INTRODUCTION TO MANIFOLDS — II

Tangent Bundles

1. Tangent vectors, tangent space.

Let M^n be a smooth n-dimensional manifold, endowed with an atlas of charts $x\colon U \to \mathbb{R}^n$, $y\colon V \to \mathbb{R}^n$, …, where $M = U \cup V \cup \cdots$ are domains of the corresponding charts.

♡ **Definition.** Two smooth curves $\varphi_i\colon (-\varepsilon, \varepsilon) \to M$, $i = 1, 2$, passing through the same point $p \in M$, are said to be `1-equivalent`, $\varphi_1 \sim \varphi_2$, if in some chart $x\colon U \to \mathbb{R}^n$

$$\|x(\varphi_1(t)) - x(\varphi_2(t))\| = o(t) \qquad \text{as } t \to 0^+. \tag{1}$$

♣ **Problem 1.** Prove that the condition (1) is actually independent of the choice of the chart.

♡ **Definition.** The `tangent space` to the manifold M at the point p is the quotient space $C^1(\mathbb{R}^1, M)/\sim$ by the equivalence (1).

Notations: the equivalence class of a curve φ will be denoted by $[\varphi]_p$. Instead of saying that a curve φ belongs to a certain equivalence class $v = [\cdot]_p$, we say that the curve φ is tangent to the vector v.

♣ **Problem 2.** Prove that the tangent space at each point is isomorphic to the arithmetic space $\mathbb{R}^n$.

Solution. Fix any chart x around the point p and consider the maps $\widetilde{\mathrm{iso}}, \mathrm{iso}$ defined as

$$\widetilde{\mathrm{iso}}\colon \mathbb{R}^n \to C^\infty(\mathbb{R}, M), \quad v = (v_1, \dots, v_n) \mapsto \varphi_v(\cdot), \quad \varphi_v(t) = x^{-1}(x(p) + tv) \tag{2}$$

$$\mathrm{iso}(v) = [\widetilde{\mathrm{iso}}(v)]_p. \tag{3}$$

This map is injective (prove!). To prove its surjectivity, for any smooth curve φ consider its x-coordinate representation,

$$x(\varphi(t)) = x(p) + tv + \cdots,$$

existing by virtue of differentiability of the latter. Then $\widetilde{\mathrm{iso}}(v) \sim \varphi$.

Remark. This is a good example of abstract nonsense! The idea is that you associate with each curve its linear terms, the coordinate system being fixed. Then any curve is uniquely defined by its linear terms up to the 1-equivalence, since the definition (1) was designed especially for this purpose!

♡ **Definition.** The string of real numbers $(v_1, \dots, v_n)$ is called the coordinate representation of the tangent vector $[\varphi]_p$ in the coordinate system x.

♣ **Problem 3.** If M^n is a hypersurface in $\mathbb{R}^{n+1}$, then the tangent space is well defined by geometric means. Prove that this "geometric" tangent space is isomorphic to the one defined by the abstract definition above. A good exercise for practicing in abstract nonsense!

`Important note:` The coordinate system x occurs in the construction of isomorphisms (2), (3) in the most essential way! If another coordinate system is chosen, then the isomorphisms will be completely different.

♣ **Problem 4.** Let $v = [\varphi]_p \in T_pM$ be a tangent vector associated with a tuple $(v_1, \dots, v_n)$ in a coordinate system x, and y another coordinate system around the same point, with the transition functions h: $y = h(x) \iff x = h^{-1}(y)$. Find the coordinate representation of `the same vector` in the new coordinates y.

♣ **Problem 5.** Prove that the tangent space possesses the natural linear structure. (Warning: you have to formalize, what does this mean!)

Notations: the tangent space at a point $p \in M$ is denoted by T_pM.

Remark. If some two points p, q belong to the same coordinate neighborhood x and $(v_1, \dots, v_n)$ is a tuple of reals, then one can take two vectors with the same coordinates, but attached to different points p and q. These two vectors must be considered as tt different! In other words, $T_pM \cap T_qM = \varnothing$, if $p \neq q$. This seemingly contradicts the geometric intuition, but is much more convenient for other means.

♣ **Problem 6.** Prove that in general two vectors attached to different points but having the same coordiantes in a certain chart, become different in another coordinate system.

♡ **Definition.** The `tangent bundle` of the manifold M is another manifold TM, which is defined (as the point set) as

$$TM = \bigcup_{p \in M} T_pM.$$

The structure of a smooth manifold is defined on TM explicitly. If $\{ U_\alpha \}$ is an atlas of charts, $x^\alpha : U_\alpha \to \mathbb{R}^n$, with the transition functions $h_{\alpha\beta}$, then we define the charts V_α covering TM in the following way,

$$V_\alpha = \bigcup_{p \in U_\alpha} T_pM \subseteq TM,$$

and introduce the coordinate functions $X^\alpha : V_\alpha \to \mathbb{R}^{2n}$ by the formulas

$$v \in T_pM, \ p \in U_\alpha \implies X^\alpha(v) = (x_1^\alpha, \dots, x_n^\alpha, v_1^\alpha, \dots, v_n^\alpha), \tag{4}$$

where $(v_1^\alpha, \dots, v_n^\alpha)$ is the coordinate representation of the vector v in the coordinate system x^α.

Immediately the following series of questions arises.

♣ **Problem 7.** Prove that each of the maps X^α is one-to-one, and all the charts V_α constitute a covering of TM.

♣ **Problem 8.** Write down explicitly the transition functions between the charts X^α, $H_{\alpha\beta} = X^\beta \circ (X^\alpha)^{-1}$, and prove that they are differentiable. What is the guaranteed order of their smoothness?

♢ *Example.* The tangent bundle to the Euclidean space $\mathbb{R}^n$ is the Euclidean space $\mathbb{R}^{2n}$. More generally, if $U \subseteq \mathbb{R}^n$ is an open domain, then $TU \simeq U \times \mathbb{R}^n$.

♢ *Example.* The tungent bundle $T\mathbb{S}^1$ to the circle $\mathbb{S}^1$ is the cylinder $\mathbb{S}^1 \times \mathbb{R}^1$.

♣ **Problem 9.** Is it true that TM is always diffeomorphic to $M \times \mathbb{R}^n$?

2. Tangent maps

Let M^m, N^n be two smooth manifolds, TM and TN their tangent bundles, and $f\colon M \to N$ a smooth map.

♣ **Problem 10.** If there are two 1-equivalent curves, φ_1 and φ_2 passing through the same point $p \in M$, then the two curves $f \circ \varphi_i$, $i = 1, 2$, are also 1-equivalent. Prove this. Is the converse true?

♡ **Definition.** The `differential` of the map f at the point $p \in M$ is the map taking a tangent vector $[\varphi]_p \in T_pM$ to the vector $[f \circ \varphi]_q \in T_qN$, where $q = f(p) \in N$ is the image of the point p. The differential is denoted by

$$f_{*p}\colon T_pM \to T_{f(p)}N, \qquad v \mapsto f_{*p}v.$$

♣ **Problem 11.** Prove that the differential is a `linear` map (in which sense?). This explains why in the previous formula we did not use parentheses around the argument v.

♡ **Definition.** The `tangent map` (sometimes it is also called differential) to (of) the map f is the map

$$f_*\colon TM \to TN, \qquad (p, v) \mapsto (f(p), f_{*p}v).$$

Another possible notations for the tangent map: Df, Tf, $\frac{\partial f}{\partial p}$, f', in short, almost all symbols used for derivatives in elementary calculus.

♣ **Problem 12.** Prove that if f is a diffeomorphism between M and N, then f_* is a diffeomorphism between the corresponding tangent bundles.

♣ **Problem 13.** Let $\mathbb{S}^1 \simeq \{\, z \in \mathbb{C}\colon |z| = 1 \,\}$ be the unit circle, and $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ the map $z \mapsto z^2$. Compute its differential.

The tangent bundle is a certain way to associate with any point of a smooth manifold M, a linear space T_pM which differentiably depends on the point p. The linear structure on this space is canonical, that is, defined without any reference to the coordinates. All additional geometric structures on manifolds (differential forms, Riemann metric, volumes, symplectic structure *etc* are based on such a fundamental structure associated with a manifold.

3. Vector fields

There exists the natural projection $\pi\colon TM \to M$, which is a smooth map (prove!) taking each pair (p, v), $v \in T_pM$ into the point p at which the latter is attached.

♡ **Definition.** A `section` of the tangent bundle is a smooth map $u\colon M \to TM$ which satisfies the identity $\pi \circ u = \mathrm{id}_M$.

(Another excellent example of jabberwocky: a section u is a certain way to associate a vector $u(p)$ with any point p in such a way that it would depend differentiably on the point p.)

A section of the tangent bundle is usually called a vector field on the manifold M. The notion of a vector field is a substitute to the notion of an ordinary differential equation on manifolds.

♡ **Definition.** An `integral curve` (sometimes `phase curve`) of a vector field u is a smooth map

$$\varphi\colon \mathbb{R}^1 \supseteq I \to M, \qquad t \mapsto \varphi(t),$$

such that

$$\forall t \in I\ [\varphi]_p = u(p), \qquad \text{where } p = f(t).$$

Again an abstract nonsense: an integral curve is a curve which is tangent to the vector field u at all points of the former.

♣ **Problem 14.** Prove that a smooth vector field u in any coordinate neighborhood is determined by n smooth functions $u_1(x_1, \dots, x_n), \dots, u_n(x_1, \dots, x_n)$, and a curve determined by its coordinate representation as

$$t \mapsto x(t) = (x_1(t), \dots, x_n(t)), \qquad t \in I$$

is integral if and only if it satisfies the differential equation

$$\dot{x}_i = u_i(x_1, \dots, x_n), \qquad i = 1, \dots, n. \tag{5}$$

♣ **Problem 15.** How would you define a nonautonomous differential equation?

The same vector field in different coordinates gives rise to different differential equations, though proprties of these equations remain the same up to a diffeomorphism. So a great idea comes: one has to study vector fields in a coordinate system in which the corresponding differential equations would have the simplest form. In short, it is useful to look for possible transformations of a given ODE.

♣ **Problem 16.** Find a formula for change of variables in ordinary differential equations.

The principal result concerning differential equations of the form (5) is the existence/uniqueness theorem.

Theorem. *If the right hand side parts of a differential equation* (5) *in a certain domain* $U \subseteq \mathbb{R}^n$ *are sufficiently smooth, then for any point* $a \in U$ *and for all sufficiently small values of* $t \in (\mathbb{R}^1, 0)$ *there exists a unique solution to* (5) *starting at* a*, that is, a smooth vector function*

$$\varphi_a \colon t \mapsto (x_1(t), \dots, x_n(t)) \in U$$

satisfying the equation (5) *and the initial condition*

$$\varphi_a(0) = a.$$

The function

$$F \colon \mathbb{R} \times U \ni (t, a) \mapsto \varphi_a(t) \in U$$

is defined on an open subset of $\mathbb{R}^{n+1}$ *and smooth on it (whenever defined). Such a function is called the* `flow map` *of the equation* (5).

♣ **Problem 17.** Prove that for any smooth vector field aand any point on the manifold, there exists a unique (up to change of the domain) integral curve passing through the point.

♣ **Problem 18.** Prove that if M is a `compact` manifold, then such a curve is defined globally: there exists a map $\varphi \colon \mathbb{R}^1 \to M$ with $\varphi(0) = a$ and tangent to the field everywhere.

The principal uniqueness/existence theorem can be reformulated for manifolds in another, much more spectacular way.

Rectification Theorem. *If* u *is a smooth vector field on a manifold, and* $p \in M$ *is a point such that* $u(p) \neq 0$*, then there exists a coordinate system* x *around the point* p *such that in this coordinate system the field* u *is* `parallel`*: all the vectors* $u(p')$ *for* p' *in the range of the chart, correspond to the same tuple of reals, say,* $(1, 0, \dots, 0)$.

Remark. A point at which a field vanishes, is called `singular point` of the vector field.

♣ **Problem 19.** Is the notion of singuar point invariantly defined?

The Rectification theorem in turn implies the existence theorem: one should apply it to the vector field on $\mathbb{R} \times M = \widetilde{M}$ given by the formula $\widetilde{v}(t,p) = (1, u(p))$ (how do you write the last formula in Jabberwocky?)

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