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A Theorem on Matrices of Analytic Functions.

By

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A function $l(x)$ which is single-valued and analytic for $|x|$ sufficiently large, but which is not analytic at $x = \infty$, may be expanded in a Laurent series in the vicinity of $x = \infty$. This series consists of the sum of a function $a(x)$ analytic at $x = \infty$ and vanishing there, given by a power series in negative powers of x, and of an entire function $e(x)$, expressed as a power series in positive powers of x beginning with a constant term; that is, we have

(1) $$l(x) = a(x) + e(x).$$

Thus $l(x)$ is decomposed into a *sum* of two functions, one of which is analytic except at $x = \infty$, and the other analytic at $x = \infty$, vanishing there.

If the additional restriction be imposed on $l(x)$ that this function does not vanish for $|x|$ sufficiently large, we can similarly express $l(x)$ in the form of a *product*

(2) $$l(x) = a(x)\, e(x)\, x^k,$$

where $a(x)$ is some function analytic but not zero at $x = \infty$, where $e(x)$ is an entire function that nowhere vanishes, and where k is an integer.

To prove this statement we note that $\log l(x)$ is analytic for $|x|$ sufficiently large. Furthermore $\log l(x)$ alters only by a constant $-2k\pi\sqrt{-1}$, k being an integer, when x makes a positive circuit of $x = \infty$. Accordingly the function $\log l(x) - k \log x$ is single-valued and analytic in the vicinity of $x = \infty$, and by (1) can be expressed in the form $a_1(x) + e_1(x)$ where $a_1(x)$ is analytic at $x = \infty$ and $e_1(x)$ is an entire function. If we put $a(x) = e^{a_1(x)}$, $e(x) = e^{e_1(x)}$ we have at once the relation (2); the functions $a(x)$ and $e(x)$ have the specified properties.

Suppose now that $(l_{ij}(x))$ is a square matrix*) of n^2 functions, each

*) The notation and the elements of the theory of matrices used in the present paper may be found in Schlesinger, *Vorlesungen über lineare Differentialgleichungen,* p. 18—19.

single-valued and analytic for $|x| \geqq R$. Each of these functions may be expressed as a Laurent series for $|x| \geqq R$. The generalization of (1) is trivial in this case and may be written

$$(l_{ij}(x)) = (a_{ij}(x)) + (e_{ij}(x)),$$

the decomposition (1) being applied to the separate elements of the matrix.

The generalization of (2) *for such a matrix is of an entirely different nature.* I was first led to a theorem effecting such a generalization by the study of the singular points of ordinary linear differential equations.*) The present article contains a completed form of the theorem and a proof of it based on the theory of the Fredholm integral equation. The theorem seems to possess an intrinsic interest quite aside from the interest of this application to ordinary linear differential equations. In the article which immediately follows the present one I have carried out the application.

The theorem is as follows: *Let* $(l_{ij}(x))$ *be any matrix of functions single-valued and analytic for* $|x| \geqq R$ *(but not necessarily analytic at* $x=\infty$*), and such that the determinant of this matrix does not vanish for* $|x| \geqq R$. *There exists then a matrix* $(a_{ij}(x))$ *of functions analytic at* $x=\infty$, *reducing to the unit matrix* (δ_{ij}) *at* $x=\infty$, *and a matrix* $(e_{ij}(x))$ *of entire functions, of determinant nowhere zero in the finite plane, such that*

$$(l_{ij}(x)) = (a_{ij}(x))\,(e_{ij}(x)\,x^{k_j}) \tag{3}$$

where $k_1, k_2, \cdots, k_n$ *are integers.*

In demonstrating the truth of this theorem let us begin by proving the existence of a solution $(\varepsilon_{ij}(x))$ of the matrix equation

$$(l_{ij}(x))\,(\varepsilon_{ij}(x)) = (b_{ij}(x)) \tag{4}$$

where $(\varepsilon_{ij}(x))$ is a matrix of entire functions of determinant not identically zero to be determined so that the elements of $(b_{ij}(x))$ are rational in character at $x=\infty$ (i. e. are analytic or have only a pole at $x=\infty$). Otherwise stated, we shall prove the existence of n sets of entire functions $\varepsilon_1(x), \cdots, \varepsilon_n(x)$ of determinant not identically zero for each of which we have

$$l_{i1}(x)\,\varepsilon_1(x) + \cdots + l_{in}(x)\,\varepsilon_n(x) = b_i(x) \qquad (i=1,\cdots,n), \tag{5}$$

$b_1(x), \cdots, b_n(x)$ being rational in character at $x=\infty$.

Let $(m_{ij}(x))$ denote the matrix inverse to $(l_{ij}(x))$ and defined by either set of equations

$$\begin{aligned} l_{i1}(x)m_{1j}(x) + \cdots + l_{in}(x)m_{nj}(x) &= \delta_{ij}, \\ m_{i1}(x)\,l_{1j}(x) + \cdots + m_{in}(x)\,l_{nj}(x) &= \delta_{ij}, \end{aligned} \qquad (i,j=1,\cdots,n). \tag{6}$$

*) Trans. Am. Math. Soc. 10 (1909), p. 436—470; in particular see p. 438—443.

The functions $m_{ij}(x)$ are clearly single-valued and analytic for $|x| \geqq R$ since the determinant of $(l_{ij}(x))$ does not vanish for such values of x.

Let us put also

$$(7) \qquad m_{i1}(x) l_{1j}(t) + \cdots + m_{in}(x) l_{nj}(t) = \delta_{ij} + k_{ij}(x, t) \quad (i, j = 1, \cdots, n)$$

and in this way define $k_{ij}(x, t)$ for $|x| \geqq R$, $|t| \geqq R$. From the second set of equations (6) it follows that each function

$$\frac{k_{ij}(x, t)}{t - x}$$

is analytic in x and t for $|x| \geqq R$, $|t| \geqq R$, since $k_{ij}(x, t)$ vanishes identically for $x = t$.

From the above equations and (7) we deduce further the equations

$$(8) \qquad l_{i1}(x) k_{1j}(x, t) + \cdots + l_{in}(x) k_{nj}(x, t) + l_{ij}(x) = l_{ij}(t) \quad (i, j = 1, \cdots, n),$$

which may also serve to define the functions $k_{ij}(x, t)$.

Now consider the set of n linear integral equations in $f_1(x), \cdots, f_n(x)$

$$(9) \qquad \frac{1}{2\pi\sqrt{-1}} \int_C \frac{\sum\limits_{j=1}^{n} k_{ij}(x, t) f_j(t)}{t - x} \, dt = f_i(x) - \sum_{j=1}^{n} m_{ij}(x) \, p_j(x) \quad (i = 1, \cdots, n),$$

where $p_1(x), \cdots, p_n(x)$ are any polynomials in x, and the variables x, t are restricted to lie on the circle C for which $|x| = R$; the integration around C is taken in a positive sense.

If we set $x = Re^{\sqrt{-1}\theta}$, $t = Re^{\sqrt{-1}\varphi}$ where θ and φ are angular variables, and put

$$\frac{1}{2\pi} \frac{k_{ij}\left(Re^{\sqrt{-1}\theta}, Re^{\sqrt{-1}\varphi}\right)}{1 - e^{\sqrt{-1}(\theta - \varphi)}} = K(2(i-1)\pi + \theta, \, 2(j-1)\pi + \varphi) \quad (0 < \theta, \, \varphi < 2\pi),$$

$$f_i\left(Re^{\sqrt{-1}\theta}\right) = F(2(i-1)\pi + \theta) \qquad (0 < \theta < 2\pi),$$

$$\sum_{j=1}^{n} m_{ij}\left(Re^{\sqrt{-1}\theta}\right) p_j\left(Re^{\sqrt{-1}\theta}\right) = P(2(i-1)\pi + \theta) \qquad (0 < \theta < 2\pi),$$

according to a device due to Fredholm, our n equations may be combined into the single one

$$(10) \qquad \int_0^{2n\pi} K(\theta, \varphi) F(\varphi) \, d\varphi = F(\theta) - P(\theta)$$

where $K(\theta, \varphi)$ and $P(\theta)$ are continuous save for the discontinuities of a finite jump. This equation is of precisely the type of linear integral equation studied by Fredholm*), and according to his theorems admits of a solution $F(\theta)$ integrable in the sense of Riemann, at least if $P(\theta)$ satisfies k conditions of the form

$$\int_0^{2n\pi} P(\theta)\psi_i(\theta)\,d\theta = 0 \qquad (i = 1, \cdots, k).$$

Here $\psi_1(\theta), \cdots, \psi_k(\theta)$ are certain integrable functions of θ.

Returning now to equations (9), we see that these equations are satisfied by a set of integrable functions $f_1(x), \cdots, f_n(x)$ for x on C, at least provided that the coefficients of the various powers of x in the polynomials $p_1(x), \cdots, p_n(x)$ are subject to certain linear homogeneous conditions. Such a solution $f_1(x), \cdots, f_n(x)$ is made up of functions analytic in x on C, for the left-hand member of any equation (9) is analytic in x, and hence the first term $f_i(x)$ in the right-hand member is also analytic in x for $i = 1, 2, \cdots, n$.

For $|x| > R$ the equations (9) serve to define $f_1(x), \cdots, f_n(x)$ as single-valued analytic functions in terms of these functions on C. Thus we may regard the equations (9) as satisfied for $|x| \geqq R$ by such a set of functions.

Let us restrict x to lie outside of C and let us consider one of the integrals

$$(11) \qquad \frac{1}{2\pi\sqrt{-1}}\int_C \frac{f_i(t)\,dt}{t-x} \qquad (i = 1, \cdots, n).$$

As we have seen, a function $f_i(x)$ may be expressed as the sum of a function $\alpha_i(x)$ analytic at $x = \infty$ *and vanishing there*, and of an entire function $\varepsilon_i(x)$. If we replace $f_i(t)$ in the above integral by this sum $\alpha_i(t) + \varepsilon_i(t)$, the integral breaks up into the sum of two others

$$\frac{1}{2\pi\sqrt{-1}}\int_C \frac{\alpha_i(t)\,dt}{t-x} + \frac{1}{2\pi\sqrt{-1}}\int_C \frac{\varepsilon_i(t)\,dt}{t-x}.$$

The second of these partial integrals has the value zero by Cauchy's integral theorem, since $\varepsilon_i(t)$ is analytic for $|t| \leqq R$. The transformation $x = \frac{1}{x'}$, $t = \frac{1}{t'}$ reduces the first partial integral to the form

$$-x'\left[\frac{1}{2\pi\sqrt{-1}}\int_{C'} \frac{\left[\frac{1}{t'}\,\alpha_i\left(\frac{1}{t'}\right)\right]dt'}{t'-x'}\right]$$

where C' is the circle $|t'| = \frac{1}{R}$ described in a positive sense. Now the

*) Acta Math. 27 (1903), p. 365—390.

function $\frac{1}{x}\alpha_i\left(\frac{1}{x}\right)$ is analytic within and on C', since $\alpha_i(x) = f_i(x) - \varepsilon_i(x)$ is analytic for $|x| \geqq R$ and vanishes at $x = \infty$. Hence, by Cauchy's integral formula, the expression in brackets reduces to $\frac{1}{x'}\alpha_i\left(\frac{1}{x'}\right)\cdot$ The integral (11) is therefore equal to $-\alpha_i(x)$.

If we add the integral (11) to both sides of the i^{th} equation (9) for $i = 1, \cdots, n$, we obtain the following equations

$$\frac{1}{2\pi\sqrt{-1}}\int_C \frac{f_i(t) + \sum_{j=1}^{n} k_{ij}(x, t) f_j(t)}{t - x}\, dt = \varepsilon_i(x) - \sum_{j=1}^{n} m_{ij}(x) p_j(x) \quad (i = 1, \cdots, n).$$

Multiply the first of these equations by $l_{i1}(x)$, the second by $l_{i2}(x)$, and so on, and add the n resulting equations. By the aid of the equations (8) and (6) we may reduce the resulting equations to

$$\frac{1}{2\pi\sqrt{-1}}\int_C \frac{\sum_{j=1}^{n} l_{ij}(t) f_j(t)}{t - x}\, dt = \sum_{j=1}^{n} l_{ij}(x)\varepsilon_j(x) - p_i(x) \quad (i = 1, \cdots, n).$$

The left-hand member of such an equation is clearly a function $a_i(x)$, analytic for $|x| > R$ and also at $x = \infty$ where it vanishes; hence we have

(12) $\quad l_{i1}(x)\varepsilon_1(x) + \cdots + l_{in}(x)\varepsilon_n(x) = p_i(x) + a_i(x) = b_i(x) \quad (i = 1, \cdots, n)$

where $b_1(x), \cdots, b_n(x)$ are rational in character at $x = \infty$.

We are thus led to formulate the following conclusion: if each function $b_i(x)$ in (5) is rational in character at $x = \infty$, and is represented by the sum of a given polynomial $p_i(x)$ in x and of an unspecified function $a_i(x)$, analytic at $x = \infty$ and vanishing there, a set of corresponding entire functions $\varepsilon_1(x), \cdots, \varepsilon_n(x)$ will exist satisfying (5), at least provided that the coefficients of the powers of x in the polynomials $p_1(x), \cdots, p_n(x)$ satisfy a certain set of k linear homogeneous conditions.

In order that the existence of a solution of the corresponding matrix equation (4) be established, it remains only to be shown that it is possible to choose n sets $\varepsilon_1(x), \cdots, \varepsilon_n(x)$ of determinant not identically zero.

But this determinant will be zero if and only if the determinant of the corresponding sets $b_1(x), \cdots, b_n(x)$ is zero, since the product of the determinant $|l_{ij}(x)|$ (not identically zero) and the determinant of the sets $\varepsilon_1(x), \cdots, \varepsilon_n(x)$ is the determinant of the sets $b_1(x), \cdots, b_n(x)$.

If the polynomials $p_i(x)$ are all of degree m or less, this last mentioned determinant may be expanded in descending powers of x, the expansion beginning with terms of degree mn or less in x. The term in

x^{mn} here is clearly precisely the same as the corresponding term in the expansion of the determinant of the n sets $p_1(x), \cdots, p_n(x)$. It follows that if the determinant of the sets $\varepsilon_1(x), \cdots, \varepsilon_n(x)$ is to be identically zero, the determinant formed by the coefficients of x^{mn} in the n sets $p_1(x), \cdots, p_n(x)$ must vanish.

But each set of polynomials $p_1(x), \cdots, p_n(x)$ is subject to the same k conditions in order for a solution of (5) to exist. These conditions have been seen to be linear and homogeneous in the constant terms and coefficients of $x, x^2, \cdots, x^m$ occurring in the polynomials. It may be assumed that, as m increases indefinitely, these k equations of condition do not remain linearly dependent; otherwise all of the k-rowed determinants formed from the matrix of the k equations vanish, and a customary algebraic reduction in the number of conditions may be made. Choose m so large that the equations of conditions are linearly independent; if then m is changed to $m+1$, it is clear that the new unknowns introduced may be taken entirely arbitrary. Hence the determinant of the leading coefficients referred to above need not vanish, although all the n sets $p_1(x), \cdots, p_n(x)$ satisfy the k conditions. A solution of (4) with the specified properties must therefore exist.

This completely establishes the first and most important step in the proof of the theorem.

The second step is to show that by three series of modifications the solution of (4) may be made to have the three following additional properties: first, that the determinant $\Delta(x)$ of $(\varepsilon_{ij}(x))$ is nowhere zero in the finite plane; secondly, that $(b_{ij}(x))$ has the form $(a_{ij}(x)x^{\varkappa_j})$, where $\varkappa_1, \cdots, \varkappa_n$ are integers such that $\varkappa_1 \leqq \varkappa_2 \leqq \cdots \leqq \varkappa_n$, and $(a_{ij}(x))$ is a matrix of functions analytic at $x=\infty$ of determinant not zero there; thirdly, that for a suitable rearrangement $k_1, \cdots, k_n$ of $\varkappa_1, \cdots, \varkappa_n$ every function $\varepsilon_{ij}(x)$ vanishes at $x=0$ at least to the order $\varkappa_j - k_i$ if $k_i < \varkappa_j$.

Let us observe that $\Delta(x)$ is an entire function which vanishes only a finite number of times in the finite plane. In fact $\Delta(x)$ only vanishes a finite number of times for $|x| \leqq R$ since this function is then analytic. Also it vanishes a finite number of times for $|x| \geqq R$; for the product of the determinants of $|l_{ij}(x)|$ and $\Delta(x) = |\varepsilon_{ij}(x)|$ is $|b_{ij}(x)|$, analytic for $|x| \geqq R$ and rational in character at $x=\infty$ so that $|b_{ij}(x)|$ only vanishes a finite number of times for $|x| \geqq R$.

We may replace the solution $(\varepsilon_{ij}(x))$ of (4) by a new matrix solution, and at the same time remove one of the zeros of $\Delta(x)$, these being counted with their proper multiplicity. Suppose that $\Delta(x)$ vanishes at $x=a$. In this case we may choose a set of constants $c_{11}, \cdots, c_{n1}$, not all zero, so that

$$\varepsilon_{i1}(a)c_{11} + \cdots + \varepsilon_{in}(a)c_{n1} = 0 \qquad (i=1, 2, \cdots, n). \tag{13}$$

Choose now further $c_{12}, \cdots, c_{n2}, c_{13}, \cdots, c_{n3}, c_{1n}, \cdots, c_{nn}$ in any way so that the determinant $|c_{ij}|$ is not zero. If we multiply the equation (4) through by (c_{ij}) on the right, the matrix $(\varepsilon_{ij}(x))$ is replaced by $(\varepsilon_{ij}(x))(c_{ij})$. From (13) we infer that the elements of the first column of this last matrix all vanish at $x = a$. The right hand side still consists of a matrix of functions rational in character at $x = \infty$.

In this equation

$$(l_{ij}(x))\,[(\varepsilon_{ij}(x))\,(c_{ij})] = (b_{ij}(x))\,(c_{ij})$$

we may remove a factor $x - a$ in the first column of the matrix forming the left-hand side by removing it from the first column of $[(\varepsilon_{ij}(x))\,(c_{ij})]$. This operation changes $[(\varepsilon_{ij}(x))\,(c_{ij})]$ to a matrix $(\varepsilon'_{ij}(x))$ of entire functions, since all the elements in the first column of $[(\varepsilon_{ij}(x))\,(c_{ij})]$ vanish at $x = a$. Furthermore the determinant of $(\varepsilon'_{ij}(x))$ is clearly $\frac{c\Delta(x)}{x-a}$. The same operation of division replaces $(b_{ij}(x))$ by a matrix $(b'_{ij}(x))$ of functions rational in character at $x = \infty$.

Thus we obtain a new matrix solution of (4), and its determinant is the same as that of the first solution except that a factor $x - a$, corresponding to the zero at $x = a$, has been removed.

By a succession of such steps we may remove all of the finite set of zeros of $\Delta(x)$, and finally obtain a solution $(\varepsilon_{ij}(x))$ of (4) whose determinant $\Delta(x)$ never vanishes.

We shall now show that not only may we make $\Delta(x) \neq 0$, but also that by further modification we may reduce $(b_{ij}(x))$ to the form $(a_{ij}(x)x^{\nu_j})$ where $(a_{ij}(x))$ is a matrix of functions analytic at $x = \infty$, of determinant not zero there.*)

To prove the existence of such a solution we note that in (4) we may make the same interchange of columns in $(\varepsilon_{ij}(x))$ and in $(b_{ij}(x))$ to obtain new matrices $(\varepsilon'_{ij}(x))$ and $(b'_{ij}(x))$ which satisfy (4); the determinants $\Delta(x)$ and $\Delta'(x) = |\varepsilon'_{ij}(x)|$ are the same or at most differ only in sign. Likewise we may add to the elements of any column of $(\varepsilon_{ij}(x))$ a sum of the corresponding elements of other columns multiplied by polynomials in x, and thus obtain a matrix $(\varepsilon'_{ij}(x))$ of entire functions whose determinant $\Delta'(x)$ is equal to $\Delta(x)$; the corresponding matrix $(b'_{ij}(x))$ is obtained by the same operation on the columns of $(b_{ij}(x))$.

We may start therefore with any matrix $(b_{ij}(x))$ corresponding to a solution $(\varepsilon_{ij}(x))$ of (4), and be led to new solutions for which $(b_{ij}(x))$ is modified either by a succession of arbitrary interchanges of columns, or

*) Cf. Hensel und Landsberg, Theorie der algebraischen Funktionen einer Variabeln, p. 158—173.

by adding to any given columns a sum of multiples of other columns by arbitrary polynomials. At each step $\Delta(x)$ remains unaltered or changes to $-\Delta(x)$, so that the property $\Delta(x) \neq 0$ is preserved.

These two transformations of $(b_{ij}(x))$ suffice to give it the stated form. In the first place we may put in evidence in each column of $(b_{ij}(x))$ the lowest power of x which may be removed and leave all the elements of the column analytic at $x=\infty$, by writing it in the form $\left(a_{ij}(x)x^{\varkappa_j}\right)$ where the functions $a_{ij}(x)$ are analytic at $x=\infty$ and do not all vanish for any fixed value of j. A proper interchange of columns makes $\varkappa_1 \leqq \varkappa_2 \leqq \cdots \leqq \varkappa_n$. If the determinant of $(a_{ij}(x))$ does not vanish at $x=\infty$ our statement is proved, so that we can limit ourselves to the case when this determinant does vanish at $x=\infty$. Since $|a_{ij}(x)|$ vanishes at $x=\infty$, we may choose quantities $c_1, \cdots, c_n$, not all zero, so that

$$a_{i1}(x)c_1 + \cdots + a_{in}(x)c_n = 0 \quad \text{at} \quad x=\infty \quad (i=1, 2, \cdots, n).$$

Let c_α be the last of the quantities $c_1, \cdots, c_n$ which is not zero. On division of the above equations by c_α, these take the form

$$a_{i1}(x)\,d_1 + \cdots + a_{i\,\alpha-1}(x)\,d_{\alpha-1} + a_{i\alpha}(x) = 0 \quad \text{at} \quad x=\infty \quad (i=1, 2, \cdots, n).$$

Consequently if we multiply the first column of $\left(a_{ij}(x)x^{\varkappa_j}\right)$ by the polynomial $d_1 x^{\varkappa_\alpha - \varkappa_1}$, the second by the polynomial $d_2 x^{\varkappa_\alpha - \varkappa_2}, \cdots$, and add, element for element, the result to the α^{th} column, that column becomes

$$\left(a_{i1}(x)\,d_1 + \cdots + a_{i\alpha}(x)\right) x^{\varkappa_\alpha} \qquad (i=1, \cdots, n).$$

Hence we obtain a matrix $(b'_{ij}(x))$ the same as $(b_{ij}(x))$ except for its α^{th} column where we have a factor $x^{\varkappa_\alpha - 1}$ in evidence since the expression in parenthesis vanishes at $x=\infty$ for all values of i. That is, we have changed $\varkappa_\alpha$ to $\varkappa_\alpha - 1$, leaving $\varkappa_1, \cdots, \varkappa_{\alpha-1}, \varkappa_{\alpha+1}, \cdots, \varkappa_n$ unchanged.

We may proceed by a finite succession of such steps until this process comes to an end. For, the value of the determinant $|b_{ij}(x)|$ is $x^{\varkappa_1 + \cdots + \varkappa_n}|a_{ij}(x)|$ and is unaltered in the process above given. If $\varkappa_1 + \cdots + \varkappa_n$ could be indefinitely *decreased*, then $|b_{ij}(x)|$ would vanish to an arbitrarily high order at $x=\infty$, and hence would be identically zero. When this process comes to an end, $(b_{ij}(x))$ has the form $\left(a_{ij}(x)x^{\varkappa_j}\right)$ where $a_{ij}(x)$ is a matrix of functions analytic at $x=\infty$ of determinant not zero there.

A rearrangement of the columns of $(\varepsilon_{ij}(x))$ and $(b_{ij}(x))$ may be made so that $\varkappa_1 \leqq \varkappa_2 \leqq \cdots \leqq \varkappa_n$, and the second series of modification is complete.

Finally we make the third and last series of modifications which, for some arrangement $k_1, \cdots, k_n$ of $\varkappa_1, \cdots, \varkappa_n$, makes $\varepsilon_{ij}(x)$ vanish at $x=0$ to at least the order $\varkappa_j - k_i$ for all values of i and j such that $k_i < \varkappa_j$. This reduction is to be effected by modifications which do not impair the two properties already secured for the solution of (4).

On account of the first of these properties the determinant $|\varepsilon_{ij}(x)|$ is not zero at $x=0$. Consequently the $(n-1)$-rowed minor formed by striking some α_n^{th} row and the n^{th} column of this determinant does not vanish at $x=0$; likewise an $(n-2)$-rowed minor formed by further striking out some α_{n-1}^{th} row and the $(n-1)^{\text{th}}$ column will not be zero at $x=0$. Continuing in this way we obtain a set of integers $\alpha_1, \cdots, \alpha_n$ forming a permutation of the series of natural integers $1, 2, \cdots, n$ such that the determinant formed by striking out all the last $n-\mu$ columns and the $\alpha_{\mu+1}^{\text{th}}, \cdots, \alpha_n^{\text{th}}$ rows is not zero at $x=0$.

Let us first consider the case in which we can take

$$\alpha_1 = 1,\ \alpha_2 = 2,\ \cdots,\ \alpha_n = n$$

so that none of the series of determinants

$$\varepsilon_{11}(x),\ \varepsilon_{11}(x)\,\varepsilon_{22}(x) - \varepsilon_{12}(x)\,\varepsilon_{21}(x),\ \cdots,\ \Delta(x)$$

vanishes at $x=0$. The argument in the more general case is then disposed of without difficulty.

If we multiply equation (4) through on the right by a matrix

$$(p_{ij}(x)) \equiv \begin{pmatrix} 1 & p_{12}(x) & p_{13}(x) & \cdots & p_{1n}(x) \\ 0 & 1 & p_{23}(x) & \cdots & p_{2n}(x) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdots & 1 \end{pmatrix}$$

where we have $p_{ij}(x) = 0$ for $i > j$, $p_{ii}(x) = 1$, and where $p_{ij}(x)$ for $i < j$ is an arbitrary polynomial of degree $\varkappa_j - \varkappa_i$, the matrix $(\varepsilon_{ij}(x))$ is replaced by another matrix $(\varepsilon_{ij}(x))\,(p_{ij}(x))$ of entire functions of determinant the same as that of $(\varepsilon_{ij}(x))$ since we have $|p_{ij}(x)| = 1$. Moreover

$$(b_{ij}(x)) = \left(a_{ij}(x)x^{\varkappa_j}\right)$$

is replaced by $\left(a_{ij}(x)x^{\varkappa_j}\right)(p_{ij}(x))$ which may be written

$$\begin{pmatrix} x^{\varkappa_1}a_{11}(x), & x^{\varkappa_2}(a_{11}(x)\,p_{12}(x)\,x^{\varkappa_1-\varkappa_2} + a_{12}(x)), & \cdots \\ x^{\varkappa_1}a_{21}(x), & x^{\varkappa_2}(a_{21}(x)\,p_{12}(x)\,x^{\varkappa_1-\varkappa_2} + a_{22}(x)), & \cdots \\ \cdot & \cdot & \cdot \\ x^{\varkappa_1}a_{n1}(x), & x^{\varkappa_2}(a_{n1}(x)\,p_{12}(x)\,x^{\varkappa_1-\varkappa_2} + a_{n2}(x)), & \cdots \end{pmatrix}.$$

This is a matrix of the form $\left(a_{ij}(x)x^{\varkappa_j}\right)$ where every function $a_{ij}(x)$ is analytic at $x=\infty$ and where the determinant $|a_{ij}(x)|$ retains its former value. Thus we have found a further modification which preserves the two properties already secured for the solutions of (5): namely, we may replace $(\varepsilon_{ij}(x))$ by $(\varepsilon_{ij}(x))\,(p_{ij}(x))$ and $(b_{ij}(x))$ by $(b_{ij}(x))\,(p_{ij}(x))$.

Now consider the first element $\varepsilon_{11}(x)\,p_{12}(x) + \varepsilon_{12}(x)$ in the second column and first row of $(\varepsilon_{ij}(x))\,(p_{ij}(x))$. Since $e_{11}(x)$ is not zero at $x=0$, it is clear that by properly choosing the polynomial $p_{12}(x)$ we may make this element vanish to at least the order $\varkappa_2 - \varkappa_1$ at $x = 0$.

Next consider the first two elements

(14) $\varepsilon_{11}(x)p_{13}(x) + \varepsilon_{12}(x)p_{23}(x) + \varepsilon_{13}(x)$ and $\varepsilon_{21}(x)p_{13}(x) + \varepsilon_{22}(x)p_{23}(x) + \varepsilon_{23}(x)$

in the third column of the same matrix. It is possible to make both of these vanish to the degree $\varkappa_3 - \varkappa_2$ by a proper choice of the first $\varkappa_3 - \varkappa_2$ coefficients in the polynomials $p_{13}(x)$ and $p_{23}(x)$. For we have to make vanish the expressions obtained by putting $x = 0$ in the above and in the $\varkappa_3 - \varkappa_2 - 1$ pairs of expressions obtained by sucessive differentiations at $x = 0$. The first pair of equations determines the constant terms in $p_{13}(x)$ and $p_{23}(x)$ since by hypothesis the determinant $\varepsilon_{11}(x)\varepsilon_{22}(x) - \varepsilon_{12}(x)\varepsilon_{21}(x)$ is not zero at $x = 0$. The second pair of equations determines the coefficients of the first powers of x in these polynomials, since these equations are linear in these coefficients with the same determinant as the first pair of equations. Continuing in this way we make both elements (14) vanish to the order $\varkappa_3 - \varkappa_2$ at $x = 0$. Furthermore we may, in addition, make the first of these vanish to the degree $\varkappa_3 - \varkappa_1$, by choosing properly the remaining $\varkappa_2 - \varkappa_1$ arbitrary coefficients in $p_{13}(x)$ of degree $\varkappa_3 - \varkappa_1$; we note that the coefficient $\varepsilon_{11}(x)$ of $p_{13}(x)$ is not zero at $x = 0$.

Determining the functions $p_{ij}(x)$ by a series of steps like the above we may modify $(\varepsilon_{ij}(x))$ so that for any $i < j$ the element $\varepsilon_{ij}(x)$ vanishes to the order $\varkappa_j - \varkappa_i$ at least at $x = 0$. That is, the solution will have the stated third property if we take $\varkappa_1 = k_1, \varkappa_2 = k_2, \cdots, \varkappa_n = k_n$.

If we cannot take $\alpha_1 = 1, \cdots, \alpha_n = n$ we can nevertheless by the same process make the second element in the α_1^{th} row of $(\varepsilon_{ij}(x))(p_{ij}(x))$ vanish to the order $\varkappa_2 - \varkappa_1$, and then make the third elements in the α_1^{th} and α_2^{th} row vanish to the order $\varkappa_3 - \varkappa_1$ and $\varkappa_3 - \varkappa_2$ respectively at $x = 0$, and so on. In other words, if $k_1, k_2, \cdots, k_n$ is the same permutation of $\varkappa_1, \cdots, \varkappa_n$ as $\alpha_1, \cdots, \alpha_n$ is of $1, \cdots, n$, then the modified $(\varepsilon_{ij}(x))$ will be such that whenever we have $\varkappa_j > k_i$ the element $(\varepsilon_{ij}(x))$ will vanish at least to the order $\varkappa_j - k_i$ at $x = 0$.

This completes the reduction of the solution of (4) to a form possessing the stated three properties.

The third and final step is now made at once. On account of the last mentioned property of $(\varepsilon_{ij}(x))$, we may write it in the form $(\varepsilon'_{ij}(x)\, x^{\varkappa_j - k_i})$ where any function $\varepsilon'_{ij}(x)$ will certainly be entire if $\varkappa_j \geqq k_i$; if $\varkappa_j < k_i$ the function $\varepsilon'_{ij}(x)$ is also entire since it is the same as $x^{k_i - \varkappa_j}\varepsilon_{ij}(x)$. The determinant $|\varepsilon'_{ij}(x)|$ is the same as that of $(\varepsilon_{ij}(x))$. Hence $|\varepsilon'_{ij}(x)|$ is a matrix of entire functions of determinant nowhere zero.

The matrix $(x^{\varkappa_j - k_i}\varepsilon'_{ij}(x))$ may be written $(\delta_{ij}x^{-k_j})(\varepsilon'_{ij}(x))(\delta_{ij}x^{\varkappa_j})$ and

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the matrix $\left(a_{ij}(x)x^{\varkappa_j}\right)$ may be written $\left(a_{ij}(x)\right)\left(\delta_{ij}x^{\varkappa_j}\right)$. Consequently from (4) we obtain

$$\left(l_{ij}(x)\right)\left(\delta_{ij}x^{-k_j}\right)\left(\varepsilon'_{ij}(x)\right)\left(\delta_{ij}x^{\varkappa_j}\right)=\left(a_{ij}(x)\right)\left(\delta_{ij}x^{\varkappa_j}\right),$$

whence we find

$$\left(l_{ij}(x)\right)=\left(a_{ij}(x)\right)\left(\varepsilon'_{ij}(x)\right)^{-1}\left(\delta_{ij}x^{k_j}\right).$$

Since the matrix $(\varepsilon'_{ij}(x))^{-1}$ is also a matrix of entire functions of determinant the reciprocal of $|\varepsilon'_{ij}(x)|$ and not zero for any finite x, this equation may be written

$$\left(l_{ij}(x)\right)=\left(a_{ij}(x)\right)\left(e_{ij}(x)x^{k_j}\right)$$

where $(a_{ij}(x))$, $(e_{ij}(x))$, and $k_1, \cdots, k_n$ have all the properties specified in the theorem except that $(a_{ij}(x))$ need not reduce to (δ_{ij}) at $x=\infty$. However if $(a_{ij}(x))$ reduces to (c_{ij}) of determinant not zero it is possible to secure this final property for $a_{ij}(x)$ by replacing $(a_{ij}(x))$ and $\left(e_{ij}(x)x^{k_j}\right)$ by

$$\left(a_{ij}(x)\right)\left(c_{ij}\right)^{-1} \quad \text{and} \quad \left(c_{ij}\right)\left(e_{ij}(x)x^{k_j}\right)$$

respectively. Thus the theorem is demonstrated.

One is naturally led to inquire to what extent the matrices $(a_{ij}(x))$, $(e_{ij}(x))$, and the integers $k_1, \cdots, k_n$ are determined for a given matrix $(l_{ij}(x))$. In the case $n=1$ the explicit solution obtained at the outset is unique.

For $n>1$, a first simple fact is that for a given $k_1, \cdots, k_n$ the matrices $(a_{ij}(x))$, $(e_{ij}(x))$ are uniquely determined in case such matrices exist. In fact, if a second distinct choice $(\bar{a}_{ij}(x))$ and $(\bar{e}_{ij}(x))$ of these matrices were possible, we should have

$$\left(a_{ij}(x)\right)\left(e_{ij}(x)\right)=\left(\bar{a}_{ij}(x)\right)\left(\bar{e}_{ij}(x)\right)$$

or

$$\left(\bar{a}_{ij}(x)\right)^{-1}\left(a_{ij}(x)\right)=\left(\bar{e}_{ij}(x)\right)\left(e_{ij}(x)\right)^{-1}.$$

But the matrix $(\varphi_{ij}(x))$ on the left side of this last equation is a matrix of functions analytic at $x=\infty$ and reducing to (δ_{ij}) at $x=\infty$; and at the same time is a matrix of entire functions, as the right hand side of the same equation shows. Hence $(\varphi_{ij}(x))$ must reduce identically to the unit matrix, and thus we have $(a_{ij}(x))=(\bar{a}_{ij}(x))$ and $(e_{ij}(x))=(\bar{e}_{ij}(x))$ contrary to hypothesis.

Secondly, we observe that if a second choice $(\bar{a}_{ij}(x))$ and $(\bar{e}_{ij}(x))$, of $(a_{ij}(x))$ and $(e_{ij}(x))$, be at hand for a second choice $\bar{k}_1, \cdots, \bar{k}_n$ of $k_1, \cdots, k_n$, then we must have

$$k_1+\cdots+k_n=\bar{k}_1+\cdots+\bar{k}_n.$$

For we have the obvious relation between determinants

$$\psi(x)=x^{k_1+\cdots+k_n}\left|a_{ij}(x)\right|\left|e_{ij}(x)\right|=x^{\bar{k}_1+\cdots+\bar{k}_n}\left|\bar{a}_{ij}(x)\right|\left|\bar{e}_{ij}(x)\right|$$

where $\psi(x)$ is clearly single-valued and analytic in the vicinity of $x = \infty$, and does not vanish there. Also $|a_{ij}(x)|$ and $|\bar{a}_{ij}(x)|$ are analytic at $x = \infty$ and do not vanish there, while $|e_{ij}(x)|$ and $|\bar{e}_{ij}(x)|$ are entire functions nowhere zero in the finite plane. But $\psi(x)$ admits of a separation into factors of this form in only one way, as has been noted. Thus we obtain the stated relation.

For a given matrix $(l_{ij}(x))$, the applications that I have made elsewhere show that the values of $k_1, \cdots, k_n$ are restricted by one or more relations of the above form. The *general* case appears to be that in which there is no other condition but the one obtained above, but in particular cases $k_1, \cdots, k_n$ may be wholly determined. Thus the distribution of the possible sets of values of $k_1, \cdots, k_n$ may be complicated, although for each set there is but one choice of $(a_{ij}(x))$ and $(e_{ij}(x))$. It is also not difficult to show that it is possible to pass from one solution $(a_{ij}(x))$, $(e_{ij}(x))$ and $k_1, \cdots, k_n$ to any other by means of matrices of polynomials in $\frac{1}{x}$ of determinant identically 1. That is, the transcendental part of the problem is completely solved when a single determination of $(a_{ij}(x))$, $(e_{ij}(x))$ and $k_1, \cdots, k_n$ has been effected.