♣ **Problem 6.** Prove that the set of matrices $M_r \subset M$ of the rank $r \leqslant n$ is a smooth submanifold in M. Is this submanifold closed? Compute its dimension. (Answer: $n^2 - (n-r)^2 = 2nr - r^2$.)

Partition of unity

Everywhere below M stands for a smooth n-dimensional manifold.

♣ **Problem 7.** Construct a C^∞-smooth function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) > 0$ if $x > 0$ and $f(x) = 0$ when $x \leqslant 0$. Why such an example is impossible in the analytic category?

♣ **Problem 8.** Construct a function which will be positive only on the interval $(0,1) \subset \mathbb{R}$, and identically zero outside.

♣ **Problem 9.** Construct a smooth nonnegative function which is equal to 1 on $(-1,1)$ and vanishes outside $(-2,2)$.

♣ **Problem 10.** The same question about a function $\varphi : \mathbb{R}^n \to \mathbb{R}_+$, equal to 1 for $\|x\| < 1$ and vanishing for $\|x\| \geqslant 2$.

♣ **Problem 11.** Prove that on a smooth manifold M^n for any two points $x \neq y$ there exists a nonnegative function which is identically equal to 1 near x and to zero near y.

♣ **Problem 12.** Prove that for any point x there is a neighborhood U, $M \supseteq U \ni x$, diffeomorphic to $\mathbb{R}^n$.

♣ **Problem 13.** Construct a diffeomorphism $\mathbb{R}^{n+1} \supset \mathbb{S}^n -$ (North Pole) $\to \mathbb{R}^n$. (Answer: stereographic projection.)

♣ **Problem 14.** Prove that for any point $x \in M$ there exists a neighborhood U and a smooth map $f \colon M \to \mathbb{S}^n$ which is a diffeomorphism between U and $\mathbb{S}^n -$ (North Pole).

♣ **Problem 15.** Prove that for any compact manifold M there exists an injective smooth map $f \colon M \to \mathbb{R}^N$ for a sufficiently big N, which has rank n everywhere on M (**the Whitney embedding theorem in the weakest form**).

♣ **Problem 16.** Prove that for any discrete set of points $x_1, x_2, \dots \in M$ there exist a smooth function $f \colon M \to \mathbb{R}$ which has nondegenrate local minima at these points, $f(x_i) = 0$, and positive outside of the set.

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