ADVANCES IN SOVIET MATHEMATICS
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Around Hilbert Sixteenth Problem

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> Im Anshuß ... die Frage nach der Maximalzahl und Lage der Poincaréschen Grenzzyklen für eine Differentialgleichung erster Ordnung und erster Grades von der Form:
> $$\frac{dy}{dx} = \frac{Y}{X}$$
> wo X, Y ganze rationale Funktionen n-ten Grades in x, y sind.
>
> DAVID HILBERT, *Mathematische Probleme,* 1900

Two principal subjects of this Volume are bifurcations of limit cycles of planar vector fields and desingularization of singular points, both for individual vector fields and for analytic families of the latter. These subjects are closely related to the second part of the Hilbert Sixteenth Problem. The goal of this introductory paper is to introduce the general context and outline connections between the various results obtained in the five research papers constituting this Volume. We had no intention, however, to give a complete survey of the area: the recent collection of lecture notes [S] covers a broader field and gives a panorama of the current state. Among the notes in [S] the papers [I2], [D2], [R3] and [Rs] are the most close in scope to the subjects treated below.

In this introduction we refer to the papers constituting the Volume, as PAPER 1, ... ,PAPER 5.

§1. The Hilbert problem and finiteness theorems for limit cycles of polynomial vector fields

The shortest (the original) way of formulating this problem is to ask *what is the number and position of Poincaré limit cycles* (isolated periodic solutions) *for a polynomial differential equation* $dy/dx = P(x, y)/Q(x, y)$, *where* P *and* Q *are polynomials of degree* n.

1.1. Different forms of the Hilbert's question. The formulation given by Hilbert admits several specifications, described in the following three subsections; the connections between them are schematically shown on Table 1 (the statements of other problems will appear later).

Recall that a limit cycle of the differential equation

$$\frac{dy}{dx} = \frac{P_n(x\,,\,y)}{Q_n(x\,,\,y)}\,, \qquad P_n\,,\,Q_n \in \mathbb{R}[x\,,\,y]\,,\ \ \deg P_n\,,\,Q_n \leqslant n \tag{1}$$

is a periodic solution which has an annulus-like neighborhood free of other periodic solutions on the $(x\,,\,y)$-plane.

INDIVIDUAL FINITENESS PROBLEM. *Prove that a polynomial differential equation* (1) *may have only a finite number of limit cycles.*

This problem is known also as *Dulac problem* since the pioneering work of Dulac (1923) who claimed to solve it, but gave an erroneous proof.

EXISTENTIAL HILBERT PROBLEM. *Prove that for any finite* $n \in \mathbb{N}$ *the number of limit cycles is uniformly bounded for all polynomial equations* (1) *of degree* $\leqslant n$ *.*

If we denote

$$H(n) = \left\{ \begin{array}{l} \text{the uniform upper bound for the number} \\ \text{of limit cycles occurring in polynomial} \\ \text{differential equations of degree } \leqslant n \end{array} \right\} \tag{2}$$

then the existential problem consists in proving that

$$\forall n \in \mathbb{N} \qquad H(n) < \infty.$$

CONSTRUCTIVE HILBERT PROBLEM. *Give an upper estimate for* $H(n)$ *or suggest an algorithm for computing such an estimate.*

The solution of this last problem would obviously imply the solution of the previous ones (the individual and the existential versions). However, in the next section we consider other statements which are not reduced to the constructive Hilbert problem.

1.2. Nonaccumulation Theorem. Out of three forms of the Hilbert problem, only the first one (the weakest) is proved. Two independent and rather different proofs were given almost simultaneously by Yu. Ilyashenko [I1] and J. Écalle [É]. The preliminary stages of both proofs include the following preprocessing which was known already to Dulac.

Note that a polynomial differential equation makes sense also at infinity. To make this statement precise, consider the line field on $\mathbb{R}^2$ whose slope at a point $(x\,,\,y)$ is $P(x\,,\,y)/Q(x\,,\,y)$ (we assume that the pair of polynomials has no common factors, thus the points of indeterminacy are isolated; the precise definition of a line field with singularities can be found in PAPER 2).

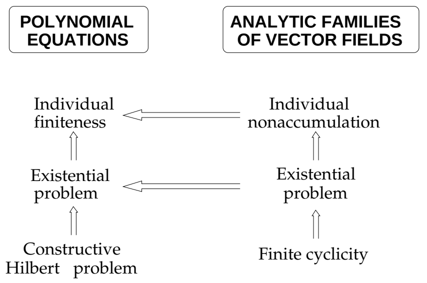

TABLE 1. Relations between different versions of the Hilbert problem: arrows stand for implications.

Then this line field extends analytically onto the projective compactification $\mathbb{R}P^2 \supset \mathbb{R}^2$: in a neighborhood of the infinite line $\mathbb{R}P^1$ one can multiply the vector field $P_n\,\partial/\partial x + Q_n\,\partial/\partial y$ by a meromorphic nonzero factor in such a way that the resulting vector field, which spans the same line field, would admit an analytic extension onto $\mathbb{R}P^1$ with at most $n+1$ singular points on the infinite line.

To obtain an orientable phase space, one can consider the sphere $\mathbb{S}^2$ together with the canonical covering $\pi\colon \mathbb{S}^2 \to \mathbb{R}P^2$; the pullback of the line field from $\mathbb{R}P^2$ is a symmetrical line field on $\mathbb{S}^2$ with isolated singularities. Note that both the sphere and the projective plane are compact.

Assume for a moment that a polynomial vector field possesses an infinite number of limit cycles. By the Poincaré–Bendixson theorem, limit cycles of any differential equation should be "nested" around singular points. Since a polynomial differential equation may have only a finite number of such points, then limit cycles must accumulate (in the sense of Hausdorff metric) to a certain object which consists of some singular points and regular arcs (infinite trajectories) connecting them. Modulo a certain terminological discord, such objects are called *polycycles*. The precise definition follows.

DEFINITION. A polycycle of a line field is a cyclycally ordered collection of singular points $p_1, p_2, \dots, p_k$ (eventually with repetitions) and arcs (integral curves) connecting them in the specified order: the jth arc connects p_j with p_{j+1} for $j = 1, \dots, k$.

A trivial example of a polycycle is a periodic solution of the differential equation, without singular points and only one arc. Another case of polycycle having no arcs and just one singular point, is also admissible but far from being trivial.

The above arguments reduce the individual finiteness problem to the *nonac-*

cumulation problem: prove that limit cycles cannot accumulate to a polycycle of a polynomial differential equation. This problem is *semilocal*: its assertion concerns a small neighborhood of a polycycle rather than the whole (x, y)-plane. The following theorem proved by Ilyashenko and Écalle solves the individual finiteness problem (the Dulac problem).

NONACCUMULATION THEOREM. *For any analytic vector field on a two-dimensional real analytic manifold* (*surface*)*, limit cycles cannot accumulate to a polycycle.*

Thus the nonaccumulation theorem is analytic rather than algebraic assertion, and it implies the individual finiteness for some analytic differential equations as well.

COROLLARY (individual finiteness theorem for analytic vector fields on the sphere). *An analytic vector field on the* 2*-dimensional sphere* $\mathbb{S}^2$ *may have only a finite number of limit cycles.*

1.3. Desingularization. The singular points occurring on a polycycle, may be of any degree of degeneracy, for instance, the polycycle may consist of just one very degenerate singular point. Nevertheless there is known a procedure which allows for investigation of polycycles having only relatively simple singularities, the *elementary* ones.

DEFINITION. A singular point of a planar differential equation is *elementary*, if the linearization of the equation at this point at least one nonzero characteristic number (the eigenvalue of the linearization matrix).

The simplest example of a non-elementary singularity is the *cuspidal point*, the singular point with the nonzero nilpotent linearization matrix. Linearization of a vector field at the cuspidal point has the form $\dot x = y$, $\dot y = 0$.

The procedure of simplification of singular points of a differential equation is known under several names: *desingularization*, *blowing-up*, *σ-process*, *resolution of singularities*. In any case, the idea is to delete a singular point from its small neighborhood and replace it by a one-dimensional curve, a projective line or a circle. For example, to make the *polar blow-up* of the origin, one introduces polar coordinates,

$$(r, \varphi) \xrightarrow{\rho} (x, y) = (r\cos\varphi, r\sin\varphi), \qquad r > 0,\ 0 \leqslant \varphi < 2\pi.$$

The differential equation put into the polar coordinates admits an analytic extension for (small) negative values of r and division by a factor of r^ν, where ν is determined by principal terms of the Taylor expansion of the right hand sides at the origin. After such a division one may consider the system in a narrow annulus $-h < r < h$, $0 \leqslant \varphi < 2\pi$. The whole circle

$r = 0$ is the preimage of what formerly was a singular point of the equation, and singularities of the new field on this circle are in some sense simpler then the original singularity at the point $x = y = 0$ on the (x, y)-plane. If necessary, the procedure may be iterated (the new points are in turn blown up) until *all singularities become elementary*. The possibility of blowing up any singular point (satisfying the Lojasiewicz condition in the smooth case or isolated in the analytic category) into elementary singularities is the assertion of Bendixsson–Seidenberg–Dumortier theorem, see [D1], [VdE].

The polar blow-up has some disadvantages. First, it involves trigonometric functions and thus leads to the loss of algebraicity. Second, the points (r, φ) and $(-r, \varphi+\pi)$ correspond to the same point on (x, y)-plane, thus after the resolution the number of singular points is doubled. There exists an algebraic version, the *σ-process*, which operates with polynomial expressions and does not produce twin singularities: from the geometrical point of view it amounts to replacing the annulus around $r = 0$ by the Möbius band, the quotient space of the annulus by the equivalence $(r, \varphi) \sim (-r, \varphi + \pi)$. The central circle becomes then the projective line. In more details these procedures are explained in Paper 2 and Paper 3.

In any case, when proving the nonaccumulation theorem, one may consider only *elementary polycycles*, that is polycycles carrying only elementary singularities on some analytic 2-dimensional surface.

1.4. Analytic nature of the monodromy map. After this preprocessing one has to study the *Poincaré return map*, or *monodromy* Δ around the elementary polycycle which we denote by γ. This map is defined exactly as in the case of a (nonsingular) periodic orbit: choose a small segment Σ transversal to an arc of the polycycle and let Δ be the map of this segment into itself along solutions of the equation provided that they never leave the small neighborhood of γ. Unlike the case of a periodic orbit, the map Δ is not analytic at the point $p = \gamma \cap \Sigma$ and is defined usually only from one side of p. The problem of investigating analytic properties of the map Δ is very complicated (requires hundreds of pages in both known versions), and as a result one comes to the conclusion that fixed points of the map $\Delta\colon \Sigma \to \Sigma$ cannot accumulate to p.

The analysis carried in [I1] is based on using functional cochains and superexact asymptotic expansions. The main tool in [É] is the theory of resurgent functions and resummability. Both tools currently do not allow for any generalization of the proof for the case of vector fields depending on parameters.

§2. Analytic families of vector fields and cyclicity of polycycles

2.1. Universal polynomial family. The natural way to look at *all* polynomial equations (1) at once is to consider them as an *analytic family of line fields on the sphere*; the parameters of this family are coefficients of the polynomials P, Q. The parameter space thus introduced is the Euclidean space with the deleted origin (since the case $P=Q=0$ does not correspond to a line field), but in fact the simultaneous multiplication of both P and Q by a common factor $\lambda \in \mathbb{R}$, $\lambda \neq 0$, does not change the line field. Thus the existential Hilbert problem becomes a particular case of the following global conjecture.

GLOBAL FINITENESS CONJECTURE. *For any analytic family of line fields on the two-dimensional sphere $\mathbb{S}^2$ with a compact finite-dimensional parameter space B, the number of limit cycles for all fields is uniformly bounded over all parameter values.*

In what follows we refer to the family obtained by compactification of (1) as the *universal polynomial family* of degree n.

2.2. Cyclicity. Suppose that the global finiteness conjecture is wrong for a family of line fields $\alpha(\varepsilon)$, $\varepsilon \in B$. Then, since it is known that each individual field from the family has only a finite number of limit cycles (see §1.2), there must exist an infinite sequence of parameter values $\varepsilon_k \in B$, $k = 1, 2, \dots$, such that the corresponding line fields have monotonously increasing numbers of limit cycles. Since the base B is compact, without loss of generality we may assume that the sequence ε_k converges to a certain point $\varepsilon_* \in B$. Next, since the sphere is compact, the cycles of $\alpha(\varepsilon_k)$ should accumulate to a certain compact subset of the sphere, invariant by the field $\alpha(\varepsilon_*)$. Such sets were introduced by Françoise and Pugh [FP] under the name of *limit periodic sets*. The precise definition looks as follows.

DEFINITION. A subset $\gamma_* \subset \mathbb{S}^2$ is a limit periodic set for the family of line fields $\alpha(\varepsilon)$, $\varepsilon \in B$, at a point ε_* if there exists a sequence of points $\varepsilon_k \to \varepsilon_*$ and the corresponding line fields $\alpha_k = \alpha(\varepsilon_k)$ have limit cycles γ_k which converge to γ_* in the sense of Hausdorff distance.

The structure of limit periodic sets admits a simple description.

PROPOSITION [FP]. *A limit periodic set either is either a polycycle, or contains an arc of nonisolated singularities of the field $\alpha(\varepsilon_*)$.*

The following definition intoduces an important characteristics of a limit periodic set occurring in a certain family of (line, vector) fields.

DEFINITION. We say that a limit periodic set γ_* occurring in a family of line fields on the sphere for a certain parameter value ε_*, *has cyclicity* $\leqslant \mu$ if there exist neighborhoods U, V, $\mathbb{S}^2 \supseteq U \supset \gamma$, $B \supseteq V \ni \varepsilon_*$ such that for any $\varepsilon \in V$ the field $\alpha(\varepsilon)$ has no more than μ limit cycles in U. In other words, the polycycle generates no more than μ limit cycles after bifurcation in the family $\alpha(\cdot)$.

The minimal such μ (if it exists) is called the *cyclicity* of the limit periodic set, otherwise the cyclicity is said to be infinite.

REMARK. The notions of limit periodic set and cyclicity are defined for families rather then for individual line fields. Still if we have a polycycle for an analytic individual line field α_*, then cyclicity of such polycycle may sometimes be estimated from above for *any analytic family* $\alpha(\varepsilon)$ unfolding the field α_*. In this case the term *absolute cyclicity* is used.

Returning back to the global finiteness conjecture, we see that the assumption on the unboundedness of the number of limit cycles would lead to contradiction if the following assertion were proved.

FINITE CYCLICITY CONJECTURE (Roussarie). *Any limit periodic set occurring in an analytic family of line fields on the sphere, has finite cyclicity in this family.*

The above parametric localization procedure can be formulated in the form of an implication.

THEOREM [R2], SEE ALSO [Aea].

Finite cyclicity of any limit periodic set in a family of (vector, line) fields on the sphere with a compact parameter space	$\Longrightarrow$	Existence of a universal bound for the number of limit cycles occurring in this family

2.3. Examples of cyclicity estimates. There are many different examples of limit periodic sets whose cyclicity is known to be finite or even explicitly computed. For example, a hyperbolic periodic orbit (a trivial polycycle) has absolute cyclicity 1. The simplest nontrivial example of a polycycle (with at least one arc and at least one singularity) is a separatrix loop of a hyperbolic saddle: continuations of stable and unstable invariant curves form a closed loop. If the divergence of the vector field at the saddle point is different from zero, then the loop has absolute cyclicity 1 (Andronov and Leontovich). In those (and in many other cases) the results are valid even for smooth families, and they were in fact established in the classical bifurcation theory. In §3 we give a brief summary of the cyclicity results given *gratis* by that theory and its recent developments.

On the other hand, there are some results which establish finite (not absolute) cyclicity for polycycles occurring in analytic families. In these results no nondegeneracy-type assumptions is made, so they may be applied to polycycles with the identical monodromy.

The simplest case of a periodic orbit was studied in [FP]. The first really nontrivial case of a separatrix loop of a nondegenerate saddle was analyzed by Roussarie [R1] together with the closely related case of cuspidal singular points.

But historically the first case of effective computation of cyclicity which is not absolute, is due to Bautin [B]. Bautin studied bifurcations of limit cycles from an *elliptic singular point*, i.e. the point at which the eigenvalues are complex conjugate, $\sigma \pm i\omega$, $\omega \neq 0$, in the universal family of quadratic vector fields.

BAUTIN THEOREM. *An elliptic singular point has cyclicity* $\leqslant 3$ *in the universal polynomial family of degree* 2 *.*

The original proof is very complicate and involves heavy computations. In PAPER 5 a new proof of this result is suggested. This proof follows to a certain extent the original Bautin's proof, but on the final steps instead of almost incomprehensible manipulations with integrals, a simple geometric reasoning based a hidden $\mathbb{Z}_3$-symmetry allows to arrive to the conclusion.

REMARK. Recently yet another proof of this result was suggested by H. Żołądek [Z]. That proof is based on some rotational symmetry of the problem.

2.4. Quadratic vector fields. Recently an intense attack was launched by Dumortier, Roussarie and Rousseau [DRR] to solve the existential Hilbert problem for the family of quadratic vector fields, that is to prove that

$$H(2) < \infty.$$

There was composed a list of 121 polycycles and degenerate limit periodic sets which may occur after compactification. Out of this list, a substantial number of cases has been analyzed. The principal difficulties in investigating all these cases occur when a polycycle is identical (which means that from one side its neighborhood is filled with closed periodic orbits), or when a limit periodic set with nonisolated singularities occurs. The same applies to investigation of other universal polynomial families, and it is not very likely that without involving some essentially new ideas, the general existential Hilbert problem could be solved.

Still there exists a natural way to change settings in the existential Hilbert problem so that those pathologies would be ruled out. This reformulation is known as the *Hilbert–Arnold problem*.

§3. Generic smooth families of vector fields and the Hilbert–Arnold problem

3.1. Generic smooth families of vector fields on the sphere. There can be posed a question, why Hilbert had chosen polynomial families as the subject for investigation concerning limit cycles. Perhaps, the reason was that the universal polynomial family is the only constructive family of line fields on the plane which extends to a family of line fields with singularities on the sphere. Towards the second half of this century, the ideology changed and typical smooth objects (vector fields and their families) became a legal subject of consideration.

If we replace in the formulation of the existential Hilbert problem the universal family (1) by a *generic* family of vector fields on the 2-sphere $\mathbb{S}^2$, then at least some of the difficulties mentioned at the end of the previous section disappear.

Recall that for any subset $B \subseteq \mathbb{R}^n$ a function $\varphi\colon B \to \mathbb{R}$ is said to be smooth if it admits a C^∞-smooth extension onto some open neighborhood U of B.

Let $B \subset \mathbb{R}^n$ be a finite-dimensional compact. Then the space of C^∞-smooth families of vector fields $v(\cdot)\colon \mathbb{S}^2 \times B \to T\mathbb{S}^2$ admits the natural topology, induced by the metric $d(v_1, v_2) = \sum_k 2^{-k}\|v_1 - v_2\|_k$, where

$$\|v\|_k = \max_{\substack{x\in\mathbb{S}^2,\ \varepsilon\in B,\\ |\alpha|+|\beta|=k}} |D_x^\alpha D_\varepsilon^\beta v(x, \varepsilon)|. \tag{3}$$

Definition. We say that a generic n-parameter family of vector fields on the sphere possesses a certain property $\mathcal{P}$, if this property holds for a residual subset of the total space of all n-parameter families.

The property is said to hold for a generic family (without indicating the number of parameters n explicitly), if it holds for any generic n-parameter family, whatever a finite number n is.

Proposition. *In a generic family only isolated singularities occur, and their multiplicity is bounded over any compact subset in the space of parameters.*

Corollary. *Any limit periodic set occurring in a generic family, is a polycycle.*

In general, one may expect that generic families of vector fields in many respects resemble analytic families, but this is an observation rather than a formal claim. Still there are many reasons to believe that the following basic conjecture holds.

HILBERT–ARNOLD PROBLEM. *Prove that in a generic family of vector fields on the sphere $\mathbb{S}^2$ with a compact base B, the number of limit cycles is uniformly bounded.*

The Hilbert–Arnold problem was implicitly formulated in [AI] as a particular case of a conjecture stating that for generic n-parameter families of vector fields on the sphere, only a finite number of local bifurcation diagrams can be realized. In fact, the full conjecture as it was formulated in [AI], is wrong: the counterexample is given in PAPER 4 below. However, this example does not disprove the Hilbert–Arnold conjecture.

For small $n = 2$ and 3, that is, for few-parametric families of vector fields, the Hilbert–Arnold problem admits investigation by case studies, see §3.2, §3.3 for more information.

REMARK. The topology generated by the family of norms $\|\cdot\|_k$, is not the only possible: for example, one may take into consideration only derivatives in the x-variables and disregard those in the parameters ε. Other modifications are also available. In fact, to supply the word *generic* in the formulation of Hilbert–Arnold problem with a precise meaning, is a part of solution of the problem.

The same localization technique which was used when reducing investigation of analytic families to the study of small neighborhoods of limit periodic sets, works also for smooth families. The difference with the analytic case is in the presence of the natural index n, the number of parameters. It turns out that at least for small n, cyclicity of polycycles occurring in generic n-parameter families, admits an upper estimate in terms of n.

EXAMPLE. In a generic n-parameter family, the maximal multiplicity of a limit cycle does not exceed $n+1$: for example, for a structurally stable vector field only hyperbolic limit cycles occur, in codimension 1 may appear semistable limit cycles of multiplicity 2 *etc.* Thus cyclicity of a trivial polycycle (without singularities) in a generic n-parameter family, does not exceed $n+1$. For some reasons we exclude trivial (poly)cycles without vertces from consideration when giving the following definition.

DEFINITION. The *bifurcation number* $B(n)$ is the maximal cyclicity of *nontrivial* polycycles occurring in generic n-parameter families.

The definition of the number $B(n)$ does not depend on the choice of the base of the family, but only on its dimension n.

LOCAL HILBERT–ARNOLD PROBLEM (for n-parametric families). *Prove that for any finite n, the number $B(n)$ is finite.*

Solution of this problem would imply solution of the global Hilbert–Arnold problem by virtue of the same compactness arguments as in §2.2.

CONSTRUCTIVE HILBERT–ARNOLD PROBLEM. *Compute explicitly or give an explicit upper estimate for the bifurcation number* $B(n)$.

3.2. Hilbert–Arnold problem for few-parametric families. Bifurcations of limit cycles in generic few-parameter families of vector fields were the subject of studies since late thirties. The information accumulated about simplest bifurcations in generic one-parameter families, may be compressed into a single equality.

THEOREM B_1 (Andronov–Leontovich, 1930s; Hopf, 1940s).

$$B(1) = 1.$$

This result summarizes results of investigation of three classical bifurcations, separatrix loop of a saddle with nonzero divergence, saddle-node loop and an elliptic point.

The next number in the series, $B(2)$, has longer history and the corresponding equality summarizes numerous results.

THEOREM B_2 (Takens, Bogdanov, Leontovich–Cherkas, Mourtada, Grozovskiĭ, early 1970s–1993).

$$B(2) = 2.$$

The proof of this result involves consideration of bifurcations of 8 different polycycles which may occur in generic two-parameter families. Out of this list, four bifurcations were already studied by different authors before the problem of computing $B(2)$ was explicitly formulated, and the cyclicity found to be at most 2 . Out of the four remaining cases, three are very simple and correspond to cyclicity 1 , and the last case, the *half-apple*, was studied recently by a graduate student T. Grozovskiĭ. It consists of a polycycle with 2 singularities, a nondegenerate saddle and a saddle-node, and its cyclicity does not exceed 2 .

The list of polycycles occurring in dimensions 2 and 3 is given in PAPER 4. A complete investigation of all polycycles occurring in codimension 3 , would yield an upper estimate for the bifurcation number $B(3)$, though several cases from this list seem to be very hard (for example, a loop carrying a degenerate cuspidal point). At the same time it is clear that an attempt to obtain a sharp upper estimate for $B(4)$ by a similar case study, is hopeless.

3.3. Lips and other ensembles. Investigation of generic 3-parametric families revealed some very simple still surprisig facts concerning bifurcation of limit cycles in such families.

The definition of bifurcation numbers starts from the notion of a polycycle. There might be an alternative approach. For any smooth family $v(x, \varepsilon)$,

$x \in \mathbb{S}^2$, $\varepsilon \in B \Subset \mathbb{R}^n$ of vector fields on the 2-sphere one may define an integer-valued counting function $\mathfrak{c}(\cdot) = \mathfrak{c}_v(\cdot) \colon B \to \mathbb{Z}_+$,

$$\mathfrak{c}(\varepsilon) = \text{the number of limit cycles of the field } v(\cdot\,,\varepsilon) \text{ on } \mathbb{S}^2 .$$

This function is defined for all values of the parameter $\varepsilon \in B$ and clearly even for generic families there cannot be any natural bound for $\mathfrak{c}$ in terms of $n = \dim B$, though for such families this function presumably has finite values.

However, if we introduce the *oscillation* function $\mathfrak{o}(\cdot) = \mathfrak{o}_v(\cdot)$ constructed for the family v as

$$\mathfrak{o}(\varepsilon) = \operatorname{osc} \mathfrak{c}(\varepsilon) = \lim_{r \to 0+} \left(\sup_{|\varepsilon' - \varepsilon| < r} \mathfrak{c}(\varepsilon') - \inf_{|\varepsilon' - \varepsilon| < r} \mathfrak{c}(\varepsilon') \right) \geqslant 0 ,$$

then it turns out that the function $\mathfrak{o}(\cdot)$ admits an upper estimate for all generic 1- and 2-parametric families. Note that the oscillation function is zero for a generic point $\varepsilon \in B$, since generic vector fields are structurally stable.

THEOREM. *For a generic n-parameter family of smooth vector fields on the sphere, the oscillation function does not exceed 2 if $n = 1$, and is everywhere less or equal to 3 for $n = 2$.*

The proof of this theorem is also obtained by studying separate cases: now one has to take into account the possibility of simultaneous formation of several polycycles and multiple limit cycles. It is clear that if two polycycles are disjoint, then their simultaneous occurrence is an event of the codimension equal to the sum of degeneracy codimensions of each polycycle independently, and the oscillation function is equal to the sum of the terms corresponding to bifurcations of each polycycle. In generic one-parameter families this is the only possibility, and the upper bound equal to 2 appears because a semistable (double) limit cycle may disappear or generate two close hyperbolic limit cycles.

In codimension 2, however, appear *ensembles* of polycycles, that is graphs formed by several polycycles with a common singularity or an arc. Such ensembles may have degeneracy codimension *strictly smaller* than the sum of codimensions of polycycles constituting them. On the other hand, different polycycles constituting an ensemble, sometimes cannot *simultaneously* generate the maximal number of limit cycles, so that the total number of limit cycles born from an ensemble is only a subadditive function.

EXAMPLE. Consider an ensemble composed by two separatrix loops of the same hyperbolic saddle point with a nonzero divergence. Then this ensemble is a union of *three* polycycles, two simple loops and the eight-shaped contour. Each one of them has cyclicity 1, but their union has cyclicity 2 in a generic 2-parametric family, since the simultaneous generation of limit cycles by all three polycycles is impossible (D. Seregin, E. Malgina, in preparation).

In codimension 2 there are possible 9 different types of ensembles, (see PAPER 4), and their investigation yields the above theorem. One can express the upper bound for the oscillation function $\mathfrak{o}(\cdot)$ in terms of bifurcation numbers similar to the numbers $B(n)$.

DEFINITION. The global bifurcation number $C(n)$ is the upper bound for the number of limit cycles which can be born from *all polycycles* which may simultaneously occur in a generic n-parameter family of vector fields on the sphere.

After introducing this number, an evident estimate holds for the oscillation function,

$$\mathfrak{o}_v(\cdot) \leqslant C(n) \qquad \text{for a generic } n\text{-parameter family } v\,,$$

and the above theorem can be formulated as folows:

$$C(1) = 2\,, \qquad C(2) = 3.$$

One might expect that for generic 3-parameter families there also should be such a universal bound, and a problem for estimating the number $C(3)$ should be the next in a row. But the counterexample below shows that it is not the case: though for any generic 3-parameter family the function $\mathfrak{o}(\cdot)$ is locally bounded, there cannot be an upper bound common for all generic 3-parameter families, as it was in the cases $n = 1\,, 2$. The reason for that is a simultaneous occurrence of a continuum of polycycles for isolated values of parameters in generic 3-parameter families of vector fields on the sphere.

The simplest example of a generic 3-parameter family of vector fields on the 2-sphere with a continuum of coexisting polycycles was found by A. Kotova and baptized *lips*. Consider two saddle-nodes $S_\pm$ of multiplicity 2 (topologically equivalent to the vector field $x^2\,\partial/\partial x \pm y\,\partial/\partial y$); their simultaneous occurrence is an event of codimension 2. Suppose that the (uniquely defined) trajectory emanating from S_- and the (uniquely defined) trajectory which tends to S_+ are continuations of each other, together forming a heteroclinic orbit η. This means an additional degeneracy of the field, thus the whole picture may occur for isolated values of parameters in a generic 3-parameter family.

Each saddle-node has a parabolic sector entirely filled by trajectories tending to (from) the singular point S_- (resp., S_+). Without increasing the codimension, we may assume that *there exists a trajectory* χ_0 *passing through* *interiors of both saddle-nodes.* If there is at least one such trajectory, then all sufficiently close trajectories χ_s, $s \in (\mathbb{R}^1, 0)$, also are bi-asymptotic to both saddle-nodes, see Figure 1. Finally we obtain a continuum of polycycles

$$S_+ \to \chi_s \to S_- \to \eta \to S_+ , \qquad s \in (\mathbb{R}^1, 0).$$

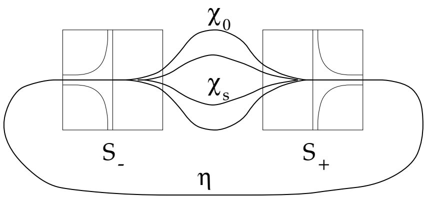

FIGURE 1. Lips.

Bifurcations of lips in generic 3-parametric families are studied in PAPER 4; as a byproduct of this investigation, the *generalized Legendre duality* was constructed.

KOTOVA THEOREM. *For any natural N there exists a 3-parametric family $v(\cdot, \varepsilon)$ of vector fields on the sphere, such that the oscillation function $\mathfrak{o}(\varepsilon)$ constructed for this family, takes a value $> N$ at a certain point, and the same is true for all sufficiently C^k-close three-parametric families of vector fields, if k is large enough.*

In other words,

$$C(3) = +\infty.$$

COROLLARY (Kotova, Stanzo). *For generic 3-parameter families of smooth vector fields on the sphere, there exists an infinite number of pairwise locally topologically nonequivalent bifurcation diagrams.*

The bifurcation diagram for the lips is constructed in PAPER 4 by V. Stanzo.

The lips are not the only possible "pathology" which may occur in generic 3-parametric families. For example, somewhere "between" the two saddle-nodes, an arbitrary number of nondegenerate saddles may occur without rising the codimension of the whole picture. Those saddles may participate in creation of polycycles with three singular points, and it is clear that any finite number of such polycycles may coexist. For further details see PAPER 4.

3.4. Elementary polycycles and their finite cyclicity in generic families. The local Hilbert–Arnold problem was solved under an additional assumption that all singular points occurring on a polycycle, are elementary (see the definition in §1.3).

DEFINITION. The *elementary bifurcation number* $E(n)$ is the maximal cyclicity of a nontrivial *elementary* polycycle occurring in a (smooth) generic n-parameter family.

The only nonelementary polycycle which may occur in a 2-generic family, is a cuspidal point whose bifurcations were studied by Bogdanov and Takens, and whose cyclicity was found to be 1. Thus Theorems B_i, $i = 1, 2$, imply that

$$E(1) = 1, \qquad E(2) = 2.$$

However, the nature of the function $n \mapsto E(n)$ is now understood much better than that of $B(\cdot)$.

THEOREM (Ilyashenko and Yakovenko, 1992). *For any n the elementary bifurcation number $E(n)$ is finite.*

Moreover, the function $n \mapsto E(n)$ admits a primitive recursive majorant.

COROLLARY. *The global Hilbert–Arnold problem has a positive solution for families of vector fields in which only elementary singularities occur: any generic family of vector fields on the sphere with a compact finite-dimensional base of parameters and with elementary singular points only, has a uniformly bounded number of limit cycles.*

REMARK. A primitive recursive function is an integer function of a natural argument n, which admits an algorithmically effective computation for any specified value of the argument. The formal definition is given in PAPER 1. In fact, it is very likely that this majorant is elementary, that is, an explicit expression could be written for it.

This theorem was announced in [IY1]. A complete demonstration of this result is given in PAPER 1. It consists of the four principal steps:

1. *C^k-smooth normalization* of the family near each elementary singularity. The main tool for that is provided by the classification theorems from [IY2]. The normal forms are polynomial and integrable. We perform an explicit integration of normal forms in the class of Pfaffian functions introduced by A. Khovanskiĭ [K] and show that the correspondence maps near each singular point in the normalized coordinates can be expressed through elementary transcendental functions which satisfy some algebraic Pfaffian equations. The degree and the total number of these equations can be estimated in terms of n.

2. *"Algebraization"* of the system of equations obtained on the previous step: the reduction procedure suggested in [K] allows for elimination of transcendental functions from the equations determining fixed points of the monodromy map. After this elimination there appears a system of equations having the form of a *chain map*, a composition of a polynomial map and a jet extension of a generic smooth map.

3. *Gabrielov-type finiteness conditions* are established for a smooth map $F\colon \mathbb{R}^k \to \mathbb{R}^k$ to have a uniformly bounded number of regular preimages $\#F^{-1}(y)$ when the point y varies over a compact subset of $\mathbb{R}^k$. These conditions are automatically satisfied if a map F is real analytic. We introduce a topological complexity characteristics, the *contiguity number*, in terms of which an upper estimate for the number of preimages can be expressed.

4. *Thom–Boardmann-type construction* allows to prove that the above finiteness conditions can be expressed in terms of transversality of the jet extension of F to some semialgebraic subsets of the jet space. Moreover, this construction can be generalized to cover *chain maps* of the form $P \circ (j^\ell F)$, where P is a polynomial, and $j^\ell F$ is the ℓ-jet extension of a generic smooth map. This is exactly the class of maps which appear after the Khovanskiĭ elimination procedure (step 2 above). The contiguity number of a chain map is expressed through the integer data (degree of the polynomial P, order of the jet ℓ and dimension of the domain and target spaces).

REMARK. Though the upper estimate obtained in the above theorem, is *not family-dependent* (common for all n-parameter families), still the cyclicity established is *not absolute*: we require the family considered as a map $v\colon \mathbb{S}^2 \times (\mathbb{R}^n, 0) \to T\mathbb{S}^2$ to be transversal to some algebraic stratified subset in an appropriate jet space after the jet extension.

§4. Parametric desingularization

In §1.2 we explained the role of elementary singularities in proving the nonaccumulation theorem. If something similar could be proved for families rather than for individual vector (or line) fields, then the theorem on finite cyclicity of generic elementary polycycles would imply the Hilbert–Arnold problem in full generality. Unfortunately, such a straightforward policy does not yield immediate results.

4.1. Two approaches to parametric desingularization. From the point of analytic geometry, locally a family of (vector, line) fields may be considered as a single (vector, line) field on the *total space of the bundle* $\pi\colon (\mathbb{R}^2, 0) \times (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\pi(x, y; \varepsilon) = \varepsilon$, *tangent to the fibers of this bundle*. In this section we consider only the case of analytic families.

Isolated singular points occurring in the family of vector fields in such settings correspond to an analytic at most n-dimensional set which intersects the fibers of the bundle π by discrete subsets.

One approach to parametric desingularization, suggested by Z. Denkowska and R. Roussarie [DR], was applied to investigation of families of vector fields which have nonelementary singularity occurring for isolated values of the parameter ε (a typical example is an unfolding of a cuspidal point), which we assumed to be the origin $x = y = 0$, $\varepsilon = 0$. The idea is to blow-up the origin *in the total space of the bundle*, deleting it and pasting in an $(n+1)$-dimensional sphere instead. One of the possible ways to do this is to consider a quasihomogeneous mapping

$$\begin{gathered} \sigma\colon \mathbb{S}^{n+1} \times (\mathbb{R}^1_+, 0) \to (\mathbb{R}^2, 0) \times (\mathbb{R}^n, 0), \\ (\overline{x}, \overline{y}, \overline{\varepsilon}_1, \dots, \overline{\varepsilon}_n; r) \overset{\sigma}{\longmapsto} (\overline{x}\, r^{\nu_1}, \overline{y}\, r^{\nu_2}, \overline{\varepsilon}_1\, r^{\mu_1}, \dots, \overline{\varepsilon}_n\, r^{\mu_n}) \end{gathered} \tag{4}$$

with the weights ν_i, μ_j chosen in an appropriate way.

After such a procedure the total space becomes not a bundle but rather a singular "foliation", whose fibers have different dimensions; the fiber over a nonzero value of ε is still two-dimensional, while the fiber over $\varepsilon = 0$ is diffeomorphic to the sphere $\mathbb{S}^{n+1}$. The investigation of the pullback of the original family onto a neighborhood of the pasted in sphere proceeds further in different charts which play different role.

We do not intend to expose the procedure in full details, referring the reader to the papers [DR], [R3].

The other approach suggested by S. Trifonov is explained in Paper 2. The main idea is to blow up not just one point in the total space, but rather the entire *singular locus*, the set of all singular points for all fields in the family. A brief explanation of this approach is given in the next section.

4.2. Desingularization after Trifonov. Suppose that in an analytic family of vector fields a singular point depends analytically on the parameters (like in the case when the singularity is nondegenerate). Then without loss of generality one may assume that locally the singularity remains at the origin $x = y = 0$ for all values of the parameters $\varepsilon \in (\mathbb{R}^n, 0)$. Thus we consider a line field on the total space $(\mathbb{R}^2, 0) \times (\mathbb{R}^n, 0)$, parallel to the vertical direction (the second factor), with singular points on the horizontal plane $\{x = y = 0\} \subset (\mathbb{R}^{n+2}, 0)$.

Then the mapping

$$\begin{gathered} \sigma\colon \mathbb{S}^1 \times (\mathbb{R}^1_+, 0) \times (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0) \times (\mathbb{R}^n, 0), \\ \sigma(\cos\theta, \sin\theta; \varepsilon) = (r\cos\theta, r\sin\theta; \varepsilon) \end{gathered}$$

which in fact is a trivial suspension of the standard polar blow-up, performs a simultaneous blow-up of the singularity at $x = y = 0$ for all vector fields from the family. In a similar way the algebraic σ-process can be applied to all vector fields of the complexified family: as a result, we obtain a new total space which is obtained from the original one by deleting the plane $x = y = 0$ and pasting in the cylinder $\mathbb{C}P^1 \times (\mathbb{C}^n, 0)$.

The real problems begin when the singularity looses its analytic dependence on the parameters. In this case the two basic additional arguments are used.

First, one should consider not all singularities, but only the *essential* ones. To explain the term, note that in analytic families for some exceptional values of the parameters the singular locus may well be nonisolated as a subset of the corresponding fiber. Since the fiber is two-dimensional (and this property will be maintained when iterating the construction, in contrast to the approach of Denkowska and Roussarie), the line field may be extended to the analytic curve of singular points by cancelling nontrivial common factors in the right hand side of the differential equations. After such cancellation the curve carries only discrete singular points of the extension of the line field. They are called *essential singularities*.

The second idea is to make singular reparametrizations of the family, or what is called in Paper 2 *unfoldings of the base*. The simplest (and in a sense the principal) case of such unfoldings is the spherical blow-up of $(\mathbb{R}^n, 0)$ at the origin, as in (4).

Trifonov proves that after making an appropriate blow-up of the parameter space, all essential singular points of the pullback of the original family may be placed on a finite number of analytic sections of the foliation. However, the latter looses its locally trivial topological nature: after performing all required steps the fibers become not globally diffeomorphic, though still two-dimensional and locally diffeomorphic.

The final form of the principal result of Paper 2 can be formulated as follows.

Definitions. 1. An *analytic family of two-dimensional surfaces* is a triplet $M \xrightarrow{\pi} B$, where M and B are analytic manifolds, $\dim M = \dim B + 2$ and the map π has the constant rank equal to $\dim B$.

2. Let M be a manifold covered by an atlas of charts $\{U_\alpha\}$ and in each chart U_α a vector field v_α is defined. We say that this family of vector fields defines a *line field with singularities on* M, if on each nonempty intersection $U_\alpha \cap U_\beta$ there exist a nonvanishing smooth function $\varphi_{\alpha\beta}$ such that $v_\alpha = \varphi_{\alpha\beta} v_\beta$.

3. An *analytic family of line fields with singularities* is a line field with singularities on M, tangent to all fibers $\pi^{-1}(\varepsilon)$, $\varepsilon \in B$. The family is *proper*, if the restriction of π on the singular locus of the family is proper.

Definition. A *simple blowing-up* of a two-dimensional surface M is a map $\sigma\colon \tilde M \to M$ which is biholomorphic except for the preimage $\Sigma \subset \tilde M$ of just one point $p \in M$; the exceptional set Σ is biholomorphic to the projective line $\mathbb{C}P^1$ and the germ $\sigma\colon (\tilde M, \Sigma) \to (M, p)$ is left-right equivalent to the standard blow-up described above.

The last condition means that in a small neighborhood of Σ two local charts can be introduced, (x, u) and (y, v), with the transition functions between them $y = xu$, $v = 1/u$, such that the map σ has in these charts the form

$$(x, u) \mapsto (x, ux), \qquad (y, v) \mapsto (yv, y).$$

A *resolution* of the surface M is an analytic map $\theta : \tilde M \to M$ which is the composition of a finite number of simple blow-ups.

TRIFONOV THEOREM. *For a proper family α of line fields on an analytic family of two-dimensional surfaces, there exists another family of surfaces $\tilde M \xrightarrow{\tilde\pi} \tilde B$ and a pair of analytic maps $H\colon \tilde M \to M$, $\rho\colon \tilde B \to B$ such that:*

(1) $\pi \circ H = \rho \circ \tilde\pi$;
(2) *The map H restricted on any two-dimensional fiber $\tilde\pi^{-1}(\tilde\varepsilon)$, $\tilde\varepsilon \in \tilde B$, is a resolution (a finite composition of simple blowing-ups;)*
(3) *All essential singularities of the pullback family $H^*\alpha$ (defined in the natural way) are elementary.*

REMARKS. 1. The assertion of the theorem holds for both real and complex analytic categories.

2. The new base $\tilde B$ of the new family may be not connected.

4.3. Singular perturbations. Trifonov theorem claims that after blowing up in a family of analytic line fields, a new family can be constructed in such a way that all *essential* singularities of this new family are elementary. However, this result says nothing about singularities that are not essential. They correspond to the dynamical phenomenon called *singular perturbation* in the theory of differential equations: one needs to study families of, say, vector fields on the plane, which for certain values of parameters exhibit a whole curve of nonisolated singularities. A comprehensive discussion on this subject can be found in PAPER 2. *The appearance of singular perturbations after parametric desingularization constitutes now the main gap between the theorem on finite cyclicity of elementary polycycles together with the parametric desingularization theorem on one side and the general Hilbert–Arnold problem on the other.*

4.4. Complexity of desingularization. When proving Trifonov theorem on parametric desingularization, one needs effective means to estimate the number of blow-ups necessary to resolve completely an isolated singularity.

Besides, there is a general question: given a germ of vector field at a singular point, how many terms of its Taylor expansion determine completely the topology of the phase portrait of the field? The constructive procedure which gives an answer to that question, is based on the desingularization technique. The topology of an elementary singular point is determined (in the degenerate case) by the principal term of the restriction of the vector field on the center manifold (curve). Knowing the sequence of steps resolving a nonelementary singularity into elementary ones, one may "glue up" the local phase portraits into the global phase portrait of the original field, provided that the trajectories are not spiralling around the singularity. The latter condition may be guaranteed by assuming that there exists at least one *characteristic orbit*, a trajectory tending to the singularity with a certain limit slope. The procedure of blowing up involves only lower degree Taylor terms, so to estimate the order of the Taylor polynomial which would determine completely the topology of the phase portrait, one needs to control the number of blow-ups.

These matters constitute the subject of PAPER 3 from the present volume. To formulate the main result, recall the principal definition.

DEFINITION. Let $v(x\,,\,y) = v_1(x\,,\,y)\,\partial/\partial x + v_2(x\,,\,y)\,\partial/\partial y$ be the germ of a smooth vector field at the origin. The *multiplicity* of the germ v is the dimension of the local algebra,

$$\mu_0(v) = \dim_{\mathbb{R}} Q_v\,, \qquad Q_v = \mathbb{R}[[x\,,\,y]]/\left\langle \hat v_1\,,\,\hat v_2\right\rangle\,,$$

where $\mathbb{R}[[x\,,\,y]]$ is the ring of all formal power series over the reals in two variables $(x\,,\,y)$, $\hat v_i$ are Taylor series of the coordinate functions v_i, $i = 1\,,\,2$, and $\langle \hat v_1\,,\,\hat v_2\rangle$ is the ideal generated by the two series.

DEFINITION. The jet of a vector field at the singular point is called *topologically sufficient*, if any two vector fields with that same jet, are topologically equivalent in a certain small neighborhood of the singular point.

KLEBAN THEOREM. *The order of a topologically sufficient jet for a vector field with a singular point of multiplicity $\mu < \infty$ having a characteristic orbit, does not exceed $2\mu + 2$.*

The existence or absence of the characteristic orbit is a fact that can be also established by analyzing the $(2\mu + 2)$-jet of the vector field.

This theorem gives a quantitative version of the general result by Dumortier [D1]. The proof is based on controlling the number of blow-ups sufficient to resolve a degenerate isolated singularity of multiplicity μ into elementary singularities. Such an approach was suggested in [VdE] by Van den Essen.

LEMMA (see PAPER 2). *1. If the linear part of a vector field v at the singular point is zero, then the sum of multiplicities of all singular points appearing after one blow-up, is strictly less than the multiplicity μ of the original singularity*;

2. If the linear part is nonzero (and the singular point is still nonelementary), then after no more than $[\frac{\mu}{2}]+2$ blow-ups it can be resolved into elementary singularities.

Another result of similar nature, estimating the number of *quasihomogeneous blow-ups* in terms of the multiplicity of the germ of a vector field, was announced recently by M. Pelletier [P].

4.5. Concluding remarks. Concluding this short survey, we would like to return to the general formulation of the Hilbert–Arnold problem. In order to achieve further progress in that direction, after proving the theorem on finite cyclicity of elementary polycycles in generic families and the theorem on parametric desingularization, one needs to study bifurcation of limit cycles in singular perturbations. A particular case of a singular perturbation is the family of vector fields

$$\begin{cases} \dot x = h(x\,,\,y)\,f_0(x\,,\,y) + \displaystyle\sum_{k=1}^{n} \varepsilon_k\, f_k(x\,,\,y\,,\,\varepsilon)\,, \\ \dot y = h(x\,,\,y)\,g_0(x\,,\,y) + \displaystyle\sum_{k=1}^{n} \varepsilon_k\, g_k(x\,,\,y\,,\,\varepsilon)\,, \end{cases} \qquad \begin{array}{c} (x\,,\,y) \in \mathbb{C}^2\,, \\ \varepsilon = (\varepsilon_1\,,\,\dots\,,\,\varepsilon_n) \in \mathbb{C}^n\,, \end{array}$$

such that the vector field $f_0\,\partial/\partial x + g_0\,\partial/\partial y$ has only elementary singularities (for example, has no singularities at all). In a more general context the definition of a singular perturbation is given in PAPER 2.

The equation of Van der Pol is a specific example of singularly perturbed vector field, and in that example limit cycles are known to be born.

To specify the problem, consider a *simple cusp*, the cuspidal singular point of maximal nondegeneracy: its linear part is nilpotent $y\,\partial/\partial x$ and nonlinear terms are generic. It is known that after three blowing-up steps such point is resolved into elementary ones. Thus applying the technique of parametric desingularization, one may try to prove an analog of the result obtained in PAPER 1. Denote by $EC(n)$ the maximal cyclicity of a polycycle carrying only elementary singularities and simple cusps and occurring in generic n-parameter families.

Conjecture. *The number $EC(n)$ is finite for any $n < +\infty$ and the function $n \mapsto EC(n)$ admits a primitive recursive majorant.*

Proving this conjecture would be a first step towards obtaining the complete solution of the Hilbert–Arnold problem.

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