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# ON ISOMONODROMIC DEFORMATIONS OF FUCHSIAN SYSTEMS

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*Dedicated to Professor R. V. Gamkrelidze*

Abstract. Isomonodromic deformations of Fuchsian systems are considered. A description of all possible forms of such isomonodromic deformations is presented.

## 1. Introduction

An isomonodromic deformation of a Fuchsian system

$$\frac{dy}{dx} = \Big(\sum_{i=1}^{n} \frac{B_i^0}{x - a_i^0}\Big) y$$

is an analytic family

$$\frac{dy}{dx} = \Big(\sum_{i=1}^{n} \frac{B_i(a)}{x - a_i}\Big) y$$

of Fuchsian equations with the same monodromy as the initial one for each fixed $a$ and such that

$$B_i(a)\big|_{a^0} = B_i^0.$$

Each isomonodromic deformation is completely determined by a Pfaffian system

$$dy = \omega y$$

on $P^1(\mathbb{C}) \times D(a^0)$ (where $D(a^0)$ is a small disk with the center at $a^0$ on the space $\mathbb{C}^n$ of parameters) such that (Theorem 2)

(i) $\omega = \sum\limits_{i=1}^{n} \dfrac{B_i(a)}{x - a_i}\, dx$ for each fixed $a \in D(a^0)$;

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(ii) $d\omega = \omega \wedge \omega$ (the condition of integrability of the Pfaffian system).

In Sec. 3 we describe the general form of such a form $\omega$. It turns out that (Theorem 3) *each matrix differential* 1*-form* $\omega$ *on* $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^{n}\{x - a_i = 0\}$ *presenting an isomonodromic deformation of the original system has the form*

$$\omega = \sum_{i=1}^{n} \frac{B_i(a)}{x-a_i} d(x-a_i) + \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{t=1}^{m_l} \frac{\gamma_{t,k,l}(a)}{(x-a_l)^t} da_k + \sum_{r=1}^{n} \gamma_r(a) da_r,$$

*where* $\gamma_{t,k,l}(a), \gamma_r(a)$ *are holomorphic in* $D(a^0)$, *and*

$$m_i \le \max\big([\mathrm{Re}(\beta_i^s)] - [\mathrm{Re}(\beta_i^q)]\big),$$

*where the maximum is taken over all differences between integer parts of real parts of all eigenvalues* $\beta_i^j$ *of* $B_i^0$.

The most well-known type of such deformations is the Schlesinger deformation. It is defined by

$$\omega_s = \sum_{i=1}^{n} \frac{B_i(a)}{x-a_i} d(x-a_i).$$

And condition (ii) for this form has the form

$$dB_i(a) = -\sum_{j=1, j\neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j} d(a_i - a_j).$$

This relation is called the *Schlesinger equation.* It was investigated by Schlesinger [16], Jimbo and Miwa [11], B. Malgrange [13], Its and Novokshenov [10], Sibuya [17], and other mathematicians from different points of view.

But Schlesinger deformations do not cover all kinds of isomonodromic deformations. The corresponding examples are also presented in Sec. 3.

## 2. Preliminary information

A Fuchsian system with singular points $a_1, \dots, a_n$ on $P^1(\mathbb{C})$ has the following form:

$$\frac{dy}{dx} = \Big(\sum_{i=1}^{n} \frac{B_i}{x-a_i}\Big) y. \tag{1}$$

Let $\infty$ be nonsingular for (1). This condition is equivalent to the following one:

$$\sum_{i=1}^{n} B_i = 0. \tag{2}$$



Denote by $p$ the size of the matrices $B_i$. System (1) determines the *monodromy representation*

$$\chi : \pi_1(P^1(\mathbb{C}) \setminus \{a_1, \dots, a_n\}, x_0) \longrightarrow \mathrm{GL}(p, \mathbb{C}) \tag{3}$$

as follows.

Let $g_i$ be a loop with the beginning at $x_0$ going around the singular point $a_i$ along a small circle (without going around other singular points). Denote the homotopic class of $g_i$ again by $g_i$. The elements $g_1, \dots, g_n$ generate $\pi_1(P^1(\mathbb{C}) \setminus \{a_1, \dots, a_n\}, x_0)$ and satisfy only one relation $g_1 \cdot \ldots \cdot g_n = e$.

Under an analytic continuation along $g_i$ a fundamental matrix (a base of solutions) $Y(x)$ of system (1) goes to a fundamental matrix $\tilde{Y}(x)$ of the same system. Thus, $Y(x) = \tilde{Y}(x) G_i$ with some constant nondegenerate matrix $G_i$. The correspondence $g_i \mapsto G_i$ generates the monodromy representation (3) of system (1).

Denote by $E_i$ the matrix $1/(2\pi i) \ln G_i$ with eigenvalues $\rho_i^j$ satisfying the condition

$$0 \le \mathrm{Re}\, \rho_i^j < 1. \tag{4}$$

In what follows we need the following statement about the structure of the space of solutions to a Fuchsian system (1) (see [5], [12], or [2], [1] for details).

**Theorem 1.** *For a fundamental matrix* $Y(x)$ *of system* (1) *and for each* $a_i$ *there exists a constant nondegenerate matrix* $S_i$ *such that the following factorization holds in a neighborhood of* $a_i$:

$$Y_i(x) = Y(x) S_i = U_i(x)(x-a_i)^{\Lambda_i}\,(x-a_i)^{E_i'}, \tag{5}$$

*where*:

(i) $U_i(x)$ *is holomorphically invertible at* $a_i$,

(ii) $E_i' = S_i^{-1} E_i S_i$ *has a block diagonal form* $E_i' = \mathrm{diag}(E_i^1, \dots, E_i^k)$ *with upper-triangular matrices* $E_i^j$, *where, in turn, each* $E_i^j$ *has a unique eigenvalue* $\sigma_i^j$ *from the set* $\{\rho_i^s\}$ *defined by* (4) *and* $\sigma_i^j \neq \sigma_i^s$ *if* $j \neq s$,

(iii) $\Lambda_i$ *has the similar block diagonal structure* $\Lambda_i = \mathrm{diag}(\Lambda_i^1, \dots, \Lambda_i^k)$ *with diagonal integer valued matrices* $\Lambda_i^j$ *whose entries* ${}^s\lambda_i^j$ *satisfy the following inequalities*:

$${}^s\lambda_i^j \ge {}^{s+1}\lambda_i^j. \tag{6}$$

Denote by $X$ a space of solutions to our system at $a_i$. Then $X = X_1 \oplus \ldots \oplus X_k$, where $X_s$ is a root subspace of dimension $d_s$ for the monodromy operator $g_i^*$ corresponding to the eigenvalue $\exp(2\pi i \sigma_i^s)$. (The latter decomposition of $X$ implies the block diagonal forms of $E_i$ and $\Lambda_i$ mentioned in the theorem.) Decomposition (5) defines a weighted filtration of every subspace $X_s$ as follows. Let

$$^{1}\lambda_i^s = \ldots = {}^{s_1}\lambda_i^s = \psi_1, \ldots, {}^{s_1+\ldots+s_{t-1}+1}\lambda_i^j = \ldots = {}^{s_t}\lambda_i^j = \psi_t$$

with $\psi_1 > \ldots > \psi_t$. Denote by $X_s^l$ the space spanned by the first columns with numbers $d_1 + \ldots + d_{s-1} + 1, \ldots d_1 + \ldots + d_{s-1} + s_1 + \ldots + s_l$ of the matrix $Y_i(x)$ from (5). The spaces $X_s^l$ form the filtration

$$0 \subset X_s^1 \subset X_s^2 \subset \ldots \subset X_s^t = X_s \tag{7}$$

with the weights $\psi_1, \ldots, \psi_t$ of the space $X_s$, and this filtration is stable under the action of the monodromy operator $g_i^*$ (as immediately follows from (5)). We call such filtrations *Levelt's filtrations*. The matrices $S_i$ are called *connection matrices*, because they connect $Y(x)$ with Levelt's bases $Y_i$ at each $a_i$.

*Remark* 1. Theorem 1 means that *locally each Fuchsian system is completely determined* (*up to a change of the dependent variable holomorphically invertible at* $a_i$) *by its monodromy, and Levelt's filtration at the point* $a_i$.

Let us return to Theorem 1. Denote the numbers ${}^t\lambda_i^j$ (in order of their appearance on the diagonal of $\Lambda_i$) by $\lambda_i^1, \ldots, \lambda_i^p$ respectively. The numbers $\lambda_i^j + \rho_i^j = \beta_i^j$ are called *exponents of system* (1). As immediately follows from decomposition (5), they give asymptotics of the solutions to the Fuchsian system at $a_i$ (see [4]). Indeed, it follows from (5) that

$$y_j = (x - a_i)^{\beta_i^j} u_j + \ldots \tag{8}$$

for the $j$th columns $y_j$ and $u_j$ of the matrices $Y_i$ and $U_i$ respectively. Since $u_j$ is holomorphic at $a_i$ and does not vanish there, we obtain from (5)

$$y_j \sim (x - a_i)^{\beta_i^j} r(x),$$

where for $x$ in some neighborhood of $a_i$ one has

$$c_1 \le |r| < c_2 |\ln(x - a_i)|^{p-1}, \quad c_1 > 0, \; c_2 > 0.$$

It follows from (1), (5) that

$$B_i = \lim_{x \to a_i} ((x - a_i)A(x)) = \lim_{x \to a_i} \left( (x - a_i) \frac{dY_i}{dx} Y_i^{-1} \right) =$$

$$= \lim_{x \to a_i} \left[ (x - a_i) \frac{dU_i}{dx} U_i^{-1} + U_i(\Lambda_i + (x - a_i)^{\Lambda_i} E_i (x - a_i)^{-\Lambda_i}) U_i^{-1} \right] =$$

$$= U_i(a_i)(\Lambda_i + E_i^0) U_i^{-1}(a_i), \tag{9}$$

where $A\,dx = \omega$, $E_i^0 = \lim_{x \to a_i} L_i$, $L_i = (x - a_i)^{\Lambda_i} E_i (x - a_i)^{-\Lambda_i}$ is holomorphic at $a_i$, because of (ii) and of the fact that $E_i$ is upper-triangular.



Equality (9) implies that $\beta_i^j$ *are the eigenvalues of the matrices* $B_i$.

The important characteristic of the behavior of solutions to (1) at $a_i$ are the numbers $\lambda_i^j = [\mathrm{Re}\beta_i^j]$, which are called *valuations of system* (1) *at* $a_i$.

## 3. General form of isomonodromic deformations

Consider a small disk $D(a^0)$ with the center $a^0 = (a_1^0, \ldots, a_n^0)$ on the space $\mathbb{C}^n \setminus \cup_{i \ne j}\{(a_i - a_j) = 0\}$. Consider the space $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n \{(x - a_i) = 0\}$. As well as in Theorem 1 consider the loops $g_1^0, \ldots, g_n^0$ on the space $P^1(\mathbb{C}) \setminus \{a_1^0, \ldots, a_n^0\}$ with the beginning at the point $x_0$. Let $g_1^a, \ldots, g_n^a$ be loops on the space $P^1(\mathbb{C}) \setminus \{a_1, \ldots, a_n\}$ with the beginning at the same point $x_0$ (where $a$ denotes the point $(a_1, \ldots, a_n)$) and such that on the space $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n \{(x - a_i) = 0\}$ each loop $t_a \cdot g_i^a \cdot t_a^{-1}$ is homotopic to $g_i$ (here $t_a$ is the straight line path from the point $(x_0; a_1^0, \ldots, a_n^0)$ to $(x_0; a_1, \ldots, a_n)$.

Consider a family of Fuchsian systems

$$\frac{dy}{dx} = \left( \sum_{i=1}^n \frac{B_i(a)}{x - a_i} \right) y, \qquad \sum_{i=1}^n B_i(a) = 0 \tag{10}$$

depending holomorphically on the parameter $a = (a_1, \ldots, a_n) \in D(a^0)$. Family (10) is called *isomonodromic* (or an *isomonodromic deformation* of the initial Fuchsian system corresponding to $a = a^0$) if for each fixed $a$ the corresponding system from (10) has the same monodromy as for $a = a^0$ (with respect to the homotopic classes of the loops $g_i^a$ and $g_i^0$ respectively). This means that for each value $a$ there exists a fundamental matrix $Y(x, a)$ of the corresponding system from (10) such that this $Y(x, a)$ has the same monodromy matrices (with respect to the corresponding $g_i^a$) for all $a \in D(a^0)$. We call such a family of matrices an *isomonodromic family of matrices*, or simply, an *isomonodromic matrix*.

**Proposition 1.** *For any isomonodromic family* (10) *there exists an isomonodromic family of matrices analytic simultaneously in* $x$ *and* $a$ *on* $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n \{(x - a_i) = 0\}$.

*Proof.* It follows from the theorem of analytic dependence on parameters for a system of ODE that there exists a family $\tilde{Y}(x, a)$ of fundamental matrices to system (10) analytic in $a$ for each fixed $x$. Since it is also an analytic function of $x$, for each fixed $a$, we have that $\tilde{Y}(x, a)$ is an analytic function of its both arguments. Unfortunately, this matrix function is not necessarily isomonodromic. Its monodromy matrices $G_i(a)$ (with respect to the corresponding $g_i^a$) may depend on $a$. Since for every fixed $a$ one has

$$\tilde{Y}(x, a) = Y(x, a) C(a) \tag{11}$$

for some matrix function $C(a)$ with values in $\mathrm{GL}(p,\mathbb{C})$, the following identity holds:

$$C^{-1}(a)G_iC(a) = G_i(a)$$

for the monodromy matrices $G_i$ of an isomonodromic fundamental matrix $Y(x,a)$. Note here that because of analyticity of $\tilde{Y}(x,a)$ the matrices $G_i(a)$ are holomorphic in $a$.

Our goal is to show that the matrix function $C(a)$ can be chosen as an analytic function of $a$. Then the corresponding isomonodromic fundamental matrix $Y(x,a)$ from (11) will be analytic in $a$ and we are done.

Consider a map

$$\gamma : \mathrm{GL}(p,\mathbb{C}) \to \mathrm{GL}(p,\mathbb{C}) \times \ldots \times \mathrm{GL}(p,\mathbb{C}),$$

defined as follows:

$$\gamma(X) = (X^{-1}G_1X), \ldots, (X^{-1}G_nX).$$

Clearly this is the restriction of an analytic (even algebraic) action of the Lie group $\mathrm{GL}(p,\mathbb{C})$ on $\mathrm{GL}(p,\mathbb{C}) \times \ldots \times \mathrm{GL}(p,\mathbb{C})$. Thus:

(i) as an analytic map, at each point $\gamma$ has the same rank (more precisely, its differential at each point has the same rank); see [15];

(ii) the image $\mathrm{Im}\,\gamma$ is an analytic submanifold of $\mathrm{GL}(p,\mathbb{C}) \times \ldots \times \mathrm{GL}(p,\mathbb{C})$ (not necessarily a close submanifold); see again [15];

(iii) a centralizer $H$ of the set $G_1, \ldots, G_n$ is an analytic Lie subgroup of $\mathrm{GL}(p,\mathbb{C})$ (even algebraic closed subgroup, see the same theorem in the cited book).

Thus, the triple $(\mathrm{GL}(p,\mathbb{C}), \mathrm{Im}\,\gamma, \gamma)$ is a locally trivial holomorphic bundle with the fiber $H$ ( see [7]).

So, there is a covering $\{V_i\}$ of $\mathrm{Im}\,\gamma$ such that for each $V_i$ there exists a local holomorphic section of the bundle:

$$\psi_i' : V_i \to \mathrm{GL}(p,\mathbb{C}).$$

We have an analytic map

$$\chi : D \to \mathrm{Im}\,\gamma,$$

which is defined by the monodromy of the analytic matrix $\tilde{Y}(x,a)$. Consider a covering $\{D_i\}$ of $D$ such that $\chi(D_i) \subset V_i$. Now for each $i$ we have the map

$$\psi_i : D_i \to \mathrm{GL}(p,\mathbb{C})$$

such that $\psi_i = \psi_i' \circ \chi$, and, by construction,

$$(\psi_i(a))^{-1}G_i\psi_i(a) = G_i(a)$$

for all $j, i$.



Therefore for all $D_i, D_j$ with nonempty intersections we get

$$g_{ij} = \psi_i(a)(\psi_j(a))^{-1} \in H.$$

Thus we have the cocycle $\{g_{ij}\}$, which determines a principal analytic bundle with the structure group $H$ over $D$. Since each such bundle is topologically trivial (see [8], Corollary 10.3), now by Theorem I of [6] we get that this bundle is analytically trivial. So, for every $D_i$ there exists an analytic map

$$s_i(a) : D_i \to H$$

such that in $D_i \cap D_j$ one has

$$s_i(a) = g_{ij}s_j = \psi_i(a)\psi_j^{-1}(a)s_j(a).$$

Denote by $C^{-1}(a) = \psi_i^{-1}(a)s_i(a)$ the analytic map from the whole $D$ into $\mathrm{GL}(p,\mathbb{C})$. Then, by construction,

$$\tilde{Y}x, a)C^{-1}(a)$$

has the same monodromy $G_i(a_0)$ for all $a$. Thus, it can be chosen as an analytic isomonodromic fundamental matrix. The proposition is proved.[1] □

The matrix function $Y(x,a)$ has some monodromy which due to the analyticity of $Y(x,a)$ depends only on the homotopic classes of the paths on $P^1(\mathbb{C}) \setminus \cup_{i=1}^n\{(x-a_i)=0\}$ with the beginnings at $(x_0, a^0)$ (but this monodromy, in principle, may depend on the location of this initial point). Because of the isomonodromy condition, the monodromy matrices as the functions of the initial point $(x_0, a^0)$ for fixed $x_0$ are locally constant with respect to $a^0$. Due to the definition of the monodromy of a linear system of ODE, these matrices are locally constant with respect to $x_0$ for each fixed $a^0$.

So $Y(x,a)$ defines a monodromy representation

$$\pi_1(P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n\{(x-a_i)=0\}, (x_0,a^0)) \longrightarrow \mathrm{GL}(p;\mathbb{C}). \quad (12)$$

(Note here that the space $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n\{(x-a_i)=0\}$ can be retracted to $P^1(\mathbb{C}) \setminus \{a_1^0, \ldots, a_n^0\}$, so its fundamental group is generated by the same elements $g_1^0, \ldots, g_n^0$.)

And the matrix differential 1-form $\omega = dY(x,a)Y^{-1}(x,a)$ is single valued and can be considered as a form on $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n\{(x-a_i)=0\}$. Indeed, for all $g \in \pi_1(P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n\{(x-a_i)=0\}, (x_0,a^0))$ we have

$$g^*\omega = dg^*Y(x,a)g^*Y^{-1}(x,a) = \big(dY(x,a)\big)G_gG_g^{-1}Y^{-1}(x,a) = \omega.$$

[1] Another version of the proof of the proposition was suggested by D. V. Anosov and it will appear in the Preprint Series of the University of Ulm.

By construction, the Pfaffian system

$$dy = \omega y \tag{13}$$

on $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n \{(x - a_i) = 0\}$ is completely integrable (this means that $d\omega = \omega \wedge \omega$) and for each fixed $a$ it coincides with the corresponding Fuchsian system from (10). As the result of the above considerations, we get the following statement.

**Theorem 2.** *Family* (10) *of Fuchsian systems is isomonodromic if and only if there exists a matrix differential* 1*-form* $\omega$ *on* $P^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^n \{(x - a_i) = 0\}$ *such that*

(i) $\omega = \sum_{i=1}^n \frac{B_i(a)}{x - a_i} dx$ *for each fixed* $a \in D(a^0)$;

(ii) $d\omega = \omega \wedge \omega$.

From the theorem it follows that an isomonodromic family (10) is completely determined by the corresponding form $\omega$ with properties (i), (ii). Our goal now is to describe the general form of such $\omega$. For this purpose consider at first some special case.

For the initial Fuchsian system

$$\frac{dy}{dx} = \left(\sum_{i=1}^n \frac{B_i^0}{x - a_i^0}\right) y, \qquad B_i^0 = B_i(a^0) \tag{14}$$

consider its isomonodromic deformation, which is described by

$$\omega_s = \sum_{i=1}^n \frac{B_i(a)}{x - a_i} d(x - a_i). \tag{15}$$

First of all it is necessary to prove that such a special isomonodromic deformation of the original system does exist. To prove this is the same as to prove that the following problem has a solution:

$$d\omega_s = \omega_s \wedge \omega_s,$$

$$B_i^0 = B_i(a^0), i = 1, \dots, n, \quad \sum_{i=1}^n B_i(a) = 0. \tag{16}$$

But a straightforward calculation shows that condition (ii) for the form $\omega_s$ is equivalent to the following relation:

$$dB_i(a) = -\sum_{j=1, j \neq i}^n \frac{[B_i(a), B_j(a)]}{a_i - a_j} d(a_i - a_j), \tag{17}$$

which is called the *Schlesinger equation*.



Indeed,

$$d\omega_s = d\left(\sum_{i=1}^n \frac{B_i(a)}{x - a_i} d(x - a_i)\right) = \sum_{i=1}^n dB_i \wedge \frac{d(x - a_i)}{x - a_i} =$$

$$= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial B_i(a)}{\partial a_j} d(a_j - x + x)\right) \wedge \frac{d(x - a_i)}{x - a_i} =$$

$$= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial B_i(a)}{\partial a_j}\right) dx \wedge \frac{d(x - a_i)}{x - a_i} -$$

$$- \sum_{i=1}^n \sum_{j=1}^n \frac{\partial B_i(a)}{\partial a_j} \frac{d(x - a_j) \wedge d(x - a_i)}{x - a_i}.$$

On the other hand,

$$\omega_s \wedge \omega_s = \left(\sum_{i=1}^n \frac{B_i(a)}{x - a_i} d(x - a_i)\right) \wedge \left(\sum_{j=1}^n \frac{B_j(a)}{x - a_j} d(x - a_j)\right) =$$

$$= \sum_{i=1}^n \sum_{j=1}^n B_i B_j \frac{d(x - a_i) \wedge d(x - a_j)}{(x - a_i)(x - a_j)} =$$

$$= \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{B_i B_j}{a_i - a_j} \left(\frac{1}{x - a_i} - \frac{1}{x - a_j}\right) d(x - a_i) \wedge d(x - a_j).$$

Since the differentials $dx$, $d(x - a_1), \dots, d(x - a_n)$ are linearly independent, one gets that the equation $d\omega_s = \omega_s \wedge \omega_s$ is equivalent to the system

$$\sum_{j=1}^n \frac{\partial B_i(a)}{\partial a_j} = 0, \qquad i = 1, \dots, n; \tag{18}$$

$$\frac{\partial B_i(a)}{\partial a_j} = \frac{[B_i, B_j]}{a_i - a_j}, \quad i, j = 1, \dots, n; \quad i \neq j. \tag{19}$$

Multiplying the parts of (19) by $da_j$ and adding them for all $j \neq i$, we get

$$dB_i - \frac{\partial B_i}{\partial a_i} da_i = -\sum_{j=1, j \neq i}^n \frac{[B_i, B_j]}{a_i - a_j} d(-a_j).$$

Multiplying both parts of (19) by $-da_i$ and adding them for all $j \neq i$, we have

$$-\sum_{j=1, j \neq i}^n \frac{\partial B_i(a)}{\partial a_j} da_i = -\sum_{j=1, j \neq i}^n \frac{[B_i, B_j]}{a_i - a_j} da_i.$$

If we add the latter two equations and take into account (18), then we get (17).

On the other hand, it is easy to see that Eqs. (18), (19), in turn, follow from (17). Thus, condition (ii) is really equivalent to Eq. (17).

It is well known that Eq. (17) is completely integrable (see, e.g., [13]) (the corresponding verification consists in checking the Frobenius condition). Thus, this special kind of isomonodromic deformations of a Fuchsian system really exists. Let us call the isomonodromic deformations defined by forms (15) the *Schlesinger deformations* (SD).

Denoting by $Y^s(x,a)$ the fundamental matrix of Pfaffian system (13) with the form $\omega=\omega_s$, we conclude that

$$Y^s(\infty),a)\equiv \text{const}. \tag{20}$$

Indeed,

$$d_aY^s(\infty),a)(Y^s(\infty),a)^{-1}=\omega_s(\infty,a)\equiv 0.$$

Thus, relation (20) holds.

Below there follows a simple but a very useful example of a Schlesinger isomonodromic deformation.

**Example 1.** The Fuchsian family

$$\frac{dy}{dx}=\left(\sum_{i=1}^n\frac{B_i}{x-aa_i^0}\right)y$$

with constant coefficient matrices satisfying the equality $\sum_{i=1}^n B_i=0$ is the Schlesinger isomonodromic deformation (with respect to the parameter $a$ of the original system for $a\in D(0)$, where $D(0)$ is a small disk with the center at 0 on $\mathbb{C}$).

Indeed, for our case the Schlesinger equation (17) has the following form:

$$0=dB_i=-\sum_{j=1,j\neq i}^n\frac{[B_i,B_j]}{aa_i^0-aa_j^0}d(aa_i^0-aa_j^0)=$$
$$=-\sum_{j=1}^n[B_i,B_j]\frac{da}{a}=\left[B_i,\sum_{j=1}^n B_j\right]\frac{da}{a}=0,$$

because of the identity $\sum_{i=1}^n B_i=0$.

The following theorem presents a general form of $\omega$ from Theorem 2.



**Theorem 3.** *Each matrix differential 1-form $\omega$ on $P^1(\mathbb{C})\times D(a^0)\setminus\cup_{i=1}^n\{x-a_i=0\}$ responsible for an isomonodromic deformation* (10) *(see Theorem* 2*) of the original system* (14) *has the form*

$$\omega=\sum_{i=1}^n\frac{B_i(a)}{x-a_i}d(x-a_i)+\sum_{l=1}^n\sum_{k=1}^n\sum_{t=1}^{m_l}\frac{\gamma_{t,k,l}(a)}{(x-a_l)^t}da_k+\sum_{r=1}^n\gamma_r(a)da_r, \tag{21}$$

*where $\gamma_{t,k,l}(a),\gamma_r(a)$ are holomorphic on $D(a^0)$,*

$$m_i\le\max([\mathrm{Re}(\beta_i^s)]-[\mathrm{Re}(\beta_i^q)]), \tag{22}$$

*and the maximum is taken over all differences between integer parts of real parts of all eigenvalues of $B_i^0$ (in other words, the maximum is taken over the differences of the valuations of the original system at $a_i$).*

*Proof.* For Pfaffian system (13) with a given $\omega$ consider an isomonodromic fundamental matrix $Y(x,a)$. Let $Y_s(x,a)$ be a fundamental matrix of (13) with $\omega=\omega_s$ of form (15) such that $\omega|_{a^0}=\omega_s|_{a^0}$ and $Y_s(x,a)$ has the same monodromy as $Y(x,a)$. Then the matrix $\Gamma(x,a)=Y(x,a)Y_s^{-1}(x,a)$ is meromorphic in $P^1(\mathbb{C})\times D(a^0)$ with poles at $\cup_{i=1}^n\{x-a_i=0\}$ only. Since for each fixed $a$ it must have form (i) from Theorem 2, $\omega$ has the required form (21). So, it remains to prove inequality (22).

Each isomonodromic deformation preserves the eigenvalues $\beta_i^j$ of the matrices $B_i(a)$ from (10). Indeed, by definition, each isomonodromic deformation must preserve the numbers $\rho_i^j$ (the logarithms of the eigenvalues of the monodromy matrices normalized due to (4)). But due to the continuity of $B_i(a)$, the integer numbers $\lambda_i^j=\beta_i^j-\rho_i^j$ (the valuations of the systems, see Sec. 2) are also preserved by the deformations. So, the two systems under consideration have the same valuations with the same multiplicities for each fixed $a$. But their Levelt's filtrations can be different. Let Levelt's filtrations for the systems (more precisely, for the given fundamental matrices of the systems) be defined by the matrices $\Lambda_1,\dots,\Lambda_n$ and by $R_1(a),\dots,R_n(a)$ and $S_1,\dots,S_n$ respectively. Due to [3], the matrices $S_1,\dots,S_n$ are independent of $a$, while the matrices $R_i$, in principle, depend on $a$. We omit here the proof of the fact that this dependence is holomorphic (this can be deduced from an analog of Sauvage's lemma (see [9]) with a parameter).

Recall here that due to the definition of the connection matrices for each fixed $a$ we have Levelt's decompositions (5)

$$Y(x,a)R_i^{-1}(a)=U_i(x,a)(x-a_i)^{\Lambda_i}(x-a_i)^{E_i'(a)},$$
$$Y_s(x,a)S_i^{-1}=U_i^s(x,a)(x-a_i)^{\Lambda_i}(x-a_i)^{E_i^s},$$

where $E_i'(a)$ and $E_i^s$ have a block diagonal structure and $R_i^{-1}(a)E_i'(a)R_i(a)=S_i^{-1}E_i^sS_i=E_i$ because the monodromies of the matrices $Y(x,a)$ and

$Y_s(x,a)$ are equal by the construction. This means that at each $\{x-a_i=0\}$ the matrix $\Gamma(x,a)$ has the form

$$\Gamma(x,a) = U_i(x,a)(x-a_i)^{\Lambda_i} R_i S_i^{-1} (x-a_i)^{-\Lambda_i} (U_i^s(x,a))^{-1},$$

where the matrix $R(a) = R_i S_i^{-1}$ has the same block diagonal structure as both $E_i'(a)$ and $E_i^s$.

So,

$$Y(x,a) = \Gamma(x,a) Y_s(x,a) = U_i(x,a)(x-a_i)^{\Lambda_i} R(a)(x-a_i)^{E_i^s}$$

and

$$\omega = dY(x,a)Y^{-1}(x,a) = \omega_1 + \omega_2,$$

where

$$\omega_1 =$$
$$= dU_iU_i^{-1} + \frac{U_i}{x-a_i}\left(\Lambda_i + (x-a_i)^{\Lambda_i} R_i E_i R_i^{-1}(x-a_i)^{-\Lambda_i}\right) U_i^{-1} d(x-a_i) =$$
$$= \omega_s + \text{holomorphic form},$$

because $\omega$ has a logarithmic singularity at $a_i$ for each fixed $a_i$ by the condition of the theorem. And

$$\omega_2 = U_i(x-a_i)^{\Lambda_i} d_a R R^{-1} (x-a_i)^{-\Lambda_i} U_i^{-1} \qquad (23)$$

has at most a pole of order $\max_{r,q}(\lambda_i^r - \lambda_j^q)$ at $\{x-a_i=0\}$. □

**Corollary 1.** *Let a form $\omega$ define an isomonodromic deformation* (10)*. Then the Pfaffian system* (13) *with the indicated $\omega$ has regular singularities at the divisor $\cup_{i=1}^n \{x-a_i=0\}$.*

In this sense, all isomonodromic deformations of a Fuchsian system are *regular deformations* (RD).

**Corollary 2.** *Assume that each monodromy matrix of system* (14) *can be transformed to a Jordan normal form consisting of one Jordan block only. Then for each isomonodromic deformation of this system there exists a form $\omega$ which defines this deformation and has the form*

$$\omega = \sum_{i=1}^n \frac{B_i(a)}{x-a_i} d(x-a_i) + \gamma(a) da. \qquad (24)$$

*Proof.* Let isomonodromic deformation (10) of system (14) be determined by a form $\omega$ from (21). For each fixed $a_i$ consider the Jordan filtration (unique for the case under the consideration) for the space of solutions of (10). But any Levelt's filtration, as a filtration, preserved by the monodromy operator $g_i^*$ must be obtained from the Jordan filtration by the union of some of its subspaces. Since the transformation $\Gamma$ defined by the



theorem must preserve the weights, in our case it must also preserve the whole Levelt's filtrations. Therefore the matrices $R_i(a)$ can be chosen equal to the corresponding $S_i$. So the matrix $\Gamma(x,a)$ is holomorphically invertible in $P^1(\mathbb{C}) \times D(a^0)$, and, thus, it depends only on $a$. Therefore

$$\omega = dY(x,a)Y^{-1}(x,a) = d\Gamma(a)\Gamma^{-1}(a) + \Gamma\omega_s\Gamma^{-1}$$

has form (24). Let us call the isomonodromic deformations determined by form (24) *unnormalized Schlesinger deformations* (USD). □

**Corollary 3.** *If for each $i$ the matrix $B_i$ of system* (14) *is nonresonant* (*this means that there are no nonzero integer differences between eigenvalues of $B_i$*)*, then each form $\omega$ defining an isomonodromic deformation of* (14) *has the form* (24).

*Proof.* Under the conditions of the corollary matrices $R(a)$, $d_aR(a)$ from (23) commute with $\Lambda_i$. Indeed, by construction, both $R(a)$ and $\Lambda_i$ have the same block diagonal structure and due to the conditions of the corollary each block $\Lambda_i^j$ is a scalar matrix (see decompositions (5)). Thus, $R(a)$, $d_aR(a)$ commute with $\Lambda_i$; and we get from (23) that

$$\omega_2 = U_i d_a R R^{-1} U_i^{-1}$$

is holomorphic at $\{x-a_i=0\}$. Since the same is true for all $i$, $\omega_2$ does not depend on $x$. □

The corollary shows that the term $\sum_{l=1}^n \sum_{k=1}^n \sum_{t=1}^{m_l} \frac{\gamma_{t,k,l}(a)}{(x-a_l)^t} da_k$ in expression (21) for $\omega$ responsible for the isomonodromic deformation appears only in the resonant case.

Consider the following example of a regular, but not Schlesinger (neither normalized, nor unnormalized), deformation.

**Example 2.** The family

$$\frac{dy}{dx} = \left( \begin{pmatrix} 1 & 0 \\ -\frac{2a_1}{a_1^2-1} & 0 \end{pmatrix} \frac{1}{x+a_1} + \begin{pmatrix} 0 & -6a_1 \\ 0 & -1 \end{pmatrix} \frac{1}{x} + \right.$$
$$\left. + \begin{pmatrix} 2 & 3+3a_1 \\ \frac{1}{1+a_1} & -1 \end{pmatrix} \frac{1}{x-1} + \begin{pmatrix} -3 & -3+3a_1 \\ \frac{1}{a_1-1} & 2 \end{pmatrix} \frac{1}{x+1} \right) y \qquad (25)$$

of Fuchsian equations is isomonodromic, and it is determined by the following completely integrable form:

$$\omega = \begin{pmatrix} 1 & 0 \\ -\dfrac{2a_1}{a_1^2-1} & 0 \end{pmatrix} \frac{d(x+a_1)}{x+a_1} + \begin{pmatrix} 0 & -6a_1 \\ 0 & -1 \end{pmatrix} \frac{dx}{x} +$$
$$+ \begin{pmatrix} 2 & 3+3a_1 \\ \dfrac{1}{1+a_1} & -1 \end{pmatrix} \frac{d(x-1)}{x-1} + \begin{pmatrix} -3 & -3+3a_1 \\ \dfrac{1}{a_1-1} & 2 \end{pmatrix} \frac{d(x+1)}{x+1} +$$
$$+ \begin{pmatrix} 0 & 0 \\ \dfrac{2a_1}{a_1^2-1} & 0 \end{pmatrix} \frac{da_1}{x+a_1}. \quad (26)$$

This $\omega$ is of form (21), but it is not of form (24). Since the term $\gamma(a)da$ is absent here, this example presents the so-called *normalized regular deformation* (NRD).

Every differential form $\omega$ responsible for an isomonodromic deformation (14) has the form $\omega = \omega_s + \sum_{i=1}^n \psi_i(x,a)da_i$ (see (21), (15)). Is the part $\sum_{i=1}^n \psi_i(x,a)da_i$ determined uniquely by the "principal part" $\omega_s = \sum_{i=1}^n \frac{B_i(a)}{x-a_i} d(x-a_i)$ (we call it "principal," because for every fixed $a$ it coincides with the coefficient form of our Fuchsian family)? The answer to this question is negative.

The freedom in choosing $\sum_{i=1}^n \psi_i(x,a)da_i$ is the following: one can replace the isomonodromic matrix $Y(x,a)$ (see above), describing our deformation by $Y(x,a)R(a)$, where $R(a)$ belongs to the centralizer of the monodromy matrices $G_1, \dots, G_n$ of our system (more precisely, of the matrix $Y(x,a)$). Clearly this change does not influence $\omega_s$, but it probably changes $\sum_{i=1}^n \psi_i da_i$. Nevertheless the following statement holds.

**Proposition 2.** *Let the monodromy of isomonodromic Fuchsian family* (10) *be irreducible. Then the corresponding differential form $\omega$ responsible for the deformation* (*see Theorem* 3) *is determined uniquely by* (10) *up to addition of a term $df(a)f^{-1}(a)I$, where $f(a)$ is an analytic function.*

*Proof.* Let $Y(x,a)$ be an isomonodromic matrix of our family. Then due to conditions of the proposition and due to Shur's lemma, any other isomonodromic matrix $Y'(x,a)$ must have the form $Y'(x,a) = Y(x,a)R(a)$ with $R(a) = f(a)I$ for some scalar function $f(a)$. Thus,

$$\omega' = dY'(x,a)(Y')^{-1}(x,a) = dY(x,a)Y^{-1}(x,a) +$$
$$+ Y(x,a)dR(a)R^{-1}(a)Y^{-1}(x,a) = \omega + df(a)f^{-1}(a)I.$$



If the monodromy of system (10) is reducible, it may have nontrivial symmetries of the form $\Gamma(x,a) = Y(x,a)C(a)Y^{-1}(x,a)$ with $C(a)$ from the centralizer of the monodromy matrices of $Y(x,a)$ (even for the case where the equation itself is irreducible). This leads to different differential forms responsible for the deformation (10). Some of these forms can be simple (for example, Schlesinger forms), others can have additional poles at $\cup_{i=1}^n \{x - a_i = 0\}$ (of orders which are not greater than Theorem 3 says).

For the resonant case an interesting question is how to get the simplest possible form of $\omega$ for a given system (10).

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