THE STURM THEOREMS AND SYMPLECTIC GEOMETRY

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UDC 517.9

The theorems of Sturm on oscillation and nonoscillation of solutions to a second-order differential equation have a topological nature: they describe the rotation of a straight line in the phase space of the equation [1, 11].

A straight line is just a Lagrangian subspace of the phase plane. The higher-dimensional generalization of Sturm's theory, the account of which is given below, describes the evolution of a Lagrangian plane in the symplectic phase space of a linear Hamiltonian system, for example, of the system of n Newton equations

$$\ddot{x} = -A(t)x,\ x \in \mathbf{R}^n,\ A' = A \tag{1}$$

with potential energy U = (Ax, x)/2, or of the more general system of Lagrange equations

$$(B(t)\dot{x})^{\cdot} = -A(t)x,\ x \in \mathbf{R}^n,\ A' = A,\ B' = B \tag{2}$$

with positive-definite kinetic energy $T = (B\dot{x}, \dot{x})/2$.

Instead of zeros of solutions, one must consider moments at which the Lagrangian plane is vertical. A Lagrangian plane is said to be vertical if it contains a nonzero vector with x = 0 (i.e., if it is not transverse to a fiber of the cotangent bundle).

The generalizations of Sturm's theorem, formulated below, are distinguished also by a "correction term" n, equal to the number of degrees of freedom (instead of "between two...," as in Sturm's theory, we shall have "between n + 1...," etc.).

In such a classical area as Sturm's theory it is hard to follow all the predecessors, and I can only say, like Bott and Edwards, that I do not make any claim as to the novelty of the results. In connection with this I remark that numerous authors writing on the Maslov index, symplectic geometry, geometric quantization, Lagrangian analysis, etc., starting with [2], have not noticed the earlier works of Lidskii, as well as the earlier works of Bott [3] and Edwards [4], in which was constructed a Hermitian version of the theory of the Maslov index and Sturm intersections.

1. Symplectic Analogs of the Sturm Theorems

We consider first the evolution of Lagrangian planes under the action of system (1) or (2) (this particular case was already considered by Morse [9, 10]).

Nonoscillation Theorem. If the potential energy is nonpositive, then the number of moments of verticality does not exceed the number n of degrees of freedom.

Theorem on Zeros. On a segment containing n + 1 moments of verticality of one Lagrangian plane any other Lagrangian plane becomes vertical at least once. Moreover, the difference between the moments of verticality of two arbitrary Lagrangian planes, evolving under the same system, on any segment of the time axis does not exceed n.

All these results follow, as in the usual Sturm theory, from the fact that the evolution of Lagrangian planes undergoes a particular acceleration when the Hamiltonian is increased.

Consider a linear system of Hamiltonian equations (with variable coefficients)

$$\dot{q} = \partial H/\partial p,\ \dot{p} = -\partial H/\partial q,\ (p, q) \in \mathbf{R}^{2n}. \tag{3}$$

Alternation Theorem. Suppose the Hamiltonian H is positive-definite on the Lagrangian planes α and β. Then the numbers ν_α and ν_β of moments at which a Lagrangian plane, evolving under system (3), is not transverse to the planes α and β, respectively, do not differ, on any segment, by more than the number of degrees of freedom:

M. V. Lomonosov State University, Moscow. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 19, No. 4, pp. 1-10, October-December, 1985. Original article submitted June 11, 1985.

0016-2663/85/1904-0251$09.50

$$|\nu_\alpha - \nu_\beta| \leqslant n.$$

COROLLARY. On any segment containing n + 1 moments of nontransversality to α there is a moment of nontransversality to β.

This theorem applies, in particular, to all Lagrangian planes for Eqs. (1) and (2) with positive potential energy.

Suppose now that the Hamiltonians of the system studied are positive-definite only on one Lagrangian plane α. In systems (1) and (2) this condition is always satisfied for the plane x = 0 (fiber of the tangent bundle), and if the potential energy is positive-definite, then it is satisfied for every Lagrangian plane; here $H = (B^{-1}p, p)/2 + (Aq, q)/2$, $q = x$, $p = B\dot{x}$.

We denote by $\nu(H)$ the number of moments on some segment of the time axis at which a Lagrangian plane, evolving under the action of system (3), is not transverse to α.

Comparison Theorem. If $H' \geqslant H$, then $\nu(H') \geqslant \nu(H) - n$.

COROLLARY 1. On a segment containing n + 1 moments of verticality of some Lagrangian plane, evolving under system (2), every Lagrangian plane, evolving under system (2) with a Hamiltonian (not necessarily strictly) larger than the original one, becomes vertical at least once.

Moreover, the difference between the numbers of moments of verticality for the systems with the larger and the smaller Hamiltonian is not larger than the number of degrees of freedom.

COROLLARY 2 (Oscillation Theorem). If the potential energy in Newton's equation (1) is uniformly positive-definite $[U(x, t) \geqslant \omega^2 x^2/2]$, then the number of moments of verticality of an arbitrary Lagrangian plane on any segment of the time axis of duration t is not smaller than $[\omega t/\pi] + 1 - n$. In particular, this number grows unboundedly as $t \to \infty$.

2. Geometry of the Lagrangian Grassman Manifold

We consider the manifold of (nonoriented) Lagrangian subspaces of the symplectic space $(\mathbf{R}^{2n}, \omega)$, denoted by Λ_n. In [2] it is shown that $\Lambda_n \approx \mathbf{U}(n)/\mathbf{O}(n)$.

Definition. The train of a given point of the manifold Λ_n is the set of all Lagrangian planes which are not transverse to the given one. The given point is called the vertex of the train.

Every train is a codimension one (algebraic) variety in Λ_n whose singularities form a set of codimension 2 in the train (see [2]). Every train is transversely oriented by the velocity field of some (and then of any) one-parameter positive definite Hamiltonian (see [2] or Sec. 5 below).

The transverse orientation of the train permits us to define the intersection index for oriented curves on it (under the assumption that the initial and final points are not lying on the train).

In the neighborhood of the vertex the train is diffeomorphic with the variety of nondegenerate quadratic forms (generating functions) in $\mathbf{R}^n$. Consequently, it divides the neighborhood of the vertex into n + 1 subsets (corresponding to the inertia indices of nondegenerate forms — generating functions of Lagrangian planes transverse to the vertex and close to it). The transverse orientation of the train defines an order of these subsets (according to increasing order of the positive inertia index). In particular, in the neighborhood of the vertex there is distinguished the "positive" domain maximal with respect to the indicated order (the corresponding quadratic forms are positive-definite quadratic functions).

The index of any curve (with extremities outside the train) lying in the neighborhood of the vertex is equal to the increment of the positive inertial index of the generating function and hence does not exceed n.

Definition. The Maslov index of the path starting at a point α, which does not belong to the train of the point β, and ending at β, is by definition the index of intersection with the train of β of a close path starting at α and ending at a point β' close to β and lying in the positive domain (Fig. 1).

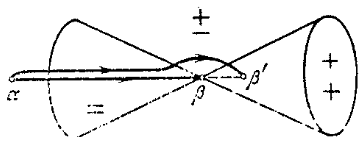

Fig. 1

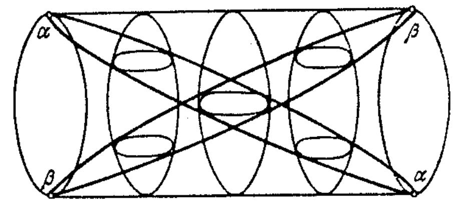

Fig. 2

The index of the path $(\alpha\beta)$ shown in Fig. 1 equals 2. The index of a path is invariant under homotopies of the path in which the extremities of the path remain always transverse Lagrangian planes, i.e., as long as α does not intersect the train of β.

The homotopy class of a path connecting α to β can be represented by a pair of points $(\tilde{\alpha}, \tilde{\beta})$ on the universal covering $\tilde{\Lambda}_n$ of the manifold Λ_n. We denote by $m(\tilde{\alpha}, \tilde{\beta})$ the Maslov index of such a path.

The fundamental group $\mathbf{Z}$ of Λ_n acts on $\tilde{\Lambda}_n$ by translations. We let +1 denote the action of the generator which intersects the train with index 1. In these notations the results of [2, 13] can be stated as follows:

THEOREM. The Maslov index $m(u, v)$ of pairs of points on the covering $\tilde{\Lambda}_n$ has the following properties:

1°. $m(u, v + 1) = m(u, v) + 1$.

2°. $m(u, v) + m(v, u) = n$.

3°. $m(u, v) + m(v, w) + m(w, u) = n + I(\pi u, \pi v, \pi w)$,

where $\pi:\tilde{\Lambda}_n \to \Lambda_n$ is the covering and the index $I(\alpha, \beta, \gamma)$ of a triplet of pairwise transverse Lagrangian planes is defined by the next construction.

Definition. For each pair α, β of transverse Lagrangian planes in $(\mathbf{R}^{2n}, \omega)$ we consider the adjoined quadratic form $\Phi[\alpha, \beta]$ in $\mathbf{R}^{2n}$, whose value on any vector is equal to the value of the symplectic form ω on its components in α and β:

$$\Phi[\alpha, \beta](\zeta) = \omega(\xi, \eta), \text{ where } \zeta = \xi + \eta, \quad \xi \in \alpha, \quad \eta \in \beta.$$

Definition. The index $I(\alpha, \beta, \gamma)$ of the triplet (α, β, γ) is the inertia index of the restriction to γ of the form $\Phi[\alpha, \beta]$ adjoined to α and β.

Remark 1. Two arbitrary transverse Lagrangian planes can be transformed into two arbitrary others by a symplectic transformation. Consequently, one can always introduce Darboux coordinates relative to which $\omega = dp \wedge dq$, α is the coordinate p plane, and β is the coordinate q plane. Then $\Phi[\alpha, \beta] = pq$.

In particular, the Lagrangian planes α, β can be uniquely reconstructed given the adjoined form Φ. In fact, they are stable and unstable manifolds, respectively, for the Hamilton equations with Hamiltonian Φ.

Remark 2. The index of a triplet of planes enjoys the following properties:

1°. $I(\alpha, \beta, \gamma) + I(\beta, \alpha, \gamma) = n$.

2°. $I(\alpha, \beta, \gamma) = I(\beta, \gamma, \alpha)$.

The first is a consequence of the equality $\Phi[\beta, \alpha] = -\Phi[\alpha, \beta]$, while the second follows from the fact that $\Phi[\alpha, \beta]|\gamma$ is taken into $\Phi[\beta, \gamma]|\alpha$ by the projection of γ onto α along β (incidentally, both properties follow from the theorem).

Proof of the Theorem. 1°. The path corresponding to the pair $(u, v + 1)$ is obtained from the path corresponding to the pair (u, v) by adding a loop whose index of intersection with the train of the plane πv equals 1.

2°. The index $m(u, v)$ does not change under deformations of the path as long as πu does not intersect the train of πv. When intersection does occur, and in positive direction, $m(u, v)$ decreases by 1, whereas $m(v, u)$ increases by 1. It therefore suffices to verify property 2° for a single yet arbitrary path. To this end we can choose the path $(\alpha\beta)$ shown in Fig. 1 contained in a single chart (then α corresponds to a negative-definite form, and β to the null form). In this case $m(\tilde{\alpha}, \tilde{\beta}) = n$ (the inertia index grows from 0 to n). For the inverse path (which can be chosen so that it will not intersect the train), $m(\tilde{\alpha}, \tilde{\beta}) = 0$.

3°. The sum is invariant under deformations of the triplet (u, v, w) provided that the planes remain transverse. The substitution of u by u + 1 does not affect the sum since, according to 1° and 2°, the first (third) component decreases (respectively, increases) by one.

Shifts of v and w also do not affect the sum. Thus, the sum depends only on the triplet of Lagrangian planes, and not on the paths joining them.

For special paths the sum can be calculated explicitly. Fix two transverse Lagrangian planes α and β. We identify the variety of Lagrangian planes γ transverse to β with a space of quadratic forms on α (to the plane γ we attach its generating function $S_\gamma = \pi^*_{\alpha\gamma}\Phi[\alpha, \beta]/2$, where $\pi_{\alpha\gamma}:\alpha \to \gamma$ is the projection along β: in Darboux coordinates, γ has the equation $q = \delta S/\delta p$).

To the plane α there corresponds the form $S_\alpha = 0$, while to the planes of the train of α there correspond degenerate forms. Let γ be a Lagrangian plane transverse to both α and β. We connect α and γ to β by the paths corresponding to the form

$$S_\alpha + \lambda P, \quad S_\gamma + \lambda P \qquad (0 \leqslant \lambda \leqslant \infty),$$

where P is some positive-definite form. Now connect γ to α by the straight line λS_γ in the space of forms. The indices of the constructed paths are calculated directly according to the definition: $m(\tilde{\beta}, \tilde{\alpha}) = 0$, $m(\tilde{\beta}, \tilde{\gamma}) = 0$, $m(\tilde{\gamma}, \tilde{\alpha})$ = the negative inertia index of the form S_γ (the increment of the positive inertia index on the path from S_γ to εP, $\varepsilon > 0$). By 1° and 2°, $m(\tilde{\alpha}, \tilde{\beta}) + m(\tilde{\beta}, \tilde{\gamma}) + m(\tilde{\gamma}, \tilde{\alpha}) = n + 0 + I(\alpha, \beta, \gamma)$, as asserted.

3. An Example: The Case of Two Degrees of Freedom

Proposition. The train of a point in Λ_2 is diffeomorphic to a quadratic cone in RP^3 (homeomorphic to a sphere with its poles identified).

Proof. The manifold of degenerate quadratic forms in the plane is a quadratic cone as well as the representation of the train in the chart $\{S_\gamma\}$ centered at the point in question.

Proposition. Λ_2 is the nonoriented total space of the nontrivial bundle with fiber the sphere S^2 and with base the circle, glued from $S^2 \times [0, 1]$ by means of the antipodal involution of the sphere.

Since the incorrect statement $\Lambda_2 = S^2 \times S^1$ is "proved" twice in the textbook of Guillemin and Sternberg [6], we give the proof of the proposition. The bundle $\Lambda_2 \to S^1$ is defined by the mapping $u0 \to \det^2 u$. Its fiber is $SU(2)/SO(2) = S^3/S^1 = S^2$ (Hopf bundle). Multiplication of u by the scalar $e^{i\varphi}$ results in multiplication of $\det^2 u$ by $e^{4i\varphi}$. For $0 \leqslant \varphi < \pi/2$ the image of the fiber fills out the bundle's total space. For $\varphi = \pi/2$ $e^{i\varphi} = i$. Multiplication by i takes every Lagrangian plane into its orthogonal complement. Therefore, the gluing map $S^2 \to S^2$ is the antipodal involution. It changes the orientation of the sphere. Consequently, the bundle is not trivial and Λ_2 is not orientable.

In terms of this bundle, the disposition of the train of some point, say, of the North pole α of the fiber over zero, can be described as follows.

As the point of the base moves along the circle, the trace of the train on the fiber looks first like a small parallel near the North pole, which then passes through the equator to eventually shrink, when the point on the base completes its trip around the circle to the South pole of the fiber (which pole is taken by the gluing mapping into the North pole).

To visualize the relative position of the train of two points α and β, we shall assume that β is the South pole of the same fiber (the general case can be reduced to this one by the action of the symplectic group). The train of the point β consists of parallels symmetric to the parallels forming the train of α relative to the equator of the fiber. Both trains intersect transversally along the equator of the fiber opposite to the original one (Fig. 2).

None of the trains divides the total bundle space Λ_2. The complement of the train of a point in Λ_2 is diffeomorphic to the three-dimensional Euclidean space.

In fact, fixing a pair of transverse Lagrangian planes α and β, we have earlier identified the set of Lagrangian spaces γ transverse to β with the space of quadratic forms S_γ on the plane α.

On passing to the covering $\tilde{\Lambda}_2$, the train of the point β is covered by the infinitely sheeted surface obtained by rotating a sinusoid around the abscissa axis. This surface divides the covering into domains diffeomorphic to the Euclidean space [and labelled by the values of the index $m(\cdot, \tilde{\alpha})$].

The union of the trains of two transverse planes α and β divides Λ_2 into three regions. Under the identification of planes γ transverse to β with their generating functions S_γ (quadratic forms on α) these domains pass into the domains consisting of the forms of signature (+, +), (+, —), and (—, —) (Fig. 3). Figure 4 schematically shows the transverse orientations of the two trains and the contiguity of domains [as in the Kerr solution (of Einstein's equation — translator's note)].

4. Quadratic Forms and the Rayleigh—Gårding Inequalities for Hyperbolic Polynomials

Sturm's theory is based on the following simple lemma.

LEMMA 1. Suppose some quadratic form in $\mathbf{R}^n$, depending smoothly on a parameter, is degenerate for some critical value of the parameter, and its derivative with respect to the parameter is positive-definite on the kernel of the degenerate form. Then the positive inertia index of the form increases at the passage of the parameter through the critical value (and the increment is precisely the dimension of the kernel).

Proof. For the case where the degenerate form is the null form the lemma is obviously true. The general case is reduced to this one by isolating the nondegenerating part as a component smooth in the parameter (see also Lemma 2 below).

THEOREM. If a degenerate quadratic form varies so that the velocity of its variation is positive-definite on the kernel of the form, then the indicated velocity vector does not belong to the tangent cone to the variety of degenerate forms at the given point.

Proof. Every vector of the tangent cone is tangent to a smooth curve lying completely in the variety of degenerate forms (this is obvious in the case of the null form, to which the general case reduces again by isolating the nondegenerating part). If the velocity vector would belong to the tangent cone, the corresponding curve would provide a deformation contradicting Lemma 1.

Remark 1. The cone of positive-definite forms, translated at a point of the hypersurface of degenerate forms in the space of all forms, does not intersect this hypersurface in a sufficiently small neighborhood of the given point.

This is a consequence of the Rayleigh—Fischer—Courant inequalities for eigenvalues. As a matter of fact, the same property is enjoyed by the hypersurface given by any homogeneous hyperbolic polynomial (the discriminant of a quadratic form is a hyperbolic polynomial) with respect to the convex cone constituting any of the connected components of the domain of hyperbolic vectors. This follows, for example, from the generalized Rayleigh—Fischer—Courant inequalities carried out by Gårding [5] to arbitrary hyperbolic polynomials.

Remark 2. The following sharpening of Lemma 1 is often useful:

LEMMA 2. Suppose the derivative of the quadratic form with respect to the parameter is nonnegative, and for the critical value of the parameter the restriction of the derivative of the form to its kernel has rank k. Then at the passage through the critical value the number of positive squares increases at least by k.

Proof. We use an auxiliary Euclidean structure. Consider the positive eigensubspace P of the form A(t) for the critical value t = 0 of the parameter and the positive eigensubspace Q of the restriction of the derivative to the kernel of A(0). For small t > 0 the form A(t) is positive-definite on P + Q. In fact, for $\xi \in P$, $\eta \in Q$, and t > 0, we have, using that $A(0)\eta = 0$,

$$(A(t)\xi, \xi) \geqslant c_1 \|\xi\|^2, \quad (A(t)\eta, \eta) \geqslant c_2 t \|\eta\|^2, \quad |(A(t)\eta, \xi)| \leqslant c_3 t \|\xi\| \|\eta\|,$$

whence $(A(t)(\xi+\eta), (\xi+\eta)) \geqslant c_4 (\|\xi\|^2 + t\|\eta\|^2)$ (because $t^{1/2}\|\xi\| \, t^{1/2}\|\eta\| \leqslant t^{1/2}\|\xi\|^2 + t^{1/2}\|\eta\|^2$). Therefore, the form A(t) increases by at least k on passing from t = 0 to t = +ε. This is the more so true on passing from t = —ε to t = +ε, since this number is a monotone function of t (the Rayleigh—Fischer—Courant inequalities).

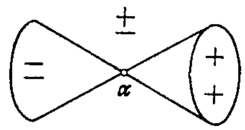

Fig. 3

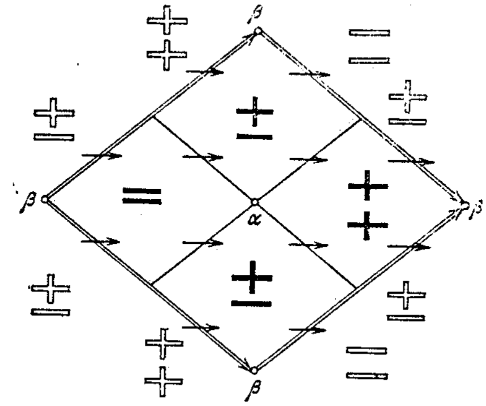

Fig. 4

5. Transverse Orientation of Trains

Definition. We call positive vectors on the Lagrangian Grassman manifold the velocity vectors of motions of Lagrangian planes under the action of systems with positive-definite Hamiltonians.

THEOREM. A positive vector does not belong to the tangent cone of any train.

The positive vectors at an arbitrary point form a convex (open) cone. We show that this cone is entirely contained in one of the subsets into which the tangent cone to the train divides the tangent space to the Grassman manifold.

Let $\Phi[\alpha, \beta]$ be the quadratic form in phase space adjoined to the pair (α, β) of transveserse Lagrangian planes (see Sec. 2). We shall denote with a dot the derivative in the direction of the Hamiltonian vector field with Hamiltonian H.

LEMMA. $\dot{\Phi}[\alpha, \beta]|\alpha = 2H$.

Proof. We choose Darboux coordinates in which α is the p plane and β is q plane. Then $\Phi = pq$, and hence $\dot{\Phi} = \dot{p}q + p\dot{q} = pH_p - qH_q$. For $q = 0$ Euler's theorem yields $\dot{\Phi} = 2H$.

Proof of the Theorem. Let γ be a Lagrangian plane which is not transverse to α. Pick a Lagrangian plane β transverse to both α and γ. The phase flow g^t of the Hamiltonian H takes γ into the Lagrangian planes $\gamma(t) = g^t\gamma$. Consider the generating functions of these planes, i.e., the following quadratic forms given on α:

$$\varphi_t = \pi^* \, i^* \, g^{t*} \, \Phi \, [\alpha, \beta]/2,$$

where $i:\gamma \to R^{2n}$ is the imbedding and $\pi:\alpha \to \gamma$ designates the projection along β.

The form φ_0 is degenerate, with kernel $\gamma \cap \alpha$. The form $d\varphi/dt$ is positive-definite on the kernel of φ_0 (this follows from the lemma, because the Hamiltonian is positive-definite). By the theorem of Sec. 4, $d\varphi/dt$ does not belong to the tangent cone to the variety of degenerate forms. Hence, $d\gamma/dt$ does not belong to the tangent cone to the train of the point α.

COROLLARY 1. Every train is transversely oriented by the directions of positive vectors.

COROLLARY 2. If a Lagrangian plane evolves under the action of a system with (possibly time-dependent) positive-definite Hamiltonian, then the index of intersection of the corresponding path on the Lagrangian Grassman manifold with the train of any point is nonnegative.

6. Proof of the Nonoscillation Theorem

Equation (1) or (2) is equivalent to the Hamiltonian system (3).

LEMMA 1. The derivative of the function $\Phi = pq$ along the Hamiltonian field is equal to 2L, where $L = T - U$ is the Lagrangian.

Proof.

$$(pq)^{\cdot} = pq^{\cdot} + p^{\cdot}q = pB^{-1}p - qAq = 2\,(T - U).$$

Suppose the evolving Lagrangian plane $\gamma(t)$ is vertical (i.e., not transverse to the plane $q = 0$) for $t = t_0$. Suppose the Lagrangian is nonnegative (the potential energy is nonpositive). Let $g^t_{t_0}$ denote the phase transformation from time t_0 to time t. Consider on $\gamma = \gamma(t_0)$

the quadratic form $\psi_t = i^*(g_{t_0}^t)^*\Phi$.

Its derivative $d\psi/dt$ is, according to Lemma 1, nonnegative (because $L \geqslant 0$). Hence, the number of its positive squares may only increase as t grows.

LEMMA 2. At the passage of t through any moment of verticality the number of positive squares of the form ψ_t increases at least by 1.

Proof. The kernel of the form ψ_{t_0} contains the intersection of γ with the plane $q = 0$. On this intersection the derivative is positive-definite (for $q = 0$ we have $L = T$). By Lemma 2 of Sec. 4, the number of positive squares increases, on crossing the value t_0, by at least the dimension of the intersection. The lemma is proved.

Since the number of positive squares of an increasing form in $\mathbf{R}^n$ cannot increase more than n times, the number of moments of verticality does not exceed n. The theorem is proved.

Remark 1. We have simultaneously proved the following symplectic generalization of the theorem: if the Poisson bracket of the quadratic forms H and $\Phi[\alpha, \beta]$ is positive-definite, then a Lagrangian plane evolving under the system with Hamiltonian H becomes transverse to α at most n times (actually, it suffices that the Poisson bracket be nonnegative, but its restriction to α must be positive-definite).

Remark 2. The number of zeros of an individual solution of the Newton equation with (time-dependent) negative-definite potential energy can be infinite, as shown by the integrable example of a uniformly rotating potential hump in the plane (in a rotating coordinate system one obtains an autonomous system).

7. Proof of the Theorem on Zeros

We consider the evolution $\gamma(t)$ of a Lagrangian plane γ under system (3), the Hamiltonian of which is positive-definite on the Lagrangian plane α.

For example, in the case of systems (1) or (2) the last condition is always fulfilled on the vertical plane $q = 0$, since the kinetic energy is positive-definite.

LEMMA 1. The contribution of the moment τ in the index of intersection of the path $\{\gamma(t)\}$ with the train of the point α is equal to the dimension of the intersection $\delta = \gamma(\tau) \cap \alpha$.

Proof. Pick a Lagrangian plane β transverse to both α and $\gamma(\tau)$. The generating function for $\gamma(\tau)$ has the kernel δ. Its derivative is positive-definite on δ (on δ it coincides, by the lemma in Sec. 5, with the Hamiltonian). Hence, on passing through the point τ the number of positive squares increases, according to Lemma 1 of Sec. 4, by $\dim\delta$, as claimed.

Consider a family, depending continuously on t, of paths $\Gamma(t)$ from $\gamma(t)$ to α. We assume that $\gamma(t_1)$ and $\gamma(t_2)$ are transverse to α, and label the objects related to t_1 (t_2) by the index 1 (respectively, 2).

LEMMA 2. The number ν of points of verticality of the plane $\gamma(t)$ in the interval between t_1 and t_2, counting multiplicities [i.e., the sum of the intersections of all planes $\gamma(t)$ with α], is equal to the difference between the Maslov indices of the paths $\Gamma_1 = \Gamma(t_1)$ and $\Gamma_2 = \Gamma(t_2)$.

Proof. Each time the plane becomes vertical, the Maslov index of the path decreases by the dimension of the respective intersection (by Lemma 1 and the definition of the index).

Proof of the Theorem. Let now $\gamma'(t)$ be the evolution of a different Lagrangian plane under the action of the same system, and let $\Gamma'(t)$ be the corresponding system of paths. In the covering space we pick a point $\tilde{\alpha}$ over α and we cover the paths $\Gamma(t)$ and $\Gamma'(t)$ by paths $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ which end at $\tilde{\alpha}$; their origins are denoted by $\tilde{\gamma}$ and $\tilde{\gamma}'$, respectively. In the notations of Sec. 2, the assertion of Lemma 2 is:

$$\nu = m(\tilde{\alpha}, \tilde{\gamma}_2) - m(\tilde{\alpha}, \tilde{\gamma}_1).$$

By the theorem of Sec. 2,

$$m(\tilde{\alpha}, \tilde{\gamma}(t)) + m(\tilde{\gamma}(t), \tilde{\gamma}'(t)) - m(\tilde{\alpha}, \tilde{\gamma}'(t)) = I(\alpha, \gamma(t), \gamma'(t)). \quad (4)$$

The middle term does not depend on t, since the index is symplectically invariant. The right-hand side lies in the interval between 0 and n for all t. Consequently, the increment of the left-hand side between t_1 and t_2 is bounded in modulus by n. Hence,

$$|[m(\tilde{\alpha}, \tilde{\gamma}_1) - m(\tilde{\alpha}, \tilde{\gamma}_1')] - [m(\tilde{\alpha}, \tilde{\gamma}_2) - m(\tilde{\alpha}, \tilde{\gamma}_2')]| \leqslant n.$$

By Lemma 2, $|\nu' - \nu| \leqslant n$, as claimed.

We next turn to the proof of the alternation theorem. In the above notations, the n ber of moments at which $\gamma(t)$ is not transverse to α is

$$\nu_\alpha = I(\alpha, \gamma_2, \gamma_1) - m(\tilde{\gamma}_2, \tilde{\gamma}_1)$$

[by (4) and Lemma 2 of Sec. 7]. In exactly the same way

$$\nu_\beta = I(\beta, \gamma_2, \gamma_1) - m(\tilde{\gamma}_2, \tilde{\gamma}_1)$$

[it is important that the subtrahend does not depend on the way in which the path $\{\gamma(t)\}$ lifted to the covering]. Since $0 \leqslant I \leqslant n$, the modulus of the difference does not exceed as asserted.

8. Proof of the Comparison Theorem

Repeating the reasoning of the proof of the theorem on zeros we obtain for arbitrary t relation (4) for the planes γ and γ' evolving under the systems with Hamiltonians H and but now $M(t) = m(\tilde{\gamma}(t), \tilde{\gamma}'(t))$ depends on t.

LEMMA. If $H' \geqslant H$, then M(t) does not decrease as t grows.

Proof. Suppose now that $H' - H$ is positive-definite. Let τ be a moment of nontransversality of the planes $\gamma(\tau)$ and $\gamma'(\tau)$. The symplectic diffeomorphism $(g_\tau^t)^{-1}$, defined by the system with Hamiltonian H, takes $\gamma(t)$ into $\gamma(\tau) = \alpha$ and γ' into a Lagrangian plane $\sigma(t$ The Hamiltonian controlling the motion of σ is positive-definite on α (for $t = \tau$ it is equ to $H' - H$). But $M(t) = m(\tilde{\alpha}, \tilde{\sigma}(t))$, since the Maslov index is symplectically invariant.

According to Lemma 1 of Sec. 7 (and the definition of m given in Sec. 2) the last ind grows (does not decrease) as t is increased. In the case $H' \geqslant H$ the lemma is proved by pa ing to the limit.

Subtracting relation (4) with $t = t_2$ from the same relation with $t = t_1$, we get

$$\nu' - \nu = I_1 - I_2 + M_1 - M_2.$$

By the lemma, $M_2 \geqslant M_1$. Since $0 \leqslant I \leqslant n$, we get $\nu' \geqslant \nu - n$, and the theorem is proved.

Remark 1. The fact that the rotation of the Lagrangian plane $\gamma(t)$ is accelerated on increasing the Hamiltonian H(t) follows from the following Liouville-type formula for the angular velocity of the squared determinant: $\Delta(t) = \det^2 \gamma(t)$, where $\det^2\colon U(n)/O(n) \to S^1$ (see [2]).

Rotation Theorem. $\dot{\Delta} = 2ih\Delta$, where $h = \operatorname{tr} H|_{\gamma(t)}$ [the trace of the quadratic form $(Ax, x)/2$ in Euclidean space is the trace of the operator A].

Proof. Let $\alpha = \gamma(t_0)$, $\beta = i\alpha$. In the plane α we choose an orthonormal base. We denote the corresponding coordinates in α and in β by p and q, respectively. The Lagrangian plane $\gamma(t)$ has a generating function S(t), i.e., is given by the equation $q = \partial Z/\partial p$.

From Hamilton's equation we get $\partial S/\partial t|_{t_0} = H|_\alpha$. We denote by s this quadratic form. The operator $E + i\varepsilon\nabla s$ takes the indicated orthogonal basis in α into an orthonormal basis in $\gamma(t_0 + \varepsilon)$ with error $O(\varepsilon^2)$. Consequently,

$$\Delta(t_0 + \varepsilon) = \det{}^2(E + i\varepsilon\nabla s) + O(\varepsilon^2) = 1 + 2i\varepsilon \operatorname{tr} s + O(\varepsilon^2),$$

as asserted.

Remark 2. This paper is the outcome of an attempt of understanding the result of Pet [7] on nonoscillation of complete elliptic integrals. Although in this way I have not obtained a proof of Petrov's theorem, I believe that the symplectic [8] and variational aspec of the Gauss–Manin connection deserve a careful investigation, which may lead to generaliza tions of Petrov's theorem (at least to the hyperelliptic case) and hence to new estimates of the number of limit cycles arising in Hamiltonian systems under polynomial perturbations

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FILTERING BASES, COHOMOLOGY OF INFINITE-DIMENSIONAL LIE ALGEBRAS, AND LAPLACE OPERATORS

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UDC 513.836

We denote by $L_k(n)$ the Lie algebra of polynomial vector fields on n-dimensional space over the field **Q** of rational numbers, having trivial k-jet at the point 0. In this paper we are mainly concerned with the algebras $L_k(1)$, whose notation is sometimes abbreviated to L_k. We note that the vector fields $e_i = x^{i+1}d/dx$ $(i = k, k + 1,\ldots)$ constitute a basis in L_k, and the commutation operation is given by the formula $[e_i, e_j] = (j - i)e_{i+j}$.

The basic result of the present paper is the construction of a special basis in the exterior complex of the algebra L_k. We call this basis filtering since it defines a filtration in the exterior complex of the algebra L_k, which is very useful, in particular, for calculating the cohomology. Thus, we get a proof of the famous theorem of Goncharova on the cohomology of the algebras L_k.

The problem of calculating the cohomology of these Lie algebras (with trivial coefficients) was posed by I. M. Gel'fand at the 1970 mathematical congress and he formulated the conjecture there that

$$\dim H^q(L_k) = \binom{q+k-1}{k-1} + \binom{q+k-2}{k-1}. \tag{1}$$

The motivation for posing this problem was the key role which the cohomology of the algebras L_k plays in the calculation of the cohomology of various Lie algebras with coefficients in a broad class of modules (cf. [1, 2]). Subsequently it turned out that this cohomology is of great value in the theory of representations of infinite-dimensional Lie algebras (cf. [3-5]) and in algebraic topology (cf. [6]).

The first proof of Gel'fand's conjecture was published in 1973 by Goncharova [7]. Although a very interesting idea lies at its foundation, it is very complicated and it is not easy to understand it. Hence in the following years a number of attempts to get different proofs of Goncharova's theorem were made. Four papers relating to this theme are known to

TsNIIPromzdanii Gosstroya SSSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 19, No. 4, pp. 11-22, October-December, 1985. Original article submitted July 26, 1984.