- Motivation: $L_x Y$ is not a directional derivative:
  \[ L_{fx} \neq f L_x \] (as vector fields).
  
  Explanation: depends on the field $X$ rather than its value $X(p)$ at any given point.
  
  Derivation requires parallel transport, albeit for small distances.
  Easy to do in the scalar case, problematic in general.
  
  Principal example: Embedded submanifolds.
  
  Way out: "infinitesimally small" steps:
  \[ \text{Parallel transport} = \text{v.f. of a constant velocity rotation.} \]
  
  Example 1.
  Circle:
  \[ \text{Large circle} \]
  \[ \text{Closed path} \rightarrow \text{Rotation by the angle} = \frac{\pi}{2} \]
  
  Example 2. Sphere.
How the parallel transport can be described?

Linear spaces attached to each point \( a \in M \)

\[ X = \text{direction} \]
\[ Y = \text{vector to translate} \]
\[ \Pi = \text{plane of deformation} \]

\[ \dim \Pi = \dim M; \quad \Pi \text{ is a "graph"} \]

\[ F \times M = \text{Total space} \]

\[ \Pi \in T_y (F \times M) \quad \text{Distribution on } F \times M \]

Equations defining the distribution:

\[ dy_i = \sum_k A_{ik} (x, y) dx_k \quad \text{Equation defining } \Pi \]

If we want that the transport to an infinitely close "fiber" to be \underline{LINEAR}: \( \theta \) should be linear in \( Y \): \( \theta_{xy} = \sum A_{ik} (x, y) dx_k \)

\[ \downarrow \]

One can form a \underline{MATRIX 1-FORM} \( \Omega = \begin{pmatrix} \omega_1 & \ldots & \omega_m \\ \omega_{m+1} & \ldots & \omega_{2m} \end{pmatrix} \)

and write the equation in the vector form:

\[ dy = \Omega dy \]

(\( \Omega = \text{matrix form} \) )

Giving such form is equivalent to defining a \underline{connection}.

Problem: \underline{When the parallel transport is independent of the path? \( \iff \) Distribution is integrable?}

Digression: Frobenius theorem in the language of 1-forms:

\[ \Pi = \sum \omega_i = 0; \ldots; \omega_m = 0 \]

It is called involutive, if \( d\omega_i \leq \text{ideal in } \Lambda^2 (\cdot) \), spanned by \( \omega_i \).

Involatility "old sense" = involutivity; "new sense"

\[ d\omega_k (X_i, X_j) = X_i \omega_k (X_j) - X_j \omega_k (X_i) - \omega_k (\{X_i, X_j\}) \]
Applying this to \( dy - dy \) (vector 1 form, comp. unit):

\[
d\omega \cdot y = \omega \wedge dy = 0 \quad \text{and} \quad \int_{\omega} y = 0
\]

\[
\Rightarrow \quad d\omega = \omega \wedge \omega = 0
\]

\[\Rightarrow \quad \omega \wedge \omega = 0 \quad \text{N.B.: Why } d\omega \neq 0?\]

Matrix 2 form:

\[
\begin{pmatrix} \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{23} \end{pmatrix}
\]

\[\begin{pmatrix} \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{23} \end{pmatrix} = 0 \]

Curvature tensor: the result of parallel translation over the parallelogram \((\varepsilon X, \varepsilon Y)\)

\[
\Delta = \text{identity} + \varepsilon^2 R(X, Y)
\]

Dual approach: instead of parallel translation, - differential operators

Covariant derivative

\[
\nabla_X Y \quad \text{is linear in } Y
\]

\[
\nabla_X Y \quad \text{is linear in } X
\]

Christoffel symbols:

\[
\Gamma^k_{ij} = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)
\]

\[
\Gamma^k_{ij} = \frac{\partial}{\partial x^j} \quad \text{basic vector fields}
\]

Thus:

\[
R(X, Y) \cdot Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z
\]

\( \Box \) Verify that \( R \) is tensorial con.t. \( X, Y, Z \):

\[
R(fX, Y) Z = f \nabla_X \nabla_Y Z - \nabla_Y (f \nabla_X Z) - \nabla_{[fX, Y]} Z
\]

\[
= f R(X, Y) Z + L_X f \nabla_Y Z - L_Y f \nabla_X Z
\]

\[\text{tensor involving } df\]
Once torsionality is proven, choose any two vectors and extend them as coordinate vector fields:

\[ \mathbf{X} = \frac{\partial}{\partial x}, \quad \mathbf{Y} = \frac{\partial}{\partial y} \]

\[ \Omega = A \, dx + B \, dy \]

In these coordinates, \([X, Y] = 0\) matrix functions

\[ R = \nabla_x \nabla_y - \nabla_y \nabla_x = \left[ \begin{array}{c} \frac{\partial}{\partial x} - A \\ \frac{\partial}{\partial y} - B \end{array} \right] \]

\[ R \cdot \mathbf{Z} = \left( \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \right) \mathbf{Z} + (BA - AB) \mathbf{Z} \]

\[ d\Omega = \left( \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \, dx \, dy \]

\[ \Omega \wedge \Omega = (AB - BA) \, dx \, dy \]

\[ \nabla : \text{Sections} \rightarrow \text{fiber-valued } 1\text{-forms} \]

\[ \nabla \text{ a derivation:} \]

\[ \nabla (f \mathbf{X}) = f \, \nabla \mathbf{X} + \mathbf{X} \wedge \eta \]

\[ \nabla_x = i_x \nabla - \text{section } \eta \text{-form} \]

\[ \text{Iteration of } \nabla \text{ (Similar to the de Rham complex)} \]

\[ \Lambda^k (\mathfrak{g}) \otimes \Gamma (\pi) \]

\[ \nabla_k : \Lambda^k \otimes \Gamma (\pi) \rightarrow \Lambda^{k+1} \mathbf{B} \otimes \Gamma (\pi) \]

\[ \nabla_k (\alpha \wedge S^k) = d\alpha \wedge S^k + \alpha \wedge \nabla S^k_{(k)} \]
Curvature via the complex:

\[ \Gamma(\pi) \xrightarrow{\nabla} \Gamma(\pi) \otimes \Lambda^1(B) \xrightarrow{\nabla} \Gamma(\pi) \otimes \Lambda^2(B) \xrightarrow{\cdot} \ldots \]

\[ \nabla (\alpha \wedge s) = d\alpha \wedge s + (-1)^{\text{degree}} \alpha \wedge \nabla s \]

1 section

(Leibniz rule)

\[ \nabla^2 (d^2 s) = \nabla (d^2 f \cdot s + d^2 s) = d^2 f \cdot \nabla s + d^2 s - d^2 f \cdot s \]

\[ = d^2 \nabla s \]

Hence \( \nabla^2 \) is a tensor:

\[ i_X i_Y \nabla^2 s = R(X,Y) \cdot s \]

Linear automorphism of the fiber into itself

Curvature tensor

\[ \nabla^2 = \text{Curvature (matrix) 2-form} \]
Connections on the tangent bundle. Lecture 10

\( \mathcal{D}_X Y - \mathcal{D}_Y X \) can be computed with \( [X, Y] \)

**Definition:** Connection is symmetric, if its difference is zero.

**Computation:** if \( \Gamma^k_{ij} = \) \( i \) th component of \( \nabla^2 \frac{\partial^2}{\partial x^i \partial x^j} \), then

\[
\text{Symmetric} \implies \Gamma^k_{ij} = \Gamma^k_{ji}
\]

**Riemannian manifolds:** Additional structure

\( \langle \cdot, \cdot \rangle \) - positive definite bilinear form on the tangent bundle \( TM \)

Defined in local coordinates by \( i \) smooth function \( g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle \)

**Scalar product \( \Rightarrow \) lengths, angles, ...**

**Explore:** Connections between Riemannian structures and connections

**Axiom of compatibility (Leibnitz rule)**

\[
\mathcal{D}_X \langle Y, Z \rangle = \langle \mathcal{D}_X Y, Z \rangle + \langle Y, \mathcal{D}_X Z \rangle
\]

**Explanation:** if two fields are parallel along a curve, then their scalar product is constant.

\( \Rightarrow \) all parallel transports are isometric operators.

**Principal Theorem of Riemannian geometry.**

There exists a unique LC connection for any RM.

**Write a system of eqs for \( \Gamma^k_{ij} \):**

\[
\sum_k \frac{\partial}{\partial x^k} g_{ij} = \sum_k \left< \nabla_{x^k} e_i, e_j \right> + \left< e_i, \nabla_{x^k} e_j \right> = \sum_k \left( \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{li} \right)
\]

**Why it is solvable?** Pointwise!

- WLOG \( g_{ij} = \delta_{ij} \) - system becomes
Corollary: There exists a unique covariant derivative which preserves vectors tangent to a hypersurface in $\mathbb{R}^n$.

How to construct this LCC explicitly?

Let $\nabla = \text{"vector d"}$ be the standard flat connection on $\mathbb{R}^n$.

$M \subset \mathbb{R}^n$ a hypersurface, $N$ = normal vector field on it.

Problem: find a connection $\overline{\nabla}$ s.t. $N \times Y$ tangent to $M$,

- add a matrix valued $1$-form $\alpha$ with the usual image

$$\overline{\nabla} = \nabla + \alpha \cdot N$$

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X,Y).N$$

(a symmetric, bilinear form)

To be tangent, again to $M$, we must postulate

$$\forall X, Y \quad \langle \overline{\nabla}_X Y, N \rangle = 0.$$

This determines $\alpha(X,Y)$ uniquely:

$$0 = \langle \overline{\nabla}_X Y, N \rangle = \overline{\nabla}_X \langle Y, N \rangle - \langle Y, \overline{\nabla}_X N \rangle$$

$$+ \alpha(X,Y) \overline{\nabla}_X \langle Y, N \rangle$$

$$\Rightarrow \alpha(X,Y) = - \langle Y, \overline{\nabla}_X N \rangle$$

unique possibility.

Why it is symmetric?

Why it is compatible with $\langle \cdot, \cdot \rangle$?
First check symmetry:
\[- \langle \nabla_x N, Y \rangle = \langle [\nabla_x, \nabla_y] N, X \rangle \]
\[- \nabla_x \langle N, Y \rangle + \langle N, \nabla_x Y \rangle \]
\[- \nabla_x \langle N, X \rangle + \langle N, \nabla_x X \rangle \]

But \( [\nabla_x, \nabla_y] N = 0 \) (since \( \nabla \) is symmetric).

\( X, Y \) tangent to \( M \) \( \Rightarrow \) \([X, Y]\) also tangent to \( M \)
\( \langle N, [X, Y] \rangle = 0 \).

Second, check the compatibility: easier.

\[ \nabla_x \langle Y, Z \rangle = \nabla_x \langle Y, Z \rangle = \langle \nabla_x Y, Z \rangle + \langle Y, \nabla_x Z \rangle \]

Since \( \nabla_\theta \) is \( \langle \cdot, \cdot \rangle \)-compatible.

\[ \nabla_x \langle Y, Z \rangle = \langle \nabla_x Y, Z \rangle + \langle Y, \nabla_x Z \rangle \]

Since \( \nabla - \nabla \) is a normal to \( M \), hence to \( Y \) and \( Z \).

Geodesic curves:
\[ \ddot{y} + \mathbf{R} \dot{y} = 0 \]
(parallel transport)