INTRODUCTION TO MANIFOLDS — V

Algebraic language in Geometry (continued).

Everywhere below $F\colon M \to N$ is a smooth map, and $F^*\colon C^\infty(M) \to C^\infty(N)$ the associated homomorphism of commutative algebras, $F^* g = g \circ F \iff (F^* g)(x) = g(F(x))$.

Let $x \in M$ be a point of a smooth manifold, and $\mathfrak{m}_x \subseteq C^\infty(M)$ the corresponding maximal ideal:

$$\mathfrak{m}_x = \{\, f \in C^\infty(M)\colon f(x) = 0 \,\}.$$

♡ **Definition.**

$$\mathfrak{m}_x^2 := \left\{ \sum_\alpha f_\alpha g_\alpha, \quad f_\alpha, g_\alpha \in \mathfrak{m}_x \right\}.$$

In the same way higher powers $\mathfrak{m}_x^k$ of a maximal ideal are defined.

♣ `Problem 1.`

$$\mathfrak{m}_x^2 = \{\, f \in C^\infty(M) : f(y) = O(|y-x|^2) \,\} = \boxed{\text{functions without free and linear terms in the Taylor expansion centered at } x.} \qquad \square$$

♣ `Problem 2.` If $F\colon M \to N$ a smooth map, $F(a) = b$, then $F^* \mathfrak{m}_b^k \subseteq \mathfrak{m}_a^k$ for any natural k. $\square$

♣ `Problem 3.` $(F^*)^{-1} \mathfrak{m}_a^k = \mathfrak{m}_b^k$. $\square$

♣ `Problem 4.` A tuple of functions $f_1, \dots, f_k \in \mathfrak{m}_a \subseteq C^\infty(M)$ has rank[1] $< k$ at the point $a \iff \exists c_1, \dots, c_k \in \mathbb{R}\colon \sum_k c_k f_k \in \mathfrak{m}_a^2$. $\square$

♣ `Problem 5.`

$$\operatorname{rank}_a (F^* f_1, \dots, F^* f_k) \leqslant \operatorname{rank}_{F(a)} (f_1, \dots, f_k). \quad \square$$

♣ `Problem 6.` Give an example of the sharp inequality in the above formula. $\square$

[1]The rank of a system of functions at a certain point is by definition the rank of the Jacobian matrix evaluated at this point.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Theorem. *If the morphism F^* is surjective, then the corresponding map is an immersion*[2],

$$\forall a \in M \qquad \operatorname{rank}_a F = \dim M,$$

and $a \neq b \implies F(a) \neq F(b)$. $\square$

♣ `Problem 7.` Prove that the inverse is true provided that M is compact. $\square$

♣ `Problem 8.` Give a counterexample if M is not compact. $\square$

♣ `Problem 9.` If F is a surjective map (i.e. $F(M) = N$), then F^* is an injective morphism. Prove. $\square$[3]★

♣ `Problem 10.` Is the inverse true? Prove that it is, provided that M is compact. $\square$

Inspired by the above Theorem, one could think that *if the morphism F^* is* `surjective`*, then the map F is a* `submersion`, that is, the rank of its differential at any point is equal to $\dim N$.

♣ `Problem 11.` Prove that such a naiveness is unjustified. $\square$

Cotangent space

Let $a \in M, b \in N$ be a pair of points, $F(a) = b$.

♣ `Problem 12.` Prove that the `quotient spaces`

$$T_a^*M = \mathfrak{m}_a/\mathfrak{m}_a^2, \qquad T_b^*N = \mathfrak{m}_b/\mathfrak{m}_b^2$$

are linear spaces, their dimensions are equal to the dimensions of M (resp., N), and F^* induces the `linear map`

$$T_b^*F \colon T_b^*N \to T_a^*M.$$

♡ **Definition.** The space $T_a^*M = \mathfrak{m}_a/\mathfrak{m}_a^2$ is called the `cotangent space` to the manifold M at the point a. The union of all cotangent spaces,

$$T^*M = \bigcup_{a \in M} T_a^*M,$$

is the `cotangent bundle` of M.

♣ `Problem 13.` Differentials of smooth functions at a point a are in one-to-one correspondence with elements of `the` cotangent space T_a^*M[4].

♣ `Problem 14.` Prove that a derivative $D \in \operatorname{Der}(C^\infty(M))$ induces a linear functional on any cotangent space:

$$D \quad \rightsquigarrow \quad D_a \colon T_a^*M \to \mathbb{R},$$
$$D_a \colon df(a) \mapsto (L_v f)(a), \; v \leftrightsquigarrow D.$$

[2] The rank of a map is the rank of its differential.

[3] $g_1(b) \neq g_2(b),\ F^*g_1 = F^*g_2 \implies a \notin F(M)$.

[4] But globally this is not so, beware!

♡ **Definition.** A `tangent space` to a manifold M at a point a is the dual space,

$$T_a M = (\mathfrak{m}_a/\mathfrak{m}_a^2)^*.$$

The tangent and cotangent bundles and all other elements of geometric picture of the World can be introduced in terms of the `structural ring` $C^\infty(M)$ `of a manifold` M.

Looking forward...

Let $A = C^\infty(M)$ be the structural algebra, and $I \subseteq A$ an ideal consisting of functions which vanish on a closed subset $Z \subseteq M$. Assume whatever regularity you want about Z and prove ...

♣ `Problem 15.` The space $C^\infty(Z)$ is isomorphic to the quotient space $C^\infty(M)/I$. This isomorphism is an isomorphism of algebras. □

♣ `Problem 16.` Let $I = \mathfrak{m}_a$ be a maximal ideal. What is then the `local ring`

$$C^\infty(M)/\mathfrak{m}_a = A_a?$$

Prove that it is a one-dimensional linear space. □

♣ `Problem 17.` Let $M = \mathbb{R}^n$, and $F\colon \mathbb{R}^n \to \mathbb{R}^k$ a smooth map, $F = (f_1, \dots, f_k)$, and $\operatorname{rank}_a F = k$ everywhere. What is the ideal $I = \langle f_1, \dots, f_k\rangle$? and the quotient space A/I? □

♣ `Problem 18.` If $Z = \{\, g_1 = \dots = g_s = 0 \,\} \subseteq N$ is a smooth submanifold, then what is the quotient space

$$C^\infty(M)/\left\langle F^* g_1, \dots, F^* g_s\right\rangle$$

and which conditions you should impose for your statement to be true? □

♡ **Definition.** The `local algebra` of a map $F\colon M \to N$ at a point $b \in N$ is the quotient space

$$A_b = C^\infty(M)/F^*\mathfrak{m}_b.$$

♣ `Problem 19.` Prove that if $F^{-1}(b)$ consists of isolated nondegenerate preimages, then their number is equal to the dimension of the local algebra. □

A very instructive example: compute

$$\dim_{\mathbb{R}} C^\infty(\mathbb{R})/F^*\mathfrak{m}_0, \qquad F\colon x \mapsto x^2.$$

How can you explain the answer?

All these matters will be discussed later!

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