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On Functions and Curves Defined by Ordinary Differential Equations

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Dedicated to V. I. Arnold on his 60th birthday

Abstract. These notes constitute a substantially extended version of a talk given in the Fields Institute (Toronto) during the semester "Singularities and Geometry", that culminated by *Arnoldfest* in celebration of V. I. Arnold's 60th anniversary.

We give a survey of different results showing how an upper bound for the number of isolated zeros for functions satisfying ordinary differential equations, may be obtained without solving these equations. The main source of applications is the problem on zeros of complete Abelian integrals, one of the favorite subjects discussed on Arnold's seminar in Moscow for over quarter a century.

Data æquatione quotcunque fluentes quantitæ involvente fluxiones invenire et vice versa.

Isaac Newton

It is useful to solve differential equations.

Translation by *Vladimir Arnold*

§1. Introduction

1.1. Equations and solutions. One of the illusions that are pleasant to nourish is the claim that simple equations cannot have complicated solutions. Though completely refuted by the recent progress in the dynamical systems, this principle

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still holds in a more restricted context. For example, a planar real algebraic curve of some known degree d cannot have too many real ovals on the real plane and cannot intersect straight lines by too many (more than d) isolated points. This example can be easily generalized to algebraic varieties of higher dimensions. Thus at least in the context of elementary real or complex algebraic geometry simple descriptions cannot lead to perverse objects.

The requirement of algebraicity is too restrictive, as was relatively recently discovered by A. Khovanskii [Kh]. One can in fact allow all elementary functions (some provisions made for sine and cosine) and their compositions to participate in describing the *real* loci: still the result could not be too complicate. For example, the number of connected components of such a locus can sometimes be explicitly majorized in terms of *computational complexity* of the equations describing the locus.

The situation apparently changes completely when differential equations enter the scene, though not from the very beginning. First order *Pfaffian* differential equations can be incorporated within the fewnomials theory and even constitute the core of this approach: all elementary functions are allowed precisely because they are determined by simple Pfaffian equations. The section §1.4 below contains an ultra-brief synopsis of the *Fewnomials theory* [Kh].

But this optimism cannot last for long. The (second part of the) Hilbert Sixteenth Problem, that asks about the maximal number of limit cycles (periodic solutions) for a planar polynomial vector field of degree n in terms of n, is still a great challenge. Even in the weakest form (the existential finiteness) for the simplest class of vector fields (quadratic) the question remains open despite continuing efforts. Thus the question about the number of zeros (or fixed points, what is the same) of a function (the Poincaré return map) defined by an ordinary differential equation, is highly nontrivial.

Yet somewhere halfway to this *Ulthima Thule* one encounters problems that are substantially more treatable. One such problems, on the order of contact between integral trajectories of polynomial vector fields and algebraic hypersurfaces, was recently solved by A. Gabrielov [G1]. The result and some connections are described in §1.5 below.

1.2. Abelian integrals. Another problem is more closely related to the Hilbert 16th problem: it is sometimes referred to as *weakened, infinitesimal* or *tangential* Hilbert problem. Consider a planar Hamiltonian vector field (that has no limit cycles, as all closed phase curves are nonisolated) and a one-parameter polynomial perturbation of this field. In the Pfaffian form such perturbation can be written as $dH + \varepsilon\omega = 0$, where the Hamiltonian $H = H(x, y)$ is a polynomial in two variables and $\omega = P(x, y)\,dx + Q(x, y)\,dy$ a differential 1-form with polynomial coefficients P, Q. The first variation of the Poincaré return map (with respect to the parameter ε) is given then by the *complete Abelian integral*

$$I(t) = \oint_{\delta_t \subset \{H(x,y)=t\}} P(x, y)\,dx + Q(x, y)\,dy \tag{1.1}$$

over a continuous family of ovals δ_t belonging to the level curves $\{H(x, y) = t\}$.

The general problem is to place an upper bound for the number of *isolated* zeros of Abelian integrals of the form (1.1) in terms of only the degrees $\deg H$ and $\deg \omega = \max(\deg P, \deg Q)$. It admits several particular cases: these of low degrees (usually 3 or 4, in particular the *elliptic integrals* corresponding to $H(x,y) = y^2 + x^3 - 3x$), the case of hyperelliptic functions (when $H(x,y) = y^2 + p(x)$, p being a polynomial in *one* variable), etc. The first results appeared in 1976 for the elliptic case, and it took about 10 years to achieve complete clarity in this particular case. The ultimate result, a sharp bound for the number of zeros, was achieved by G. Petrov around 1990 [Pe].

There are surprisingly few general results, despite the continuous flow of publications on this subject. Besides a long list of low-degree particular cases (already very difficult to study), there is a general existential finiteness theorem proved by A. Varchenko and A. Khovanskii in 1984, and a series of asymptotic estimates valid for a fixed H as $d = \deg \omega$ increases to infinity, that began by a double exponential in d upper bound and ended with a bound of the form $O(d)$. Unfortunately, the best (linear) bound due to Petrov and Khovanskii (1996, unpublished) is purely asymptotical: the constants that occur in the majorant, are only existentially finite. (The previous results due to Novikov, Ilyashenko and the author are somewhat more explicit, but the bound is certainly not uniform over all Hamiltonians H).

Thus the problem of obtaining *explicit* and *constructive* upper bounds in the problem on zeros of Abelian integrals remains as challenging as twenty years ago. The problem may be generalized for perturbations of *integrable* rather than simply Hamiltonian systems, but this question is apparently much more complicated compared even to the unsolved problem on (usual) Abelian integrals.

1.3. Solutions and equations. Apart from several particular results, the study of Abelian integrals is based on investigation of a linear ordinary differential equation with rational coefficients, called *Picard–Fuchs equation*, that is satisfied by these integrals.

In the elliptic case this equation can be derived explicitly, as in [Pe]. But in general the possibility of writing such equation is explained by the nature of analytic continuation of the function $I(t)$ into the complex domain. After such continuation one obtains a multivalued function with ramification points at the critical values of the Hamiltonian. The monodromy group describing transformations of branches after circumventing the ramification points, is completely determined by the topology of foliation of the complex plane $\mathbb{C}^2$ by the (complex) level curves of H. It turns out that the linear space spanned by all branches of I, is at most μ-dimensional, where μ is the number of critical values of H, properly counted with multiplicities, and the monodromy transformations are given by *Picard–Lefschetz formulas* (generically). As all derivatives of $I(t)$ with respect to t have the same monodromy, the space of such functions over the field of rational functions can be at most μ-dimensional, so some linear combinations of these derivatives with rational coefficients must vanish: this is the Picard–Fuchs equation.

Unfortunately, this construction is very implicit. Apart from the order of this equation and the degree of its coefficients, almost nothing can be said. In particular, this equation would probably depend in a rather singular way on the parameters (H and ω).

The hyperelliptic case is exceptional in this sense. Consider n forms $\omega_k = x^k y\,dx$, $k = 0, 1, \ldots, n-1$ and their integrals $I_k(t)$ over the level curves of the hyperelliptic Hamiltonian $H(x, y) = y^2 + p_n(x)$, where $p_n(x) = x^n + \cdots$ is a polynomial of degree n. One may show that restrictions of these forms generate the first cohomology of every fiber $\{H = t\}$ and, moreover, they generate the whole space of hyperelliptic integrals as a $\mathbb{C}[t]$-module. Each of these integrals I_k solves a certain Picard–Fuchs equation by itself, but together they satisfy a *system* of linear first order equations of the form

$$(t + A)\dot{I}(t) = BI(t), \qquad A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \tag{1.2}$$

with constant matrices A, B depending on p_n in a rather regular way. This result can be found in [Gi], where the symplectic structure of the system (1.2) was discovered from topological observations, and in [R], where the explicit description of the matrices was obtained.

In any case, it is clear now that differential equations provide valuable information for the problem on zeros of Abelian integrals. Notice that this approach is in some sense contrary to the general ideology of calculus: starting from "known" (explicitly expressed by quadratures) solution, one tries to reconstruct the corresponding differential equation and then eventually prove something that would be automatically valid for *all* solutions of this equation. Despite its apparent paradoxality, this approach works in certain cases.

1.4. First versus higher order. An elementary function of one or several arguments is an expression built by using arithmetic operations from constants, independent variables, exponents, trigonometric functions and their inverses. The combinatorial complexity of an elementary function is the number of building operations, and it does not depend on the choice of constants. Every elementary function has its natural (perhaps, void) domain of definition (the values of arguments for which all intermediate results in the construction process make sense). One could expect that the number of isolated zeros of an elementary function in its natural domain admits an explicit bound in terms of the combinatorial complexity of this function. A good example is the Descartes rule for the number of (positive) roots of a univariate polynomial in terms of the number of monomials entering with nonzero coefficients. Indeed, the combinatorial complexity of $x^n = \exp(n \ln x)$ is uniformly bounded over all natural n, and what remains is to build the sum of k monomials using $k-1$ additions and k multiplications by constants.

The picture, however, becomes not so simple if one allows for the trigonometric functions. Even the simplest of them, $\sin x$, has the infinite number of zeros in its natural domain. Yet, as one can show, if we define the natural domain of sine and cosine to be the interval $[-\pi, \pi]$ rather than $\mathbb{R}$, the general principle is restored.

The reasons for this exception and the proof of the general statement are explained by the fewnomials theory [Kh]. The basic fact is that the graph of any elementary function can be obtained as an integral surface of a *Pfaffian differential equation* with rational or algebraic coefficients: indeed, if $y = \exp ax$, then $dy - ay\,dx = 0$, and the similar Pfaffian equations can be written for other basic elementary functions. The building step corresponds in writing a *system* of Pfaffian equations after properly introducing new variables. These systems possess certain topological properties excluding spiral-type behavior, and the main technical step consists in eliminating Pfaffian equations replacing them by rational expressions built from the coefficients of the Pfaffian forms [Kh].

The exceptional nature of sine and cosine roots in the fact that they satisfy not the *first* but rather the *second* order differential equation $d^2y/dx^2 + y = 0$ and as such cannot be included into the theory. However, on the restricted domain $(-\pi, \pi)$ the function $\sin x$ satisfies the Pfaffian equation $dy - \sqrt{1-y^2}\,dx = 0$ with a correctly defined branch of the root. This representation already allows to use the "restricted sine" for building purposes.

The power of the fewnomials theory extends far beyond the class of elementary functions: one can allow *any function* whose graph is an integral surface for an algebraic Pfaffian 1-form to occur in the building process. The detailed exposition can be found in [Kh], but the general principle remains the same: *the number of zeros is determined by combinatorial complexity of the determining equation*, and not by the absolute values of constants occurring in the formulas.

Yet there is a drastic difference between equations of the first and higher orders. The simplest example $d^2y/dx^2 + \omega y = 0$ shows that if a function is defined by a higher order differential equation (even linear with constant coefficients), then it may have arbitrarily many zeros on a given (finite) interval, if the magnitude of coefficients is not restricted. The following section §2 of this survey contains a brief exposition of an approach that allows to establish an explicit bound for the number of zeros of *any* solution of a linear ordinary differential equation (with variable coefficients) in terms of the magnitude of the latter, both in the real and complex analytic settings. This bound depends in general on the size of the domain, real or complex, on which the equation is considered.

Within this approach, one has to distinguish (as usual) between nonsingular and singular cases. The function $y(x) = \frac{1}{2}(x^i + x^{-i}) = \operatorname{Re}\exp(i \ln x) = \cos\ln x$ satisfies the linear equation $x^2y'' + xy' + y = 0$ that has a regular (Fuchsian) singularity at $x = 0$, yet possesses an infinite number of zeros on the semiinterval, say, $(0, 1]$. One can show that this phenomenon would be impossible for a Fuchsian singularity with the real spectrum (see §4 for the details).

1.5. Linear and nonlinear. All the said refers to linear equations. Yet one often encounters systems of equations, and these systems may well be nonlinear, say, polynomial. This case is substantially more difficult.

In [G1] A. Gabrielov achieved a striking result, an upper bound for the *order of contact* between an integral trajectory of a polynomial vector field in $\mathbb{R}^n$ and an algebraic hypersurface. The bound, originally double exponential in the dimension n of the space and polynomial in the degree d of the field and the hypersurface, was recently improved by himself to become simple exponential.

The last section §5 contains a brief summary of the technique that allows to reduce the problem on zeros of functions defined by systems of polynomial (or rational) differential equations to that satisfying certain (very) high order linear equations. From the geometric point of view this reduction provides an upper bound for the number of *isolated intersections*, as established recently by D. Novikov and the author. The detailed exposition is given elsewhere [NY1, NY2].

Note that unlike the maximal order of contact that turns out to be of a "fewnomial" nature (the answer depends on the degrees of the right hand sides), the number of intersections between an integral curve of a polynomial vector field and an algebraic hypersurface cannot be majorized in terms of the dimension and degree only, but depends on the coefficients of the polynomial data. The answer is polynomial in the magnitude of the coefficients and the size of the curve, but the power is given by a tower of four exponents with the combination of dimension and degrees on the top floor.

1.6. Functions versus curves. Return for the moment to the case of one linear nth order equation. Its general solution is given by a linear combination of n independent solutions $y_1(x), \dots, y_n(x)$. The corresponding theorem from §2 implies an upper bound for the number of isolated intersections between the (parameterized) curve $x \mapsto (y_1, \dots, y_n)$, $x \in I = (x_0, x_1) \subset \mathbb{R}$, and an arbitrary linear hyperplane in $\mathbb{R}^n$. Conversely, one may start with a curve and ask about the maximal number of isolated intersections between it and an arbitrary linear or affine hyperplane. Then one can reduce this question to that about an appropriate linear equation. Yet the freedom to choose the parametrization and, what is more important, a fundamental system of solutions, leads to an invariant geometric formulation of the result: the *meandering index* of a curve (the maximal number of isolated intersections with affine hyperplanes) can be majorized by a weighted sum of integral Frenet curvatures of the curve and the number of generalized inflection points on it, see §3.

This result can be easily incorporated into the general scheme, if we associate with a curve the Frenet differential equations describing dynamics of its osculating frame. Then the main result of §3 has the following sense: for Frenet equations with bounded "coefficients" (Frenet curvatures) the geometric (shape) complexity of solutions admits an explicit upper bound.

1.7. Alternative Rolle theories. One fragment of construction is invariably present in demonstrations of various theorems about zeros. The classical Rolle theorem asserts that between any two subsequent roots of a differentiable function at least one root of its derivative occurs. This immediately implies the inequality $N_I(f) \leqslant N_I(f') + 1$ between the number of isolated zeros $N_I(f)$ of a function f and its derivative f' on a real interval $I \subset \mathbb{R}$. This inequality may be improved to $N_I(f) \leqslant N_I(f')$ for I-periodic functions, and besides the roots may be considered with multiplicities.

It is well known that no such inequality exists for complex analytic functions (compare $f(x) = \exp i\omega x - 1$ and its derivative on any subset of $\mathbb{C}$ containing a piece of the real axis for large real values of ω). Yet one can replace the number of zeros by certain relevant functionals so that the inequality will be restored in the form close to the original. One such choice is the total (absolute) variation of argument of an analytic function along the boundary of a complex domain. The corresponding inequality is described in §3 together with corollaries.

In addition to changing the functional, one may also change the operator, replacing the derivative in the Rolle inequality by several appropriate differential, difference or monodromy operator. Such modifications lead to meaningful results allowing for further applications.

1.8. The structure of the paper. This paper was based on the lecture notes containing a survey of several recent results about zeros of functions defined by differential equations. However, some of them are given below in a substantially improved form (e.g., Theorem 4.1, cf. [RY] or Theorem 3.7, cf. [NY4]) or with a simplified proof (Lemma 2.2, cf. [Ki]). In such cases we tried to give as complete proofs as was possible without reproducing big parts of preceding articles. Otherwise, the style is sufficiently informal to allow for an easy digestion.

* * *

I would like to conclude this introductory section with a tribute to **Vladimir Igorevich Arnold**. Most of the subjects mentioned above and below, arose many times in discussions on Arnold seminar in Moscow during the last twenty or more years that I remember. It was the stimulating atmosphere of these discussions and an outstanding personality of the leader that made me and perhaps many of us devoted to this branch of Mathematics for life.

§2. Zeros of functions defined by nonsingular linear ordinary differential equations in the real and complex domain

We start with the simplest case of a linear equation

$$\begin{gathered} y^{(n)} + c_1(t)\, y^{(n-1)} + \cdots + c_n(t)\, y = 0, \\ t \in I = [t_0, t_1] \subseteq \mathbb{R},\ \ c_i(t) \in \mathbb{R}, \end{gathered} \tag{2.1}$$

of order n with variable bounded coefficients. The choice of the leading coefficient equal 1, reflects the fact that the equation is *nonsingular* on the interval I. Obviously, for any subinterval of I there always exists a nonzero solution of this equation, that has at least $n-1$ zeros on this subinterval, counted with multiplicities. The equation (2.1) is called *disconjugate* (sometimes also *nonoscillating* or *Chebyshev*) on I, if any nontrivial (not identically zero) solution has at most $n-1$ roots on I.

2.1. Linear equations with real bounded coefficients on the real interval: the paradigm. Assume that the coefficients of the equation (2.1) are explicitly bounded on the interval I,

$$\forall t \in I \quad |c_i(t)| \leqslant C_i < \infty, \qquad i = 1, 2, \dots, n. \tag{2.2}$$

Prototheorem 2.0. *If $\sum_{k=1}^n C_k \ell^k/k! < 1$, then the equation (2.1) is disconjugate on any subinterval of length ℓ inside I.*

Proof. Assume that on the contrary, there exists a solution $y(x)$ having at least n isolated roots, counted with multiplicities.

1. By the (classical) Rolle inequality the derivative $y'(x)$ must have at least $n-1$ root, so at least $n-2$ roots for y'', ... , until the last nth derivative $y^{(n)}$ that should possess at least one root on the considered subinterval. Denote by x_k any one of the roots of the kth derivative $y^{(k)}(x)$.

2. Using the Newton–Leibnitz formula, one can restore the function from its derivative: for any k between 1 and n and any point a,

$$y^{(k-1)}(x) = y^{(k-1)}(a) + \int_a^x y^{(k)}(t)\, dt.$$

The choice of the base point a in each case can be arbitrary, so we put it at x_{k-1}, one of the roots of $y^{(k-1)}$. Then the first term disappears, and majorizing the integral, we conclude with the recurrent inequalities $\|y^{(k-1)}\| \leqslant \ell \cdot \|y^{(k)}\|$ for all $k = 1, \dots, n$, between the sup-norms of the derivatives, resulting in the estimates $\|y^{(n-k)}\| \leqslant \ell^k \cdot \|y^{(n)}\|$.

In fact, the inequality is stronger: writing the expression for $y^{(n-k)}$ as the multiple integral

$$y^{(n-k)}(t_{n-k}) = \int_{x_{n-k}}^{t_{n-k}} dt_{n-k+1} \cdots \int_{x_{n-1}}^{t_{n-1}} dt_n \, y^{(n)}(t_n)$$

we can majorize it by $\|y^{(n)}\|$ times the volume of the k-symplex,

$$\int_0^{\ell} d\tau_k \int_0^{\tau_k} d\tau_{k-1} \int_0^{\tau_{k-1}} d\tau_{k-2} \cdots \int_0^{\tau_2} d\tau_1 = \ell^k / k!,$$

which finally yields an additional factor of $1/k!$:

$$\|y^{(n-k)}\| \leqslant \frac{\ell^k}{k!} \|y^{(n)}\|. \tag{2.3}$$

3. Plugging these estimates into the original equation we notice that, unless $\|y^{(n)}\| = 0$, the leading term is overtaking (in the sense of the sup-norm) the sum of all other terms and hence the equality cannot hold everywhere—a contradiction. The case $y^{(n)} \equiv 0$ is equally impossible, since a polynomial of degree $n - 1$ cannot have n roots, as this was assumed. $\square$

It is convenient to minimize the number of parameters, assuming that all coefficients are bounded by *the same* constant which is greater or equal to 1.

Corollary 2.1. *Let $C = \max(C_1, \dots, C_n, 1)$ be the maximal absolute value allowed for the coefficients of the equation (2.1) on any interval of length ℓ.*

Then any nontrivial solution may have at most

$$(n-1) + \tfrac{1}{\ln 2} n \ell C \tag{2.4}$$

isolated roots on this interval.

Proof. Obviously, our choice of C implies that $C_k \leqslant C^k$, and therefore for any interval of length h the inequality $\sum_1^n C_k h^k / k! \leqslant \exp Ch - 1 < 1$ guarantees that the equation on this interval is disconjugate: resolved with respect to h, this gives $h < h_0(C) = \ln 2 / C$. Subdividing the given interval into $\lfloor \ell / h_0 \rfloor + 1 \leqslant \frac{1}{\ln 2} \ell C + 1$ subintervals of disconjugacy, we establish the required upper bound for the number of roots. $\square$

Remark 1. There exists an alternative way [IY] of obtaining a linear in the magnitude of the coefficients upper bound for the number of isolated zeros for solutions of a linear equation. This alternative approach requires the coefficients of the equation to be analytically extendable onto some complex open neighborhood of the real interval I, and the dimensions of this complex neighborhood explicitly enter into the bound. The idea is to exploit the relation between growth and distribution of zeros of an analytic function. Yet despite the obvious complexity of this approach, it can sometimes produce better bounds for the number of zeros, especially when the order of the equation is highly superior to the total magnitude of the coefficients (something like "fewnomial" linear equations). The reason is rather obvious, as the approach based on subdivision of long intervals into domains of disconjugacy ignores any connection between these domains.

Remark 2. The condition of disconjugacy, instead of the form $\exp Ch - 1 < 1$ that results in the bound (2.4), may well be written in the form $h < \ln(1 + C^{-1})$, also valid for $C < 1$. This gives a better bound $(n-1)+n\ell/\ln(1+C^{-1})$ for the number of zeros, but the expression (2.4) is certainly shorter. As we will be mostly interested in applications of the Prototheorem and Corollary 2.1 to the cases when the coefficients are bounded by very large numbers, these differences become negligible.

Remark 3. The inequality (2.3) can be alternatively obtained as a corollary to known results on polynomial interpolation. Indeed, the uniform accuracy of a n-point interpolation of a C^n-smooth function by a polynomial of degree $n-1$ can be expressed via the sup-norm of the nth derivative of this function. If f has n zeros, then its interpolating polynomial is zero, thus the required estimate can be obtained. This argument was communicated to us by P. Milman and Y. Yomdin.

2.2. Disconjugacy of ordinary linear equations in the complex domain after W. Kim. The above construction does not work for complex zeros of holomorphic functions, mainly because the Rolle theorem is not valid in this context and therefore one cannot guarantee that the derivatives have roots. However, one can bypass this difficulty: an upper bound for the sup-norm of an analytic function f in a convex compact subdomain $D \subset \mathbb{C}$ can be estimated in terms of the norm of its nth derivative in this subdomain, provided that there are sufficiently many zeros of f in D. This approach was taken by W. Kim [Ki], but his proof of Lemma 2.2 refers to advanced results from approximation theory in the complex domain (cf. with Remark 3 above). We give a direct elementary proof, found jointly with D. Novikov.

Lemma 2.2 [Ki]**.** *Let $D \subset \mathbb{C}$ is a convex bounded domain with $\ell = \operatorname{diam} D$, and f a function analytic and bounded in D. If f has n isolated roots in D, then*

$$\|y^{(n-k)}\| \leqslant \|y^{(n)}\| \cdot \ell^k/k!, \quad k = 1, \dots, n,$$

where $\|\cdot\|$ is the sup-norm with respect to D.

We start with a formal identity between differential operators with rational coefficients. Let $a_1, \dots, a_n \in \mathbb{C}$ be any n points and $\partial = \frac{d}{dz}$ the differential operator.

Proposition 2.3.

$$\partial^n = \big((z-a_n)\partial + n\big) \cdots \big((z-a_1)\partial + 1\big) \frac{1}{(z-a_1)\cdots(z-a_n)}.$$

Proof of the Proposition. Both parts of the identity are nth order differential operators with coinciding leading terms. Thus it is sufficient to show that their null spaces coincide. Taken into account that any ratio $p_{n-1}(z)/\prod_j(z-a_j)$ with a polynomial of degree $\leqslant n-1$ in the numerator can be expanded as a linear combination of simple fractions $(z-a_j)^{-1}$, it is sufficient to show that each such fraction belongs to the null space of the composition of the linear differential operators $L_j = (z-a_j)\partial + j$ in the specified order. It obviously follows from the identities

$$((z-a_j)\partial + j)(z-a_k)^{-j} = \begin{cases} 0, & \text{if } j = k, \\ c_{kj}(z-a_k)^{-(j+1)} & \text{otherwise,} \end{cases}$$

so that every fraction $(z-a_k)^{-1}$ is taken by the operators L_j in succession, up to a scalar factor, to $(z-a_k)^{-2}$, then $(z-a_k)^{-3}$ etc., until it is killed by L_k. $\square$

"Inverting" formally the *singular* differential operator $L_k = (x-a_k)\partial + k$, i.e. solving the corresponding linear ordinary differential equation, we immediately derive the following corollary.

Corollary 2.4. *Let I_k be the integral operator, defined as*

$$(I_k f)(z) = \frac{1}{(z-a_k)^k}\int_{a_k}^{z}(t-a_k)^{k-1}f(t)\,dt.$$

Then for f analytic in D the image $I_k f$ is also analytic, and if f has zeros at each point a_k, then

$$f = \prod_{j=1}^{n}(z-a_j)\times(I_1 I_2\cdots I_n)(f^{(n)}). \tag{2.5}$$

Proof of the Corollary. From the construction of I_k it follows that $L_k \circ I_k = \mathrm{id}$. Applying both parts of the above formal identity to the right hand side of the formula (2.5), we see that they coincide and hence (2.5) is valid up to a polynomial of degree $\leqslant n-1$ as an additive term. But the assumption on zeros immediately implies that this polynomial is zero. $\square$

Proof of the Lemma. Assume now that the domain D is convex and all roots a_k are inside D. Then each integration from the definition of I_k can be performed along a straight line segment from a_k to z, and then we have the inequality for the sup-norms

$$\|I_k f\| \leqslant \tfrac{1}{k}\cdot\|f\|, \qquad k=1,\dots,n.$$

Indeed, $|I_k f(z)| \leqslant |(z-a_k)|^{-k}\int_{a_k}^{z}|(t-a_k)|^{k-1}\|f\|\,|dt| = \|f\|\cdot|(z-a_k)|^{-k}\cdot|(z-a_k)|^k/k$. This together with (2.5) implies the bound

$$\|y\| \leqslant \|y^{(n)}\|\cdot\ell^n/n!, \qquad \ell = \operatorname{diam} D.$$

Differentiating (2.5) and applying the same arguments, one can establish the similar inequalities for the derivatives:

$$\|y^{(n-k)}\| \leqslant \|y^{(n)}\|\cdot\ell^k/k!, \quad k=1,\dots,n. \quad \square$$

The inequalities just established can be plugged into the proof of the Prototheorem in the same way as in the real case. The result is the following theorem.

Theorem 2.5 (W. J. Kim [Ki]**).** *Assume that the coefficients of the linear nth order differential equation*

$$w^{(n)} + c_1(z)\,w^{(n-1)} + \cdots + c_n(z)\,w = 0,$$
$$z\in D\subseteq\mathbb{C}, \quad |c_i(z)|\leqslant C_i.$$

are analytic in a convex domain D and the diameter of D is related to the bounds C_j by the inequality

$$\sum_{k=1}^{n} C_k \ell^k / k! < 1,$$

then the equation is disconjugate in D, i.e. any solution of this equation may have at most $n-1$ isolated zeros in this domain. $\square$

In the same way as with the Prototheorem 2.0, one may derive from this complex disconjugacy condition an explicit upper bound for the number of zeros of any solution in any domain where the coefficients of the equation are explicitly bounded: one has to subdivide this domain into smaller domains satisfying the disconjugacy condition. If the diameter of the domain is bounded by $\ln 2/C$, where $C \geqslant 1$ is the upper bound for the coefficients of the equation in this domain, then the equation is disconjugate there.

However, unlike the real case, the number of such domains will grow approximately as C^2, i.e. the *square* of the magnitude of the coefficients, which is contrary to the expected growth rate. The asymptotically accurate bound can be obtained using a properly generalized argument principle.

2.3. Argument principle. In this section we consider a linear ordinary differential equation with *real time* but *complex valued* coefficients (and solutions),

$$\begin{gathered} w^{(n)} + c_1(t)\, w^{(n-1)} + \cdots + c_n(t)\, w = 0, \\ t \in I = [t_0, t_1] \subseteq \mathbb{R}, \;\; c_i(t) \in \mathbb{C}, \end{gathered} \tag{2.6}$$

(the dependent variable is denoted differently to stress that it takes complex rather than real values). For simplicity only we assume that $\operatorname{Re} c_i$ and $\operatorname{Im} c_i$ are real analytic on I.

Generalizing the notion of disconjugacy of real equations, we introduce the following definition.

Definition. A complex-valued equation (2.6) is disconjugate on I, if the variation of argument of any solution $w(t)$ without zeros in I is less than $\pi(n+1)$.

Assume that the coefficients of the equation are bounded by some known constants, exactly as in (2.2).

Theorem 2.6. *If the interval is sufficiently short with respect to the magnitude of coefficients, so that*

$$\sum_{k=1}^{n} C_k \ell^k / k! < \tfrac{1}{2}, \quad \ell = t_1 - t_0, \tag{2.7}$$

then the equation (2.6) is disconjugate on I.

The proof of this theorem is postponed until the next section. Now we turn to corollaries.

Completely analogous to the real case, one can derive from the above disconjugacy condition (2.7) an explicit upper bound for the variation of argument of any solution along any line segment in the complex plane, provided that the moduli of coefficients of the equation are explicitly bounded along this segment. Denoting as before $C = \max(C_1, \dots, C_n, 1)$, we see that the condition $Ch \leqslant \ln \frac{3}{2}$ is sufficient for an equation to be complex disconjugate on an interval of length h. Together with invariance of all input data by rotations $t \mapsto \alpha t$, $|\alpha| = 1$, this immediately implies the following corollary.

Corollary 2.7. *If the coefficients $c_i(t)$ of the equation (2.6) are analytic on a finite line segment $I \subset \mathbb{C}$ of length ℓ and $|c_i(t)| \leqslant C$ for some $C \geqslant 1$, then the variation of argument of any nontrivial solution of this equation is explicitly bounded:*

$$\operatorname{Var} \operatorname{Arg} w(t)|_I \leqslant \pi(n+1)(1 + \tfrac{1}{\ln \frac{3}{2}} \ell C). \quad \square$$

This last claim allows to place effective upper bounds for the number of complex roots of any solution of a linear ordinary differential equation. If the equation has holomorphic coefficients in a polygonal complex domain D and the upper bounds for the absolute value of these coefficients in $\overline{D}$ are explicitly known, then one can apply Corollary 2.7 to each side of the boundary of D and use the classical argument principle.

Example. Let D be a rectangle of perimeter ℓ and the coefficients of the equation are bounded by $C \geqslant 1$ in this rectangle. Then any nontrivial solution of this equation may have at most

$$2(n+1) + \tfrac{1}{\ln(9/4)}(n+1)\ell C$$

isolated zeros in the rectangle. This gives an upper bound that is linear in the diameter of D and the magnitude of the coefficients of the equation. One should be careful, though, when replacing the rectangle by polygonal domains with more sides: the first term in the above sum should be properly increased.

2.4. Petrov argument and proof of Theorem 2.6. The proof of the above theorem is based on the following argument due to German Petrov [Pe], remarkable both for its simplicity and power.

Let $\gamma \subset \mathbb{C}$ be a sufficiently regular (say, polygonal or piecewise real analytic) curve on the complex plane and $f \colon \gamma \to \mathbb{C}$ a complex-valued function on γ; for simplicity we assume that f has no zeros on γ. Then a continuous branch of argument of f can be chosen and the variation of argument $|\operatorname{Var} \operatorname{Arg} f_\gamma|$ well defined independently of the orientation of γ.

Petrov Argument. *Let $N_-(f)$ be the number of zeros of the imaginary part* $\operatorname{Im} f$ *on γ. Then*

$$|\operatorname{Var} \operatorname{Arg} f_\gamma| \leqslant \pi(1 + N_-(f)).$$

Proof. For a function whose imaginary part remains, say, positive, the variation of argument cannot exceed π. Thus between any two points at which the values of argument differ by π, there should be at least one point on γ, where $\operatorname{Im} f$ vanishes. $\square$

Remarks. Obviously, the inequality remains true if N_- is replaced by the number $N_+(f)$ of zeros of $\operatorname{Re} f$ on γ. Besides, one can easily show that this principle remains valid for functions with zeros on γ, provided that they are analytic on γ which was already assumed sufficiently regular.

Notice the resemblance of this inequality with the Rolle theorem: for a closed curve γ the left hand side of the inequality majorizes the number of zeros of f inside the domain bounded by the curve, whereas the right hand side is a certain operator applied to f (an analog of the derivation for the classical Rolle theorem). This resemblance is not complete, since $\operatorname{Im} f$ is not an analytic function even if f were. However, if $\gamma = \bigcup_i \gamma_i$ is the boundary of a polygonal domain D, then $\operatorname{Im} f$ can be obtained as a restriction of an analytic function f_i on each side γ_i of γ, that under certain circumstances can be analytically extended onto the whole of D. This allows to majorize the number of zeros of a function analytic in a polygonal domain and admitting analytic continuation into sufficiently large domains, in terms of the number of zeros of some auxiliary functions that may eventually be simpler than f. This scheme will be fully exploited later: now we use the Petrov Argument to prove Theorem 2.6.

Proof of the Theorem 2.6. Assume that the equation is not disconjugate on the interval I, and the variation of argument of a solution $w(t) = x(t) + iy(t)$ is greater or equal to $\pi(n+1)$. Then the Petrov Argument implies that *both* the real part $x(t)$ and the imaginary part $y(t)$ have at most n real zeros on I. This immediately implies the inequalities (2.3) for the imaginary part $y(t)$ and the same for $x(t)$.

Plugging these estimates into the equation and assuming without loss of generality that $\|x^{(n)}\| \geqslant \|y^{(n)}\|$, we obtain a majorant for the sum of all non-leading terms:

$$\left|\sum_1^n c_k(t) w^{(n-k)}(t)\right| \leqslant \sum_1^n \frac{C_k \ell^k}{k!}(\|x^{(n)}\| + \|y^{(n)}\|) \leqslant 2\|x^{(n)}\| \cdot \sum\nolimits_1^n C_k \ell^k/k!.$$

On the other hand, the leading term $w^{(n)}(t)$ at some point $t \in I$ achieves the absolute value at least as large as $\|x^{(n)}\|$. This contradicts to the inequality (2.7) if $\|x^{(n)}\| \not\equiv 0$. But in the latter case $w(t)$ must be a polynomial of degree $\leqslant n-1$. $\square$

We will once again use the Petrov argument while studying zeros in a neighborhood of a Fuchsian singular point in §4.

2.5. Apparent singularities. Until now all considerations concerned only the *nonsingular* linear differential equations: after normalizing the leading coefficient to 1 the other coefficients were assumed bounded in the domain where the zeros of solutions were counted. But very often in applications one has to consider equations (with analytic coefficients) possessing one or more *singular points*. Since we consider only homogeneous equations whose coefficients can be simultaneously multiplied by any nonzero meromorphic function, without loss of generality one can always assume that the equation is in the form

$$c_0(t)w^{(n)} + c_1(t)w^{(n-1)} + \cdots + c_n(t)w = 0, \tag{2.8}$$

with entire functions $c_0(t), \ldots, c_n(t)$, $c_0 \not\equiv 0$. In fact, we will restrict ourselves to the case when the coefficients $c_k(t)$ are *polynomials* in one complex variable. By normalization one may always achieve the situation when the leading coefficient c_0 is a *unitary* polynomial,

$$c_0(t) = t^\nu + \sum_{j=0}^{\nu-1} c_{0j} t^j, \qquad c_{0j} \in \mathbb{C}. \tag{2.9}$$

The zero locus $\Sigma = \{c_0(t) = 0\}$ is in general a singular locus for solutions of the equation (poles, ramification points or even essential singularities). But some solutions may well extend analytically through this polar set. A simplest example is that of the Euler equation $t(tw')' - 3tw' + 2w = 0$, which has two independent solutions t and t^2. Both are analytic at $t = 0$, and the singularity of the equation is due to the degeneracy of the Wronskian of t and t^2 at this point. Such points are usually called *apparent* singularities. An intermediate possibility is to have some solutions analytic at the singular point, while some others not. The construction outlined below applies to *analytic* solutions of linear equations eventually having singularities in the specified domain $D \subset \mathbb{C}$, that we assume for simplicity polygonal.

The argument principle in the form of Theorem 2.6 cannot be directly applied to the boundary of D, since some of the roots of c_0 may be on or near this boundary. However, it is known that for any positive $h > 0$ one can construct a system of circular disks in $\mathbb{C}$, having the sum of diameters h and such that outside their union the unitary polynomial c_0 admits a *lower* bound: $|c_0(t)| \geqslant (h/4e)^\nu$ (the Cartan inequality, see [IY]). Now one can replace each side of the boundary of D by its parallel translate (in the inwards direction) by no more than h and on the boundary of this reduced polygonal domain one can explicitly majorize the ratios c_i/c_0, which is sufficient to apply Theorem 2.6. Thus we see that if the domain D is allowed to be slightly decreased (or if the bounds for the coefficients c_i are known in a larger domain $\widetilde{D}$ containing an h-neighborhood of D), then the apparent singularities do not create additional difficulties.

The real problems occur when solutions are truly singular: this subject is discussed in §4.

§3. Geometry of spatial curves

3.1. Factorization of differential operators. All previous results on zeros related to differential equations of the form $Ly = 0$, where $L = \sum_{k=0}^{n} c_k(t)\partial^{n-k}$, $c_0 \equiv 1$, is a differential operator in the "expanded" form. Yet, in full analogy with the algebraic case, the "factored" operators are much more easy to analyze. Of course, to "factor" a differential operator, one needs to know its "roots". The corresponding decomposition formula is probably due to Frobenius, but certainly appears in Pólya's paper [Po]. Let

$$L = \partial^n + c_1(t)\partial^{n-1} + \cdots + c_{n-1}(t)\partial + c_n(t)$$

be a differential operator, $\partial = \frac{d}{dt}$, and $f_1(t), \dots, f_n(t)$ its fundamental system of solutions on the interval I, written in any order.

We construct a sequence of Wronskians $W_0(t), \dots, W_n(t)$, starting from $W_0 \equiv 1$, $W_1 = f_1$, $W_2 = f_1 f_2' - f_1' f_2$ etc, W_k being the Wronski determinant of the first k functions.

Lemma 3.1 [Po], see also [NY3].

$$L = \frac{W_n}{W_{n-1}} \cdot \partial \cdot \frac{W_{n-1}^2}{W_n W_{n-2}} \cdot \partial \cdot \frac{W_{n-2}^2}{W_{n-1} W_{n-3}} \cdot \partial \cdots \partial \cdot \frac{W_1^2}{W_2 W_0} \cdot \partial \cdot \frac{W_0}{W_1} \tag{3.1}$$

A short elementary proof of this result can be found in [NY3]. It is this form of a differential operator, in which the disconjugacy conditions become absolutely transparent.

Theorem 3.2 [Po]. *If all Wronskians are nonvanishing on the interval I, then the equation $Ly = 0$ is disconjugate on it.*

Proof. Let f be a solution. Then, integrating, we obtain $\frac{W_{n-1}^2}{W_n W_{n-2}} \cdot \partial \cdot \frac{W_{n-2}^2}{W_{n-1} W_{n-3}} \cdot \partial \cdots \partial \cdot \frac{W_1^2}{W_2 W_0} \cdot \partial \cdot \frac{W_0}{W_1} f$ is a constant that without loss of generality may be assumed nonzero (if not, then one or several more integrations would be required).

From the assumptions on W_i it follows that each multiplication by the cross-ratio of the Wronskians does not change the number of zeros, while each differentiation may reduce it at most by one (by the usual Rolle theorem). As the right hand side is nonvanishing, the number of zeros of f can be at most $n-1$. $\square$

In fact, in [Po] it is shown that the assumptions on the Wronskians are almost necessary in the following sense: if the equation is disconjugate, then one can construct a fundamental system of solutions that would produce Wronskians without zeros inside I.

If some of the Wronskians W_i have zeros on I, then the disconjugacy may not hold. Yet, knowing $\nu_i \geqslant 0$, the number of zeros of each W_i, one may produce an upper bound for the number of zeros of any solution on I. The most straightforward way would be to subdivide the interval I by $N = \nu_1 + \cdots + \nu_n$ points into $N+1$ intervals, each of them by construction being an interval of disconjugacy. As each of the division points itself may be a root of multiplicity at most $n-1+\delta$, with the sum of all δ's being ν_n, we obtain the following upper bound.

Proposition 3.3. *The number of roots of any nonzero solution can be at most $(2N+1)(n-1)$, where $N = \sum_1^n \nu_k$ is the total number of roots of all Wronskians.* $\square$

A more detailed analysis carried out in [NY4] allows to write an upper bound for the number of zeros in the form of a weighted combination of ν_k with the weights ranging from 1 to $n-1$ (and not all equal to $2n-2$, as in the above formula). Yet even this improved bound is too excessive. Under the additional assumption of real analyticity one can prove the following much finer bound.

Theorem 3.4. *Suppose that the coefficients (or solutions) of the equation are real analytic and, as before, ν_k is the number of isolated zeros of the corresponding Wronskian W_k, counted with multiplicities.*

Then the number of zeros of any nonzero solution does not exceed

$$(n-1) + 4(\nu_1 + \cdots + \nu_{n-2}) + 3\nu_{n-1} + \nu_n.$$

In other words, the bound can be made as sharp as $(n-1)+4N$, using the notation above. The proof of this result announced previously in [NaY] is outlined in the subsequent sections.

Remark. Appearance of differential equations in this context is essential but transitory: the results can be formulated as bounds for the number of zeros in linear envelopes. Let $f_1, \ldots, f_n$ be a tuple of real analytic functions, how many isolated zeros may exhibit their arbitrary linear combination $\lambda_1 f_1 + \cdots + \lambda_n f_n$? The answer is given in terms of the number of isolated zeros of the Wronskians built from f_j.

Notice that the equation satisfied by all f_j may well be singular on I, as we do not assume that the last Wronskian W_n is nonvanishing (the assumption required is $W_n \not\equiv 0$, but this can be always satisfied after eliminating linear dependencies between the initial functions). But all these singularities are obviously apparent ones: this is a typical example of how apparent singularities may occur.

3.2. Voorhoeve index, Rolle–Voorhoeve inequality and demonstration of Theorem 3.4. To prove Theorem 3.4, we will replace the number of zeros by another, suitably chosen, index that has similar behavior under multiplications and derivations. This index was introduced by M. Voorhoeve [V] for a special case of real analytic functions (the case we actually need here), and later generalized in [KY] for spatial curves. The general construction is more easy to explain.

Let $\gamma\colon [0,1] \to \mathbb{R}^n$, $t \mapsto x(t)$, be a closed smooth parameterized curve in space, avoiding the origin (i.e. $x(t) \neq 0$ for all $t \in [0,1]$ and $x(0) = x(1)$). Denote by $S(\gamma)$ the (Euclidean) length of the central projection of γ on the unit sphere $t \mapsto x(t)/\|x(t)\|$. As γ is nonsingular, the velocity vector $\|\dot{x}(t)\|$ never vanishes, and hence the velocity curve $\dot{\gamma}\colon t \mapsto \dot{x}(t)$ is also closed and avoids the origin. Let $S(\dot{\gamma})$ be the length of its central projection on the same sphere.

Theorem 3.5 (Rolle theorem for closed spatial curves).

$$S(\gamma) \leqslant S(\dot{\gamma}). \tag{3.2}$$

There are known several proofs for this fact: two of them are given in [KY], some other were communicated to us by Yu. G. Reshetnyak, see also [Re]. Probably the most instructive one, explaining the connections with the standard Rolle theorem, is based on the Buffon needle principle from integral geometry: the spherical length of a curve is proportional to the number of intersections between this curve and a random hyperplane passing through the origin. As this holds equally for the curve $\dot{\gamma}$, we have to compare the number of isolated intersections of γ and $\dot{\gamma}$ with all hyperplanes. But this is already the standard Rolle inequality, as between any two consecutive points of intersection of γ with an arbitrary hyperplane, there should be a point at which the velocity vector is parallel to that hyperplane. Thus the inequality (3.2) is the claim that the number of zeros of a periodic function on the period is no greater than the number of zeros of its derivative, in an averaged form.

Remark. The inequality (3.2) has a counterpart valid for non-closed curves [KY]. The right hand side should be replaced then by $S(\dot{\gamma})+\varPhi_1-\varPhi_0$, where $\varPhi_i$ is the angle between $x(t)$ and $\dot{x}(t)$ for $t = 0$ and $t = 1$ respectively. Note that the difference $\varPhi_1 - \varPhi_0$ never exceeds π.

As the case $\mathbb{R}^2 \simeq \mathbb{C}$ is not excluded, the inequality (3.2) can be applied to planar curves, in particular, to curves having the form $f(\partial D)$, where ∂D is the boundary of a planar domain D and f a function meromorphic in D without zeros or poles on ∂D. The result can be expressed in terms of *absolute variation of argument* of analytic functions. Since f has no zeros on the boundary ∂D, the branch of $\varphi(t) = \operatorname{Arg} f(t)$ can be selected for a suitably chosen parametrization of the boundary by, say, a segment $[0,1]$. We define the *Voorhoeve index*

$$V_D(f) = \int_0^1 |\dot{\varphi}(t)|\,dt.$$

Notice that $V_D(f)$ is a majorant for the topological index of f on the boundary. The inequality (3.2) for planar curves implies the following generalization of the Rolle theorem for any function $f = f(z)$ meromorphic in D and its derivative $f'(z)$.

Theorem 3.6 [KY] (Rolle–Voorhoeve inequality).

$$V_D(f) \leqslant V_D(f') + \varkappa(\partial D), \tag{3.3}$$

where $\varkappa(\partial D)$ is the integral curvature of the boundary (equal to 2π for convex domains).

Together with (3.3) we have the triangle inequality that can be easily verified for any two meromorphic functions,

$$V_D(fg) \leqslant V_D(f) + V_D(g). \tag{3.4}$$

If f is a *real* meromorphic function on some segment $I \subset \mathbb{R}$, then the Voorhoeve index $V_D(f)$ can be easily computed for an infinitesimally small neighborhood of I: if D_ε is a convex ε-neighborhood of I, then, assuming that f is neither zero nor infinity at the endpoints of I, we have

$$V_I(f) := \lim V_{D_\varepsilon}(f) = 2\pi(N_I(f) + P_I(f)),$$

where $N_I(f)$ and $P_I(f)$ are the respective numbers of zeros and poles of f on I, counted with their multiplicities.

Now the inequalities (3.3), (3.4) (in their limit form for D_ε, as in [V]) can be applied to all solutions of (3.1) in the same way as the standard Rolle theorem was applied. Each differentiation reduces the Voorhoeve index by 2π, each multiplication (except for the last one) by $2\pi(\nu_{k-1} + 2\nu_k + \nu_{k+1})$, and adding these together we obtain an upper bound for the Voorhoeve index of any solution f. As $V(f) \geqslant 2\pi N(f)$, this implies the upper bound for the number of zeros as claimed.

Remark 1. The Rolle–Voorhoeve inequality (3.3) in an example in the spirit of an "alternative" Rolle theory, in which the key role is played by the argument of an analytic function. One might expect existence of a "dual" theory based on properties of the modulus of analytic functions. Such theory can be indeed developed, see, e.g. [NY5] and [RYo], and the key notion of this theory is that of Bernstein classes. Yet the two theories are essentially isomorphic, as shows the last section of [KY].

Remark 2. For a linear differential equation with constant (complex) coefficients the decomposition (3.1) takes especially simple form, since all intermediate cross-ratios have the form $\exp(\lambda_j t - \lambda_{j+1} t)$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues of the corresponding differential operator (arbitrarily ordered). Computation of the Voorhoeve index of an exponential function is an easy exercise. Applying the "alternative Rolle inequalities" (3.3)–(3.4), one can produce an explicit upper bound for the number of complex zeros of *quasipolynomials* $\sum_{\lambda \in \Lambda} c_{k\lambda} t^k \exp \lambda t$ in a bounded domain $D \Subset \mathbb{C}$ in terms of the spectrum $\Lambda \subset \mathbb{C}$ and the diameter of D [KY].

3.3. Oscillation of spatial curves. A geometric reformulation of the question about zeros of arbitrary linear combinations of several given functions, is that about meandering of spatial curves. Any collection of real analytic functions $f_1(t), \dots, f_n(t)$ of one variable defines a spatial parameterized curve $\Gamma\colon t \mapsto f(t) = (f_1(t), \dots, f_n(t))$. The maximal number of isolated intersections between this curve and an arbitrary *affine* hyperplane is the natural measure of *meandering*, or sinuosity $\Omega(\Gamma)$ of the curve Γ. (The choice of affine rather than linear hyperplanes reflects the natural desire to have this meandering characteristics be translation invariant).

The construction exposed in §3.2 gives an explicit bound for $\Omega(\Gamma)$ in terms of the coordinate functions f_k and their derivatives. However, this answer is not geometric: among other things, it depends on the choice of the coordinates in the ambient space $\mathbb{R}^n$. It turns out that after a proper *averaging* of this answer over all coordinate systems, the resulting expression has a clear geometric meaning of the *weighted sum of integral Frenet curvatures*.

To explain the answer, recall that if the curve is not hyperplanar (what can be always assumed without loss of generality), then the successive vector derivatives $\dot f(t), \ddot f(t), \dots, f^{(n)}(t)$ are linear independent at almost any point of Γ. Together they constitute the *osculating frame* of the curve, that can be orthogonalized in such a way that the span of the first k vectors $e_1(t), \dots, e_k(t)$ of this orthonormal *Frenet frame* coincides with the span of the first k derivatives of f. Besides, we may always assume the parameter be natural (the arclength along the curve). Then the evolution of the Frenet frame can be described by a nonautonomous system of linear equations, known as *Frenet formulas*:

$$\dot e_i(t) = \sum_{j=1}^{n} A_{ij}(t) e_j(t), \qquad i = 1, \dots, n.$$

As the frame remains all the time orthonormal, the matrix $A(t) = \|A_{ij}(t)\|$ should be antisymmetric. Besides, each vector $\dot e_i(t)$ should belong to the space spanned by $e_1(t), \dots, e_{i+1}(t)$. This leaves only one possibility $A_{i,i+1} = -A_{i+1,i} = \varkappa_i$, $i = 1, \dots, n-1$, all other entries being zeros. The values $\varkappa_i = \varkappa_i(t)$ are usually refereed to as (generalized, for $n > 3$) *Frenet curvatures*: for three-dimensional curves $\varkappa_1$ is the usual curvature and $\varkappa_2$ the torsion. Using the freedom of the orthonormalization, one may always assume that all these curvatures, except for the last one $\varkappa_{n-1}$, are positive almost everywhere, while the last one, $\varkappa_{n-1}(t)$ will generically change sign at isolated points of Γ. Such points are natural multidimensional analogues of the inflection points for flat curves: as there is no common name, we use the term "hyperinflection points" for them. Obviously, the point is a hyperinflection point if and only if the osculating frame degenerates. This condition can be rewritten in the form

$$W(1, f_1, \dots, f_n)(t) = 0,$$

where $W(\cdot)$ is the Wronskian of $n+1$ function $f_0 = 1, f_1, \dots, f_n$.

Now the main result concerning oscillation of spatial curves can be formulated. Let $K_j(\Gamma)$ be the integral Frenet curvatures,

$$K_j = \int_0^\ell |\varkappa_j(t)|\, dt, \qquad j = 1, \dots, n-1,$$

(recall that t is the natural parameter), and let $\nu(\Gamma)$ be the number of hyperinflection points of Γ.

Theorem 3.7 [NaY]**.**

$$\Omega(\Gamma) \leqslant (n-1) + \nu(\Gamma) + 4\pi^{-1} \sum_{j=1}^{n-1} K_j(\Gamma).$$

The proof is based on the following generalization of the Fáry theorem [F]. Consider a curve $\Gamma \subset \mathbb{R}^n$ and let Π be an orthogonal projection of $\mathbb{R}^n$ onto some

k-dimensional subspace. Then $\Pi(\Gamma)$ will be a k-dimensional curve, and all elements of the above construction could be repeated relative to the k-plane containing Γ. In particular, the hyperinflection points can be identified with zeros of the corresponding determinant: as one can easily see, if an orthogonal coordinate system in $\mathbb{R}^n$ is chosen Π is the projection on the first k axes, then (preserving the notation above) the hyperinflection points are zeros of the Wronskian $W(1, f_1, \dots, f_k)(t)$.

The collection of all k-planes in $\mathbb{R}^n$ (the Grassmanian) carries the natural measure invariant by the action of the orthogonal group. One may average different characteristics of $\Pi(\Gamma)$ with respect to this measure. The following result generalizes theorems by J. Milnor [Mi] (for $n = 3$, $k = 2$) and I. Fáry [F] (for $n = 3$, $k = 1$, see also the remark below).

Lemma 3.8 [NaY], see also [NY4]. *The average number of hyperinflections of k-dimensional projections of a real analytic curve $\Gamma \subset \mathbb{R}^n$ is $\pi^{-1}K_k(\Gamma)$.*

Using this principle, one can easily derive Theorem 3.7 from Theorem 3.4. The idea is to find the "best" coordinate system, or rather the flag of subspaces. Lemma 3.8 implies that for a given curve Γ one can always find a coordinate subspace in such a way that the number of hyperinflections of the projection of Γ on that subspace is majorized in terms of the integral curvature $K_k(\Gamma)$. This choice of subspaces can be made coherently, as in [NY4], and as a result we prove the existence of the flag of subspaces,

$$\{0\} = \Lambda^0 \subset \Lambda^1 \subset \cdots \subset \Lambda^{n-1} \subset \Lambda^n = \mathbb{R}^n$$

in such a way that the number of hyperinflections of the projection of Γ onto Λ^k is majorized by the kth integral curvature for $k < n$. This gives the upper bounds for the number of zeros of all Wronskians $W(1, f_1, \dots, f_k)$. It remains only to refer to Theorem 3.4.

Remark. As an easy corollary to Lemma 3.8, we conclude that averaging the kth integral curvature over all uniformly distributed ℓ-dimensional projections of a curve in $\mathbb{R}^n$ for any $\ell \geqslant k$, we obtain the kth integral curvature of the original curve. This precisely coincides with the assertion of Fáry theorem.

3.4. Variations and ramifications. Theorem 3.7 proved above, is a full geometric counterpart to theorems of §2 for linear nth order equations. Indeed, the Frenet equations can be considered as a (nonautonomous linear) system of equations on the orthogonal group. The sum of integral curvatures of a curve that appears in the majorant is simply an integral L^1-norm of the matrix of coefficients of this system. It remains only to notice that, despite its appearance, the upper bound for the linear equations is also given in terms of the integral L^1-, rather than L^∞-norm of the coefficients. Indeed, let $C(t) \geqslant 1$ be *variable* maximum of 1 and $c_j(t)$ for the equation (2.1). Then the upper bound for the number of zeros on an interval $I \subset \mathbb{R}$ is $n - 1$ times the number of rectangles of area $< \ln 2$, covering the subgraph of the function $C(t)$ on I. The latter is obviously an approximation to the integral of $C(t)$, and the lost similarity is thus restored.

To complete the analogy, it remains only to notice that hyperinflection points correspond to (apparent) singularities of a linear equation and as such, are not allowed for the regular case (2.1). Thus the term ν_{n-1} has no analogues in the bounds of §2.

In fact, one can show that if the integral curvatures of Γ are all sufficiently small and there are no hyperinflections, then the curve is nonoscillating in the obvious sense: no affine hyperplane can intersect it by more than n isolated points. The bounds are explicit and can be derived, see [NY4], from the topological result of B. Shapiro [Sh]: if a curve is *not* nonoscillating, then its osculating flag (spanned by the osculating frame) becomes nontransversal to any specified flag at some point of the curve. This latter claim can be also considered as a form of the Rolle theorem for spatial curves. In turn, the nonoscillation conditions thus obtained can be used to produce upper bounds for the meandering (as before, by subdividing the given curve into nonoscillating pieces), but the resulting bound would be much worse than that given by Theorem 3.7.

Theorem 3.7 can be generalized in several directions. First, it can be reformulated for spherical or projective curves, where the curvatures are to be understood as *geodesic curvatures*, and a new term (the geodesic length) appears in the majorizing expression [NY4]. Second, for curves in the Euclidean space one may introduce rotation around subspaces of codimension > 1, as the spherical length of the orthogonal projection of the curve parallel to the subspace. Then one can show [NY4] that rotation around any k-dimensional affine subspace is majorized by a weighted sum of the first $k+1$ integral curvatures. Thus Theorem 3.5 (the Rolle inequality for closed curves) becomes a particular case of this more general statement, as $S(\dot\gamma) = K_1(\gamma)$. Another particular case is due to J. Milnor [Mi] who majorized the linking number of a closed curve in $\mathbb{R}^3$ with an arbitrary line (obviously, the above introduced rotation majorizes the linking in the same way as the variation of argument majorizes the topological index in §3.2). Finally, one may formulate a complex analog of Theorem 3.7: given a compact piece of a holomorphic curve in $\mathbb{C}^n$ with the real boundary γ, one asks about the maximal number of isolated intersections with *complex* affine hyperplanes. The upper bound may be given by a sum of certain complex counterparts of Frenet integral curvatures of the boundary γ, see [NaY].

§4. Singular points and singular perturbations

Everywhere in this section we consider linear differential equations with meromorphic (in particular, rational) coefficients and complex time, near a singular point (supposed to be at $t = 0$). According to the general classification, one distinguishes between tame (Fuchsian, or regular) singularities and wild (irregular, non-Fuchsian) ones. In the latter case not too much can be said: infinite number of zeros of a nontrivial solution can accumulate to the singularity, and the best thing one can hope is to study the accumulation rate. On the contrary, the Fuchsian case can be sometimes dealt with.

4.1. Fuchsian equations, monodromy, spectrum. We consider now the case of an equation possessing a "true" isolated singular point at the origin. Assuming this singularity be of the Fuchsian type, we can (see [I]) write the equation in the form

$$z^n w^{(n)} + a_1(z)\, z^{n-1} w^{(n-1)} + \cdots + a_{n-1}(z)\, z w' + a_n(z)\, w = 0. \qquad (4.1)$$

Yet the natural time for such equations is the logarithmic one $z = \ln t$ so that $z\frac{d}{dz} = \frac{d}{dt}$. In this chart the equation takes the standard form (2.1) with coefficients $c_i(t)$ analytic, *bounded* and $2\pi i$*-periodic* in the left half-plane $\mathbb{H} = \{\operatorname{Re} t \leqslant 0\} \subset \mathbb{C}$. The new coefficients $c_i(t)$ can be easily recomputed into the old ones $a_j(z)$ and vice versa.

Solutions of the equation (4.1) are in general multivalued, so their zeros should be counted on different branches separately. It will be technically convenient to consider the domain covered by the infinite semistrip $\Pi = \{\operatorname{Re} t \leqslant 0, |\operatorname{Im} t| \leqslant 2\pi\}$.

The classical monodromy of the equation is the linear operator Δ taking a branch of a solution into the result of its analytic continuation along the small loop around the singularity. In the logarithmic chart the monodromy corresponds to the *shift operator* T, taking any $f(t)$ analytic in $\mathbb{H}$, into

$$Tf(z) = f(z + 2\pi i).$$

Restricted on the n-dimensional (complex) space of solutions of a given Fuchsian equation, it can be identified with an $n \times n$-matrix Δ. The spectrum of this matrix can be computed from the coefficients of the equation as follows.

Denote by $L = \partial^n + c_1(t)\partial^{n-1} + \cdots + c_{n-1}(t)\partial + c_n(t)$ the differential operator corresponding to the equation written in the logarithmic chart. The coefficients $c_j(t)$, being $2\pi i$-periodic and bounded in $\mathbb{H}$, have definite limits c_j^* as $\operatorname{Re} t \to -\infty$. We consider the limit operator L^* with constant coefficients,

$$L^* = \partial^n + c_1^*\partial^{n-1} + \cdots + c_{n-1}^*\partial + c_n^*,$$

and let $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ be the spectrum of L^*: $\lambda \in \Lambda \iff L^* \exp \lambda t = 0$ (each eigenvalue is counted with its multiplicity). We refer to Λ as the *spectrum* of the Fuchsian singular point as well.

Frobenius theorem [I]. *Any solution of the Fuchsian equation can be represented in the form*

$$w = \sum_{\lambda \in \Lambda} \sum_{k=0}^{n_\lambda - 1} c_{k\lambda}(t)\, t^k \exp \lambda t,$$

where $n_\lambda \geqslant 1$ *is the multiplicity of each point* $\lambda \in \Lambda$*, and* $c_{k\lambda}(t)$ *are bounded* $2\pi i$*-periodic analytic in* $\mathbb{H}$ *coefficients.*

Corollary. *The spectrum of the monodromy operator consists of the exponentials* $\{\exp 2\pi i\lambda,\ \lambda \in \Lambda\}$*, counted with their multiplicities.*

Note that the above representation in the initial chart z turns into a convergent Dulac series $w(z) = \sum_{\lambda\in\Lambda} \sum_{k=0}^{n_\lambda - 1} b_{k\lambda}(z)\, z^\lambda \ln^k z$.

4.2. Fuchsian singularities with the real spectrum: upper bound for zeros. According to the general principle, one should expect that knowing an upper bound for the coefficients of the Fuchsian equation in the logarithmic chart would be sufficient to produce a uniform upper bound for the number of zeros of solutions, at least in the semistrip Π. Without additional assumptions this expectation is wrong, as shows the simplest case of the Euler equations (the equations having constant coefficients in the logarithmic chart): the equation $y'' + y = 0$ has solutions with an infinite number of zeros in the semistrip.

It turns out, however, that if the spectrum of the singularity belongs completely to the real axis, then one can indeed establish an explicit upper bound for the number of zeros. (Note that in the above counterexample the spectrum consists of two points $\pm i$). For Euler equations this was well known, and one can find relevant estimates in [KY]. The general case of equations with variable coefficients was treated in [RY], where the following result was proved.

Assume that the coefficients of the equation $Ly = 0$ (in the logarithmic chart, as usual) are bounded by a constant $C \geqslant 1$. Then from the previously established result it follows that the variation of argument of any nonzero solution along any interval of length 4π (the width of the semistrip) can be at most

$$B = B(L) = \pi(n+1)(1+\beta C), \qquad \beta = 4\pi/\ln \tfrac{3}{2}.$$

Theorem 4.1 [RY]. *If the spectrum of the Fuchsian singular point is completely real, then the number of zeros of any nontrivial solution in the semistrip* Π *is finite and does not exceed:*

(1) $n(2B+1)$, *if the specified solution is real on the real axis* $\mathbb{R}$,
(2) $(n+1)(2B+1)$, *if the coefficients of the equation are themselves real on* $\mathbb{R}$.

To prove this result, we replace Π by a very large rectangle $\Pi_\varepsilon = \Pi \cap \{\operatorname{Re} t \geqslant -1/\varepsilon\}$ and try to bound the number of roots in Π_ε uniformly over all small $\varepsilon > 0$. The direct application of the argument principle as given by Corollary 2.7 is impossible, since the length of the horizontal sides of Π_ε is very large. The proof is based on a version of the Petrov Argument.

Let T be the shift operator and μ a complex number. Consider the difference operators, introduced in [RY] under the name of *Petrov operators*:

$$P_\mu = \mu^{-1}T - \mu T^{-1}.$$

Clearly, each such operator preserves equations with T-invariant coefficients and hence acts on the space of their solutions.

Denote by $N(f)$ the number of isolated zeros of an analytic function f in the rectangle Π_ε.

Lemma 4.2 (Rolle Lemma for the difference operators). *If f is real on* $\mathbb{R}$ *and* $|\mu| = 1$, *then $P_\mu f$ is also real on* $\mathbb{R}$, *and*

$$N(f) \leqslant 2B + 1 + N(P_\mu f). \tag{4.2}$$

Proof of the Lemma. We apply the argument principle to f. The variation of argument along each vertical side of Π_ε is bounded by B. It remains to bound the variation of argument of f or, what is the same, of $\mu^{-1}f$, as $\mu \neq 0$, along the horizontal sides. By the Petrov Argument, one has to majorize the number of zeros of $\operatorname{Im} \mu^{-1} f$ along each horizontal side. Being real for real t, the function f takes complex conjugate values at symmetric points $t \pm 2\pi i$, that can be expressed as the identity $\overline{Tf} = T^{-1} f$ valid on the real axis. This together with the assumption on μ guaranteeing that $\overline{\mu} = \mu^{-1}$, means that $\operatorname{Im} \mu^{-1} f(t + 2\pi i)$ for real values of t coincides with the function $P_\mu f(t)$ that is obviously another analytic solution of the same linear equation.

Thus the number of zeros of imaginary part of f on each horizontal side of the rectangle is equal to the number of real zeros of $P_\mu f$ which does not exceed $N(P_\mu f)$, as there can be other zeros in Π_ε. Collecting the inequalities, we conclude with the required estimate provided that $P_\mu f \not\equiv 0$.

This last case is obvious: $P_\mu f \equiv 0$ means that $f(t)$ is $4\pi i$-periodic after multiplication by an appropriate exponent $\exp \lambda t$ with real λ. Thus the variations of argument along the two horizontal sides cancel each other. On the other hand, we have in this case $N(P_\mu f) = 0$, as there are no isolated zeros. It remains only to pass to limit $\varepsilon \to 0^+$. $\square$

Remark. The inequality (4.2) relates the number of zeros of an analytic function and its first difference $P_\mu f$ in the same sense as the Rolle inequality does.

4.3. Proof of the Theorem 4.1. Each Petrov operator P_μ is an automorphism of the linear space of solutions of the given equation. Moreover, if μ is an eigenvalue of the monodromy operator Δ of multiplicity ν and S is the null subspace of $(\Delta - \mu E)^\nu$ (the corresponding root subspace for Δ), then P_μ^ν also vanishes on S and leaves all other root subspaces invariant, since

$$(\mu^{-1}\Delta - \mu\Delta^{-1})^\nu = (\Delta^{-1} + \mu^{-1}E)^\nu (\Delta - \mu E)^\nu.$$

Thus the composition $P_{\mu_1}^{\nu_1} \circ \cdots \circ P_{\mu_k}^{\nu_k}$ extended over all points of the spectrum (with multiplicities ν_k as indicated, so that $\sum_k \nu_k = n$) vanishes on all solutions of the differential equation. The inductive application of the Rolle inequality (4.2) proves the Theorem in the first case (when the solution is real): the constant B remains one and the same for all steps.

In the case when only the equation is real (on $\mathbb{R}$) we observe that the real and imaginary parts of any complex solution on any horizontal line $\mathbb{R} + 2\pi i k$, are themselves solutions. Thus applying one more time the Rolle Lemma at the beginning, we reduce the second case of the Theorem to the first one. $\square$

4.4. Variations. The above theorem is the simplest version and can be modified if necessary. One such modification is important for further applications.

Suppose that we are given a multivalued function $f(z)$, which is a solution of some nth order linear equation $Lf = 0$ having a Fuchsian singularity at the origin $z = 0$, but *the equation itself is not known*. This situation is typical in certain cases, e.g. the Picard–Fuchs equation for Abelian integrals is non-constructive. What is known is the sectorial asymptotics of growth of f, i.e. the leftmost point of the (real) spectrum λ, so that

$$\forall \delta > 0 \; |f(z)| = O(|z|^{\lambda+\delta}) \qquad \text{as } z \to 0$$

remaining in any sector. Without loss of generality we may assume that $z = 0$ is the only singularity of f in the unit disk $\{|z| \leqslant 1\}$.

Instead, we suppose that another equation $Df = 0$ for the same function f is known,

$$y^{(N)}(z) + r_1(z) y^{(N-1)}(z) + \cdots + r_N(z) y(z) = 0 \tag{4.3}$$

of order $N \geqslant n$, which has rational coefficients $r_k(z)$ bounded on the unit circle $\{|z|=1\}$:

$$|r_k(z)| \leqslant \tilde{C} \quad \forall k = 1, \dots, N, \qquad C \geqslant 1.$$

No information on the number and type of singular points of the operator D is available, in particular, we do not assume that $z=0$ is a Fuchsian singularity. Yet in combination with the above qualitative data and sectorial asymptotics this information is sufficient to produce an explicit upper bound for the number of isolated roots in sectors.

The result is again easier formulated in the logarithmic chart. The equation (4.3) in the logarithmic chart $t = \ln z$ transforms into an equation with meromorphic (not necessarily rational) coefficients bounded by some constant C on the imaginary axis. The value of C can be easily recomputed from $\tilde{C}$.

Theorem 4.3. *If the spectrum of L at $z=0$ is real and f itself is real on the real axis, then the number of isolated roots of f does not exceed $(B + 4\pi|\lambda| + 1)n$, where B is given by the same formula as before, and λ is the exponent of sectorial growth.*

The proof is much shorter than the formulation. In the demonstration of Theorem 4.1 the variation of argument of f along the right vertical side of Π_ε is estimated as before by B by virtue of the equation. On the contrary, as we do not know the bound for coefficients on the left vertical side $\{\operatorname{Re} t = -\frac{1}{\varepsilon}\}$, the variation of argument of f along this side is majorized by $4\pi|\lambda| + \delta$ for all sufficiently small $\varepsilon > 0, \delta > 0$, since $f(t) = t^\nu \exp \lambda t \cdot (1 + o(1))$ as $\operatorname{Re} t \to -\infty$. $\square$

4.5. Some open problems. At this moment we stop discussing the questions related to linear differential equations of a high order: the next section is devoted to *systems* of first order (eventually nonlinear) polynomial differential equations. Thus it would be appropriate to conclude this section by several open problems. Besides their natural appearance in this context, each of them is motivated by the general problem on zeros of Abelian integrals, the main source of inspiration.

Theorem 4.1 can be applied to a Fuchsian equation in the form (2.8) having the leading coefficient c_0 normalized to a unitary polynomial as in (2.9), in a global way. The resulting claim can be described as follows: if the magnitude of all coefficients $c_{ij} \in \mathbb{R}$ of all polynomials $c_i(t) = \sum_j c_{ij} t^j$ is explicitly bounded by some number $C \geqslant 1$, all singular points $t_0 = +\infty, t_1, \dots, t_\nu$ are Fuchsian with real spectrum and not too close to each other (so that $|t_i - t_j| \geqslant \delta$, $|t_i^{-1}| \geqslant \delta$ for some $\delta > 0$), then one can produce an explicit upper bound for the number of zeros of any branch of any solution. This bound would obviously depend, besides the order of equation and the degrees of polynomials $c_j(t)$, also on C and δ. As $\delta \to 0^+$, the bound explodes to $+\infty$.

Thus one is naturally led to studying *parametric families* of Fuchsian equations, exhibiting confluence of singular points, in an attempt to establish upper bounds for the number of zeros of solutions, that would be *uniform* over the parameter(s). This subject is complicated, as the result of confluence of two Fuchsian singularities is in general a non-Fuchsian one. Probably, no results can be achieved in such general settings, as the first examples show. Yet there is some evidence that for equations with certain monodromy groups, in particular, with monodromy independent of parameters, explicit bounds can be achieved despite all complications.

The indispensable tool of study is the Petrov argument, and the first steps of the analysis reproduce the constructions exposed in [Pe].

Another question becomes quite natural as one replaces individual equations by analytic families. The normalization making the leading coefficient $c_0 = c_0(t, \varepsilon)$ a unitary polynomial $c_0 = t^\nu + \sum_{j=0}^{\nu-1} c_{0j}(\varepsilon)t^j$, is always possible provided that originally $c_0(\cdot, 0) \not\equiv 0$ for the limit value of the parameter. If this last case occurs, then we are in fact dealing with *singular perturbation*: the order of the equation drops down for $\varepsilon = 0$. One could start looking at the simplest example when $c_0(t, \varepsilon) = \varepsilon \in (\mathbb{R}, 0)$.

Once again, the full theory of singularly perturbed equations would be probably too difficult to build. Quite obviously (already at the level of families of equations with constant coefficients) one can have examples of singularly perturbed equations which do not allow any bound on the number of zeros in terms of C (even without singular points). Yet there are certain indications that such solutions (exhibiting too many zeros for small ε) would explode in an attempt to continue them for $\varepsilon = 0$.

More precisely, suppose that we are given a family of ordinary differential operators $L_\varepsilon = \sum_{j=0}^{n} c_j(t, \varepsilon)\partial^{n-j}$ on a real interval $I \Subset \mathbb{R}$ analytically depending on the real parameter $\varepsilon \in (\mathbb{R}, 0)$, and this family is normalized by the requirement that L_0 is a differential operator of some order $k < n$ with a unitary leading coefficient $c_k(t, 0) = t^\nu + \cdots$. Assume that all coefficients are explicitly bounded, $|c_j(t, \varepsilon)| \leqslant C$ in $I \times (\mathbb{R}, 0)$.

We believe that, eventually under reasonable additional assumptions, for any family $y(t, \varepsilon)$ of solutions of the equation $Ly = 0$ analytically depending on ε at $\varepsilon = 0$, one can produce an explicit upper bound for the number of isolated zeros of $y(\cdot, \varepsilon)$ on I, depending on C, I and n and valid uniformly over all values of ε.

For the present, this conjecture is proved for families of equations with constant coefficients. Note that this result is not covered by results from [KY] described in Remark 2, §3.2, since the spectrum Λ_ε of the corresponding operator explodes as $\varepsilon \to 0$.

§5. Meandering of trajectories of polynomial vector fields

Everywhere in §2–§4 only the linear theory was discussed: all equations were linear (eventually, singular). Below we briefly explain how the nonlinear case may be treated. The exposition is based on [NY1,NY2].

5.1. Meandering of trajectories of polynomial vector fields in $\mathbb{R}^n$. From the geometrical point of view instead of higher order scalar equations (linear or not) one should rather consider systems of first order equations, interpreted as vector fields. Then the problem on majorizing the number of zeros is to be replaced by that of majorizing the meandering index as introduced in §3. It would be natural to expect that the integral trajectories of the vector field given by the system of equations

$$\dot{x}_j = v_j(t, x_1, \ldots, x_n), \qquad j = 1, \ldots, n, \tag{5.1}$$

would have the meandering index bounded in terms of the magnitude of the right hand side parts $|v_j(\cdot)|$. As the equations (5.1) are in general nonlinear (so that in particular the trajectories may blow up in finite time), one has to exercise some care in formulating the conjecture. The easiest way would be to choose a box

$$B_R = \{|x_j| \leqslant R,\ |t| \leqslant R\} \subset \mathbb{R}^{n+1} \tag{5.2}$$

and try to prove that any piece of any integral trajectory of the system (5.1) entirely lying inside the box B_R, has the meandering index bounded in terms of R (the size of the box) and $R' = \max\{|v_j(t,x)| : (t,x) \in B_R,\ j = 1,\dots,n\}$.

Unfortunately, no results are known for the problem posed in such generality. Moreover, there are reasons to believe that the bound cannot be given in the above terms, that is, one can construct real analytic functions v_j with $R' = 1$ in such a way that the meandering of orbits of the corresponding system will be arbitrarily large. (What one can prove is that for any given v and R the meandering of *all* orbits inside the box B_R will be uniformly bounded).

Remark. The reason why the results of §3 cannot be directly applied, is very simple: even provided that one can explicitly integrate the system (5.1) and find an n-parametric family of integral curves (which would be already too ambitious to assume), we cannot majorize the integral curvatures uniformly over all curves from the family. Indeed, to compute the kth curvature function $\varkappa_k(t)$ along a solution, we need to take the ratio of two minors formed by derivatives of solutions, and though they are bounded from above, no lower bound (for the denominator) may exist in principle. To get more convinced, consider a family of curves in $\mathbb{R}^3$ (a fibration) containing a straight line as one of the fibers. Though the curvature of nearby curves must be small (by continuity), there is no reason why the torsion should be bounded.

Thus from the very beginning we restrict ourselves to the *polynomial* case, assuming that the functions v_j are polynomials of degree d in $n+1$ variables,

$$\begin{aligned} v_j &\in \mathbb{R}[t, x_1, \dots, x_n]\ \deg v_j = d, \\ v_j(t,x) &= \sum_{k+|\alpha|\leqslant d} v_{jk\alpha}\, t^k x^\alpha, \end{aligned} \tag{5.3}$$

and we assume that the *height*, the maximal absolute value of the coefficients of all these polynomials, is known. For convenience we assume that it is bounded by the same constant R as the size of the box:

$$|v_{jk\alpha}| \leqslant R. \tag{5.4}$$

The question remains the same: give an upper bound for meandering of all trajectories of (5.1) in the box (5.2), knowing the dimension n of the problem, the degree d of the polynomial vector field and assuming that its height is at most R. In fact, one may even go one step further and ask about the maximal possible number of intersections between phase curves of a given vector field and an arbitrary algebraic hypersurfaces of degree $\leqslant d$.

5.2. Polynomial growth. In this section we formulate the first result concerning the above problem.

Prototheorem 5.0. *For any dimension n and degree d there exists a finite number $N = N(n,d)$ with the following property.*

The number of isolated intersections between an integral trajectory of a polynomial vector field (5.1) of height R and an arbitrary algebraic hypersurface of degree d in the box of size R can be at most $(2+R)^N$, uniformly over all such vector fields and all their trajectories in the box.

Of course, one could disentangle in this formulation the size of the box and the height of the polynomials, in the same way as the degrees of the hypersurface and the field should not necessarily be equal. The formulation is aimed at reducing the number of parameters, and the bottom message of it is as follows: dependence of the meandering on all *magnitudes* (the parameters expressed as *real* numbers) is *polynomial*, and the exponent depends only on the algebraic complexity of the input data.

The proof of this (simple) theorem is outlined below.

5.3. Universal equation. First we reduce the problem to that for *one* system and *one* hypersurface. This is a mere change of language. In the expanded notation (5.3) we treat the coefficients $v_{jk\alpha}$ as *new variables* governed by the trivial equations

$$\dot v_{jk\alpha} = 0, \qquad 1 \leqslant j \leqslant n,\ k + |\alpha| \leqslant d.$$

The same procedure is applied to the algebraic surface: writing its equation in the form $\{p(t,x) = 0\}$ and expanding $p(t,x) = \sum_{k+|\alpha|\leqslant d} p_{k\alpha}\, t^k x^\alpha$, we simply add $p_{k\alpha}$ to the list of independent variables in the same way. Notice that without loss of generality we may assume that the height of p is bounded by the same constant R.

Thus all polynomial equations of the same degree become incorporated into one *universal* polynomial equation, and in the same way the algebraic surface also becomes universal. Note that the original restrictions on the magnitude of coefficients are transformed into restrictions on the phase variables, so that we may consider the box B_R of the same size in the new phase space (of much larger dimension).

5.4. Ascending chains of ideals. Returning to the previous notation, we consider the vector field $\dot x = v(t,x)$ and an algebraic hypersurface $\{p(t,x) = 0\}$ in the space of some dimension m with the box B_R of size R at the center of that space.

Let D be the Lie derivation of the ring of polynomials $\mathbb{R}[t,x]$, taking any polynomial into its derivative along the system $\dot x = v(t,x)$. Starting from the polynomial $p = p_0$ defining the hypersurface, iterations of D generate the infinite sequence of polynomials,

$$p_{k+1} = Dp_k, \qquad k = 0, 1, 2, \ldots \tag{5.5}$$

As the ring of polynomials is Noetherian, the chain of ideals generated by the first several polynomials p_k, must eventually stabilize. This means that at a certain moment ℓ one may find a representation

$$p_{\ell+1} = \sum_{k=0}^{\ell} h_k p_k, \qquad h_k \in \mathbb{R}[t,x] \tag{5.6}$$

with some polynomial multipliers h_k.

5.5. Demonstration of the Prototheorem. Being polynomials, the multipliers h_k admit an upper bound on any centered box, polynomial in the size R of the box: there exists $C < \infty$ such that the maximum of all h_k on the box B_R does not exceed $(2+R)^C$.

Consider an arbitrary integral curve γ of the vector field v, parameterized as $t \mapsto x(t)$, and restrict the identity (5.6) on that curve. By construction (5.5), p_k restricted on γ, i.e. the function $p_k(t, x(t))$, is the kth derivative of the function $f(t) = p_0(t, x(t))$, whose zeros exactly correspond to intersections of γ with the algebraic surface $p_0 = 0$. Denote the restrictions of h_k on γ by $a_k(t) = h_k(t, x(t))$. Then the *nonlinear* identity (5.6) is transformed into the "linear nth order differential equation"

$$y^{(\ell+1)} = \sum_{k=0}^{\ell} a_k(t) y^{(k)},$$

whose solution is f. As soon as we consider the curve lying in the box B_R, the coefficients a_k are bounded by $(2+R)^C$, the length of the interval is at most $2R$, and hence any of the results of §2 produces an upper bound for the number of zeros of $f = p_0|_\gamma$, that would be polynomial in R, bounded by $(2+R)^N$ for some N. As the equation v is "universal" (depends only on n and d), so are: the chain (5.5), the number ℓ, the collection $\{h_k\}_{k=0}^{\ell}$, and finally the bounds C and N. This proves our prototheorem. $\square$

Remark. Of course, the miracle of "linearization" is easily explained: for another choice of γ the coefficients a_k will be totally different. What is important that they still admit the same upper bounds for the same boxes.

5.6. Discussion. The above construction proves also the following purely existential claim: for a real analytic vector field the meandering of trajectories contained in any finite box, is uniformly bounded over all such trajectories. This follows from the Noetherianity of the ring of analytic functions on polydisks. Of course, this fact can be alternatively derived from the Gabrielov finiteness principle [G2].

What is much more important is that one can proceed further, trying to find or estimate explicitly the constant N as the function of n, d. For this, it is necessary to determine all elements of the decomposition (5.6), starting from the number ℓ at which the stabilization of the chain of polynomial ideals

$$(p_0) \subset (p_0, p_1) \subset \cdots \subset (p_0, \ldots, p_k) \subset \cdots$$

occurs. But in fact knowing only ℓ would be already sufficient, as we will explain in a moment.

Knowing ℓ one knows an upper bound for the degrees of all polynomials p_k participating in the decomposition (5.6). As follows from the classical theorem of G. Hermann, one can estimate the degrees of the multipliers h_k. Then the multipliers themselves can be found using the method of indeterminate coefficients, i.e. rewriting the identity (5.6) as a system of linear (algebraic) nonhomogeneous equations, knowing apriori that the solution exists.

This is a crucial step: in general, nothing can prevent the heights of the polynomials h_k found in this way, from being very large: without analyzing the structure of the matrix of coefficients of the linear algebraic system, one cannot estimate how ill-posed it might be. As this height enters explicitly into the final answer, this should be avoided.

But our case is different, due to the previous universalization. Indeed, declaring all coefficients of the vector field and the equation of the hypersurface new independent variables, we arrive to the new, universal field and hypersurface, whose coefficients take only value 0 and 1, in other words, our problem is over the ring of polynomials with *integral* coefficients $\mathbb{Z}[t, x]$ rather than $\mathbb{R}[t, x]$. This, in particular, means that all p_k also have integral coefficients, well bounded from above by an explicit expression involving k, n, d. Of course, the polynomials h_k may have non-integral coefficients, but in any case they will be rational with numerators explicitly bounded (say, by the Cramer rule). As the denominator cannot be smaller than 1, this implies an upper bound for the height of all multipliers h_k. This allows to complete the proof effortlessly.

Unfortunately, the problem of determining the last remaining ingredient, the length of the ascending chain of ideals, is by far more difficult. From results of A. Seidenberg [Se] it follows that there exists an algorithm computing ℓ starting from the number of variables n and the known degrees of generators p_k, but the complexity of this algorithm was recently discovered to be "infinite". In particular, as was shown by G. Moreno [Mo], without additional assumptions the length of the chain can be as large as the Ackermann generalized exponential of n, the function growing more rapidly than anything that can be explicitly written.

It is the additional property of p_k being iterated derivatives, that makes the chain of ideals stabilize much faster than in the general case. However, this subject goes too far beyond the scope of this survey. The answer looks as a tower of four exponents: the value $N(n,d)$ can be majorized by a primitive recursive function growing asymptotically at most as the tower of four exponents,

$$N(n,d) \leqslant \exp\exp\exp\exp(3n\ln d + O(1)).$$

The detailed exposition can be found in [NY2].

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