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Counting Real Zeros of Analytic Functions Satisfying Linear Ordinary Differential Equations

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We suggest an explicit procedure to establish upper bounds for the number of real zeros of analytic functions satisfying linear ordinary differential equations with meromorphic coefficients. If the equation

$$a_0(t)\, y^{(\nu)} + a_1(t)\, y^{(\nu-1)} + \cdots + a_\nu(t)\, y = 0$$

has no singular points in a small neighborhood U of a real segment K, all the coefficients $a_j(t)$ have absolute value $\leqslant A$ on U and $a_0(t) \equiv 1$, then any solution of this equation may have no more than $\beta(A+\nu)$ zeros on K, where $\beta = \beta(U, K)$ is a geometric constant depending only on K and U. If the principal coefficient $a_0(t)$ is nonconstant, but its modulus is at least $a > 0$ somewhere on K, then the number of real zeros on K of any solution analytic in U, does not exceed $(A/a + \nu)^\mu$ with some $\mu = \mu(U, K)$.

0. INTRODUCTION: MOTIVATIONS AND RESULTS

0.1. *Fewnomials Theory and Its Implications.* If $f(t)$ is an elementary function built from the functions const, id, log, exp, arctan, arcsin, using algebraic operations and superpositions, then on the natural domain of definition this function admits an effective upper estimate for the number of (real) isolated zeros. This fundamental result belongs to A. Khovanskiĭ, and the estimate can be given in terms of the *combinatorial complexity* of the explicit expression for *f*. In particular, this result explains why an upper

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estimate for the number of isolated zeros of a polynomial may be given not in terms of its degree, but rather through the number of monomials occurring with nonzero coefficients (Descartes rule). This observation is the reason why the theory is called the *fewnomials theory* [K].

Under some additional restrictions the trigonometric functions (sine and cosine) may be also allowed to occur in the expression for f. In this case one should take not the natural domain of f, but rather make it narrower so that the arguments of $\sin(\cdot)$ and $\cos(\cdot)$ would vary over some bounded intervals; the length of those intervals measured in the wavelengths of sine and cosine plays the role of additional combinatorial complexity of the pair (f, I), where I is the domain on which the zeros of f are counted.

In fact, the natural class of functions in the context of the fewnomials theory is the class of *Pfaffian functions*. Without going deeply into the subject (for the full exposition of this theory see the book [K], where also the multivariate case is considered), we introduce the class of simple Pfaffian functions as follows.

DEFINITION. A real analytic function f defined on an interval $I \subseteq \mathbb{R}$ is called *simple Pfaffian function*, if it satisfies on I some first order algebraic differential equation $y'(t) = R(t, y(t))$, where R is a real analytic branch of an algebraic function. The maximal interval I with such properties is called the *natural domain* of the simple Pfaffian function f and denoted by $\operatorname{dom} f$.

The condition on R means that there exists a polynomial in three real variables, $P(t, y, w) \in \mathbb{R}[t, y, w]$, such that $P(t, y, R(t, y)) \equiv 0$, and the analytic branch of $R(t, y)$ may be continued over the curve $\{(t, f(t)) : t \in I\}$.

EXAMPLES. All elementary functions from the first list are simple Pfaffian functions in the above sense. Indeed, if we put

$$\operatorname{dom} \exp(t) = \mathbb{R}, \qquad \operatorname{dom} \log(t) = \mathbb{R}_+,$$
$$\operatorname{dom} \arctan(t) = \mathbb{R}, \qquad \operatorname{dom} \arcsin(t) = (-1, 1),$$

then on those natural domains one has

$$(\exp at)' = a \exp at, \qquad (\log t)' = 1/t,$$
$$(\arctan t)' = \frac{1}{1+t^2}, \qquad (\arcsin t)' = \frac{1}{\sqrt{1-t^2}},$$

(prime stands for the derivative in t). On the other hand, the functions $\sin at$ and $\cos at$ form a natural system of solutions of the *second order* equation $y'' + a^2 y = 0$ and as such are *not simple Pfaffian functions* on $\mathbb{R}$.

However, they may be considered as Pfaffian after a proper restriction of their domain: the function $\sin t$ restricted on the interval $(-\pi, \pi)$ is simple Pfaffian, since it satisfies the algebraic equation of the first order:

$$y' = \sqrt{1-y^2}, \qquad t \in (-\pi, \pi).$$

We summarize this part as follows. There exists an effective procedure for estimating the number of real zeros of functions satisfying first order differential equations, and this procedure applies also to expressions built from such functions using algebraic operations and superpositions. At present, one cannot extend this theory for functions satisfying equations of higher orders. The goal of the present Note is to fill this gap for functions satisfying *linear* ordinary differential equations of any order with *meromorphic* coefficients.

0.2. *Abelian Integrals.* Besides the above exposed intrinsic reasons, there is another challenging problem which can be reduced to investigation of real zeros of solutions of analytic linear differential equations. Consider a polynomial $H(x, y)$ in two real variables, and denote by $\delta(t)$ a real oval (closed connected component) of the nonsingular level curve $H(x, y) = t$. As t varies between two critical values of H, the oval $\delta(t)$ varies continuously and in a certain sense analytically. Thus for any polynomial 1-form $\omega = P(x, y)\,dx + Q(x, y)\,dy$ the value

$$I(t) = \oint_{\delta(t)} \omega$$

is well-defined, and the function $t \mapsto I(t)$ is real analytic on any interval entirely consisting of regular values of H. This function is called the *complete Abelian integral*, and the question about the number of isolated roots of Abelian integrals is of extreme importance for bifurcation theory. The problem is to find an upper estimate in terms of degrees of the polynomials P, Q, H. Up to now, except for some specific cases of H of degree 3 or 4, when the integrals can be expressed through elliptic functions, only the general existence result is known, due to A. Varchenko and A. Khovanskiĭ [V], [K]: *for any polynomials P, Q, H of degrees not exceeding d, the number of isolated zeros of the corresponding Abelian integrals is bounded by a certain constant $N(d) < \infty$.* The proof gives no information about the nature of the function $d \mapsto N(d)$. On the other hand, the Abelian integral $I(t)$ is known to satisfy a linear differential equation with rational coefficients, see [AVG].

Moreover, as this is shown in [Y] using some ideas from [Il], for a generic Hamiltonian one can construct a linear differential equation of order $\nu = (\deg H - 1)^2$ with rational coefficients, depending only on the

Hamiltonian, such that for any choice of the form ω the Abelian integral $I(t)$ can be expressed as

$$I(t)=\sum_{j,\,k=1}^{\nu} r_{jk}(t)\,u_j^{(k-1)},\qquad r_{jk}\in\mathbb{C}(t),$$

where $u_1(t), ..., u_\nu(t)$ constitute a fundamental system of solutions for that equation, and $r_{jk}(t)$ are rational functions of degrees $\leqslant(\deg\omega/\deg H)+O(1)$, where we put $\deg\omega=\max(\deg P,\ \deg Q)$. Clearly, to estimate the number of real zeros of $I(t)$, it is sufficient to consider only *polynomial envelopes*, combinations of the same form with polynomial coefficients $r_{jk}\in\mathbb{C}[t]$. One may easily construct a linear differential equation of order $\leqslant n^2(d+1)$, where d is the upper bound for the degrees of the polynomial coefficients, in such a way that all functions $t^\alpha u_j^{(k-1)}(t)$ will satisfy this equation for all $j, k=1, ..., n$, $\alpha=0,\ 1, ..., d$. It turns out that *if the monodromy group of the tuple of functions $u_1, ..., u_n$ is irreducible, then the coefficients of this equation can be estimated in the form suitable for application of Theorem* 2 *below*. This implies that *for almost all Hamiltonians the Abelian integral $I(t)$ may have no more than* $\exp\exp O(\deg\omega)$ *real zeros on any fixed compact segment free of critical values of H, as* $\deg\omega\to\infty$. However, this subject will be considered separately.

0.3. *Formulation of Results.* Let $K\Subset\mathbb{R}$ be a compact real segment and $U\subset\mathbb{C}$ an open neighborhood of K. Without loss of generality we may assume that U is simply connected and has a sufficiently smooth compact boundary $\Gamma=\bar U\setminus U$, $K\cap\Gamma=\varnothing$.

Consider a function $f(\cdot)\colon U\to\mathbb{C}$ analytic in U, continuous in $\bar U$ and real on K: $f(K)\Subset\mathbb{R}$. Assume that this function satisfies in U the linear ordinary differential equation

$$a_0(t)\ y^{(\nu)}+a_1(t)\ y^{(\nu-1)}+\ \cdots\ +a_{\nu-1}(t)\ y'+a_\nu(t)\ y=0 \tag{0.1}$$

with coefficients analytic in U and continuous in $\bar U$. Denote

$$A=\max_{k=0,\,...,\,\nu}\ \max_{t\in\bar U}|a_k(t)|. \tag{0.2}$$

The first theorem covers the *nonsingular* case, when the equation (0.1) has no singular points in $\bar U$, that is, the leading coefficient $a_0(t)$ has no zeros in $\bar U$. In this case without loss of generality one can assume that $a_0(t)\equiv 1$, since after division of the equation (0.1) by a_0 all coefficients would remain analytic. The definition of A implies then that $A\geqslant 1$.

THEOREM 1. *If the leading coefficient a_0 is identically equal to* 1, *then the number $N_K(f)$ of isolated roots of f on K, counted with their multiplicities,* *does not exceed $\beta(A+\nu)$, where $\beta=\beta(U, K)<\infty$ is some constant depending only on the geometry of the pair (U, K) and ν is the order of the equation* (0.1).

The second result covers the case of equations with singular points in U. In this case the analyticity requirement for f is an independent and crucial condition. We do not assume that $a_0(t)$ has no zeros in $\bar{U}$, but the natural assumption $a_0(t)\not\equiv 0$ guarantees that

$$a=\max_{t\in K}|a_0(t)|>0. \tag{0.3}$$

THEOREM 2. *There exists another constant $\mu=\mu(U, K)$ also depending only on the geometry of the pair (U, K), such that the number of isolated zeros of f in K does not exceed $(A/a+\nu)^{\mu}$.*

The constants β and μ admit explicit upper estimates in terms of the distance from K to the boundary $\Gamma=\partial U$ and the length of the segment K (see below).

0.4. *Remarks.* We want to stress the difference in the nature of estimates obtained in this paper from those peculiar to the fewnomials theory. The latter yields the estimates in terms of integer data, say, the number of algebraic operations and superpositions necessary to construct the expression for f from elementary functions (we exclude the trigonometric functions for simplicity). This means that the size of real parameters, say, the maximal absolute value of coefficients of the polynomial combinations, does not affect the estimates. In a completely different way, for functions satisfying linear differential equations, the upper estimate for the number of zeros is given in terms of the magnitude of coefficients of the equations, see Section 2.3 below.

0.5. *Regular Singularities at the Endpoints.* Theorems 1 and 2 completely cover the case of functions analytic at all points of the compact segment K. Now assume that one of the endpoints of $K=[t_0, t_1]$, say, t_0, is a singular point for a solution $f(t)$. If t_0 is a pole, then multiplying f by an appropriate factor $(t-t_0)^s$ we may restore the analyticity. On the other hand, the substitution $y(t)=(t-t_0)^{-s}z(t)$ transforms the equation (0.1) into an equation which after multiplication by $(t-t_0)^{s+\nu}$ would also have analytic coefficients so that Theorem 2 could be applied; all upper and lower bounds for the coefficients of this new equation can be easily estimated. Thus we see that the case of a pole at the endpoint gives rise to no difficulties.

The case of an essential singularity will not be discussed here. So assume that t_0 is a ramification point. In this case one cannot apply Theorems 1 and 2 directly. However, under the following two additional assumptions, one may estimate the number of isolated zeros of f on $K\setminus\{t_0\}$ (without loss of generality all other points of K may be assumed nonsingular for f).

Recall that a point $t_0 \in \mathbb{C}$ is called a regular singularity for the equation (0.1), if

$$a_j(t) = (t-t_0)^{\nu-j}\,\tilde{a}_j(t), \qquad \tilde{a}_j(\cdot) \text{ are analytic at } t_0 \text{ and } \tilde{a}_0(t_0) \neq 0.$$

The *indicial equation* associated with the regular singularity is the equation

$$c_0\lambda(\lambda-1)\cdots(\lambda-\nu+1) + c_1\lambda(\lambda-1)\cdots(\lambda-\nu+2) + \cdots + c_{\nu-1}\lambda + c_\nu = 0,$$

where $c_j = \tilde{a}_j(t_0)$, see [In].

It can be shown [H] that in a small neighborhood of a regular singular point (without loss of generality we assume that $t_0 = 0$) any solution f of the equation (0.1) can be represented in the form $f(t) = \sum_{k,\lambda} h_{k,\lambda}(t)\, t^\lambda \ln^k t$, where the finite sum is extended over a finite set of λ which coincide mod $\mathbb{Z}$ with the roots of the indicial equation, and the term $t^\lambda \ln^k t$ appears only if the multiplicity of the corresponding root is greater than k. The functions $h_{k,\lambda}$ are analytic at the point $t_0 = 0$.

Moreover, each function $h_{k,\lambda}$ can be shown to satisfy a linear equation similar to (0.1) of the order at most ν^2, see [H]; this equation may be explicitly constructed if the original equation (0.1) for f is explicitly given.

If all exponents λ in this formula are real, the Khovanskiĭ elimination procedure [K] (see also [IY] for a brief summary of this theory in the form suitable for our applications), allows to reduce the problem on the number of real zeros of the function f representable in the above form and real on some interval $(0, r) \subset \mathbb{R}_+$, to estimating the number of real isolated zeros of some auxiliary functions. These auxiliary functions can be explicitly constructed using operations of addition, multiplication and differentiation, from the functions $h_{k,\lambda}$ and the exponents λ.

Evidently, the auxiliary functions are analytic in the domain of analyticity of $h_{k,\lambda}$. On the other hand it is known [In], [P], that functions satisfying linear differential equations with meromorphic coefficients, constitute a differential ring: sums, products and derivatives of such functions again satisfy some linear differential equations; these equations can be explicitly written provided that the equation for the original functions are given, and this procedure is fairly constructive.

But this means that we can apply Theorem 2 and find an upper estimate for the number of zeros of each auxiliary function. Thus at least theoretically one may found an upper estimate for the number of zeros of any function satisfying the equation (0.1) on an interval between two real regular singularities of the latter.

Unfortunately, when performing all this constructions, we are unable to control the magnitude of the coefficients of linear equations arising in the process. Thus it is for the moment impossible to give an explicit estimate in terms of the coefficients of the original equation (0.1) only, though the above arguments prove that there exists an effective algorithm for obtaining such estimate, say, if the coefficients of the original equation were real polynomials with integer coefficients.

1. Three Lemmas on Analytic Functions

We need three results in the spirit of the classical theory of distribution of zeros of analytic functions [L]. The results are given in the form suitable for applications, and complete proofs are supplied.

1.1. *Zeros of Analytic Functions.* Let $U \Subset \mathbb{C}$ be a simply connected domain which possesses the Green function (usually we will assume even greater regularity, say, smoothness of the boundary $\Gamma = \partial U$ of this domain), and $K \Subset U$ a compact subset of U, not necessary connected.

Denote by $z_a(t)\colon U \to \mathbb{C}$ any conformal mapping taking U into the (open) unit disk D and the point $a \in U$ into its center, $z_a(a) = 0$. The modulus $|z_a(t)|$ is uniquely defined by those requirements, and its logarithm after division by 2π yields the Green function $G(t, a)$ of the domain U for the Dirichlet boundary value problem. For any $a \in U$ the image $z_a(K)$ will be a compact subset of the unit disk, its (Euclidean) distance from the boundary of the latter being strictly positive. We introduce the following geometric characteristic of the pair (U, K).

Definition. The *relative diameter* of the compact K with respect to U is the value

$$\rho(U, K) = \max_{a,\, t \in K} |z_a(t)| < 1. \tag{1.1}$$

Together with the relative diameter we introduce two derived quantities $\gamma = \gamma(U, K)$ and $\sigma = \sigma(U, K)$ as follows:

$$\gamma(U, K) = -\frac{1}{\ln \rho}, \qquad \sigma(U, K) = \frac{1+\rho}{1-\rho}, \qquad \rho = \rho(U, K). \tag{1.2}$$

Both γ and σ are positive finite conformal invariants of the pair (U, K).

Remark (O. Schramm). The quantities ρ, γ and σ are related by simple formulas to the *hyperbolic diameter* of the set $z_a(K) \Subset D$: the latter is the diameter of the image $\tilde{K} = z_a(K)$ with respect to the hyperbolic metric in the disk D. Recall that the hyperbolic metric $|dz|_h^2$ is invariant by fractional-linear transformations preserving the disk D, and $|dz|_h = |dz|/(1-|z|^2)$. Hence we have

$$\operatorname{diam}_h \tilde{K} = \frac{1}{2}\ln\frac{1+\rho}{1-\rho} = \frac{1}{2}\ln\sigma(U, K),$$

and ρ, γ can be also expressed through $\operatorname{diam}_h \tilde{K}$ in the similar way.

Let f be a function analytic in U and continuous in $\bar{U}$. Denote by $M(f)$, $m(f)$ the maxima of $|f|$ on $\bar{U}$ and K respectively,

$$M(f) = \max_{t \in \bar{U}} |f(t)|, \qquad m(f) = \max_{t \in K} |f(t)|.$$

By the maximum modulus principle $m(f) \leqslant M(f)$, and the equality holds if and only if the function f is a constant. The ratio $M(f)/m(f) \geqslant 1$ characterizes the growth of the function *f in the gap between U and K*. Clearly, this quantity remains unchanged if we multiply the function f by a nonzero constant.

Denote by $N_K(f)$ the number of isolated zeros of f on K, counted with their multiplicities. The first result is a generalization of the Jensen formula for the number of zeros of an analytic function in a disk.

LEMMA 1. *The number $N_K(f)$ of isolated zeros of the function f analytic on U and continuous on $\bar{U}$ on the compact set $K \Subset U$ admits the following estimate,*

$$N_K(f) \leqslant \gamma \ln\frac{M}{m}, \qquad M = M(f),\ m = m(f),\ \gamma = \gamma(U, K). \tag{1.3}$$

Proof. Let $a \in K$ be the point at which the maximum of $|f(t)|$ on K is achieved, and take a conformal mapping $z_a\colon U \to \mathbb{C}$ of U onto the unit disk; since the assertion of the lemma is conformally invariant, it is sufficient to estimate the number of zeros of the function $f(z)$ in the disk $|z| \leqslant \rho < 1$ provided that

$$\max_{|z| \leqslant 1} |f(z)| \leqslant M, \qquad |f(0)| = m.$$

But for such a function by the Jensen formula [J][1] for any r between ρ and 1

$$\ln m = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r\exp i\vartheta)|\, d\vartheta - \sum_{|z_j| < r} \ln\frac{r}{|z_j|},$$

[1] This formula may be obtained by applying the Green formula $\iint_U (u\,\Delta v - v\,\Delta u)\, d\mu(z) = \int_{\partial U} (u(\partial v/\partial n) - v(\partial u/\partial n)\, dS$ to the pair of functions $u = \ln|z|$, $v = \ln|f(z)|$ in the unit disk U, see [S].

where the summation is extended over all zeros z_j of $f(z)$ in the disk of radius r assuming that the boundary circle of the radius r does not carry zeros of f.

The integral does not exceed $\ln M$ by the maximum principle, hence the above formula implies the inequality

$$\sum_{|z_j|<r} \ln \frac{r}{|z_j|} \leqslant \ln \frac{M}{m}, \tag{1.4}$$

and restricting the summation in (1.4) only for zeros z_j in the disk of radius $\rho < r$, we have for the number of zeros N_ρ in this disk the inequality

$$N_\rho \left(\ln r + \ln \frac{1}{\rho} \right) \leqslant \ln \frac{M}{m} \qquad \text{for almost any } r \in (\rho, 1).$$

Since r can be taken arbitrarily close to 1, we conclude that

$$N_\rho \leqslant \frac{1}{\ln (1/\rho)} \ln \frac{M}{m}. \quad \blacksquare$$

Remarks. The formula (1.3) allows to estimate the number of isolated zeros of *any analytic function* in terms of its growth in the gap $U \setminus K$.

A. J. van den Poorten [vdP] proved the inequality (1.3) in a slightly stronger form for the pair of two concentric disks, noting that the result belongs to the realm of mathematical folklore and referring to M. Waldschmidt and R. Tijdeman; we compute the corresponding estimate in Example 1 below. Y. Yomdin established a formula equivalent to (1.3) for a special kind of domains (a real segment and its ε-neghborhood), and estimated the constant γ for small $\varepsilon > 0$ (see Example 2 below).

1.2. *Properties of the Constant γ: Examples.* The constant $\gamma = \gamma(U, K)$ is a conformal invariant of the pair (U, K). Moreover, it depends monotonously on K and U:

$$U' \supseteq U, \quad K' \subseteq K \;\Rightarrow\; \gamma(U', K') \leqslant \gamma(U, K). \tag{1.5}$$

Those properties allow for explicit estimation of the constant γ in some simple though important cases.

Recall that the *cross ratio* $\mathscr{D}(a, b, c, d) = (a-c)(b-d)(a-b)^{-1}(c-d)^{-1}$ of any four points $a, b, c, d \in \mathbb{C}$ is invariant by any fractional-linear transformation $z \mapsto (q_1 z + q_2)/(q_3 z + q_4)$, $q_i \in \mathbb{C}$.

EXAMPLE 1. Let $U=\{|t|<1\}$ be the unit disk and $K_\varepsilon=\{|t|\leqslant 1-\varepsilon\}$ a smaller concentric disk. Then

$$\gamma(U, K_\varepsilon)=\frac{2}{\varepsilon^2}(1+O(\varepsilon)) \qquad \text{as} \quad \varepsilon\to 0^+. \tag{1.6}$$

Indeed, it is clear that the maximum in the definition of ρ is achieved when the points a and t are diametrally opposite in K, say, $-1+\varepsilon$ and $1-\varepsilon$. Taking the four points $(-1, -1+\varepsilon, 1-\varepsilon, 1)$ into $(-1, 0, x, 1)$ by a fractional-linear transformation, we immediately find from the equation $\mathscr{D}(-1, -1+\varepsilon, 1-\varepsilon, 1)=\mathscr{D}(-1, 0, x, 1)$ that $\rho=x=1-\varepsilon^2/(2-2\varepsilon+\varepsilon^2)=1-\frac{1}{2}\varepsilon^2+O(\varepsilon^3)$.

EXAMPLE 2 (Y. Yomdin). Let now K be a real segment and U its small ε-neighborhood. More precisely, we consider the segment $K=[0, \ln 2]$ and a thin rectangle around it, $U_\varepsilon=\{\operatorname{Re} t\in(-\frac{1}{2}\varepsilon\pi, \ln 2+\frac{1}{2}\varepsilon\pi), |\operatorname{Im} t|<\frac{1}{2}\varepsilon\pi\}$. In order to compute the constant $\gamma(U_\varepsilon, K)$ we make a conformal transformation $t\mapsto z=\exp\varepsilon^{-1}t$. The interior of U_ε will be mapped into the U-shaped domain which in the polar coordinates r, φ on $\mathbb{C}$ is a rectangle, $r\in(\exp(-\pi/2), 2^{1/\varepsilon}\exp(\pi/2))$, $\varphi\in(-\pi/2, \pi/2)$. This domain contains the disk with the interval $(c^{-1}, 2^{1/\varepsilon}c)$, $c=\exp(\pi/2)$, as the diameter, and on the other hand is contained in the right half-plane $\operatorname{Re} z>0$. This immediately yields the two-sided estimate for the constant $\rho_\varepsilon=\rho(K, U_\varepsilon)$ by virtue of the above monotonicity; one needs to solve two following equations with respect to $\rho_\pm$,

$$\mathscr{D}(c^{-1}, 1, 2^{1/\varepsilon}, 2^{1/\varepsilon}c)=\mathscr{D}(-1, 0, \rho_-, 1),$$

$$\mathscr{D}(0, 1, 2^{1/\varepsilon}, +\infty)=\mathscr{D}(-1, 0, \rho_+, 1),$$

and then $\rho_+\leqslant\rho_\varepsilon\leqslant\rho_-$. The straitforward computation shows that the corresponding constant γ_ε has the following asymptotical behavior,

$$\gamma(U_\varepsilon, K)\sim \mathrm{const}_1\cdot\exp(\mathrm{const}_2\,|K|/\varepsilon), \qquad \text{as} \quad \varepsilon\to 0,$$

the constants const_i being universal and $|K|$ the length of the segment.

Remark (O. Schramm). The latter asymptotic representation could be obtained by using more general arguments. From the Köbe one-quarter theorem it follows that for the domain $U\Subset\mathbb{C}$ the usual Euclidean metric $|dt|^2$ and the hyperbolic metric $|dt|_h^2$ inherited from the unit disk, are related by the asymptotical equivalence $|dt|_h\sim \mathrm{const}\cdot|dt|/\operatorname{dist}(t, \partial U)$, where $\operatorname{dist}(\cdot,\cdot)$ is the Euclidean distance. From this observation it easily follows that in Example 2 the hyperbolic diameter of K is $\sim\mathrm{const}\cdot|K|/\varepsilon$, and the corresponding estimate for γ may be obtained using the relation between $\gamma(U, K)$ and $\operatorname{diam}_h \tilde{K}$.

1.3. *Lower Estimates for Analytic Nonvanishing Functions.* In this subsection we establish an estimate of the nature similar to (1.3) for a lower bound of analytic functions without zeros in a domain. Let U and $K \Subset U$ be as before, f a function analytic in U, $M = M(f)$ and $m = m(f)$ the same as above and $\sigma = \sigma(U, K)$ the constant introduced in (1.2).

LEMMA 2. *If f has no zeros in $\bar{U}$, then*

$$\forall t \in K \quad |f(t)| \geqslant M\left(\frac{m}{M}\right)^{\sigma}, \qquad M = M(f),\ m = m(f),\ \sigma = \sigma(U, K). \tag{1.7}$$

Proof. Using the same arguments of conformal invariance of the assertion of the lemma, it is sufficient to establish the estimate (1.7) in the disk of radius $\rho < 1$ for the function $f(z)$ without zeros with

$$|f(0)| = m, \qquad \max_{|z| \leqslant 1} |f(z)| \leqslant M.$$

The logarithm $\ln |f(z)|$ is a harmonic function, so the difference $u(z) = \ln M - \ln |f(z)|$ is also harmonic and positive in the unit disk. Applying the Harnack inequality to $u(\cdot)$,

$$u(z) \leqslant u(0)\,\frac{1 + |z|}{1 - |z|},$$

we immediately conclude with the estimate $\ln |f(z)| \geqslant \ln M - ((1 + \rho)/(1 - \rho))(\ln M - \ln m)$ which is equivalent to (1.7). ∎

1.4. *Cartan Inequality and Lower Estimates for Analytic Functions Away from Their Zeros.* Lemmas 1 and 2 together with the Cartan inequality below, imply that an analytic function can be estimated from below outside a certain small neighborhood of its zeros. Recall that a polynomial in one variable is *unitary*, if its principal coefficient is 1: $p(t) = t^n + \cdots$.

CARTAN INEQUALITY [L]. *For any unitary polynomial p of degree n and any $h > 0$ one may delete from the plane $\mathbb{C} = \{t\}$ no more than n open disks, the sum of their diameters being less than h, in such a way that on the complement the polynomial would admit the lower estimate $|p(t)| \geqslant (h/4e)^n$, where $e = \exp(1)$ is the Euler number.*

Let the pair of sets $K \Subset U$, the function $f: U \to \mathbb{C}$, and the bounds $M = M(f) = \max_{\bar{U}} |f|$, $m = m(f) = \max_K |f|$ be as before.

LEMMA 3. *There exist two finite positive constants* ς, τ *depending only on the geometry of the pair* (U, K) *with the following property. For any positive* $h<1$ *one may construct a finite number of disks* $D_j \subset \mathbb{C}$ *with the sum of diameters less than* h *such that on the complement* $K \setminus \bigcup_j D_j$ *the function* f *admits the lower estimate*

$$\forall t \in K \Big\backslash \bigcup_j D_j \qquad |f(t)| \geqslant m \left(\frac{m}{M}\right)^{\varsigma - \tau \ln h}. \tag{1.8}$$

Proof. Take an intermediate open set V such that $K \Subset V$, $\bar{V} \Subset U$. By Lemma 1, the number n of zeros of f in $\bar{V}$, counted with multiplicities, does not exceed $\gamma \ln(M/m)$, where $\gamma = \gamma(U, \bar{V})$, since $\max_{\bar{V}} |f| \geqslant \max_K |f| = m$. Denote these zeros by $t_1, ..., t_n$ (repetitions allowed) and consider the unitary polynomial p with roots at t_j and the ratio $\tilde{f} = f/p$ which is an analytic function without zeros in $\bar{V}$,

$$p(t) = \prod_{j=1}^{n} (t - t_j), \qquad \tilde{f}(t) = f(t)/p(t) \neq 0 \qquad \text{for} \quad t \in \bar{V}.$$

From the evident inequalities

$$\max_{\bar{V}} |p(t)| \leqslant (\operatorname{diam} \bar{V})^n, \qquad \min_{\partial U} |p(t)| \geqslant (\operatorname{dist}(\bar{V}, \partial U))^n,$$

$$\max_{t \in \bar{V}} |\tilde{f}(t)| \leqslant \max_{t \in \partial U} |\tilde{f}(t)|$$

it follows that

$$\tilde{M} = \max_{t \in \bar{V}} |\tilde{f}(t)| \leqslant M \cdot \lambda_1^n, \qquad \lambda_1^{-1} = \operatorname{dist}(\bar{V}, \partial U),$$

$$\tilde{m} = \max_{t \in K} |\tilde{f}(t)| \geqslant m \cdot \lambda_2^{-n}, \qquad \lambda_2 = \operatorname{diam} \bar{V}.$$

By Lemma 2, the function $\tilde{f}$ admits a lower estimate on $K \Subset V$,

$$\forall t \in K \quad |\tilde{f}(t)| \geqslant \tilde{m} \left(\frac{\tilde{m}}{\tilde{M}}\right)^{\sigma} \geqslant m \left(\frac{m}{M}\right)^{\sigma+\theta}, \quad \sigma = \sigma(V, K), \ \theta \leqslant \gamma \ln(\lambda_1^{\sigma} \lambda_2^{\sigma-1}).$$

Now to estimate f from below, we take n disks around the roots of p as in the Cartan inequality: then outside the disks we have

$$|p(t)| \geqslant \left(\frac{h}{4e}\right)^n \geqslant \left(\frac{h}{4e}\right)^{\gamma \ln(M/m)} \geqslant \left(\frac{m}{M}\right)^{\varkappa},$$

$$\varkappa = \varkappa(h; U, K) \leqslant \gamma(U, \bar{V}) \cdot (\ln 4 - \ln h),$$

where h is the sum of diameters of the disks. Putting the two lower estimates for $\tilde{f}$ and p together, we obtain the required inequality (1.8) with $\zeta = \sigma + \theta + \gamma \ln 4$, $\tau = \gamma$. ∎

Remark. If we take K being a (closed) unit disk, and U an open disk of radius $e = \exp 1$, then in this particular case the estimate (1.8) coincides with the lower estimate from [L], where explicit expressions for ζ and τ are given. We focus here mainly on the dependence of all the estimates (1.3), (1.7), (1.8) on the function f, considering the pair (U, K) as fixed.

2. Equations with Bounded Analytic Coefficients and the Proof of Theorem 1

2.1. *Proof of Theorem* 1. We consider the real segment K, its neighborhood U with a piecewise smooth boundary Γ and a linear equation (0.1) of order ν with the leading coefficient $a_0(t) \equiv 1$; the other coefficients are analytic in U and continuous in $\bar{U}$. Below we denote by $|L|$ the length of a piecewise smooth curve L on the t-plane.

The equation (0.1) is equivalent to the system of first order linear differential equations

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\, \mathbf{y}(t), \qquad \mathbf{y} = (y, y', ..., y^{(\nu-1)}) \in \mathbb{C}^\nu, \tag{2.1}$$

with the analytic $(\nu \times \nu)$-matrix-function $\mathbf{A}(t)$ in the so called *companion form*: $\mathbf{A}(t) = \lfloor a_{ij}(t) \rfloor_{i,j=1}^{\nu}$, $a_{i,i+1} = 1$ for $i = 1, 2, ..., \nu - 1$, $a_{\nu,j} = -a_{\nu-j+1}$, all other elements are zeros.

If we introduce the norm on $\mathbb{C}^\nu$ as $\|\mathbf{y}\| = |y_1| + \cdots + |y_\nu|$, then the associated matrix norm of $\mathbf{A}(t)$ will be bounded for $t \in U$: $\|\mathbf{A}(t)\| < A + 1$, A being defined by (0.2). The equation (2.1) implies then that the result of analytic continuation of any solution $\mathbf{y}(\cdot)$ along any path Γ' on the t-plane, connecting two points t_0 and t_1, satisfies the inequality

$$\|\mathbf{y}(t_1)\| \leqslant \|\mathbf{y}(t_0)\| \cdot \exp(|\Gamma'|(A+1)).$$

Let l_0 be the distance from K to $\Gamma = \partial U$, and l_1 the length of Γ. Take a path $\Gamma_* = \Gamma' + \Gamma$ which connects K with the boundary $\partial U = \Gamma$ and then makes the closed loop along Γ. Denote by $t_0 \in K$ the starting point of this path. Then for the analytic vector-function $\mathbf{f}(t) = (f(t), ..., f^{(\nu-1)}(t))$ constructed from the solution f of the original equation (0.1), we have

$$\max_{t \in \partial U} \|\mathbf{f}(t)\| \leqslant \|\mathbf{f}(t_0)\| \cdot \exp |\Gamma_*|\, (A+1),$$

since any point on the boundary will be reached along the path.

On the other hand, there exists a number k between 0 and $\nu-1$ such that

$$|f^{(k)}(t_0)| \geqslant \frac{1}{\nu} \|\mathbf{f}(t_0)\|.$$

Since for this value of k we have at the same time the trivial inequalities

$$m_k = \max_{t \in K} |f^{(k)}(t)| \geqslant |f^{(k)}(t_0)|, \qquad M_k = \max_{t \in \Gamma} |f^{(k)}(t)| \leqslant \max_{t \in \Gamma} \|\mathbf{f}(t)\|,$$

they together imply that

$$\frac{M_k}{m_k} \leqslant \nu \exp |\Gamma_*| (A+1).$$

By the Lemma 1, this implies that the number $N_K(f^{(k)})$ of real zeros of the kth derivative $f^{(k)}$ on K does not exceed $\gamma(U, K)(|\Gamma_*| (A+1) + \ln \nu)$.

It remains only to remark that by virtue of the classical Rolle lemma, the real function $f(t)$ itself may have no more than $N_K(f^{(k)}) + k$ real zeros. Since $k \leqslant \nu - 1$, we conclude that the number of zeros of f does not exceed $\gamma |\Gamma_*| (A+1) + \gamma \ln \nu + \nu \leqslant \gamma |\Gamma_*| (A+1) + (\gamma+1) \nu \leqslant \beta(A+\nu)$. Thus the estimate asserted by Theorem 1 is established if we put

$$\beta = (\gamma+1)(|\Gamma_*|+1), \qquad |\Gamma_*| = |\Gamma| + \operatorname{dist}(K, \Gamma). \tag{2.2}$$

The constants $|\Gamma_*|$, γ depend only on the pair (U, K), hence $\beta = \beta(U, K)$ also is a geometric constant, though not a conformal invariant anymore: it depends also on the Euclidean distance from K to Γ. ∎

2.2. *Generalization.* The above proof of Theorem 1 does not use the fact that the coefficients a_j, $j = 1, ..., \nu$ of the equation (0.1) are bounded everywhere in U. The arguments exposed above prove in fact a more general statement which will be used in the proof of Theorem 2.

Definition. We say that a closed subset $L \subset U$ *encircles* a subdomain $U' \subset U$, if for any function φ analytic in U, $\max_{t \in U'} |\varphi(t)| \leqslant \max_{t \in L} |\varphi(t)|$.

Let K be a real segment and U its neighborhood with a smooth boundary Γ. Assume that $f(t)$ is a function analytic in U, continuous in $\bar{U}$, real on K and satisfying the equation (0.1) with the constant leading coefficient $a_0(t) \equiv 1$ in the same way as in Theorem 1. But instead of the analyticity of the coefficients $a_j(t)$, $j = 1, ..., \nu$ everywhere in U, we assume only that there exist a subdomain $U' \ni K$ and a piecewise smooth path Γ_* starting on K and encircling the domain U' such that

$$\forall t \in \Gamma_* \qquad |a_j(t)| \leqslant A. \tag{2.3}$$

THEOREM 1′. *Under the above hypotheses the number of isolated zeros of the function f on K does not exceed $\beta'(A+\nu)$, where $\beta'=\beta(U',K)$ is a constant depending only on the geometry of the pair (U',K) and the length of the path Γ_*: $\beta(U',K)=(\gamma(U',K)+1)(|\Gamma_*|+1)$.*

Indeed, integrating the norm inequality along the path Γ_*, we obtain an upper estimate for $f^{(k)}(t)$ for all $t\in\Gamma_*$ in the same form as before. But since Γ_* encircles U', the same upper bound is valid for U'. Applying Lemma 1 to the function $f^{(k)}$ and the pair (U',K), we estimate in the same way as before the number of zeros of the kth derivative. The rest of the proof goes without any changes.

2.3. *Remark.* The upper estimate for the number of zeros may be slightly improved if the equation (0.1) has no term $y^{(\nu-1)}$, since in this case the reduction from the equation (0.1) to the system (2.1) may be modified to yield a smaller norm of the coefficients matrix $\|\mathbf{A}(\cdot)\|$.

EXAMPLE. Consider the second order equation $y''+a_2(t)\,y=0$ with the analytic coefficient $a_2(t)$ bounded in some neighborhood U of a real segment $K\Subset\mathbb{R}$. If $\max_{t\in\bar U}|a_2(t)|\leqslant A$, then Theorem 1 gives an upper estimate for the number of zeros of any solution f of this equation in the form $N_K(f)\leqslant O(A)$ as $A\to+\infty$, while the Sturmian theorems (based on completely different arguments) yield much better estimate $N_K(f)\leqslant O(\sqrt{A})$. However, if instead of the transformation $y\mapsto\mathbf{y}=(y,y')$ reducing the equation to the system of first order equations, we make the transformation $y\mapsto\mathbf{y}^*=(y,\sqrt{1/A}\,y')$, then the resulting system would have the coefficients matrix $\mathbf{A}^*(\cdot)$ with the norm $\|\mathbf{A}^*(t)\|\leqslant\sqrt{A}$, and the resulting estimate for the number of zeros would be asymptotically Sturmian-like.

Clearly, this remark refers to equations of any order, since by an appropriate transformation one may always eliminate the term with the $(n-1)$st derivative from the equation. However, this reduction affects other coefficients of the equation.

3. EQUATIONS WITH GENERAL MEROMORPHIC COEFFICIENTS AND THE PROOF OF THEOREM 2

In this section we consider differential equations of the form (0.1) with general analytic coefficients a_j, assuming only that the leading coefficient does not vanish identically, so that ν is the true order of the equation. Instead of proving Theorem 2, we prove here a slightly more general assertion.

THEOREM 2′. *Assume that K is a real segment, $U\Supset K$ is its neighborhood with a smooth boundary, and $K_*\Subset U$ is another compact subset of U.*

Consider a function f analytic in U, bounded on $\bar{U}$, real on K and satisfying in U the linear equation (0.1) *with analytic coefficients. Let*

$$a = \max_{t \in K_*} |a_0(t)| > 0, \qquad A = \max_{j=0,\dots,\nu} \max_{t \in \bar{U}} |a_j(t)| \geqslant a.$$

Then there exists $\mu = \mu(U, K, K_*) < \infty$ such that

$$N_K(f) \leqslant (\nu + A/a)^{\mu}.$$

Theorem 2 as it was formulated in the introductory section, follows from Theorem 2′ if we put $K_* = K$.

3.1. *Contours Avoiding Singularities.* The proof of Theorem 2′ is based on the following simple geometrical argument. Let $K \Subset U$ be as usual a pair of sets, but assume now that K is a segment.

LEMMA 4. *If U has a smooth boundary Γ, and K is a segment containing more than one point, $|K| > 0$, then there exist a positive constant $\delta = \delta(U, K) > 0$ and a subdomain $U' \subset U$ containing K strictly inside, $K \Subset U'$, such that for any finite number of disks D_j with the sum of their diameters less than δ, there exists a path Γ_* starting on K, encircling U' and not intersecting any of those disks. The length $|\Gamma_*|$ of this path does not exceed* $|\Gamma| + \operatorname{dist}(K, \Gamma) + 1$.

Proof of Lemma 4. The idea of constructing the path with the required properties should be clear from Fig. 1. We construct a continual family of

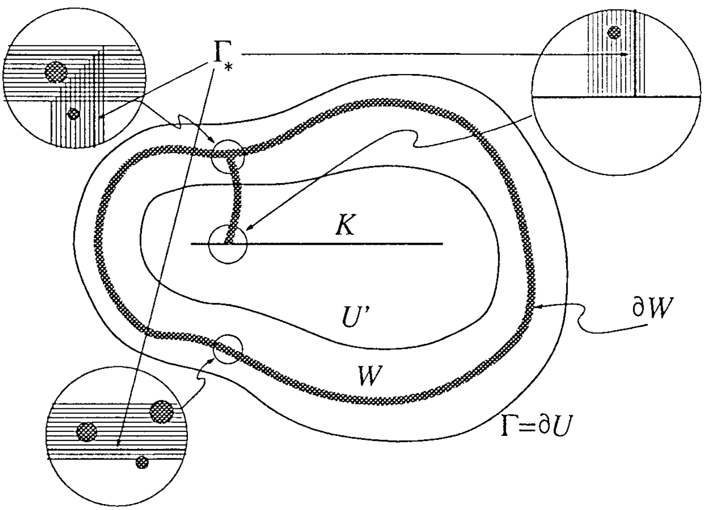

FIG. 1. Construction of paths avoiding small disks and encircling the domain U'.

"parallel" contours, together forming a band that follows along Γ; if the sum of diameters of the disks is less than the "width" of this band, then at least one of the contours will not intersect the disks. In the same way in another band connecting K and Γ one may find a path avoiding the disks. Then we are able to start on K, then reach the contour and make a full turn around. The domain U' bounded by the innermost contour, is encircled by the path.

More formally, assume that the boundary Γ is given by an equation $F(z)=0$, where $F\colon \mathbb{C}\to\mathbb{R}$ is a *smooth* function having 0 as a regular value. Then there exists an interval $I=(-\varepsilon,\varepsilon)\subset\mathbb{R}$ which consists of regular values, and for any $c\in I$ the level curve $F=c$ is a smooth curve, all of them encircling some domain U' with the required properties. Since F is Lipschitz, the F-image of any disk of diameter $r>0$ is contained in some interval of length $\leqslant Cr$, C being the Lipschitz constant. Thus we see that if $\delta<\varepsilon/C$, then the images of any number of disks with the sum of diameters less than δ cannot cover the whole interval I, hence there will be a contour Γ_1 close to Γ and encircling U'. The value ε may be chosen so small that the length of this contour will be less than $|\Gamma|+1/2$.

Next, one can choose a smooth path Γ' with $|\Gamma'|=\operatorname{dist}(K,\Gamma)+1/4$ connecting an interior point of K with Γ and transversal to both K and Γ. Then one can find another smooth function $\tilde{F}$ such that $\Gamma'=\tilde{F}^{-1}(0)\cap\bar{U}$ and for all sufficiently small c the level curves $\tilde{F}^{-1}(c)$ would connect K with Γ.

If δ is sufficiently small, then in the same way as before one can find another path Γ_2 close to Γ', with $|\Gamma_2|\leqslant\operatorname{dist}(K,\Gamma)+1/2$, which would still connect K and Γ and would avoid the disks. The union $\Gamma_*=\Gamma_1+\Gamma_2$ of this path and the contour constructed on the first step, can be interpreted as a path that satisfies all requirements. ∎

3.2. *Proof of Theorem* 2. The proof consists of two steps. First, using Lemma 3, we find a sufficiently thick subset on which the principal coefficient admits a lower estimate, and then using Lemma 4, we reduce the situation to the case when Theorem 1′ can be applied.

Step 1. Take any set $W\Supset K\cup K_*$, $\bar{W}\Subset U$ with the smooth boundary and apply Lemma 4 to the pair (W,K). This application yields a positive δ (the "thickness" of the boundary band) and an open subset $U'\Supset K$ in such a way that after deleting any number of disks with the sum of diameters less than δ there still can be found a path $\Gamma_*\subset W$ starting on K, encircling U' and avoiding the disks.

Step 2. Apply Lemma 3 to the function $a_0(t)$ and the pair of sets $(U,\bar{W})$, taking h equal to δ: since

$$a=\max_{t\in K_*}|a_0(t)|\leqslant\max_{t\in\bar{W}}|a_0(t)|,\qquad A=\max_j\max_{t\in\bar{U}}|a_j(t)|\geqslant\max_{t\in\bar{U}}|a_0(t)|,$$

we conclude that there exists $\varkappa = \zeta - \tau \ln \delta$ such that everywhere on W except for a finite union of disks with the sum of diameters less than δ, the leading coefficient a_0 admits the lower estimate $|a_0(t)| \geqslant a(a/A)^{\varkappa}$.

Divide now the equation (0.1) by a_0: the leading coefficient becomes equal to 1, and outside the union of disks the coefficients $\tilde{a}_j = a_j/a_0$ would be bounded by $(A/a)^{\varkappa+1}$. Consider the piecewise smooth path Γ_* encircling U' which starts on K and avoids all the disks (see Step 1). Along this path the coefficients $\tilde{a}_j$ admit the above upper estimate, hence application of Theorem 1′ yields an upper estimate for the number of zeros in the form

$$N_K(f) \leqslant \beta(U', K) \cdot ((A/a)^{\varkappa+1} + v),$$

where $\varkappa$ is a geometric constant depending only on the relative position of the sets U, W, U', K_* and K; since the choice of W and U' was made according to the relative position of U, K_* and K, one may say that $\varkappa = \varkappa(U, K, K_*)$, $\beta = \beta(U, K, K_*)$.

This estimate is already of an effective nature. However, if we want to represent the result in the form C^{μ}, where μ depends only on the geometry of the sets U, K, K_*, and C is determined by the size of the coefficients of the equation (0.2) and its order, then by letting $\mu = \varkappa + 2 + \log_2 \beta$ we can majorize the former expression. Indeed, the inequalities $v \geqslant 1$ and $A/a \geqslant 1$ imply that $(v + A/a)^{\mu} \geqslant \beta(1 + (A/a)^{\varkappa+1})$. The proof of Theorem 2 is complete. ∎

3.3. *Concluding Remark.* Formally in Theorem 1 we may not assume that the coefficients of the equation (0.1) are analytic: it is essential that the *solution* f is analytic in U. In Theorems 2 and 2′ we need analyticity of the leading coefficient $a_0(t)$ and the solution f. However, we do not know whether this generalization makes sense for applications.

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