Lect. 6  Differential forms

Covector = dual to vector:  \( L = \mathbb{R}^n \) linear space
\( L^* = \{ \xi : L \to \mathbb{R} \text{ linear functional} \} \)

\( M \) manifold
\[ TM = \bigcup_{a \in M} T_a M \text{ Tangent space (tangent bundle - explained later)} \]

\( T^* M = \bigcup_{a \in M} (T_a M)^* = \bigcup_{a \in M} T^*_a M \text{ Cotangent bundle} \)

Vector field:  \( X : M \to TM \) such that \( X(a) \in T_a M \)

Covector field  \( \xi : M \to T^* M \)  \( \xi(a) \in T_a^* M \)

Smoothness:  \( \forall X \text{ smooth } \in \mathcal{C}(M) \)
\( \langle \xi, X \rangle \text{ smooth function, } a \mapsto \langle \xi(a), X(a) \rangle \in \mathbb{R} \)

\( \xi \text{ Covector field} = \text{ map } \xi : \mathcal{D}(M) \to \mathcal{C}^\infty(M) \text{ "Algebraic description"

additive + linear over } \mathcal{C}(M) : \xi[fX] = f \xi[X] \]

Symmetry:  Differential 1-form  "Tensorial behavior"

(Form = ancient name for "functionals" with empty slots to be filled by
vector arguments 1-form may have free argument.
0-forms = smooth functions) \( k \)-form will be introduce in due trim.

Action by maps  \( \xi \text{ (functoriality)} \)

\( F : M \to N \) smooth map
\( F^* : \Lambda^k(N) \to \Lambda^k(M) \text{ "pullback" } \)
\( \langle (F^* \xi)(a), X(a) \rangle = \langle \xi(F(a)), (F_* X)(F(a)) \rangle \)

This behavior is much better than that of vector fields!
Example:

\[ f \in C^\infty(M) \Rightarrow df \in \Lambda^1(M) \]

\[ \langle df(a), X(a) \rangle = Xf(a) \]

directional derivative.

Exercise: \( F : M \to N, \quad \phi \in C^\infty(N) \)

\[ d(F^*\phi) = F^*d\phi. \]

\[ \square \text{ By definition } \square \]

Obvious observation:

\[ \sum_{i=1}^n \xi_i \in \Lambda^1(M) \]

\[ \xi_i \equiv \xi_{i_1} + \ldots + \xi_{i_k} \in C^\infty(M) \]

[ the structure of a module over the same algebra \( \Omega^1 = C^\infty(M) \) ]

Example: locally in a chart \( \leftrightarrow \) in a domain of \( \mathbb{R}^n \)

\( x_1, \ldots, x_i \) coordinates = functions on \( M \)

\( dx_1, \ldots, dx_i \) their differentials

Claim: any 1-form can be represented as \( \sum_i a_i(x) dx_i \)

In particular, \( df = \sum_i \frac{\partial f}{\partial x_i} dx_i \).

\[ \square \text{ Warning: the inverse is not true: not any 1-form is a differential } \square \]

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Lie derivatives act on 1-forms:

\( \mathcal{X} \) \( \mathcal{X} \) 1-form automorphisms of \( M \)

\( \mathcal{X}(M) \)

\[ L_X \omega = \left. \frac{d}{dt} \bigg|_{t=0} \right. (g^t)^*\omega. \]

Linear operators; Leibnitz rule holds (the same proof as for vectors).
Rules of derivation

\[ L_x \langle \frac{\partial}{\partial y}, Y \rangle = \langle L_x \frac{\partial}{\partial y}, Y \rangle + \langle \frac{\partial}{\partial y}, L_x Y \rangle \]

(follows from definition) (exercise)

Hence \[ L_x (f \cdot \frac{\partial}{\partial y}) = f \cdot L_x \frac{\partial}{\partial y} + (L_x f) \cdot \frac{\partial}{\partial y} = f \cdot L_x \frac{\partial}{\partial y} + X(f) \cdot \frac{\partial}{\partial y} \]

Proof: Take an arbitrary \( Y \). Use the Leibniz rule for \( L_x \). \( \Box \)

\[ L_x \frac{\partial}{\partial y} = ? \]

\[ \text{Tensorial property?} \]

(Chance that it is a \( 2 \)-form)

Hope: \( L_x \frac{\partial}{\partial y} \) again: a \( 1 \)-form

Verification:

\[ \langle L_x \frac{\partial}{\partial y}, fY \rangle = L_x \langle \frac{\partial}{\partial y}, fY \rangle - \langle \frac{\partial}{\partial y}, L_x (fY) \rangle \]

\[ \quad = L_x f \cdot \langle \frac{\partial}{\partial y}, Y \rangle + f L_x \langle \frac{\partial}{\partial y}, y \rangle - \langle \frac{\partial}{\partial y}, f \cdot L_x Y \rangle - \langle \frac{\partial}{\partial y}, L_x (fY) \rangle \] (Leibniz)

\[ \quad = f L_x \frac{\partial}{\partial y}, Y \rangle. \]

Second argument wrong! How to correct?

Claim (answer) \[ L_x \frac{\partial}{\partial y} - d \langle \frac{\partial}{\partial y}, X \rangle \]

\[ \text{tensor:} \quad \omega(x, \cdot) \]

\[ \omega(fX, \cdot) = f \omega(x, \cdot) \]

\[ \text{Correction} \]
Computation of
\[ \langle f_x, y \rangle = \int_{\mathbb{R}} \langle \frac{\partial}{\partial x}, y \rangle = \langle \frac{\partial}{\partial x}, L_x y \rangle \]
\[ = \int \langle \frac{\partial}{\partial x}, y \rangle + \langle \frac{\partial}{\partial x}, L_x (f_x) \rangle \]
\[ = \int L_x \langle \frac{\partial}{\partial x}, y \rangle + f \langle \frac{\partial}{\partial x}, L_x y \rangle + \langle \frac{\partial}{\partial x}, x \rangle \cdot L_x f \]
\[ = \int L_x \langle \frac{\partial}{\partial x}, y \rangle + \langle \frac{\partial}{\partial x}, x \rangle \langle df, y \rangle \]

"Wrong" term (non-tensorial)

\[ - \partial \langle \frac{\partial}{\partial x}, f_x \rangle = - \int \partial \langle \frac{\partial}{\partial x}, x \rangle - \langle \frac{\partial}{\partial x}, x \rangle df \] (L6)

Cumulatively,
\[ \omega(f_x, y) = \int \omega(x, y), \quad \Rightarrow \omega \text{ is 1 form in each argument.} \]

"Symmetry",
\[ \omega(x, y) = - \omega(y, x) \]

Direct computation ... (related to antisymmetry of \( L_x y \))

Notation: \( \omega = d \xi \), \( \xi \) Exterior Differential
\[ \xi \in \Lambda^1(M) \Rightarrow d \xi \in \Lambda^2(M) = \{ \text{bilinear antisymmetric forms} \} \]
\[ \{ \omega : \mathcal{X}(M) \times \mathcal{X}(M) \to C^0(M) \} \]
How to differentiate in public

\[ \omega(b) \rightarrow b = f^t(a) \]

\[ \nabla_b = (f^t)_* Y \]

\[ L_x \langle \omega, Y \rangle = ? \] function

\[ L_x \omega = ? \]

\[ \langle (f^t)_* \omega \cdot Y \rangle_a = \langle \omega_b \cdot (f^t)_* Y \rangle \]

\[ L_x \langle \omega, Y \rangle \approx \frac{1}{t} \left\{ \langle \omega_b \cdot Y_b \rangle - \langle \omega_a, Y_a \rangle \right\} \] (1)

\[ \langle L_x \omega, Y \rangle \approx \frac{1}{t} \left\{ \langle \omega_b \cdot \nabla_b \rangle - \langle \omega_a, Y_a \rangle \right\} \] (2)

\[ (L_x Y)_b \approx \frac{1}{t} \left\{ Y_b - \nabla_b \right\} \] (3)

Taking \[ \langle \omega_b \cdot (3) \rangle + (2), \] we obtain (1), therefore

\[ L_x \langle \omega, Y \rangle = \langle L_x \omega, Y \rangle + \langle \omega, L_x Y \rangle \]

(Leibniz's rule for pairing \[ \langle \cdot, \cdot \rangle \])
Further obvious properties of the Lie derivation:

1. \( L_x (f \omega) = f (L_x \omega) + (L_x f) \cdot \omega \)  
   (easy)

2. \( L_x \omega = f L_x \omega + (L_x f) \cdot \omega \)  
   (via Leibniz-dual)

3. \( L_x df = df \)  
   (acts on \( \mathfrak{g} \))

\[ L_x \langle df, Y \rangle = \langle df, L_x Y \rangle = XYf - L_{[x,Y]}f \]
\[ = (XY - XY + YX)f = YXf = \langle d(\omega f), Y \rangle = \langle df, Y \rangle \]

Let \( \Omega \) be a 2-form, define by the equation

\[ \Omega(x, \cdot) = L_x \omega - d\langle \omega, x \rangle \]

Direct computation: bilinearity in each vector argument separately. More precisely, by Leibniz-dual,

\[ \Omega(x, y) = L_x \langle \omega, y \rangle - \langle \omega, [x, y] \rangle - L_y \langle \omega, x \rangle \]

immediately antisymmetric...

This independence proves also linearity in \( x \).

Notation: \( \Omega = d\omega \) (the same symbol \( d \), different meaning).

\( \mid \) To distinguish, sometimes denote \( d^\omega f \), differential of \( \omega \).

Properties:

1. \( d (\omega_1 + \omega_2) = d\omega_1 + d\omega_2 \) (linear)

2. \( d_{\partial f} f = 0 \) : By definition,
   \[ ddf = L_x df - dL_x f = 0. \]
What about $d_1(f\omega)$, $f \in C^\infty(M)$?

- New operation, exterior (wedge) product of 1-forms:

$\alpha, \beta \in \Lambda^1(M), \quad X, Y \in \mathfrak{X}(M)$

$\alpha \wedge \beta \in \Lambda^2(M); \quad (\alpha \wedge \beta)(X,Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$

- Automatically antisymmetric & bilinear.

- $d_1(f\omega) = df \wedge \omega + f \cdot d\omega$

- L.H.S. =

$L_x <f\omega, Y> - L_Y <f\omega, X> - <f\omega, [X,Y]>$

$= fd_x \omega + L_x f \cdot <\omega, Y> - L_Y f \cdot <\omega, X>$

$= f \cdot d_x \omega + <df, X> <\omega, Y> - <df, Y> <\omega, X>$

- (Exercise) $L_x(\alpha \wedge \beta) = (L_x \alpha) \wedge \beta + \alpha \wedge d_x(L_x \beta)$
Reference table

Forms $\omega_i \in \Lambda^i(M)$

Operations

- $\omega_1 \pm \omega_2 \in \Lambda^i(M)$
- $f \omega$, $f \in C^\infty(M)$
- Wedge product $\omega \wedge \beta \in \Lambda^{k+i}(M)$
- $\alpha \in \Lambda^k$, $\beta \in \Lambda^\ell$, antisymmetric

Defined on 1-forms:

\[
(\omega \wedge \cdots \wedge \omega_k)(X_1, \ldots, X_k) = \det \begin{bmatrix} \omega_1(X_1) & \cdots & \omega_k(X_1) \\ \vdots & \ddots & \vdots \\ \omega_1(X_k) & \cdots & \omega_k(X_k) \end{bmatrix}
\]

Properties:

- Associative
- Multilinear
- Antisymmetric

In general:

\[
\omega \wedge \beta = (-1)^{\deg \omega \cdot \deg \beta} \beta \wedge \omega
\]

(Via transpositions of factors of degree 1)

Operators:

- Lie derivative $L_X \omega$
- Antiderivative $i_X \omega$

\[
id : \Lambda^k \to \Lambda^{k+1}, \quad k = 0, 1, \ldots, n-1
\]

Table of identities:

- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$
- $d(f) = \text{differential of } f$

Theorem: $d^2 = 0$

Homotopy formula: (can be used to define $d$ inductively)

\[
L_X = i_X d + d i_X
\]
Thus: \( \exists! \) operation \( d : \Lambda^k \rightarrow \Lambda^{k+1} \) such that

\[
d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-i)^{k+1} \wedge d\alpha \wedge (\text{Leibniz})
\]

If two forms coincide on an open set \( U \subseteq M \), then their differentials there coincide.

\[
\text{Let } \alpha \in U \text{ be an arbitrary pt and } f = \begin{cases} 1 & \text{near } a \\ \mathbf{0} & \text{outside } U \end{cases}
\]

\[
0 = d(f\alpha) = f \cdot d\alpha + df \wedge \alpha \Rightarrow d\alpha \text{ near } a = 0
\]

(b) Computation in charts: (uniquely)

\[
(\star) \quad d = \sum_i f_i \, df_i \wedge \ldots \wedge df_{k+i} \Rightarrow dd = \sum_i df_i \wedge df_i \wedge \ldots \wedge df_{k+i}
\]

(c) Existence: Define \( d \) by (\( \star \)). One has to verify the Leibniz rule, which amounts to checking that

\[
d \left( f \, dx_1 \wedge \ldots \wedge dx_k \wedge dy_1 \wedge \ldots \wedge dy_l \right) =
\]

\[
= (f \, df + g \, df \wedge \ldots \wedge df) \wedge \ldots
\]

\[
= \left( \text{the correct answer after the re-arrangement of terms} \right)
\]
Integration of forms.

(a) Integration over domains in $\mathbb{R}^n$

\[ \omega = \int \omega \, d\mu = \int \omega \, dx_1 \wedge ... \wedge dx_n \]

Define $F$ such that $V = F(U)$.

\[ \int \omega = \text{Riemann integral of } \omega \text{ against the measure } d\mu = dx_1 \cdots dx_n \]

Sign $\pm$ depends on the sign of \( \det \left( \frac{\partial (x)}{\partial (y)} \right) \)

= "orientation" of the domain.

= choice of a "positive" coordinate system.

(b) Integration over submanifolds in $\mathbb{R}^n$

\[ \int \omega = \int F^* \omega \]

Thus: "Doesn't depend on the parametrization" (modulo sign $\pm$).

Proper definition:

Oriented submanifold.

Examples:

Oriented curves.
Orientation of a boundary

"Extra first" rule
(Exterior normal
+ positive frame of body
= positive frame of the body itself.

Addition of pieces:
\[ \int_S \omega = \int_{S_1} \omega + \int_{S_2} \omega \]

Boundary of the cube (example)

\[ \partial (\text{Cube}) = \sum_{\text{faces}} \text{faces} \]

\[ \partial \partial (\text{Cube}) = \sum_{\text{faces}} \sum_{\text{edges}} \text{edges} ; \]

Each edge enters twice, with two opposite signs, \[ \Rightarrow \partial \partial (\text{Cube}) = 0. \]

Stokes theorem:

\[ \int_S \omega = \int_{\partial S} d\omega. \]

Proof:

(a) \[ S = \text{image of a cube} \]

\[ \omega = \int dx_2 \wedge \ldots \wedge dx_n \]

\[ d\omega = \frac{\partial \omega}{\partial x_1} dx_1 \wedge \ldots \wedge dx_n \]

(b) \[ S = \text{image of a cube} : \]