RESEARCH REPORT

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§1. General.

The problems I am currently interested in mainly concern the *Hilbert Sixteenth problem* (on the number of limit cycles of planar vector fields). This subject, in turn, is a subdomain of the vast realm of *dynamical systems*, strongly connected with real and complex analysis, geometry, differential equations.

The formulation of this problem is very simple, yet the progress achieved towards a complete solution of it is quite moderate. However, in recent years we are witnessing a flurry of activity concerning *bifurcations* of limit cycles involving new geometric and analytic tools. These bifurcations (changes in the topology of phase portraits as the parameters of the field change) can be studied in two different contexts: (semi)local and global. In the first approach, one has to investigate *polycycles*, or *separatrix polygons* in the phase plane, consisting of at least one singular point and one or more arcs connecting the singular points in the specified order. The second approach deals with generation of limit cycles from non-isolated periodic trajectories of integrable (in particular, *Hamiltonian*) vector fields.

In this report I tried to put a stress on two of my results that I venture to consider as rendering substantial progress in each direction, namely:

(1) the *general finiteness theorem for the number of limit cycles that can be created from a separatrix polygon with an arbitrary number of hyperbolic or semi-hyperbolic singular points* (see §3C), and
(2) the *explicit exponential upper bound for the number of limit cycles that can be created from periodic orbits of a generic Hamiltonian vector field by a small polynomial perturbation of any degree* (see §3E).

The first result seems to be the only known general assertion about bifurcation of limit cycles from polycycles that covers the case of an arbitrarily high codimension and an arbitrary number of vertices and arcs on the polycycle. The second theorem solves a problem that stood open for about twenty years, despite continuous efforts.

Both results in fact are obtained as the last steps in the long chain of works focused on developing appropriate tools and/or solving intermediate problems. In order to present a (necessarily truncated) view of the whole picture, I have discussed briefly in several subsequent paragraphs the stage where the action will take place (§2) giving basic definitions and explaining modern approaches to the problem. Then the main body of the report (§3) follows, with a synopsis of my personal contributions. The final part (§4) is devoted to an unfinished work and possible ways of attacking some other yet unsolved problems, also closely related to The Hilbert Sixteenth.

Besides the above two results I would like also to put a stress on the quite recent result,

(3) *upper bounds for rotation of spatial* (*Euclidean or projective*) *curves around linear subspaces in terms of integral curvatures*, a purely geometric result that is closely related to the second topic above (see §3F).

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This report is concerned only with my recent activity (beginning essentially in 1992, though one paper is from 1991). By no means should it be considered as a comprehensive survey of the area: the references to other people's works are very occasional. On the contrary, I tried to outline my personal contribution as clearly as possible. The numbered citations refer to the list of my publications at the end of the report; other papers are referenced by letters in square brackets (also listed at the end).

As for my previous works on nonlinear dynamic optimization, the results of that period were summarized in [1] and will not be discussed here.

§2. The Hilbert 16th problem and related problems from analytic theory of differential equations.

2.1. Limit cycles. We consider dynamical systems defined by planar vector fields. After the choice of a coordinate system the vector field becomes a pair of first order autonomous differential equations

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \qquad x, y \in \mathbb{R}, \tag{1}$$

in two variables, and due to the simple topology of the 2-plane, the only possible recurrent motions are either singular points (stationary solutions) or closed invariant curves (periodic orbits, cycles).

If the vector field is polynomial, i.e. the right hand sides f, g of the differential equations are real polynomials, then singular points can be obtained by solving a system of algebraic equations, hence their number is given by the Bezout theorem. All the way around, closed invariant curves for polynomial vector fields are usually non-algebraic, and their determination is highly nontrivial. In his 16th Problem (1900) Hilbert asks to *determine the number and position of isolated periodic orbits of a polynomial planar vector field in terms of the degrees of the polynomials* f, g. By definition, a *limit cycle* for a vector field is an *isolated* closed periodic orbit of that field.

2.2. First return map and its analytic properties. The first steps towards solution of the Hilbert problem were taken by Poincaré, who proved that limit cycles of an analytic vector field cannot accumulate to another limit cycle. The main idea of the proof was to consider a line segment I transversal to the vector field and investigate the *first return map* $\Delta \colon I \to I$. By definition, the limit cycles correspond to isolated fixed points of the first return map. On the other hand, the latter is analytic at a point $a \in I$, if the trajectory passing through a does not tend to a singular point, hence the uniqueness theorem for analytic functions can be applied.

If, on the contrary, the trajectory passing through a tends to a singular point, then the first return map Δ (eventually defined for other points of the segment I) looses its analyticity. Thus the Hilbert problem is reduced to studying *analytic properties of functions defined by differential equations*. Though these differential equations are in general nonlinear, it is important to consider also the linear equations with analytic variable coefficients; see §3E.

2.3. Bifurcations of vector fields. Almost all existing approaches to investigation of limit cycles of vector fields are based on the deformation techniques: the vector fields are considered in families depending on parameters so that for some values of the parameters the limit cycles can be easily located, and then limit cycles are monitored in their dependence on the parameters. This approach leads to the *bifurcation theory* for vector fields. There are two essentially different sets of assumptions for this theory: one may consider either *generic* cases of gradually increasing codimensions, or alternatively the *polynomial* families can be studied using global tools of algebraic geometry.

2.4. Normal forms and versal deformations of singular points. It makes sense to split the problem of studying bifurcations of vector fields in two parts. Since singular points are, in general, more easy to analyze, one may:

- try to describe bifurcations of vector fields near singular points, and then
- glue together the results obtained on the previous step.

The results of the local investigation can be formulated in terms of *local models*, or *normal forms* of (parameter dependent) vector fields near singular points. A normal form is usually a simple (parameter dependent) vector field that admits complete investigation by elementary methods (e.g., complete integration), and on the other hand, any other vector field can be transformed into this simple local model by an appropriate change of variables.

In fact, even after the first step is implemented, gluing the local normal forms can be a tremendously difficult problem, cf. [Mo].

2.5. Perturbations of integrable systems and complete Abelian integrals. The approach described above is based on passing from local to global description in the phase plane. An alternative strategy (also originally due to Poincaré and later developed into the so-called Petrovskiĭ–Landis program) is to study the global phenomena occurring in nearly integrable systems.

Assume that the differential equations (1) have a particular form: there exists a function $H = H(x,y)$ (the Hamiltonian, or full energy) such that

$$f(x,y) = \frac{\partial H}{\partial y}(x,y), \quad g(x,y) = -\frac{\partial H}{\partial x}(x,y). \tag{2}$$

Then the equations (1)–(2) still may possess periodic solutions, but these solutions will not be isolated (consider an example of $H = x^2+y^2$ corresponding to the mathematical pendulum: all solutions are periodic non-isolated), hence by definition there are no limit cycles. However, if we introduce a small parameter $\varepsilon \in \mathbb{R}$ and consider a nonconservative polynomial perturbation of (1)–(2) of the form

$$\dot x = \frac{\partial H}{\partial y}(x,y) + \varepsilon\, q(x,y), \qquad \dot y = -\frac{\partial H}{\partial x}(x,y) - \varepsilon\, p(x,y), \tag{3}$$

then all but a finite number of periodic solutions will be destroyed for $\varepsilon \neq 0$ due to the dissipation of energy H. Those that survive must satisfy the condition of energy balance. The linearization (in ε) of the balance condition yields the identity

$$\boldsymbol{I}_{H,p,q}(h) := \oint_\gamma p(x,y)\,dx + q(x,y)\,dy = 0, \qquad \gamma \subseteq \{H(x,y) = h\}, \tag{4}$$

where γ is a non-perturbed solution of the conservative system which generates a limit cycle after the perturbation (3).

Note that the condition (4) is of a more algebraic nature: the left hand side is the integral of a polynomial 1-form over an algebraic level curve γ. However, as a function of h, this integral (called the *complete Abelian integral*) is a transcendental function, so that the problem of describing its zeros is also far from being trivial.

§3. Survey of the principal results achieved in my papers.

In this section I give a (necessarily brief) synopsis of my results obtained in connection with various problems related to The Hilbert Sixteenth. The parts typeset in small print contain more technical information, or sometimes describe digressions into adjacent areas of studies.

3A. Normal forms (see §2.4)

In the theory of normal forms there are two main factors/parameters:

- the smoothness of transformations of the classification group (homeomorphisms, diffeomorphisms, analytic conjugacies, formal power series), and
- the degeneracy index (codimension) of the vector field (or a map) whose deformation is considered: generic vector fields have codimension 0, singularities of codimension 1 occur in generic one-parameter families for isolated values of the parameter etc.

In the papers [2,3] (jointly with Yu. Ilyashenko) we established C^k-smooth normal forms (for any finite $k \in \mathbb{N}$) for families of vector fields in $\mathbb{R}^n$ near a singular point and families of local diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ in codimensions 0 (the generic case) and 1 (typical one-parametric families). In [2] we give a complete list of cases when C^k-smooth polynomial integrable normal forms exist and show how the integrability can be used to investigate bifurcations of the planar separatrix loop: our theorem in [2, §3] simplifies the proof of an earlier result by R. Roussarie [R2] and shows that the upper estimates for the number of limit cycles are in fact sharp.

The paper [3] deals with the other cases of codimension 1 when there arise *functional moduli*, i.e. obstructions preventing even C^1-smooth transformation to any polynomial normal form. In this case we show that the smooth normalization is still possible in sectorial domains of a special form, containing the origin in their closure and semi-invariant by the flow (resp., iterations). We analyze then the uniqueness of the sectorial normalizing transformations and show that the obstruction is of a nature similar to the Stokes multipliers for linear systems, namely, that there can be defined a canonical atlas of normalizing charts and the transition functions of that atlas represent obstructions to integrability. The main positive result is the theorem on sectorial embedding in C^k-category ($k \leqslant \infty$), which generalizes an earlier result by J.-C. Yoccoz [Yo1] (that guaranteed only low smoothness of the embedding). This sectorial embedding theorem implies necessity of the Malta–Palis sufficient conditions [MP] for non-appearance of simultaneous saddle connections in deformation of a semistable limit cycle. Other applications of this result can be found in the paper by L. J. Diaz, M. Viana and J. Rocha [DRV].

A generic vector field (generic diffeomorphism) has hyperbolic nonresonant linearization matrix at a singular (resp., fixed) point. In codimension 1 these two conditions can be violated as follows:

(1) One (and only one) resonance between the eigenvalues of the linearization matrix occurs while the hyperbolicity persists;
(2) The singular (fixed) point looses its hyperbolicity without appearance of resonances other than implied by the following conditions:
  (2.1) One eigenvalue of a singular point moves to the imaginary axis (necessarily at the origin);
  (2.2) A pair of complex conjugate eigenvalues moves to the imaginary axis;
  (2.3) One multiplicator of the fixed point of a diffeomorphism moves to the unit circle (necessarily at the points ± 1);
  (2.4) A pair of complex conjugate multiplicators moves to the unit circle.

Note that without parameters a generic singularity of a vector field (diffeomorphism) can be linearized by a C^∞-smooth transformation (S. Sternberg–K. T. Chen, see [H]), and in general the transformation cannot be chosen analytic (complete investigation of necessary and sufficient conditions of analytic linearizability was achieved recently by J.-C. Yoccoz [Yo2]).

In the presence of parameters the situation changes.

Theorem [2]. *An arbitrary finite-parameter deformation of a generic vector field* (*diffeomorphism*) *can be* C^k*-smoothly linearized by a transformation smoothly depending on the parameters for any finite* k *but in general not for* $k = \infty$.

An arbitrary finite-parameter deformation of codimension 1 *singularities in the cases* (1) *and* (2.1) *can be* C^k*-smoothly transformed to a polynomial integrable form. The smoothness order* $k \in \mathbb{N}$ *cannot in general be made infinite.*

The "nonlinear Stokes phenomenon" is shown in [3] to occur in the cases (2.2) and (2.3). We formulate the result only for the case (2.2) with the multiplicator $+1$. In this case it is sufficient to consider 1-parameter deformations of scalar maps of the form $x \mapsto f(x,\varepsilon) = x + x^2 - \varepsilon + \cdots$. For $\varepsilon > 0$ the map possesses two fixed points $a_\pm(\varepsilon)$, and in general in any neighborhood of the origin on the (x,ε)-plane, containing both a_+ and a_-, the map cannot be transformed into a polynomial even by a C^1-smooth transformation.

Theorem [3]. *For $\varepsilon < 0$ the map $x \mapsto f(x,\varepsilon)$ is C^∞-conjugate to the time 1 flow of the standard integrable vector field $\dot x = -\mu(\varepsilon) + x^2 + c(\varepsilon)x^3$. The conjugating map $H = H(x,\varepsilon)$ extends C^∞-smoothly across positive (negative) semi-axis of the axis $\varepsilon = 0$ to the domain $\{x > -\varepsilon,\ 0 < \varepsilon < \varepsilon_0\}$ (resp., $\{x < \varepsilon,\ 0 < \varepsilon < \varepsilon_0\}$). The co-chain map $H_+ \circ (H_-)^{-1}$ defined for $\varepsilon > 0$, $-\varepsilon < x < \varepsilon$ is flat in ε and uniquely defined by f modulo flow maps of the standard vector field. Any flat function commuting with the time 1 map of the standard vector field, can be realized in this way.*

3B. Flow versus orbital classification of vector fields

When looking for standard models for vector fields, one may either allow for time rescaling or not. The first classification, called *orbital classification*, is coarser than the second one, but still the model field carries all information about the *phase portrait* of the vector field. However, sometimes one needs to preserve the parameterization of trajectories by time. We refer to the second classification model vector fields as the *flow normal forms*.

The two theories are very similar, though the orbital classification naturally has less invariants. Flow normal forms of vector fields at singular points are well known (F. Takens [T]–G. Belitskiĭ [Be]). As for the flow normal forms near limit cycles, somewhat unexpectedly only the orbital classification was known: two vector fields are orbitally equivalent near a limit cycle if and only if their Poincaré maps are conjugated. In particular, a theorem by F. Takens [T] says that a planar vector field near a limit cycle of finite multiplicity is orbitally equivalent to the vector field $\dot r = R(r) = \pm r^{k+1} + a\, r^{2k+1},\ \dot\theta = 1$ on the cylinder $\mathbb{R} \times \mathbb{S}^1$.

In [4] I analyzed the flow classification problem for such vector fields in the simplest nonhyperbolic case $k = 1$ and showed that besides the natural additional parameter T, the period of the cycle, there is only one additional invariant, and the flow normal form can be written as $\dot r = (T^{-1} + b\theta)R(r),\ \dot\theta = (T^{-1} + b\theta)$ in C^∞-differentiability class. The invariant b is given a geometric description in terms of the difference of periods of two cycles born from the semistable cycle by a small perturbation. For $k > 1$ there is not one but k additional invariants [4].

The case of a limit cycle with multiplicator -1 (such a vector field cannot be planar, but can be realized on the Möbius band) was analyzed by my graduate student D. Novikov in 1994 (M.Sc. thesis, the Weizmann institute, unpublished), who also explained the geometric nature of the Belitskiĭ invariants in the case of a planar vector field with the eigenvalues $\pm i\omega$.

3C. Bifurcations of limit cycles and the Hilbert–Arnold problem (see §2.3)

This section surveys the result I am especially fond of, solution of the *Hilbert–Arnold problem* (see below) for vector fields having only elementary singularities. The next several subsections are based on the expository article [5].

3C1. Bifurcation of polycycles. As was already remarked, limit cycles of planar vector fields depending on parameters, can bifurcate from singular points, other limit cycles or more complicated objects, the so-called *limit periodic sets* [FP], of which the most important are *polycycles*. Roughly speaking, a polycycle is a limit cycle of infinite period, having singular points on it. More precisely, a polycycle is an ordered collection of singular points and heteroclinic connections (entire phase trajectories with compact closure) joining them in the specified order [5].

The problem of determining the maximal number of limit cycles that can be created from a polycycle by a small variation of parameters, is known as the problem of estimating the *cyclicity* (an analog of multiplicity) of the polycycle [R1]. In fact, several most simple cases of cyclicity estimates were analyzed even before the problem was posed in full genericity [R2], [AI].

Note that in this problem the vector fields need not necessarily be polynomial: one may consider arbitrary smooth families of vector fields on the sphere $\mathbb{S}^2$.

3C2. The Hilbert–Arnold problem. Analyzing known results on bifurcation of simplest polycycles and comparing them to results on the Poincaré map for polynomial vector fields, V. Arnold observed [AI] that in order to exhibit a polycycle of a high cyclicity, the vector field must be rather degenerate (satisfying many independent equality-type restrictions). This led him to the natural conjecture that *generic* finite-parameter families cannot exhibit polycycles of very large cyclicity if the number of parameters is small. A more formal assertion follows.

The Arnold Conjecture. 1 (global). *A generic finite-parameter family* $\{v_\varepsilon\}_{\varepsilon\in B}$ *of smooth vector fields with a compact parameter space* $B \subset \mathbb{R}^n$ *on the two-dimensional sphere* $\mathbb{S}^2$ *admits a uniform upper bound for the number of limit cycles for all values of the parameter.*

2 (local). *A generic n-parameter family of vector fields on the sphere may have polycycles of cyclicity at most* $C = C(n) < \infty$.

The local conjecture implies the global one [R1].

3C3. First steps. Generic one-parameter families were studied in the late thirties by A. Andronov and H. Hopf, the main result being the equality $C(1) = 1$. The case of 2 parameters has a longer history, and the equality $C(2) = 2$ was finally achieved only in 1993. This is a concise summary of results by E. Leontovich, F. Takens, R. Bogdanov, A. Mourtada and T. Grozovskiĭ.

The case $n = 3$ constitutes already an open problem, where some unexpected phenomena were observed [KS] that disproved a more general original version of the Arnold conjecture. However, after a proper modification, the conjecture $C(3) = 4$ seems to be realistic. As before, this conjecture is proved by studying separate cases (the list of polycycles occurring in generic 3-parameter families, consists of several tens of items). The most difficult cases had been attacked by F. Dumortier, R. Roussarie and J. Sotomayor [DRS]. The main obstruction on the way to the complete solution is the occurrence of highly degenerate singular points on the polycycle.

However, even if all points on the polycycle are not too degenerate (see below), the study of different configurations of vertices and arcs gives rise to very substantial difficulties. R. Roussarie in [R2] completely analyzed the simplest case of the separatrix loop ("one singularity and one arc"), and a simplification of his proof can be found in [2]. Then came the turn of loops with degenerate singularities, lenses (diangles), triangles, eight-shaped figures etc., when heroic concerted efforts of the Dijon team (R. Roussarie, A. Mourtada, A. Jebrane, M. El-Morsalani, see [KS] for numerous references) led to a series of impressive results on cyclicity estimates. They started with obtaining explicit bounds, but as the difficulties mounted, the results became less sharp and less explicit. One of the results crowning this assault is a theorem by A. Mourtada [M] giving an algorithm ("*un algorithme de finitude*") for estimating cyclicity of a polygon with hyperbolic saddles, provided that there are no nontrivial multiplicative relations between the hyperbolicity ratios of those saddles. Unfortunately, this theorem does not cover even all generic one-parameter families, though it provides a deep insight into the analytic structure of the Poincaré map of such a polycycle.

3C4. Elementary polycycles. A different strategy was chosen in my joint work with Yu. Ilyashenko. Instead of explicitly writing the asymptotic expansion for the Poincaré map, we reduced the question about its fixed points to a quite general system of equations, part of them being polynomial (and the degrees of the polynomials were under full control), while the other equations involved smooth functions with the only assumption of n-genericity (only functions that may occur in generic n-parameter families are allowed to enter the equations).

Further investigation of the resulting system required some new general methods (see §3D below). A brief exposition of the main ideas of the proof was published in [6], and the full demonstrations appeared in a necessarily rather voluminous paper [7]. However, the main result can be formulated in one line after the following definition is introduced.

Recall that a singular point of a planar vector field is *elementary*, if at least one eigenvalue of its linearization is nonzero. A polycycle is called *elementary*, if it contains only elementary singularities. The notion of elementarity formalizes the property of being not too degenerate: nonelementary singularities may occur in codimension at least 2. There are many reasons why elementary singularities play an essential role, the principal reason being that any non-elementary singular point by a finite number of blow-ups can be resolved into a number of elementary ones (Bendixsson–Seidenberg). Define the number $E(n)$ as the upper bound for cyclicities of all elementary polycycles that may occur in generic n-parameter families ("$E(n)$ is $C(n)$ for elementary polycycles").

Principal Theorem [6,7]. *For any $n \in \mathbb{N}$ the number $E(n)$ is finite. Furthermore, the correspondence $n \mapsto E(n)$ is majorized by a primitive recursive function.*

The algorithmic nature of this primitive recursive function is completely described in the proof. Quite recently (in October 1995) V. Kaloshin (Moscow State Unniversity) extracted an explicit double exponential estimate of the form $E(n) \leqslant \exp\exp O(n)$ from our proof (unpublished). The problem involves deep considerations from the constructive commutative algebra and effective algebraic geometry over the field of reals $\mathbb{R}$.

3C5. Bautin theorem revisited [16]. This work is concerned with bifurcations of limit cycles in polynomial systems. In 1932 N. Bautin proved a result [B] which gradually became one of the most cited references in papers on limit cycles and their bifurcations. The Bautin theorem claims that within the class of quadratic vector fields a singular point can generate no more than 3 small limit cycles after a small variation of parameters.

Despite its popularity, the result remained to be somewhat mysterious, and the original proof of Bautin, based on very cumbersome and enigmatic calculations, was reproduced almost literally in all existing re-expositions. In an attempt [16] to unveil the mystery, I discovered that all formal manipulations and the labor-consuming calculus can be avoided due to the presence of a hidden $\mathbb{Z}_3$-symmetry: simple geometrical arguments prove the result of Bautin and eventually allow for further generalizations.

3D. Gabrielov-type theorem for generic smooth functions and related results from singularity theory

This section briefly describes some auxiliary results that were developed while proving the main theorem on finite cyclicity of elementary polycycles, but have an independent value.

It is well known that analytic maps possess many finiteness properties, in particular, the following general principle (A. Gabrielov–B. Teissier) is valid: if $F = (f_1, \dots, f_m)$ is a map real analytic on the unit cube $Q^n = [0,1]^n$, then the number connected components of preimages $F^{-1}(y) \cap Q^n$ of a variable point $y \in \mathbb{R}^m$, is uniformly bounded over all y. In particular, if $m = n$, then the number of regular (=nondegenerate) preimages of y is bounded.

This principle is apparently false for smooth maps, as the simplest counterexamples show. However, it turns out that the exceptions have an infinite degree of degeneracy, so the finiteness holds for all maps that occur in generic finite-dimensional families.

Theorem [7, §5]. *For a generic finite-parameter family of smooth maps $F = F(x, \varepsilon)\colon Q^n \times B \to \mathbb{R}^n$ with the compact parameter space $\varepsilon \in B \subset \mathbb{R}^k$ the number of nondegenerate preimages of a point $y \in \mathbb{R}^n$ is uniformly bounded over all $y \in \mathbb{R}^n$, $\varepsilon \in B$.*

The result is also valid for chain maps of the form $\widetilde{F} = P \circ F$, where F is as before, and P is an arbitrary polynomial map.

There is a local counterpart of this theorem. For the germ of a smooth map $\boldsymbol{F}\colon (\mathbb{R}^n, 0) \to \mathbb{R}^n$ we define the *Gabrielov multiplicity* as the upper bound for the number of nondegenerate preimages of a variable point $y \in \mathbb{R}^n$ in a sufficiently small neighborhood of the origin:

$$g = \limsup_{\varepsilon \to 0^+} \sup_{y \in \mathbb{R}^n} \#\boldsymbol{F}^{-1}(y) \cap B_\varepsilon \cap \{x : \det \boldsymbol{F}_*(x) \neq 0\}, \qquad B_\varepsilon = \{\|x\| < \varepsilon\} \subset \mathbb{R}^n.$$

To simplify the formulation below, we say that the germ $\boldsymbol{F}$ is k-generic, if it may occur in a generic k-parameter family of maps (in the same way as in the above theorem), or equivalently, the degeneracy codimension of $\boldsymbol{F}$ is at most k.

Theorem [7, §6]. *For a k-generic smooth germ $\boldsymbol{F}\colon (\mathbb{R}^n,0)\to\mathbb{R}^n$ the Gabrielov multiplicity can be at most $C=C(k,n)$, where C is a primitive recursive function of its integer arguments. The Gabrielov multiplicity of the composition $P\circ\boldsymbol{F}$ with the polynomial map P of degree d can be at most $C'=C'(k,n,d)$.*

Here and before "smooth" means C^ℓ-differentiable, with ℓ sufficiently large.

3E. Zeros of complete Abelian integrals and related questions

3E1. Zeros of Abelian integrals. The problem of determining the number of zeros of complete Abelian integrals is also one of the toughest challenges related to the Hilbert 16th problem (see §2.5). Even particular cases of this problem, say, when $H(x,y)$ is a cubic potential and $\omega=p\,dx+q\,dy$ is a low-degree form, are very difficult to analyze if one questions the sharp bounds: see the survey [Rs] for a partial list of references. In particular, in [8] I proved that for $H(x,y)=y^2+U_3(x)$ and $\omega=p_3(x)y\,dx$, where U_3,p_3 are real polynomials of the third degree, the Abelian integral can have at most 3 isolated zeros.

As for the general case, there were only a few results: lower bounds, linear in the degree d of the form, due to Ilyashenko [I], an existential finiteness theorem by Varchenko–Khovanskiĭ [Va], [Kh] and (a relatively simple) upper bound for the *multiplicity* of each zero by Mardesic [Ma], also linear in d. The particular case of Morse hyper-elliptic Hamiltonians $H=y^2+U(x)$, $U\in\mathbb{R}[x]$ a Morse polynomial in one variable, was completely investigated by G. Petrov in a series of publications [P]. However, no explicit upper bound for the general case was known, despite approximately twenty years of continuing efforts.

I returned to studying Abelian integrals in 1994. The general idea was to exploit connections between the number of zeros and the growth of Abelian integrals in the complex domain. The growth properties can be analyzed using the fact that Abelian integrals extended as multivalued analytic functions for all regular values of the Hamiltonian H, satisfying Picard–Fuchs type linear ordinary differential equations.

As a result, the first explicit upper bound for the number of zeros of complete Abelian integrals was obtained in the joint work with Yu. Ilyashenko [11] using analytic tools developed earlier in [9], [10] (see below).

Theorem [11]. *If we fix almost any Morse polynomial H and consider all polynomial forms of degree $\leqslant d$, then the number N of isolated zeros of the corresponding Abelian integrals on a compact real segment I free from critical points of H, can be at most double exponential in d:*

$$N=N(H,I,d)\leqslant\exp\exp(Kd)\quad\textit{as } d\to\infty,\qquad K=K(H,I)<\infty.$$

This bound certainly was unrealistic, and the subsequent progress in improving it was achieved in the work done together with my Masters (now Ph.D.) student D. Novikov [12], [13]. In these papers we improved the upper bound to become simple exponential in d and at the same time proved that the constant K can be chosen valid uniformly for all segments I (containing zeros arbitrarily close to the critical values of H). In the previous notations, the final result can be formulated as the (asymptotic) inequality.

Principal Theorem [12], [13].

$$N(H,I,d)\leqslant\exp(K^*d)\quad\textit{as } d\to\infty,\qquad K^*=K^*(H)<\infty.$$

The result was later generalized for complex zeros of the Abelian integrals (with the same simple exponential bound) [14].

These results (which I also consider as my highest achievement) were proved using a combination of ideas stemming from various fields. Following is a brief description of separate contributions.

3E2. Zeros of real functions satisfying linear ordinary differential equations. The general principle relating the growth rate of the modulus $|f(z)|$ of an analytic function f with the distribution of zeros of f, as $|z|$ tends to infinity, is well known for entire functions and has been developed into very fine results within the Nevanlinna theory [L]. We change the settings and fix some domain $U \Subset \mathbb{C}$, but try to get uniform upper bounds for the number of zeros for some classes of functions analytic in U, in particular, for the spaces of solutions of linear ODE's with analytic coefficients of the form

$$\begin{gathered} a_0(z)y^{(n)} + a_1(z)\,y^{(n-1)} + \cdots + a_{n-1}(z)\,y' + a_n(z)\,y = 0, \\ a_j(z) \text{ analytic in } U \subset \mathbb{C}, \qquad |a_j(z)| < C \text{ for } z \in U. \end{gathered} \tag{5}$$

In this case the growth of all solutions can be easily controlled in terms of the determining equation, and an analog of the Jensen inequality yields an upper bound for the number of zeros.

Theorem [10]**.** *If the equation* (5) *has no singularities in* U (*then without loss of generality one may assume* $a_0 \equiv 1$), *then any solution* f *of* (5) *may have at most* $O(C+n)$ *real isolated zeros on any segment* $I \Subset \mathbb{R} \cap U$, *provided that* f *is real on* I.

If the equation has singular points, but all of them are apparent (so that all solutions extend analytically across these points), then the linear upper bound can be replaced by a polynomial one after appropriate normalization of the principal coefficient [10]. All asymptotics are explicitly computable.

This result in principle gives an opportunity to place an upper bound on the number of real isolated zeros of any function in any bounded domain, as soon as the determining equation is explicitly written.

3E3. Polynomial and rational envelopes. As was remarked, an Abelian integral of a polynomial 1-form $\omega = p\,dx + q\,dy$ over regular level curves of a Hamiltonian H satisfies a certain Picard–Fuchs linear equation. However, the considerations of the previous subsection cannot be directly applied to this equation, firstly because it is practically impossible to place a bound on the coefficients of this equation, and secondly, since the equations for different forms are different, so no common upper bound can be obtained.

Instead one may consider a universal linear differential equation depending only on the Hamiltonian, in such a way that the space of all Abelian integrals can be represented as the *rational envelope* of this universal equation, that is, linear combinations of solutions of the latter and their derivatives with rational coefficients. In [9] I introduced this universal equation and studied its properties, among which the most important is *almost irreducibility* of the monodromy group. If the degree of the form ω is known, then the degrees of rational coefficients of the rational envelope can be explicitly computed [9]. If we are interested in the number of zeros, then the rational envelopes can be reduced to *polynomial envelopes* (the definition is natural), and one can easily write down a linear equation of a high order, whose solution space will be the polynomial envelope of the original (universal) equation: the order of this new equation grows with the degrees of the polynomial coefficients that are allowed to occur in the envelope.

3E4. Polynomial envelopes of linear differential equations with an irreducible monodromy group. The final step in proving the double exponential estimate [11] was in establishing a *quantitative* version of the known fact that solutions of an irreducible equation and their derivatives are linear independent over the field of rational functions. We obtain a qualitative measure of this independence which allows us to estimate the coefficients of the linear equation corresponding to the polynomial d-envelope (i.e. with coefficients of degree $\leqslant d$).

Further progress and improvement [12], [13] was based on an alternative representation of the equation for the polynomial envelopes and a local analysis near singular points. The main result of [12], [13] is the following theorem.

Theorem [12]**,** [13]**.** *If a linear equation with an irreducible monodromy group has no singular points on the closure of a real interval $I \subset \mathbb{R}$, then any function from the polynomial (or rational) d-envelope of that equation, real on I, may have at most* $\exp Kd$ *real isolated zeros on I, where $K = K(I) < \infty$.*

If all characteristic exponents corresponding to a singular point $z = z_0$ are real, then the above estimate is valid also for intervals containing z_0 in their closure.

3E5. Complex zeros. One of the key ingredients in the proof of the theorems from §3E2, is the Rolle theorem which asserts that the number of real isolated zeros of a smooth function can be at most that of its derivative plus 1. This result is invalid if we consider complex analytic functions. However, in [14] we established a different quantity, called the *Bernstein index*, which majorizes the number of zeros of an analytic function, but unlike the former, takes comparable values for a function and its derivative: if f is a function analytic in some open domain U, then the difference between the Bernstein indices of f and f' is bounded from above by a number depending only on U. This index is introduced in terms of the growth of the function in the gap between two nested domains. We developed an elementary theory of the Bernstein index and applied the conclusions to the problem of counting *complex* zeros of functions defined by linear differential equations (cf. §3E2) and Abelian integrals. As a result, we were able to prove complex analogs of theorems from §3E2 and §3E4, in particular, the same simple exponential upper bound for polynomial envelopes in the complex case.

Theorem [14]**,** cf. with §3E2. *Fix a compact subset D of the domain U of analyticity of coefficients of the equation* (5)*. Then any solution of the equation* (5) *may have at most $O_{D,U}(C + n \ln n)$ complex analytic zeros in D.*

There were previously known some results on *disconjugacy* of linear differential equations in the complex domain (see [Ki] and references therein), guaranteeng that any solution may possess no more than $n-1$ complex zero, provided that the coefficients are sufficiently small, i.e. $C < C_0(U)$. However, these results give a worser asymptotical bound of the order $O(n\,C^2)$ as $C \to \infty$ for the number of zeros.

Combining the above theorem with the technique described in §3E4, we obtained a complex analog of the theorem from [12], [13].

Theorem [14]**,** cf. with §3E4. *For any compact subset D on a positive distance from the singular locus of a linear equation with an irreducible monodromy group, a function from a polynomial d-envelope may have at most* $\exp O_D(d)$ *complex isolated zeros in D.*

Corollary. *If $f(z)$ is an analytic function satisfying an irreducible linear differential equation, and D is a domain free from singular points of that equation, then the number of zeros of the νth derivative $f^{(\nu)}(z)$ in D cannot exceed* $\exp O_D(\nu)$ *as $\nu \to \infty$.*

The most recent work in this series, jointly with A. Khovanskiĭ [15], concerns another similar index that we called provisionally the *Voorhoeve index*, after the publications [V1], [V2], where a specific particular case of that index was introduced. The construction is closely related to the argument principle. Namely, if f is a function analytic in the open simply connected domain U, we put $V_U(f)$ being the total variation of $\operatorname{Arg} f$ along the boundary of U. Then the analog of the Rolle inequality [15] asserts that $V_U(f) \leqslant V_U(f) + \varkappa(U)$, where the $\varkappa(U)$ is the absolute integral curvature of the boundary.

This result has an immediate application to counting complex roots of quasipolynomials $\sum e^{\lambda_k z} p_k(z)$, $p_k \in \mathbb{C}[z]$.

Theorem [15]. *For any function $f(z)$ from the linear space of all quasipolynomials with* $\deg p_k \leqslant d$, *the number of roots in a convex domain U can exceed the dimension of that space by no greater than* $\operatorname{diam} U \cdot L$, *where L is the length of the shortest polyline visiting all points representing the exponents λ_k on the plane $\mathbb{C}$.*

This seems to be the best known estimate. Finally we establish two-sided inequalities between the Voorhoeve and the Bernstein indices, showing that the corresponding generalized Rolle theories are essentially isomorphic.

3E6. Zeros near Fuchsian singularities. The results described in §3E2 deal with zeros of analytic functions defined by linear equations in the nonsingular or apparently singular case. Together with my M. Sc. student M. Roitman, quite recently we obtained a similar upper estimate for zeros of such functions arbitrarily close to the Fuchsian singular points, provided that the spectrum of the singularity is completely real.

Without loss of generality one can assume that the singularity is at the origin $z = 0$ and the equation already has the form

$$Ly = 0, \qquad L = \sum_{j=0}^{n} c_j(z)\, E^{n-j}, \quad E = z\frac{d}{dz} \text{ the Euler operator,} \tag{6}$$

where $c_j(z)$ are analytic in the unit disk and $c_0(z) \equiv 1$, $|c_j(z)| \leqslant C < \infty$ for $|z| < 1$ (any Fuchsian equation can be reduced to the form (6) by regrouping terms).

Theorem [17]. **1.** *The number of zeros of any solution $f = f(z)$ of the equation* (6), *real on the semiinterval* $[0, 1/2)$, *does not exceed $k_n C + k'_n$, where k_n, k'_n are some explicitly computable constants, provided that all roots of the* indicial equation $\lambda^n + \sum_{j=1}^{n} c_j(0)\lambda^{n-j} = 0$ *are real.*

2. *The similar upper bound (but with different k_n, k'_n) holds for the number of* complex *zeros of* any (*not necessarily real*) *solution of equation* (6) *in the sector* $0 < |z| < 1/2$, $|\operatorname{Arg} z| < \pi$, *provided that the coefficients $c_j(z)$ are real on the real axis and all roots of the indicial equation are real.*

It is essential to remark that the Fuchsian singularities with the real spectrum are characteristic for the theory of Abelian integrals (see §2.5 above): the theorem just formulated allows for an explicit estimate of the number of limit cycles born by polynomial perturbation of a Hamiltonian field, provided that the corresponding Picard–Fuchs equation can be *explicitly written* (there exists an algorithm of doing this in any particular case).

3F. Oscillation of spatial curves

From the geometrical point of view the question about the number of zeros of an arbitrary solution to a given linear differential equation, is the question about the maximal number of intersections of a certain curve in $\mathbb{R}^n$ (or $\mathbb{C}^n$, depending on the context) with an arbitrary affine hyperplane. In the paper [18] we constructed an upper bound for the latter number in terms of geometric characteristics of the curve, namely, by the weighted sum of the *generalized integral curvatures* of the curve.

Recall that for a Riemannian manifold M^n and any sufficiently smooth generic curve $\Gamma\colon [0,\ell] \to M^n$ parameterized by the arc length s, the Frenet curvatures $\varkappa_j = \varkappa_j(s)$, $j = 1, \dots, n-1$ can be defined in the same way as for smooth curves in $\mathbb{R}^3$, where $\varkappa_1$ is the standard curvature and $\varkappa_2$ the torsion. The points where the last curvature $\varkappa_{n-1}(\cdot)$ vanishes, are called *generalized inflections*. Define the generalized integral curvatures $K_j(\mathit{\Gamma})$ of the $\mathit{\Gamma}$ for $j=-1,0,1,\dots,n$ as follows:

$$
\begin{aligned}
K_{-1}(\mathit{\Gamma}) &= \tfrac12\pi\times\{\text{the number of endpoints of } \mathit{\Gamma}\} = && \begin{cases} 0, & \text{if } \mathit{\Gamma} \text{ is closed,}\\ \pi, & \text{otherwise,}\end{cases}\\
K_0(\mathit{\Gamma}) &= \int_0^\ell c^{1/n}(s)\,ds && \text{(the weighted length of } \mathit{\Gamma}\text{),}\\
K_j(\mathit{\Gamma}) &= \int_0^\ell |\varkappa_j(s)|\,ds, \qquad j=1,\dots,n-1, && \text{(the "true" integral curvatures),}\\
K_n(\mathit{\Gamma}) &= \pi\times\#\{s\colon \varkappa_{n-1}(s)=0\} && \text{(the number of inflections),}
\end{aligned}
\tag{7}
$$

where $c=c(s)$ is the Gaussian curvature of the ambient manifold M at the corresponding point of the curve.

Theorem [18]**.** *Let M^n be one of the three manifolds*:

1. *the Euclidean space $\mathbb{R}^n$ with the standard flat metric and $c\equiv 0$,*
2. *the sphere $\mathbb{S}^n_r\subset\mathbb{R}^{n+1}$ of radius $r>0$ with the induced metric of constant positive curvature $c\equiv r^n$, or*
3. *the real projective space $\mathbb{P}^n=\mathbb{S}^n_1/\{\pm1\}$ with the induced* (*Fubini–Studi*) *metric of constant positive curvature $c\equiv 1$,*

and $\mathit{\Gamma}$ a generic curve with generalized integral curvatures $K_j(\mathit{\Gamma})$, $j=-1,0,1,\dots,n$, given by (7)*. Then this curve may intersect any "hyperplane"* (*an affine hyperplane in $\mathbb{R}^n$, an equator of $\mathbb{S}^n_r$ or a projective hyperplane in $\mathbb{P}^n$, depending on the type of M*) *by no more than*

$$\tfrac{1}{\pi}\cdot\bigl(w_{n+1}K_{-1}(\mathit{\Gamma})+w_nK_0(\mathit{\Gamma})+\dots+w_0K_n(\mathit{\Gamma})\bigr)$$

isolated points, where the weights w_k are unversal (*do not depend neither on the space M from the list of three, nor even on its dimension*):

$$(w_0,w_1,w_2,w_3,w_4,\dots)=(1,1,2,2,3,\dots)\qquad \text{i.e.}\quad w_k=k-1\quad \textit{for } k\geqslant 4.$$

In the Euclidean case this result can be generalized. Define the *rotation of the curve $\mathit{\Gamma}$ around the origin* as the spherical length of the central projection of this curve on the unit sphere centered at the origin. Then *rotation of $\mathit{\Gamma}$ around an affine subspace L of an arbitrary dimension $k<n-1$ in $\mathbb{R}^n$* can be defined as the rotation of the orthogonal projection of $\mathit{\Gamma}$ on the orthogonal comlement $L^\perp$ along L around the point that is the projection of L.

Theorem [18]**.** *Rotation of a generic smooth curve $\mathit{\Gamma}\subset\mathbb{R}^n$ around any k-dimensional affine subspace does not exceed the weighted sum of the first $k+1$ integral curvatures*

$$w_{k+2}K_{-1}(\mathit{\Gamma})+w_kK_1(\mathit{\Gamma})+\dots+w_0K_{k+1}(\mathit{\Gamma})$$

(*note that in the Euclidean space $K_0(\mathit{\Gamma})$ is always zero*).

Note. The assertion of the above theorem for $n=3$ and $k=1$ was proved by J. Milnor in [Mi]. The case $k=0$ (and n arbitrary) was established in [15] and for $n=2$ implies the Rolle theorem for the Voorhoeve index (see §3E5). It is interesting to observe that this upper estimate does not depend on the dimension n of the ambient Euclidean space, hence one could expect a similar bound for curves in the Hilbert space.

§4. Research currently in progress

In this section I briefly outline the problems I am currently working on, together with some unfinished or incomplete results.

4.1. Critical points of the period function. Consider a Newtonian mechanical system with one degree of freedom with a polynomial potential. If the system is asynchronous, then the period of oscillations as a function of energy is a nonconstant function. The question about the number of intervals of monotonicity in terms of the degree of the potential, is a long-standing problem where only partial results were obtained. I hope to be able to achieve progress in this direction using the theory developed in [14], [17]–[18], see also below.

4.2. Integrable non-Hamiltonian systems. The Abelian integrals arise whenever a polynomial perturbation of a Hamiltonian system is considered. However, if the system is only integrable, then the integrating factor (or the first integral) is not necessarily polynomial. However, in several important cases the first integrals are of the so-called Darboux type (a product of linear factors in real noninteger powers). I hope that the technique established in the algebraic settings can be adjusted to work in this simple transcendental case as well, yielding thus new results on bifurcation of limit cycles in perturbed integrable systems.

4.3. Holomorphic curves and curves defined by systems of first order linear equations. It would be natural to apply the geometric theorems described in §3F to curves defined by systems of first order linear equations with variable coefficients in $\mathbb{R}^n$. In particular, if we were able to answer the question about the number of intersections with hyperplanes for an arbitrary solution of a rather special system of the form

$$(tE + A)\dot{\boldsymbol{x}}(t) = B\boldsymbol{x}(t), \qquad \boldsymbol{x}(t) \in \mathbb{R}^n,\ A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R}), \tag{8}$$

in terms of the matrices A, B, this would solve the problem described in §4.1: one can show that the period function from §4.1 is the first coordinate of one of the solutions to (8) in which the matrices A and B are built in a simple way from critical points/values of the potential.

However, if the linear system has nontrivial invariant subspaces, it is impossible to place upper bounds on the Frenet curvatures $\varkappa_j(\cdot)$ for the trajectories close to these subspaces. One could hope that this difficulty could be circumvented by moving into the complex domain, thus avoiding singularity. Apparently, this approach would require a counterpart of theorems established in §3F for holomorphic curves. I already know one such result for holomorphic curves in $\mathbb{C}^2$, yet the total picture is still obscure.

References

1. S. Yakovenko and L. Kontorer, *Nonlinear semigroups and infinite horizon optimization*, Idempotent Analysis (Maslov V. P. and Samborskiĭ S. N., eds.), AMS publ., Providence, RI, 1992, pp. 167–210.
2. Yu. Ilyashenko and S. Yakovenko, *Finite-differentiable normal forms for local families of diffeomorphisms and vector fields*, Russian Mathematical Surveys **46** (1991), no. 1, 1–43.
3. ———, *Nonlinear Stokes phenomena in smooth classification problems*, Nonlinear Stokes phenomena (Yu. Ilyashenko, ed.), AMS publ., Providence, RI, 1992, pp. 235–287.
4. S. Yakovenko, *Smooth normalization of a vector field around a semistable limit cycle*, Annales de l'Institut Fourier, Grenoble **43** (1993), no. 3, 893–903.
5. Yu. Ilyashenko and S. Yakovenko, *Concerning the Hilbert Sixteenth problem*, Advances in Mathematical Sciences, vol. 23, AMS publ., Providence, RI, 1995, pp. 1–20.
6. ———, *Cyclicité finie des polycycles élémentaires*, Comptes Rendus Acad. Sci. Paris, tome **316** Série **I** (1993), 1081–1086.
7. ———, *Finite Cyclicity of Elementary Polycycles in Generic Families*, Concerning the Hilbert XVIth problem, Advances in Mathematical Sciences, vol. 23, AMS publ., Providence, RI, 1995, pp. 21–95.
8. S. Yakovenko, *The real zeros of a class of Abelian integrals arising in Bifurcation theory*, Selecta Matematica Sovietica, **9** (1990), no. 3, 255–262.
9. ———, *Abelian Integrals as rational envelopes*, Nonlinearity **7** (1994), 1237–1250.
10. Yu. Ilyashenko and S. Yakovenko, *Counting real zeros of analytic functions satisfying linear ordinary differential equations*, Journal of Diff. Equations (1995), accepted for publication.
11. ———, *Double exponential estimate for the number of zeros of complete Abelian integrals*, Inventiones mathematicae **121** (1995), no. 3, 613–650.

12. D. Novikov and S. Yakovenko, *Une borne simplement exponentielle pour le nombre de zéros réels isolés des intégrales complètes abéliennes*, Comptes Rendus Acad. Sci. Paris, tome **320** Série I (1995), 853–858.
13. ———, *Simple exponential estimate for the number of zeros of complete Abelian integrals*, Annales de l'Institut Fourier, Grenoble **45** (1995), no. 4, 897–927.
14. ———, *A Complex Analog of Rolle Theorem and Polynomial Envelopes of Irreducible Differential Equations in the Complex Domain*, J. London Math. Society (1995), accepted for publication.
15. A. Khovanskiĭ and S. Yakovenko, *Generalized Rolle theorem in* $\mathbb{R}^n$ *and* $\mathbb{C}$, Journal of Dynamical and Control Systems **2** (1996), no. 2, 16 pp. (to appear).
16. S. Yakovenko, *A Geometric Proof of the Bautin Theorem*, Concerning the Hilbert Sixteenth problem, Advances in Mathematical Sciences, vol. 23, AMS publ., Providence, RI, 1995, p. 203–219.
17. M. Roitman and S. Yakovenko, *Zeros of Fuchsian functions near singularities with a real spectrum*, Mathematical Research Letters (1996), based on the M. Sc. thesis of M. Roitman, The Weizmann Institute, 1995 (to appear).
18. D. Novikov and S. Yakovenko, *Integral curvatures, oscillation and rotation of spatial curves around affine subspaces*, Journal of Dynamical and Control Systems **2** (1996), 35 pp. (to appear).

Literature cited

[AI] V. Arnold and Yu. Ilyashenko, *Ordinary differential equations*, Dynamical systems–1, Encyclopedia of Mathematical Sciences, vol. 1, Springer, 1988, pp. 1–148.

[Ba] N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium state of the type focus or center*, American Math. Society Translations **100** (1954), Providence, Rhode Island, 1–19; Reprinted in: Stability and Dynamical Systems, American Mathematical Society Translations Series 1, vol. **5** (1962), 396–413.

[Be] G. Belitskiĭ, *Equivalence and normal forms of germs of smooth mappings*, Russian Math. Surveys **33** (1978), no. 1.

[DRV] L. J. Diaz, J. Rocha, M. Viana, *Saddle-node cycles and prevalence of strange attractors*, Preprint (1994).

[DRS] F. Dumortier, R. Roussarie, J. Sotomayor, *Generic* 3*-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case*, Ergodic theory and Dynamical systems **7** (1987), 375–413; *Generic* 3*-parameter families of vector fields on the plane, unfoldings of saddle, focus and elliptic singularities with nilpotent linear parts*, Lecture Notes in Mathematics **1480** (1991), 1–164.

[FP] J.-P. Françoise and C. Pugh, *Keeping track of limit cycles*, J. of Differential Equations **65** (1986), 139–157.

[H] P. Hartman, *Ordinary Differential Equations.*

[I] Yu. Ilyashenko, *Appearance of limit cycles in perturbation of the equation* $\frac{dw}{dz} = -\frac{R_z}{R_w}$, *where* $R(z,w)$ *is a polynomial. USSR Math. Sbornik,* **78** (1969), 360–373; An example of equation $\frac{dw}{dz} = \frac{P_n(z,w)}{Q_n(z,w)}$ having a countable number of limit cycles and an arbitrarily large genre after Petrovskiĭ–Landis. Ibid. **80** (1969), 388–404.

[Kh] A. Khovanskiĭ, *Real analytic varieties with the finiteness property and complex Abelian integrals*, Funct. Anal. Appl. **18** (1984), 119–127.

[Ki] W. J. Kim, *The Schwarzian derivative and multivalence*, Pacific J. of Math. **31** (1969), no. 3, 717–724.

[KS] A. Kotova and V. Stanzo, *On few-parametric families of vector fields on the two-dimensional sphere*, Concerning the Hilbert Sixteenth problem, Advances in Mathematical Sciences, vol. 23, AMS publ., Providence, RI, 1995, p. 155–201.

[L] S. Lang, *Introduction to Complex Hyperbolic Spaces*, Springer, 1987.

[MP] I. P. Malta and J. Palis, *Families of vector fields with finite modulus of stability*, Lecture Notes Math., vol. 898, 1981, pp. 212–229.

[Ma] P. Mardešić, *An explicit bound for the multiplicity of zeros of generic Abelian integrals*, Nonlinearity **4** (1991), 845–852.

[Mi] J. Milnor, *On total curvature of closed space curves*, Mathematica Scandinavica **1** (1953), 289–296.

[Mo] A. Mourtada, *Cyclicité finie des polycycles hyperboliques de champs de vecteurs du plan. Algorithme de finitude*, Annales de l'Institut Fourier de l'Université de Grenoble, tome **41** fascicule 3 (1991), 719–753.

[P] G. Petrov, *On the number of zeros of complete Abelian integrals*, Funct. anal. appl. **18** (1984), 73–74; *Complex zeros of an elliptic integral*, Ibid. **20** (1986), 46–49; *Nonoscillation of elliptic integrals*, Ibid. **24** (1990), 45–50.

[R1] R. Roussarie, *A note on finite cyclicity and Hilbert's 16th problem*, Dynamical systems (proceedings of the Chilean symposium, Valparaiso, 1986, R. Bamon, R. Labarca and J. Palis, eds), Lecture Notes Math., vol. 1331, Springer Verlag, Berlin, 1988, pp. 161-188.

[R2] ———, *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields. Bol. Soc. Brasil Mat.* **17** (1986), 67–101.

[Rs] C. Rousseau, *Bifurcation methods in Polynomial systems*, In: Bifurcations and periodic orbits of vector fields, Kluwer, 1993, pp. 383–428.

[T] F. Takens, *Normal forms for certain singularities of vector fields*, Ann. Inst. Fourier, Grenoble **23** (1973), no. 2, 163–195.

[Va] A. Varchenko, *Estimate of the number of zeros of Abelian integrals depending on parameters and limit cycles*, Funct. Anal. Appl. **18** (1984), no. 2, 98–108.

[V1] M. Voorhoeve, *On the oscillation of exponential polynomials*, Mathematische Zeitschrift **151** (1976), 277–294.

[V2] ———, *A generalization of Descartes' rule*, Journal of London Math. Society (2 Series) **20** (1979), 446–456.

[Yo1] J.-Ch. Yoccoz, *Centralizateurs et conjugaisons différentiables des difféomorphismes du cercle*, Thèse, Université d'Orsay, 1985.

[Yo2] J.-C. Yoccoz, *Linearisation des germes de diffeomorphismes holomorphes de* $(\mathbb{C}, 0)$, Comptes Rendus Acad. Sci. Paris, Série I, t. **306** (1988), no. 1, 55–58.