HOMOTOPY FORMULA. COHOMOLOGY.

Analysis *versus* topology

1. Homotopy formula.

♡ **Definition.** The `Lie derivative` of a differential form $\omega \in \Lambda^d(M)$ of any degree d on a manifold M^n along a vector field v is

$$\boldsymbol{L}_V\,\omega = \lim_{t\to 0}\frac{1}{t}(g^{t*}\omega - \omega),$$

where g^t is the flow of the field v, and g^{t*} is the associated pullback action:

$$g^{t*}\omega(v_1,\dots,v_d) = \omega(g^t_* v_1,\dots,g^t_* v_d).$$

♡ **Definition.** If v is a vector field, then for any singular d-dimensional polyhedron σ its v-trace $\boldsymbol{H}_v(\sigma)$ is the saturation of σ by pgase curves of the field v defined for $t\in[0,1]$:

$$\boldsymbol{H}_v(\sigma) = \bigcup_{\substack{x\in\sigma,\\ t\in[0,1]}} g^t(x),$$

where g^t is the flow of v.

♣ `Problem 1.` Prove that $\boldsymbol{H}_v(\sigma)$ is a $(d+1)$-dimensional singular polyhedron.

♡ **Definition.** We supply $\boldsymbol{H}_v(\sigma)$ with the orientation in the following way: if $e_1,\dots,e_d$ is the declared-to-be-positive basis of vectors tangent to σ, then the $(d+1)$-tuple $v,e_1,\dots,e_d$ is the basis orienting $\boldsymbol{H}_v(\sigma)$.

Fubini Theorem for differential forms. *If $\sigma^d \subseteq M^n$ is a d-dimensional chain, and $\omega\in\Lambda^{d+1}$ a differential $(d+1)$-form, then*

$$\int_{\boldsymbol{H}_v(\sigma)} = \int_0^1 dt\int_\sigma g^{t*}(\boldsymbol{i}_v\,\omega).$$

Proof. It is sufficient to prove this formula for a single "cell" $(D, f\colon D\to M)$, D being a convex polytop in $\mathbb{R}^d$. Let $\widetilde{D} = [0,1]\times D$ be the Cartesian product in $\mathbb{R}^{d+1}$, oriented according to the above definition, and $F\colon \widetilde{D}\to M$ be the map,

$$F\colon (t,u)\mapsto g^t(f(u)),\qquad u\in D.$$

Then $F(\widetilde{D}) = \boldsymbol{H}_v(\sigma)$, where $\sigma = f(D)$, and

$$\int_{\boldsymbol{H}_v(\sigma)}\omega = \int_{\widetilde{D}} F^*\omega.$$

Applying the Fubini theorem to the function $a(t,u)$ which is the only coefficient of the form $F^*\omega = a(t,u)\,dt\wedge du_1\wedge\dots\wedge du_d$, we obtain the required identity. $\square$

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Corollary.

$$\int_{\boldsymbol{H}_{\varepsilon v}(\sigma)} \omega = \int_\sigma \boldsymbol{i}_v\, \omega + O(\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

♣ `Problem 2.` What is geometrically the set $\boldsymbol{H}_{\varepsilon v}(\sigma)$ for small ε?

The geometric homotopy formula. *For any chain* σ

$$g_v^1\sigma - \sigma = \partial(\boldsymbol{H}_v(\sigma)) + \boldsymbol{H}_v(\partial\sigma).$$

Proof. The boundary of the polytop $\widetilde{D} = [0,1] \times D$ (see the proof above) is the side "surface" $[0,1] \times \partial D$ and the two copies of D. The orientation convention implies that

$$\partial\widetilde{D} = \{\, 1 \,\} \times D - \{\, 0 \,\} \times D - [0,1] \times \partial D,$$

which immediately yields the above formula after application of the map $F\colon (t,u) \mapsto g^t \circ f(u)$. □

The analytic homotopy formula.

$$\boldsymbol{L}_V\, \omega = \boldsymbol{i}_v\, d\omega + d\, \boldsymbol{i}_v\, \omega.$$

Proof. Consider the flow map of the field εv for small ε and apply the above results. □

Corollary. *If $h^t\colon M \to M$ is a family of smooth maps of a manifold M to itself differentiably depending on the parameter $t \in [0,1]$ (a* `homotopy between` h^1 `and` *h^0), and ω is a closed form ($d\omega = 0$), then*

$$(h^1)^*\omega - (h^0)^*\omega = \textit{an exact form.}$$

Corollary: the Poincaré lemma. *If M is a star-shaped domain in $\mathbb{R}^n$, then any closed form is exact.*

Indeed, there exists a homotopy between the identical map id and a constant map $M \to O$.

2. Cohomology.

♡ **Definition.** The k-th `de Rham cohomology` of a smooth manifold is the quotient space

$$H^k(M) = Z^k(M)/B^k(M) = (\text{closed } k\text{-forms})/(\text{exact } k\text{-forms}),$$

where

$$Z^k(M) = \left\{\, \omega \in \Lambda^k(M)\colon d\omega = 0 \,\right\},$$
$$B^k(M) = \left\{\, \omega \in \Lambda^k(M)\colon \exists\theta \in \Lambda^{k-1}(M),\ d\theta = \omega \,\right\}.$$

By definition, we put $H^0(M) = Z^0(M) = \{\, f \in C^\infty(M)\colon df = 0 \,\}$. Each $H^k(M)$ is a linear space over reals.

♣ `Problem 3.` Prove that for a smooth manifold M

$$\dim H^0(M) = \text{the number of connected components of } M.$$

♣ `Problem 4.` Prove that $\dim H^1(\mathbb{S}^1) = 1$.

♣ `Problem 5.` Prove that $\dim H^1(C) = \dim H^1(\mathbb{S}^1)$, where $C = \mathbb{R}^1 \times \mathbb{S}^1$ is the standard cylinder.

♣ `Problem 6.` Compute the cohomology of the Möbius band.

♣ `Problem 7.` Prove that $\dim H^1(\mathbb{S}^2) = 0$, $\dim H^2(\mathbb{S}^2) = 1$.

♣ `Problem 8.` Compute the cohomology of the projective plane $\mathbb{R}P^2$.

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