Supplementary problems

Matrix manifolds

Problem 1. Let $M = \text{Mat}_{n \times n} \simeq \mathbb{R}^{n^2}$ be the set of all square matrices. Prove that the map

$$\text{Ad}_{C} : A \mapsto C^{-1} A C, \quad \det C \neq 0,$$

defines a diffeomorphism of the manifold $M$ onto itself.

Problem 2. The same question about the manifold $N \subset M$ of matrices of determinant 1.

Problem 3. The same question but for the map

$$M \ni A \mapsto B^{-1} A C, \quad \det B, \det C \neq 0.$$  \hfill (1)

Problem 4. Prove that for any two matrices of the same rank there exists a diffeomorphism $M \to M$ of the form (1) taking one into the other.

Problem 5. Prove that

$$\det(E + \epsilon B) = 1 + \epsilon \text{tr} B + O(\epsilon^2).$$

Using the previous problem, deduce the formula for the first order term in the expansion $\det(A + \epsilon B)$, when $\det A \neq 0$.

Problem 6. Prove that the set of matrices $M_r \subset M$ of the rank $r \leq n$ is a smooth submanifold in $M$. Is this submanifold closed? Compute its dimension. (Answer: $n^2 - (n - r)^2 = 2nr - r^2$.)

Partition of unity

Everywhere below $M$ stands for a smooth $n$-dimensional manifold.

Problem 7. Construct a $C^\infty$-smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) > 0$ if $x > 0$ and $f(x) = 0$ when $x \leq 0$. Why such an example is impossible in the analytic category?

Problem 8. Construct a function which will be positive only on the interval $(0,1) \subset \mathbb{R}$, and identically zero outside.

Problem 9. Construct a smooth nonnegative function which is equal to 1 on $(-1,1)$ and vanishes outside $(-2,2)$.
Problem 10. The same question about a function $\varphi : \mathbb{R}^n \to \mathbb{R}_+$, equal to 1 for $\|x\| < 1$ and vanishing for $\|x\| \geq 2$.

Problem 11. Prove that on a smooth manifold $M^n$ for any two points $x \neq y$ there exists a nonnegative function which is identically equal to 1 near $x$ and to zero near $y$.

Problem 12. Prove that for any point $x$ there is a neighborhood $U, M \supset U \ni x$, diffeomorphic to $\mathbb{R}^n$.

Problem 13. Construct a diffeomorphism $\mathbb{R}^{n+1} \supset S^n - \text{(North Pole)} \to \mathbb{R}^n$. (Answer: stereographic projection.)

Problem 14. Prove that for any point $x \in M$ there exists a neighborhood $U$ and a smooth map $f : M \to S^n$ which is a diffeomorphism between $U$ and $S^n - (\text{North Pole})$.

Problem 15. Prove that for any compact manifold $M$ there exists an injective smooth map $f : M \to \mathbb{R}^N$ for a sufficiently big $N$, which has rank $n$ everywhere on $M$ (the Whitney embedding theorem in the weakest form).

Problem 16. Prove that for any discrete set of points $x_1, x_2, \cdots \in M$ there exist a smooth function $f : M \to \mathbb{R}$ which has nondegenerate local minima at these points, $f(x_i) = 0$, and positive outside of the set.

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