Journal of Differential Equations **177**, 267–306 (2001)
doi:10.1006/jdeq.2000.3967, available online at http://www.idealibrary.com on IDEAL®

Redundant Picard–Fuchs System for Abelian Integrals

D. Novikov[1]

Laboratoire de Topologie, Université de Bourgogne, Dijon, France
E-mail: dmitry@math.toronto.edu

and

S. Yakovenko

Department of Mathematics, The Weizmann Institute of Science, Rehovot, Israel
E-mail: yakov@wisdom.weizmann.ac.il

Received February 1, 2000; revised September 7, 2000

We derive an explicit system of Picard–Fuchs differential equations satisfied by Abelian integrals of monomial forms and majorize its coefficients. A peculiar feature of this construction is that the system admitting such explicit majorants appears only in dimension approximately two times greater than the standard Picard–Fuchs system. The result is used to obtain a partial solution to the tangential Hilbert 16th problem. We establish upper bounds for the number of zeros of arbitrary Abelian integrals on a positive distance from the critical locus. Under the additional assumption that the critical values of the Hamiltonian are distant from each other (after a proper normalization), we were able to majorize the number of all (real and complex) zeros. In the second part of the paper an equivariant formulation of the above problem is discussed and relationships between spread of critical values and non-homogeneity of uni- and bivariate complex polynomials are studied.

Key Words: Abelian integrals; Picard–Fuchs systems.

Contents.

[1] Current address: Department of Mathematics, Toronto University, Toronto, Canada.

0022-0396/01 $35.00

1. TANGENTIAL HILBERT SIXTEENTH PROBLEM, COMPLETE ABELIAN INTEGRALS AND PICARD–FUCHS EQUATIONS

The main result of this paper is an explicit derivation of the Picard–Fuchs system of linear ordinary differential equations for integrals of polynomial 1-forms over level curves of a polynomial in two variables, regular at infinity.

The explicit character of the construction makes it possible to derive upper bounds for the coefficients of this system. In turn, application of the bounded meandering principle [16, 18] to the system of differential equations with bounded coefficients allows to produce upper bounds for the number of complex isolated zeros of these integrals on a positive distance from the ramification locus.

1.1. *Abelian integrals and tangential Hilbert 16th problem.* If $H(x, y)$ is a polynomial in two real variables, called the Hamiltonian, and $\omega = P(x, y)\,dx + Q(x, y)\,dy$ a real polynomial 1-form, then the problem on limit cycles appearing in the perturbation of the Hamiltonian equation,

$$dH + \varepsilon\omega = 0, \qquad \varepsilon \in (\mathbb{R}, 0), \tag{1.1}$$

after linearization in ε (whence the adjective "tangential") reduces to the study of *complete Abelian integral*

$$I(t)=I(t;H,\omega)=\oint_{H=t}\omega, \tag{1.2}$$

where the integration is carried over a continuous family of (real) ovals lying on the level curves $\{H=t\}$.

Problem 1 (Tangential Hilbert 16th problem). Place an upper bound for the number of real zeros of the Abelian integral $I(t;H,\omega)$ on the maximal natural domain of definition of this integral, in terms of $\deg H$ and $\deg \omega=\max(\deg P, \deg Q)+1$.

A more natural version appears after complexification. For an arbitrary complex polynomial $H(x, y)$ having only isolated critical points, and an arbitrary complex polynomial 1-form ω, the integral (1.2) can be extended as a multivalued analytic function ramified over a finite set of points (typically consisting of critical values of H). The problem is to place an upper bound for the number of isolated complex roots of any branch of this function, in terms of $\deg H$ and $\deg \omega$.

1.2. *Abelian integrals and differential equations.* Despite its apparently algebraic character, the tangential Hilbert problem still resists all attempts to approach it using methods of algebraic geometry. Almost all progress towards its solution so far was based on using methods of analytic theory of differential equations.

In particular, the (existential) general finiteness theorem by Khovanskiĭ–Varchenko [13, 25] claims that for any finite combination of $d=\deg\omega$ and $n=\deg H$ the number of isolated zeros is indeed uniformly bounded over all forms and all Hamiltonians of the respective degree. One of the key ingredients of the proof is the so called *Pfaffian elimination*, an analog of the intersection theory for varieties defined by Pfaffian differential equations [14].

Another important achievement, an explicit upper bound for the number of zeros in the *elliptic case* when $H(x, y)=y^2+p(x)$, $\deg p=3$ and forms of arbitrary degree, due to G. Petrov [22], uses the fact that the *elliptic integrals* $I_k(t)=\oint x^{k-1}y\,dx$, $k=1, 2$, in this case satisfy an explicit system of linear first order system of differential equations with rational coefficients. This method was later generalized for other classes of Hamiltonians whose level curves are elliptic (i.e., of genus 1), see [8, 11, 28] and references therein.

In [17] the authors constructed a linear differential equation satisfied by all Abelian integrals of 1-forms of degree $\leqslant d$ and obtained using the tools from [12] an asymptotically exponential in d upper bound for tangential Hilbert problem.

The ultimate achievement in this direction is a theorem by Petrov and Khovanskii, placing an *asymptotically linear* in $\deg \omega$ upper bound for the number of zeros of arbitrary Abelian integrals, with the constants being uniform over all Hamiltonians of degree $\leqslant n$ (unpublished). However, one of these constants is purely existential: its dependence on n is totally unknown.

It is important to remark that all the approaches mentioned above, require a very basic and easily obtainable information concerning the differential equations (their mere existence, types of singularities, polynomial or rational form of coefficients, in some cases their degree).

1.3. *Meandering of integral trajectories.* A different approach suggested in [15] consists in an attempt to apply a very general principle, according to which integral trajectories of a polynomial vector field (in $\mathbb{R}^n$ or $\mathbb{C}^n$) have a controllable meandering (sinuosity), [16, 18]. More precisely, if a curve of known size is a part of an integral trajectory of a polynomial vector field whose degree and the magnitude of the coefficients are explicitly bounded from above, then the number of isolated intersections between this curve and any affine hyperplane in the ambient space can be explicitly majorized in terms of these data. The bound appears to be very excessive: it is polynomial in the size of the curve and the magnitude of the coefficients, but the exponent as the function of the degree and the dimension of the ambient space, grows as a tower (iterated exponent) of height 4.

In order to apply this principle to the tangential Hilbert problem, we consider the curve parameterized by the *monomial integrals*,

$$t \mapsto (I_1(t), ..., I_N(t)), \qquad I_i(t) = \oint_{H=t} \omega_i,$$

where ω_i, $i = 1, ..., N$ are all *monomial* forms of degree $\leqslant d$. Isolated zeros of the Abelian integral of an arbitrary polynomial 1-form $\omega = \sum_i c_i \omega_i$ correspond to isolated intersections of the above curve with the hyperplane $\sum c_i I_i = 0$. If this monomial curve is integral for a system of polynomial differential equations with *explicitly bounded* coefficients, then the bounded meandering principle would yield a (partial) answer for the tangential Hilbert 16th problem.

The system of polynomial (in fact, linear) differential equations can be written explicitly for the case of *hyperelliptic integrals* corresponding to the Hamiltonian $H(x, y) = y^2 + p(x)$ with an arbitrary univariate *potential* $p(x) \in \mathbb{C}[x]$, see Section 2 below and references therein. Application of the bounded meandering principle allowed us to prove in [15] that the number of zeros of hyperelliptic integrals is majorized by a certain tower function depending only on the degrees of $n = \deg H = \deg p$ and $d = \deg \omega$. (Actually, it was done under an additional assumption that all critical values of p are real, but we believe that this restriction is technical and can be removed).

1.4. *Picard–Fuchs equations and systems of equations.* In order to generalize the construction from [15] for the case of arbitrary (not necessarily hyperelliptic) Hamiltonians it is necessary, among other things, to write a system of polynomial differential equations for Abelian integrals and estimate explicitly the magnitude of its coefficients.

The mere existence of such a system is well known since times of Riemann if not Gauss. In today's language, the monodromy group of any form depends only on the Hamiltonian. Denote by μ the rank of the first homology group of a typical affine level curve $\{H = t\} \subset \mathbb{C}^2$. Then for any collection of 1-forms $\omega_1, ..., \omega_\mu$ the *period matrix* $X(t)$ can be formed, whose entries are integrals of ω_i over the cycles $\delta_1(t), ..., \delta_\mu(t)$ generating the homology. If the determinant of this matrix if not identically zero, then $X(t)$ satisfies a linear ordinary differential equation of the form

$$\dot{X}(t) = A(t)\, X(t), \qquad A(\cdot) \in \mathrm{Mat}_{\mu \times \mu}(\mathbb{C}(t)), \tag{1.3}$$

with a rational matrix function $A(t)$. This system of equations is known under several names, from *Gauss–Manin connection* [21, especially p. 18] to *Picard–Fuchs system* (of linear ordinary differential equations with rational coefficients, in full). We shall systematically use the last name.

The rank of the first homology can be easily computed: for a generic Hamiltonian of degree $n+1$ it is equal to n^2. The degree $\deg A(t)$ can be relatively easily determined if the degrees of the forms ω_i are known. However, the choice of the forms ω_i may also be a difficult problem for some Hamiltonians. The matrix $A(t)$ apriori may have poles not only in the ramification points of the Abelian integrals, which leads to additional difficulties. But worst of all, this topological approach gives absolutely no control over the magnitude of the (matrix) coefficients of the rational (matrix) function $A(t)$.

1.5. *Regularity at infinity and Gavrilov theorems.* Part of these problems can be resolved. In particular, if the Hamiltonian is sufficiently regular at infinity, then all questions concerning the degrees, can be answered.

Definition 1. A polynomial $H(x, y) \in \mathbb{C}[x, y]$ of degree $n+1$ is said to be *regular at infinity*, if one of the three equivalent conditions holds:

(1) its principal homogeneous part $\hat{H}$, a homogeneous polynomial of degree $n+1$, is a product of $n+1$ pairwise different linear forms;

(2) $\hat{H}$ has an isolated critical point (necessarily of multiplicity $\mu = n^2$) at the origin $(x, y) = (0, 0)$;

(3) the level curve $\{\hat{H} = 1\} \subset \mathbb{C}^2$ is nonsingular.

This condition means that after the natural projective compactification of the (x, y)-plane $\mathbb{C}^2$, all "interesting" things still happen only in the finite part of the compactified plane. In particular, for a polynomial regular at infinity:

(1) all level curves $\{H = t\}$ intersect the infinite line $\mathbb{C}P^1_\infty \subset \mathbb{C}P^2$ transversally,

(2) all critical points $\{(x, y)\colon dH(x, y) = 0\}$ are isolated and their number is exactly $\mu = n^2$ if counted with multiplicities,

(3) the rank of the first homology of any regular affine level curve $\{H = t\}$ is $\mu = n^2$,

(4) the map $H\colon \mathbb{C}^2 \to \mathbb{C}^1$ is a topological bundle over the set of the regular values of H, hence the Abelian integrals can be ramified only over the critical values of H.

In [6, 7] L. Gavrilov proved that for polynomials regular at infinity, the space of Abelian integrals is finitely generated as a $\mathbb{C}[t]$-module by μ *basic integrals* that can be chosen as integrals of any ν forms ω_i of degree $\leqslant 2n$ whose differentials form the basis of the quotient space $\Lambda^2 / d\hat{H} \wedge \Lambda^1$, where Λ^k is the space of polynomial k-forms on $\mathbb{C}^2$. This assertion is a global analog of the local result due to E. Brieskorn and M. Sebastiani [4, 24].

As a corollary, it follows that the collection of these basic integrals satisfies a system of equations (1.3) of size $\mu \times \mu$ with $\mu = n^2$, and it is easy to place an upper bound for the degree of the corresponding matrix function $A(t)$. This system is minimal (irredundant): generically (for Morse Hamiltonians regular at infinity), all branches of full analytic continuation of an Abelian integral span exactly μ-dimensional linear space.

From this theorem one can also derive further information concerning the Picard–Fuchs system. Namely, one can prove that if in addition to being regular at infinity, H is a Morse function on $\mathbb{C}^2$, then the matrix $A(t)$ of the Picard–Fuchs system (1.3) has only simple poles (Fuchsian singularities) at the critical values of the Hamiltonian and only at them (the point $t = \infty$ is a regular though in general non-Fuchsian singularity).

However, these results do not yet allow an explicit majoration of the coefficients (e.g., the residue matrices) of the matrix function $A(t)$ in (1.3).

1.6. *Redundant Picard–Fuchs system*: *The first main result*. We suggest in this paper a procedure of explicit derivation of the Picard–Fuchs system of equations, based on the division by the gradient ideal $\langle H_x, H_y \rangle \subset \mathbb{C}[x, y]$ in the polynomial ring. It turns out that if instead of choosing

$\mu = n^2$ forms of degree $\leqslant 2n$ constituting a basis modulo the gradient ideal, one takes all $\nu = n(2n-1)$ cohomologically independent monomial forms of degree $\leqslant 2n$, then the resulting Picard–Fuchs system can be written in the (generalized) *hypergeometric* form [2]

$$(tE - A)\,\dot{X}(t) = BX(t), \qquad A, B \in \mathrm{Mat}_{\nu\times\nu}(\mathbb{C}), \tag{1.4}$$

where E is the identity matrix, and $X(t)$ is the rectangular period $\nu \times \mu$-matrix.

The procedure of deriving the system (1.4), being completely elementary, can be easily analyzed and upper bounds for the matrix norms $\|A\|$ and $\|B\|$ derived. These bounds depend on the magnitude of all non-principal terms $H - \hat{H}$ of the Hamiltonian, relative to the principal part $\hat{H}$.

More precisely, we introduce a normalizing condition (*quasimonicity*) on the homogeneous part: this condition plays the same role as the assumption that the leading term has coefficient 1 for univariate polynomials. The quasimonicity condition can be always achieved by an affine change of variables, provided that H is regular at infinity, hence it is not restrictive. Theorem 2, our first main result, allows to place an upper bound for the norms $\|A\| + \|B\|$ in terms of the norm (sum of absolute values of all coefficients) of the non-leading part $H - \hat{H}$, assuming that H is quasimonic.

1.7. *Corollaries*: *Theorems on zeros.* The above information on coefficients of Picard–Fuchs system already suffices to apply the bounded meandering principle and obtain *an explicit upper bound* for the number of zeros of complete Abelian integrals *away from the critical locus of the Hamiltonian* (Theorem 3), which seems to be the first known explicit result of that kind.

In addition to this bound valid for *some* zeros and *almost all* Hamiltonians, one can apply results (or rather methods) from [23]. If in addition to the quasimonicity and bounded lower terms, all *critical values* $t_1, \ldots, t_\mu$ of the Hamiltonian H are far away from each other (i.e., a *lower* bound for $|t_i - t_j|$ is known for $i \neq j$), then one can majorize the number of zeros on any branch of the Abelian integral by a function depending only on n, d and the minimal distance between critical values. The accurate formulation is given in Theorem 4.

1.8. *Equivariant formulation.* However, the description given by Theorem 2, is not completely sufficient for further advance towards solution of the tangential Hilbert problem by studying zeros of Abelian integrals *near the critical locus* when the latter (or some part of it) shrinks to one point of high multiplicity.

One reason is that in order to run an inductive scheme similar to that constructed in [15], one has to make sure that the Hamiltonian

$H: \mathbb{C}^2 \to \mathbb{C}^1$ can be rescaled (using affine transformations in the preimage $\mathbb{C}^2$ and the image $\mathbb{C}^1$) so that *simultaneously*:

(1) the critical values of H do not tend to each other (e.g., their diameter is bounded from below by 1), and

(2) the "non-homogeneous part" $H - \hat{H}$ is bounded by a constant explicitly depending on n

(each of the two conditions can be obviously satisfied separately).

Another, intrinsic reason is the equivariance (or, rather precisely, non-invariance of neither Theorem 2 nor Theorems 3 and 4) by the above affine group action. In order to be geometrically sound, all assertions should be related to a certain privileged *affine* chart on the t-plane. Since our future goal is to study a neighborhood of the critical locus, it is natural to choose the privileged chart so that the critical locus will not shrink to one point.

More detailed explanations and motivations are given in Section 4 below, where we formulate several problems all in the following sense: for a polynomial whose principal homogeneous part is normalized (in a certain sense) and whose critical values are explicitly bounded, it is required to place an upper bound for the "non-homogeneous" part, eventually after a suitable translation (which does not affect the principal part, naturally).

1.9. *Geometry of critical values of polynomials.* The reason why several problems of the above type were formulated instead of just one, is very simple: *we do not know a complete solution*, so partial, existential or limit cases were considered as intermediate steps towards the ultimate goal. In Section 5 we prove that:

- if a monic complex polynomial $p(x) = x^{n+1} + \cdots \in \mathbb{C}[x]$ has all critical values in the unit disk, then its roots form a point set of diameter < 11 (Theorem 6) and hence by a suitable translation the norm of the non-principal part can be made $\leqslant 12^{n+1}$ (this gives a complete solution in the univariate and hyperelliptic cases);
- all critical values of a Hamiltonian regular at infinity, cannot simultaneously coincide unless the Hamiltonian is essentially homogeneous (Theorem 5);
- for any normalized principal part $\hat{H}$ there exists an upper bound for $H - \hat{H}$ (eventually after a suitable translation), provided that the critical values of H are all in the unit disk (Corollary to Theorem 5).

All these are positive results towards solution of the problem on critical values. It still remains to compute the upper bound from the last assertion explicitly: the proof below does not provide sufficient information for that.

However, it can be shown by simple examples that this bound *cannot be uniform* over all homogeneous parts. As some of the linear factors from $\hat{H}$ approach too closely to each other, an explosion occurs and the non-principal part may be arbitrarily large without affecting the "moderate" critical values. The phenomenon can be seen as "almost occurrence" of atypical values, ramification points for Abelian integrals that are not critical values of H: such points are known to appear if the principal part $\hat{H}$ has a non-isolated singularity.

2. PICARD–FUCHS SYSTEM IN THE HYPERELLIPTIC CASE

2.1. *Gelfand–Leray residue.* The derivative of an Abelian integral $\oint_{H=t} \omega$ can be computed as the integral over the same curve of another 1-form θ called the Gelfand–Leray derivative (residue). More precisely, if a pair of polynomial 1-forms ω, θ satisfies the identity $d\omega = dH \wedge \theta$, then for any continuous family of cycles $\delta(t)$ on the level curves $\{H = t\}$

$$\frac{d}{dt} \oint_{\delta(t)} \omega = \oint_{\delta(t)} \theta, \qquad \forall \delta(t) \subset \{H = t\} \tag{2.1}$$

(the Gelfand–Leray formula). The identity remains true if θ is only meromorphic but has zero residues after restriction on each curve $H = t$.

The identity between ω, dH and θ explains the standard notation $\theta = d\omega / dH$: to find θ, one has to divide $d\omega$ by dH. In general this division is not possible in the class of polynomial 1-forms, but one can always divide $d\omega$ by dH *with remainder*: the corresponding identity after integration will give a differential equation relating Abelian integrals with their derivatives.

We illustrate this idea by deriving explicitly the Picard–Fuchs system for hyperelliptic Hamiltonians. In the hyperelliptic case the outlined approach yields a complete and in some sense minimal (irredundant) system that could be in principle derived by a number of different ways, e.g., as in [9]. Moreover, using the explicit nature of Euclid's algorithm of division of univariate polynomials, one can produce explicit upper bounds for the magnitude of the coefficients of the resulting equations, that are difficult (if possible at all) to obtain applying methods from [9]. The constructions from this section serve as a paradigm for further exposition in Section 3.

2.2. *Division by polynomial ideals and* 1-*forms.* Let $q_1, q_2 \in \mathbb{C}[x, y]$ be a pair of polynomials generating the ideal $\langle q_1, q_2 \rangle \subset \mathbb{C}[x, y]$ that has a finite codimension μ. By definition, this means that there exist μ polynomials $r_1, \ldots, r_\mu \in \mathbb{C}[x, y]$ (the remainders) such that any polynomial

$f \in \mathbb{C}[x, y]$ admits representation $f = q_1 u_2 - q_2 u_1 + \sum_1^\mu \lambda_i r_i$ with polynomials $u_1, u_2 \in \mathbb{C}[x, y]$ and constants $\lambda_i \in \mathbb{C}$.

It is convenient to interpret this identity as a division formula for polynomial 2-forms: any polynomial 2-form $\Omega = f(x, y)\, dx \wedge dy$ can be divided by the given 1-form $\xi = q_1\, dx + q_2\, dy$ with the "incomplete ratio" $\eta = u_1\, dx + u_2\, dy$ and the remainder that is a linear combination of the 2-forms $\Omega_i = r_i\, dx \wedge dy$,

$$\Omega = \xi \wedge \eta + \sum_{i=1}^{\mu} \lambda_i \Omega_i.$$

Denoting by Λ^k, $k = 0, 1, 2$, the modules (over the ring $\mathbb{C}[x, y]$) of polynomial k-forms on $\mathbb{C}^2$, we say that the tuple of 2-forms $\{\Omega_i\}_1^\mu$ generates the quotient $\Lambda^2/\xi \wedge \Lambda^1$.

The *gradient ideal* $\langle H_x, H_y \rangle$ has a finite codimension provided that H is regular at infinity. Applying the division formula in the particular case $\xi = dH$ and $\Omega = d\omega$, where ω is a differential polynomial 1-form and representing the generators explicitly as $\Omega_i = d\omega_i$ for appropriate polynomial primitives $\omega_i \in \Lambda^1$ yields the divisibility

$$d\omega = dH \wedge \eta + \sum_{i=1}^{\mu} \lambda_i\, d\omega_i. \tag{2.2}$$

This means that the Gelfand–Leray derivative of the form $\omega - \sum_1^\mu \lambda_i \omega_i$ can be found in the class of *polynomial* 1-forms, $\eta \in \Lambda^1$.

2.3. *Appearance of Picard–Fuchs systems*: *The general scheme.* The primitive remainder 1-forms ω_i from (2.2) are determined (non-uniquely) by the Hamiltonian H. The 2-forms $H\, d\omega_i$ are polynomial 2-forms that in turn can be divided by dH as above, yielding the system of identities

$$H\, d\omega_i = dH \wedge \eta_i + \sum_{j=1}^{\mu} a_{ij}\, d\omega_j, \qquad i = 1, ..., \mu,\ a_{ij} \in \mathbb{C}. \tag{2.3}$$

After taking the Gelfand–Leray residues and integration over any cycle $\delta(t)$ on the level curve, we obtain a system of linear identities relating *derivatives* of the integrals $I_i = \oint \omega_i$ with integrals of some other polynomial forms $J_i = \oint \eta_i$:

$$t\dot{I}_i = J_i + \sum_j a_{ij} \dot{I}_j. \tag{2.4}$$

From Gavrilov theorems it already follows that the "quotient" integrals J_i can be expressed as combinations of "remainder" integrals I_i with coefficients polynomial in t; after substitution into the system (2.4) this would already yield a linear system of differential equations with rational coefficients on I_i as functions of t. In some cases (e.g., in the hyperelliptic case considered below) both the division and the representation of J_i via I_i can be performed explicitly and bounds on the coefficients of the resulting system obtained.

Alternatively, if H is regular at infinity then the polynomial forms η_i have the same degree as ω_i. If we increase the number of the forms ω_i including all monomial forms of a given degree $n+1$, then η_j can be always represented as the linear combinations of ω_i with *constant* coefficients. After minor modifications this yields the redundant system (1.4).

2.4. *Derivation of the Picard–Fuchs system in the hyperelliptic case.* Throughout this section we assume that $H(x, y)=\frac{1}{2}y^2+p(x)$, where $p\in\mathbb{C}[x]$ is a *monic* polynomial of degree $n+1$ in one variable without the term x^n: $p(x)=x^{n+1}+c_{n-1}x^{n-1}+\cdots+c_1x+c_0$. Denote by c the ℓ^1-norm of the string of its non-principal coefficients, $c=|c_0|+\cdots+|c_{n-1}|$.

The gradient ideal and the corresponding quotient algebra in this case can be easily computed,

$$\langle H_x, H_y\rangle=\langle p'(x), y\rangle,$$

$$\mathbb{C}[x, y]/\langle H_x, H_y\rangle\simeq\mathbb{C}[x]/\langle x^n\rangle\simeq\bigoplus_{k=1}^{n}\mathbb{C}x^{k-1},$$

so that the quotient algebra is an algebra of truncated univariate polynomials of degree $\leqslant n-1$. This observation motivates the following computation.

Denote by $\omega_i=x^{i-1}y\,dx$, $i=1, ..., n$, the differential 1-forms whose derivatives $d\omega_i=x^{i-1}\,dx\wedge dy$ generate $\Lambda^2/dH\wedge\Lambda^1$. Then

$$\begin{aligned}
H\,d\omega_i&=(\tfrac{1}{2}y^2+p(x))\,x^{i-1}\,dx\wedge dy\\
&=[\tfrac{1}{2}x^{i-1}yH_y+b_i(x)\,H_x+a_i(x))]\,dx\wedge dy\\
&=(\tfrac{1}{2}x^{i-1}y\,dx-b_i(x)\,dy)\wedge dH+a_i(x)\,dx\wedge dy\\
&=\left[\tfrac{1}{2}\omega_i+\sum_{j=1}^{n}b_{ij}\omega_j+d(yb_j(x))\right]\wedge dH+\sum_{j=1}^{n}a_{ij}\,d\omega_j,
\end{aligned}$$

where we used the following identities:

(i) division with remainder: the polynomial $x^{i-1}p(x)$ of degree $n+i$ is divided out by $p'(x)=H_x$ as

$$x^{i-1}p(x)=b_i(x)\,p'(x)+a_i(x),\qquad \deg b_i\leqslant i,\qquad \deg a_i\leqslant n, \tag{2.5}$$

(ii) the form $b_i(x)\,dy$ is represented as a linear combination of the basic forms modulo an exact term

$$b_i(x)\,dy = d(yb_i) - b_i'(x)\,y\,dx = \sum_{j=1}^{i} b_{ij} x^{j-1} y\,dx + dF_i, \tag{2.6}$$

since the degree of $b_i' \in \mathbb{C}[x]$ never exceeds $i-1$;

(iii) the remainders $a_i(x)\,dx \wedge dy$ can be represented as linear combinations of $d\omega_j$:

$$a_i(x)\,dx \wedge dy = \sum_{j=1}^{n} a_{ij} x^{j-1}\,dx \wedge dy = \sum_{j=1}^{n} a_{ij}\,d\omega_j. \tag{2.7}$$

Integrating over closed ovals of the level curves $H=t$ (so that the exact forms dF_i disappear) and using the Gelfand–Leray formula (2.1), we conclude with the system of linear ordinary differential equations

$$t\dot{I}_i - \sum_{j=1}^{n} a_{ij}\dot{I}_j = \tfrac{1}{2}\,I_i + \sum_{j=1}^{n} b_{ij} I_j \tag{2.8}$$

or, in the matrix form,

$$(tE - A)\,\dot{I} = BI, \qquad I \in \mathbb{C}^n, \qquad A, B \in \operatorname{Mat}_{n\times n}(\mathbb{C}), \tag{2.9}$$

where, obviously, $I_j(t) = \oint \omega_j$ are the Abelian integrals and $I = (I_1, ..., I_n)$ the column vector.

Remark 1. The computation above does not depend on the choice of the cycle of integration, therefore the system of equations will remain valid if we replace the column vector I by the *period matrix* $X(t)$ obtained by integrating all forms ω_i over all *vanishing cycles* $\delta_j(t)$, $j=1, ..., n$ (see [1]) on the hyperelliptic level curves.

2.5. *Spectral properties of matrices A and B.* The matrices A, B can be completely described using the division algorithm. The identities (2.5) imply the following claim, which gives a complete description (eigenbasis and eigenvalues) of A.

PROPOSITION 1. *Let $x_* \in \mathbb{C}$ be a critical point of p and $t_* = p(x_*)$ the corresponding critical value. Then the column vector $(1, x_*, x_*^2, ..., x_*^{n-1}) \in \mathbb{C}^n$ is the eigenvector of A with the eigenvalue t_*.*

COROLLARY 1. *If the potential p is a Morse polynomial, then A is diagonalizable and its eigenvalues are the critical values of H.*

Entries of the matrix B can be described similarly: $b_{ij}=0$ for $j>i$ because of the assertion about degrees of $b_j(x)$, so B is triangular. The diagonal entries can be easily computed by looking at the leading terms: since p is monic, $b_i(x)=x^i/(n+1)+\cdots$, hence $-b_i'(x)=-\frac{i}{n+1}x^{i-1}+\cdots$. Substituting this to the formula (2.6), we obtain a complete spectral description of the matrix B. Notice that the eigenvalues of B coincide with the growth exponents of the hyperelliptic integrals $\oint_{y^2+p(x)=t} x^i y\,dx$ as $t\to+\infty$ along, say, the positive semiaxis.

Proposition 2. *The matrix B is always diagonalizable. Its spectrum consists of the numbers $\frac{1}{2}-\frac{i}{n+1}$, $i=1, ..., n$.*

However, knowledge of the critical values of H is not yet sufficient to produce an upper bound for the norms $\|A\|$, $\|B\|$, since the conjugacy by the Vandermonde matrix (whose columns are the above eigenvectors $(1, x_j, x_j^2, ..., x_j^{n-1})^T$, $j=1, ..., n$) may increase arbitrarily the norm of the diagonal matrix $\operatorname{diag}(t_1, ..., t_n)$, where t_j are all critical values of H (or p, what is the same). On the contrary, a linear change in the space of 1-forms that makes A diagonal, can increase in an uncontrollable way the norm of the matrix B, whose eigenbasis differs from the standard one by a triangular transformation. It is the explicit division procedure that allows to majorize the matrix norms.

2.6. *Bounds for the matrix norms.* For a polynomial $p\in\mathbb{C}[x]$ let $\|p\|$ be the sum of absolute values of its coefficients (we will refer to it as the *norm*, or ℓ^1-norm of p). It has the advantage of being multiplicative, $\|pq\|\leqslant\|p\|\cdot\|q\|$.

Proposition 3. *If $q=x^n+\cdots\in\mathbb{C}[x]$ is a monic polynomial with $\|q-x^n\|=c$, then any other polynomial $f\in\mathbb{C}[x]$ of degree $d\geqslant n$ can be divided with remainder,*

$$f(x)=b(x)\,q(x)+a(x), \qquad \deg a\leqslant n-1, \tag{2.10}$$

so that

$$\|b\|+\|a\|\leqslant K\,\|f\|, \qquad K=1+C+C^2+\cdots+C^{d-n}, \qquad C=1+c=\|q\|. \tag{2.11}$$

Proof. The proof goes by direct inspection of the Euclid algorithm of univariate polynomial division. The assertion of the Proposition is trivial for $q=x^n$: in this case the string of coefficients of f has to be split into two, and immediately we have the decomposition $f=bx^n+a$ with $\|b\|+\|a\|=\|f\|$.

The general nonhomogeneous case is treated by induction. Suppose that the inequality (2.11) is valid for any polynomial $\tilde{f}$ of degree $\leqslant d-1$ (for

$d=n-1$ it is trivially satisfied by letting $b=0$ and $a=\tilde{f}$). Take a polynomial f of degree d and write the identity $f=bx^n+a=bq+b(x^n-q)+a=bq+\tilde{f}$, where the polynomial $\tilde{f}=a+b(x^n-q)$ is of degree $\leqslant d-1$ and has the norm explicitly bounded: $\|\tilde{f}\|\leqslant c\,\|b\|+\|a\|\leqslant(1+c)(\|b\|+\|a\|)\leqslant C\,\|f\|$. By the induction assumption, $\tilde{f}$ can be divided, $\tilde{f}=\tilde{b}q+\tilde{a}$, with the norms satisfying the inequality (2.11). Collecting everything together, we have $f=(b+\tilde{b})\,q+\tilde{a}$ and $\|b+\tilde{b}\|+\|\tilde{a}\|\leqslant\|f\|+C\,\|f\|\,(1+C+\cdots+C^{d-1-n})\leqslant\|f\|\,(1+\cdots+C^{d-n})$. ∎

As a corollary to this Proposition and the explicit procedure of the division, we obtain upper bounds for norms of the matrices A, B. Recall that we use the ℓ^1-norm on the "space of columns," so the norm of a matrix $A=(a_{ij})_{i,\,j=1}^n$ is

$$\|A\|=\max_{j=1,\,\ldots,\,n}\sum_{i=1}^{n}|a_{ij}|. \tag{2.12}$$

THEOREM 1. *Suppose that* $p(x)=x^{n+1}+\sum_{i=0}^{n-1}c_ix^i$ *is a monic polynomial of degree* $n+1$ *and the non-principal part of* p *is explicitly bounded*: $\sum_{i=0}^{n-1}|c_i|\leqslant c$.

Then the entries of the matrices A, B *determining the Picard–Fuchs system* (2.9) *are explicitly bounded*:

$$\|A\|+\|B\|\leqslant n^2(1+C+\cdots+C^{n+1}),\qquad C=1+c=\|p\|. \tag{2.13}$$

Proof. The derivative $p'(x)$ is not monic, but the leading coefficient is explicitly known: $p'(x)=(n+1)(x^n+\cdots)$, with the non-principal part denoted by the dots bounded by c in the sense of the norm. Applying Proposition 3 to $q=p'/(n+1)$, we see that any polynomial can be divided by p' and the same inequalities (2.11) would hold (since $n+1\geqslant 1$).

Thus we have $\|b_i\|+\|a_i\|\leqslant K\,\|x^{i-1}\|\,\|p\|=KC$, where $K=1+C+\cdots+C^n$, then obviously $\|b_j'\|\leqslant n\,\|b_j\|$ and finally for the sum of matrix elements A, B occurring in the ith line, we produce an upper bound $\frac{1}{2}+\sum_j|b_{ij}|+\sum_j|a_{ij}|\leqslant nC(1+C+\cdots+C^n)+\frac{1}{2}\leqslant n(1+C+\cdots+C^{n+1})$. Clearly, this means that every entry of these matrices is majorized by the same expression and therefore for the matrix ℓ^1-norms on $\mathbb{C}^n$ we have the required estimate. ∎

2.7. *Digression*: *Doubly hyperelliptic Hamiltonians.* The algorithm suggested above, works with only minor modifications for *doubly hyperelliptic* Hamiltonians having the form $H(x,y)=p(x)+q(y)$ (the hyperelliptic case corresponds to $q(y)=\frac{1}{2}y^2$). Assume that $n+1=\deg_x p$, $m+1=\deg_y q$ (there is no reason to require that $n=m$).

In this case the quotient algebra by the gradient ideal is generated by nm monomials x^iy^j, $0\leqslant i\leqslant n-1$, $0\leqslant j\leqslant m-1$. We claim that any collection of monomial primitives ω_{ij} to the monomial 2-forms $d\omega_{ij}=x^iy^j\,dx\wedge dy$

satisfies a system of nm equations having the same form (2.9) though a different size. Indeed,

$$H\,d\omega_{ij} = p(x)\,x^i\,dx \wedge y^j\,dy - q(y)\,y^j\,dy \wedge x^i\,dx.$$

Dividing the 1-form $p(x)\,x^i\,dx$ with remainder by the 1-form $dp(x)$, we express the former as $b_i(x)\,dp(x) + a_i(x)\,dx$ with $\deg b_i \leqslant i+1 \leqslant n+1$, $\deg a_i \leqslant n-1$ and multiply the result by $y^j\,dy$. The second term can similarly be rewritten involving the representation $q(y)\,y^j = b_j^*(y)\,dq + a_j^*(y)\,dy$. Putting everything together, we conclude that

$$\begin{aligned} H\,d\omega_{ij} &= [\,dp(x) \wedge\ b_i(x)\,y^j\,dy - dq(y) \wedge\ b_j^*(y)\,x^i\,dx\,] \\ &\quad + [\,a_i(x)\,y^j - x^i a_j^*(y)]\,dx \wedge\ dy. \end{aligned}$$

Since $dH = dp(x) + dq(y)$, we see that the first bracket is actually the wedge product $dH \wedge \eta_{ij}$, where $\eta_{ij} = b_i(x)\,y^j\,dy + b_j^*(y)\,x^i\,dx$ is a polynomial 1-form whose differential

$$d\eta_{ij} = \left(\frac{\partial b_i}{\partial x}\,y^j - \frac{\partial b_j^*}{\partial y}\,x^i \right) dx \wedge dy$$

has the coefficient of degree $\leqslant i$ in x and $\leqslant j$ in y and hence can be expanded as a linear combination of the forms ω_{ij} modulo an exact form. The second bracket, being a 2-form with coefficient of degrees $\leqslant n-1$ in x and $\leqslant m-1$ in y, is a linear combination of the forms $d\omega_{ij}$. Thus we have the equations

$$\begin{gathered} H\,d\omega_{ij} = dH \wedge \left(\sum_{k,\,l=0} B_{ij,\,kl}\,\omega_{kl} + dF_{ij} \right) + \sum_{k,\,l} A_{ij,\,kl}\,d\omega_{kl}, \\ i, k = 0, ..., n-1, \qquad j, l = 0, ..., m-1. \end{gathered}$$

Rescaling x and y by appropriate factors *independently*, we can assume that the polynomials $p(x)$ and $q(y)$ are both monic. Then all divisions will be bounded provided that the norms $\|p\|$ and $\|q\|$ are explicitly bounded, and in a way completely similar to the arguments from Section 2.6, we can derive upper bounds for the matrix coefficients $A_{ij,\,kl}$, $B_{ij,\,kl}$.

Thus the case of doubly hyperelliptic Hamiltonians does not differ much from the ordinary hyperelliptic case, at least as far as the Picard–Fuchs systems for Abelian integrals are concerned.

2.8. *Discussion.* The Picard–Fuchs system written in the form (2.9) for a generic hyperelliptic Hamiltonian (with the potential $p(x)$ being a Morse function on $\mathbb{C}$), is a system remarkable for several instances:

- it possesses only Fuchsian singularities (simple poles) both at all finite singularities $t = t_j$, $j = 1, ..., n$, and at infinity;
- it has no apparent singularities: all points t_j are ramification points for the fundamental system of solutions $X(t)$ that is obtained by integrating all forms ω_i over all vanishing cycles $\delta_j(t)$ (the period matrix);
- it is minimal in the sense that analytic continuations of any column of the period matrix $X(t)$ along all closed loops span the entire space $\mathbb{C}^n$.
- its coefficients can be explicitly bounded in terms of $\|H\|$.

(All these observations equally apply to doubly hyperelliptic Hamiltonians.)

In the next section we generalize this result for arbitrary bivariate Hamiltonians. It will be impossible to preserve simultaneously *all* properties, and we shall concentrate on the derivation of the *redundant* system, eventually exhibiting apparent singularities, but all of them (including that at infinity) Fuchsian and with explicitly bounded coefficients.

3. DERIVATION OF THE REDUNDANT PICARD–FUCHS SYSTEM

3.1. *Notations and conventions.* Recall that Λ^k denotes the space of polynomial k-forms on $\mathbb{C}^2$ for $k = 0, 1, 2$. They will be always equipped with the ℓ^1-norms: the norm of a form is always equal to the sum of absolute values of all its coefficients. This norm behaves naturally with respect to the (wedge) product: for any two forms $\eta \in \Lambda^k$, $\theta \in \Lambda^l$, $0 \leqslant k + l \leqslant 2$, we always have $\|\eta \wedge \theta\| \leqslant \|\eta\| \cdot \|\theta\|$.

It is also convenient to grade the spaces of polynomial forms so that the degree of a k-form is the maximal degree of its (polynomial) coefficients plus k. Under this convention the exterior derivation is degree-preserving: $\deg d\theta = \deg \theta$ (unless $d\theta = 0$). An easy computation shows that $\|d\theta\| \leqslant \deg \theta \cdot \|\theta\|$ for any 0- and 1-form θ. On several occasions the finite-dimensional linear space of k-forms of degree $\leqslant d$ will be denoted by Λ^k_d.

If $\omega \in \Lambda^1$ is a polynomial 1-form and $H \in \Lambda^0$, then by $d\omega/dH$ is always denoted the Gelfand–Leray derivative (2.1), while by $\frac{d\omega}{dx \wedge dy}$ we denote the polynomial coefficient of the 2-form $d\omega$.

The space Λ^2 sometimes will be identified with $\Lambda^0 \simeq \mathbb{C}[x, y]$, the submodule $dH \wedge \Lambda^1$ with the gradient ideal $\langle H_x, H_y \rangle \subset \mathbb{C}[x, y]$, and the quotient algebra as a linear space over $\mathbb{C}$ with the quotient $\Lambda^2/dH \wedge \Lambda^1$.

3.2. *Normalizing conditions and quasimonic Hamiltonians.* In the ring $\mathbb{C}[x]$ of univariate polynomials division by the principal ideal $\langle p \rangle$ is a linear operator whose norm can be controlled in terms of $\|p\|$ provided that the leading term of p is bounded from below, in particular when the polynomial is monic (see the proof of Proposition 3). The definition below introduces a generalization of this condition for ideals in the ring $\mathbb{C}[x, y]$ of bivariate polynomials.

Recall that two *homogeneous* polynomials $a, b \in \mathbb{C}[x, y]$ of the same degree n have no common linear factors if and only if their resultant is non-zero and hence the Sylvester matrix is invertible. In this case an arbitrary homogeneous polynomial f of degree $2n-1$ can be represented as $f = au + bv$ with uniquely defined homogeneous polynomials u, v of degree $n-1$ each.

DEFINITION 2. A pair of homogeneous polynomials $a, b \in \mathbb{C}[x, y]$ of degree n is said to be *normalized* if the linear operator $(u, v) \mapsto au + bv$ restricted on the subspace of pairs of homogeneous polynomials of degree $n-1$, has the inverse of the unit norm, in other words, if any homogeneous polynomial f of degree $2n-1$ can be represented as $f = au + bv$ with an explicit control over norms of the homogeneous "ratios" u, v of degree $n-1$:

$$f = au + bv, \qquad \|u\| + \|v\| \leqslant \|f\|. \tag{3.1}$$

DEFINITION 3. A homogeneous polynomial 1-form $\eta = a\,dx + b\,dy$ of degree $n+1$ is normalized if its coefficients $a, b \in \mathbb{C}[x, y]$ form a normalized pair.

For nonhomogeneous objects we impose normalizing conditions on their principal homogeneous part.

DEFINITION 4. A polynomial 1-form $\xi \in \Lambda^1$ of degree n is *normalized at infinity*, if its principal homogeneous part $\hat{\xi}$ is normalized.

A Hamiltonian $H(x, y) \in \mathbb{C}[x, y]$ of degree $n+1$ is said to be normalized at infinity or *quasimonic*, if dH is normalized at infinity in the sense of the previous definition.

Remark 2. To be normalized at infinity has nothing to do with the ℓ^1-norm of a form or Hamiltonian. We will mostly use the term "quasimonic".

3.3. *Balanced Hamiltonians.* In order to simplify the calculations below, we impose additional normalizing condition on H meaning that the non-principal (low degree) terms are not dominating the principal part.

DEFINITION 5. A Hamiltonian $H \in \mathbb{C}[x, y]$ will be called *balanced*, if it is quasimonic (the principal homogeneous part $\hat{H}$ is normalized) and $\|H - \hat{H}\| \leqslant 1$.

For a balanced Hamiltonian, its differential dH is a 1-form that is (by definition) normalized at infinity and differs from its principal homogeneous part $d\hat{H}$ by the form of degree n and $\|dH - d\hat{H}\| \leqslant n$.

The two conditions, normalization at infinity and that of balance between principal and non-principal parts, can be obtained simultaneously by suitable affine transformations. If the Hamiltonian H is regular at infinity, then after a suitable choice of $\lambda \in \mathbb{C}$ one can make any of the two polynomials, $\lambda H(x, y)$ or $H(\lambda x, \lambda y)$ being normalized at infinity (the same refers to 1-forms). Furthermore, if H is already quasimonic, one can always choose a suitable $\lambda \in \mathbb{C}$ so that $\lambda^{n+1} H(\lambda^{-1}x, \lambda^{-1}y)$ will be balanced while remaining quasimonic.

3.4. *Lemma on bounded division.* Division by a balanced 1-form is a linear operator whose norm can be easily controlled.

Let $\xi \in \Lambda^1$ be a polynomial 1-form of degree $n+1$ normalized at infinity, with the principal homogeneous part denoted by $\hat{\xi}$.

LEMMA 1. *Any polynomial 2-form $\Omega \in \Lambda^2$ can be divided with remainder by ξ,*

$$\Omega = \xi \wedge \eta + \Theta, \tag{3.2}$$

where the remainder $\Theta \in \Lambda^2$ is a 2-form of degree $\leqslant 2n$ and the "incomplete ratio" $\eta \in \Lambda^1$ is a 1-form of degree $\deg \Omega - \deg \xi$.

The decomposition (3.2) *is in general non-unique. However, one can always find η and Θ so that if $\|\xi - \hat{\xi}\| = c$, then*

$$\|\eta\| + \|\Theta\| \leqslant K \|\Omega\|, \qquad K = (1 + C + \cdots + C^{d-2n}), \tag{3.3}$$

where $C = c + 1$ and $d =$ $\deg \Omega$.

Proof. The proof reproduces almost literally the division algorithm for univariate polynomials, see Proposition 3.

1. For a homogeneous form $\Omega = f\,dx \wedge dy$ of degree $2n+1$ the divisibility $\Omega = \tilde{\xi} \wedge \eta$ by the homogeneous form $\tilde{\xi} = a\,dx + b\,dy$ is the same as the representation (3.1) (recall that our convention concerning the degrees of the form means that in this case $\deg f = 2n - 1$). From the normalization condition it follows then that $\|\eta\| = \|u\| + \|v\| \leqslant \|f\| = \|\Omega\|$ simply by definition.

2. Writing the division identities for all monomial forms of degree $2n+1$, multiplying them by arbitrary monomials and adding results we see then that any polynomial 2-form Ω *containing no terms of degree $2n$ and less*, can be divided by $\hat{\xi}$ and the norm of the "ratio" $\tilde{\eta}$ does not exceed

$\|\Omega\|$. Finally, *any* form can be represented as the sum of a "remainder" $\hat{\Theta}$, the collection of terms of degree $\leqslant 2n$, and the higher terms divisible by $\hat{\xi}$.

All together this means that if $\hat{\xi}$ is a homogeneous normalized 1-form of degree $n+1$, then any polynomial 2-form Ω can be divided out as

$$\Omega = \hat{\xi} \wedge \tilde{\eta} + \tilde{\Theta}, \qquad \|\tilde{\eta}\| + \|\tilde{\Theta}\| \leqslant \|\Omega\|, \qquad \deg \tilde{\eta} \leqslant \deg \Omega - \deg \hat{\xi}. \tag{3.4}$$

3. To divide by a nonhomogeneous form ξ normalized at infinity, we first divide by its principal part $\hat{\xi}$ as in (3.4). Then

$$\Omega = \xi \wedge \tilde{\eta} + (\hat{\xi} - \xi) \wedge \tilde{\eta} + \tilde{\Theta} = \xi \wedge \tilde{\eta} + \tilde{\Omega}. \tag{3.5}$$

It remains to notice that $\|\tilde{\eta}\| \leqslant \|\tilde{\eta}\| + \|\tilde{\theta}\| \leqslant \|\Omega\|$ and $\tilde{\Omega}$ is a new 2-form whose degree is strictly less than $d = \deg \Omega$, provided that $d > 2n$. Since the norm of $\xi - \hat{\xi}$ is explicitly bounded by c, we have

$$\|\tilde{\Omega}\| \leqslant c\, \|\tilde{\eta}\| + \|\tilde{\Theta}\| \leqslant (1+c)(\|\tilde{\eta}\| + \|\tilde{\Theta}\|) \leqslant C\, \|\Omega\|.$$

We may now continue by induction, accumulating the divided parts $\tilde{\eta}$ and reducing the degrees of "incomplete remainders" $\tilde{\Omega}$ until the latter become less or equal to $2n$. More accurately, we use the inductive assumption to divide out $\tilde{\Omega} = \xi \wedge \eta' + \Theta$ with $\|\eta'\| + \|\Theta\| \leqslant \|\tilde{\Omega}\|\, (1 + C + \cdots + C^{d-1-2n}) \leqslant \|\Omega\|\, (C + C^2 + \cdots + C^{d-2n})$ and put $\eta = \tilde{\eta} + \eta'$ so that $\Omega = \xi \wedge \eta + \Theta$. Since $\|\tilde{\eta}\| \leqslant \|\Omega\|$, we have $\|\eta\| + \|\Theta\| \leqslant \|\eta\| + \|\eta'\| + \|\Theta\| \leqslant (1 + C + \cdots + C^{d-2n})\, \|\Omega\|$. ∎

COROLLARY 3.2. *If H is a balanced Hamiltonian of degree $n+1$, then any polynomial 2-form Ω of degree $\leqslant 3n$ can be divided by dH,*

$$\Omega = dH \wedge \eta + \Theta, \qquad \|\eta\| + \|\Theta\| \leqslant (n+1)^{n+1} \cdot \|\Omega\|. \tag{3.6}$$

Proof of the corollary. It is sufficient to remark that for a balanced Hamiltonian the form dH is normalized at infinity and the difference between dH and its principal homogeneous part $d\hat{H}$ is of norm $\leqslant n$. ∎

3.5. *Derivation of the redundant Picard–Fuchs system.* Now we can write explicitly a system of first order linear differential equations for Abelian integrals, with coefficients explicitly bounded provided the Hamiltonian is balanced (i.e., its lower order terms do not dominate the principal homogeneous part). The reason why this system is called redundant, will be explained below.

Consider $v = n(2n-1)$ monomial 2-forms Ω_i spanning Λ^2_{2n}, and let $\omega_i \in \Lambda^1_{2n}$ be their monomial primitives (arbitrary chosen), $d\omega_i = \Omega_i$, with unit coefficients so that $\|\omega_i\| = 1$ and $\|d\omega_i\| \leqslant 2n$. Then any 2-form of degree $\leqslant 2n$ can be represented as a linear combination of $d\omega_i$, hence any 1-form of degree $\leqslant 2n$ admits representation as a linear combination of ω_i, $i = 1, \ldots, v$ modulo an exact differential.

THEOREM 2. *Let H be a balanced Hamiltonian of degree $n+1$.*

Then the column vector $I=(I_1(t), ..., I_\nu(t))$ of integrals of all monomial 1-forms ω_i of degree $\leqslant 2n$ over any cycle on the level curves $\{H(x, y)=t\}$ satisfies the system of linear ordinary differential equations

$$(tE-A)\,\dot{I}=BI, \qquad I=I(t)\in\mathbb{C}^\nu, \qquad A, B\in \mathrm{Mat}_{\nu\times\nu}(\mathbb{C}). \tag{3.7}$$

The norms of the constant matrices A, B are explicitly bounded:

$$\|A\|+\|B\|\leqslant 6n(n+1)^{n+1}. \tag{3.8}$$

Remark 3. We use here the norms of matrices (2.12), associated with ℓ^1-norms on the spaces of polynomials, as defined in (2.12).

Remark 4. As was already mentioned, the assumption that H is balanced, does not involve loss of generality, since any Hamiltonian regular at infinity can be balanced by appropriate affine transformation (see however the discussion below).

Proof of the Theorem. We start with a computation showing that the system can be indeed written in the form (3.7): this derivation will be later slightly modified to produce explicit bounds.

For any $i=1, ..., \nu$ the 2-form $H\,d\omega_i$ of degree $\leqslant n+1+2n$ can be divided out with remainder by the form dH (which is, by assumption, normalized at infinity):

$$H\,d\omega_i=dH\wedge\eta_i+\Theta_i, \qquad \deg\eta_i\leqslant\deg d\omega_i\leqslant 2n, \qquad \deg\Theta_i\leqslant 2n. \tag{3.9}$$

Since 2-forms $d\omega_i$ span the whole space of 2-forms of degree $\leqslant 2n$, every $d\eta_i$ and Θ_i are linear combinations of $d\omega_j$,

$$d\eta_i=\sum_{j=1}^{\nu} b_{ij}\,d\omega_j, \qquad \Theta_i=\sum_{j=1}^{\nu} a_{ij}\,d\omega_j,$$

with appropriate complex coefficients a_{ij}, b_{ij} forming two $\nu\times\nu$-matrices A, B respectively, and certain polynomials $F_j\in\mathbb{C}[x, y]$.

The first identity implies that $\eta_i=\sum_j b_{ij}\omega_j+dF_i$ for suitable polynomials F_i. Integrating over cycles on the level curves $\{H=t\}$ and using the Gelfand–Leray formula for derivatives, we conclude that

$$t\dot{I}_i=\sum_j b_{ij}\,I_j+\sum_j a_{ij}\dot{I}_j, \qquad i, j=1, ..., \nu,$$

which is equivalent to the matrix form (3.7) claimed above.

In order to place the upper bounds on the matrix norms $\|A\|$ and $\|B\|$, we can use the bounded division lemma, but additional efforts are required. Indeed, the normalization at infinity *does not imply* any upper bound on the norm of the principal part $\|\hat{H}\|$, so the norm of the left hand side in (3.9) is apriori unbounded and does not allow for application of Lemma 1.

To construct a system satisfying the inequalities (3.8), we decompose H into the principal part $\hat{H}$ and the collection of lower terms $h = H - \hat{H}$ and treat two parts, $\hat{H}\,d\omega$ and $h\,d\omega_i$ separately.

Let $\rho \in \Lambda_2^1$ be the 1-form $x\,dy - y\,dx$ with $\|\rho\| = 2$. Then by the Euler identity,

$$(n+1)\,\hat{H}\,dx \wedge dy = d\hat{H} \wedge \rho, \tag{3.10}$$

and therefore

$$\hat{H}\,d\omega_i = \frac{d\omega_i}{dx \wedge dy} \cdot \hat{H}\,dx \wedge dy = \frac{d\omega_i}{(n+1)\,dx \wedge dy} \cdot d\hat{H} \wedge \rho = (dH - dh) \wedge \eta'_i, \tag{3.11}$$

where $\|\eta'_i\| \leqslant \|d\omega_i\|\ \|\rho\|/(n+1) \leqslant 2 \cdot 2n/(n+1) \leqslant 4$. Now the term $H\,d\omega_i$ can be explicitly expanded as

$$H\,d\omega_i = \hat{H}\,d\omega_i + h\,d\omega_i = dH \wedge \eta'_i - dh \wedge \eta'_i + h\,d\omega_i = dH \wedge \eta'_i + \Omega'_i,$$

where $\|\Omega'_i\| \leqslant \|\eta'\|\ \|dh\| + \|h\|\ \|d\omega_i\| \leqslant 4n + 2n = 6n$ and $\deg \Omega'_i \leqslant 3n$. Applying Corollary 2, we write

$$\Omega'_i = dH \wedge \eta''_i + \Theta_i$$

with $\|\eta''_i\| + \|\Theta_i\| \leqslant 6n(n+1)^{n+1}$ which together with the previous bounds for $\|\eta'_i\|$ would imply the inequality

$$\|\eta_i\| + \|\Theta_i\| \leqslant 6(n+1)^{n+2}. \tag{3.12}$$

for the identities (3.9)

Since all forms ω_i, $d\omega_i$ are monomial with norms $\geqslant 1$, expanding η_i and Θ_i leads to coefficients satisfying the conditions

$$\sum_{j=1}^{\mu} |a_{ij}| \leqslant \|\theta_i\|, \qquad \sum_{j=1}^{\mu} |b_{ij}| \leqslant \|\Theta_i\|,$$

which gives the required bounds on $\|A\|$ and $\|B\|$. ∎

In order to incorporate the case of quasimonic but not balanced Hamiltonians, we derive an obvious corollary.

COROLLARY 3. *If H is quasimonic and the difference between H and its principal homogeneous part $\hat{H}$ is explicitly bounded,*

$$\|H - \hat{H}\| \leqslant c, \tag{3.13}$$

then one can choose the monomial forms so that the system (3.7) *for their integrals involves the matrices A, B satisfying the inequality*

$$\|A\| + \|B\| \leqslant 6(n+1)^{n+2} \cdot c^{n+1}. \tag{3.14}$$

Proof. It is sufficient to make a transformation replacing the initial Hamiltonian $H(x, y)$ by $c^{-(n+1)}H(cx, cy)$. This will make H balanced and the main theorem applicable. Notice that such transformation implies the change of time (the independent variable) $t \mapsto c^{-(n+1)}$ for the resulting system (3.7). With respect to the original variable the system (3.7) will take the form with the same matrix B, and A multiplied by c^{n+1}. ∎

Remark 5. Note that the system in the non-balanced case is written for forms ω_i in general not satisfying the condition $\|\omega_i\| = 1$, as was the case with balanced Hamiltonians: the linear rescaling $(x, y) \mapsto (cx, cy)$ results in a diagonal transformation that is in general non-scalar on the linear space of differential forms.

3.6. *Abelian integrals of higher degrees.* The system of differential equations (3.7) holds for integrals of the basic monomial forms ω_i generating all polynomial differential 1-forms of degree $\leqslant 2n$. To write an analogous system for integrals of 1-forms of higher degrees, one can use the fact that the integrals $\oint \omega_i$ generate the space of all Abelian integrals as a free $\mathbb{C}[t]$-module, provided that H is Morse and regular at infinity, see [6]. More precisely, if $\deg \omega = d$, then for any cycle $\delta(t)$ on the level curve $\{H = t\}$ one can represent

$$\oint_{\delta(t)} \omega = \sum_{i=1}^{\nu} p_i(t) \oint_{\delta(t)} \omega_i,$$

$$p_i \in \mathbb{C}[t], \qquad (n+1) \deg p_i + \deg \omega_i \leqslant \deg \omega \tag{3.15}$$

(in fact, it is even sufficient to take any $\mu = n^2$ forms ω_i whose differentials span $\Lambda^2/dH \wedge \Lambda^1$).

Thus the linear span of all functions $t^k I_j(t)$, $j = 1, ..., \nu$, $0 \leqslant k \leqslant m = \lfloor d/(n+1) \rfloor$, contains all Abelian integrals of forms of degree $\leqslant d$. The generators $\{t^k I_j(t)\}_{0\leqslant k\leqslant m}^{1\leqslant j\leqslant n}$ of this system satisfy a block upper triangular system of linear first order differential equations obtained by derivation of (3.7):

$$(tE-A)\frac{\mathrm{d}}{\mathrm{d}t}(t^k I)=B\,t^k I+k(tE-A)\,t^{k-1}\,I,\qquad k=1,\ldots,m. \tag{3.16}$$

This system can be written in the matrix form involving two constant $(m+1)\,\nu\times(m+1)\,\nu$-matrices exactly as (3.7) and the entries of these matrices will be explicitly bounded, though this time the bounds and the size of the system will depend explicitly on d. Nevertheless this allows to treat integrals of forms of arbitrary fixed degree d exactly as integrals of the basic forms.

3.7. *Properties of the redundant Picard–Fuchs system.* Directly from the form in which the system (3.7) was obtained, it follows that it has singular points at all critical values $t=t_j$ of the Hamiltonian; the eigenvector corresponding to the eigenvalue t_j has coordinates $(d\omega_i/(dx\wedge dy))(x_j,y_j)$, $i=1,\ldots,\nu$. However, in general (since $\mu<\nu$) these eigenvalues do not exhaust the spectrum of A.

The other eigenvalues of A actually depend on the division with remainder, that is non-unique because the forms ω_i are linear dependent in $\Lambda^2/dH\wedge\Lambda^1$. Thus no invariant meaning can be associated with this part of the spectrum, and the corresponding singularities are apparent for the Abelian integrals (though other solutions can apriori have singularities at these "redundant" points).

PROPOSITION 4. *If H is a Morse function on $\mathbb{C}^2$, then the system* (3.7) *can be constructed so that the matrix A has a simple spectrum while satisfying the same inequalities as before.*

Note that singularities of the system (3.7) are Fuchsian even if A has multiple eigenvalues. This can be seen by inspection of the inverse $(tE-A)^{-1}$ for A being in the Jordan normal form.

Proof of the Proposition. Assume that the enumeration of the forms ω_i is arranged so that the first μ of them constitute a basis in $\Lambda^2/dH\wedge\Lambda^1$. The procedure of division of the forms $H\,d\omega_i$ by dH can be altered to produce a unique answer, if we require that the remainder is always a linear combination of only the first μ forms. Moreover, instead of dividing the forms $H\,d\omega_i$ with $\mu+1\leqslant i\leqslant\nu$, we will divide the forms $(H-\lambda_i)\,d\omega_i$ with arbitrarily chosen constants $\lambda_i\in\mathbb{C}$, $i=\mu+1,\ldots,\nu$:

$$H\,d\omega_{\mathrm{i}}-\sum_{j=1}^{\mu}a_{ij}\,d\omega_j\in dH\wedge\Lambda^1,\qquad i=1,\ldots,\mu,$$

$$(H-\lambda_i)\,d\omega_{\mathrm{i}}-\sum_{j=1}^{\mu}a_{ij}\,d\omega_j\in dH\wedge\Lambda^1,\qquad i=\mu+1,\ldots,\nu.$$

After division organized in such a way, the matrix A of the system (3.7) obtained after expanding the incomplete fractions, will have block lower-triangular form. The upper-left block of size $\mu\times\mu$ has as before the eigenvalues $t_1, ..., t_\mu$, while the lower-right block of size $(\nu-\mu)\times(\nu-\mu)$ is diagonal with λ_i being the diagonal entries. Note that in this alternative derivation we lost control over the magnitude of the coefficients of remainders and incomplete ratios.

Thus for the same column vector of Abelian integrals we have constructed *two* essentially different systems of the same form (3.7) but with different pairs (A, B) of $\nu\times\nu$-matrices (the first bounded in the norm, the second with a predefined spectrum). By linearity, any linear homotopy between the two systems will also admit all Abelian integrals as solutions.

Consider such a homotopy parameterized by $s\in[0, 1]$. The eigenvalues of the matrix A do depend algebraically on the parameter s. For $s=1$ they are equal to the critical values $t_1, ..., t_\mu$ of H and arbitrarily prescribed values $\lambda_{\mu+1}, ..., \lambda_\nu$. Since $t_i\neq t_j$ and λ_i can be also chosen different from all t_j and from each other, the eigenvalues are simple for $s=1$ and hence they remain pairwise different for almost all values of s, in particular, for arbitrarily small positive s when the system is arbitrarily close to the first system (of explicitly bounded norm). Perturbing in that way achieves simplicity of the spectrum of the matrix A while changing the norms of A, B arbitrarily small. ▮

4. ZEROS OF ABELIAN INTEGRALS AWAY FROM THE SINGULAR LOCUS AND RELATED PROBLEMS ON CRITICAL VALUES OF POLYNOMIALS

In this section we show how the explicitly derived hypergeometric system for Abelian integrals allows to obtain partial solution of the tangential Hilbert 16th problem. First we recall the general results on zeros of functions defined by polynomial ordinary differential equations.

4.1. *Meandering theorem and upper bounds for zeros of Abelian integrals.* A (scalar) linear ordinary differential equation with explicitly bounded coefficients admits an explicit upper bound for the number of isolated (real or complex) zeros of all its solutions, see [12, 26].

The *system* of equations (3.7) can be reduced to one linear equation of degree $\leqslant\nu^2$ with rational in t coefficients in such a way that any linear combination $u(t)=\sum_{i=1}^{\nu} c_i I_i(t)$ of the integrals $I_i(t)$ with constant coefficients $c_1, ..., c_\nu\in\mathbb{C}$, will be a solution to this equation: it is sufficient to find a linear dependence between any fundamental $\nu \times \nu$-matrix $X(t)$ and its derivatives up to order $\nu^2 - 1$ over the field $\mathbb{C}(t)$ of rational functions.

Unfortunately, this procedure does not allow to place any bound on the magnitude of coefficients of the resulting equation. Instead, in [16, Appendix B] we described an algorithm of derivation of another linear equation of much higher order, whose coefficients are polynomially depending on the coefficients of the initial system (3.7). This algorithm is explicit, so that all degrees and coefficients admit explicit upper bounds. As a result, the system (3.7) is reduced to a *Fuchsian linear differential equation* of the form

$$\Delta^{\ell}(t)\, u^{(\ell)} + h_{\ell-1}(t)\, \Delta^{\ell-1}(t)\, u^{(\ell-1)} + \cdots + h_1(t)\, \Delta(t)\, u' + h_0(t)\, u = 0, \qquad (4.1)$$

where $\Delta(t) = (t - t_1) \cdots (t - t_\nu)$ is the characteristic polynomial of the matrix A and all polynomial coefficients $h_i(t) \in \mathbb{C}[t]$, $i = 0, ..., \ell - 1$, have degrees $\deg h_i$ and heights $\|h_i\|$ explicitly bounded by elementary functions of n. It is important to note here that the bounds, though completely explicit, are enormously excessive, being towers (iterated exponents) of height 4.

The coefficients of the equation (4.1) are explicitly bounded from above on the complement to sublevel sets $\{|\Delta(t)| \geqslant \varepsilon\}$ for every given positive $\varepsilon > 0$. At the roots of Δ (eigenvalues of A) the equation (4.1) has Fuchsian singularities, but the eigenvalues of A that are *not* critical values of H, are *apparent singularities* for all linear combinations of the Abelian integrals I_j (see Sect. 4).

Recall that Σ is the critical locus (collection of all critical values) of the Hamiltonian H. Let R be a finite positive number and $K_R \Subset \mathbb{C} \setminus \Sigma$ the set obtained by cutting the set

$$\{t \in \mathbb{C} : \forall j = 1, ..., \mu \ \ |t - t_j| > 1/R,\ |t| < R\} \qquad (4.2)$$

along no more than μ line segments to produce a *simply connected* compact "on the distance $1/R$ from both Σ and infinity."

Applying a general theorem on oscillations of solutions of linear equations with bounded coefficients [18, 16, 26], we arrive to the following theorem.

THEOREM 3 (see [16]). *Let H be a balanced Hamiltonian of degree $n+1$ and K_R a compact on distance $1/R$ from the critical locus of H in the sense of* (4.2).

Then the number of zeros inside K_R of any Abelian integral of a form of degree d does not exceed $(2+R)^N$, where $N = N(n, d)$ is a certain elementary function depending only on n and d.

The function $N(n, d)$ can be estimated from above by a tower (an iterated exponent) of four stories and certainly gives a very excessive bound. Yet we would like to remark that this is absolutely explicit answer, involving no undefined (existential) constants.

Remark 6. The necessity of cutting in the definition of K_R is due to the fact that Abelian integrals are multivalued and a choice of branch should be specified each time when zeros are counted.

The coefficients of the equation (4.1) blow up as $t \to \Sigma$, so no upper bound for zeros can be derived from the general theorem [12]. However, if the singularity t_i is apparent and distant from all other points, say, at least by 1, then one can place an upper bound on the coefficients of (4.1) on the boundary of the disk $\{|t-t_i| = \frac{1}{2}\}$ and then by [26, Corollary 2.7] the variation of argument of any solution along the boundary can be explicitly bounded and by the argument principle, this would imply an upper bound for the number of zeros also *inside* the disk, where the coefficients are very large.

It turns out that a similar construction can be also carried out when t_i is a true (non-apparent) singularity, provided that it is of Fuchsian type and the spectrum of the monodromy operator is on the unit circle.

Suppose that a function $u(t)$ analytic in the punctured disk $\{0 < |t-t_i| \leqslant 1\}$ admits a finite representation $u(t) = \sum_{\lambda, k} f_{k,\lambda}(t)(t-t_i)^{\lambda} \ln^k(t-t_i)$ with coefficients $f_{k,\lambda}$ analytic in the closed disk $\{|t-t_i| \leqslant 1\}$, involving *only real* exponents λ. If this function satisfies a linear ordinary differential equation (with a Fuchsian singularity at $t = t_i$) whose coefficients are explicitly bounded on the boundary circumference of this disk, then it is proved in [26, Theorem 4.1] that any branch of u admits an upper bound for the number of zeros in this disk in terms of the magnitude of the coefficients on the boundary and the order of the equation (the first result of this type was proved in [23]).

The assumption on the spectrum always holds for Abelian integrals, since the above exponents λ are always rational [1] (in particular, equal to 1 for a Morse critical value). Thus the above result (together with the bounded meandering principle) can be applied to the tangential Hilbert problem provided that all critical values of the Hamiltonian are at least 1-distant from each other. An arbitrary Morse Hamiltonian one can rescaled to such form, yet the number $\min_{i \neq j} |t_i - t_j|$ will enter then into the expression for the bound.

By analogy with the previous result, denote by K_∞ a simply connected open set obtained by slitting $\mathbb{C} \setminus \Sigma$ along rays connecting critical values with infinity.

THEOREM 4. *Let H be a balanced Hamiltonian of degree $n+1$, whose critical values $t_1, ..., t_\mu$ satisfy for some positive $R<\infty$ the condition*

$$|t_i - t_j| \geqslant 1/R, \qquad |t_i| \leqslant R \qquad \forall i \neq j.$$

Then the number of zeros inside K_∞ of any Abelian integral of a form of degree d does not exceed $(2+R)^{N'}$, where $N' = N'(n, d)$ is a certain elementary function depending only on n and d.

Sketch of the proof. Multiplying the Hamiltonian by R and applying the bounded meandering principle to the Picard–Fuchs system (3.7), we construct a scalar linear equation of a very large order, satisfied by all Abelian integrals, so that its coefficients are explicitly bounded on distance $\geqslant 1$ from the critical locus by an expression polynomial in R as above.

To count zeros of Abelian integral inside the set $K_{1/2}$, one can use Theorem 3. The remaining part $K_\infty \backslash K_{1/2}$ consists of *disjoint* disks of radius 1/2 centered at the critical values t_i and slit along radii. Theorem 4.1 from [26] applies to ever such disk and gives an upper bound for the number of zeros in these disks, thus completing the proof. ∎

Note the difference between two apparently similar results: Theorem 3 gives a *uniform* upper bound for the number of zeros in a *certain domain* (depending on the Hamiltonian, but always nonvoid for Morse Hamiltonians regular at infinity).

On the contrary, Theorem 4 formally solves the tangential Hilbert problem for *all* Morse Hamiltonians (giving an upper bound for the number of *all* zeros, wherever they occur), but the bound is *not uniform* and explodes when the Hamiltonian approaches the boundary of the set of Morse polynomials regular at infinity.

4.2. *Discussion.* The group of affine transformations of variables x, y acts naturally on the space Hamiltonians and polynomial 1-forms, hence to be geometrically sound, upper bounds for the number of zeros of Abelian integrals should be invariant by this action. In particular, the above mentioned "positive distance to the critical locus" (resp., "distance between the critical values") occurring in the formulation of Theorems 3 and 4 should be invariant by affine rescaling of Hamiltonians. Besides intrinsic considerations, the need for the bounds invariant by this action is motivated by the future study of zeros of Abelian integrals near singularities (cf. with [15]).

From the analytic point of view, the problem is in the choice of normalization on the variety of Hamiltonians regular at infinity. The geometric invariance requires this normalization to be imposed in terms of geometry of *configurations of the critical values* of the Hamiltonians. On the other hand, the assertion of Theorems 3 and 4 derived from the explicit form of the system (3.7), uses the pre-normalization in terms of the *coefficients* of the Hamiltonian, more precisely, the ℓ^1-norms of its nonhomogeneity (the difference between H and its principal homogeneous part).

Thus in a natural way the problem on equivalence of the two normalizing conditions arises. The rest of this section contains an accurate formulation of this problem.

4.3. *Affine group action and equivariant problem on zeros of Abelian integrals.* Consider the affine complex space of Hamiltonians $\mathscr{H} = \Lambda^0_{n+1}$ and the space of 1-forms $\mathscr{F} = \Lambda^1_d$ of a given degree d. The Abelian integrals are multivalued functions on $((\mathbb{C} \times \mathscr{H}) \setminus \mathbf{\Sigma}) \times \mathscr{F}$, where $\mathbf{\Sigma}$ is the *global discriminant*,

$$\mathbf{\Sigma} \subset \mathbb{C} \times \mathscr{H}, \qquad \mathbf{\Sigma} = \{(t, H) \colon t \text{ is a critical value of } H\}.$$

The group G_2 of affine transformations of $\mathbb{C}^2$ and the group G_1 of affine transformations of $\mathbb{C}^1$ act naturally on $\mathbb{C} \times \mathscr{H}$,

$$(H, t) \xmapsto{g_2,\, g_1} (g_1 \circ H \circ g_2,\, g_1 t),$$

leaving $\mathbf{\Sigma}$ invariant. The problem of counting zeros of Abelian integrals should be also formulated for subsets in $(\mathbb{C} \times \mathscr{H}) \setminus \mathbf{\Sigma}$ that are invariant by this action.

To achieve this equivariant formulation, we follow the ideology of normal forms and choose a convenient representative from each orbit of the group action. To factorize by the action of G_1, we notice that any point set $t_1, ..., t_\mu$ not reducible to one point, can be put by a suitable affine transformation of $\mathbb{C}^1$ (or, what is equivalent, by the choice of a chart t) to a configuration satisfying two conditions,

$$t_1 + \cdots + t_\mu = 0, \qquad \max_{t = 1, ..., \mu} |t_j| = 1, \tag{4.3}$$

and such transformation is determined uniquely modulo rotation of $\mathbb{C}$, preserving the Euclidean metric on $\mathbb{C} \simeq \mathbb{R}^2$. Any set $\mathscr{K}$ in $(\mathbb{C} \times \mathscr{H}) \setminus \mathbf{\Sigma}$ invariant by the G_1-action, leaves its trace on the t-plane as a subset K disjoint from the points $\Sigma = \{t_j\}_1^\mu$ and the distance from K to Σ measured in this privileged chart, is the natural equivariant distance between $\mathscr{K}$ and $\mathbf{\Sigma}$.

We arrive thus to the following equivariant formulation of the problem on zeros of Abelian integrals, restricted in the sense that it concerns only zeros distant from singularities (this terminology was recently suggested by Yu. Ilyashenko).

Problem 2 (Equivariant restricted tangential Hilbert 16th problem). Let H be a Hamiltonian of degree $n+1$ regular at infinity, whose critical values $t_1, ..., t_\mu$, $\mu = n^2$, satisfy the normalizing conditions (4.3).

For any finite $R > 0$ it is required to place an upper bound for the number of isolated zeros of Abelian integrals $\oint_{H=t} \omega$ of any form of degree $\leqslant d$ in the sets K_R as in (4.2). The bound should depend only on n, d and R.

4.4. *From Theorem* 3 *to equivariant problem.* In order to derive from Theorem 3 a solution to the equivariant problem, one should try to find in the orbit of the G_2-action on $\mathscr{H}$ a Hamiltonian as close to be balanced as possible.

Indeed, if for some affine transformation $g \in G_2$ the Hamiltonian $\tilde{H} = H \circ g$ is already balanced, then integrals of any form ω over any level curve $H = t$ are equal to integrals of the form $g^*\omega$ over the curve $\tilde{H} = t$ (by the simple change of variables in the integral). But as $g: \mathbb{C}^2 \to \mathbb{C}^2$ is an affine map, the form $g^*\omega$ is again a polynomial 1-form of the same degree as ω, while the new Hamiltonian $\tilde{H}$ is balanced. Hence Theorem 3 can be applied to produce the upper bound for the number of zeros exactly in the form we need to solve the equivariant problem: the result will be automatically a bound which is polynomial in R with the exponent depending only on d and n.

In fact, it is sufficient to find in the G_2-orbit of H a Hamiltonian $\tilde{H}$ that would be quasimonic and whose difference from its principal homogeneous part $\hat{H}$ would be of norm explicitly bounded in terms of n. Indeed, if $\tilde{H}$ is such a polynomial and $\|\tilde{H} - \hat{H}\| \leqslant \tau = \tau(n)$, then the transformation

$$\tilde{H}(x, y) \rightsquigarrow H^*(x, y) = \tau^{-(n+1)}\, \tilde{H}(\tau x, \tau y) \tag{4.4}$$

will preserve the principal homogeneous part $\hat{H}$ while dividing all other terms by appropriate positive powers of τ so that in any case $\|H^* - \hat{H}\| \leqslant 1$. This means that H^* is balanced and Theorem 3 can be applied and will give a bound on zeros $1/R$-distant from the critical locus of H^* in terms of R, n, d as required. The transformation (4.4) does not preserve the normalizing conditions (4.3), but the *conclusion* of Theorem 3 can be rescaled to produce an upper bound on zeros $1/\tau^{n+1}R$-distant from the (normalized) critical locus of H, by a suitable power of $R\tau^{n+1}$, which will give a solution to the equivariant problem.

Recall that the balance condition consists of the two parts: the (quasimonic) normalization of the principal homogeneous terms and the unit bound for the norm of all non-principal terms. The first part can be easily achieved by a suitable G_2-action. Indeed, replacing $H(x, y)$ by $H(\tau x, \tau y)$, one can effectively multiply the principal homogeneous part $\hat{H}$ by τ^{n+1} and thus achieve the required normalization.

It will be convenient in the future not to change the principal part any more, once it was made quasimonic. This means that the only remaining degree of freedom to use is the group of translations of $\mathbb{C}^2$ (and rotations that do not affect norms).

Summarizing this discussion, we see that in order to derive from Theorem 3 the equivariant restricted tangential Hilbert 16th problem (Problem 2), it would be sufficient to solve the following problem.

Definition 6. For a quasimonic polynomial H with the principal part $\hat{H}$ we call its *effective nonhomogeneity* the lower bound

$$\varkappa(H) = \inf_{T \in G_2} \|H \circ T - \hat{H}\|, \qquad T \text{ a translation of } \mathbb{C}^2. \tag{4.5}$$

Problem 3. Given a quasimonic Hamiltonian H of degree $n+1$, whose critical values satisfy the normalizing conditions (4.3), place an upper bound for the effective nonhomogeneity $\varkappa(H)$.

This and related problem, completely independent from all previous considerations, is discussed and partially solved in the next section.

5. CRITICAL VALUES OF POLYNOMIALS

It can be shown relatively easily that a quasimonic polynomial whose non-principal part is bounded from above (in the sense of the norm), has all critical values inside a disk of known radius shrinking to a point as the non-principal part tends to zero (Proposition 6 below).

One might hope that a converse statement is also true: if all critical values of a Hamiltonian H come very close to each other, then (eventually after appropriate translations in the preimage and the image) H differs from its principal homogeneous part $\hat{H}$ by a small polynomial.

This fact indeed holds true for univariate (and hence hyperelliptic) polynomials, where we were able to produce explicit inequalities between the diameter of the critical locus $\operatorname{diam} \Sigma = \max_{i, j=1, \dots, \mu} |t_i - t_j|$ and the nonhomogeneity $\|H - \hat{H}\|$, see Theorem 6 and Corollary 7.

Yet for the truly bivariate polynomials the problem turned out to be considerably harder, and the best we were able to do is to show that *for any fixed principal part $\hat{H}$ the above two normalizations are equivalent*, but as different linear factors of $\hat{H}$ approach each other, the equivalence explodes.

5.1. *Geometric consequences of quasimonicity.* The normalizing condition at infinity (for 1-forms and Hamiltonians) was introduced in purely algebraic terms as an inequality imposed on the principal homogeneous part of a 1-form (resp., Hamiltonian). However, one can provide a simple geometric meaning to this condition.

Recall that if H is regular at infinity, then its principal homogeneous part $\hat{H}$ has an isolated critical point at the origin. This means that the gradient $\nabla\hat{H}$ never vanishes outside the origin, and in particular its minimal (Hermitian) length on the boundary of the unit bidisk $\mathbb{B}=\{|x|\leqslant 1,|y|\leqslant 1\}\subset\mathbb{C}^2$ is strictly positive. Because of the homogeneity, this is sufficient to place a lower bound on the length of $\nabla\hat{H}$ everywhere on $\mathbb{C}^2\backslash\{0\}$.

PROPOSITION 5. *If $\hat{H}$ is normalized (quasimonic), then everywhere on the boundary of the unit bidisk $\mathbb{B}$ the Hermitian length of $\nabla\hat{H}$ is no smaller than* 1.

Proof. Consider the part $\partial\mathbb{B}_1$ of the boundary $\partial\mathbb{B}$ which is given by the inequalities $|x|=1,\ |y|\leqslant 1$ (the other part is treated similarly). The homogeneous polynomial x^{2n-1} can be represented as $a\hat{H}_x+b\hat{H}_y$ with $\|a\|+\|b\|\leqslant 1$. Restricting this on $\partial\mathbb{B}_1$ we see that the Hermitian product of the gradient $\nabla\hat{H}=(\hat{H}_x,\hat{H}_y)$ and the vector field V with coordinates $(\bar{a},\bar{b})$ is everywhere equal to 1 in the absolute value. The Hermitian length of V at any point of $\partial\mathbb{B}_1$ can be easily majorized by $\sqrt{|\bar{a}(x,y)|^2+|\bar{b}(x,y)|^2}$ which is no greater than $\sqrt{\|a\|^2+\|b\|^2}\leqslant 1$ on $\partial\mathbb{B}_1=\{|x|=1,|y|\leqslant 1\}$. But then by the Cauchy inequality, the length of $\nabla\hat{H}$ cannot be smaller than 1 on $\partial\mathbb{B}_1$. ∎

5.2. *Almost-homogeneity implies close critical values.* We begin by showing that a quasimonic Hamiltonian whose non-principal part is bounded, admits an upper bound for the moduli of critical values. This solves the problem inverse to Problem 3.

PROPOSITION 6. *If H is a quasimonic Hamiltonian of degree $n+1$ with the principal part $\hat{H}$ and $\|H-\hat{H}\|\leqslant 1/n\sqrt{2}$, then the critical values of H are all in the disk $\{|t|\leqslant 3/n\}$.*

Proof. Denote $H=\hat{H}+h$. The gradient of each monomial of degree $\leqslant n$ has the Hermitian length bounded by $n\sqrt{2}$ on the unit bidisk $\mathbb{B}$. Thus if $\|h\|<1/n\sqrt{2}$, then ∇h has its length strictly bounded by 1 everywhere in $\mathbb{B}$.

By Proposition 5, the length of $\nabla\hat{H}$ is at least 1 everywhere on the boundary of $\mathbb{B}$, so by the topological index theorem, all $\mu=n^2$ critical points of H must be be inside $\mathbb{B}$.

Note that a quasimonic principal part $\hat{H}$ admits no apriori upper bound on $\mathbb{B}$ hence $\sup_{\mathbb{B}} |\hat{H}|$ can be arbitrary large. However, the *critical values* of $H = \hat{H} + h$ can be explicitly majorized by the absolute value. Indeed, at any critical point (x_*, y_*), the gradient of H vanishes so $\nabla\hat{H}(x_*, y_*) = -\nabla h(x_*, y_*)$. By the Euler identity, $(n+1)\,|\hat{H}(x_*, y_*)| = |(x_*, y_*)\cdot\nabla\hat{H}(x_*, y_*)| = |(x_*, y_*)\cdot\nabla h(x_*, y_*)| \leqslant \sqrt{2}$, since the Hermitian length of $\nabla h(x, y)$ is explicitly bounded by 1 in $\mathbb{B}$. Finally, since $|h(x, y)| \leqslant \|h\| \leqslant 1/n\sqrt{2}$ in $\mathbb{B}$, we conclude that $|H(x_*, y_*)| \leqslant |\hat{H}(x_*, y_*)| + |h(x_*, y_*)| \leqslant \sqrt{2}/(n+1) + 1/n\sqrt{2} \leqslant 3/n\sqrt{2} \leqslant 3/n$. ∎

Thus when discussing the equivariant restricted Hilbert problem, only the other direction (Problem 3) is interesting.

5.3. *Dual formulation, limit and existential problems.* Problem 3 that can be considered as a constrained minimization problem in the space of polynomials in two variables, admits reformulation in dual terms as follows.

Problem 4 (dual to Problem 3). Given a quasimonic Hamiltonian H of effective nonhomogeneity $\varkappa(H) = 1$, place a *lower* bound on the diameter of its critical values

$$\operatorname{diam}\Sigma = \max_{1 \leqslant i \neq j \leqslant \mu} |t_i - t_j|.$$

Having solved this problem, one can easily derive from it by the rescaling arguments as above a solution to Problem 3 and vice versa.

The dual formulation of Problem 4 allows a limit version: one is required to show that if H cannot be reduced to a homogeneous polynomial by a translation, i.e., $\varkappa(H) > 0$, then $\operatorname{diam}\Sigma > 0$, i.e., not all critical values coincide.

This limit problem can be settled.

THEOREM 5. *If a polynomial $H(x, y)$ regular at infinity has only one critical value (necessarily of multiplicity $\mu = n^2$), then by a suitable translations in the preimage and the image H can be made homogeneous: $H(x, y) = \hat{H}(x+\alpha, y+\beta) + \gamma$, where $\hat{H}$ is the principal homogeneous part of H.*

We postpone the proof of Theorem 5 until Section 5.6, deriving first as a corollary an existential solution of either of the two equivalent Problems 3 and 4.

COROLLARY 4. *For a quasimonic Hamiltonian $H = \hat{H} + h$ of degree $n+1$ there exist two positive finite constants, $\alpha = \alpha(\hat{H})$ and $\beta = \beta(\hat{H})$, depending* *only on the principal part* $\hat{H}$, *such that the critical locus* $\Sigma = \Sigma(H)$ *and the effective non-homogeneity* $\varkappa(H)$ *are related as follows*:

$$\begin{aligned} \varkappa(H) \geqslant 1 &\Rightarrow \Sigma \cap \{|t| > \alpha\} \neq \varnothing, \\ \Sigma \subseteq \{|t| \leqslant 1\} &\Rightarrow \varkappa(H) \leqslant \beta. \end{aligned} \tag{5.1}$$

Proof of the Corollary. Consider the affine space $\mathscr{H}_n \simeq \mathbb{C}^{(n+1)(n+2)/2}$ of polynomials of degree $\leqslant n$ in two variables, and define two nonnegative functions on it,

$$f(h) = \kappa(\hat{H} + h) = \inf_{T \in \mathbb{C}^2} \|T^*(\hat{H} + h)\|, \qquad g(h) = \sum_{t \in \Sigma(\hat{H} + h)} |t|^2,$$

where T ranges over all translations of the plane T^2 and $\Sigma(H)$ is the collection of all critical values of the Hamiltonian $H = \hat{H} + h$ with the fixed principal part $\hat{H}$.

Both functions, as one can easily see, are semilagebraic on $\mathbb{C}^2 \simeq \mathbb{R}^4$. From Theorem 5 it follows that $f(h)$ must vanish if $g(h) = 0$, i.e., that the zero locus of g is contained in that of f.

By the Łojasiewicz inequality, there exist two positive finite constants $C, \rho > 0$, such that

$$f(h) \leqslant C g^{\rho}(h), \qquad \forall h \in \mathscr{H}_n.$$

From this inequality the assertion of the Corollary easily follows if we let $\beta = C n^{\rho}$ and $\alpha = (nC)^{-1/2\rho}$. Since C, ρ depend only on the construction of f, g, that is, on $\hat{H}$, the Corollary is proved. ∎

Unfortunately, the proof gives no means to compute explicitly the bounds α and β. Moreover, below we will show that they *cannot be chosen uniformly* over all quasimonic principal parts.

5.4. *Parallel problems for univariate polynomials.* One can easily formulate analogs of all the above problems for univariate polynomials, in which case monic rather than quasimonic polynomials are to be considered. Note that the critical values of the hyperelliptic Hamiltonian $H(x, y) = y^2 + p(x)$ coincide with that of the univariate potential $p \in \mathbb{C}[x]$, and also the effective non-homogeneity (more accurately, non-quasihomogeneity) of H coincides with that $\varkappa(p)$. Thus all results proved below, are valid not only for univariate polynomials, but also for hyperelliptic bivariate Hamiltonians.

The limit problem for this case is fairly elementary. It was solved by A. Chademan [5] as a step towards the existential solution of Problem 3 for univariate polynomials (Corollary 6 below).

PROPOSITION 7 (Chademan [5]). *A complex polynomial that has only one critical value at* $t=0$, *is a translated monomial* $\alpha(x-a)^{n+1}$.

Proof. Assuming without loss of generality that the polynomial $p(x)$ is monic, we can always write the derivative

$$p'(x)=(n+1)(x-a_1)^{\nu_1}\cdots(x-a_k)^{\nu_k},$$

where $a_1, ..., a_k$ are geometrically distinct critical points and all $\nu_k>0$. For any $j=1, ..., k$ the polynomial p can be expressed as the primitive of p' integrated from a_j,

$$p(x)=p(a_j)+\int_{a_j}^{x} p'(s)\,ds=0+(x-a_j)^{\nu_j+1}\,q_j(x), \qquad q_j\in\mathbb{C}[x],$$

in other words, p is divisible by $(x-a_j)^{\nu_j+1}$. As this holds for all points a_j, $j=1, ..., k$, hence $\deg p\geqslant \deg p'+k$ and therefore only one ν_j can be different from zero. ∎

In the standard way (see the proof of Corollary 5 above) the following existential majorant can be derived.

COROLLARY 5 (Chademan [5]). *If* $p(x)=x^{n+1}+p_{n-1}x^{n-1}+\cdots+p_1x+p_0$ *is a monic polynomial of degree* $n+1$ *without the term* x^n, *and all complex critical values of* p *lie in the unit disk* $\{|t|\leqslant 1\}$, *then*

$$|p_{n-1}|+\cdots+|p_1|+|p_0|\leqslant C_n,$$

where C_n *is a constant depending only on* n. ∎

However, in the same way as before, the proof based on solution of the limit problem gives no possibility of effectively computing the constant C_n. We compute it using a completely different approach.

5.5. *Spread of roots vs spread of critical values for univariate monic complex polynomials.*

THEOREM 6. *If all critical values* $\{t_1, ..., t_n\}$ *of a monic univariate polynomial* $p(x)=\prod_{j=0}^{n}(x-x_j)$ *are in the unit disk, then the diameter of the set of its roots is no greater than* $4e$:

$$\Sigma\subset\{|t|\leqslant 1\}\Rightarrow \forall j,k=0, ..., n\ |x_j-x_k|\leqslant 4e.$$

Proof. Consider the real-valued function $f: \mathbb{C} \to \mathbb{R}$, $f(x) = |p(x)|$. It is smooth outside the roots of the polynomial p. Moreover, its critical values (different from zero) coincide with $|t_j|$, as the critical points for f and p are the same.

By the main principle of the Morse theory, all sublevel sets $M_s = \{x \in \mathbb{C} : f(x) \leqslant s\}$ for $0 < s < \infty$ of the function f remain homeomorphic to each other until s does not pass through a critical value of f. One can easily verify that the set M_s is simply connected for all large s (it differs only slightly from the disk $\{|t| \leqslant s^{1/n+1}\}$). Our assumption on the critical values guarantees that the set $M_1 = \{|p(x)| \leqslant 1\} \subset \mathbb{C}$ corresponding to $s = 1$ is therefore also connected (though its shape can be very non-circular anymore).

On the other hand, by the famous Cartan lemma [20] for any positive ε one can delete from $\mathbb{C}$ one or several disks with the sum of diameters less than ε so that on the complement the monic polynomial of degree $n+1$ satisfies a lower bound $|p(x)| \geqslant (\varepsilon/4e)^{n+1}$. This lemma implies that the set M_1 can be covered by one or several circular disks with the sum of diameters $\leqslant 4e$.

But the set M_1 (like all sets M_s with positive s) contains all roots of p, so if there are two roots x_i, x_j on the distance more than $4e$, then the union of disks covering these two roots simultaneously, cannot be connected (it is sufficient to project all the disks on the line connecting these roots and reduce the assertion to one dimension). This contradiction proves the theorem. ∎

COROLLARY 6. *By a suitable translation $p(x) \mapsto p(x+a)$ a monic polynomial $p(x) = x^{n+1} + \cdots$ whose critical values are normalized by the conditions* (4.3), *can be reduced to the form $p(x) = x^{n+1} + \sum_{j=0}^{n} p_j x^j$ with $\sum_j |p_j| = \|x^{n+1} - p\| \leqslant 8^{n+1}$.*

Proof. By Theorem 6, the roots of p form a point set of diameter $d \leqslant 4e$ in the x-plane. Any such set can be covered by a regular hexagon with the opposite sides being at the distance d [3, 10]. Shifting the origin at the center of this hexagon makes all roots x_j satisfying the inequality $|x_j| \leqslant d/\sqrt{3}$.

A monic polynomial of degree $n+1$ with all roots inside the disk of radius $r > 0$ has all its coefficients bounded by the respective coefficients of the polynomial $(x+r)^{n+1}$, by the Vieta formulas. For the latter polynomial the sum of (absolute values of) all coefficients is the value at $x = 1$ (since all these coefficients are nonnegative). Putting everything together, we conclude that after shifting the origin at the center of the hexagon, $\|p(x)\| \leqslant (1 + 4e/\sqrt{3})^{n+1} \leqslant 8^{n+1}$. ∎

Remark 7. Simply shifting the origin to one of the roots makes all of them being in the circle of radius $4e$, which finally yields an upper bound $\|p\| \leqslant (1+4e)^{n+1} \leqslant 12^{n+1}$ without referring to the claim on hexagonal cover.

The assertion of Theorem 6 for *real* polynomials having bounded *real* critical values, can be proved in a completely different way. The following proposition gives an insight as to how accurate the bound established in Theorem 6 is.

PROPOSITION 8. *A monic real polynomial of degree* $n+1$ *whose real critical values are all in the interval* $[-1, 1]$, *has all its real roots in some interval of the length* 4.

Proof. Between any two roots the polynomial satisfies the condition $-1 \leqslant p(x) \leqslant 1$, since extrema of p between these points are achieved at real critical points and hence both are real critical values of p.

Among monic polynomials of degree $n+1$ on the unit interval $-1 \leqslant x \leqslant 1$ the smallest uniform upper bound $c_n = 2^{-(n+1)}$ is achieved for the Chebyshev polynomial $T_n(x) = 2^{-(n+1)} \cos(n+1) \arccos x$: for any other monic polynomial of this degree, the C^0-norm $\max_{-1 \leqslant x \leqslant 1} |p(x)|$ will be greater or equal to c_n. Applying this assertion to the polynomial $2^{n+1} p(x/2)$ we conclude that the largest real interval on which the monic polynomial can satisfy the condition $|p| \leqslant 1$, is of length 4 (twice the length of $[-1, 1]$). ∎

Thus Theorem 6 can be considered as generalizing (in some sense) the extremal property of the Chebyshev polynomials to the complex domain.

5.6. *Demonstration of Theorem* 5. The proof of Theorem 5 is an immediate corollary to the two following lemmas.

LEMMA 2. *A polynomial regular at infinity and having only one complex critical value, has a unique critical point.*

This lemma is in fact valid for polynomials of any number of variables. The second claim is dimension-specific.

LEMMA 3. *A bivariate polynomial regular at infinity and having a unique complex critical point at the origin, is homogeneous.*

Proof of Lemma 2. Let H_ε be an analytic one-parameter perturbation of the polynomial $H_0 = H$, such that for all $\varepsilon \neq 0$ the polynomial H_ε is Morse.

Consider the monodromy group of the bundle $H_\varepsilon : \mathbb{C}^2 \to \mathbb{C}^1$ for an arbitrary small ε. It is known [1] that vanishing cycles form the basis of the homology of all fibers, each being a cyclic vector (i.e., all continuations of any vanishing cycle span the entire first homology of the typical fiber $\{H_\varepsilon = t\}$.

Suppose that there are at least two critical points $a_1 \neq a_2 \in \mathbb{C}^2$ for H_0. Then for all sufficiently small ε the polynomial H_ε will have two disjoint groups of critical points with close critical values. Moreover, these groups of critical points are well apart (say, the distance between them is never smaller than half the distance between a_1 and a_2).

But then the vanishing cycles "growing" from critical points not belonging to the same group, are also disjoint, therefore their intersection index must be zero.

But then the Picard–Lefschetz formulas imply that the subspaces generated by each group of vanishing cycles, must be both invariant, which contradicts the fact that each group must consist of cyclic elements for the monodromy. ∎

Proof of Lemma 3. Consider the one-parameter analytic (polynomial) homotopy between H and its principal part, $H_\varepsilon(x, y) = \varepsilon^{n+1} H(\varepsilon^{-1} x, \varepsilon^{-1} y)$. Then for $\varepsilon = 0$ H_0 coincides with the principal homogeneous part, while $H_1 = H$.

The germ of H_ε at the origin $x = y = 0$ has a multiplicity μ_ε (the Milnor number) that is equal to n^2 for any ε. Indeed, by the Bézout theorem, the total number of critical points of H counted with multiplicities in the projective plane $\mathbb{C}P^2$, is n^2; the condition of nondegeneracy at infinity implies that all of them are in the finite (affine) part $\mathbb{C}^2$. The uniqueness assumption means that all these n^2 points coincide at the origin.

By the famous theorem due to D. T. Lê and C. P. Ramanujam [19], the topological type of an analytic germ is constant along the stratum $\mu = \text{const}$, therefore the germs of H_0 and H_1 at the origin are topologically equivalent, in particular, the germs of analytic curves $\{H_0 = 0\}$ and $\{H_1 = 0\}$ in $(\mathbb{C}^2, 0)$ are homeomorphic.

But by the Zariski theorem [27], the order of a planar analytic curve (i.e., the order of the lowest order terms which occur in the Taylor expansion of the local equation defining this curve) is a topological invariant. For the curve $H_0 = 0$ this order is $n+1$, as the polynomial H_0 is homogeneous. But this means that the lowest order of terms that may occur in H_1, is also $n+1$, that is, $H_0 = H_1$ and H coincides in fact with its principal homogeneous part. ∎

Remark 8. Consider the gradient vector field ∇H. Its principal homogeneous part, $\nabla \hat{H}$, is a homogeneous vector field on the plane that has an isolated singularity of multiplicity n^2 at the origin.

Assertion of Lemma 3 means that adding any nontrivial lower order terms to H would necessarily create singular points of the gradient vector field outside the origin, thus changing the multiplicity of what remains at the origin.

However, this assertion about *arbitrary* (not necessarily gradient) polynomial vector fields is *false*, as the following example shows.

EXAMPLE 1 (Lucy Moser–Jauslin). The nonhomogeneous vector field

$$(x^3 - y^3 + x)\frac{\partial}{\partial x} + (2x^3 - y^3 + x)\frac{\partial}{\partial y}$$

has a unique singular point of the maximal multiplicity 9 at the origin, and the principal homogeneous part has an isolated singularity.

5.7. *Existential bounds cannot be uniform.* As was already noted, the proof of Corollary 5 gives no indication on how to compute the bounds $\alpha(\hat{H})$ and $\beta(\hat{H})$ for a given homogeneous part $\hat{H}$. However, the folowing example shows that there cannot be the bound uniform over all principal parts: as some of the linear factors approach each other, the values of β and α^{-1} may grow to infinity.

EXAMPLE 2. The form $\hat{H}_a(x, y) = ax^{n+1}/(n+1) + y^{n+1}/(n+1)$ is normalized for $a \geqslant 1$, as one can easily see by comparing the operator of division by $d\hat{H} = \langle ax^n, y^n \rangle$ on $2n$-forms with that by the ideal $\langle x^n, y^n \rangle$.

The polynomial $H_a(x, y) = \hat{H}_a(x, y) - x$ has critical points at $y = 0$, $x = 1/\sqrt[n]{a}$ (of multiplicity n for every choice of branch of the root). The corresponding critical values all converge to zero asymptotically as $a^{-1/n}$ as $a \to \infty$.

On the other hand, the effective nonhomogeneity of the univariate polynomial $p_a(x) = a(x^{n+1}/(n+1)) - x$ (and hence the value $\varkappa(H_a)$) remain bounded away from zero as $a \to \infty$. Indeed, if after shifting the polynomial p_a by $r = r(a) \in \mathbb{C}$ the coefficient before x^n goes to zero, then necessarily $ar(a) \to 0$. On the other hand, the coefficient before the linear term is equal to $1 + ar(a)^n$ and hence is bounded away from zero.

Thus the bounds established in Corollary 5, cannot be made uniform over all homogeneous parts. Of course, the reason is that the space of quasimonic principal parts is not compact (e.g., the polynomials $\hat{H}_a$ have no limit points as $a \to \infty$). In turn, this is related to the fact that some of the linear factors entering $\hat{H}_a$, tend to each other (as points on the projective line $\mathbb{C}P^1$).

5.8. *Discussion: Atypical values and singular perturbations.* The phenomenon occurring in the above example, might be characteristic. When the Hamiltonian is not regular at infinity, the Abelian integrals may have ramification points that are not critical values of H. Such points, called *atypical values*, must necessarily be singular for any system of Picard–Fuchs equations, and are studied mostly by topological means.

On the other hand, the fact that entries of the matrices A, B may grow to infinity as the principal part of $\hat{H}$ degenerates, means that the system (3.7) (written in the privileged chart to make the assertion equivariant) undergoes a *singular perturbation* (appearance of a large parameter in the right hand side that is equivalent to putting a small parameter before some of the higher order derivatives).

Thus we see that "atypical singularities" in the Picard–Fuchs system can appear as a result of singular perturbation. The analytic approach based on studying division by dH and arguments involving geometry of critical values, may be a complementary tool for the study of singularities "coming from infinity".

ACKNOWLEDGMENTS

We are grateful to J. M. Aroca, F. Cano, J.-P. Françoise, L. Gavrilov, Yu. Ilyashenko, A. Khovanskii, P. Milman, R. Moussu, R. Roussarie and Y. Yomdin for numerous discussions and many useful remarks. Bernard Teissier suggested an idea that finally developed into the proof of Lemma 3. Lucy Moser provided us with a counterexample (Section 5.6).

We are grateful to all our colleagues from Laboratoire de Topologie, Université de Bourgogne (Dijon) and Departamento de Algebra, Geometría y Topología, Universidad de Valladolid, where a large part of this work was done. They made our stays and visits very stimulating.

REFERENCES

1. V. I. Arnold, S. M. Guseĭn-Zade, and A. N. Varchenko, "Singularities of Differentiable Maps. Vol. II, Monodromy and Asymptotics of Integrals," Birkhäuser, Boston, MA, 1988.
2. W. Balser, "Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations," Springer-Verlag, New York, 2000.
3. T. Bonnesen and W. Fenchel, "Theorie der Konvexen Körper," Chelsea, Bronx, NY, 1971.
4. E. Brieskorn, Die Monodromie der Isolierten Singularitäten von Hyperflächen, *Manuscripta Math.* **2** (1970), 103–161.
5. A. Chademan, Bounded algebraic critical values, *J. Sci. Univ. Tehran Int. Ed.* **1** (1996), 43–50.
6. L. Gavrilov, Petrov modules and zeros of Abelian integrals, *Bull. Sci. Math.* **122** (1998), 571–584.
7. L. Garvrilov, Abelian integrals related to Morse polynomials and perturbations of plane Hamiltonian vector fields, *Ann. Inst. Fourier, Grenoble* **49** (1999), 611–652.
8. F. Girard and M. Jebrane, Majorations affines du nombre de zéros d'intégrales abéliennes pour les hamiltoniens quartiques elliptiques, *Ann. Fac. Sci. Toulouse Math. (6)* **7** (1998), 671–685.
9. A. B. Givental, Sturm's theorem for hyperelliptic integrals, *Algebra Anal.* **1** (1989), 95–102; transl. *Leningrad Math. J.* **1** (1989), 1157–1163.

10. B. Grünbaum, Borsuk's problem and related questions, *in* "Proc. Sympos. Pure Math.," Vol. VII, pp. 271–284, Amer. Math. Soc., Providence, RI, 1963.
11. E. Horozov and I. D. Iliev, Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians, *Nonlinearity* **11** (1998), 1521–1537.
12. Yu. Il'yashenko and S. Yakovenko, Counting real zeros of analytic functions satisfying linear ordinary differential equations, *J. Differential Equations* **126** (1996), 87–105.
13. A. Khovanskii, Real analytic manifolds with the property of finiteness, and complex abelian integrals, *Funkt. Anal. Prilozhen.* **18** (1984), 40–50. [In Russian]
14. A. Khovanskii, "Fewnomials," Translations of Mathematical Monographs, Vol. 88, American Mathematical Society, Providence, RI, 1991.
15. D. Novikov and S. Yakovenko, Tangential Hilbert problem for perturbations of hyperelliptic Hamiltonian systems, *Electron. Res. Announc. Amer. Math. Soc.* **5** (1999), 55–65 .
16. D. Novikov and S. Yakovenko, Trajectories of polynomial vector fields and ascending chains of polynomial ideals, *Ann. Inst. Fourier* **49** (1999), 563–609.
17. D. Novikov and S. Yakovenko, Simple exponential estimate for the number of real zeros of complete Abelian integrals, *Ann. Inst. Fourier (Grenoble)* **45** (1995), 897–927.
18. D. Novikov and S. Yakovenko, Meandering of trajectories of polynomial vector fields in the affine n-space *in* "Proceedings of the Symposium on Planar Vector Fields (Lleida, 1996)," *Publ. Mat.* **41** (1997), 223–242.
19. Lê Dung Tráng and C. P. Ramanujam, The invariance of Milnor's number implies the invariance of the topological type, *Amer. J. Math.* **98** (1976), 67–78.
20. B. Ya. Levin, "Distribution of Zeros of Entire Functions," Transl. of Math. Monographs, Vol. 5, Amer. Math. Soc., Providence, RI, 1964.
21. F. Pham, Singularités des systèmes différentiels de Gauss–Manin, *in* "Progress in Mathematics," Vol. 2, Birkhäuser, Boston, 1979.
22. G. Petrov, Complex zeros of an elliptic integral, *Funkt. Anal. Prilozhen.* **21** (1987), 87–88.
23. M. Roitman and S. Yakovenko, On the number of zeros of analytic functions in a neighborhood of a Fuchsian singular point with real spectrum, *Math. Res. Lett.* **3** (1996), 359–371.
24. M. Sebastiani, Preuve d'un conjecture de Brieskorn, *Manuscripta Math.* **2** (1970), 301–308.
25. A. Varchenko, Estimation of the number of zeros of an Abelian integral depending on a parameter, and limit cycles, *Anal. Prilozhen* **18** (1984), 14–25. [In Russian]
26. S. Yakovenko, On functions and curves defined by ordinary differential eqiations *in* "The Arnol'dfest (Proceedings of the Fields Institute Conference in Honour of the 60th Birthday of Vladimir I. Arnol'd)" (E. Bierstone, B. Khesin, A. Khovanskii and J. Marsden, Eds.), Amer. Math Soc., Providence, RI, 1999.
27. O. Zariski, Studies in equisingularity, III. Saturation of local rings and equisingularity, *Amer. J. Math.* **90** (1968), 961–1023.
28. Zhao Yulin and Zhang Zhifen, Linear estimate of the number of zeros of abelian integrals for a kind of quartic Hamiltonians, *J. Differential Equations* **155** (1999), 73–88.