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INTEGRAL CURVATURES, OSCILLATION AND ROTATION OF SPATIAL CURVES AROUND AFFINE SUBSPACES

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ABSTRACT. The main result of the paper is an upper bound for the oscillation of spatial curves around geodesic subspaces of the ambient space in terms of the integral geodesic curvatures of the curves.

Let $\mathbb{M}^n$ be the Euclidean space $\mathbb{R}^n$, the projective space $\mathbb{P}^n$, or the sphere $\mathbb{S}^n$ equipped with the Riemannian metric of Gaussian curvature $c(\mathbb{M}) = 0, 1$ or $r^{-n} > 0$, respectively, and $\Gamma \subset \mathbb{M}$ be a smooth curve parametrized by the arc length $s \in [0, \ell]$.

For these curves the (geodesic) Frenet curvatures $\varkappa_1(s), \dots, \varkappa_{n-1}(s)$ can be defined, the last one up to the choice of sign in the nonorientable case of $\mathbb{P}^n$. The *generalized inflection points* are defined by the condition that the last curvature $\varkappa_{n-1}(s)$ vanishes.

We prove that the number of intersections of Γ with an arbitrary affine hyperplane $L_{n-1} \subset \mathbb{R}^n$ (respectively, any equator of codimension 1 in the sphere or a projective hyperplane in $\mathbb{P}^n$) can be at most $1/\pi$ times the sum $w_0 K_n(\Gamma) + w_1 K_{n-1}(\Gamma) + \cdots + w_{n-1} K_1(\Gamma) + w_n K_0(\Gamma) + w_{n+1} K_{-1}(\Gamma)$, where

$K_1(\Gamma), \dots, K_{n-1}(\Gamma)$ are (absolute) integral Frenet curvatures of Γ,
$K_n(\Gamma) = \pi \times$ (number of generalized inflection points),
$K_0(\Gamma) = c^{1/n}(\mathbb{M}) \cdot |\Gamma|$, where $|\Gamma|$ is the Riemannian length of Γ,
$K_{-1}(\Gamma) = 0$ or π is $\pi/2$ times the number of endpoints of Γ,
$w_0 = w_1 = 1$, $w_2 = 2$, $w_j = j - 1$ for $j \geqslant 3$ are the universal weights.

For curves in the Euclidean case $\mathbb{M} = \mathbb{R}^n$ a similar estimate can be found for properly defined *rotation around affine subspaces* of arbitrary dimension k between 0 and $n-2$. We show that this rotation can be at most $w_0 K_{k+1}(\Gamma) + \cdots + w_{k-1} K_1(\Gamma) + w_{k+1} K_{-1}(\Gamma)$, where the term $w_k K_0(\Gamma)$ is missing since $c(\mathbb{R}^n) = 0$.

The proof is based on arguments from integral geometry (also called geometric probability) and nonoscillation theory for ordinary linear equations.

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1. Oscillation and rotation around affine subspaces

The principal question posed in this paper can be formulated as follows: given a smooth curve Γ in the Euclidean space $\mathbb{R}^n$ with the known integral Frenet curvatures $\int_\Gamma |\varkappa_j(s)|\,ds$, $j = 1,\dots,n-1$, find an upper bound for the number of intersections of Γ with any affine hyperplane $A \subset \mathbb{R}^n$, $\dim A = n-1$, and, more generally, estimate from above the "rotation" of Γ around any affine subspace of codimension greater than 1 (the notion of rotation needs yet to be defined).

This problem was studied in various settings in numerous publications (especially the hyperplanar case). Our work was inspired by the beautiful paper by John Milnor [1], in which many ideas developed below were already present.

We start with the set of definitions and list some known results concerning oscillatory properties of curves, referring to them as Facts. One of these facts, a theorem by Shapiro [11], [12], is a topological rather than metric assertion that implies a metric result that we formulate as Theorem 1.

Then the main result of the paper, Theorem 2, is formulated. The proof of Theorem 2 is given in Sec. 2. It rests on two auxiliary results from integral geometry (a multidimensional generalization of the Fáry theorem and the averaging property of the rotation index), and a variation on the theme of the Pólya theorem. The proof of Theorem 2 is derived from these auxiliary results, which, in turn, are proved in Sec. 3 and Sec. 4 respectively. In Subsec. 1.5 we formulate and in Subsec. 2.5 prove the counterpart of Theorem 2 for spherical and projective curves. The last section, Sec. 5, contains an alternative proof of Theorem 2 for curves in $\mathbb{R}^3$ using isoperimetric inequalities on the sphere.

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1.1. Settings and definitions. Everywhere below $|X|$ means the natural measure of the set X (the number of points if X is a discrete set, the length of a curve, the area of a surface, etc).

We consider a smooth curve Γ in the Euclidean space $\mathbb{R}^n$, parametrized as $t \mapsto x(t)$, $t \in I = [0,\ell]$, $x(t) = \big(x_1(t),\dots,x_n(t)\big) \in \mathbb{R}^n$. Sometimes we assume that the parametrization is *natural*, i.e., the parameter is the arc length measured along Γ; in this case we denote it by s.

1.1.1. *Integral curvatures and integral inflection.* We denote by $v_k(t)$, for $k = 1, \ldots, n$, the vectors of the *osculating frame*: by definition,

$$v_k : I \to \mathbb{R}^n, \quad t \mapsto v_k(t) = \frac{d^k}{dt^k} x(t), \qquad k = 1, \ldots, n.$$

Definition 1 (regular curves, inflections, hyperconvexity). A parametrized curve Γ is *regular* if the first $n-1$ vectors $v_k(t)$, $k = 1, \ldots, n-1$, are linearly independent for all $t \in I$, whereas the complete set $\{v_k(t)\}_{k=1}^n$ is linearly dependent only at isolated points of Γ. The points where the osculating frame degenerates are called (generalized) *inflection points*, by nalogy with the planar case. A curve without inflection points is said to be *hyperconvex*.

Remark. By the Thom transversality theorem, a *generic* smooth curve is regular.

For the regular curve Γ parametrized by the natural parameter s (the arc length), the osculating frame $\{v_k(s)\}_1^n$ admits orthogonalization; in other words, the tuple of smooth vector functions $e_1(s), \ldots, e_n(s)$ can be constructed in such a way that

(1) for any k between 1 and $n-1$ the vectors $v_1(s), \ldots, v_k(s)$ and $e_1(s), \ldots, e_k(s)$ span the same k-dimensional space and the angle between $e_k(s)$ and $v_k(s)$ is always acute,
(2) all vectors $e_1(s), \ldots, e_n(s)$ taken together constitute a positively oriented orthonormal frame for all $s \in I$.

Since the frame $\{e_k\}$ is orthonormal, the vectors $e_k = e_k(s)$ satisfy the system of linear equations with the antisymmetric matrix of coefficients $A(s) = \{a_{ij}(s)\}_{i,j=1}^n$. The additional observation that the derivative $\frac{d}{ds} e_k(s)$ must belong to the subspace spanned by $e_1(s), \ldots, e_{k+1}(s)$ implies that $a_{ij}(s) \equiv 0$ for all i, j such that $|i - j| \geqslant 2$, and the system must have the form of the *Frenet formulas*

$$\frac{d}{ds} e_k(s) = -\varkappa_{k-1}(s)\, e_{k-1}(s) + \varkappa_k(s)\, e_{k+1}(s), \qquad k = 1, \ldots, n, \quad (1.1)$$

(assuming $\varkappa_0 = \varkappa_n \equiv 0$, $e_0 = e_{n+1} \equiv 0$). The numbers $\varkappa_j = \varkappa_j(s)$, *defined*, in fact, by the Frenet formulas (1.1), are called *Frenet curvatures*. For regular curves the first $n-2$ curvatures are everywhere positive whereas the last one, $\varkappa_{n-1}(s)$, changes sign at inflection points.

Definition 2 (integral curvatures, integral inflection). For $k = 1, \dots, n-1$ we define the kth integral curvature of Γ as

$$K_k(\Gamma) = \int_0^\ell |\varkappa_j(s)|\, ds$$

if the curve is parametrized by the arc length s; for an arbitrary parametrization t this naturally gives the value $K_j(\Gamma) = \int_0^\ell |\varkappa_j(t)| \cdot \|v_1(t)\|\, dt$. Note that for all $k \leqslant n-2$ the Frenet curvatures $\varkappa_j$ are positive, and, hence $|\varkappa_j(s)| = \varkappa_j(s)$.

The *integral inflection* is defined as

$$K_n(\Gamma) = \pi \bar{K}_n(\Gamma) = \pi \cdot \bigl|\{t \in I \colon \varkappa_{n-1}(t) = 0\}\bigr|,$$

where $\bar{K}_n(\Gamma) := \bigl|\{t \colon \varkappa_{n-1}(t) = 0\}\bigr|$ is the number of inflection points on Γ.

1.1.2. *Oscillation and rotation around affine subspaces.* Let L be a linear subspace in $\mathbb{R}^n$ and $A = a + L$ be an affine subspace of dimension $\dim L = \dim A = k$, $0 \leqslant k \leqslant n-1$. We define the oscillation of Γ around A if $\dim A = n-1$ and rotation around A if $\dim A \leqslant n-2$ as the *angular length* of Γ as seen from the origin in the direction orthogonal to L (resp., as seen from the point $A \cap A^\perp$ in the direction $A^\perp$). The formal definition is given first for linear subspaces and then for affine subspaces.

Definition 3 (spherical indicatrix, angular length). If $\Gamma \in \mathbb{R}^n$, $n \geqslant 2$, and $0 \notin \Gamma$, then the *spherical indicatrix* of Γ is the spherical projection of Γ on the unit sphere $\mathbb{S}^{n-1}$ from the origin. The *angular length* of Γ is the (spherical) length of its indicatrix.

Definition 4 (rotation around linear subspaces). If $L \subset \mathbb{R}^n$ is a linear subspace of dimension $k = \dim L \leqslant n-2$, disjoint from Γ, then the rotation $\Omega(\Gamma, L)$ of Γ around L is the angular length of the orthogonal projection of Γ on $L^\perp$ along L.

If $\dim L = n-1$ (i.e., L is a linear hyperplane) and L is transversal to Γ, then $\Omega(\Gamma, L)$ is defined as $\pi \cdot |\Gamma \cap L|$.

Definition 5 (rotation around affine subspaces). If $A = L + a$ is an affine subspace transversal to Γ, then $\Omega(\Gamma, L + a)$ is defined as $\Omega(\Gamma - a, L)$, where $\Gamma - a$ is the parallel translate of Γ.

For an arbitrary affine subspace $A \subset \mathbb{R}^n$ of positive codimension we define $\Omega(\Gamma, A)$ as the upper limit of $\Omega(\Gamma, A_\varepsilon)$ as $\varepsilon \to 0^+$, where A_ε is an affine subspace parallel to A and ε-close to it.

Finally, we introduce the characteristics $\Omega_k(\Gamma)$ for all $k = 0, 1, \dots, n-1$,

$$\Omega_k(\Gamma) = \sup_{\dim A = k} \Omega(\Gamma, A),$$

where the supremum is taken over all affine subspaces of dimension k.

1.1.3. *Remarks.* For a plane curve the angular length is the total variation of the argument along the curve. If $\Gamma \subset \mathbb{R}^2$ is closed, then $(2\pi)^{-1}\Omega(\Gamma, 0)$ gives an upper bound for the topological index of Γ with respect to the origin.

Therefore, for any subspace A of codimension 2 and a closed curve Γ the value $(2\pi)^{-1}\Omega(\Gamma, A)$ majorizes the *linking number* between Γ and A.

For hyperplanes, the definition of rotation is given separately and may seem somewhat artificial. The reason for this is rather simple: for the "unit sphere" $\{\pm 1\} = \mathbb{S}^0 \subset \mathbb{R}^1$ the geodesic distance between antipodal points is not defined whereas for all other spheres $\mathbb{S}^k \subset \mathbb{R}^{k+1}$, $k > 0$, it is equal to π. If we extend the definition of the geodesic distance for the sphere $\mathbb{S}^0$ in the appropriate way, then the definition of $\Omega(\Gamma, L)$ will become more uniform. Similar reasons motivate the occurrence of the factor π in the definition of the integral inflection.

Besides, in Sec. 3 we prove several results concerning average values of integral curvatures and rotations. It turns out that they remain valid for the last "curvature" K_n and the hyperplane "rotation" Ω_{n-1} if the factor π is properly placed in their definitions.

1.2. Nonoscillation theorems: starting points. After all the definitions are given, we list some known results in the spirit of the implication "bounded curvatures $\Longrightarrow$ bounded rotation." In order to write in a similar way the inequalities concerning both closed and nonclosed curves, we introduce the notation $|\partial\Gamma|$ for the number of endpoints of the curve Γ: this number is zero if Γ is closed, 2 if Γ is simply connected, etc.

1.2.1. *Rotation around hyperplanes.* The simplest case, in fact, an elementary exercise, concerns an oscillation of plane curves $\Gamma \subset \mathbb{R}^2$ around straight lines.

Fact 1 (see, e.g., [1]).

$$\Omega_1(\Gamma) \leqslant K_1(\Gamma) + K_2(\Gamma) + \pi\,|\partial\Gamma|. \tag{1.2}$$

The proof is based on the Rolle theorem and the following observation: if the tangent vectors at two endpoints of the curve are parallel to the same line, then either the integral curvature of the curve is at least π, or there should be an inflection point.

Most other results concerning the oscillation around hyperplanes are formulated on the assumption that the curve is hyperconvex. Perhaps, the most general among them is a corollary from the theorem by Shapiro [11] (see also [12]). The Shapiro theorem is topological and concerns *oscillating curves* in $\mathbb{R}^n$. In an obvious way, for any curve in $\mathbb{R}^n$ the lower bound $\Omega_{n-1}(\Gamma) \geqslant \pi n$ is valid since there always exists a hyperplane passing through n arbitrary points of the curve. The curves for which $\Omega_{n-1}(\Gamma) = \pi n$ are *nonoscillating* (for obvious reasons) and *oscillating* otherwise. The Shapiro theorem asserts that if a curve is *oscillating*, then its osculating frame makes, in some sense, a full turn in the flag space (see 2.2.3 below for the exact formulation; we derive this theorem as a corollary of our main result). Since the velocity of rotation of the osculating frame is naturally measured by the Frenet curvatures, this observation implies the following sufficient condition for nonoscillation.

Theorem 1. *If the curve $\Gamma \subset \mathbb{R}^n$ is hyperconvex and*

$$\int_\Gamma \sqrt{\varkappa_1^2(s) + \cdots + \varkappa_{n-1}^2(s)}\, ds < \frac{1}{n\sqrt{2}}, \tag{1.3}$$

then the curve is nonoscillating, i.e., $\Omega_{n-1}(\Gamma) = \pi n$.

The proof of this theorem is given below in Subsec. 4.3. Since any regular curve can be partitioned into $\pi^{-1}K_n(\Gamma) + 1$ hyperconvex pieces and any one of these pieces can, in turn, be subdivided into sufficiently short arcs satisfying (1.3), we can give the following corollary.

Corollary 1.

$$\Omega_{n-1}(\Gamma) \leqslant \pi n + \pi\sqrt{2}\,n^2 \cdot [K_1(\Gamma) + \cdots + K_{n-1}(\Gamma)] + n\,K_n(\Gamma). \tag{1.4}$$

This inequality, however, cannot be accurate enough since any connection between nonoscillating arcs is lost. The main result of this paper, Theorem 2, gives a better value (linear in n) for the coefficients (note that as n grows, the largest coefficients on the right-hand side have the order of magnitude of n^2). Some other particular results concerning the oscillation of hyperconvex curves around hyperplanes are mentioned in [11].

1.2.2. *Rotation of subspaces of codimension* $\geqslant 2$. This group of results is less numerous. Two main examples are the Milnor theorem on a linking number between closed curves and straight lines in $\mathbb{R}^3$ [1] and a theorem by Khovanskiĭ and the second author [6] on the rotation around points.

Fact 2 (cf. Milnor [1, Theorem 3]).

$$\forall \Gamma \subset \mathbb{R}^3 \qquad \Omega_1(\Gamma) \leqslant K_1(\Gamma) + K_2(\Gamma) + \pi\,|\partial\Gamma|. \tag{1.5}$$

In fact, Milnor writes an upper estimate for the linking number between a closed spatial curve and any straight line in $\mathbb{R}^3$, but his arguments also prove a stronger result concerning the rotation, which can be derived using the averaging principles described below, from estimate (1.2).

The following result gives an upper bound for the rotation of Γ around any point and is valid in all dimensions.

Fact 3 (Khovanskiĭ and Yakovenko [6]).

$$\forall \Gamma \subset \mathbb{R}^n \qquad \Omega_0(\Gamma) \leqslant K_1(\Gamma) + \frac{\pi}{2}\,|\partial\Gamma|. \tag{1.6}$$

1.3. Formulation of the main result. Inequalities (1.2) compared to (1.5) and (1.6) suggest that in the general case an upper bound for $\Omega_k(\Gamma)$ can be found in the form of a weighted sum of the integral curvatures $K_1(\Gamma)$ through $K_{k+1}(\Gamma)$, with a constant term added if the curve is nonclosed; besides, one can hope that the weights can be chosen *independently of the dimension n of the ambient space.* Our main result claims that this hope can indeed be justified.

Let $\Gamma \subset \mathbb{R}^n$ be a smooth regular curve with $|\partial\Gamma|$ endpoints (i.e., not necessarily connected).

Theorem 2 (main). *The oscillation of any regular curve Γ satisfies the following inequalities:*

$$\begin{aligned}
\Omega_0(\Gamma) &\leqslant \tfrac{1}{2}\pi|\partial\Gamma| + K_1(\Gamma),\\
\Omega_1(\Gamma) &\leqslant \pi|\partial\Gamma| + K_1(\Gamma) + K_2(\Gamma),\\
\Omega_2(\Gamma) &\leqslant \tfrac{3}{2}\pi|\partial\Gamma| + 2K_1(\Gamma) + K_2(\Gamma) + K_3(\Gamma),\\
\Omega_3(\Gamma) &\leqslant 2\pi|\partial\Gamma| + 2K_1(\Gamma) + 2K_2(\Gamma) + K_3(\Gamma) + K_4(\Gamma),
\end{aligned}$$

and, in general, for any $k \leqslant n-1$ we have

$$\Omega_k(\Gamma) \leqslant \frac{1}{2}\pi(k+1)|\partial\Gamma| + \sum_{j=1}^{k+1} w_{k+1-j}K_j(\Gamma), \tag{1.7}$$

where the sequence of weights

$$w_0 = w_1 = 1, \quad w_2 = w_3 = 2, \quad w_j = j-1 \qquad \text{for } j = 4,5,\dots \tag{1.8}$$

is universal.

The inequality for Ω_{n-1} implies the sufficient condition for nonoscillation.

Corollary 2. *A hyperconvex curve whose integral Frenet curvatures are small enough to satisfy the inequality*

$$\sum_{j=1}^{n-1} w_{n-j}K_j(\Gamma) < \pi \tag{1.9}$$

is nonoscillating.

This inequality is stronger than (1.3). In fact, one could well add to the left-hand side the last "curvature" and drop the hyperconvexity assumption since $K_n(\Gamma) \geqslant \pi$ for curves with inflections

1.4. Remarks. Inequalities (1.7) give an *upper bound* for the number of intersections of Γ with *any* affine hyperplane. However, most hyperplanes intersect Γ at substantially smaller number of points. In particular, a random uniformly distributed hyperplane passing, say, through the origin (in order to avoid noncompactness of the set of all affine hyperplanes), intersects Γ at no more than $\pi^{-1}K_1(\Gamma) + 1$ points.

It is interesting to observe that the estimate established by Theorem 2 is stable with respect to the dimension of the ambient space: if we fix k, then the upper bound for $\Omega_k(\Gamma)$ is *independent* of the dimension n. This fact suggests that the analog of Theorem 2 also holds for curves in infinite-dimensional Hilbert spaces.

Inequalities (1.7) are sharp for small $k = 0, 1$ and perhaps for $k = 2$. All the way around, inequality (1.3) is *not* sharp even for small n. It is clear from Subsec. 4.3 how the latter can be improved, although at the price of rather sophisticated computations. However, one can relatively easily establish the implication

$$K_1(\Gamma) + K_2(\Gamma) < \frac{\pi}{2} \implies \Gamma \text{ is nonoscillating,} \tag{1.10}$$

for three-dimensional hyperconvex curves (see Subsec. 4.4). But even this result is inferior to the inequality $2K_1(\Gamma) + K_2(\Gamma) < \pi$ guaranteeing nonoscillation of hyperconvex three-dimensional curves, a special case of (1.9) for $n = 3$ and $K_3(\Gamma) = 0$.

The final remark concerns the choice of the weights w_j in (1.8); obviously, without discussing this matter it is not possible to analyze the sharpness of the inequalities obtained. This choice is defined by Lemma 4 on the roots of solutions to linear ordinary differential equations; see Sec. 4. From a geometrical point of view the integrand appearing in (1.3) seems to be more natural (it admits interpretation as the angular velocity of rotation of the osculating orthogonal frame of a curve). However, reasonably sharp estimates involving $\int_\Gamma (\varkappa_1^2(s) + \cdots + \varkappa_{n-1}^2(s))^{1/2}\, ds$ are not yet available.

1.5. Spherical and projective curves. The main result admits reformulation for spherical and projective curves. Recall that for any Riemannian n-dimensional manifold M^n and any sufficiently smooth curve $\Gamma\colon [0, \ell] \to M$ one can define the osculating frame $v_j(t) \in T_{x(t)}M$, $j = 1, \ldots, n$, in the same way as for curves in a Euclidean space, and this frame can be similarly orthogonalized with the only exception that the last vector $e_n(t)$ is defined modulo multiplication by ± 1 for the nonorientable manifold M.

1.5.1. *Geodesic curvatures.* We denote by ∇ the operator of covariant differentiation with respect to the natural (Levi-Civita) connection compatible with the metric on M. Then, assuming that the curve is parametrized by the arc length, one can easily see that

$$\nabla_{e_1(t)} e_j(t) = \widetilde{\varkappa}_{j-1}(t) e_{j-1}(t) + \widetilde{\varkappa}_j(t) e_{j+1}(t) \qquad j = 1, \dots, n, \tag{1.11}$$

with the standard convention that $e_0(t) \equiv e_{n+1}(t) \equiv 0$. The functions $\widetilde{\varkappa}_j(t)$, defined by (1.11), are called *geodesic Frenet curvatures* (in the nonorientable case only the modulus of the last curvature $\pm\widetilde{\varkappa}_{n-1}(t)$ is defined). However, the integral geodesic curvatures $\widetilde{K}_j(\Gamma)$, $j = 1, \dots, n$, relative to M, make sense: the last one is π times the number of points, where $\widetilde{\varkappa}_{n-1}$ vanishes.

We will be interested in the two simplest Riemannian manifolds:

- the sphere $r \cdot \mathbb{S}^n$ of radius $r > 0$ that inherits its metric from the embedding in the Euclidean space $\mathbb{R}^{n+1}$, and
- the real projective space $\mathbb{P}^n$ obtained as a quotient space of the *unit* sphere $\mathbb{S}^n$ by identifying the opposite points $\pm x$. The spherical metric induces the *Fubini-Study* metric on $\mathbb{P}^n$: the length of each line is equal to π.

1.5.2. *Oscillatory behavior on $\mathbb{S}^n$ and in $\mathbb{P}^n$.* For a spherical curve it makes sense to ask how many times it can intersect an *equator*, the nearest analog of a hyperplane. For projective curves one may look for an upper bound for the number of intersections with any projective hyperplane of codimension 1 in $\mathbb{P}^n$.

The following two corollaries of Theorem 2 giving answers to these questions are proved in Subsec. 2.5.

Theorem 3. *Let $\Gamma \subset r\cdot\mathbb{S}^n$ be a spherical curve with the geodesic integral curvatures $\widetilde{K}_j(\Gamma)$, $j = 1, \dots, n$, of the spherical length $\widetilde{K}_0(\Gamma) = |\Gamma|$. Then Γ can intersect any equator (embedded sphere $r \cdot \mathbb{S}^{n-1}$) at no more than*

$$\frac{1}{2} n |\partial\Gamma| + w_n \widetilde{K}_0(\Gamma)/\pi r + \sum_{j=1}^{n} w_{n-j} \widetilde{K}_j(\Gamma)/\pi \tag{1.12}$$

isolated points, where w_j are the same as before (1.8).

The upper bound (1.12) turns into (1.17) in the limit $r \to +\infty$, as one could expect. As a natural corollary, we obtain a similar result for projective curves.

Theorem 4. *The projective curve $\Gamma \subset \mathbb{P}^n$ of length $\widetilde{K}_0(\Gamma) = |\Gamma|$ intersects any projective hyperplane $P^{n-1} \subset \mathbb{P}^n$ at no more than $\frac{1}{2} n |\partial\Gamma| + \sum_{j=0}^{n} w_{n-j} \widetilde{K}_j(\Gamma)/\pi$ points.*

Remark. The boundary term can be incorporated into the sum of integral curvatures without changing the law (1.8) for the weights if we set

$$K_0(\Gamma) = c^{1/n}|\Gamma|, \qquad K_{-1}(\Gamma) = \frac{\pi}{2}|\partial\Gamma|, \tag{1.13}$$

where $c = c(M) > 0$ is the Gaussian curvature of the ambient manifold M in each of the three cases, $M = \mathbb{R}^n$, $r \cdot \mathbb{S}^n$, or $\mathbb{P}^n$. The universal relation embracing all these cases will then have the form

$$\pi \times \{\text{number of intersections with any "hyperplane"}\} \leqslant$$
$$\leqslant w_{n+1}K_{-1}(\Gamma) + w_nK_0(\Gamma) + \cdots + w_0K_n(\Gamma) = \sum_{j=-1}^{n} w_{n-j}K_j(\Gamma). \tag{1.14}$$

However, this relation is justified only *a posteriori*. It seems to be an intriguing problem to find a direct proof of (1.14) for other classes of manifolds.

2. Demonstration of the main result

This section is the core of the paper. The two main components of the proof are the averaging lemmas for rotation and integral curvatures belonging to the realm of geometric probability (also called integral geometry), and a variation on the theme of Pólya [8] on zeros of solutions of linear ordinary differential equations that admits reformulation in geometric terms. In this section we derive Theorem 2 from these results. The demonstration of the two integral geometric lemmas is postponed until Sec. 3, and the Pólya theorem and its generalizations are discussed in Sec. 4.

2.1. Geometric Probability. From now on we will deal with only linear (not affine) subspaces of the ambient Euclidean space $\mathbb{R}^n$. By $\mathbb{S}^{n-1}$ we denote the standard unit sphere with the Lebesgue $(n-1)$-dimensional measure $d\sigma_{n-1}$. For $p \in \mathbb{S}^{n-1}$ we denote by $\mathbb{R}p$ the line spanned by p; if $L \subset \mathbb{R}^n$ is a linear subspace, then $L^\perp$ is its orthogonal complement, and $P_L \colon \mathbb{R}^n \to L^\perp$ is the orthogonal projection on $L^\perp$ along L. If $L = \mathbb{R}p$, then we write $p^\perp$ and P_p instead of $(\mathbb{R}p)^\perp$ and $P_{\mathbb{R}p}$ respectively. Sometimes instead of $L^\perp$ and $p^\perp$ we will write $\mathbb{R}^{n-k}_L$ and $\mathbb{R}^{n-1}_p$, where $k = \dim L$.

2.1.1. *Averaging integral curvatures.* The first main result means that each integral curvature $K_j(\Gamma)$ can be restored by averaging the corresponding integral curvatures of the orthogonal projections $P_p(\Gamma)$ in a random direction. Note that $P_p(\Gamma)$ is a hyperplane curve and as such possesses a complete set of integral curvatures $K_j(P_p(\Gamma))$, $j = 1, \dots, n-1$, relative to the hyperplane $p^\perp$ (as usual, the last one is the integral inflection of the corresponding projection).

Lemma 1.

$$\forall j = 1, \dots, n-1 \qquad K_j(\Gamma) = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} K_j\big(P_p(\Gamma)\big)\, d\sigma_{n-1}(p). \quad (2.1)$$

Here $|\mathbb{S}^{n-1}|$ stands for the $(n-1)$-dimensional volume of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

Remark. The factor π was introduced in the definition of K_n to make this relation valid for $j = n-1$. However, the proof for this case requires special considerations.

Remark. The case $n = 3$, $j = 1$ (averaging property of integral curvature for spatial curves), is known as the Fáry theorem [4]. In fact, minor modifications can be made to extend the proof of Fáry for any n (and $j = 1$, as before).

The case $n = 3$, $j = 2$ (integral torsion of spatial curves) was proved by Milnor in [1]. The proof given by Milnor with only minor modifications works for any n and $j = n-1$.

The averaging property for intermediate curvatures seems to be a new result.

Remark. In general, the projection of a regular curve is *not* a regular curve: it can even be nonsmooth, as the simplest examples already show. However, from the Sard theorem it follows that the Lebesgue measure of directions $p \in \mathbb{S}^{n-1}$ corresponding to "bad" projections is zero, and, hence one may disregard these pathologies when computing the average.

2.1.2. *Averaging rotation.* Let L be a linear subspace of dimension $k \leqslant n-2$ and $p \in \mathbb{S}^{n-1}$ be a vector on the unit sphere. Then, for almost all p, the linear sum $L + \mathbb{R}p$ is a $(k+1)$-dimensional subspace.

Lemma 2.

$$\Omega(\Gamma, L) = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} \Omega(\Gamma, L + \mathbb{R}p)\, d\sigma_{n-1}(p). \qquad (2.2)$$

Remark. The factor π is introduced in the definition of $\Omega(\Gamma, \cdot)$ for hyperplanes in order that this equality would remain valid for subspaces L of dimension $n-2$.

As before, the fact that for a metrically negligible set of directions the subspace $L + \mathbb{R}p$ degenerates does not affect the integral.

2.2. Flags, inflections and oscillation around hyperplanes. The second (analytic) component of the proof of Theorem 2 is introduced in this section. The principal result of it is an inequality relating the number of intersections of a space curve with an arbitrary affine hyperplane, with a number of inflection points of orthogonal projections of this curve onto a family of (linear) subspaces of all intermediate directions.

2.2.1. *Flags.* Recall that a (complete) flag $\mathcal{L}$ in the linear n-dimensional space L is a chain of embedded linear subspaces of L of increasing dimensions:

$$\mathcal{L} = \{L_j\}_{j=0}^n, \quad 0 = L_0 \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_n = L, \quad \dim L_j = j.$$

If $v_1, \dots, v_n$ is any *ordered* tuple of vectors in n, then the flag *spanned* by this tuple is the flag whose jth subspace is spanned by the first j vectors. Any *orthogonal* frame $e_1, \dots, e_n$ can be almost uniquely restored from the flag it spans: the map

$$(e_1, \dots, e_n) \mapsto (\operatorname{Span}(e_1), \operatorname{Span}(e_1, e_2), \dots, \operatorname{Span}(e_1, \dots, e_n))$$

is a covering of the *flag variety* by the orthogonal group $\mathrm{SO}(n)$ with the discrete fiber of 2^{n-1} points.

2.2.2. *Inflections relative to a flag and the Third Principal Lemma.* Let $\Gamma \subset \mathbb{R}^n$ be a regular curve and $\mathcal{L} = \{L_j\}_{j=1}^n$, $L_n = \mathbb{R}^n$, be a complete flag. We denote by Γ_j the orthogonal projection of Γ on L_j parallel to $L_j^\perp$ for all $j = 1, \dots, n$, so that $\Gamma_n = \Gamma$. Each Γ_j is a j-dimensional curve and as such possesses j integral curvatures $K_i(\Gamma_j)$, $i = 1, \dots, j$. The last of them is the (relative) integral inflection $K_j(\Gamma_j) = \pi\nu_j$, $\nu_j = \nu_j(\Gamma, \mathcal{L}) \in \mathbb{Z}_+$, where ν_j is the number of inflection points of the projection Γ_j.

Remark. We have extended the notion of an inflection point for parametrized "curves" in $\mathbb{R}^1$: by definition, the inflection point of the "curve" $t \mapsto x(t) \in \mathbb{R}^1$ is the point where the Wronski determinant $\langle v_1 \rangle = |x'(t)|$ vanishes, in other words, the critical point of the map $x(\cdot)\colon [0, \ell] \to \mathbb{R}^1$. The "curve" is hyperconvex if it has no "inflections." This convention will be adopted from now on.

Now we can formulate the main result of this subsection. Let $w_0, w_1, w_2, \dots$ be the sequence of weights introduced in (1.8).

Lemma 3. *If $\Gamma \subset \mathbb{R}^n$ is a regular curve and $\mathcal{L} = \{L_j\}_{j=1}^n$ is a complete flag such that the orthogonal projection of Γ onto L_j has $0 \leqslant \nu_j = \nu_j(\Gamma, \mathcal{L}) < \infty$ inflection points, then*

(1) *the curve Γ intersects any affine hyperplane at no more than*

$$\frac{1}{2}n|\partial\Gamma| + \sum_{j=1}^{n} w_{n-j}\nu_j$$

isolated points, so that

$$\Omega_{n-1}(\Gamma) \leqslant \tfrac{1}{2}\pi n|\partial\Gamma| + \sum_{j=1}^{n} w_{n-j}\,\pi\nu_j(\Gamma,\mathcal{L}), \tag{2.3}$$

(2) *the velocity curve $\dot{\Gamma}$: $[0,\ell] \ni t \mapsto \dot{x}(t) \in \mathbb{R}^n$ intersects any linear hyperplane at no more than $\frac{1}{2}(n-1)|\partial\dot{\Gamma}| + \sum_{j=1}^{n} w_{n-j}\nu_j$ isolated points.*

This result is a geometric version of a theorem by Pólya [8], [9] on the zeros of solutions of linear ordinary differential equations (1922). The proof of Lemma 3 is given in Sec. 4 together with a discussion of related topics and some historical notes.

2.2.3. *Digression*: *Shapiro theorem.* As a corollary of Lemma 3 we obtain a condition describing *oscillating* (i.e., nonnonoscillating) curves.

Corollary 3. *If $\Gamma \subset \mathbb{R}^n$ is oscillating, then, for any complete flag $\mathcal{L}$, the projections $\Gamma_j \subset L_j$ cannot be all hyperconvex*:

$$\nu_1(\Gamma,\mathcal{L}) + \cdots + \nu_n(\Gamma,\mathcal{L}) > 0. \tag{2.4}$$

In fact, the assertion of Corollary 3 can be formulated more naturally, using the notions of an *osculating flag* and the *transversality* of flags.

Definition 6. Two flags $\mathcal{L}$ and $\mathcal{L}' = \{L'_j\}$ in the same space are *transversal* if, for any pair of indices i, j such that $i + j \geqslant n$, the subspaces L_i and L'_j are transversal.

The flag $\mathcal{L}'$ is said to be *orthogonal* to the flag $\mathcal{L}$ if its subspaces are orthogonal complements of the subspaces of $\mathcal{L}$ (naturally, taken in the reverse order): $L'_j = L^{\perp}_{n-j}$, $j = 1, \ldots, n$. The flag orthogonal to $\mathcal{L}$ is denoted by $\mathcal{L}^{\perp}$.

If $L = \mathbb{R}^n$ and an orthogonal coordinate system is fixed by specifying an orthogonal frame $e_1, \ldots, e_n$, then the *standard flag* $\mathcal{E} = \{E_j\}_{j=0}^{n}$ is spanned by the basis vectors. The orthogonal flag $\mathcal{E}^{\perp}$ is sometimes referred to as the *antipodal flag*.

Definition 7. The *osculating flag* $\mathcal{L}_\Gamma(t)$ of the regular hyperconvex curve Γ is a (variable) flag spanned by the osculating frame.

Remark. If a curve has inflection points, then at these points the osculating frame does not span a flag (or, more precisely, the flag spanned by the frame is not complete) since the vectors from the osculating frame are linearly dependent. However, since we have assumed regularity of the curve, all k-dimensional subspaces of the osculating flag are well defined for $k < n$, and for $k = n$ we assume that the last subspace is always $\mathbb{R}^n$.

From these definitions it is almost obvious that if $x \in \Gamma$ is a point on the curve, which becomes an inflection point of the projection Γ_j, then the jth subspace of the osculating flag $\mathcal{L}_\Gamma(x)$ is *non*transversal to the subspace $L_j^\perp$ of the orthogonal flag $\mathcal{L}^\perp$. Thus we arrive at the reformulation of Corollary 3.

Corollary 4 (Shapiro theorem [11], [12]). *If the hyperconvex regular curve $\Gamma \subset \mathbb{R}^n$ is oscillating, then, for any complete flag $\mathcal{L}$, there exists at least one point $x \in \Gamma$ such that $\mathcal{L}_\Gamma(x) \not\pitchfork \mathcal{L}$.*

In fact, this theorem is valid also for projective curves. It should be pointed out that the Shapiro theorem generalizes, to a certain extent, the Rolle theorem (consider the case of plane convex curves). There are some other Rolle-type theorems; see [6], [7]. In addition, the (classical) Rolle theorem is the key tool in the demonstration of Lemma 3; see Sec 4.

2.3. Demonstration of Theorem 2 for hyperplanes. To prove Theorem 2 for hyperplanes and estimate $\Omega_{n-1}(\Gamma)$, we construct a flag $\mathcal{L} = \{L_j\}_{j=1}^n$ in such a way that the weighted sum of integral curvatures for the projection Γ_j of $\Gamma = \Gamma_n$ on *each* subspace L_j, $1 \leqslant j \leqslant n$, is bounded in terms of the weighted sum of curvatures of the original curve. This automatically provides upper bounds for the corresponding integral inflections of the projections, and therefore it remains to apply Lemma 3 in order to estimate the number of intersections.

The construction of the flag $\mathcal{L}$ is carried out by induction on the *co*dimension of the subspaces. Let $w_0, w_1, \dots, w_{n-1}$ be the weights (1.8) and

$$\begin{aligned}T(\Gamma) &= w_{n-1}K_1(\Gamma) + \cdots + w_1 K_{n-1}(\Gamma) + w_0 K_n(\Gamma)\\ &= w_{n-1}K_1(\Gamma) + \cdots + w_1 K_{n-1}(\Gamma) + \pi w_0 \nu_n\end{aligned}$$

be the weighted sum of the integral curvatures, where ν_n is the number of inflection points of Γ.

The sum of the first $n-1$ terms admits averaging: by Lemma 1 the value $\sum_{j=1}^{n-1} w_{n-j}K_j(\Gamma)$ is equal to the average value of the function $\chi(p) = \sum_{j=1}^{n-1} w_{n-j}K_j(P_p(\Gamma))$ on the sphere $\mathbb{S}^{n-1}$, where $P_p(\Gamma)$ is the orthogonal projection of Γ onto the hyperplane $p^\perp$ parallel to a random vector $p \in \mathbb{S}^{n-1}$.

The function $\chi(p)\colon \mathbb{S}^{n-1} \to \mathbb{R}$ does not exceed its average value at some point $p \in \mathbb{S}^{n-1}$. We denote the corresponding normal hyperplane by $L_{n-1} = p^\perp$ (it will play the role of the $(n-1)$-dimensional subspace of the flag $\mathcal{L}$), the projection of Γ onto L_{n-1} by Γ_{n-1}, and the number of inflections of this projection by ν_{n-1}. Then

$$w_{n-1}K_1(\Gamma_{n-1}) + \cdots + w_2K_{n-2}(\Gamma_{n-1}) + \pi w_1\nu_{n-1} + \pi w_0\nu_n \leqslant T(\Gamma).$$

The procedure of averaging can be repeated once again, this time applied to the truncated sum $\sum_{j=1}^{n-2} w_{n-j}K_j(\Gamma_{n-1})$, and a subspace $L_{n-2} \subset L_{n-1}$ of codimension 1 in L_{n-1} and, hence, of codimension 2 in $\mathbb{R}^n$, can be found such that for the projection Γ_{n-2} of Γ onto L_{n-2} one has the inequality

$$w_{n-1}K_1(\Gamma_{n-2}) + \cdots + w_3K_{n-3}(\Gamma_{n-2}) + \pi w_2\nu_{n-2} + \\ + \pi w_1\nu_{n-1} + \pi w_0\nu_n \leqslant T(\Gamma),$$

where, obviously, ν_{n-2} is the number of inflections of the curve $\Gamma_{n-2} \subset L_{n-2}$.

Iterating these arguments n times, we construct all subspaces L_{n-1}, L_{n-2}, ..., L_2, L_1 of the flag $\mathcal{L}$, and for the number of inflection points of the corresponding projections we have the inequality

$$\pi(w_{n-1}\nu_1 + \cdots + w_1\nu_{n-1} + w_0\nu_n) \leqslant T(\Gamma).$$

Applying the first assertion of Lemma 3, we infer that the number of points of intersection of Γ with any affine hyperplane does not exceed $n|\partial\Gamma|/2 + T(\Gamma)/\pi$, and, hence the oscillation satisfies the inequality

$$\Omega_{n-1}(\Gamma) \leqslant \frac{1}{2}\pi n|\partial\Gamma| + T(\Gamma) = \frac{1}{2}\pi n|\partial\Gamma| + \sum_{j=1}^{n} w_{n-j}K_j(\Gamma).$$

The proof for the codimension 1 case is complete.

2.4. Demonstration of Theorem 2 in the general case. The proof in the general case is carried out by induction on the codimension of subspaces, the hyperplane case being the base of induction.

Suppose that the inequalities of Theorem 2 are already established for all codimension c subspaces and any dimension of the ambient space ($c = 1$ corresponds to hyperplanes). Take a linear subspace L of dimension $k - 1$ and codimension $c + 1$, so that $n = c + k$. Now let p be a variable vector on the unit sphere in $\mathbb{S}^{n-1}$ and P_p be the corresponding orthogonal projection onto $H_p := p^{\perp}$. Then for almost all p the projection $L_p := P_p(L) \subset H_p$ has dimension $k - 1$, and, hence, codimension c, and therefore, for the projection $\Gamma_p = P_p(\Gamma)$ we know by the induction assumption that the oscillation (relative to H_p) satisfies the inequality

$$\forall p \in \mathbb{S}^{n-1} \qquad \Omega(\Gamma_p, L_p; H_p) \leqslant \frac{1}{2}\pi k|\partial\Gamma_p| + \sum_{j=1}^{k} w_{k-j}K_j(\Gamma_p; H_p), \quad (2.5)$$

where $\Omega(\cdot,\cdot;H_p)$ is the oscillation around $k-1$-dimensional subspace $L_p \subset H_p$ and w_j are the weights introduced in (1.8).

Note that

$$\Omega(\Gamma_p, L_p; H_p) = \Omega(\Gamma, L + \mathbb{R}p; \mathbb{R}^n) \tag{2.6}$$

since both sides are, by definition, the angular lengths of the *same* curve in $L_p^\perp \cap H_p = (L + \mathbb{R}p)^\perp$.

The number of endpoints $|\partial\Gamma_p|$ is the same as $|\partial\Gamma|$ for almost all p. After averaging inequality (2.5), we infer from (2.6) and (2.2) that the left-hand side turns into $\Omega(\Gamma, L)$ by Lemma 2, and the average of the right-hand side by Lemma 1 and the linearity is the weighted sum of integral curvatures of the original curve Γ. Thus the inequality

$$\Omega(\Gamma, L) \leqslant \frac{1}{2}\pi k|\partial\Gamma| + \sum_{j=1}^{k} w_{k-j} K_j(\Gamma)$$

is established for subspaces of codimension $c+1$ as well, and, hence, by induction, Theorem 2 is proved in full generality. □

2.5. Oscillation of spherical and projective curves: proof of Theorems 3 and 4. The basic case is that of curves on the unit sphere $\mathbb{S}^{n-1}$.

Consider a regular curve Γ parametrized by the arc length. Its velocity curve $\dot\Gamma \colon t \mapsto \widetilde{x}(t) = \dfrac{d}{dt}x(t)$ belongs to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. If at the end of the proof of Theorem 2 (the hyperplane case, Subsec. 2.3) we replace the reference to the first claim of Lemma 3 by the second one, we shall obtain an upper bound for the oscillation of $\dot\Gamma$ in terms of the integral curvatures of the primitive curve Γ. To obtain the proof of Theorem 3, it remains to recompute the integral curvatures of Γ in terms of the geodesic curvatures of $\dot\Gamma$.

If $e_1(t), \dots, e_n(t)$ is the Frenet frame for Γ, then $\widetilde{e}_1(t) = e_2(t), \dots, \widetilde{e}_{n-1}(t) = e_n(t)$ is the Frenet frame for $\dot\Gamma$ considered as a spherical curve; to obtain the full Frenet frame for the same curve considered as a space curve, one needs to add $\widetilde{e}_n(t) = \pm e_1(t)$, the (unitary) radius-vector of $\dot\Gamma$. However, the parameter t *is not* the natural parameter on $\dot\Gamma$ since

$$\frac{d}{dt}\widetilde{x}(t) = \frac{d}{dt}e_1(t) = \varkappa_1(t)e_2(t) = \varkappa_1(t)\widetilde{e}_1(t).$$

The covariant derivative on a submanifold of the Euclidean space (equipped with the induced metric) admits the following simple description (see [2, Ch. 2, 3.1]): one should take the (usual) derivative with respect to the ambient Euclidean space and project the result orthogonally onto the space tangent to the submanifold. Since the Frenet formulas have to be written with respect to the arc length parameter s on $\dot{\Gamma}$, we arrive at the set of relations

$$\nabla_{\widetilde{e}_1(s)}\widetilde{e}_j(s) = \Pi\Big(\frac{d}{dt}e_{j+1}(t)\Big)\cdot\frac{dt}{ds}, \qquad j=1,\dots,n-1,$$

where $\Pi = \Pi_t$ stands for the orthogonal projection onto the tangent subspace to the sphere $\mathbb{S}^{n-1}$ (at the corresponding point $\widetilde{x}(t)$). The Frenet formulas (3.15) for the derivatives $\frac{d}{dt}e_j(t)$ together with the identity $dt/ds = 1/\varkappa_1(s)$ immediately yield

$$\nabla_{\widetilde{e}_1(s)}\widetilde{e}_j(s) = \varkappa_1^{-1}(s)\Pi(-\varkappa_j\widetilde{e}_{n-1} + \varkappa_j\widetilde{e}_{j+1}), \qquad j=1,\dots,n-1,$$

where $\widetilde{e}_0(s) = \widetilde{x}(s)$ is the radius-vector of $\dot{\Gamma}$ and $\widetilde{e}_n \equiv 0$ by definition. In fact, the projection Π leaves all right-hand sides unchanged, except for $j=1$, where Π kills the term proportional to $\widetilde{x}(s)$, normal to the sphere. Comparing the remainder with equalities (1.11), we infer that the geodesic curvatures $\widetilde{k}_1,\dots,\widetilde{\varkappa}_{n-2}$ can be expressed as

$$\widetilde{\varkappa}_1(t) = \frac{\varkappa_2(t)}{\varkappa_1(t)},\ \widetilde{\varkappa}_2(t) = \frac{\varkappa_3(t)}{\varkappa_1(t)},\ \dots\ \widetilde{\varkappa}_{n-2}(t) = \pm\frac{\varkappa_{n-1}(t)}{\varkappa_1(t)}. \tag{2.7}$$

Since the arc length element is $ds = \varkappa_1(t)\,dt$, we finally arrive at the identities relating $K_j(\Gamma)$ to the integral characteristics of $\dot{\Gamma}$ expressed in terms of the induced spherical metric on $\mathbb{S}^{n-1}$:

$$K_1(\Gamma) = |\dot{\Gamma}|, \qquad K_j(\Gamma) = \widetilde{K}_{j-1}(\dot{\Gamma}), \quad j=2,\dots,n \tag{2.8}$$

(recall that $|\dot{\Gamma}|$ is the length of $\dot{\Gamma}$). The last equality $K_n(\Gamma) = \widetilde{K}_{n-1}(\dot{\Gamma})$ expresses the fact that the vanishing points of $\varkappa_{n-1}(t)$ and $\widetilde{\varkappa}_{n-2}(t)$ are the same. The reference to the formula (1.7) completes the proof for spherical curves on the unit sphere since *any such curve* $t \mapsto \widetilde{x}(t)$ *is the velocity curve of any of its vector primitives* $t \mapsto x(t) = \int \widetilde{x}(t)\,dt$ (note that the primitive curve needs not be closed).

If $r \neq 1$, then the obvious rescaling $x \mapsto x/r$ brings the sphere $r\cdot\mathbb{S}^{n-1}$ into the unit sphere. After this rescaling the length is multiplied by $1/r$, while the other curvatures $\widetilde{K}_j$ remain unchanged.

Finally, if $\dot{\Gamma} \subset \mathbb{P}^{n-1}$ is a projective curve, then we can consider the canonical $\mathbb{S}^{n-1} \to \mathbb{P}^{n-1}$ which is an isometric two-sheet covering. For the preimage of $\dot{\Gamma}$ on $\mathbb{S}^{n-1}$ everything will be doubled, namely, the length, integral curvatures, the number of endpoints etc., but the number of intersections with the equators (hyperplanes) will also be. Thus one arrives at the same formula for $\mathbb{P}^{n-1}$ as for the unit sphere. □

3. Demonstration of averaging properties for curvatures and rotations

3.1. The principal formula of geometric probability. The proof of the two key lemmas, 2 and 1, is based on the main principle of geometric probability: the *k-dimensional measure of a smooth k-dimensional submanifold $M \subseteq \mathbb{S}^{n-1}$ can be obtained by averaging the $(k-1)$-dimensional measures of its slices*. More precisely, the following identity holds for any smooth submanifold:

$$\frac{\sigma_k(M)}{|\mathbb{S}^k|} = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} \frac{\sigma_{k-1}(M \cap \mathbb{S}_p^{n-2})}{|\mathbb{S}^{k-1}|} \, d\sigma_{n-1}(p), \tag{3.1}$$

where $\mathbb{S}_p^{n-2} = p^\perp \cap \mathbb{S}^{n-1}$ is the $(n-2)$-dimensional equator orthogonal to the direction p, $\sigma_k(M)$ is the Lebesgue k-measure of M, and $\sigma_{k-1}(M \cap \mathbb{S}_p^{n-2})$ is the $(k-1)$-measure of the slice cut from M by $\mathbb{S}_p^{n-2}$.

The general discussion of this fact can be found in [10]. We will need this formula in two special cases, where it can be justified by one-line arguments.

Proposition 1. *The length $|\gamma|$ of the spherical curve $\gamma \subset \mathbb{S}^{n-1}$ is π times the average number of intersections with the random equator $\mathbb{S}_p^{n-2}$:*

$$|\gamma| = \frac{\pi}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} |\gamma \cap \mathbb{S}_p^{n-2}| \, d\sigma_{n-1}(p), \tag{3.2}$$

which, taking into account the definitions of rotation around the origin and the hyperplane $p^\perp$, is the same as the identity

$$\Omega(\Gamma, 0) = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} \Omega(\Gamma, p^\perp) \, d\sigma_{n-1}(p). \tag{3.3}$$

Sketch of the proof. The proof of (3.1) for $k = 1$, which coincides with (3.2), is obvious if γ is a piece of a large circle: then the integrand on the right-hand side is a function equal to 1 in the spherical sector between two medians with the opening proportional to the length of γ. Hence this result is valid for spherical polygons, and the case of a general smooth curve is obtained by approximation. □

Proposition 2. *For any measurable function $\chi : \mathbb{S}^{n-1} \to \mathbb{R}$ we have*

$$\frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} \chi(r) \, d\sigma_{n-1}(r) = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} d\sigma_{n-1}(p) \left\{ \frac{1}{|\mathbb{S}^{n-2}|} \int\limits_{\mathbb{S}_p^{n-2}} \chi(q) \, d\sigma_{n-2}(q) \right\}. \tag{3.4}$$

Sketch of the proof. The formula (3.4) coincides with (3.1) for χ being the indicator function of a full-dimensional submanifold and $k = n - 1$. To justify (3.1) in this extreme case, it is sufficient to note that the measure given by the right-hand side of (3.1) is $(n-1)$-dimensional and rotation invariant. Thus it must coincide with the Lebesgue area $\sigma_{n-1}(\cdot)$ modulo a constant factor. By taking $M = \mathbb{S}^{n-1}$, one can easily check that this factor is in fact equal to 1. □

3.2. Rotation around random subspaces and the proof of Lemma 2. We start with elementary properties of rotation. In order to avoid confusion, we use the extended notation $\Omega(\Gamma, L; L')$ for the rotation of $\Gamma \subset L'$ around $L \subset L'$.

3.2.1. *Rotation and projections.* Rotation of Γ along any subspace L is equal to the rotation around $P^{-1}(P(L))$ for any projection $P = P_p$. This follows from the following identity.

Proposition 3. *If $L \subset \mathbb{R}^n$ is a linear subspace, $p \in L$ is a direction (as usual, identified with a point on the sphere $\mathbb{S}^{n-1}$), and $P = P_p$ is the orthogonal projection from $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}_p = p^\perp$, then*

$$\Omega(\Gamma, L; \mathbb{R}^n) = \Omega(P(\Gamma), P(L); \mathbb{R}^{n-1}_p). \tag{3.5}$$

Remark. This construction can be iterated as many times as necessary, so that for any pair of subspaces $L \subseteq L' \subsetneq \mathbb{R}^n$ we have

$$\Omega(\Gamma, L'; \mathbb{R}^n) = \Omega(P(\Gamma), P_L(L'); L^\perp). \tag{3.6}$$

Assertion (3.6) in the case $L = L'$ coincides with the *definition* of $\Omega(\Gamma, L)$.

Proof. Indeed, both parts of (3.5) are the angular (spherical) length of the projection of Γ onto $L^\perp \subset \mathbb{R}^{n-1}_p$ since $P_L = P_{P(L)} \circ P$ (recall that $P_L \colon \mathbb{R}^n \to L^\perp$ stands for the orthogonal projection along L). □

3.2.2. *Random one-dimensional extensions of subspaces.* There are two equivalent ways to parametrize $(k+1)$-dimensional linear subspaces containing the given k-dimensional subspace L. This implies an integral identity that will be used later.

Proposition 4. *For any $\Gamma \subset \mathbb{R}^n$ and any $L \subset \mathbb{R}^n$, $\dim L = k < n - 1$,*

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \Omega(\Gamma, L + \mathbb{R}p)\, d\sigma_{n-1}(p) = \\ = \frac{1}{|\mathbb{S}^{n-k-1}|} \int_{\mathbb{S}^{n-k-1}_L} \Omega(\Gamma, L \oplus \mathbb{R}q)\, d\sigma_{n-k-1}(q), \tag{3.7}$$

where, as usual, $\mathbb{S}^{n-k-1}_L = \mathbb{S}^{n-1} \cap L^\perp$.

Proof. Note first that for almost all $p \in \mathbb{S}^{n-1}$ the subspace $L + \mathbb{R}p$ is $(k+1)$-dimensional, while $L \oplus \mathbb{R}q$ is *always* $(k+1)$-dimensional for $q \in L^\perp$. Obviously, $L+\mathbb{R}p = L\oplus\mathbb{R}\cdot(P_L(p))$ for $p \notin L$, and for p uniformly distributed over $\mathbb{S}^{n-1}$ the normalized projection $P_L(p)/\|P_L(p)\|$ is uniformly distributed in $\mathbb{S}_L^{n-k-1}$. □

3.2.3. *Proof of Lemma 2 for zero-dimensional subspaces.* Let $L = \{0\}$ be a zero-dimensional subspace. Then the assertion of Lemma 2 can be formulated as follows:

$$\Omega(\Gamma, 0) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \Omega(\Gamma, \mathbb{R}p)\, d\sigma_{n-1}(p). \tag{3.8}$$

Substitute into (3.4) the function $\chi(r) = \Omega(\Gamma, r^\perp)$, the oscillation around the hyperplane $r^\perp$. Then, by virtue of (3.3), the left-hand side is just $\Omega(\Gamma, 0)$. On the other hand, denoting $P = P_p$, we have

$$\begin{aligned}
\Omega(\Gamma, \mathbb{R}p) &= \Omega(P(\Gamma), 0; \mathbb{R}_p^{n-1}) && \text{by (3.5)}\\
&= \frac{1}{|\mathbb{S}^{n-2}|} \int_{\mathbb{S}_p^{n-2}} \Omega(P(\Gamma), q^\perp \cap \mathbb{R}_p^{n-1}; \mathbb{R}_p^{n-1})\, d\sigma_{n-2}(q) && \text{by (3.3)}\\
&= \frac{1}{|\mathbb{S}^{n-2}|} \int_{\mathbb{S}_p^{n-2}} \Omega(\Gamma, q^\perp; \mathbb{R}^n)\, d\sigma_{n-2}(q) && \text{by (3.5),}
\end{aligned}$$

which, after integration over all $p \in \mathbb{S}^{n-1}$, yields the right-hand side of (3.4). The proof in the case $\dim L = 0$ is complete.

3.2.4. *Proof of Lemma 2 in the general case.* Consider the orthogonal projection $P_L : \mathbb{R}^n \to \mathbb{R}_L^{n-k} = L^\perp$. We have

$$\begin{aligned}
\Omega(\Gamma, L; \mathbb{R}^n) &= \Omega(P_L(\Gamma), 0; L^\perp) && \text{by definition}\\
&= \frac{1}{|\mathbb{S}^{n-k-1}|} \int_{\mathbb{S}^{n-k-1}} \Omega(P_L(\Gamma), \mathbb{R}p; L^\perp)\, d\sigma_{n-k-1}(p) && \text{by (3.8)}\\
&= \frac{1}{|\mathbb{S}^{n-k-1}|} \int_{\mathbb{S}^{n-k-1}} \Omega(\Gamma, L \oplus \mathbb{R}p; \mathbb{R}^n)\, d\sigma_{n-k-1}(p) && \text{by (3.6)}\\
&= \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \Omega(\Gamma, L + \mathbb{R}p; \mathbb{R}^n)\, d\sigma_{n-1}(p) && \text{by (3.7).}
\end{aligned}$$

Thus the assertion of Lemma 2 is proved in full generality. □

3.3. Integral curvature of a random projection and the proof of Lemma 1. Let Γ be a curve with an arbitrary (not necessary natural) parametrization $[0,\ell] \ni t \mapsto x(t) \in \mathbb{R}^n$. We start with an analytic expression for Frenet curvatures.

3.3.1. *Analytic expression for curvatures.* Let $v_1(t), \dots, v_n(t)$ be successive vector derivatives $v_1(t) = \dot{x}(t)$, $v_2(t) = \ddot{x}(t)$, $\dots$, $v_n(t) = \frac{d^n}{dt^n}x(t)$. Taken together, they constitute the osculating frame.

We denote by $V_k(t) = \langle v_1(t), \dots, v_k(t)\rangle$, for $k = 1, \dots, n$, the k-dimensional volume of the tuple $v_1, \dots, v_k$:

$$\langle v_1\rangle = \|v_1\|, \quad \langle v_1, v_2\rangle = \det{}^{1/2}\begin{bmatrix}(v_1,v_1) & (v_1,v_2)\\ (v_2,v_1) & (v_2,v_2)\end{bmatrix},$$

and, in general, for any $k = 1, 2, \dots, n$,

$$\langle v_1, \dots, v_k\rangle = \det{}^{1/2}\begin{bmatrix}(v_1,v_1) & (v_1,v_2) & \cdots & (v_1,v_k)\\ (v_2,v_1) & (v_2,v_2) & \cdots & (v_2,v_k)\\ \vdots & \vdots & \ddots & \vdots\\ (v_k,v_1) & (v_k,v_2) & \cdots & (v_k,v_k)\end{bmatrix}$$

(obviously, $(\cdot,\cdot)$ is the Euclidean scalar product in $\mathbb{R}^n$).

The Frenet curvatures $\varkappa_j(t)$, originally defined via the orthogonalization of the osculating frame, admit the following representation:

$$\varkappa_k(t) = \frac{\langle v_1, \dots, v_{k-1}\rangle\,\langle v_1, \dots, v_{k+1}\rangle}{\langle v_1, \dots, v_k\rangle^2\,\langle v_1\rangle} = \frac{V_{k-1}(t)\,V_{k+1}(t)}{V_k^2(t)\,V_1(t)}, \qquad v_j = v_j(t), \qquad k = 1, \dots, n-1. \tag{3.9}$$

Indeed, let $L_k = L_k(t) = \operatorname{Span}(v_1, \dots, v_k) = \operatorname{Span}(e_1, \dots, e_k)$ be the kth subspace of the osculating flag. Since the frame $\{e_k\}$ is obtained from the frame $\{v_k\}$ by orthogonalization, $v_k = \frac{V_k}{V_{k-1}}\, e_k \bmod L_{k-1}$. Differentiating this equality and denoting the natural parameter by s, we obtain, from the Frenet formulas (3.15),

$$v_{k+1} = \frac{d}{dt}v_k = \frac{V_k}{V_{k-1}}\cdot\frac{de_k}{ds}\cdot\frac{ds}{dt} \bmod L_k = \frac{V_k}{V_{k-1}}\,\varkappa_k e_{k+1}\,|v_1| \bmod L_k.$$

On the other hand, we should have $v_{k+1} = \frac{V_{k+1}}{V_k}\, e_{k+1} \bmod L_k$. This immediately implies that $V_1\varkappa_k = |v_1|\varkappa_k = \frac{V_{k+1}}{V_k} : \frac{V_k}{V_{k-1}}$, which coincides with (3.9).

After integration, identity (3.9) takes the form independent of the choice of the parametrization:

$$K_k(\Gamma) = \int_0^\ell \frac{\langle v_1, \dots, v_{k-1}\rangle \, \langle v_1, \dots, v_{k+1}\rangle}{\langle v_1, \dots, v_k\rangle^2} \, dt = \\ = \int_0^\ell \frac{V_{k-1}(t)\, V_{k+1}(t)}{V_k^2(t)} \, dt. \tag{3.10}$$

If $P = P_p : \mathbb{R}^n \to \mathbb{R}_p^{n-1} = p^\perp$ is the orthogonal projection along the direction $p \in \mathbb{S}^{n-1}$, then, by linearity, the osculating frame of $P(\Gamma)$ is the frame $P(v_1), \dots, P(v_n)$, and this identity yields explicit formulas for Frenet curvatures of the projected curve. If we denote by $\mathcal{L}_\Gamma(t) = \{L_j(t)\}_{j=1}^{n-1}$ the osculating flag of Γ spanned by the frame $\{v_j\}$, then, for any $k = 1, \dots, n-1$, we have

$$\begin{gathered} \langle P(v_1), \dots, P(v_k)\rangle = \sin(p, L_k) \cdot \langle v_1, \dots, v_k\rangle, \\ P = P_p, \ v_j = v_j(t), \ L_k = L_k(t) = \operatorname{Span}(v_1(t), \dots, v_k(t)), \end{gathered} \tag{3.11}$$

and $\sin(p, L_k)$ is the sine of the angle between p and L_k defined as the Euclidean angle in $\mathbb{R}^n$ between p and its orthogonal projection on L_k.

3.3.2. *Demonstration of Lemma* 1 *for* $k \leqslant n-1$. Substituting (3.11) into (3.10) for any k between 1 and $n-2$, we obtain

$$K_k(P(\Gamma)) = \int_0^\ell \frac{V_{k+1}(t) V_{k-1}(t)}{V_k^2(t)} \cdot \frac{\sin(p, L_{k+1}(t)) \sin(p, L_{k-1}(t))}{\sin^2(p, L_k(t))} \, dt.$$

Denoting $\kappa_k = V_{k+1}V_{k-1}/V_k^2 = \|v_1\| \cdot \varkappa_k$ and averaging this equality over $p \in \mathbb{S}^{n-1}$, we obtain

$$\frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} K_k(P_p(\Gamma))\,dp =$$

$$= \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} d\sigma_{n-1}(p)\left\{\int\limits_0^\ell \kappa_k(t)\cdot\frac{\sin(p, L_{k+1}(t))\cdot\sin(p, L_{k-1}(t))}{\sin^2(p, L_k(t))}\,dt\right\} =$$

$$= \int\limits_0^\ell \kappa_k(t)\,dt\left\{\frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{\sin(p, L_{k+1}(t))\cdot\sin(p, L_{k-1}(t))}{\sin^2(p, L_k(t))}\,d\sigma_{n-1}(p)\right\} \overset{(!)}{=}$$

$$\overset{(!)}{=} \int\limits_0^\ell \kappa_k(t)\,dt\left\{\frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{\sin(p, E_{k+1})\cdot\sin(p, E_{k-1})}{\sin^2(p, E_k)}\,d\sigma_{n-1}(p)\right\} =$$

$$= \left(\frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{\sin(p, E_{k+1})\cdot\sin(p, E_{k-1})}{\sin^2(p, E_k)}\,d\sigma_{n-1}(p)\right) \times \left(\int_0^\ell \kappa_k(t)\,dt\right) =$$

$$= \mathrm{const}_{k,n-1} \cdot K_k(\Gamma), \tag{3.12}$$

where E_j are subspaces of the standard flag $\mathcal{E} = \{E_j\}_1^{n-1}$. The transformation marked by (!), the key point of all the computation, holds by the rotation symmetry: any three subspaces of the flag $\mathcal{L}_\Gamma$ can be simultaneously transformed into three subspaces of any other flag by an appropriate rotation of $\mathbb{R}^n$.

In order to carry out the proof, it remains to show that the constant factor, denoted by $\mathrm{const}_{k,n-1}$ in (3.12), converges and is, in fact, equal to 1. This is done by a straightforward computation in the spherical coordinates on $\mathbb{R}^{n-1}$. For convenience, we replace the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ by $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ endowed with the Euclidean coordinates $p = (p_1, \dots, p_{n+1})$. Our goal is to prove the identity

$$\mathrm{const}_{k,n} = \frac{1}{|\mathbb{S}^n|}\int\limits_{\mathbb{S}^n} \frac{\sin(p, E_{k+1})\cdot\sin(p, E_{k-1})}{\sin^2(p, E_k)}\,d\sigma_n(p) \tag{3.13}$$

The sphere can be parametrized by the angles $\phi_1, \dots, \phi_n$ as

$$\begin{aligned}
p_1 &= \sin\phi_1, \\
p_2 &= \cos\phi_1 \sin\phi_2, \\
p_3 &= \cos\phi_1 \cos\phi_2 \sin\phi_3, \\
&\dots \quad \dots \quad \dots \\
p_n &= \cos\phi_1 \cdots \cos\phi_{n-1} \sin\phi_n, \\
p_{n+1} &= \cos\phi_1 \cdots \cos\phi_{n-1} \cos\phi_n, \\
\phi_i &\in (-\pi/2, \pi/2) \quad \forall\, i = 1, \dots, n-1, \quad \phi_n \in (-\pi, \pi)
\end{aligned}$$

so that the n-volume element on the sphere has the form

$$d\sigma_n(p) = \cos^{n-1}\phi_1 \cos^{n-2}\phi_2 \cdots \cos\phi_{n-1}\, d\phi_1 \cdots d\phi_{n-1} d\phi_n.$$

Introducing the notation

$$B_k = \int\limits_{-\pi/2}^{\pi/2} \cos^k\theta\, d\theta = \sqrt{\pi}\, \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)},$$

where Γ is the Euler gamma-function, we have the identity

$$1 = \frac{2\pi}{|\mathbb{S}^n|} B_{n-1} B_{n-2} \cdots B_2 B_1, \tag{3.14}$$

following from the definition $\int_{\mathbb{S}^n} d\sigma_n(p) = |\mathbb{S}^n|$.

The angle between the point p and the coordinate plane $E_k \subset \mathbb{R}^{n+1}$ spanned by the first k coordinate vectors can be easily measured: the squared sine of this angle is equal to the squared length of the projection of p onto the remaining (complementary) coordinate subspace $E_k^\perp$. In other words, we have

$$\begin{aligned}
\sin^2(p, E_k) &= \cos^2\phi_1 \cdots \cos^2\phi_k \times \\
&\times (\sin^2\phi_{k+1} + \cos^2\phi_{k+1} \sin^2\phi_{k+2} + \cdots + \cos^2\phi_{k+1} \cdots \cos^2\phi_n^2) = \\
&= \cos^2\phi_1 \cdots \cos^2\phi_k.
\end{aligned}$$

Since $k \leqslant n-1$ (the case we are interested in), all cosines are positive in the domain of parametrization, and, hence, for the integral (3.13) we have the expression

$$\begin{aligned}
\mathrm{const}_{n,k} &= \frac{1}{|\mathbb{S}^n|} \int\limits_{\mathbb{S}^n} \frac{(\cos\phi_1 \cdots \cos\phi_{k-1})(\cos\phi_1 \cdots \cos\phi_{k+1})}{(\cos\phi_1 \cdots \cos\phi_k)^2}\, d\sigma_n(p) = \\
&= \frac{2\pi}{|\mathbb{S}^n|} \int \cdots \int\limits_{-\pi/2 \leqslant \phi_j \leqslant \pi/2} \frac{\cos\phi_{k+1}}{\cos\phi_k} \cdot \cos^{n-1}\phi_1 \cdots \cos\phi_{n-1}\, d\phi_1 \cdots d\phi_{n-1}.
\end{aligned}$$

Now it is immediately clear that the integral converges for $n-1 \geqslant k \geqslant 1$ and is equal to the same product (3.14) with the terms B_k and B_{k+1} transposed. Hence we have proved the identity $\mathrm{const}_{n,k} = 1$, so that the assertion of Lemma 1 is proved for all integral curvatures K_k, except for the last one, the integral inflection.

3.3.3. *Averaging integral inflection.* The idea of the proof is the same as in [1].

Let $e_1(s), \dots, e_n(s)$ be the orthogonal osculating frame of the naturally parametrized regular curve $\Gamma \subset \mathbb{R}^n$ obtained by the orthogonalization of the frame $v_1, \dots, v_n$. The last Frenet formula takes the form

$$\dot{e}_n(s) = -\varkappa_{n-1}(s)\, e_{n-1}(s). \tag{3.15}$$

Consider the curve Γ^*, parametrized as $s \mapsto e_n(s)$. Since $\|e_n(s)\| \equiv 1$, Γ^* is a spherical curve, and from (3.15) it follows that

$$|\Gamma^*| = \int_0^\ell |\varkappa_{n-1}(s)|\, ds = K_{n-1}(\Gamma). \tag{3.16}$$

Applying the formula (3.2), we conclude that

$$K_{n-1}(\Gamma) = \frac{\pi}{|\mathbb{S}^{n-1}|} \cdot |\Gamma^* \cap p^\perp|\, d\sigma_{n-1}(p).$$

Now it remains only to note that if the vector $e_n(s)$ is orthogonal to the vector $p \in \mathbb{S}^{n-1}$ at some point $s \in [0, \ell]$, then the curve $P_p(\Gamma) \subset \mathbb{R}^{n-1}_p = p^\perp$ has an inflection point at $P_p(x(s))$ since the projection P_p restricted to the subspace L_{n-1} of the osculating flag $\mathcal{L}_\Gamma$ is *degenerate* (the rank is not full). Therefore $|\Gamma^* \cap p^\perp| = K_{n-1}(P_p(\Gamma))$, and, taking the coefficient π into account, we obtain the equality

$$K_{n-1}(\Gamma) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} K_{n-1}(P_p(\Gamma))\, d\sigma_{n-1}(p).$$

The proof of Lemma 1 is complete. □

4. Pólya theorem and demonstration of Lemma 3

4.1. Roots of linear combinations. We start with a seemingly irrelevant question. Given a tuple of sufficiently smooth functions $f_1(t), \dots, f_n(t)$, all defined on the common interval $I \doteq [\alpha, \beta] \subset \mathbb{R}^1$, how many isolated zeros can the (nontrivial) linear combination $\lambda_1 f_1 + \dots + \lambda_n f_n$ have on that interval for an arbitrary choice of the coefficients $\lambda_j \in \mathbb{R}$? We shall formulate this problem in *periodic* and *nonperiodic* contexts, the former meaning that all f_j extend as a $(\beta - \alpha)$-periodic functions on $\mathbb{R}^1$. To formulate the results in the uniform way, we introduce the number $\delta \in \{0,1\}$ equal to 0 if the functions are periodic and to 1 otherwise. The periodic context corresponds, in fact, to functions defined on the circle $\mathbb{S}^1 = I/(\alpha \sim \beta)$ rather than on the interval I. Thus, in some sense, $2\delta = |\partial I|$ is the number of endpoints of I.

We fix the order of functions f_j and introduce (following Pólya [8], [9]) $n+1$ functions $W_k \colon I \to \mathbb{R}$ as the Wronski determinants of the first k functions $f_1, \dots, f_k$:

$$W_0(t) \equiv 1, \quad W_1(t) = f_1(t), \quad W_2(t) = f_1'(t)f_2(t) - f_1'(t)f_2(t), \quad \dots$$

$$W_k(t) = \det \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_k(t) \\ f_1'(t) & f_2'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(t) & f_2^{(k-1)}(t) & \cdots & f_k^{(k-1)}(t) \end{bmatrix}, \tag{4.1}$$

$$k = 1, 2, \dots, n.$$

4.1.1. *Chebyshev systems and Pólya theorem.* In the simplest case the non-vanishing of the Wronskians implies the Chebyshev property (nonoscillation) of the linear spaces of functions.

Theorem 5 (Pólya theorem [8]). *If all Wronskians $W_1, \dots, W_n$ are nonvanishing on I, then any linear combination $\sum_{j=1}^{n} \lambda_n f_n$ can have at most $n-1$ isolated root on I, counting with multiplicities.*

4.1.2. *The general case.* If the Wronskians W_j have zeros on I, not vanishing identically, then Theorem 5 does not hold any longer. However, in this case one can find an upper bound for the number of isolated zeros occurring in linear combinations. We denote by $\nu_j \geqslant 0$ the number of zeros of W_j on I.

The simplest way is to consider the partition of I by the roots of Wronskians into $\delta + \sum_{j=1}^{n} \nu_j$ subintervals. Then Theorem 5 can be applied to each of them, yielding an upper bound of $(n-1)(\delta + \sum \nu_j)$ for the number of zeros *outside* the partition points. Adding the number of zeros eventually occurring at these points, we arrive at the upper bound $(n-1)\delta + n\sum \nu_j$. However, this estimate can be substantially improved. The idea of such improvement was given in [7], where a result, though inferior to the inequality below, but still sufficient for the purposes of [7], was obtained.

Lemma 4. *If ν_j is the number of zeros of W_j on the interval I, then the number of isolated zeros occurring in any nontrivial linear combination of the functions $f_1, \dots, f_n$, does not exceed*

$$(n-1) \cdot \delta + \sum_{j=1}^{n} w_{n-j} \nu_j, \tag{4.2}$$

where the sequence of w_j is the same as in (1.8).

Note that if $\nu_1 = \cdots = \nu_n = 0$, then the assertion of Lemma 4 coincides with that of Theorem 5.

Proof. The proof is based on the fact that all linear combinations satisfy the following nth order linear ordinary differential equation with variable coefficients:

$$D_n D_{n-1} \cdots D_1 \, y = 0, \qquad D_i = \frac{W_i}{W_{i-1}} \, \partial \, \frac{W_{i-1}}{W_i}, \quad \partial = \frac{d}{dt} \tag{4.3}$$

(this form is due to Frobenius, and a simple proof of this fact can be found in [7]). Equation (4.3) can be transformed as

$$\Delta_{n-1} \cdot \frac{1}{W_{n-3}} \cdot \Delta_{n-2} \cdots \Delta_3 \cdot \frac{1}{W_1} \cdot \Delta_2 \cdot \frac{1}{W_0} \cdot \Delta_1 \, y = \text{const} \cdot W_{n-2} W_n, \tag{4.4}$$

where $\Delta_j = W_j^2 \cdot \partial \cdot (W_j)^{-1}$ is the differential operator transforming the function φ into $W_j^2(\varphi/W_j)'$, and, without loss of generality, one can assume that const $\neq 0$ (otherwise the order of the equation can be further reduced). We need first to modify the Rolle theorem to allow for functions with poles and differential operators other than ∂.

If $g\colon I \to \mathbb{R}$ is a smooth function and $N(g) < \infty$ is the number of its zeros on I, then $\Delta_j g$ is also smooth, and

$$N(g) \leqslant N(\Delta_j g) + \nu_j + \delta, \tag{4.5}$$

where δ is 0 or 1, depending on whether g is periodic or not. Indeed, g/W_j is a function that is smooth on $\nu_j + \delta$ intervals between zeros of W_j. The application of ∂ can decrease the number of zeros on each interval at most by one (Rolle theorem), and the multiplication by W_j^2 restores the smoothness.

If the function g itself has p poles (e.g., g is a fraction whose denominator has p isolated zeros), then (4.5) should be replaced by

$$N(g) \leqslant N(\Delta_j g) + \nu_j + p + \delta \tag{4.6}$$

since the number of intervals of continuity will in this case be $\nu_j + p + \delta$.

Let $f = \sum_{j=1}^{n} \lambda_j f_j$ be a nontrivial linear combination of functions f_j. Consider the sequence of functions occurring in the evaluation of the left-hand side of (4.4):

$$F_0 = f,\ F_1 = \Delta_1 F_0,\ F_2 = \Delta_2 W_0^{-1} F_1,\ F_3 = \Delta_3 W_1^{-1} F_2,\ \ldots,$$
$$F_{n-2} = \Delta_{n-2} W_{n-4}^{-1} F_{n-3},\ F_{n-1} = \Delta_{n-1} W_{n-3}^{-1} F_{n-2} = \text{const} \cdot W_{n-2} W_n.$$

The number of poles of F_k is at most $\nu_{n-2} + \nu_{n-3} + \nu_{n-4} + \cdots$, assuming that $\nu_0 = \nu_{-1} = \cdots = 0$.

Iterating inequalities (4.6), we obtain the chain of $(n-1)$ inequalities

$$\begin{aligned}
N(F_1) &\geqslant N(F_0) - \nu_1 - \delta,\\
N(F_2) &\geqslant N(F_1) - \nu_2 - \nu_0 - \delta,\\
N(F_3) &\geqslant N(F_2) - \nu_3 - (\nu_0 + \nu_1) - \delta,\\
N(F_4) &\geqslant N(F_3) - \nu_4 - (\nu_0 + \nu_1 + \nu_2) - \delta,
\end{aligned}$$

and so on. Adding up all these inequalities, we arrive at the estimate

$$\begin{aligned}
N(W_n W_{n-2}) = N(F_{n-1}) \geqslant N(f) - (n-1)\delta - (\nu_1 + \nu_2 + \cdots + \nu_{n-1}) -\\
- (n-2)\nu_0 - (n-3)\nu_1 - \cdots - \ nu_{n-3}.
\end{aligned}$$

The left-hand side of this inequality is equal to $\nu_{n-2} + \nu_n$ by (4.4), while $\nu_0 = 0$. Therefore the number of zeros $N(f)$ is estimated from above by the combination

$$\begin{aligned}
\nu_n + \nu_{n-1} + 2\nu_{n-2} + 2\nu_{n-3} + 3\nu_{n-4} + \cdots +\\
+ (n-3)\nu_2 + (n-2)\nu_1 + (n-1)\delta.
\end{aligned}$$

The proof is complete. □

Remark. In the above proof we assumed that the roots of W_j are disjoint from the roots of the corresponding factor, so that each division by W_j increases the number of intervals of continuity by ν_j but does not change the number of zeros. As a matter of fact, the coincidence may happen, so that cancellation occurs, but then the number of intervals of continuity will be smaller on *each* step from that moment on. One can easily check that the overall estimate in this case will be even *better*. Alternatively, one can use a small perturbation to move zeros of W_j away and then use the semicontinuity arguments for the number of zeros.

4.2. Demonstration of Lemma 3. Let $\mathcal{L} = \{L_j\}_{j=1}^n$ be a complete flag and $\nu_j = \nu_j(\Gamma, \mathcal{L}) < \infty$ be the number of inflection points of the orthogonal projection of Γ onto L_j.

We choose the orthogonal coordinate system such that the frame spans the flag $\mathcal{L}$ (this choice is essentially unique, modulo change of signs of the coordinates). Then the curve Γ corresponds to the smooth vector function $t \mapsto x(t) = (x_1(t), \ldots, x_n(t))$. We denote $f_j(t) = \dfrac{d}{dt}x_j(t)$ and consider the Wronskians corresponding to the ordered tuple $f_1, \ldots, f_n$.

Then the vanishing of the jth Wronskian W_j corresponds to the inflection point of the projection of Γ onto L_j (the first j coordinates); see 3.3.1. Note that the velocity curve $\dot{\Gamma}$ is closed if and only if the functions f_j are periodic.

By Lemma 4, the number of isolated roots of any linear combination $\sum_{j=1}^{n}\lambda_j f_j(t)$ can be at most $N = \frac{1}{2}(n-1)|\partial\dot{\Gamma}| + \sum_{j=1}^{n} w_{n-j}\nu_j$. In geometric terms, this means that the *velocity vector* $\dot{x}(t) = (\dot{x}_1(t), \dots, \dot{x}_n(t))$ intersects any *linear* hyperplane $\{\sum \lambda_j x_j = 0\} \subset \mathbb{R}^n$ at most at N points. This is exactly the second claim of Lemma 3. But then, by the Rolle theorem, the curve Γ itself can intersect any *affine* hyperplane at most at $N + \frac{1}{2}|\partial\dot{\Gamma}|$ points, which is the inequality asserted by the Lemma since $|\partial\dot{\Gamma}| \leqslant |\partial\Gamma|$ (the velocity curve $\dot{\Gamma}$ may be closed whereas Γ may not, but the inverse is impossible).

An alternative (direct) way to prove the second assertion of Lemma 3 is to consider a system of $n+1$ functions $1, x_1(t), \dots, x_n(t)$ and expand the corresponding Wronskians by the elements of the first row, thus proving an upper bound for the number of intersections of Γ with any affine hyperplane $\lambda_0 + \sum \lambda_j x_j = 0$. □

4.3. Demonstration of Theorem 1. The idea of demonstration is straightforward: if the hyperconvex curve Γ parametrized by the arc length $t \in [0, \ell]$ is oscillating, then its osculating flag $\mathcal{L}_\Gamma(t)$ must become nontransversal to any other flag, in particular, to $\mathcal{L}' = \mathcal{L}_\Gamma(0)^\perp$. But $\mathcal{L}_\Gamma(0)$ is in some sense "maximally transversal" to $\mathcal{L}'$, and to take a nontransversal position, $\mathcal{L}_\Gamma(t)$ should go a sufficiently long way in the flag variety. On the other hand, the velocity of the "curve" $t \mapsto \mathcal{L}_\Gamma(t)$ in this variety is controlled by the instant curvatures $\varkappa_j(t)$, $j = 1, \dots, n-1$, which means that the integral curvatures cannot be too small.

For practical reasons it is more convenient to work not in the flag variety, but rather in its covering space identified with the orthogonal group.

Let $\mathfrak{G} : I = [0, \ell] \to \mathrm{SO}(n)$, $t \mapsto E(t)$ be the *associated curve*: the point t is mapped into the orthogonal matrix whose columns are the vectors $e_j(t)$ of the orthogonalized Frenet frame. Then the Frenet formulas take the form $\dot{E}(t) = A(t)E(t)$, where $A(t)$ is an antisymmetric matrix function with the entries $\pm\varkappa_j(t)$, $j = 1, \dots, n-1$, occurring on the principal sub- (resp., super-) diagonal. Without loss of generality, we can assume that $E(0)$ is an identity matrix (the corresponding flag is the coordinate one $\mathcal{E}$).

We embed the orthogonal group $\mathrm{SO}(n)$ into the Euclidean space $\mathbb{R}^{n^2}$ of square matrices with the norm $\|X\|^2 = \sum_{i,j=1}^{n} x_{ij}^2$, where x_{ij} are the entries of X.

By the Cauchy–Bunyakovskii inequality, $\|AX\| \leqslant \|A\| \cdot \|X\|$. Evidently, $\|A(t)\|^2 = 2\varkappa_1^2(t) + \dots + 2\varkappa_{n-1}^2(t)$, and $\|E(t)\|^2 \equiv n$ since $E(t)$ is an orthogonal matrix. Therefore

$$\|\dot{E}(t)\| \leqslant \sqrt{2n}\cdot\sqrt{\textstyle\sum_{j=1}^{n-1}\varkappa_j^2(t)}, \qquad |\mathfrak{G}| \leqslant \sqrt{2n}\cdot\int_0^{\ell}\sqrt{\textstyle\sum_{j=1}^{n-1}\varkappa_j^2(t)}\,dt, \tag{4.7}$$

where $|\mathfrak{G}|$ is the length of $\mathfrak{G}$ in $\mathbb{R}^{n^2}$.

On the other hand, all matrices sufficiently close to the identity matrix correspond to the flags transversal to the antipodal flag $\mathcal{E}^{\perp}$. More precisely, if all upper-left $(k\times k)$-minors of $E(t)$ are nonzero, then the above transversality holds. In particular, if $\|X\| < 1/\sqrt{n}$, then $E(0)+X$ corresponds to the flag still transversal to $\mathcal{E}^{\perp}$. Indeed, in this case, for any $i=1,\dots,n$, we have $\left(\sum_{j=1}^{n}|x_{ij}|\right)^2 \leqslant n\sum_{j=1}^{n}x_{ij}^2 \leqslant n\|X\|^2 < 1$ and any row of the matrix $E(0)+X$ has a dominant diagonal element. Thus all minors are nonzero, and the required transversality holds.

In other words, we have proved that the ball $\{E(0)+X\in\mathbb{R}^{n^2}: \|X\| < 1/\sqrt{n}\}$ of radius $1/\sqrt{n}$ centered at $E(0)$ consists of matrices that span flags transversal to $\mathcal{E}^{\perp}$. If the curve Γ is oscillating, then, by the Shapiro theorem, the associated curve $\mathfrak{G}$ should leave this ball, and, hence, its length should be at least $1/\sqrt{n}$. Taking into account the inequality (4.7), we arrive at the final estimate (1.3). $\square$

It is clear that the measuring of lengths in $\mathbb{R}^{n^2}$ rather than in the group $\mathrm{SO}(n)$ results in the loss of sharpness. To get the best results from this approach, one should use a left-invariant metric on $\mathrm{SO}(n)$ and estimate the distance from $E(0)$ to the nearest non-transversal matrix in this metric. However, we do not want to discuss the general case, but will rather consider three-dimensional curves, where a similar computation is relatively easy.

4.4. Nonoscillating curves in $\mathbb{R}^3$ via the Shapiro theorem. As an illustration of the Shapiro theorem (Corollary 4), we prove that a hyperconvex (nonclosed) curve with $K_1(\Gamma)+K_2(\Gamma) < \frac{\pi}{2}$ is nonoscillating, $\Omega_2(\Gamma) \leqslant 3\pi$.

Consider the osculating flag $\mathcal{L}(s) = \mathcal{L}_{\Gamma}(s)$. Without loss of generality, we can assume that $\mathcal{L}(0) = \mathcal{E}$ (the standard flag). If the curve is oscillating, then $\mathcal{L}(s)$ should become nontransversal to the antipodal flag $\mathcal{E}^{\perp}$ at some point, by virtue of Corollary 4. This may happen in one of the two possible scenarios:

- either the tangent $e_1(s)$ intersects the plane spanned by $e_2(0)$ and $e_3(0)$,
- or the vector $e_3(s)$ intersects the plane spanned by $e_1(0)$ and $e_2(0)$.

In both cases the length of the path made by the corresponding vector on the sphere before the intersection occurs is at least $\frac{\pi}{2}$ (the spherical distance from the north pole to the equator).

On the other hand, from the Frenet formulas $\dot{e}_1(s) = \varkappa_1(s)e_2(s)$, $\dot{e}_3(s) = -\varkappa_2(s)e_2(s)$ it follows that the path made by $e_1(s)$ for $s \in [0, \ell]$ is exactly $K_1(\Gamma)$ and that made by $e_3(s)$ is $K_2(\Gamma)$. Therefore the inequality $K_1+K_2 < \frac{\pi}{2}$ excludes both possibilities, and the contradiction obtained proves that the curve Γ is nonoscillating.

5. Isoperimetric inequalities on $\mathbb{S}^2$ and nonoscillation

In this section we consider *three-dimensional hyperconvex* curves, primarily the closed ones, and prove inequality (1.7) for $\Omega_2(\Gamma)$ based on a completely different set of arguments. We will always assume that Γ is parametrized by the arc length $s \in [0, \ell]$, so that the velocity curve (*hodograph*) $\dot{\Gamma}\colon s \mapsto \dot{x}(s)$ is a spherical curve. We also return locally to the classical terminology, referring to $\varkappa(s) = \varkappa_1(s) > 0$ as the curvature and to $\theta(s) = \varkappa_2(s)$ as the torsion.

As follows from identities (2.7), the arc length element on $\dot{\Gamma}$ is $\varkappa(s)\,ds$ and the (first and unique) geodesic curvature of $\dot{\Gamma}$ is $\widetilde{\varkappa}(s) = \theta(s)/\varkappa(s)$. If Γ is hyperconvex (without inflection points), then $\dot{\Gamma}$ is *geodesically convex*.

5.1. Isoperimetric inequalities on a sphere. First we consider *hemispheric convex lobes*, closed piecewise smooth curves formed by a piece A of a smooth geodesically convex curve and an arc of a large circle (equator) E, entirely belonging to one hemisphere. We denote by α and α' the exterior angles at the vertices of the lobe and let $\widetilde{K}(A)$ be the integral *geodesic* curvature of the arc A (the integral of the geodesic curvature $\widetilde{k}(s)$ against the arc length). Our local aim is to prove the inequality

$$\alpha + \alpha' + \widetilde{K}(A) + 2|A| \geqslant 2\pi. \tag{5.1}$$

Note that $\alpha+\alpha'+\widetilde{K}(A)$ is the integral of the geodesic curvature of the *entire lobe* since $\widetilde{K}(E) = 0$. To prove (5.1), we first note that by the spherical excess theorem (Gauss–Bonnet formula),

$$\alpha + \alpha' + \widetilde{K}(A) = 2\pi - S, \tag{5.2}$$

where S is the area of the lobe [2, Ch. 1, 2.7]. Now the problem is to majorize S in terms of $|A|$.

The standard isoperimetric inequality between the length $|\gamma|$ of a simple closed spherical curve and S, the area bounded by this curve, has the form

$$|\gamma|^2 \geqslant 4\pi S - S^2, \tag{5.3}$$

the equality being attained only if γ has a constant geodesic curvature. In a similar way the *Dido problem* of finding the shortest curve with endpoints on the equator, bounding together with the piece of equator the largest possible area, has only constant curvature solutions, normally crossing the equator. For the corresponding solution A one has the isoperimetric inequality

$$|A|^2 \geqslant 2\pi S - S^2, \tag{5.4}$$

where S is now the area bounded by the curve and the equator taken together. From (5.4) it follows that within the range $0 < S \leqslant \frac{8}{5}\pi$ the length and the area are related by the inequality

$$|A| \geqslant \frac{1}{2}S, \qquad 0 \leqslant S \leqslant \frac{8}{5}\pi. \tag{5.5}$$

Unfortunately, without additional considerations (5.4) does not imply any lower bound for $|A|$ if S approaches the area of the hemisphere 2π. In order to analyze the region of large areas, we apply inequality (5.3) to the closed arc formed by A and E taken together: since $S \leqslant 2\pi$ (the lobe belongs to a hemisphere), the inequality $(|A| + |E|)^2 \geqslant 4\pi S - S^2$ implies

$$|A| + |E| \geqslant \pi + \frac{1}{2}S, \qquad \frac{2}{5}\pi \leqslant S \leqslant 2\pi. \tag{5.6}$$

Now it remains to point out that for a *convex* lobe $|E| \leqslant \pi$. Indeed, if we rotate a half-equator inside the lobe while keeping its endpoints fixed, we can obtain an *inner* tangency between $|A|$ and a geodesic curve, which is impossible. Substituting this into (5.6), we conclude that the inequality $|A| \geqslant S$ holds on two overlapping intervals, $\left[0, \frac{8}{5}\pi\right]$ and $\left[\frac{2}{5}\pi, 2\pi\right]$, covering the entire range of admissible areas $[0, 2\pi]$. Together with (5.2) this proves (5.1).

Our next aim is to extend inequality (5.1) for geodesically convex hemispheric curves with endpoints on an equator, but eventually self-intersecting. It turns out to be even easier.

If the curve A is self-intersecting, forming a number of "petals," then one can break A into (oriented) smooth pieces and reconnect them in such a way that together with the arc E of the equator they will form $\nu > 1$ closed curves, only one of them containing E (the case $\nu = 1$ corresponds to simple A). The domains bounded by these curves may overlap, but in any case their areas S_i will not exceed 2π, and at least one of them will be convex disjoint from E.

The spherical excess theorem can be applied to each domain; if we add the corresponding relations together, then the resulting relation will take the form

$$\alpha + \alpha' + \widetilde{K}(A) = 2\pi\nu - (S_1 + \cdots + S_\nu).$$

For each domain with the area S_j and bounded by the arc of the length $|A_j|$ (except for one of them whose boundary has the length $|A_j| + |E|$), the isoperimetric inequality (5.3) gives the inequality $|A_j|^2 \geqslant 4\pi S_j - S_j^2$ which, since all S_j are less than 2π, implies that $|A_j| \geqslant S_j$ (for the exceptional domain the latter takes the form $|A_j|+|E| \geqslant S_j$). Adding these inequalities together and noting that $|A_1|+\dots+|A_\nu| = |A|$, we arrive at the inequality $\alpha+\alpha'+\widetilde{K}(A)+|A|+|E| \geqslant 2\pi\nu$. It remains to observe that $|E| \leqslant 2\pi$ and $\nu > 1$ and to conclude that

$$\alpha+\alpha'+\widetilde{K}(A)+|A| \geqslant 2\pi(\nu-1) \geqslant 2\pi,$$

which is even stronger than asserted by (5.1).

5.2. Length-curvature inequality for geodesically convex spherical curves oscillating around an equator. Now assume that A is a geodesically convex spherical curve intersecting some equator at $n \geqslant 2$ points. We show that in this case

$$\widetilde{K}(A)+2|A| \geqslant \pi(n-|\partial A|). \tag{5.7}$$

To prove this, we break A into smooth pieces between subsequent intersections with the equator; their number is n if A is closed and $n-1$ otherwise. We denote by α_i and α_i' the exterior angles of the lobes formed by A_i together with the corresponding arcs of the equator. Then, as one can easily see,

$$\alpha_{i+1}+\alpha_i' = \pi \qquad i=1,\dots,n-1$$

(if A is closed, then the subscript i is cyclical modulo n).

We apply the inequality (5.1) to each of these lobes and add the results together. Then, since all curvatures $\widetilde{K}(A_i)$ are of the same sign, the resulting inequality takes the form

$$\begin{cases} \pi n+\widetilde{K}(A)+2|A| \geqslant 2\pi n, & \text{if } A \text{ is closed,} \\ \alpha_1+\alpha_{n-1}'+\pi(n-2)+\widetilde{K}(A)+2|A| \geqslant 2\pi(n-1), & \text{otherwise.} \end{cases}$$

Since both α_1, α_{n-1}' are less than π, we arrive at inequality (5.7).

5.3. Hyperconvex curves in $\mathbb{R}^3$ and on $\mathbb{S}^2$. Inequality (5.7) is, in fact, the upper bound for oscillation of spherical curves that coincides with (1.12) for $n = 2$ and $r = 1$. To prove the inequality $\Omega_2(\Gamma) \leqslant 2K_1(\Gamma)+K_2(\Gamma)+\frac{3}{2}|\partial\Gamma|$, which is a special case of Theorem 2 for three-dimensional hyperconvex curves, we apply once again the Rolle theorem to the spherical curve $A = \dot{\Gamma}$, as in the proof of Theorem 2 for hyperplanes: the resulting estimate will then be

$$\Omega_2(\Gamma) \leqslant \frac{\pi}{2}|\partial\Gamma| + \pi|\partial\dot{\Gamma}| + 2|\dot{\Gamma}| + \widetilde{K}(\dot{\Gamma}) \leqslant$$
$$\leqslant \frac{3}{2}\pi|\partial\Gamma| + 2K_1(\Gamma) + K_2(\Gamma)$$

by virtue of relations (2.8). □

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