COMMENTARIES TO ARNOL′D PROBLEMS

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1. Infinitesimal Hilbert 16th problem

Problem 1978-6. Relaxed Hilbert 16th problem.

Problem 1979-16. Study the number of zeros of the integral $I(h) = \oint_{\gamma_h}(P\,dx + Q\,dy)$, where γ_h is a closed curve from the (continuous) family of periodic orbits of a polynomial vector field [e.g., $\gamma_h = \{x, y\ H(x,y) = h\}$, say, for $H = y^2 + x^3 - x$] — an infinitesimal version of the Hilbert 16th problem on cycles. What can be the maximal number of roots of $I(h)$ when $I(h)$ is not identically zero?

Problem 1980-1. $I(h) = \oint_{H=h}(P\,dx + Q\,dy)$. Place an upper bound for the number of zeros of I.

Problem 1983-11. Is that true that the integrals $I(h) = \oint_{H=h}(P\,dx+Q\,dy)$ with varying polynomials P, Q form a Chebyshev system (or, at worst, the number of zeros is not too much bigger)? Here, for instance, H is a cubic polynomial y^2+x^3-x. A similar question about perturbations of other integrable polynomial systems of the Lotka–Volterra type [where $H = x^\alpha y^\beta z^\gamma$, $z = 1-x-y$, with the corresponding (non-polynomial) P, Q].

Problem 1989-17. How many limit cycles can be born by a polynomial perturbation of degree n of an integrable polynomial system of degree n?

The problem reduces to investigation of the number of zeros of the integral

$$I(h) = \oint \frac{P\,dx + Q\,dy}{M}$$

along ovals $H = h$ of the system $\dot x = X(x,y)$, $\dot y = Y(x,y)$ with the integrating factor M, where X, Y, P, Q are polynomials of degree n. It is not solved even for $n = 2$ and even when $M = 1$ and H is a polynomial. When $M = 1$ and H, P, Q are polynomials of a fixed degree, there exists a uniform upper bound for the number of zeros (A. N. Varchenko, A. G. Khovanskiĭ), but it is non-effective.

Problem 1990-24. How large can be the number of isolated zeros of the complete Abelian integral

$$I(h) = \oint_{\gamma_h}(P\,dx + Q\,dy)$$

where γ_h is a closed component of the level curve $\{(x,y)\ H(x,y) = h\}$, if P, Q, H are polynomials of given degrees?

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Problem 1990-25. Let g be a natural number ≥ 2 and $U(x)$ a fixed polynomial of degree $2g+2$. Consider the family of hyperelliptic integrals of the first kind,

$$I(h) = \oint_{\gamma_h} \frac{P(x)}{y}\, dx,$$

where γ_h is a closed component of the level curve $\{(x,y)\, y^2 + U(x) = h\}$, and $P(x)$ an arbitrary polynomial of degree $\leq g$. Is this family of integrals a Chebyshev one (i.e., is it true that for any P the number of isolated zeros of the function I is at most $g-1$)?

Problem 1994-51. Infinitesimal version of the Hilbert 16th problem. Assume that a polynomial vector field on the plane admits a first integral whose level curves are cycles (filling at least some annulus on the plane). Consider small polynomial perturbations (of prescribed degree) of this vector field. Location of the limit cycles [appearing in this perturbation] are given in the first approximation by zeros of a certain integral (found by Poincaré) along non-perturbed closed curves (which are the level curves of the first integral).

Is the number of zeros of the Poincaré integral bounded (by a constant depending only on the degree of the perturbation)?

Problem 1994-52. A particular case of the previous problem: consider the complete Abelian integral

$$I(h) = \oint (P\, dx + Q\, dy)$$

along an oval of an algebraic curve $H(x,y) = h$. The polynomials $P(x,y)$ and $Q(x,y)$ describe a polynomial perturbation of the Hamiltonian vector field, and $I(h)$ is the Poincaré integral.

Find an upper bound for the number of isolated real zeros of the function I for all polynomials P, Q of the given degree.

Comments. The problem on zeros of Poincaré integrals, known also as the infinitesimal Hilbert 16th problem, is one of the most recurring in the Arnol'd's lists. It was published in [1], reappeared in the list [3] and rather recently the expanded formulation was again given in [4]. Two very closely related problems, **1979-26** and **1980-3** which could well be included in the list, are singled out because they are essentially solved.

Origins, preliminary remarks. The problems on zeros of the Poincaré integral

$$I = I(h; H, \omega) = \oint_{\gamma_h} M^{-1}\omega, \qquad \gamma_h \subseteq \{H = h\}, \qquad \omega = P(x,y)\, dx + Q(x,y)\, dy, \tag{1}$$

for the polynomial perturbation

$$M\, dH + \varepsilon\omega = 0, \tag{2}$$

in particular, *complete Abelian integrals* corresponding to $M = 1$ and a polynomial H, appeared as an attempt to find an amenable relaxation of the Hilbert problem on limit cycles.

As a function of h, $I(h)$ is the first variation of the Poincaré return map with respect to the small parameter ε, at $\varepsilon = 0$. Thus the problem on zeros of integrals of the form (1) becomes a localized (better to say, linearized or *infinitesimal*) version of the Hilbert 16th problem on the number of limit cycles of planar polynomial vector fields, for systems infinitesimally close to integrable ones.

Probably, the question was also inspired by the works by I. Petrovskiĭ and E. Landis [32, 33, 34] who tried to reduce the Hilbert 16th problem stated in full generality, to perturbations of integrable systems.

It should be stressed that vanishing of the Poincaré integral is only a necessary condition for appearance of limit cycles, and it works only for limit cycles born out of *nonsingular* level curves of the first integral. Description of limit cycles born from *separatrix polygons* (carrying singular points of the non-perturbed vector field) is a considerably more delicate subject, which admits a satisfactory solution only in the simplest case of a separatrix loop carrying one nondegenerate saddle (R. Roussarie [36, 37]).

Besides, identical vanishing of the Poincaré integral (1) does not mean in general that the family (2) consists of integrable systems only: higher variations in ε may still be nonzero and it is their zeros that will determine the number and location of limit cycles born in the perturbation. However, for Abelian integrals this is impossible: in [15] Yu. Ilyashenko proved that for a sufficiently generic polynomial Hamiltonian H the integral of a form of degree $\deg\omega = \max(\deg P, \deg Q) + 1$ no greater than $\deg H$ vanishes identically if and only if ω itself is exact on $\mathbb{R}^2$. Clearly, in this case the system is Hamiltonian for all ε. This result, generalized by L. Gavrilov [9] for higher degree forms and by I. Pushkar′[35] for higher dimensions, provides an effective criterion for nontriviality of the perturbation (2). Everywhere below only *isolated* zeros of the Abelian integrals are counted.

From the very beginning it should be said that *general* results for perturbations of conservative non-Hamiltonian systems are practically absent, with few exceptions concerning perturbed Lotka–Volterra systems. Therefore we will mostly discuss the problem on zeros of Abelian integrals with $H \in \mathbb{R}[x, y]$ and $M \equiv 1$.

Brief history. The first nontrivial case (for H quadratic the Abelian integrals are rational functions of h) corresponds to cubic Hamiltonians. R. Bogdanov studied the complete *elliptic integral*

$$I(h) = \oint_{\{H=h\}} (a + bx)y\,dx, \qquad H(x, y) = \tfrac{1}{2}y^2 + \tfrac{1}{3}x^3 - x \tag{3}$$

and proved that it has at most one real isolated zero [6]. This problem appeared in connection with construction of the versal deformation of what is known today as the cuspidal singularity of Bogdanov–Takens [5]. Later, in [16] Ilyashenko suggested another proof of the same result, based on the complexification of the Abelian integral as a function of $t \in \mathbb{C}$ ramified over the collection of critical values of the complexified Hamiltonian $H(x, y) \in \mathbb{C}[x, y]$. Since then, complexification became a primary tool in investigation of complete Abelian integrals.

Shortly after that a number of different particular cases of elliptic integrals was studied, but the major breakthrough occurred in the works by G. Petrov. He proved that for the standard elliptic Hamiltonian as in (3), integrals of *all polynomial forms* of arbitrarily high degree form a *nonoscillating*, or *Chebyshev* family: the maximal number of real isolated zeros is by one less than the dimension of this family considered as a linear space over $\mathbb{R}$ [30]. Later Petrov proved that the same non-oscillatory property holds also for *complex* isolated zeros counted in a slit plane [31]. The proofs rely substantially on the fact that the elliptic integrals $\oint y\,dx$

and $\oint xy\,dx$ satisfy an explicitly written system of *Picard–Fuchs* linear ordinary differential equations with rational coefficients, so that their ratio satisfies a Riccati equation. On the other hand, these two integrals generate the space of all Abelian integrals over the ring of polynomial functions of h. The results of Petrov settle the particular question raised in **1979-16** and give an affirmative answer in the problem **1983-11** in the part related to the elliptic integrals.

Earlier, simultaneously and independently, A. Khovanskiĭ [17] and A. Varchenko [38] proved the general finiteness result: for any combination of degrees n and d, the number of isolated zeros of all Abelian integrals of forms of degree $\le d$ over the level curves of Hamiltonians of degree $\le n$ is uniformly bounded by a constant $C(n,d)$ depending only on n and d. Their proofs, however, gave no idea of how to estimate the constant $C(n,d)$: its mere existence is ultimately derived from compactness arguments.

This result remains until nowadays the only general assertion valid for all Hamiltonians and all forms without restriction. Since it was achieved, the accents were shifted to *computability* of the bounds.

Digression: fewnomials theory and Pfaffian manifolds. The proof of Khovanskiĭ–Varchenko theorem is based on a beautiful geometric theory of Pfaffian manifolds, developed by Askol′d Khovanskiĭ. The central idea behind this theory can be described roughly as follows: a real affine variety defined by a mixture of algebraic and Pfaffian equations, shares many properties of real algebraic varieties provided that it "looks like an algebraic variety" topologically. A simplest example is that of integral trajectories of planar polynomial vector fields. If these trajectories are not spirals (they should subdivide the real plane into two parts, in particular, being limit cycles), then the number of isolated intersections of these trajectories, say, with straight lines is explicitly bounded in terms of the degree of the planar vector field. This observation immediately allows to solve the problem **1976-2**.

The constructions in the Pfaffian manifolds theory, especially the *Pfaffian elimination*, are explicit and efficient. Geometrically they could be described as a multidimensional generalization of the Rolle theorem on alternation between roots of a smooth function of one real variable, and roots of its derivative.

One of the most spectacular achievements of this theory is an upper bound for the number of isolated solutions of a system of algebraic equations, given *not* in terms of the degrees of this equation like in the Bézout theorem, but rather through the number of different *monomial terms* occurring in the equations, uniformly over all degrees. This explains the alternative code name "*fewnomials theory*" used to designate the entire toolkit. A typical fewnomials theory result is described in the problem **1979-22**.

Applications of the Pfaffian manifolds theory can sometimes be very unexpected. Thus, if the resonant Poincaré–Dulac formal normal form [2] for all singular saddle points of an analytic planar vector field is convergent, then any polycycle carrying only these points cannot accumulate near itself an infinite number of limit cycles of this field. This particular case of the finiteness theorem (see commentary to the problem **1981-16**) was discovered by R. Moussu and C. Roche in [21]. Their key argument is integrability of the resonant normal form which in turn implies the fact that the Poincaré map can be described by a mixture of Pfaffian and analytic equations.

This theory, together with its numerous ramifications, is exposed in the book [1]. The revised Russian edition [19] contains new applications to Hardy fields, complexity problems, Tarski problem etc.

Recent achievements: low degree cases. Despite their diversity, recent results related to the infinitesimal Hilbert 16th problem can be organized into several clusters.

The most abundant group of results deals with particular cubic or quartic Hamiltonians and special choices of low degree (usually the same) perturbation forms. If the number of essential parameters is small enough, sometimes bifurcation diagrams of zeros can be constructed. Usually problems of this type appear in connection with bifurcations of limit cycles in families of vector fields exhibiting certain resonances. Though it is impossible to mention all results, probably the most spectacular single recent achievement in this direction is due to L. Gavrilov [10], see problem **1979-26**. Gavrilov proved that for a real cubic Hamiltonian with 4 distinct (complex) critical values, the number of zeros of any integral of a quadratic 1-form can be at most 2.

The advantage of cubic Hamiltonians is that their level curves are elliptic, thus the corresponding integrals can be in some sense reduced to elliptic integrals. The Picard–Fuchs system satisfied by these integrals, admits as a factor the 2-dimensional linear system reducible to a Riccati equation similar to that from [30]. Zeros of functions obtained as rational combinations of solutions of a Riccati equation, can be produced using the "fewnomials" technique introduced by Khovanskiĭ [18]. This idea after an appropriate (rather sophisticated) elaboration allowed to prove that for any cubic Hamiltonian and any polynomial form of degree $\deg\omega \le d$ the number of isolated zeros can be at most $5d + 10$ (Horozov and Iliev [11]).

In the same paper it is shown that a generic cubic Hamiltonian admits a quartic 1-form ω yielding 5 isolated zeros to the integral (2). This gives a generally negative answer to the question raised in problem **1983-11**, whether Abelian integrals are always non-oscillating (as was the case in the standard elliptic case). Yet the conjecture from **1990-25** about non-oscillation of hyperelliptic integrals, remains open.

Note added in proof. In February 2002 Chengzhi Li and Zenghua Zhang announced a complete solution of the infinitesimal Hilbert problem for the quadratic case ($\deg H = 3$, $\deg\omega = 2$). They showed that the genericity condition appearing in the Gavrilov theorem [10] is in fact obsolete. For more details see problem **1979-26**.

Asymptotic bounds. The role played by the Hamiltonian H and the polynomial 1-form ω is clearly unequal. Ignoring the origins of the infinitesimal Hilbert problem, one may further relax it by freezing the Hamiltonian and investigating how the bound on the number of zeros may depend on the form. This suggestion is tacitly made in formulations of the problems **1994-51**, **1994-52**.

First results in this direction were obtained by Yu. Ilyashenko, D. Novikov and S. Yakovenko. Assuming that the Hamiltonian is generic, they proved in [13, 24, 25] that as $\deg\omega = d \to \infty$, the number of isolated zeros may grow at most as $O(\exp cd)$, where $c = c(H)$ is a constant depending only on H. The demonstration leaves a theoretical opportunity to compute $c(H)$ in terms of the monodromy group of H and a geometry of its critical values, but the result of the computation must necessarily explode as some of the critical values of H approach each other. The key idea behind the proof is to exploit the irreducibility of the monodromy group of the Picard–Fuchs equation in the complex domain.

An asymptotically accurate answer was obtained by Petrov and Khovanskiĭ in 1996. They proved that the number of isolated zeros can grow at most as $K_1(n)\,d+K_0(H)$, where $K_1(n)$ is an explicit constant depending only on the degree $n=\deg H$ while $K_0(H)$ is independent of ω but depends on H. Apparently, one can prove that this constant is uniformly bounded over all Hamiltonians of degree n by some $K_0(n)$, but the bound $K_0(n)$ is absolutely non-efficacious exactly as the Varchenko–Khovanskiĭ bound $C(n,d)$ mentioned above. Though the proof is not yet formally published, some of its ingredients were already incorporated in other constructions [27, 39].

This result to a certain extent answers the question as it is formulated in problem **1994-52**. Though the constant $K_1(n)$ is bigger than 1, still the relative excessiveness of this upper estimate over the lower estimate guaranteed by the dimensionality arguments, is bounded uniformly over all forms of all degrees (for fixed $\deg H$), partially corroborating thus the conjecture that appeared in the earlier problem **1983-11**.

Algorithmically constructive bounds. The fewnomials theory applies to functions defined by planar polynomial differential equations, such as the Riccati equation mentioned above, describing their zeros in terms of the *degrees* of the defining equations.

There is no such “fewnomials theory” for polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$ with $n>2$ [23]. However, one may compute an explicit upper bound on the number of isolated intersections between integral trajectories of a polynomial vector field and an arbitrary algebraic hypersurface in the n-space, not solving the equations. The answer depends (polynomially) on the *magnitude* of coefficients of the vector field, as well as on its degree and dimension (as a tower function, i.e., an iterated exponent). This result (the “*meandering theorem*”), obtained by Novikov and Yakovenko [26, 28], can be applied to Picard–Fuchs systems of linear ordinary differential equations with rational coefficients, satisfied by Abelian integrals.

A precondition for such application is an explicit knowledge of the magnitude of the coefficients of the system. An explicit derivation of Picard–Fuchs equations allowing to bound their coefficients, was achieved in [29], see also [22].

The construction in the hyperelliptic case has an especially transparent form. Application of the meandering theorem in this case allows to place an *explicitly computable* upper bound in the form of a tower function (iterated exponent) of n, on the number of zeros of hyperelliptic integrals, under the additional technical assumption that all critical values of the potential are real (Novikov and Yakovenko [27]).

Actually, the result on zeros of hyperelliptic integrals is obtained as a particular case of the following general principle. A collection of (analytic multivalued) functions $f_1(t),\dots,f_n(t)$ on the Riemann sphere, satisfying a Fuchsian system of linear equations, behaves algebraically-like if the monodromy group of this system possesses certain spectral properties. The quasialgebraicity property mentioned above means that the question on the number of (complex isolated) zeros of any function f from the differential Picard–Vessiot extension field $\mathbb{C}(f_1,\dots,f_n)$ can be explicitly answered in terms of complexity of f in this field. See [39] for the exact formulations and discussion.

Restricted problems. Various approaches to obtaining asymptotic or algorithmic bounds on the number of zeros of Abelian integrals, are based on different properties of Abelian integrals (usually in the complex domain). For instance, the exponential asymptotic bounds from [13] is based on irreducibility of the monodromy group of the Abelian integrals, whereas the key results from [39] are valid for any complex analytic functions satisfying Fuchsian systems of differential equations with bounded residue matrices.

These methods, though not giving a complete answer for the problem in full generality, sometimes allow for explicit upper bounds for almost all Hamiltonians, except for a proper semialgebraic subset of zero measure. As a rule, the *estimates* explode to infinity when approaching this exceptional "bad" subsets, while the number of zeros remains in fact bounded by the Varchenko–Khovanskiĭ theorem. Yet the explicit nature of the estimates for a "large" portion of Hamiltonians is of obvious interest. Following Yu. Ilyashenko, we call such problems *restricted* versions of the infinitesimal Hilbert problem. Expanding the meaning of the "restrictedness", one can include in this class also majorizing the number of isolated zeros of Abelian integrals in some specific domains (e.g., on a specified distance from the set of critical values of H).

In this restricted sense the infinitesimal Hilbert problem is in principle solved in [29]: for any H with a properly normalized principal homogeneous part and any $\varepsilon > 0$ one can place an explicit upper bound for the number of isolated zeros of all Abelian integrals, at least ε-distant from the critical values of H (the bound depends on H and ε). Moreover, for all Hamiltonians with the principal homogeneous part normalized as above, and pairwise distant critical values, the number of all isolated zeros of all integrals can be bounded uniformly in terms of n, d and the (inverse) minimal distance between the critical values. The bounds are given by tower functions of height 4.

Very recently A. Glutsuk and Yu. Ilyashenko achieved considerable progress towards solving the restricted infinitesimal problem for the particular class of Hamiltonians of the form $H(x, y) = p(x) + q(y)$ with two monic polynomials of the same degree $\deg p = \deg q = n + 1$. Using different ideas partly stemming from [14, 13], they obtain in [12] an explicit upper bound for the number of isolated zeros, growing as $\exp(2435n^4)$, provided that all n^2 critical points of H are in the disk of of radius 2 but at lest $1/n^2$-distant from each other.

Non-Hamiltonian case. As was already remarked, the case of general Poincaré integrals with nontrivial integrating factors is much more complicate. To begin with, merely a classification of integrable polynomial systems is very complicated. While all center conditions in the quadratic case are known since the work by Dulac [7], the analogous problem for cubic systems is not solved. Thus, as suggested in the problem **1983-11**, one should begin with a certain typical (or simplest) class of integrable systems. A natural candidate is the class of Darboux integrable systems $M\,dH = 0$, where $H(x, y) = F_1^{\alpha_1} \cdots F_n^{\alpha_n}$ is the first integral, the product of polynomials $F_i \in \mathbb{R}[x, y]$ in real powers $\alpha_i \in \mathbb{R}$, and $M = F_1 \cdots F_n H^{-1}$ is the nontrivial integrating factor. The famous Lotka–Volterra system corresponds to three linear terms $F_1 = x$, $F_2 = y$ and $F_3 = 1 - x - y$ and seems to be one of the two simplest examples (the other one is a product of two terms with F_1 linear and F_2 quadratic).

It is much more difficult to describe the analytic continuation of the Poincaré integrals, since the "level curves" $H = h$ after complexification will not be affine

Riemann surfaces continuously depending on h, but rather essentially noncompact leaves of the holomorphic foliation $\{M\,dH = 0\}$ with singularities on $\mathbb{C}P^2$. This makes it very difficult (if possible at all) to apply complex analytic methods that were the main tools of research in the Hamiltonian case. As a consequence, there is not possible to write a finite-dimensional system of Picard–Fuchs equations (an infinite system was derived for the Darbouxian case by H. Żołądek in [8]).

Concerning the particular low-degree cases, one should mention the paper by Żołądek [40], see problem **1980-3**. In most other results concerning specific perturbations of the Lotka–Volterra system, usually monotonicity of some ratios of "monomial" Poincaré integrals is obtained by using very specific methods that do not admit generalizations for the general Darbouxian case or perturbations of higher than second degree. This monotonicity implies uniqueness of zero of the corresponding "binomial" linear combination of integrals. A useful tool for establishing such monotonicity for systems with the first integral of the form $H(x, y) = \Phi(x) + \Psi(y)$ was discovered by Chengzhi Li and Zhifen Zhang [20]: despite its seemingly artificial form, it proves to be working in many independently arising particular cases.

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2. Hilbert 16th problem and ramifications

Problem 1958-3. Find a multidimensional version of the Hilbert conjecture on the number of limit cycles. *For instance, interesting is the number of integral curves connecting two algebraic or invariant manifolds and sufficiently "monotone".*

Comments. This question reappeared with some modification in [1], see also [2, 3], in connection with the problems on complexity of dynamical intersections.

In one of the formulations it is suggested to estimate the number of intersections between a fixed variety Y and the saturation of another variety X by trajectories of length $\leq N$ of a polynomial vector field in $\mathbb{R}^n$, with $\dim X + \dim Y = n - 1$.

This problem was solved for $\dim X = 0$, when it reduces to the question on the number of intersections between an integral curve of a polynomial vector field, and an algebraic hypersurface. The bound, obtained by D. Novikov and S. Yakovenko [4, 5], depends polynomially on the magnitude of the coefficients and the "size" of the integral curve, while the power exponent is a computable but enormously fast growing function of the dimension n and the degree of the field.

This result also holds in the complex space and can be applied to Picard–Fuchs equations for Abelian integrals. This yields some explicit bounds for the infinitesimal Hilbert problem, see the comments to problem **1978-6** e.a.

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Problem 1971-9. Generalize the Hilbert problem on limit cycles for discrete time systems.

Comments. One of the possible variants is the dynamics of intersections discussed in the problem **1988-6** (commentary), see also **1988-7**, **1989-2**, **1990-1**, **1990-20**, **1990-21**, **1992-12**—**1992-14**, **1994-45**—**1994-50**, where mostly the case of generic smooth maps is considered [1, 2, 3].

Yet it is the algebraicity of the discrete time dynamical system that should also play an important role. The straightforward generalization, "estimate the number of periodic points of period n in terms of the degree and n", is trivial: the union of all n-periodic orbits is an algebraic subvariety for any finite n, and its complexity can be easily estimated. For instance, if this set is discrete, than the number of its points grows exponentially in n by virtue of the Bézout theorem.

It is the nonalgebraicity of solutions (limit cycles) of planar polynomial vector fields, that makes them so difficult to track. Thus a "proper" Hilbert-type question for discrete time systems should involve infinite aperiodic orbits of polynomial maps. In particular, one might try to begin by estimating "nonalgebraicity" of infinite orbits. To do this, a numeric measure for this is to be introduced and bounded from above in terms of the degree of the polynomial map.

One such characteristics can be easily described. What can be the maximal time during which an orbit may stay on a given algebraic subvariety, without being forced to stay on it forever? This question is a discrete time analog of the question on the maximal order of tangency between trajectories of a polynomial vector field and an algebraic hypersurface, the problem posed by J.-J. Risler in connection with control problems [4].

The discrete time problem was solved by D. Novikov and S. Yakovenko in [5] for dimension-preserving polynomial maps. The continuous time problem was solved by A. Gabrielov and A. Khovanskiĭ [6] who gave an exponential bound for the maximal order of tangency.

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Problem 1976-37. Can a planar vector field defined by two quadratic polynomials, have more than 3 limit cycles?

Comments. The question is apparently motivated by Bautin's famous result [1] asserting that in a quadratic perturbation of the linear center (the Hamiltonian linear vector field corresponding to the Hamiltonian $H(x, y) = x^2 + y^2$) no more than 3 limit cycles can be born. The original proof was obtained by somewhat mysterious calculations. Simplified proofs were obtained in [3] and [4].

It was long believed that this result implies that quadratic vector fields cannot have more than 3 limit cycles. In 1980 Shi Song Ling [2] constructed a counterexample with 4 limit cycles by explicitly perturbing a quadratic system with an ultra-ultra-weak focus at the origin (generating three small limit cycles in the perturbation) and one more "large" limit cycle far away.

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Problem 1979-26. Let P, Q in the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ — polynomials of the second degree, and $H(x, y)$ a first integral of this system (not necessarily a polynomial one). How many limit cycles can be born from components of level curves of H by small variations of P, Q leaving them quadratic polynomials?

Comments. The problem was published in [1].

For perturbations of a *generic* Hamiltonian system with a real cubic polynomial H, the problem was solved by L. Gavrilov [2]. He proved that for any real cubic Hamiltonian with four distinct critical values, and any cubic differential form ω the number of limit cycles born in the corresponding *quadratic* perturbation (2) is at most 2. This technically involved theorem incorporates previously obtained results by E. Horozov and I. Iliev [3]. Gavrilov theorem solves the problem **1979-26** for perturbations of Hamiltonian quadratic systems, as well as other numerous results.

The central moment is a theorem on zeros of the corresponding Abelian integrals, see commentaries to the problem **1978-6** e.a.

Note added in proof. In February 2002 the infinitesimal Hilbert problem for the quadratic case was finally settled: *the number of isolated zeros of any Abelian integral of a real quadratic polynomial* 1*-form over closed level curves of a real cubic Hamiltonian is at most* 2. This result was achieved in a series of case study works treating the degenerate cases not covered by the Gavrilov theorem (actually, some of these results chronologically preceded [2]). Namely, when one of the critical points of H escapes to infinity, the bound was obtained in [5, 6]. The case of two coinciding critical values attained at two *distinct* critical points (in this case the Hamiltonian system exhibits a heteroclinic loop) was covered in [7]. The final blow was dealt in [4] by Chengzhi Li and Zenghua Zhang who announced the solution for the case of cubic Hamiltonians exhibiting a cuspidal singularity.

It is important to stress that in these degenerate cases a bound on the number of zeros of Abelian integrals does not imply yet a bound on the number of limit cycles. It may happen that the integral $I(t) = \oint_{H=t} \omega$ vanishes identically, whereas the system $dH + \varepsilon\omega = 0$ for any $\varepsilon \neq 0$ is non-integrable.

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Problem 1979-27. Let P, Q in the system of differential equations $\dot{x} = P(x, y)$, $\dot{Q}(x, y)$, be power series starting from homogeneous polynomials P_n, Q_n of degree n. Is that true that for almost all pairs (P_n, Q_n) the number of limit cycles born from the origin by a small perturbation of the system, is bounded by a constant depending only on n?

Problem 1983-16. Is it true that the number of limit cycles born from a singular point of an analytic system, is bounded (except for systems forming a set of codimension infinity, *integrable*)?

Comments. These two problems can be considered as an initial step in an attempt to relax the Hilbert 16th problem by localizing it on a neighborhood of singular point (another type of localization, with respect to parameters, leads to the problem on zeros of Poincaré integrals, see problem **1979-16** and its follow-up).

Yet it turned out that, besides limit cycles born from centers with nonzero linear part (Andronov–Hopf) and cuspidal points (Bogdanov, [1]), there were only a few results, some of them incomplete. Perhaps, the main reason is that all difficulties characteristic of global problems, reappear in the local problem after a suitable blow-up procedure.

Considerably better is the situation with other types of polycycles (graphics, separatrix polygons) which can also generate limit cycles by small perturbations. (One singular point, degenerate or not, is a particular case of a polycycle). *Cyclicity* of a polycycle is the number of limit cycles that can be born this way.

The known results can be arranged according to the number and the types of singular points on the polycycle, that is, ultimately, according to the codimension of occurrence of polycycles in generic families of planar vector fields. For all polycycles of small codimension 1 and 2 the cyclicity is known. The list of polycycles of cyclicity 3, the *Kotova Zoo*, is composed and for many beasts from this zoo the cyclicity is known or at least estimated from above. These and other results can be found with appropriate references in the book [2].

Results of general nature are scarce. It is known that cyclicity of a *generic* polycycle of any finite codimension n, carrying only *elementary* singularities (with non-nilpotent linear parts), is finite and bounded by an algorithmically computable function $E(n)$. This result by Ilyashenko and Yakovenko [3] was recently improved by V. Kaloshin, who simplified some parts of the construction and achieved the transparency that allowed him to prove that $E(n) \leq 2^{25n^2}$ [4, 6, 5]. To get rid of the elementariness assumption, a parametric desingularization procedure is required. There were attempts to construct such theory (Roussarie–Denkowska, Trifonov), but all failed to reach the level of applicability required for further progress in this direction: even for bifurcations of a cuspidal loop, an upper bound the number of limit cycles is proved only modulo an assertion on monotonicity of a certain transcendental function [7].

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Problem 1980-2. The boundary value problem for $\dot x = P(x, e^{it})$, $x(2\pi) = x(0)$: the number of solutions.

Problem 1984-16. Study the equation $\frac{dy}{dx} = f(x, y)$, where x, y are angular coordinates on the circle and f a trigonometric polynomial. How many cycles can it have for a given Newton polygon?

Comments. This is one more attempt to relax the Hilbert problem on limit cycles, this time modifying the class of admissible differential equations. The answer for the trigonometric problem as it is stated, is known only for $\deg f = 1$, where it is shown that at most 2 limit cycles can occur [1]. For $\deg f = 2$ at least 6 cycles are possible (ibid.).

The "polynomial" periodic problem **1980-2** is somewhat better understood. The differential equation of the form $\dot x = P(x, t)$ with the monic polynomial $P = x^d + \sum_0^{d-1} a_k(t)\, x^k$ and arbitrary dependence on t, the so called *Abel equation*, was studied in [3]. For $d = \deg_x P \le 3$ there can be at most d solutions with $x(0) = x(1)$, while for $d \ge 4$ the number of cycles can be arbitrary, depending on the coefficients of the polynomial f [6, 4, 5].

Very recently Yu. Ilyashenko constructed an upper bound for the number of limit cycles of the *periodic* Abel equation (as in the initial formulation) in terms of the *magnitude* of the coefficients, $C = \max_k \max_t |a_k(t)|$. In [2] he proved that the number of cycles can be majorized by an explicit expression double exponential in C. Results of similar nature were also obtained for limit cycles of the Liénard equation.

A different approach to studying the cubic Abel equation

$$y' = p(x)y^2 + q(x)y^3, \qquad p(x), q(x) \in \mathbb{R}[x], \tag{4}$$

with polynomial $p(x), q(x)$ was suggested recently by J.-P. Françoise, Y. Yomdin and coauthors. Very roughly, the idea is to solve this equation in formal series and study the algebra and geometry of coefficients of these series.

For instance, the growth of coefficients of a converging series $\sum_{k\ge 0} a_k y^k$, closely related to the growth rate of the sum of this series, is responsible for the distribution of its zeros. If the coefficients are themselves polynomials in the additional parameter(s), then *uniform* bounds on the number of zeros can be derived from analysis of the ascending chain of *Bautin ideals* $I_k = \langle a_0, a_1, \dots, a_k\rangle$ and the (infinite) descending chain of ideals $J_k = \langle a_{k+1}, a_{k+2}, \dots\rangle$ [9, 10].

In application to the cubic Abel equation (4), consider the "Green function" $G(x, y)$, defined as the value at the moment of time x of the solution of this equation, defined by the initial condition $y(0) = y$. The expansion of this function $G(x, y) = y + \sum_{k\ge 2} a_k(x)\, y^k$ has polynomial coefficients $a_k \in \mathbb{R}[x]$, and the recurrent rule for a_k can be easily written. The questions on zeros of the function G contain in a nutshell many difficulties characteristic for the Hilbert problem. The number of isolated roots of $G(1, y)$ will be an equivalent of the problem on limit cycles. Determination of the points $x = b$ for which $G(b, y) \equiv y$ is the "Poincaré center problem" for the Abel equation. In this case we say that the points $x = 0$ and $x = b$ are *conjugated* along the equation (4).

One can easily construct examples of Abel equations with conjugated points as follows: starting from an arbitrary Abel equation, make a many-to-one polynomial change of the independent variable $x = x(t)$. The result will be a "foldable" Abel equation with all points of each preimage $t^{-1}(b)$ conjugated with each other for any choice of b. The conjecture is that this is the only possibility for appearance of conjugated points. In the language of composition algebra of coefficients, this is tantamount to existence of a non-trivial compositional common factor for the primitives $P = \int p$ and $Q = \int q$ [8].

One can also formulate an infinitesimal version of this problem. If $q(x) \equiv 0$, then the equation (4) becomes integrable and the conjugate points occur only at the roots of $P(x)$. Adding a *small perturbation* $\varepsilon q(x)y^3$ to this integrable equation makes the Green function $G(x, y)$ depending on ε, and the first variation in ε, the "Poincaré integral" $F(x, y) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} G(x, y)$ (cf. with the problem **1978-6** e.a.), can be reduced to the integral

$$F(x, y) = \int_0^x \frac{q(t)\,dt}{1 - yp(t)} = \sum_{k=0}^{\infty} m_k(x)\, y^k, \qquad m_k(x) = \int_0^x P^k(t)q(t)\,dt. \tag{5}$$

The coefficients $m_k(x)$ are the moments of q with respect to the weight $P(x)$, and their common zeros determine the "infinitesimally conjugate" points. As before, zeros of a compositional common factor of P and Q are common zeros of all the moments m_k, the problem is to describe the other such roots.

Returning to the initial problem involving the Abel equation (4) with trigonometric polynomials $p, q \in \mathbb{C}[\exp ix]$, it is proved that this equation is a center (i.e., all trajectories are 2π-periodic) when all 2-dimensional moments $\int_0^{2\pi} P^k Q^l dP(t)$ are zeros [7]. This assertion is wrong for arbitrary trigonometric functions. The reason behind this fact is that the moments can be computed as periods of polynomial 1-forms on a certain naturally arising algebraic curve, and the assertion is valid if this curve is rational.

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Problem 1980-3. The number of limit cycles born in the "Lotka–Volterra" system

$$\begin{cases} \dot{x} = x\,(\alpha + \beta x + \gamma y + \cdots), \\ \dot{y} = y\,(\delta + \varepsilon x + \zeta y + \cdots), \end{cases}$$

near $\alpha = \delta = 0$. In particular, integrals along $x^p y^q z^r = h$, $z = 1 - x - y$.

Comments. This problem is a particular low-degree case of the general infinitesimal Hilbert problem, see the commentary to problem **1978-6** e.a.

The question about the number of limit cycles born from the quadratic Lotka–Volterra system was answered by H. Żołądek [1]. He proved using tremendously heavy and absolutely mysterious computations that in the quadratic perturbation of the Lotka–Volterra system the corresponding Poincaré integral may have at most 2 isolated zeros.

References

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Problem 1981-16. Is that true that a polynomial vector field on the plane has only finitely many limit cycles? *H. Dulac committed an error.*

Comments. The assertion now commonly referred to as the *Dulac conjecture* or *Dulac problem*, was solved independently and by two completely different methods by Yu. Ilyashenko [1] and J. Ecalle [2]. In both cases the affirmative answer is derived from the *nonaccumulation theorem* asserting that limit cycles of an *analytic* vector field on the plane cannot accumulate to a *polycycle*, a separatrix polygon formed by one or more singular points of the vector field and arcs connecting these points.

Each proof is extremely involved and occupies an entire book. Ilyashenko's publication was preceded by several articles [3, 4, 5, 6] proving the nonaccumulation theorem for special classes of polycycles and containing in a nutshell the basic ingredients of the general proof.

The finiteness theorem is widely considered as an absolute peak achievement in all activity concerning the Hilbert 16th problem.

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3. Miscellaneous problems

Problem 1976-2. Let there be given two vector fields of degrees n and m on the plane. Can one estimate from above the number of intersections between their limit cycles in terms of n and m (find a sharp bound)?

Comments. This result appears in the book by Khovanskiĭ [1, p. 26] under the name *Bezout theorem for P-curves*: the number of isolated intersections can be at most $(n+m)(2n+m)+n+1$. Somewhat strangely, the bound is asymmetric. It is not known, whether the bound is sharp.

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Problem 1979-22. Estimate the number of ovals of a curve with a fewnomial equation, through the number of its terms.

Comments. This problem was first posed in [1] and solved in [2] (Theorems 4 and 5, chapter 2).

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Problem 1984-7. Construct a theory of versal deformations of Fuchsian systems. Is it true that regular singularities are isomonodromic limits of (confluent) Fuchsian points? Which matrices from the monodromy group converge to the Stokes matrices in the irregular case?

Comments. Relegated to A. Glutsuk who solved a very closely related problem.

Problem 1984-10. Describe variational and symplectic properties of Picard–Fuchs equations (the Gauss–Manin connexion). Aren't they the Euler equations for an appropriate group?

Problem 1985-12. Are the Picard–Fuchs equations Hamiltonian with respect to some natural symplectic structure, and do they possess a positive Lagrangian responsible for some kind of non-oscillatory behavior?

Comments. The problem reappeared recently in [1].

The question was motivated by Arnol'd's "Lagrangian Sturm theorem" [2] describing moments of non-transversality between Lagrangian planes moving by virtue of a linear Hamiltonian system with quadratic (nonautonomous) Hamiltonian, and a frozen (fixed) Lagrangian plane.

In [3] A. Givental proved that the system Picard–Fuchs equations for *hyperelliptic* Abelian integrals are Hamiltonian. More precisely, integrals of the forms $(x^{2g-k}/y)\,dx$, $k=1,\dots,2g$, over the level curves $\{H(x,y)=t\}$ of the Hamiltonian $H(x,y)=y^2+x^{2g+1}+\lambda_1 x^{2g-1}+\cdots+\lambda_{2g-1}x$, satisfy a linear system which is Hamiltonian with respect to the symplectic form obtained from the intersection form.

The corresponding Hamiltonian is positive for those values of t for which the real level curve $\{H=t\}$ possesses the maximal number $g+1$ of components. By the above Arnol'd theorem, this means that certain determinants involving the hyperelliptic integrals, are non-oscillating on such intervals of t. This does not imply, however, any information on zeros of the integrals themselves.

In [4] it is shown that the Picard–Fuchs system for hyperelliptic integrals has a *hypergeometric* form, $(tE+A)\dot I = BI$ with A,B constant matrices depending on H (and I is the column vector of hyperelliptic integrals). This is true also for Abelian integrals associated with a generic bivariate Hamiltonian H, if I consists of integrals of all cohomologically independent monomial 1-forms of degree $\le 2\deg H$ (ibid.).

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