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TRAJECTORIES OF POLYNOMIAL VECTOR FIELDS AND ASCENDING CHAINS OF POLYNOMIAL IDEALS

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To Yuliĭ Sergeevich Ilyashenko who taught us so much, on his 55th birthday

ABSTRACT. We give an explicit upper bound for the number of isolated intersections between an integral curve of a polynomial vector field in $\mathbb{R}^n$ and an algebraic hypersurface. The answer is polynomial in the height (the magnitude of coefficients) of the equation and the size of the curve in the space-time, with the exponent depending only on the degree and the dimension.

The problem turns out to be closely related to finding an explicit upper bound for the length of ascending chains of polynomial ideals spanned by consecutive derivatives.

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1. Introduction

The main problem that will be addressed in this paper, concerns oscillatory properties of functions defined by polynomial ordinary differential equations. Geometrically the question is about the number of isolated intersections between an integral curve of a polynomial vector field and an algebraic hypersurface in the Euclidean n-space.

Despite the fact that, to the best of our knowledge, this problem was first discussed on Arnold's seminar in Moscow in the seventies, only a limited progress in this direction has been achieved so far. The most advanced contribution to this area, an upper bound for the multiplicity of contact between an integral curve and an algebraic hypersurface, solving the *Risler problem* [23], is due to A. Gabrièlov [4] (the two-dimensional case was studied earlier in [6]). Another very recent result [26] concerns the maximal number of infinitesimally close but distinct intersections ("*cyclicity*"): Y. Yomdin shows that the above bound established by Gabrièlov for the order of tangency, holds also for the number of intersections that can coalesce as the parameters of the problem vary. Still this does not give an explicit answer for the global number of intersections.

This paper was preceded by the conference paper [22], an extended abstract in which the main result was announced and the principal ideas of the construction have been already exposed together with motivations, but the long technical proof of the main (algebraic) Theorem 4, the cornerstone of the whole construction, was barely indicated.

Below we give a complete demonstration of the results announced in [22], focusing more on the issue of chains of polynomial ideals and algebraic varieties, that plays the key role in the proof. Besides, we improved slightly the estimates and simplified the proof in several instances. However, for the sake of readability some parts of the announcement [22] had to be reproduced below in an abridged form.

1.1. **Meandering of integral curves: formulation of the problem and the main result.** Consider a polynomial vector field in the Euclidean space $\mathbb{R}^n$, defined by a system of n first order polynomial ordinary differential equations, and let $\Gamma \subset \mathbb{R}^n$ be a compact connected piece of a phase trajectory of this field. Since Γ is a real analytic curve, for any algebraic hypersurface $\Pi \subset \mathbb{R}^n$ the following alternative holds: either $\Gamma \subset \Pi$, or the number of intersections $\#\Gamma \cap \Pi$ is finite and all of them are isolated on Γ. The problem is to place an explicit upper bound on the number of *isolated* intersections between Γ and an arbitrary algebraic hypersurface of degree $\leqslant d$ (e.g., an arbitrary affine hyperplane). This bound characterizing the curve Γ, is a natural measure for its meandering in the ambient space.

It is clear that the bound must depend on several parameters of the problem, namely:

- the dimension of the phase space,
- the degrees of the polynomial differential equation and the hypersurface,
- the size of the integral trajectory, both with respect to the ambient space $\mathbb{R}^n$ and with respect to the natural parameter ("time"),
- the magnitude of the coefficients of the differential equation.

(These parameters are not all independent, due to the possibility of various rescalings). In order to make the formulation more transparent, it is convenient to minimize their number, using *common* bounds, as follows.

Consider the system of polynomial ordinary differential equations of degree d in n variables with a polynomial right hand side $v = (v_1, \dots, v_n)$:

$$\dot{x} = v(t,x) \iff \frac{dx_j}{dt} = \sum_{|\alpha|+k=0}^{d} v_{jk\alpha} t^k x^\alpha, \qquad j = 1, \dots, n \tag{1.1}$$

(the standard multiindex notation is assumed). Suppose that the *height* (the maximal absolute value of the coefficients) of the polynomials $v_j(t,x) \in \mathbb{R}[t,x]$ is explicitly bounded from above, i.e. all $v_{jk\alpha} \in \mathbb{R}$ in (1.1) satisfy the inequality $|v_{jk\alpha}| \leqslant R$ for some known $R < \infty$.

Consider an arbitrary integral trajectory Γ of the system (1.1) entirely belonging to the centered box $\mathbb{B}_R = \{|x_j| < R,\ |t| < R\} \subset \mathbb{R}^{n+1}$ of size R in the space-time, i.e. a solution $t \mapsto (x_1(t), \dots, x_n(t))$ defined on some interval $t \in [t_0, t_1] \subseteq [-R, R]$ and satisfying the inequalities $|x_j(t)| < R$ on it.

Finally, let $\Pi \subset \mathbb{R}^{n+1}$ be an algebraic hypersurface determined by an equation $\{p(t,x) = 0\}$ in the space-time, where $p \in \mathbb{R}[t,x]$ is a polynomial of degree $\leqslant d$:

$$\Pi = \{p(t,x) = 0\}, \qquad p(t,x) = \sum_{k+|\alpha|=0}^{d} p_{k\alpha} t^k x^\alpha. \tag{1.2}$$

Since the polynomial p is defined modulo a nonzero constant factor, without loss of generality we may always assume that the height of p is also bounded by the same R, i.e. all coefficients $p_{k\alpha}$ satisfy the inequality $|p_{k\alpha}| \leqslant R$. Note that R is a common bound for the coefficients and for the "size" of Γ, whereas d is a common bound for the degrees of the vector field and the hypersurface.

Theorem 1. *For any integral trajectory Γ of a polynomial vector field of degree d and height $\leqslant R$ in $\mathbb{R}^n$ and any algebraic hypersurface of the same degree, the number of* isolated *intersections between Γ and Π, counted with multiplicities inside the box*

$\mathbb{B}_R$, admits an upper bound of the form

$$\#(\Gamma \cap \Pi \cap \mathbb{B}_R) \leqslant (2+R)^B, \qquad B = B(n,d) \in \mathbb{N}, \tag{1.3}$$

where $B = B(n,d)$ is a primitive recursive function of the integer arguments n and d. As $n, d \to \infty$, the function B grows no rapidly than the tower of four stories:

$$B(n,d) \leqslant \exp\exp\exp\exp(4n\ln d + O(1)). \tag{1.4}$$

The assertion of this theorem means the strongest form of effective computability of the bound. In principle, one can derive from the proof below an expression for $B(n,d)$ in the closed form (and not only the asymptotical growth rate (1.4), as above). But there are many reasons to believe that this bound is highly excessive, so we did not strive for such closed form bounds.

1.2. **Complex intersections.** The method of the proof of Theorem 1 works also in the complex settings and yields a similar upper bound for the number of isolated zeros of an arbitrary polynomial $p(t,x) \in \mathbb{C}[t,x]$ restricted on the holomorphic integral curve $\Gamma^{\mathbb{C}}$ inside a polydisk $\mathbb{B}_R = \{|x|_j \leqslant R,\ |t| \leqslant R\} \subset \mathbb{C}^n \times \mathbb{C}$ (replacing the box of size R). One has to exercise a special care concerning domains of definitions of Γ.

Notice that solutions (integral curves) of polynomial systems can blow up in finite time and hence may exhibit the so called movable singularities (ramifications at infinity). If we take the integral curve $\{t \mapsto x(t)\} \in \mathbb{C}^{n+1}$ passing through a certain initial point $(t_0, x_0) \in \mathbb{B}_R$, then the set of t in the disk $\{|t| < R\} \subset \mathbb{C}$, for which the curve remains in $\mathbb{B}_R$, may be not simply connected, therefore one has to specify the choice of the branches.

In order to avoid these complications, we formulate the complex theorem in the "dual form", namely, for every initial point $(t_0, x_0) \in \mathbb{B}_R \subset \mathbb{C}^{n+1}$ we will explicitly specify the size of a small disk in the t-plane, in which the curve has no more than the given number (large than n, in general) of intersections with a polynomial hypersurface.

Theorem 2. *For any n and d one can explicitly specify an integer number $\ell = \ell(n,d)$ and a positive radius $\rho = \rho(R,n,d)$ in such a way that the integral curve of the polynomial vector field (1.1) of height $\leqslant R$ in $\mathbb{C}^{n+1}$ through any point $(t_0, x_0) \in \mathbb{B}_R$ extends analytically on the disk $D_\rho = \{|t - t_0| < \rho\}$ and, restricted on this disk, can have no more than ℓ isolated intersections with any polynomial hypersurface of degree d in $\mathbb{C}^{n+1}$.*

The radius ρ depends polynomially on R: $\rho(R,n,d) = (2+R)^{-B(n,d)}$, where the functions $B(n,d)$ and $\ell(n,d)$ admit primitive recursive majorants growing no faster than (1.4) and $d^{n^{O(n^2)}}$ respectively.

1.3. **The discrete Risler problem and its ramifications.** A polynomial vector field is a dynamical system in continuous time, whose discrete time analog is a polynomial map (endomorphism or automorphism of $\mathbb{R}^n$ or $\mathbb{C}^n$). The following is an analog of the original problem on intersections between integral curves (orbits of the vector field) and polynomial hypersurfaces for the discrete time case. Recall that an *orbit* of P is the sequence of points $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ obtained by iterations of the map:

$$x_{i+1} = P(x_i), \qquad i = 0, 1, \ldots. \tag{1.5}$$

Let $P\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ be a polynomial map of degree d and $\Pi \subset \mathbb{R}^n$ an algebraic hypersurface defined by the equation $\{p(x)=0\}$, $p \in \mathbb{R}[x]$, $\deg p \leqslant d$. The problem is to find an upper bound (in terms of d and n) for the maximal number of consecutive zeros in the infinite numeric sequence $\{p(x_i):\ i=0,1,\dots\}$, on the assumption that not all members of this sequence are zeros. Geometrically this problem concerns with the number of intersections between an orbit of the dynamical system and the hypersurface Π. This problem is the discrete analog of the Risler problem on the maximal order of contact, and the result is largely parallel to the Gabrièlov theorem.

Definition. A polynomial map $P\colon \mathbb{C}^n \to \mathbb{C}^n$ is called *dimension-preserving*, if the image of any semialgebraic k-dimensional variety is again k-dimensional.

This property is generic, so the theorem below holds for almost all polynomial maps. If X and Y are two semialgebraic varieties with $P(X) \subset Y$, then the assumption that P is dimension-preserving, guarantees that $\dim X \leqslant \dim Y$.

Theorem 3. *Any orbit $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ of a dimension preserving dynamical system (1.5) of degree $\leqslant d$ that belongs to a polynomial hypersurface $\Pi=\{p=0\}$ of degree $\leqslant d$ for $i=0,1,\dots,\ell=\ell(n,d)$, where*

$$\ell(n,d) \leqslant \underbrace{M^{M^{\cdot^{\cdot^{\cdot^{M}}}}}}_{n \text{ times}}, \qquad M = 1+d^n, \tag{1.6}$$

necessarily remains on Π forever.

Remark. A particular case of the discrete Risler problem was studied in [3], with P being a linear map and the surface $\Pi=\{p=0\}$ being an algebraic sphere of degree d. However, this case differs radically from the general one, because *the degrees of the polynomials $p_k(x)=p(P(\cdots(P(x))\cdots))$ (k times) remain bounded by $d=\deg p$ for all k*, and hence the length of the corresponding chain of ideals (see below) is obviously bounded by the dimension of the linear space of polynomials of degree d, using linear algebraic tools only.

The discrete Risler problem appears naturally in an attempt to solve the (original) Risler problem, see Appendix B.

1.4. **Chains of polynomial ideals: effective Noetherianity.** The proof of Theorem 1 is based on two arguments. The first one is a nonoscillation condition for high order linear ordinary differential equations Lemma 1. An easy Corollary 1 to this lemma allows to place an explicit upper bound on the number of isolated zeros of any solution of a linear differential equation in terms of the magnitude of its coefficients.

To reduce our problem on nonlinear systems of differential equations to that on linear high order scalar equations, we construct an auxiliary ascending chain of ideals in the appropriate ring of polynomials and use the Noetherianity of this ring. (This reduction is explained in §2.3 below).

The second argument of the proof is an explicit upper bound on the length of ascending chains of polynomial ideals generated by adding consecutive derivatives, given in Theorem 4 (see also [22, Lemma 6]). The precise formulation follows.

Let $\mathfrak{R} = \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ be the ring of polynomials in n variables and $L\colon \mathfrak{R} \to \mathfrak{R}$ a derivation of this ring of degree d. This means that for some polynomials $v_1, \dots, v_n \in \mathfrak{R}$ of degree $d+1$ we have

$$\forall p \in \mathfrak{R} \quad Lp = \sum_{j=1}^{n} v_j \partial_j p, \qquad \partial_j = \frac{\partial}{\partial x_j}$$

In other words, L is the Lie derivative along the polynomial vector field $v = (v_1, \dots, v_n)$ in $\mathbb{C}^n$.

Take an arbitrary seed polynomial $p_0 \in \mathfrak{R}$ and the sequence $\{p_k\}_{k=1}^{\infty}$ of its derivatives obtained by iterating L,

$$p_{k+1} = Lp_k, \qquad k = 0, 1, 2, \dots \tag{1.7}$$

Having this sequence, one can construct the ascending chain of polynomial ideals

$$\begin{gathered} I_0 \subset I_1 \subset \cdots \subset I_k \subset I_{k+1} \subset \cdots \subset \mathfrak{R}, \\ I_{k+1} = I_k + (p_{k+1}), \qquad k = 0, 1, \dots, \; I_0 = (p_0). \end{gathered} \tag{1.8}$$

For simplicity we will assume that the seed polynomial also has the same degree d.

The chain of polynomial ideals (1.8) must eventually stabilize, as the ring $\mathfrak{R}$ is Noetherian: starting from some ℓ one should have $I_{\ell-1} = I_\ell = I_{\ell+1} = \cdots$. The number ℓ is referred to as the *length* of the ascending chain (1.8).

Theorem 4. *The length of any ascending chain of polynomials generated by iterated derivatives along a polynomial vector field as in* (1.7)–(1.8)*, is bounded by a primitive recursive function $\ell = \ell(n, d)$ of n* (*the number of variables*) *and d* (*the degree of the derivation L and the seed polynomial p_0*).

As n and d are large, this function grows polynomially in d and doubly exponential in $n^2 \ln n$:

$$\ell = \ell(n, d) \leqslant d^{n^{O(n^2)}}. \tag{1.9}$$

As with the inequalities given in Theorem 1, the term $O(n^2)$ can be made explicit, but the asymptotic upper bound (1.9) seems to be rather excessive.

This result, being a cornerstone for proof of the bound (1.4), deserves several comments. The rule (1.7) implies explicit bounds for the degrees of p_k: obviously, $\deg p_k \leqslant (k+1)d$. Thus one can in principle apply an algorithm by A. Seidenberg [25] to obtain a *constructive* bound for the length of the chain (1.8). However, the construction of Seidenberg proves only that the bound $\ell = \ell(n, d)$ is a *general* recursive function (not necessarily a *primitive* recursive one). It was only relatively recently established by G. Moreno Socías [20] that the estimates implied by the Seidenberg algorithm, cannot be improved.

More precisely, he constructed an example of ascending chain of polynomial ideals as in (1.8) with degrees of homogeneous generators p_k growing linearly, $\deg p_k = d + k$, whose length is given by the Ackermann generalized exponential. Recall that the latter is a classical example of a recursive (constructive) but not primitively recursive function growing faster than any primitive recursive function (hence faster than any explicit expression involving n and d). From this we can conclude the bound (1.9) indicates that the rule (1.7) forces the chains of ideals stabilize much faster (in some sense, infinitely faster) than in the general case.

Remark. Strictly formally, the results by Seidenberg and Moreno refer to *strictly ascending* chains and place bounds on the maximal length of the chain until the first equality $I_{\ell-1} = I_\ell$ is encountered. On the contrary, Theorem 4 gives an upper bound for the length of the chain until its complete stabilization occurs.

This peculiarity can be easily explained: the rules (1.7)–(1.8) are very specific and the very first equality $I_{\ell-1} = I_\ell$ of the chain constructed using this rule, implies inductively that $I_\ell = I_{\ell+1}$ and so far. The details can be found in [22] and in §4 below.

1.5. **Descending chains of algebraic varieties.** The above bound on the length of ascending chains of polynomial ideals has a geometric counterpart for descending chains of algebraic varieties, formulated below. This counterpart is remarkable for two circumstances. First, it allows for an easy visualization of the reasons why the algorithm of Seidenberg implies so slow stabilization (as this was explained in [22]) and how a condition parallel to (1.7) forces it to occur much sooner. Second, this result plays the same role in the proof of Theorem 3 as Theorem 4 does in the proof of Theorem 1 (and the former reduction is even simpler than the latter).

Preserving notation of the preceding section, consider a sequence of polynomials obtained from a seed polynomial $p_0 \in \mathfrak{R}$ by iterations of a *ring homomorphism* $H\colon \mathfrak{R} \to \mathfrak{R}$. As this is well-known, all homomorphisms of $\mathfrak{R}$ are produced by polynomial transformations: $H = P^*$, where $P\colon \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, in other words,

$$p_{k+1} = Hp_k \iff p_{k+1}(x) = p_k(P(x)) \qquad \forall k = 0, 1, 2, \dots \tag{1.10}$$

Using this sequence of polynomials, one can define the descending chain of their common zero loci

$$\mathbb{C}^n \supset X_0 \supset X_1 \supset \cdots \supset X_k \supset \cdots, \qquad X_k = \bigcap_{j=0}^{k} \{p_j = 0\}. \tag{1.11}$$

Since the ring $\mathfrak{R} = \mathbb{C}[x]$ is Noetherian, the chain must stabilize (which means that $X_{\ell-1} = X_\ell = X_{\ell+1} = \cdots = X_{\ell+k} = \cdots$ for some $\ell < \infty$). The problem is to determine the moment ℓ.

Theorem 5. *Let $P\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map of degree d, $p_0 \in \mathfrak{R}$ a polynomial of degree $\leqslant d$ and the sequence of polynomials $p_k \in \mathfrak{R}$ is defined using the rule* (1.10)*. Then the descending chain of algebraic varieties built as in* (1.11) *is strictly descending: $X_{\ell-1} = X_\ell$ implies that $X_\ell = X_{\ell+1} = X_{\ell+2} = \cdots$ forever.*

Under the additional assumption that P is dimension preserving, the length of this chain is bounded by the primitive recursive function $\ell(n, d)$ as in (1.6)*.*

This result is actually a simple reformulation of Theorem 3 (the equivalence is established in §3). It already appeared in [22] with a sketchy proof. We had to reproduce it briefly here, since the demonstration of a more technical Theorem 4 is largely parallel to that of Theorem 5. Besides, we formulate in §3 a simple theorem that places an upper bound for the length of an *arbitrary* descending chain of algebraic varieties of known (growing) degrees: it can be considered as a geometric counterpart of the Seidenberg algebraic algorithm.

1.6. **The structure of the paper.** In §2 we derive Theorem 1 and Theorem 2 from Theorem 4. The construction consists in a series of reductions. As this part was already discussed in the announcement [22], the exposition in §2 is rather brief and concise. All motivations can be found in [22]. It is worth mentioning that instead of referring to a result from [14] concerning zeros of solutions of linear equations, we prove by elementary methods a nonoscillation criterion in Lemma 1 and Corollary 1 bounding the number of roots. This makes the exposition more transparent compared to [22].

The next section §3, also rather brief, contains the proofs of the bounds concerning chains of algebraic varieties (Theorem 5 and a geometric counterpart of the Seidenberg result, Theorem 6) and a reduction of Theorem 3 to Theorem 5.

The last section of the main body, §4, contains a complete detailed proof of Theorem 4 on lengths of ascending chains of ideals. This is the core of the paper.

Two subjects somewhat aside are moved to appendices. Appendix A contains the proof of Gabrièlov theorem on maximal order of tangency between trajectories and algebraic hypersurfaces. Appendix B describes several refinements and improvements of the main results for the particular case of systems of *linear* differential equations.

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2. From nonlinear systems to linear equations

2.1. **Disconjugacy and oscillatory character of ordinary linear differential equations.** We start with a sufficient condition for a linear ordinary differential equation of order n to have no solutions with more than $n-1$ isolated roots on a given real interval I. Such equations are called *disconjugate* on I.

Consider a linear equation

$$y^{(\ell)} + a_1(t)\, y^{(\ell-1)} + \cdots + a_{\ell-2}(t)\, y'' + a_{\ell-1}(t) y' + a_\ell(t) y = 0 \tag{2.1}$$

on a real interval $I = [t_0, t_1]$ of length $r = t_1 - t_0$ with real bounded coefficients: $|a_k(t)| < c_k < \infty$ for all $t \in I$.

Lemma 1. *If*

$$\sum_{k=1}^{n} \frac{c_k\, r^k}{k!} < 1, \tag{2.2}$$

then any C^n-smooth function $f(t)$ satisfying a linear equation (2.1) *may have at most $\ell - 1$ isolated roots on I, counted with multiplicities.*

Breaking an arbitrary finite real interval on sufficiently short subintervals satisfying the above lemma, we obtain the following corollary.

Corollary 1. *If all coefficients of the equation* (2.1) *are bounded by the common constant* $C \geqslant 1$ *on* I*, then any nontrivial solution cannot have more than* $(\ell - 1) + \frac{1}{\ln 2}\,\ell r C$ *isolated zeros there.*

Proof of the Corollary. Obviously, our choice of C implies that $c_k \leqslant C^k$, and therefore for any interval of length h the inequality $\sum_1^\ell c_k h^k/k! \leqslant \exp Ch - 1 < 1$ guarantees that the equation on this interval is disconjugate: resolved with respect to h, this gives $h < h_0(C) = \ln 2/C$. Subdividing the given interval into $\lfloor r/h_0 \rfloor + 1 \leqslant \frac{1}{\ln 2} rC + 1$ subintervals of disconjugacy, we establish the required upper bound for the number of roots. $\square$

Lemma 1 has a complex counterpart. Assume that the equation (2.1) is defined on a *convex* domain $D \subset \mathbb{C}$ and the coefficients of this equation are analytic functions bounded by c_k therein. Denote by r the *diameter* of D.

Lemma 2 (W. J. Kim [15])**.** *If the diameter* r *and the magnitudes* c_k *of coefficients of the linear equation* (2.1) *with holomorphic coefficients in the complex domain* D *satisfy the inequality* (2.2)*, then any solution has no more than* $\ell - 1$ *isolated roots in* D*.*

The proof of Lemma 2 is based on a rather complicate identity from the complex interpolation theory, that allows to estimate the norm of an analytic function via the norm of its ℓth derivative, provided that it has at least ℓ zeros [15]. On the contrary, the real version is completely elementary.

Proof of Lemma 1. Assume that f has ℓ or more isolated roots on I. Then by the Rolle theorem, each derivative $f', f'', \dots, f^{(\ell-1)}$ must have at least one root on the interval I.

If $f^{(\ell)} \equiv 0$, then f is a polynomial of degree $\leqslant \ell - 1$ and the claim is obviously true. Otherwise without loss of generality we may assume that $|f^{(\ell)}| \leqslant 1$ on I, and the equality $|f^{(\ell)}(t_*)| = 1$ holds at some point t_*.

For an arbitrary point $a \in I$ and any k between 1 and n one has the identity $f^{(k-1)}(x) = f^{(k-1)}(a) + \int_a^x f^{(k)}(t)\,dt$ (the Newton–Leibnitz formula). The choice of the base point a in each case can be made arbitrary, so we put it at x_{k-1}, one of the roots of $f^{(k-1)}$. Then the first term disappears, and majorizing the integral we conclude with the recurrent inequalities $\|f^{(k-1)}\| \leqslant r \cdot \|f^{(k)}\|$ for all $k = 1, \dots, n$, between the sup-norms of the derivatives.

Iterating these inequalities, one can prove that $\|f^{(\ell-k)}\| \leqslant \|f^{(\ell)}\| r^k$. In fact, a stronger assertion holds: $\|f^{(\ell-k)}\| \leqslant \|f^{(\ell)}\| r^k/k!$. To see this, we write the expression for $f^{(\ell-k)}$ as the multiple integral and use the mean value theorem for the symplex with sides $h_j = |t_j - x_j| \leqslant r$, so that for an arbitrary $t_{\ell-k} \in I$

$$\begin{aligned} |f^{(n-k)}(t_{\ell-k})| &= \left| \int_{x_{\ell-k}}^{t_{\ell-k}} dt_{\ell-k+1} \cdots \int_{x_{\ell-1}}^{t_{\ell-1}} dt_\ell \cdot f^{(\ell)}(t_\ell) \right| \\ &\leqslant \|f^{(\ell)}\| \cdot \int_0^{h_{\ell-k}} dt_{\ell-k+1} \cdots \int_0^{h_{\ell-1}} dt_\ell \leqslant \|f^{(\ell)}\| \cdot \frac{r^k}{k!}. \end{aligned}$$

Plugging these estimates into the original equation (2.1) we notice that the leading term is overtaking at the point t_* (hence in the sense of the sup-norm) the sum of all other terms and therefore the equality (2.1) cannot hold everywhere—a contradiction. $\square$

The remaining part of this section (until §2.5) is devoted to the demonstration of Theorem 1: we derive it from the assertion of Theorem 4 and Corollary 1. The proof of Theorem 2 is given in §2.6. The demonstration of Theorem 4 is postponed until §4.

2.2. **The universal system.** The assertion of Theorems 1 and 2 concerns all integral curves inside the box $\mathbb{B}_R$ of all vector fields corresponding to a box in the parameter space (recall that the coefficients $v_{jk\alpha}$ in (1.1) and $p_{k\alpha}$ in (1.2) are the natural parameters of the problem, provided that n and d are fixed). We reduce the question about intersections to that for *one* universal system and *one* universal hypersurface.

Consider the coefficients $v_{jk\alpha}$ in (1.1) and $p_{k\alpha}$ in (1.2) as new independent variables governed by the trivial equations, and add to them the time variable t:

$$\dot v_{jk\alpha} = 0, \quad \dot p_{k\alpha} = 0, \quad \dot t = 1 \qquad \forall j = 1, \dots, n, \quad k + |\alpha| \leqslant d. \tag{2.3}$$

Together with the equations (1.1) they define a system of autonomous polynomial ordinary differential equations corresponding to a polynomial vector field in the new phase space of dimension $\leqslant (n+1)(1+(d+1)^{n+1})$ and of degree $d+1$. Each particular choice of coefficients of $v(t,x)$ and $p(t,x)$ subject to restrictions on the height means choosing a particular initial condition for the integral curves of this universal system in the box $\mathbb{B}_R$ belonging to the new phase space. Thus without loss of generality it is sufficient to prove Theorem 1 for one universal vector field (1.1)–(2.3) and one universal algebraic hypersurface (1.2) of degree $d+1$.

Note that all polynomials in the right hand side of the universal equations have only integer bounded coefficients (in fact, only 0 and 1 are allowed). The same refers to the equation of the universal hypersurface. This circumstance will play an important role later, in §2.4.

2.3. **Derivation of a linear equation.** From now on we consider only the universal vector field constructed above. To avoid cumbersome notation, we return to the original notations for state variables and consider a polynomial vector field defined by the system of autonomous polynomial equations $\dot x = v(x)$ of degree d and height R in the space $\mathbb{R}^n$ (or $\mathbb{C}^n$) and a polynomial hypersurface $\Pi = \{p(x) = 0\} \subset \mathbb{R}^n$ of the same degree and height. Let L be the Lie derivative of the ring $\mathfrak{R} = \mathbb{C}[x]$ along the vector field v, and define the sequence of polynomials $\{p_k\}_{k=0}^{\infty}$ as in (1.7), starting from $p_0 = p$. The corresponding ascending chain $\{I_k\} \subset \mathfrak{R}$ of ideals (1.8) will stabilize after at most ℓ steps (the value of ℓ is given by Theorem 4).

The equality $I_{\ell-1} = I_\ell$ means that for some polynomials $h_0, h_1, \dots, h_{\ell-1} \in \mathfrak{R}$

$$p_\ell = \sum_{k=0}^{\ell-1} h_k p_k, \qquad h_k \in \mathbb{C}[x]. \tag{2.4}$$

Now assume that $\varGamma \subset \mathbb{B}_R$ is a parameterized integral trajectory of the vector field v, entirely belonging to the box $\mathbb{B}_R$. Recall that by construction of the universal vector field, the time parameter t on $\varGamma$ coincides with one of the coordinate functions, so $\varGamma$ is parameterized by a subinterval I of the interval $[-R, R]$.

Denote by $f(t)$ the restriction of the polynomial p on $\varGamma$: $f(t) = p(x(t))$, $t \in I$. Then from the rule (1.7) it follows that the restriction $p_k(x(t))$ of p_k on $\varGamma$ is the kth derivative $f^{(k)}(t)$. Together with (2.4) this means that f solves the linear ordinary differential equation (2.1) with coefficients

$$a_k(t) = -h_{\ell-k}(x(t)), \qquad k = 1, \dots, \ell. \tag{2.5}$$

For successful application of Corollary 1 it remains to place explicit upper bounds on the magnitude of the coefficients a_k. As the latter are obtained by restriction of the polynomials h_k on the curve Γ lying in the box $\mathbb{B}_R$, it is sufficient to estimate explicitly the degree and height of all h_k. Indeed, if the height $\mathcal{H}(q)$ of a polynomial $q \in \mathfrak{R} = \mathbb{C}[x_1, \dots, x_n]$ of degree $D = \deg q$ is known, then obviously

$$\max_{x \in \mathbb{B}_R} |q(x)| \leqslant \mathcal{H}(q) \cdot (D+1)^n (1+R)^D,$$

and this applied to $q = h_k$ would yield an upper bound for all $c_k = \max_{t \in I} |a_k(t)|$ from (2.5), that can be plugged into the bound given by Corollary 1.

Remark. The identity (2.4) is indepenent of the choice of Γ, provided that the equation remains the same (as this is in our case, since we consider the universal vector field). But the linear equation (2.1) obtained by restriction of (2.4) on Γ, will depend explicitly on the choice of the latter. Still the magnitude of the coefficients of the resulting linear equation can be bounded in terms of R uniformly over all curves inside the same box $\mathbb{B}_R$, which is sufficient for our purposes.

2.4. **Bounds for degree and height.** The remaining part is purely algebraic. The degrees of the polynomials p_k grow linearly with k, $\deg p_k \leqslant (k+1)(d-1)+1$, so $\max_{k=0,\dots,\ell-1} \deg p_k \leqslant \ell d$. Knowing these degrees, one can estimate the degrees of h_k in the decomposition (2.4) by the inequality due to G. Hermann [12] shown to be essentially sharp by Mayr and Meyer [19]:

$$\deg h_k \leqslant D = (\ell d)^{2^n} \qquad \forall k = 0, 1, \dots, \ell - 1. \tag{2.6}$$

(More precisely, one can always replace the initial decomposition (2.4) by a new one satisfying the above restrictions for the degrees).

To place an upper bound for the height of h_k, we use the method of indeterminate coefficients. Expand h_k and p_k explicitly as $\sum_{|\alpha| \leqslant D} h_{k\alpha} x^\alpha$ and $\sum_{|\alpha| \leqslant \ell d} p_{k\alpha} x^\alpha$ respectively. Substituting these expansions into the identity (2.4), we obtain a non-homogeneous system of linear (algebraic) equations that is to be solved with respect to $N \leqslant \ell(D+1)^{n+1}$ unknowns $\{h_{k\alpha}\}$.

This system is known to possess at least one solution, so one can apply the Cramer rule to produce it. According to this rule, each component of the solution can be found as a ratio of two appropriately chosen minors of the extended matrix of the system, with a nonzero denominator.

All entries of the matrix of this system are integral and explicitly bounded from above. Indeed, they are expressed in terms of coefficients of the polynomials p_k. But the rule (1.7) preserves integrality of coefficients (as the derivation L has bounded integer coefficients, see §2.2) and the height of p_k grows in a controllable fashion: the formula $p_{k+1} = \sum_{j=1}^n v_j \, \partial_{x_j} p_k$ implies that

$$\mathcal{H}(p_{k+1}) \leqslant \deg p_k \times n(\deg p_k + 1)^n (d+1)^n \times \mathcal{H}(p_k) \tag{2.7}$$

for all $k = 0, \dots, \ell - 1$. (The first multiplier comes from computing the partial derivative $\partial_{x_j} p_k$, the second term majorizes the number of monomials when reducing similar terms in the products $v_j \partial_{x_j} p_k$ and adding them together, and the last multiplier is equal to the height of p_k, since the height of all v_j is 1).

The upper bound for $\mathcal{H}(p_k)$ provides an upper bound for the numerator of the ratio in the Cramer rule, as the size of the corresponding minor cannot exceed the dimension $N \leqslant \ell(D+1)^{n+1}$ of the matrix, and all entries therein are already explicitly bounded. On the other hand, the denominator of this ratio is a *nonzero integer* number, hence at least 1 in the absolute value. Thus we obtain an upper bound for every $|h_{k\alpha}|$ and hence for all the heights $\mathcal{H}(h_k)$ by a primitive recursive function of n and d.

2.5. **Computations and the end of the proof of Theorem 1.** The recursive formulas for the degrees and heights, together with the inequality asserted by Corollary 1 already prove the bound (1.3) with a primitive recursive exponent $B(n,d)$. To find the asymptotical growth of $B(n,d)$ for large n,d, all computations should be performed explicitly. In doing that, we use the fact that the bound $\ell(n,d)$ for the length of ascending chains is already very large and can be used in the "absorbing" sense (as the symbol O in the classical calculus):

$$\ell(n,d) = d^{n^{O(n^2)}} \implies \ell d = d^{n^{O(n^2)}} \asymp \ell,\ \ \ell^n = d^{n^{O(n^2)}} \asymp \ell,\ \dots$$

Using the inequality $\deg p_k \leqslant \ell d$ for all $k = 1,\dots,\ell$, we can simplify (2.7) to the form $\mathcal{H}(p_{k+1}) \leqslant n(d\ell+1)^{2n} \cdot \mathcal{H}(p_k) \asymp \ell\mathcal{H}(p_k)$. Therefore for all $k = 0,1,\dots,\ell$ we have the inequality

$$\mathcal{H}(p_k) \leqslant \ell^\ell \asymp d^\ell = d^{d^{n^{O(n^2)}}}. \tag{2.8}$$

The degrees of the polynomials h_k do not exceed $(\ell d)^{2^n} \asymp \ell$ by (2.6).

The size N of the matrix of the linear system described in §2.4 is therefore bounded by the expression $\ell^{n+1} \asymp \ell$. This matrix and the column in the right hand side are filled by integer numbers not exceeding d^ℓ by (2.8).

Each component $h_{k\alpha}$ of the solution can be found as a ratio of two minors of this matrix. But any such minor does not exceed the sum of $\ell!$ terms, each being at most $(d^\ell)^\ell$. Since the denominator of the ratio is $\geqslant 1$, we have the upper bound for the height of all h_k: $\mathcal{H}(h_k) \leqslant \ell!(d^\ell)^\ell \asymp d^\ell$.

A polynomial of degree ℓ and height d^ℓ restricted on the box $\mathbb{B}_R$ in $\mathbb{R}^n$, does not exceed $\ell^n d^\ell (1+R)^\ell \asymp (2+R)^{d^\ell}$. This expression is also the bound for the coefficients of the quasilinear equation. By Corollary 1 the number of real zeros of any solution of this equation does not exceed the upper bound asymptotically equivalent to $(2+R)^{d^\ell}$. Since d^ℓ is asymptotically overtaken by the tower of three exponents $\exp\exp\exp(n^3 + d + O(1))$, after returning to the initial (i.e. before passing to the universal equation and hypersurface) values of the parameters n and d, we arrive to the tower of four exponents (1.4) occurring in Theorem 1. $\square$

2.6. **Demonstration of Theorem 2.** Let a be the reference point in the box $\mathbb{B}_R$ of the phase space of the universal system (this point is obtained by the obvious suspension of the initial value point (x_0,t_0) for the original polynomial system). The complex solution Γ passing through the reference point can blow up in a finite time (exhibit singularities), but the distance to these singularities can be easily estimated from below: the polar radius $r = \|x\|$ grows at most as the solution of the equation $\dot r = C(n,d)r^{d+1}$, where C is a simple expression (the number of terms, since each term comes with the coefficient $\leqslant 1$). Thus the trajectory never leaves the box $\mathbb{B}_{2R}$ for all $|t| < \rho_0 = (C'(n,d)R^d)^{-1}$ (assuming that $t=0$ corresponds to the reference point, as the universal system is autonomous).

Restricting the identity (2.4) on the part of $\Gamma \subset \mathbb{B}_{2R}$ parameterized by this small disk $\{|t| < \rho_0\}$, we obtain a linear differential equation with bounded coefficients (in the same way, as before—all those arguments were independent on the ground field). It remains only to choose $\rho \ll \rho_0$ in such a way that the inequality of Lemma 2 be satisfied. Then the equation (2.1) restricted on this small disk, is disconjugate and hence has no more than ℓ roots (the number ℓ is equal to the order of the equation, i.e. to the length given by Theorem 4). The computations in this case remain the same as in the real case. $\square$

3. Discrete Risler problem and descending chains of algebraic varieties

In this section we prove Theorem 5 and show its equivalence to Theorem 3. Besides, a geometric analog of Seidenberg theorem [25] will be established. The main ideas of this section were already discussed in [22]: here we supply the proofs.

3.1. **Bézout inequalities after J. Heintz.** Recall that any complex algebraic variety $X \subset \mathbb{C}^n$ admits a unique irredundant irreducible decomposition $X = X_1 \cup \cdots \cup X_s$ into the union of irreducible algebraic varieties of various dimensions (the irredundancy means that neither component belongs to the union of the others). The following result allows to place explicit upper bounds for the number of irreducible components of different dimensions.

Lemma 3 (see [11]). *Assume that an algebraic variety in $\mathbb{C}^n$ is defined by any number of polynomial equations of degree $\leqslant d$, and has an irreducible decomposition with $m_0 \geqslant 0$ isolated points, $m_1 \geqslant 0$ one-dimensional varieties, ..., $m_{n-1} \geqslant 0$ irreducible $(n-1)$-dimensional components.*

Then $m_{n-1} + m_{n-2} + \cdots + m_{n-k} \leqslant d^k$ for all $k = 1, \dots, n$.

In particular, the number of isolated points of any such variety, regardless of its dimension and the number of determining equations, does not exceed d^n. Thus one can consider Proposition 3 as a generalization of the Bézout theorem.

3.2. **Algorithmic finiteness.** Consider a descending chain of algebraic varieties (1.11) *without* any information about the polynomials p_j except for an explicit control over their degrees. Assume that $\deg p_j = \phi(j)$, where $\phi(j)$ is a given computable (nondecreasing) function of j. We produce a recurrent formula for the length of *strictly* descending chain of varieties, that would define this length as a general recursive (but not primitive recursive) function of n already in the case of $\phi(j) = j + 2$.

Let $w_k^r \in \mathbb{Z}_+$ be the number of r-dimensional irreducible components of the variety X_k, and denote $w_k = (w_k^{n-1}, \dots, w_k^0) \in \mathbb{Z}_+$. We shall use the *strict lexicographic order* on $\mathbb{Z}_+$, writing $w_k \prec w_j$ if $w_k^{n-1} = w_j^{n-1}, \dots, w_k^{r+1} = w_j^{r+1}$ but $w_k^r < w_j^r$ for some r between $n-1$ and 1.

Proposition 1. *If the chain* (1.11) *is strictly decreasing, then $w_{k+1} \prec w_k$.*

Proof. This is obvious: since $X_{k+1} = X_k \cap \Pi_{k+1}$, where $\Pi_{k+1} = \{p_{k+1} = 0\}$, then each r-dimensional irreducible component of X_k either entirely belongs to the hypersurface Π_{k+1} and hence enters as a component of X_{k+1}, or intersects the Π_{k+1} by the union of irreducible varieties of dimensions strictly inferior to r, in which case there will be fewer r-dimensional components in the decomposition of X_{k+1}. $\square$

This is already sufficient to prove finiteness, as every lexicographically decreasing sequence of vectors ("words") from $\mathbb{Z}_+^n$ must eventually stabilize. If in addition the norm $\|w_k\| = w_k^{n-1} + \cdots + w_k^0$ admits an algorithmic *upper* bound in terms of k, then the termination moment admits an algorithmically computable bound. Notice that the assumption on the degrees $\deg p_j \leqslant \phi(j)$ together with Lemma 3 implies the bound

$$\|w_k\| \leqslant \big(\phi(k)\big)^n, \qquad \forall k = 0, 1, 2, \dots.$$

Theorem 6 (geometric version of Seidenberg theorem [25]). *The length of strictly descending chain* (1.11) *of common zero loci of a sequence of polynomials p_k of controlled degrees $\deg p_k \leqslant \phi(k)$, is majorized by a general recursive function.*

Proof. Notice that any lexicographically decreasing sequence $\{w_k\}$ can be subdivided into (finite) subintervals in such a way that the first coordinate remains constant along each subinterval. Then on this subinterval the truncations (the "subwords" of length $n-1$ containing all but the first "letter") again form a strictly decreasing sequence in $\mathbb{Z}_+^{n-1}$.

Let us introduce an auxiliary function of three arguments, $F(s, n, k)$ being the maximal length of a decreasing sequence in $\mathbb{Z}_+^n$ that begins with a word of the norm s and contains no more than k subintervals described above.

Suppose we have a sequence already comprising k subintervals, so that its length is $N = F(s, n, k)$. Allowing for one more subinterval means adding a new decreasing sequence of words in $\mathbb{Z}_+^{n-1}$ (as the first letter is fixed), that begins with a word of norm $\leqslant S = \phi^n(N)$. Thus the overall length of a sequence comprising $k+1$ interval, satisfies the recurrent inequality

$$F(s, n, k+1) \leqslant F(s, n, k) + F(S, n-1, S), \qquad S = \phi^n(F(s, n, k)). \tag{3.1}$$

The length of a sequence in $\mathbb{Z}_+^n$ starting from a word of norm s, can be estimated now by the expression $F(s, n, s)$, as the number of subintervals cannot exceed $w_0^{n-1} \leqslant \|w_0\| = s$. $\square$

Remark. The rule (3.1) defines a computable (general recursive) but not a primitively recursive function, as the right hand contains application of the defined function to itself. It is this type of recurrent formulas, that leads to the Ackermann exponential [20]. The arguments given in [20, 22] show that the rule (3.1) indeed may lead to a function growing faster than any primitive function, hence faster than any closed form expression.

In fact, it remains to show that there exists a scenario indeed leading to so long lexicographically decreasing sequences (this is relatively easy) and, moreover, that this scenario can be realized by an appropriate chain of algebraic varieties. We refer to [20] for such examples.

3.3. **Chains of varieties associated with discrete Risler problem: equivalence of Theorems 3 and 5.** Consider the dynamical system (1.5) in $\mathbb{C}^n$ and let $X_0 = \Pi$ be the hypersurface. The common locus of the first k polynomials $p_0, p_1, \dots, p_k$ defined recurrently by (1.10), is the set of points x on X_0 whose p-orbit remains on X_0 for the first k iterations:

$$X_k = \bigcap_{j=0}^{k} \{p_j = 0\} = \{x \in \mathbb{C}^n : P^{[j]}(x) \in \Pi \ \forall j = 0, 1, \dots, k\}.$$

This dynamic description immediately implies the inclusion

$$P(X_k \smallsetminus X_{k+1}) \subseteq X_{k-1} \smallsetminus X_k, \qquad k = 1, 2, \dots. \tag{3.2}$$

Indeed, the difference $X_k \smallsetminus X_{k+1}$ consists of points that remain on X_0 during the first k steps of their life, and leave it on the $(k+1)$st step. The P-image of any such point will remain on X_0 for $k-1$ more steps and then leave it.

This (trivial) observation proves equivalence of Theorem 3 and Theorem 5. $\square$

3.4. **Demonstration of Theorem 5.** The strict decrease of the chain (1.11) follows from (3.2): if the difference $X_{\ell-1} \smallsetminus X_\ell$ is empty, then it can contain the P-image of the difference $X_\ell \smallsetminus X_{\ell+1}$ only in case the latter difference is empty, and so far by induction.

To prove the bound (1.6), we observe that if the polynomial map P is dimension-preserving, then the sequence of dimensions $\dim(X_k \smallsetminus X_{k+1})$ is non-increasing.

This observation implies that the chain (1.11) can be subdivided by some moments $k_{n-1} \leqslant k_{n-2} \leqslant \cdots \leqslant k_1 \leqslant k_0$ into n segments of finite length,

$$\begin{aligned} &\underbrace{X_0 \supset X_1 \supset \cdots \supset X_{k_{n-1}}}_{\dim X_k \smallsetminus X_{k+1} = n-1} \supset \underbrace{X_{k_{n-1}+1} \supset \cdots \supset X_{k_{n-2}}}_{\dim X_k \smallsetminus X_{k+1} = n-2} \supset \cdots \\ &\qquad\qquad \cdots \supset \underbrace{X_{k_s+1} \supset \cdots \supset X_{k_{s-1}}}_{\dim X_k \smallsetminus X_{k+1} = s} \supset \cdots \supset \underbrace{X_{k_1+1} \supset \cdots \supset X_{k_0}}_{\dim X_k \smallsetminus X_{k+1} = 0} \end{aligned} \tag{3.3}$$

such that along the sth (from the right) segment the differences $X_k \smallsetminus X_{k+1}$, $k_s + 1 \leqslant k \leqslant k_{s-1}$, are exactly s-dimensional semialgebraic varieties (some segments can be eventually empty).

The length of each such segment does not exceed the number of s-dimensional irreducible components in the starting set X_{k_s+1} of this segment, since this number must strictly decrease on each step inside the segment. Indeed, inside the segment all components of dimension $> s$ must be preserved, otherwise the difference will be more-than-s-dimensional. On the other hand, if all s-dimensional components are preserved on some step, this means that the difference $X_k \smallsetminus X_{k+1}$ is at most $(s-1)$-dimensional, and one starts the next segment.

It remains only to notice that the degrees $\deg p_k$ grow exponentially $\deg p_k \leqslant d^{k+1}$, whereas the number of irreducible components of X_k can be estimated using Lemma 3 by the nth power of the maximal degree $(\deg p_k)^n = (d^n)^{k_s+1}$. Hence for the lengths $k_s - k_{s-1}$ we have the recurrent inequality, $k_{s-1} - k_s \leqslant (d^n)^{k_s+1}$ for downward going values of $s = n-1, \dots, 1, 0$ and the initial condition $k_n = d$ (the initial polynomial p_0 of degree d may have at most d factors). The solution of this recurrent inequality is majorized by the solution of a more simple one $k_{s-1} + 1 \leqslant M^{k_s+1}$, $M = d^n + 1$, that gives the tower of height n for $k_0 + 1$ with M on each level: thus for the length of the descending chain we obtain the required estimate (1.6). $\square$

Remark. The only property of the chain (1.11) used in the proof, is the monotonicity of dimensions of the differences $\dim(X_k \smallsetminus X_{k+1})$, which is much weaker than the algebraic rule (1.10).

If (still under the same assumption of monotonicity) we would assume the linear growth of degrees $\deg p_k \leqslant kd$, as follows from the rule (1.7), then the bound on the length of the chain would be much lower: the inequalities $k_{s-1} - k_s \leqslant [d(k_s+1)]^n$

would imply $k_{s-1}+1 \leqslant M(k_s+1)^n$ and finally a double exponential estimate $k_0 \leqslant d^{2n^{n+1}}$.

4. Convex ascending chains of polynomial ideals

This section contains the proof of Theorem 4. This proof is largely parallel to that of Theorem 5 from §3 and consists in monitoring components of the primary decomposition of the ideals constituting the chain. The source of additional difficulties is twofold: first, in the algebraic context one has to take care of *multiplicities* of the components and second, the construction should be modified to avoid explicit and implicit using of the *uniqueness* of the primary decomposition that is known to fail in general (in particular, this circumstance prevents one from speaking about the number of primary components).

The bound on the length of ascending chains of polynomial ideals is obtained by combining several results. First we establish the property called *convexity* of the ascending chain of ideals generated by adding consecutive derivatives, namely, we prove that in such chain the colon ratios $I_k : I_{k+1}$ constitute themselves an ascending chain of ideals, hence their (Krull) dimensions must form a non-decreasing sequence. Then we consider chains in which the ascension can be detected at the level of the *leading terms* (primary components of the maximal dimension), so that $\dim I_0 = \dim(I_k : I_{k+1})$ holds along the chain. For such chains we show that their length is majorized by the number of primary components in the leading term of the first ideal in the chain, counted with their *multiplicities*. Here we still can use the uniqueness part of the primary decomposition theorem. The final bound is obtained by a certain "surgery": as soon as the colon ratio $I_l : I_{l+1}$ became less than ν-dimensional, $\nu = \dim I_0$, we replace the chain of ideals $\{I_k\}$ starting from $k = l$ by another chain, by deleting (in an almost arbitrary fashion) all primary components of dimension ν and above. As the colon ratio, due to its monotonicity, should remain always less than ν-dimensional, such components would not have been affected when adding new derivatives p_k in any case, so the ascent of the newly constructed chain would essentially catch that of the initial one (in particular, their stabilization must occur simultaneously). Performing such "surgery" at most n times, we arrive to an upper bound for the length of any convex chain of polynomial ideals with an explicit control over the degrees of the generators.

4.1. Primary decomposition, leading terms, multiplicity.

After describing the general scheme, we proceed with a formal proof. In this subsection we collect several technical results which we will need later.

Dimension. Any algebraic subvariety in $\mathbb{C}^n$ is a stratified set [18] that has a certain dimension. If $I \subset \mathfrak{R}$ is an ideal and $X = V(I) = \{x \in \mathbb{C}^n : p(x) = 0\ \forall p \in I\}$ its *zero locus*, then we put $\dim I$ be the (complex) dimension of its zero locus. This number (between 0 and n) can be given a purely algebraic description, known as *Krull dimension* [27].

Primary decomposition and its uniqueness. One of the basic results of commutative algebra, known as the *Lasker–Noether theorem* [27, Ch. IV, §4], asserts that any polynomial ideal $I \subset \mathfrak{R}$ can be represented as a finite intersection of *primary* ideals, $I = Q_1 \cap \cdots \cap Q_s$. Recall that an ideal Q is primary, if $pq \in Q$ and $p \notin Q$ implies that $q^r \in Q$ for some natural exponent r. The radical $\sqrt{Q} = \{q \in \mathfrak{R} \colon q^r \in Q\}$ is a prime ideal called the *associated prime*, and by the Nullstellensatz it consists of all polynomials vanishing on the zero locus $V(Q) \subset \mathbb{C}^n$.

The primary decomposition in general is *not* unique, even if we assume that it is *irredundant*. However, in an irredundant primary decomposition the primary components whose associated primes are minimal (i.e. contain no prime ideals associated with other components), are *uniquely* defined [27, Theorem 8, p. 211]. In particular, the *leading term*

$$\mathrm{l.\,t.}\,(I) = \bigcap_j \{Q_j \colon \dim Q_j = \dim I\}, \tag{4.1}$$

the intersection of all upper-dimensional primary components, is uniquely defined, since the corresponding prime ideals are minimal for dimensionality reasons. As an application of the uniqueness part we have the following simple fact on monotonicity of the leading terms.

Lemma 4. *Suppose that $J \subset J'$ are two polynomial ideals of equal dimensions with the leading terms*

$$\mathrm{l.\,t.}\,(J) = Q_1 \cap \cdots \cap Q_s, \quad \mathrm{l.\,t.}\,(J') = Q'_1 \cap \cdots \cap Q'_{s'} \tag{4.2}$$

(*as usual, the decomposition is assumed to be irredundant*).

Then each component Q'_j contains one of the components Q_i.

Proof. We start with the obvious identity $J = J \cap J'$ and consider the decomposition of the leading terms:

$$Q_1 \cap \cdots \cap Q_s = Q_1 \cap \cdots \cap Q_s \cap Q'_1 \cap \cdots \cap Q'_{s'}. \tag{4.3}$$

The decomposition in the right hand side is not irredundant. However, all prime ideals associated with the primary terms Q'_j, must be among the primes associated with Q_j. Indeed, this follows from the simple fact that all m-dimensional irreducible components of the variety $X' = V(J') \subset X = V(J)$ should be among those of X.

Rearranging if necessary the components of $\mathrm{l.\,t.}\,(J')$, we can assume that Q'_j and Q_j have the same associated prime for all $j = 1, \dots, s'$ and $s' \leqslant s$. After collecting "similar terms" in the right hand side of (4.3), we observe that it becomes

$$(Q_1 \cap Q'_1) \cap \cdots \cap (Q_{s'} \cap Q'_{s'}) \cap Q_{s'+1} \cap \cdots \cap Q_s.$$

From the uniqueness theorem it follows that $Q_j = Q_j \cap Q'_j$ for all $j = 1, \dots, s'$, which implies that $Q_j \subset Q'_j$ for all such j. □

Multiplicity. The notion of multiplicity of an ideal is rather subtle. However, for our purposes it would be sufficient to use it only in a restricted environment, where the following construction works.

Let $I \subset \mathfrak{R}$ be an ideal and assume that $0 \in \mathbb{C}^n$ is an *isolated point* of its locus $V(I)$. Denote by $\mathfrak{m} = (x_1, \dots, x_n) \subset \mathfrak{R}$ the maximal ideal of the ring and let $\mathfrak{R}_\mathfrak{m}$ be the corresponding localization (the ring of rational fractions whose denominators do not vanish at the origin). Then I is *cofinite* at the origin, which means that the dimension of the quotient ring $\mathfrak{R}_\mathfrak{m}/I \cdot \mathfrak{R}_\mathfrak{m}$ over $\mathbb{C}$ is finite [1], i.e. $\mu_0(I) = \dim_\mathbb{C} \mathfrak{R}_\mathfrak{m}/I \cdot \mathfrak{R}_\mathfrak{m} < \infty$. The number $\mu_0(I)$ is called the *multiplicity* of I at the origin $0 \in \mathbb{C}^n$. In the similar way one may define the multiplicity $\mu_a(I)$ of any ideal I at any isolated point $a \in V(I)$ of its zero locus.

Let a be a regular (smooth) point of the zero locus of a polynomial ideal $I \subset \mathfrak{R}$ of some dimension r between 0 and n. Let Π be an affine subspace in $\mathbb{C}^n$ of codimension r, transversal to $V(I)$ at a, and L the corresponding ideal generated by r affine forms. Then the ideal $I+L$ is zero-dimensional at a and hence cofinite.

Definition. The *multiplicity* $\mu_a(I)$ of I at a is the multiplicity of $I+L$ at a (the complex dimension of the corresponding quotient algebra in the local ring). The multiplicity $\mu(I)$ is the generic value of $\mu_a(I)$ (the minimum over all smooth points $a \in V(I)$), and this definition will be applied to primary ideals only.

The multiplicity of an ideal in $\mathfrak{R} = \mathbb{C}[x_1, \dots, x_n]$ generated by polynomials of degree $\leqslant d$, can be easily estimated from above: by virtue of the Bézout theorem, the multiplicity of such ideal never exceeds d^n.

The following (obvious) property of multiplicity allows to control the length of ascending chains of primary ideals with the same associated prime: the multiplicities should strictly decrease along such a chain.

Lemma 5. *If $I \subset J$ are two non-equal primary ideals in $\mathfrak{R}$ with the same associated prime, then $\mu(I) > \mu(J)$.*

Proof. Denote by I_a (resp., J_a) the localizations of the two ideals (i.e. their images in the local ring $\mathfrak{R}_a$). The proof consists of two steps: first we show that if for two primary ideals with the same associated prime the equality $I_a = J_a$ holds after localization at almost all points, then in fact $I = J$, and the second observation is that if for $I \subset J$ the equality $\mu_a(I) = \mu_a(J)$ holds for almost all points, then $I_a = J_a$ for almost all points also.

1. If $p_1, \dots, p_s$ are generators of I, and q is an arbitrary polynomial in J, then the condition $I_a = J_a$ implies that $q = \sum r_j p_j$, where r_j are rational fractions with the denominators not vanishing at a, hence (by getting rid of the denominators) we arrive to the representation $hq = \sum h_j p_j$, where $h \in \mathfrak{R}$ is a polynomial not vanishing at a, and h_j are polynomials as well. Consider the colon ideal $I : q$. The above conclusion means that for almost all $a \in X$ the colon ideal $I : q$ contains a polynomial with $h(a) \neq 0$. For obvious reasons, for $a \notin X$ this is valid as well. Since I is primary, then by [27, Ch. III, §9, Theorem 14] the ideal $I : q$, if not trivial, is also primary with the same associated prime. But from the above assertion it follows that the zero locus of $I : q$ is strictly contained in X, so the only possibility left is that $I : q = \mathfrak{R}$, i.e. $q \in I$. Since $q \in J$ was chosen arbitrary, this proves the first assertion (note that we used only the fact that I is primary; the bigger ideal J could in fact be arbitrary).

2. Let $a \in X$ be a smooth point, $\mathfrak{R}_a$ and I_a being the corresponding localizations. Since the situation is local, without loss of generality we may assume that X is a coordinate subspace. Choose T being the complementary coordinate subspace and denote by L the corresponding ideal. Let (x, ε) be the associated local coordinates, so that $X = \{x = 0\}$, and $T = \{\varepsilon = 0\}$. The point a is the origin $(0,0)$. We will prove that $I_a \subsetneq J_a$ implies the inequality $\mu_{a'}(I) \neq \mu_{a'}(J)$ for all nearby points a'.

Let $\mu = \mu_a(I)$. If the germs $f_1, \dots, f_\mu \in \mathfrak{R}_{(0,0)}$ generate the local algebra of the cofinite ideal $(I+L)_{(0,0)}$, then any germ $q(x)$ from the restriction of $\mathfrak{R}_{(0,0)}$ on T can be represented as

$$q(x) = \sum_{j=1}^{\mu} c_j\, f_j(x,0) + \sum_{i=1}^{s} h_i(x) p_i(x,0), \tag{4.4}$$

where $p_i = p_i(x,\varepsilon)$ are generators of the ideal I, and $p_i(x,0)$ are their respective restrictions on the transversal L. If this representation is minimal (i.e. the number of germs f_j cannot be reduced), then μ is the multiplicity of I at the point $a=(0,0)$.

By the Preparation theorem in the Thom–Martinet version [18, Chapter I, §3], the representation (4.4) can be "extended" for all small nonzero ε: any element $q \in \mathfrak{R}_{(0,0)}$ admits a representation

$$q(x,\varepsilon) = \sum_{j=1}^{\mu} c_j(\varepsilon)\, f_j(x,\varepsilon) + \sum_{i=1}^{s} h_i(x,\varepsilon) p_i(x,\varepsilon). \tag{4.5}$$

Now assume that the localization $J_{(0,0)}$ is strictly bigger than $I_{(0,0)}$ and take an element $q \in J_{(0,0)} \smallsetminus I_{(0,0)}$. By (4.5), it can be expanded after a certain choice of the coefficients $c_j(\varepsilon)$.

The situation when all $c_j(\varepsilon)$ are identical zeros, is impossible, since this would mean that $q \in I_{(0,0)}$ contrary to our assumptions. Therefore for almost all values of ε the equality

$$\sum_{j=1}^{\mu} c_j(\varepsilon) f_j(x,\varepsilon) = q + \sum_{i=1}^{s} h_i p_i \in J_a$$

means a nontrivial linear dependence between the generators $f_j(\cdot,\varepsilon)$ of the corresponding local algebra $\mathfrak{R}_{(0,\varepsilon)}/J_{(0,\varepsilon)} \cdot \mathfrak{R}_{(0,\varepsilon)}$, so that its dimension is strictly smaller than μ. By definition this means that the multiplicity $\mu_{a'}(J)$ is strictly smaller than $\mu = \mu(I) = \mu_{a'}(I)$ for all nearby points $a' = (0,\varepsilon)$. The proof of the second step (and together with it the proof of the lemma) is achieved. $\square$

Illustration: chains of ideals of homogeneous dimension. The above two lemmas already imply an upper bound for the length of an ascending chain under rather specific restrictions. We will use this particular case as a building block for the general construction.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_\nu\}$ be a finite collection of pairwise different prime ideals, $P_i \subset \mathfrak{R}$, of the same dimension m. Consider a *strictly* ascending (finite) chain of ideals

$$J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_{\ell-1} \subsetneq J_\ell \tag{4.6}$$

under the additional assumption that *each ideal from this chain is an intersection of primary ideals with associated primes only from the predefined collection* $\mathcal{P}$. Denoting by Q_{kj} the primary component of J_k with the associated prime P_j (if there is no such component, we introduce a fictitious term $Q_{kj} = (1)$ to make our considerations uniform), we can write

$$J_k = Q_{k1} \cap \cdots \cap Q_{k\nu},$$
$$Q_{kj} \text{ is either } (1) \text{ or primary with } \sqrt{Q_{kj}} = P_j.$$

By the uniqueness part of the Noether–Lasker theorem, all ideals Q_{kj}, whether fictitious or not, are uniquely determined.

Lemma 6. *The length ℓ of the strictly ascending chain* (4.6) *does not exceed the number of primary components of J_0, counted with their multiplicities*:

$$\ell \leqslant \sum_{j=1}^{\nu} \mu(Q_{0j}).$$

Proof. This is an obvious corollary to Lemma 4 and Lemma 5. Indeed, by Lemma 4, the monotonicity of the chain (4.6) implies the monotonicity of all the chains

$$\begin{array}{c}Q_{01} \subset Q_{11} \subset \cdots \subset Q_{\ell 1},\\ Q_{02} \subset Q_{12} \subset \cdots \subset Q_{\ell 2},\\ \vdots \quad \vdots \qquad\qquad \vdots\\ Q_{0\nu} \subset Q_{1\nu} \subset \cdots \subset Q_{\ell\nu}.\end{array}$$

Moreover, in every column k at least one inclusion must be strict (otherwise $J_k = J_{k+1}$). Since all ideals in each line are primary with the same associated prime (unless they become trivial), the assertion of Lemma 5 applied to each line, shows that the numbers $\mu(Q_{kj})$ are nonincreasing (by definition we put $\mu((1)) = 0$), and their sum $\sum_{j=1}^{\nu} \mu(Q_{kj})$ is *strictly* decreasing as k ranges from 1 to ℓ. Since all multiplicities are nonnegative, this last observation proves the claim. ☐

4.2. **Effective commutative algebra.** The chain (1.8) consists of ideals given by their generators p_k. On the other hand, application of results such as Lemma 6 requires knowing some numbers (the number of primary components, their multiplicities). Moreover, our constructions below would require determining (or rather estimations) of similar numbers for ideals obtained by certain algebraic procedures (intersections, colon ratios etc) from the initial ideals I_k. Thus we need some tools for performing explicitly all these manipulations.

We agree that to *construct* (or *define*) an ideal in the polynomial ring $\mathfrak{R}$ means to construct (or specify) a set of generators of this ideal. Then many operations on ideals become algorithmically implementable. In particular, given polynomials generating some input ideals I and J, one can explicitly do the following [24]:

- construct the intersection $I \cap J$ and the colon ratio $A : B$;
- decide whether a given polynomial belongs to I and if so, construct an explicit expansion for the former in the generators of the latter;
- construct *some* primary decomposition of I and the intersection of all primary components of upper and lower dimensions,
- and many other (but not all) algebraic operations.

The algorithms performing the above mentioned operations, are discussed and perfected in a number of works. However, we are interested here not in the manipulations themselves, but rather in the upper bounds for the degrees of the generators of polynomial ideals. The result that will be used in the proof of Theorem 4, describes the algorithmical complexity of the primary decomposition.

Theorem ([12, 24, 25, 8, 9, 16, 17])**.** *If an ideal I of the polynomial ring $\mathfrak{R} = \mathbb{C}[x_1, \dots, x_n]$ is generated by polynomials of degree $\leqslant d$, then one can effectively construct polynomial bases of all ideals in the primary decomposition of I. The number of primary components and the degrees of polynomials in the bases can be majorized by primitive recursive functions of n and d, and each of these functions grows no faster than $d^{n^{O(n)}}$ as $n \to \infty$.*

This theorem in fact constitutes a synopsis of several references, see [12, 24, 25] for the primitive recursivity of the bound, [8, 9] for explicit estimates and [16, 17] for more precise bounds. Combination of this result with our definition of multiplicity and Bézout inequalities implies the following.

Corollary 2. *For an ideal I generated by polynomials of degree d in $\mathbb{C}[x_1,\dots,x_n]$, and any dimension $m<n$ one can construct a decomposition $I=I'\cap S$, where I' is of dimension $\leqslant m$ and all primary components of S are of dimension $m+1$ or more, in such a way that the number of primary components in* $\operatorname{l.t.}(I)$*, counted with their multiplicities, is majorized by a primitive recursive function $\nu(n,d)$ of variables d and n, growing no faster than $d^{n^{O(n)}}$ as $n\to\infty$.* □

4.3. **Convexity of chains generated by adding consecutive derivatives.** The key property of ascending chains, on which the proof of Theorem 4 is based, is that of convexity: the "multiplicative first differences" of the ideals in the chain monotonously "nondecrease".

Definition. An ascending chain of ideals (1.8) is called *convex* if

$$I_{k-1}:I_k\subset I_k:I_{k+1}\qquad \forall k=1,2,\dots. \tag{4.7}$$

Lemma 7. *Any chain of polynomial ideals $\{I_k\}$ generated by adding consecutive derivatives as in* (1.7)–(1.8)*, is convex.*

Proof. Obviously, $I_k:I_{k+1}=I_k:(p_{k+1})$. If $q\in I_{k-1}:(p_k)$, then $qp_k=\sum_{i=0}^{k-1}h_ip_i$. Applying L, we conclude:

$$\begin{aligned} qp_{k+1}&=q\cdot Lp_k=L(qp_k)-Lq\cdot p_k\\ &=\sum_{i=0}^{k-1}Lh_i\cdot p_i+\sum_{i=1}^{k}h_{i-1}p_i+Lq\cdot p_k=\sum_{i=0}^{k}\tilde h_ip_i\in I_k, \end{aligned}$$

which implies that $q\in I_k:(p_{k+1})$. □

Corollary 3. *A convex ascending chain is* strictly *ascending: if $I_{\ell-1}=I_\ell$ for some ℓ, then $I_\ell=I_{\ell+1}=I_{\ell+2}=\cdots=\bigcup_{j=0}^\infty I_j$.*

Proof. If $I_\ell=I_{\ell+1}$, then $I_\ell:I_{\ell+1}=(1)$ (the unit ideal, i.e. the whole ring $\mathfrak{R}$), so by (4.7) all other colon ratios $I_{\ell+s}:I_{\ell+s+1}$ for any natural $s>1$ are also trivial as they must contain the ideal (1). □

Corollary 4. *The sequence of dimensions* $\dim(I_k:I_{k+1})$ *is non-increasing.* □

4.4. **Length of convex chains with strictly ascending leading terms.** Consider the ascending chain (1.8) with an additional assumption that the colon ratios have the same dimension as the ideals I_k themselves. We show then that the length of this chain is completely determined by the leading term of the first ideal in the chain.

Lemma 8. *Consider a finite strictly ascending chain of ideals has the form* (1.7)–(1.8)*. Assume that all colon ratios have the same dimension as the starting ideal of this chain*:

$$\dim I_0=\dim I_0:I_1=\cdots=\dim I_k:I_{k+1}=\cdots=\dim I_{\ell-1}:I_\ell.$$

Then the length ℓ of this chain does not exceed the number of primary components of the leading term $J_0=\operatorname{l.t.}(I_0)$ of the starting ideal (*counted with multiplicities*).

Proof. This is a simple corollary to Lemma 6. Indeed, consider the chain of leading terms $\{J_k\}_{k=0}^{\ell}$, $J_k=\operatorname{l.t.}(I_k)$. Denote by m the common dimension of all colon ratios.

1. The chain $\{J_k\}$ is ascending by Lemma 4: all upper-dimensional primary components of I_{k+1} can be obtained by enlarging primary components of J_k (sometimes making them trivial).

2. Moreover, the chain $\{J_k\}$ is in fact *strictly* ascending and this is where we use the assumption on dimensions. Indeed, if $J_k = J_{k+1}$ for some k, then

$$p_{k+1} \in I_k + (p_{k+1}) = I_{k+1} \subset \operatorname{l.t.}(I_{k+1}) = J_{k+1} = J_k = \operatorname{l.t.}(I_k),$$

which means that p_{k+1} in fact belongs to all upper-dimensional primary components of I_k. Writing $I_k = J_k \cap R_k$, where R_k is the intersection of all primary components of dimension strictly inferior to m, we conclude that

$$I_k : I_{k+1} = I_k : (p_{k+1}) = \bigl(J_k : (p_{k+1})\bigr) \cap \bigl(R_k : (p_{k+1})\bigr) = (1) \cap \bigl(R_k : (p_{k+1})\bigr) \supset R_k,$$

therefore $\dim I_k : I_{k+1} \leqslant \dim R_k < m$. This contradicts the second assumption.

3. It remains only to verify that the collection of prime ideals associated with all primary components of the ideals J_k, is non-expanding (in particular, contained in that of J_0). We note that the upper-dimensional associated primes can be detected as ideals of upper-dimensional irreducible components of the loci $X_k = V(K_k) \subset \mathbb{C}^n$. The chain of algebraic varieties X_k is descending, $X_k \supset X_{k+1}$.

Each irreducible upper-dimensional component of X_k either belongs to X_{k+1}, if p_{k+1} vanishes identically on this component, or becomes an algebraic variety of dimension strictly inferior to m after intersection with the hypersurface $\{p_{k+1} = 0\} \subset \mathbb{C}^n$. Thus the collections of irreducible upper-dimensional components of the varieties X_k are non-expanding as k grows from 0 to ℓ, and the same holds for the collections of prime ideals associated with the leading terms J_k. Thus all assumptions of Lemma 6 are verified for the chain $\{J_k\}$. ☐

4.5. **Revealed growth.** The next step is to get rid of the assumption that the dimension of the colon ratios coincides with that of the starting ideal in the chain in Lemma 8. Using the lemma below, one can transform any chain with a constant dimension of colon ratios, into another chain with strictly increasing leading terms.

Assume that the dimension of the colon ratios $I_k : I_{k+1}$ remains the same along the strictly ascending chain (1.8):

$$\dim I_0 : I_1 = \cdots = \dim I_{\ell-1} : I_\ell = m. \tag{4.8}$$

Consider any primary decomposition of the starting ideal I_0 written in the form

$$I_0 = I_0' \cap S, \tag{4.9}$$

where I_0' is the intersection of all primary components of dimension m and below, and S is the intersection of all primary components of I_0 of dimensions $\geqslant m+1$.

Lemma 9. *If the chain of ideals $\{I_k'\}_{k=0}^{\ell}$ is built from I_0' using the same rule*

$$I_{k+1}' = I_k' + (p_{k+1}), \qquad k = 0, 1, \dots, \ell-1, \tag{4.10}$$

and the condition (4.8) *holds, then for all $k = 0, 1, \dots, \ell-1$*

$$p_{k+1} \in S, \tag{4.11}$$

$$I_k = I_k' \cap S, \tag{4.12}$$

$$I_k : I_{k+1} = I_k' : I_{k+1}'. \tag{4.13}$$

Proof. **1.** If (4.11) is not true for some k, then p_{k+1} would not belong to at least one primary component of S of dimension $m+1$ or more. But then the colon ratio will be at least $(m+1)$-dimensional, contrary to our assumption. Indeed, for a P-primary ideal $Q \subset \mathfrak{R}$ and any $p \in \mathfrak{R}$ we have the following alternative:

$$Q : (p) = \begin{cases} (1), & \text{if } p \in Q, \\ P\text{-primary}, & \text{if } p \in P \setminus Q, \\ Q, & \text{if } p \notin P. \end{cases}$$

In all nontrivial cases the dimension is preserved.

2. The proof of (4.12) goes by induction: for $k = 0$ it coincides with (4.9). The induction step is an application of the modular law [27]: since $p_{k+1} \in S$ by the previous argument, we have $(p_{k+1}) \cap S = (p_{k+1})$ and hence

$$I'_{k+1} \cap S = (I'_k + (p_{k+1})) \cap S = I'_k \cap S + (p_{k+1}) = I_{k+1}.$$

3. The last identity (4.13), equivalent to $I_k : (p_{k+1}) = I'_k : (p_{k+1})$, follows from (4.11) and (4.12). This completes the proof of of Lemma 9. □

4.6. **Demonstration of Theorem 4.** According to Corollary 4, any finite strictly ascending convex chain (1.8) can be subdivided into no more than n finite strictly ascending segments in such a way that along each segment the dimension of the colon ratios is the same and equal to $n - s$, where $s = 1, 2, \dots, n$ is the number of the segment:

$$\underbrace{I_0 \subset I_1 \subset \cdots \subset I_{k_1}}_{\dim I_k : I_{k+1} = n-1} \subset \underbrace{I_{k_1+1} \subset \cdots \subset I_{k_2}}_{\dim I_k : I_{k+1} = n-2} \subset \cdots$$
$$\cdots \subset \underbrace{I_{k_s+1} \subset \cdots \subset I_{k_{s+1}}}_{\dim I_k : I_{k+1} = n-s+1} \subset \cdots \subset \underbrace{I_{k_{n-1}+1} \subset \cdots \subset I_{k_n}}_{\dim I_k : I_{k+1} = 0} \tag{4.14}$$

The length of each segment can be majorized using Lemma 8 and Lemma 9. Let $I = I_{k_s+1}$ be the initial ideal of the sth segment. We decompose it as $I = I' \cap S$ and build an auxiliary (finite) chain I'_k for $k = k_s + 2, \dots, k_{s+1}$ using the rule (4.10), as described in Lemma 9.

This new chain satisfies all conditions of Lemma 8, so its length $k_{s+1} - k_s$ cannot exceed the number of primary components of $I' = I'_{k_s+1}$, counted with multiplicities. This number, by Corollary 2, does not exceed $\nu(n, (k_s+1)d)$, as the decomposed ideal I_{k_s+1} is generated by polynomials of degree $\leqslant (k_s+1)d$.

It remains only to remark that the length of the auxiliary chain majorizes the length of the sth segment of the initial chain, according to Lemma 9: as soon as the auxiliary colon ratios become less than m-dimensional with $m = n - s$, the same would occur for the initial chain as well, which means that the new segment in fact began.

This argument proves the following bound for the length of each segment:

$$k_{s+1} - k_s \leqslant \nu(n, (k_s+1)d) \asymp (k_s d)^{n^{O(n)}}. \tag{4.15}$$

This recurrent identity immediately proves that the length of the chain $\ell = \ell(n, d) = k_n$ is a primitive recursive function of d and n and grows as asserted in the theorem.

Indeed, the growth of the sequence $\{k_s\}$ as in (4.15) is overtaken (for large values of d and n) by the growth of the linear difference equation $l_{s+1} = M l_s^\alpha$ with $M = d^{n^{O(n)}}$, $\alpha = n^{O(n)}$, $l_0 = 1$, whose solutions can be found and estimated explicitly: since one can always assume $\alpha > 2$, we have $\log l_s \leqslant 2\log M \cdot \alpha^s$ so that $l_n \leqslant M^{2\alpha^n} \asymp M^{n^{O(n^2)}} \asymp d^{n^{O(n^2)}}$. $\square$

Appendix A. Gabrièlov theorem

A.1. **Formulation and general remarks.** Recall that the Risler problem consists in estimating the maximal order of contact between a trajectory of a polynomial vector field and an algebraic hypersurface in $\mathbb{C}^n$. The answer is to be given in terms of the dimension n and degree d of the polynomials defining the vector field and the hypersurface.

In the particular case $n = 2$ an upper bound for the order of contact was found in [6] by A. Gabrièlov, J.-M. Lion and R. Moussu. Later in [4] Gabrièlov solved the problem in the general case and proved that this order does not exceed $(2d)^{2^{n+1}}$ if the degrees of the field and hypersurface are both equal to d.

The order of contact between a parameterized trajectory $t \mapsto x(t)$ and a hypersurface $\{p = 0\}$ is the order of zero of the restriction $p(x(t))$. As this restriction was shown in §2 to satisfy a linear ordinary differential equation of order $\ell = \ell(n, d)$ (the length of an ascending chain), the multiplicity of a root cannot exceed $\ell - 1$, unless the solution is identically zero.

Note that, while solving the Risler problem, one can skip the universalization step of §2.2: indeed, the magnitude of coefficients is not important, only the order of the resulting equation (2.1) matters.

However, this approach gives the bound of order of magnitude $d^{n^{O(n^2)}}$ for the maximal order of tangency, which is substantially worse than the Gabrièlov bound. Below we show how the answer can be improved.

A.2. **Chain of varieties associated with Risler problem.** The Risler problem occupies in some sense an intermediate place between algebraic and geometric versions of the problem on ascending/descending chains.

Namely, assume that $\{I_k\}$ is the ascending chain of ideals (1.8) generated by consecutive derivations (1.7). Then one can associate with this chain of ideals the chain of their respective zero loci $\{X_k\}$, $X_k = V(I_k)$. Despite the fact that the chain of ideals must be *strictly* ascending by Corollary 3, the descent of the chain of varieties *should not necessarily* be strict. The easiest example is the sequence of derivatives of the polynomial x^μ in one variable, corresponding to $n = 1$ and $L = \partial/\partial x$. The chain of the respective zero loci drops after μ stable steps: $\{0\} = X_0 = X_1 = \cdots = X_{\mu-1} \neq X_\mu = \varnothing$.

However, one may estimate the length of the chain $\{X_k\}$ along which the *ultimate* stabilization must occur. The considerations below provide arguments sufficient for proving the result described in §A.1. We start with the following trivial observation.

Let $L \colon \mathfrak{R} \to \mathfrak{R}$ be a Lie derivation, v the corresponding polynomial vector field in $\mathbb{C}^n$ and Y a submanifold in $\mathbb{C}^n$ with the coordinate ring $\mathfrak{R}' = \mathfrak{R}/I(Y)$. Denote by $\pi : \mathfrak{R} \to \mathfrak{R}'$ the canonical projection. Then L "covers" a well-defined derivation L' of the ring $\mathfrak{R}'$ if Y is invariant by the flow of the field v. This claim admits a local reformulation with all rings being the rings of germs.

A.3. **Demonstration of Gabrièlov theorem.** Denote by X_∞ the stable limit, the intersection of all varieties X_k, and consider the decreasing chain of semialgebraic varieties $X_k' = X_k \smallsetminus X_\infty$. The chain $\{X_k'\}$ eventually vanishes. Suppose that on some step $k = k_s$ the variety X_{k_s}' is s-dimensional (recall that each semialgebraic variety has dimension), and denote by X the set of points at which the field v is transversal to X'_{k_s}.

The complement $X'_{k_s} \smallsetminus X$ is less-than-s-dimensional. Indeed, since the tangency condition is algebraic, its violation on a relatively open set would mean that this set is locally invariant by the flow of v and hence belongs to X_∞. But this contradicts the definition of X'_{k_s} as a part of the complement to X_∞.

We show that after some number m of steps, any point a on X will *not* belong to X'_{k_s+m}, and hence the latter semialgebraic variety should be less-than-s-dimensional. The number m can be explicitly majorized.

Since the integral curve of v through a is transversal to X at a, we can construct the germ of a codimension s analytic surface (Y, a) in $\mathbb{C}^n$ such that $X \cap Y = \{a\}$ and Y be invariant by the flow of v. Let $\mathfrak{R}'$ be the local ring of germs on (Y, a). The ascending chain of ideals $\{I_k\}$ restricted on Y, yields an ascending chain of ideals $\{I'_k\} \subset \mathfrak{R}'$, $k = k_s, k_s + 1, \ldots$, in the local ring, generated by adding consecutive derivatives $\pi L^k p_0 = (L')^k \pi p_0$ (by virtue of the above observation). Therefore this chain is convex and hence, by Corollary 3, *strictly* ascending.

The difference between the type of ascent of the chains $\{I_k\}$ and $\{I'_k\}$ is twofold:

1. all ideals of the latter chain are cofinite, so the numbers $\mu_k = \dim_{\mathbb{C}} \mathfrak{R}'/\mathfrak{R}' \cdot I'_k$ are finite, and
2. the chain $\{I'_k\}$ must terminate by the trivial ideal $I'_{k_s+m} = (1) \in \mathfrak{R}'$, since $a \notin X_\infty$.

Now it is obvious that the codimensions μ_k of the cofinite ideals must be *strictly* decreasing: otherwise we would have the equality $I'_k = I'_{k+1} \neq (1)$. However, this is impossible since by Corollary 3 this would mean that I_k stabilize on a non-trivial ideal. Therefore after no more than $m = \mu_{k_s}$ steps the chain of local ideals must stabilize on the trivial ideal $(1) = \mathfrak{R}'$.

To place an upper bound for the number of steps m after which the variety X'_{k_s+m} must become less-than-s-dimensional, we need to estimate the multiplicity μ_{k_s}. Note that this multiplicity does not depend on the choice of the transversal section Y as soon as the latter remains transversal to X at a. Choosing Y being an affine subspace in $\mathbb{C}^n$, we can majorize the multiplicity of the ideal (by virtue of Bézout theorem).

The easiest way to do that is by the inequality $\mu_{k_s} \leqslant (k_s d)^s \times 1^{n-s}$, since $k_s d$ is the maximal degree of $\leqslant s$ polynomials determining X_{k_s} (the equations for Y are linear, hence do not contribute to μ_{k_s}). This estimate leads to recurrent inequalities $k_{s-1} - k_s \leqslant (k_s d)^s$ for $s = n-1, \ldots, 2, 1$ and finally to an upper bound fro k_0 growing as $(2d)^{n!}$, as an easy computation shows.

More accurate arguments (using the fact that each X'_{k_s} is defined by equations of different degrees, from $k_1 d$ to $k_s d$), leads to an upper bound $(2d)^{2^{n+1}}$, exactly as in [4].

A.4. **Final remark.** Very recently, using completely different arguments, A. Gabriélov proved in [5] that the order of tangency does not exceed $(2nd)^{2n}$, which is a simply exponential in n upper bound.

Appendix B. Linear systems

The principal result of the paper, Theorem 1, is formulated for arbitrary polynomial vector fields. However, the case of *linear* (nonautonomous) systems is of a particular interest, first because linear systems often naturally arise in problems concerning the number of zeros, and second because in this case one may improve slightly the constructions compared to the general case.

Three instances are discussed in this appendix. We show that for a system of linear first order differential equations rationally depending on time, one can derive a *linear* equation satisfied by *all* linear combinations of components of *any* trajectory, with the same collection of singular points. Second, we obtain a simple exponential in n (the number of variables) bound for lengths of chains of ideals generated by *linear* forms polynomially depending on t. Finally, we notice that for linear systems the continuous Risler problem can be reduced to ite discrete counterpart and vice versa.

B.1. **Hyperelliptic integrals and Gauss–Manin system.** The main source of applications (in fact, the problem that motivated the whole research summarized in this paper) is the problem on zeros of complete Abelian integrals, sometimes called *tangential* or *weakened* Hilbert 16th problem. Recall that the problem consists in finding an upper bound for the number of real zeros of a (multivalued) function

$$I(t) = I_{H,\omega}(t) = \oint_{H(x,y)=t} \omega, \qquad \omega = P(x,y)\,dx + Q(x,y)\,dy \tag{B.1}$$

where $H, P, Q \in \mathbb{C}[x,y]$ are polynomials of known degrees, and the integration is carried over a continuous family of real ovals of the level curves $\{H = c\} \subset \mathbb{R}^2$. The bound is to be given in terms of the degrees only.

Almost all approaches to this problem use to a certain extent the fact that the function $I(t)$ satisfies certain linear differential equations, *Picard–Fuchs* equations. However, in one particular case when H has the form

$$H(x,y) = y^2 + p(x), \qquad p \in \mathbb{R}[x],\ \deg p = n+1 \geqslant 3,$$

these equations can be written absolutely explicitly. More precisely, the column vector $\boldsymbol{I}(t)$ formed by the integrals of the forms $x^k y\,dx$, $k = 0, 1, \dots, n-1$, was shown in [10] to satisfy the system of n first order linear differential equations

$$(tE - C)\,\dot{\boldsymbol{I}}(t) = B\boldsymbol{I}(t), \qquad C, B \in \operatorname{Mat}_{n\times n}(\mathbb{C}). \tag{B.2}$$

Here B, C are two constant $n \times n$-matrices depending on the polynomial p only (and E the identity matrix). By examination of the algorithm suggested in [10] one can verify that the norms of the matrices are bounded in terms of the height of p, provided that the latter is a unitary polynomial (with the leading coefficient 1), and the spectrum of C coincides with the set of critical values of the polynomials p and H. Notice that the height of p can be assumed to be bounded by 1 without loss of generality.

The system (B.2) is not polynomial, but becomes rational after a simple transformation (multiplication by the adjugate matrix to $tE - C$). By this transformation it can be reduced to the form (we replace $\boldsymbol{I}(t)$ by $x(t)$ to return to the notations used throughout the paper)

$$\varDelta(t) \cdot \dot{x}(t) = A(t)x(t), \qquad \varDelta(t) = \prod_{j=1}^{n}(t - t_j), \quad A(t) = \sum_{k=0}^{d} A_k\, t^k, \tag{B.3}$$

where $\Delta(t) \in \mathbb{C}[t]$ is the characteristic polynomial of the matrix C, and $t_1, \dots, t_n$ the critical values of p (counted with multiplicities). The right hand side contains the matrix polynomial $A(t) \in \operatorname{Mat}_{n\times n}(\mathbb{C}[t])$ of degree $d = n-1$ and controlled height. The system (B.3) has singular points $t_1, \dots, t_n$ (and $t = \infty$). These points are Fuchsian if the critical values t_j are pairwise distinct.

We show that, despite the presence of singular points and occurrence of denominators, a positive information about zeros of hyperelliptic integrals can be obtained by applying Theorem 1.

Remark. The results by L. Gavrilov [7] show that a similar system of first order equations can be derived also for a general polynomial H provided that its principal homogeneous part is generic. However, the resulting system will not be explicit, and there are almost no chances that the height of the right hand side would admit an upper bound uniform over all generic H.

B.2. **Reduction from linear systems to linear equations.** Applied to systems of linear equations, the procedure of derivation of a high order equation described in §2.3, results in a more accomplished answer. We formulate it for the case of equations of the form (B.3) eventually possessing singularities.

Theorem 7. *With any linear polynomial system* (B.3) *of degree $\leqslant d$ having at most d Fuchsian singularities in the finite plane, one can associate a single ℓth order differential equation of the form*

$$\varDelta^\ell(t) y^{(\ell)} + a_1(t)\varDelta^{\ell-1}(t) y^{(\ell-1)} + \cdots + a_{\ell-1}(t)\varDelta(t) y' + a_\ell(t) y = 0 \tag{B.4}$$

with the following properties:

1. *all finite singular points of the equation* (B.4) *are Fuchsian and coincide with the singularities of the initial system;*
2. *the coefficients $a_k(\cdot)$ of* (B.4) *are polynomial in t and polynomially depend on the parameters of the problem (matrix elements of A_k and the coordinates of singular points t_j);*
3. *the order of the equation* (B.4)*, the degrees and heights of the polynomial coefficients a_k are bounded by primitive recursive functions of n, d;*
4. *any linear combination $y(t) = \langle \xi, x(t)\rangle = \xi_1 x_1(t) + \cdots + \xi_n x_n(t)$, $\xi_i \in \mathbb{C}$, of coordinate functions of any solution $x(t)$ of the initial system* (B.3) *satisfies the equation* (B.4).

Corollary 5. *The number of isolated intersections between any trajectory $x(t)$ of a linear system* (B.3) *of height $\leqslant R$ and an arbitrary linear hyperplane $\langle \xi, x\rangle = 0$ over any simply connected subdomain of the set $\{t \in \mathbb{C} \colon |t - t_j| > 1/R,\ |t| < R\}$ is bounded by $(2+R)^B$, where $B = B(n,d)$ is a primitive recursive function of n, d growing no faster than* (1.4).

In other words, the assertion of Theorem 1 for linear polynomial systems admits direct complexification provided that zeros are counted away from eventual singular points. In this case one can also suppress all requirements on the geometric size of the trajectory, except for proximity to singular points in the t-plane.

Proof of Theorem 7. The system (B.3) after introducing the new independent variable τ can be put into the "true" polynomial form

$$\begin{cases} x' = A(t)x, & x \in \mathbb{C}^n,\ t \in \mathbb{C}^1, \\ t' = \varDelta(t), & {}' = \frac{d}{d\tau}. \end{cases} \tag{B.5}$$

As in §2.2, we consider the system (B.5) as a single polynomial vector field whose right hand sides belong to the polynomial ring $\mathbb{C}[t, x, \xi, A, \varSigma]$, where $A =$

$\{A_0, \dots, A_d\}$ is the collection of all coefficients of the matrix polynomial $A(t)$ and $\Sigma = \{t_1, \dots, t_m\}$ the variables replacing coordinates of all singular points.

Let $\xi = \xi_0 \in \mathbb{C}^{n*}$ be an arbitrary linear functional and $p_0(x) = \langle \xi, x\rangle \in \mathbb{C}[t, x, \xi]$ the corresponding polynomial. The rule (1.7) for the extended system (B.5) yields a sequence of polynomials $p_k \in \mathbb{C}[t, x, \xi, A, \Sigma]$, that are all *bilinear* in x and ξ simultaneously.

Let $\ell = \ell(n, d)$ be the moment of termination of the chain (1.8) and $h_0, \dots, h_{\ell-1} \in \mathbb{C}[t, x, \xi, A, \Sigma]$ the coefficients of the decomposition (2.4). Apriori the constructions of §4 do not guarantee that the polynomials h_k in *do not depend* on x and ξ, but one can always *achieve* this independence. Indeed, expanding all polynomials p_k and h_k in x and using the fact that all p_k are linear homogeneous, we see immediately that after replacing h_k by their free (with respect to x) terms preserves the identity (2.4): all higher order terms must cancel each other. In the same way one can get rid of the dependence of h_k on ξ. Thus we see that the identity (2.4) after substitution $p_k \mapsto d^k y/d\tau^k$, $y(t) = \langle \xi, x(t)\rangle$, yields the equation

$$\frac{d^\ell y}{d\tau^\ell} + a_1(t, A, \Sigma)\, \frac{d^{\ell-1} y}{d\tau^{\ell-1}} + \cdots + a_\ell(t, A, \Sigma)\, y = 0$$

for the function y. Returning to the initial independent variable t, we obtain the equation

$$\left(\Delta(t)\,\frac{d}{dt}\right)^{\ell} y + a_1(t)\left(\Delta(t)\,\frac{d}{dt}\right)^{\ell-1} y + \cdots + a_{\ell-1}(t)\left(\Delta(t)\,\frac{d}{dt}\right) y + a_\ell(t)\, y = 0, \qquad a_k \in \mathbb{C}[t, A, \Sigma]. \tag{B.6}$$

It is obvious that:

1. this equation has only Fuchsian singular points in the finite plane, provided that $\Delta(t)$ has only simple roots,
2. the magnitude of the coefficients $a_k = a_k(t, A, \Sigma)$ is bounded in terms of the height of the matrix polynomial $A(t)$ and the maximal modulus of the singular points: if $\|A_k\| \leqslant R$ and $|t_j| \leqslant R$ for all k, j, then $|a_j(t, A, \Sigma)| \leqslant (2+R)^{B(n,d)}$ on the disk $\mathbb{B}_R = \{|t| \leqslant R\}$ (see (1.3)–(1.4));
3. after reduction to the standard form $y^\ell + b_1(t)\, y^{(\ell-1)} + \cdots + b_{\ell-1}(t)\, y' + b_\ell(t) y = 0$ the (rational) coefficients $b_k(t)$ of the reduced equation are bounded by similar expressions on the set $|t - t_j| \geqslant 1/R$, $|t| < R$.

This allows for application of Lemma 1 or Lemma 2, implying upper bounds for the number of zeros away from singular points of the linear system (B.3). □

Remark. The singular point at infinity of the equation (B.6) is in general non-Fuchsian, as the degrees of the polynomials $a_k(t)$ are in general greater than allowed by the Fuchs conditions at infinity.

Remark. One can generalize results of this subsection for intersections between trajectories of linear systems with arbitrary polynomial hypersurfaces as follows. The collection of all monomials $\{x^\alpha\}_{|\alpha| \leqslant d}$ of degree $\leqslant d$ satisfies a Fuchsian system of linear polynomial equations, provided that (B.3) holds:

$$\Delta(t) \cdot \frac{d}{dt}\, x^\alpha = \sum_{j=1}^{n} \alpha_j \frac{x^\alpha}{x_j} \sum_{k=1}^{n} A_{jk}(t) x_k = \sum_{|\beta| = |\alpha|} A_{\alpha\beta}(t) x^\beta.$$

Theorem 7 can be applied to count zeros of arbitrary linear combinations of monomials $\xi_\alpha x^\alpha$, i.e. arbitrary polynomials in x.

B.3. **Ascending chains of ideals generated by linear forms.** The problem on length of ascending chains of polynomial ideals under additional assumption that the degrees of polynomials are bounded, belongs to linear algebra. If d is a uniform upper bound for $\deg p_k$, then the chain (1.8) must stabilize after at most $N=(d+1)^n$ steps: indeed, the space of all polynomials in n variables of degree $\leqslant d$ is less than N-dimensional, therefore at that step or before the newly added polynomial will be a linear combination of preceding ones.

If the degrees $\deg p_k$ are growing, this argument fails and the bound depends in the most heavy manner on the dimension n (the number of variables). For $n=1$, obviously, the number of points in the zero loci of ideals, counted with multiplicities, should decrease monotonously, therefore the length of the chain cannot exceed $\deg p_0$. The case $n>1$ is in general completely different, however, in one particular case the problem can be reduced to the univariate case.

We already noted that for a system of linear equations the chain of polynomials generated by the rule (1.7) will be linear in x, provided that $p_0(x)=\langle\xi,x\rangle$ is a linear form.

Proposition 2. *The chain of linear forms* $p_k(x)=\sum_{i=1}^n \xi_{k,i}(t)x_i\in\mathbb{C}[t,x]=\mathbb{C}[t,x_1,\dots,x_n]$ *whose degrees* $\deg_t p_k$ *grow at most linearly in* k,

$$\deg_t p_k\leqslant (k+1)d,\qquad \deg_{x_i} p_k=1,\quad i=1,\dots,n,$$

stabilizes after at most $(n+1)!d^n$ *steps.*

Proof. An identity $p_\ell(t,x)=\sum_{k=0}^{\ell-1}h_k(t,x)p_k(t,x)$ with $h_k\in\mathbb{C}[t,x]$ implies that $p_\ell(t,x)=\sum_{k=0}^{\ell-1}\lambda_k(t)p_k(t,x)$, if we let $\lambda_k=h_k(\cdot,0)\in\mathbb{C}[t]$, which means that the chain stabilizes if the non-homogeneous system of linear algebraic equations

$$\xi_{\ell,i}(t)=\sum_{k=0}^{\ell-1}\lambda_k\xi_{k,i}(t) \tag{B.7}$$

possesses a polynomial solution $\lambda(t)=(\lambda_1(t),\dots,\lambda_n(t))$.

Let $\Xi_k\in\operatorname{Mat}_{k\times n}(\mathbb{C}[t])$ be the rectangular matrix with n rows and k columns, formed by the first k column vectors $\xi_1(t),\dots,\xi_k(t)$: its entries are polynomials in one variable. Consider the auxiliary ideals $W_{k,i}\subset\mathbb{C}[t]$ formed by all $i\times i$-minors of the matrix Ξ_k. (For $i<k$ we put $W_{k,i}=\{0\}$ by definition). The following criterion of solvability of linear systems over the ring $\mathbb{C}[t]$ generalizes the well-known Kronecker–Capelli criterion for systems over fields.

Lemma 10 ([13, Bk I, Ch. II, §9])**.** *The system* (B.7) *possesses a solution if and only if* $W_{\ell,i}=W_{\ell-1,i}$ *for all* $i=1,\dots,n$.

In other words, application of Lemma 10 reduces the problem on chains of ideals spanned by linear forms, to that of simultaneous stabilization of n chains of univariate ideals.

Recall that the ring $\mathbb{C}[t]$ is a principal ideal domain, therefore for each ideal $W_{k,i}$ we can define the "number of points counted with multiplicities" $\mu_{k,i}$, equal to $\deg q_{k,i}$ if $\mathbb{C}[t]\supseteq(q_{k,i})=W_{k,i}$. For $W_{i,k}=\{0\}$ we put $\mu_{k,i}=+\infty$ by definition. The obvious monotonicities

$$W_{k,i}\subseteq W_{k+1,i},\qquad W_{k,i+1}\subseteq W_{k,i}$$

(the second follows from the expansion formula for minors) imply the inequalities

$$\mu_{k,i} \geqslant \mu_{k+1,i}, \qquad \mu_{k,i+1} \leqslant \mu_{k,i}, \tag{B.8}$$

and the moment when the system (B.7) is solvable, occurs when $\mu_{\ell,i} = \mu_{\ell-1,i}$ for all $i = 1, \dots, n$.

Denote by $\mu_k = (\mu_1, \dots, \mu_n) \in (\mathbb{Z}_+ \cup \{\infty\})^n$ the sequence of vectors, and let $\|\mu_k\|$ be the sum of all *finite* coordinates of the vector μ_k. The integer sequence $\|\mu_k\|$ is monotonically decreasing with k unless one of the infinite coordinates of the vector becomes finite (and hence remains finite for larger k).

Let i be between 1 and n and denote k_i the first time when $\mu_{k,i}$ becomes finite. The second inequality in (B.8) implies that $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$. On the interval between k_i and k_{i+1} the "norms" $\|\mu_k\|$ decrease monotonously by definition of the moments k_i, and at each of the moments k_i the "norm" $\|\mu_k\|$ may increase by $\mu_{k_i,i}$. As the degrees of the polynomials p_k generating the ideals $W_{k,i}$ are known, we can estimate $\mu_{k_i,i} \leqslant i(k_i+1)d$. Thus, as $\|\mu_k\|$ must remain nonnegative, we have the balance inequality

$$0 \leqslant d(k_1 + 2k_2 + \cdots + ik_i) - k_{i+1}, \tag{B.9}$$

which means that the sum of all jumps of $\|\mu_k\|$ occurring at the moments k_i, should match the number of regular steps when $\|\mu_k\|$ decreases at least by 1.

The growth of the sequence majorized by the inequalities (B.9) can be easily estimated: if we introduce $S_i = k_1 + 2k_2 + \cdots + ik_i$, then (B.9) implies

$$S_{i+1} = S_i + (i+1)k_{i+1} \leqslant S_i + (i+1)dS_i = S_i(1 + d(i+1)),$$

so that $S_n \leqslant (1+d)(1+2d)\cdots(1+nd) \leqslant 2d \cdot 3d \cdots (n+1)d = (n+1)!d^n$. This implies the upper bound claimed in the Proposition. □

Remark. Notice that the established bound is roughly simple exponential in n and polynomial in d, and does not depend on the rules determining the sequence of polynomials p_k, provided only that the latter remain linear in x an of degrees in t growing linearly with k.

Remark. The method can obviously be generalized for chains of ideals spanned by homogeneous forms of any degree in x, that are polynomial in one variable t. It is important that the degrees in all variables but one are bounded. Linear systems constitute an example when such situation occurs naturally.

Remark. One cannot in general majorize the height of polynomials $\lambda_k(t)$ occurring as the coefficients of the decomposition $p_\ell(t,x) = \sum_0^{\ell-1} \lambda_k(t)p_k(t,x)$. Unless the coefficients were integral from the very beginning (i.e. if the matrix polynomial $A(t)$ had all integer coefficients), the procedure of universalization from §2.2 is required. But this procedure increases the number of independent variables, therefore the approach developed above becomes unapplicable. Working instead over the field of rational functions of the parameters $A, \varSigma$, see §B.2, we can construct a linear equation

$$a_0(A,\varSigma)y^{(\ell)} + a_1(t,A,\varSigma)y^{(\ell-1)} + \cdots + a_\ell(t,A,\varSigma)y = 0$$

with the leading coefficient independent of t, all other polynomial in $t, A, \varSigma$ of bounded degrees and heights. The values of the parameters for which the leading coefficient a_0 is vanishing, correspond to singularly perturbed linear equations which require additional considerations for study.

B.4. **Discrete and continuous Risler problem for linear systems.** We conclude this appendix by a simple observation: for linear systems the continuous Risler problem can be reduced to discrete one and vice versa.

Consider a system of nonautonomous linear differential equations $\dot{x} = A(t)x$, polynomially depending on time (so that A is a matrix polynomial of degree d). Let $\Pi = \{p = 0\} \subset \mathbb{R}^n$ be a linear hyperplane: $p(x) = \langle \xi, x \rangle$, where $\xi \in \mathbb{R}^{n*}$. Any solution of this system can be expanded in a convergent Taylor series $x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} x_k$ where $x_k \in \mathbb{R}^n$ are vector coefficients, $k = 0, 1, \dots$. Substituting this expansion into the equation, we obtain recurrent formulas for coefficients $\{x_k\}$ of order d, $x_{k+1} = A_0 x_k + kA_1 x_{k-1} + \cdots + k(k-1)\cdots(k-d+1)A_d x_{k-d}$, that can be easily reduced to a first order linear difference scheme in $\mathbb{R}^{nd}$ with coefficients *polynomially depending* on the number k of the step. After the obvious suspension by the map $t \mapsto t+1$ we see that the Risler problem on how many Taylor coefficients of the function $p(x(t)) = \sum \langle \xi, x_k \rangle\, t^k/k!$ can vanish, gets reduced to the question on how many points of the orbit $\{x_k\}_{k=0}^{\infty}$ of the suspended polynomial map can belong to the (properly suspended) hyperplane $\{\langle \xi, x \rangle = 0\}$.

Remark. The Taylor coefficients of the restriction $p(x(t))$ for linear systems are obtained by iterating the ring homomorphism as in (1.10). If ℓ is the moment of stabilization of the corresponding chain of ideals (1.8), then vanishing of the first ℓ Taylor coefficients $c_0, c_1, \dots, c_{\ell-1}$ of $p(x(t)) = \sum c_k t^k$ implies identical vanishing of the latter series. If the expansion (2.4) is known explicitly, this in principle allows to majorize the magnitude of all coefficients c_k for any $k = \ell, \ell+1, \dots$ via the maximum among the first ℓ of them. Then the methods developed in [2, 26] would allow for some explicit bounds for the nuber of zeros of the restriction.

Remark. For systems of the form (B.2) the recurrent system is simplier than in the general case: the recurrent formulas take the form

$$Cx_{k+1} = (kE - B)x_k, \qquad k = 0, 1, \dots,$$

that are "dual" to the discrete time system (B.2).

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