

RELATIVE COHOMOLOGY AND VOLUME FORMS

J.-P. FRANCOISE

~~Université de Paris-Sud, Mathématiques, Orsay, France~~
Paris VI, France

We consider here the classification of couples (f, ω) , where f is a germ of a function $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$, and ω is a germ of an n -form, modulo the group of germs of diffeomorphisms of $(\mathbb{C}^n, 0)$.

For this classification, the relative cohomology of f appears to be the essential tool. E. Brieskorn and M. Sebastiani have proved that, in the isolated singularity case, these relative cohomology groups are free $\mathbb{C}\{f\}$ -modules of rank μ , the Milnor number of f . In the case where f is quasi-homogeneous, we establish the convergence of an effective algorithm which allows to compute the associated characteristic series.

This paper is a survey about questions which have been investigated previously ([F₁], [F₂], [F₃]).

The author expresses his gratitude to S. Denkowska and S. Łojasiewicz for inviting him to participate in the meeting on Singularities in Warsaw.

We will use the following notation:

\mathcal{O} : the local ring of germs of analytic functions at $0 \in \mathbb{C}^n$.

\mathfrak{m} : its maximal ideal.

Ω^k : the \mathcal{O} -module of germs at $0 \in \mathbb{C}^n$ of holomorphic k -forms.

$\text{Diff}_0(n)$: the group of germs of diffeomorphisms tangent to the identity at the order two.

Γ_P : the orbit of $P \in \mathcal{O}$ under $\text{Diff}_0(n)$.

$G = \Omega^n/dP \wedge d\Omega^{n-2}$ considered as a $\mathbb{C}\{t\}$ -module where t acts by multiplication by P .

$\Omega_*^n = \{\omega \in \Omega^n \mid \omega(0) \neq 0\}$.

We will call *volume forms* the elements of Ω_*^n .

All the coordinate systems are supposed to be holomorphic. By coordinate systems we actually always mean germs of coordinate systems.

1. The case of a Morse function

We assume here that P is a germ of a Morse function, hence there is a holomorphic coordinate system x around $0 \in \mathbb{C}^n$ so that $P = \sum_{i=1}^n x_i^2$. Let Γ_P be the set $\{f \in \mathcal{O} \mid f \in (P) + \mathfrak{m}^3\}$.

J. Vey and independently V. Guillemin ([V], [G]) proved the following:

THEOREM 1.1. *Given $(f, \omega) \in \Gamma_P \times \Omega_{*}^n$, there is a system of local coordinates $y = (y_1, \dots, y_n)$ such that f is a power series $\psi^*(P)$ where $P = \sum_{i=1}^n y_i^2$ and $\omega = dy_1 \wedge \dots \wedge dy_n$.*

Such a coordinate system is not unique, but the series ψ^* is a characteristic of the couple (f, ω) and can be interpreted geometrically by means of integrals on vanishing cycles of the level hypersurfaces $P = t$.

Let us observe that in dimension two we get Birkhoff's statement on the local normal form of a quadratic Hamiltonian system in the group of analytic symplectic transformations. We know that such a normal form exists in the formal symplectic category in higher dimensions but a theorem of C. L. Siegel shows its generic divergence ([Si]).

So Theorem 1.1 implies, by contrast, that everything goes well in the volume-preserving case for the classification of generic germs of functions. It can be interpreted in the following way.

Let f be a Morse germ, x a system of coordinates around $0 \in \mathbb{C}^n$, and $\omega = dx_1 \wedge \dots \wedge dx_n$ the associated volume form. Then there is a coordinate system $y = (y_1, \dots, y_n)$ so that $f = \psi\left(\sum_{i=1}^n y_i^2\right)$ and $dy_1 \wedge \dots \wedge dy_n = dx_1 \wedge \dots \wedge dx_n$. Here we will consider another version of this theorem.

THEOREM 1.2 *Given $(f, \omega) \in \Gamma_P \times \Omega_{*}^n$, there is a coordinate system $y = (y_1, \dots, y_n)$ so that $f = \sum_{i=1}^n y_i^2$ and $\omega = \psi(P) dy_1 \wedge \dots \wedge dy_n$ where $\psi(t) \in \mathbb{C}\{t\}$ is a characteristic of the couple (f, ω) .*

Such a theorem is, of course, no longer a statement about normal forms and it must be read in the following way: For each orbit of $\Gamma_P \times \Omega_{*}^n$ under $\text{Diff}_0(n)$, there is a simple representative (which we call a *local model*) of the type $(P, \psi(P)dy)$. We will just indicate how Theorem 1.2 is a consequence of Theorem 1.1 since the result can be deduced as a corollary of our general method presented in Section 2.

Proof. 1.1 \Leftrightarrow 1.2.

Let $(f, \omega) \in \Gamma_P \times \Omega_{*}^n$. Theorem 1.1 provides us with a local coordinate

system $x = (x_1, \dots, x_n)$ so that $f = \psi^*(P) = P + \dots$ and $\omega = dx_1 \wedge \dots \wedge dx_n$ with $P = \sum_{i=1}^n x_i^2$.

Let us write $\psi^*(P)^{1/2} = P^{1/2} u(P)$ with $u \in C\{t\}$, $u(0) = 1$. Then we define a new coordinate system by the relations $x'_i = x_i u(P)$. Clearly, in this new coordinate system we have

$$P' = \sum_{i=1}^n x_i'^2 = \left(\sum_{i=1}^n x_i^2\right) u^2(P) = \psi^*(P) = f.$$

Furthermore, the relations $x'_i = x_i u(P)$ turn into $x_i = x'_i v(P')$ with $v \in C\{t\}$, $v(0) = 1$. Hence,

$$\omega = dx_1 \wedge \dots \wedge dx_n = [v(P')^n + 2v(P')^{n-1} v'(P') P'] dx'_1 \wedge \dots \wedge dx'_n,$$

and

$$\omega = \psi(P') dx'_1 \wedge \dots \wedge dx'_n,$$

and we get Theorem 1.2.

Now, conversely, if we have a coordinate system $x' = (x'_1, \dots, x'_n)$ so that $f = P' = \sum_{i=1}^n x_i'^2$ and $\omega = \psi(P) dx'_1 \wedge \dots \wedge dx'_n$, then we solve the differential equation for w :

$$\frac{2}{n} t w'(t) + w(t) = \psi(t).$$

If we fix $w(0) = 1$, the variation of the constants leads to the following formula:

$$w(t) = t^{-n/2} \int_0^t \frac{1}{2} n \tau^{(n-2)/2} \psi(\tau) d\tau + 1$$

which is analytic in t .

Then, if we write $v = w^{1/n}$ and $x_i = x'_i v(P')$ in the coordinate system $x = (x_1, \dots, x_n)$, we conclude that $f = P'$ is a power series in $P = \sum_{i=1}^n x_i^2$, and furthermore we have $\omega = dx_1 \wedge \dots \wedge dx_n$, which is the statement of Theorem 1.1. □

2. The general isolated case

In this section, P will denote an element of \mathcal{O} which has an isolated singularity at $0 \in C^n$. Let us denote by χ the Lie algebra of germs of holomorphic vector fields at $0 \in C^n$.

We wish to describe the simplest possible local model in each orbit of $\Gamma_P \times \Omega_*^n$ under the action of $\text{Diff}_0(n)$.

Let us consider a couple $(f, \omega) \in \Gamma_P \times \Omega_*^n$. After a first change of local coordinates, we can assume that $f = P$ and write $\omega = a(x)dx$, $a(0) = a \neq 0$. Now in order to find a local model for the couple, we can try to reduce ω to a simple form with a diffeomorphism which preserves the function P .

So we need to understand better the action of $I(P)$, the isotropy subgroup of P for the group $\text{Diff}_0(n)$. I am indebted to J. Martinet for the following

PROPOSITION 2.1. *Let $\omega \in \Omega_*^n$. The tangent space $T(\omega)$ at ω to the orbit of ω for the group $I(P)$ is $dP \wedge d\Omega^{n-2}$.*

Proof. By definition, $T(\omega)$ is the set of Lie derivatives $L_X \omega$, where X is an element of $\mathcal{J}(P) \subset \chi$, the Lie algebra of vector fields which preserve P :

$$T(\omega) = \{L_X \omega \mid X \in \chi, L_X P = 0\}.$$

Let $X \in \mathcal{J}(P)$. Then $dP \wedge i_X \omega = (L_X P)\omega = 0$, and the theorem of De Rham ([D]) implies the existence of $\eta \in \Omega^{n-1}$ such that $i_X \omega = dP \wedge \eta$; and by the formula of H. Cartan, we have

$$L_X \omega = dP \wedge d(-\eta).$$

Conversely, let η be an element of Ω^{n-1} . Since ω is a volume form, there is an element X of χ such that

$$i_X \omega = dP \wedge d(-\eta).$$

Actually $X \in \mathcal{J}(P)$ since we have

$$0 = dP \wedge i_X \omega = (L_X P)\omega,$$

and finally

$$L_X \omega = dP \wedge d\eta. \quad \square$$

This proposition is not necessary in what follows but we can see from it why the module $G = \Omega^n/dP \wedge d\Omega^{n-1}$ is involved in our problem.

Now we use J. Moser's method for the conjugacy problem of forms ([Mo]).

THEOREM 2.1. *Let (ω, ω') be elements of Ω_*^n such that $\omega - \omega' \in dP \wedge d\Omega^{n-2}$. There is a $\varphi \in I(P)$ so that $\varphi^* \omega' = \omega$.*

Proof. Let us write $\omega - \omega' = dP \wedge d\eta$ and introduce the path $\omega_t = \omega + t(\omega' - \omega)$, $t \in [0, 1]$.

We look for a path of diffeomorphisms $\varphi_t \in \text{Diff}_0(n)$ which satisfies

$$(1) \quad \varphi_t^* \omega_t = \omega$$

and comes from the integration of a path of vector fields X_t . The equation (1) implies that

$$\varphi_t^*(L_{X_t} \omega_t + \omega_t) = 0,$$

and so

$$(2) \quad L_{X_t} \omega_t = dP \wedge d\eta.$$

Let us then define X_t by

$$(3) \quad i_{X_t} \omega_t = -dP \wedge \eta;$$

such an X_t satisfies (2) and the family of diffeomorphisms φ_t that we get by integration of X_t , fixing $f_0 = \text{Id}$, satisfies (1).

We remind the reader of the fact proved for instance in [To], that we can integrate X_t on a given ball for all t because X_t vanishes at $0 \in \mathbb{C}^n$.

Now (3) implies that

$$0 = dP \wedge i_{X_t} \omega_t = (L_{X_t} \cdot P) \omega_t.$$

So $X_t \in \mathcal{J}(P)$ and the φ_t obtained by integration preserve P . We now take $\varphi = \varphi_1$ to prove the result required. \square

We will use

THEOREM 2.3. ([B], [Se], [Ma₂]). *The $\mathbb{C}\{t\}$ -module $G = \Omega^n/dP \wedge d\Omega^{n-2}$ is free of rank μ .*

After a choice of $\gamma_\alpha \in \Omega^n$ ($\alpha \in A$, A an appropriate index set so that the classes of γ_α modulo $dP \wedge d\Omega^{n-2}$ give a $\mathbb{C}\{t\}$ -basis of G), we can prove the following

COROLLARY. *Given a couple $(f, \omega) \in \Gamma_P \times \Omega^n_*$, there is a $\varphi \in \text{Diff}_0(n)$ such that $\varphi^* f = P$ and*

$$\varphi^* \omega = \sum_{\alpha \in A} \psi_\alpha(P) \gamma_\alpha$$

where $\psi_\alpha(t) \in \mathbb{C}\{t\}$.

We now consider the question of the unicity of the functions ψ_α of the corollary.

In the following, we use the notations:

$\text{Diff}_0^\wedge(n)$: the group of formal transformations with n variables whose first order jet is the identity;

$\hat{\chi}(n)$: the Lie algebra of formal vector fields in n variables;

$\hat{\Omega}^k$: the formal completion of Ω^k ;

\hat{G} : $\hat{\Omega}^n/dP \wedge \hat{\Omega}^{n-2}$ regarded as a $\mathbb{C}[[t]]$ -module.

An analytic function P is considered as an element of $\hat{\mathcal{O}}$, the ring of formal power series, through the injection $\mathcal{O} \hookrightarrow \hat{\mathcal{O}}$.

LEMMA 2.4. *Let φ be an element of $I(P)$; φ can be interpolated by a one-parameter group of formal transformations which preserve the function P .*

Proof. We want to construct a one-parameter group φ_t contained in $\text{Diff}_0^\wedge(n)$ such that $\varphi_1 = \varphi$ and $\varphi_t^* P = P$.

We write $\varphi = \{\varphi_i; i = 1, \dots, n\}$, the component functions of φ , and

$$\varphi_i = x_i + \sum_j \sum_{|\beta|=j} \varphi_{i,\beta} x^\beta,$$

their Taylor developments at $0 \in C^n$. We construct $\varphi_t = \{\varphi_{t,i}; i = 1, \dots, n\}$,

$$\varphi_{t,i} = x_i + \sum_j \sum_{|\beta|=j} \varphi_{i,\beta}(t) x^\beta,$$

as the solutions of the differential equation (cf. [St])

$$(4) \quad \varphi'_t = \varphi'_0 \circ \varphi_t$$

with the boundary values $\varphi_0 = \text{Id}$ and $\varphi_1 = \varphi$.

It we assume that the $\varphi_{i,\beta}(t)$ are known by induction up to $|\beta| = j \leq k-1$, then we can determine the $\varphi_{i,\beta}(t)$ with $|\beta| = k$. The equation (4) imposes a condition

$$(5) \quad \varphi'_{i,\beta}(t) = \varphi'_{i,\beta}(0) + f_{i,\beta}(t)$$

where the $f_{i,\beta}(t)$ are known by induction and vanish at 0.

The equation (5) can be integrated to give

$$\varphi_{i,\beta}(t) = \varphi'_{i,\beta}(0)t + \int_0^t f_{i,\beta}(\tau) d\tau.$$

We can check that $\varphi_{i,\beta}(0) = 0$ and we need to choose $\varphi'_{i,\beta}(0)$ such that

$$\varphi_{i,\beta}(1) = \varphi_{i,\beta}.$$

We then observe that the coefficients $\varphi_{i,\beta}(t)$ are polynomials in t , and from the fact that φ_t is an interpolation of φ , $\varphi_t^* P - P = 0$ for all integers t .

If we fix a number k , the homogeneous part of degree k of $\varphi_t^* P - P$ is a polynomial in t which must vanish for all integer values of t and consequently this polynomial must vanish identically. We deduce that $\varphi_t^* P = P$ for all real values of t . \square

Let us denote by $\hat{X} \in \hat{\chi}$ the formal vector field which generates the group φ_t . From the preceding lemma we deduce that $L_{\hat{X}} \cdot P = 0$. We do not know if φ can be interpolated by an analytic one-parameter group but this has no importance because of the following

THEOREM 2.5. \hat{G} is isomorphic to $G \otimes_{C\{t\}} C[[t]]$.

This is a theorem of Bloom and Brieskorn who proved it using the Hironaka desingularisation. In [Ma₂], B. Malgrange gave another proof

using the regularity of the Gauss–Manin connection and his theorem on the analytical index.

In the special case where P is quasi-homogeneous, it is a simple consequence of our Section 3.

So if $[\gamma_\alpha]$ is a basis of G as a $C\{t\}$ -module, then $[\gamma_\alpha]$ is a basis of \hat{G} as a $C[[t]]$ -module and any $\omega \in \hat{\Omega}^n$ can be written in a unique way as

$$(6) \quad \omega = \sum_{\alpha \in A} \hat{\psi}_\alpha(P) \gamma_\alpha + dP \wedge d\hat{\eta}$$

where $\hat{\psi}_\alpha \in C[[t]]$ and $\hat{\eta} \in \hat{\Omega}^{n-2}$.

We can prove

PROPOSITION 2.6. *Let $\varphi \in I(P)$ and $\omega \in \Omega^n$. There is an $\eta \in \Omega^{n-2}$ such that*

$$\omega - \varphi^* \omega = dP \wedge d\eta.$$

Proof. We interpolate φ by the one-parameter formal group $\varphi_t = \exp t\hat{X}$ constructed in Lemma 2.4. We get

$$(7) \quad \omega - \varphi^* \omega = \int_0^1 \frac{d}{dt} \varphi_t^* \omega dt = \int_0^1 \varphi_t^* (L_{\hat{X}} \omega) dt.$$

Since we have $L_{\hat{X}} \hat{P} = 0$, we get

$$dP \wedge i_{\hat{X}} \omega = 0,$$

and De Rham's theorem implies that there is a $\hat{\varrho} \in \hat{\Omega}^{n-2}$ such that

$$i_{\hat{X}} \omega = dP \wedge \hat{\varrho}.$$

Now φ_t preserves P , so we have

$$(8) \quad \omega - \varphi^* \omega = dP \wedge d\left(\int_0^1 \varphi_t^* \hat{\varrho} dt\right) = dP \wedge d\hat{\eta}.$$

We can write $\omega - \varphi^* \omega = \sum_{\alpha \in A} \psi_\alpha(P) \gamma_\alpha + dP \wedge d\eta$, but this can be read as a decomposition in $\hat{\Omega}^n$. From (8) and the unicity of the decomposition (6) we have $\psi_\alpha = 0$ for all $\alpha \in A$, and

$$\omega - \varphi^* \omega = dP \wedge d\eta. \quad \square$$

We can summarize the results in

THEOREM 2.7. *There is a bijection between $\text{Diff}_\delta^\wedge(n) \setminus \Gamma_P \times \Omega_*^n$ and $C\{t\}^\mu$; the μ series ψ_α which are associated to the orbit of a couple $(f, \omega) \in \Gamma_P \times \Omega_*^n$ can be obtained in the following way: we choose a coordinate system in which f can be written as P and then we decompose in this system*

$$\omega = \sum_{\alpha \in A} \psi_\alpha(P) \gamma_\alpha + dP \wedge d\eta.$$

We have to prove that the functions ψ_α are effectively independent of the coordinate system that we use to write f as P , and only depend on $[\gamma_\alpha]$.

If we have two coordinate systems in which f can be written as P , then the diffeomorphism φ which transforms one system into another has to preserve P . In the first coordinate system, we write

$$[\omega] = \sum_{\alpha \in A} \psi_\alpha(P) [\gamma_\alpha].$$

In the second one, we have

$$[\varphi^* \omega] = \sum_{\alpha \in A} \psi'_\alpha(P) [\gamma_\alpha],$$

and Proposition 2.6 implies that

$$\psi_\alpha(P) = \psi'_\alpha(P), \quad \text{for all } \alpha \in A.$$

3. Algorithm of construction of the characteristic series for the quasi-homogeneous case

M. Sebastiani [Se] proved the conjecture of Brieskorn that if $P \in \mathcal{O}$ has an isolated singularity whose Milnor number is μ , then $G = \Omega^n/dP \wedge d\Omega^{n-2}$ is a free $\mathbb{C}\{t\}$ -module of rank μ where the action of t is the multiplication by P .

Here we give a new constructive proof of this result in the case where P is quasi-homogeneous, based on B. Malgrange's theorem on "privileged neighborhoods".

First of all we make precise some more notations.

Let

$$H = \sum_{i=1}^n m_i x_i \frac{\partial}{\partial x_i}, \quad m = (m_1, \dots, m_n) \in (\mathbb{Q}_+^*)^n,$$

be the weight vector field of P . So we have

$$L_H \cdot P = H \cdot P = P.$$

We choose monomials $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ ($\alpha \in A \subset \mathbb{N}^n$) whose classes modulo J_P , the Jacobian ideal of P , give a \mathbb{C} -basis of \mathcal{O}/J_P .

Let $d^n x = dx_1 \wedge \dots \wedge dx_n$ denote the standard volume form.

We can state

THEOREM 3.0. *The μ classes of $x^\alpha d^n x$ modulo $dP \wedge d\Omega^{n-2}$ give a $\mathbb{C}\{t\}$ -basis of the module G .*

We first prove that G is finitely generated as a $\mathbb{C}\{t\}$ -module.

3.1. G is finitely generated

3.1.1. *The formal case.* Let us denote in this section by $\hat{\mathcal{O}} = \mathbb{C}[[x_1, \dots, x_n]]$ the ring of formal series in the indeterminate $x = (x_1, \dots, x_n)$, $\hat{\Omega}^k$

the $\hat{\mathcal{O}}$ -module of k -forms with coefficients in $\hat{\mathcal{O}}$ and \hat{J}_P the Jacobian ideal of P considered as a part of $\hat{\mathcal{O}}$.

The choice of monomials x^α ($x \in A \subset N^n$) whose classes modulo \hat{J}_P give a \mathbb{C} -basis of $\hat{\mathcal{O}}/\hat{J}_P$ determines a decomposition of $\hat{\Omega}^n$; for $f \in \hat{\mathcal{O}}$, we write

$$(9) \quad \begin{aligned} f d^n x &= \sigma(f) d^n x + dP \wedge \pi \\ &= \sum_{\alpha \in A} \sigma_\alpha(f) x^\alpha d^n x + dP \wedge \pi \quad (\sigma_\alpha(f) \in \mathbb{C}). \end{aligned}$$

We begin with

LEMMA 3.1.1. For $H = \sum_{i=1}^n m_i x_i \frac{\partial}{\partial x_i}$ write $M = \sum_{i=1}^n m_i$. Then $H + M: F \mapsto H \cdot F + Mf$ is a bijection of $\hat{\mathcal{O}}$ onto $\hat{\mathcal{O}}$.

Proof. $(H + M)x^\beta = (\langle \beta, m \rangle + M)x^\beta$ and $\langle \beta, m \rangle + M$ is never zero. \square

We now consider $\hat{G} = \hat{\Omega}^n/dP \wedge d\hat{\Omega}^{n-2}$ as a $\mathbb{C}[[t]]$ -module where the action of t is multiplication by P . Then we can state

PROPOSITION 3.1. \hat{G} is a $\mathbb{C}[[t]]$ -module of finite type and the classes of $x^\alpha d^n x$ ($\alpha \in A$) give a system of generators.

Proof. For every element $f \in \hat{\mathcal{O}}$, we have to find $\varphi_\alpha \in \mathbb{C}[[t]]$ and $\xi \in \hat{\Omega}^{n-2}$ such that

$$f d^n x = \sum_{\alpha \in A} x^\alpha \varphi_\alpha(P) d^n x + dP \wedge d\xi.$$

In order to do this, we write

$$f d^n x = \sigma(f) d^n x + dP \wedge \pi_1$$

(where we choose a π_1 which is determined obviously modulo $dP \wedge \hat{\Omega}^{n-2}$ ($[[D]]$)). Then we have

$$d\pi_1 = u_1 d^n x;$$

and if $\varphi_1 = (H + M)^{-1} u_1$, we have

$$d\pi_1 = d(i_{\varphi_1 H} d^n x).$$

We then produce a ζ_2 such that

$$\pi_1 = i_{\varphi_1 H} d^n x + d\zeta_2,$$

and we have

$$f d^n x = \sigma(f) d^n x + dP \wedge i_{\varphi_1 H} d^n x + dP \wedge d\zeta_2,$$

$$f d^n x = \sigma(f) d^n x + P_{\varphi_1} d^n x + dP \wedge d\zeta_2.$$

We then write

$$\varphi_1 d^n x = \sigma(\varphi_1) d^n x + dP \wedge \pi_2, \quad d\pi_2 = u_2 d^n x,$$

$$\varphi_2 = (H + M)^{-1} u_2, \quad \pi_2 = i_{\varphi_2 H} d^n x + d\zeta_3$$

and so on ...

For the p -th iterate of this process, we get

$$f d^n x = \sum_{j=0}^p P^j \sigma(\varphi_j) d^n x + P^p dP \wedge \pi_{p+1} + dP \wedge d\left(\sum_{j=1}^p P^{j-1} \zeta_{j+1}\right)$$

where we write $\varphi_0 = F$.

It is easy to check the convergence in Krull's topology and we get the result. \square

We have to emphasize the fact that at each step of the iteration process the choice of π_j has some arbitrariness. Anticipating somewhat on what will follow, let us note that if we identify $\hat{\Omega}^n$ with $\hat{\mathcal{O}}$ and $\hat{\Omega}^{n-1}$ with $\hat{\mathcal{O}}^n$, then the choice of π_j can be done when we have a section λ of $u: dP \wedge \dots: \hat{\mathcal{O}}^n \rightarrow \hat{\mathcal{O}}$. In the analytic case, we need to precise the choice of a section.

3.1.2. *The analytic case.* For $r = (r_1, \dots, r_n) \in \mathbf{R}_+^{*n}$ let us introduce the polycylinder

$$D(r) = \{x \in \mathbf{C}^n \mid |x_i| \leq r_i\}.$$

For $f = \sum_{\beta} a_{\beta} x^{\beta} \in \mathcal{O}$, we write

$$|f|_r = \sum_{\beta} |a_{\beta}| r^{\beta}.$$

We denote by \mathcal{O}_r the set of elements of \mathcal{O} such that $|f|_r < \infty$. For $f = (f_1, \dots, f_p) \in \mathcal{O}_r^p$, we write

$$|f|_r = \sum_{i=1}^p |f_i|_r.$$

Let Ω_r^k denote the \mathcal{O}_r -module of exterior k -forms with coefficients in \mathcal{O}_r . We identify Ω_r^n (resp. Ω_r^{n-1}) with \mathcal{O}_r (resp. with \mathcal{O}_r^n).

Let u be an \mathcal{O}_r -linear mapping from \mathcal{O}_r^n to \mathcal{O}_r . A section λ of u is a \mathbf{C} -linear mapping from \mathcal{O}_r to \mathcal{O}_r^n such that $u = u\lambda u$.

We say that λ is *adapted* ([Ma₁]) to $D(r)$ if λ is a continuous map from the Banach space \mathcal{O}_r to the Banach space \mathcal{O}_r^n , i.e. if there is a $C_r > 0$ such that

$$|\lambda f|_r \leq C_r |f|_r, \quad \text{for all } f \in \mathcal{O}_r^n.$$

The theorem on the privileged neighborhoods of B. Malgrange has the following consequence:

PROPOSITION 3.2. *Given u , there is a section λ such that the set of $D(r)$ to which λ is adapted gives a fundamental system of neighborhoods of the origin.*

In the following, we choose such a section to write the iteration process that we described in Section 3.1.1. We now need to check the norms at each step of the computation.

LEMMA 3.1.2. *Let $m = \min_{j=1, \dots, n} (m_j)$, $r_0 = \min_{j=1, \dots, n} (r_j)$. If $\varphi \in \mathcal{O}_r$ and $\pi \in \Omega_r^{n-1} \simeq \mathcal{O}_r^n$ are such that $d(i_{\varphi_H} d^n x) = d\pi$, then we have*

$$|\varphi|_r \leq \frac{1}{mr_0} |\pi|_r.$$

Proof. We can write with $d\hat{x}_j = dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_n$

$$\pi = \sum_{j=1}^n \pi_j d\hat{x}_j, \quad \pi_j = \sum_{\beta} a_{\beta}^j a^{\beta};$$

then we have

$$\varphi = \sum_{j=1}^n \sum_{\beta} \frac{\beta_j}{\langle \beta, m \rangle - m_j + M} a_{\beta}^j x^{\beta - I_j},$$

with $I_j = (0, \dots, 1, \dots, 0)$ where the 1 is at the j -th position. \square

LEMMA 3.1.3. *Let us denote $R = \sum_{j=1}^n r_j$ and let $\theta \in \Omega_r^{n-1}$ be such that $d\theta = 0$; then there is a $\zeta \in \Omega_r^{n-2}$ such that $\theta = d\zeta$ and $|\zeta|_r \leq R|\theta|_r$.*

This is an obvious consequence of any proof of Poincaré's theorem. We can now state

THEOREM 3.2. *Let $D(r)$ be a polycylinder to which λ is adapted. Then there is a r' such that $D(r') < D(r)$ and for all $f \in \mathcal{O}_r$, there are $\xi \in \Omega_r^{n-2}$ and $\varphi_{\alpha}(P) \in \mathcal{O}_r$ satisfying the equation*

$$f d^n x = \sum_{\alpha \in A} x^{\alpha} \varphi_{\alpha}(P) d^n x + dP \wedge d\xi.$$

Proof. At the first step of the iteration, we have

$$f d^n x = \sigma(f) d^n x + dP \wedge \pi_1 \quad \text{with } |\pi_1|_r \leq C_r |f|_r;$$

then $d(i_{\varphi_1 H} d^n x) = d\pi_1$, and Lemma 3.1.2 gives

$$|\varphi_1|_r \leq \frac{C_r}{mr_0} |f|_r.$$

The relation $d\zeta_2 = i_{\varphi_1 H} d^n x - \pi_1$ and Lemma 3.1.3 then give

$$|\zeta_2|_r \leq R \left[1 + \frac{MR}{mr_0} \right] C_r |f|_r,$$

and so on. For the p -th iterate we get

$$|\varphi_j|_r \leq \left(\frac{C_r}{mr_0}\right)^j |f|_r,$$

$$|\zeta_{j+1}|_r \leq RC_r \left[1 + \frac{MR}{mr_0}\right] \left[\frac{C_r}{mr_0}\right]^{j-1} |f|_r,$$

for $j = 1, \dots, p$.

Let us choose r' so that $|P|_{r'} C_r / (mr_0) < 1$. Then since

$$\left| \sum_{j=1}^p P^{j-1} \zeta_{j+1} \right|_{r'} \leq \sum_{j=1}^p |P|_{r'}^{j-1} |\zeta_{j+1}|_{r'} \leq \sum_{j=1}^p |P|_{r'}^{j-1} |\zeta_{j+1}|_r$$

and

$$\left| \sum_{j=0}^p P^j \sigma(\varphi_j) \right|_{r'} \leq \sum_{j=0}^p |P|_{r'}^j |\varphi_j|_r,$$

the result of the theorem follows with, furthermore, the explicit estimations

$$|\xi|_{r'} \leq \frac{RC_r(1 + MR/(mr_0))}{1 - |P|_{r'} C_r / (mr_0)} |f|_r \quad \text{and} \quad |\varphi_\alpha(P)|_{r'} \leq \frac{|f|_r}{1 - |P|_{r'} C_r / (mr_0)}. \quad \square$$

We deduce as a corollary that $G = \Omega^n / dP \wedge d\Omega^{n-2}$ is a $C\{t\}$ -module of finite type. We now prove that G has no torsion.

3.2. The weight vector field $H = \sum_{i=1}^n m_i x_i \frac{\partial}{\partial x_i}$ defines a graduation of the exterior algebra $\Omega = \bigoplus_{k \geq 0} \Omega^k$.

A form $\bar{\omega}$ is *quasi-homogeneous of weight h* if $L_H \bar{\omega} = h\bar{\omega}$. Given a differential form, we denote by $\bar{\omega} = \sum_h \bar{\omega}_h$ its decomposition into quasi-homogeneous components.

In particular, an element $f \in \mathcal{O} (\sim \Omega^0)$ can be written as $f = \sum_h f_h$.

LEMMA 3.2.1. *Let f be an element of \mathcal{O} such that $f d^n x \in dP \wedge d\Omega^{n-2}$. Then every quasi-homogeneous component f_h satisfies*

$$f_h d^n x \in dP \wedge d\Omega^{n-2}.$$

Proof. We write $f d^n x = dP \wedge d\eta$, and so we have

$$f_h d^n x = dP \wedge (d\eta)_{h-1+M};$$

but

$$(d\eta)_{h-1+M} = d(\eta_{h-1+M})$$

because the Lie derivative commutes with d . □

PROPOSITION 3.3. *The module G is torsion free.*

Proof. Let $f \in \mathcal{O}$ be so that $Pf d^n x = dP \wedge d\eta$; we can write

$$dP \wedge (i_{fH} d^n x - d\eta) = 0,$$

and so ([D]) we have

$$i_{fH} d^n x - d\eta = dP \wedge \varrho \quad \text{and} \quad (H \cdot f + Mf) d^n x = dP \wedge d(-\varrho).$$

If we decompose f into its quasi-homogeneous components, we get

$$f_h d^n x = dP \wedge d\left(\frac{-\varrho h - 1 + M}{h + M}\right),$$

and from this we can deduce the existence of an analytic $(n-2)$ -form

$$\xi = \sum_h -\varrho \frac{h-1+M}{h+M}$$

such that

$$f d^n x = dP \wedge d\xi. \quad \square$$

3.3. A basis of G . Let $F = \frac{dP \wedge \Omega^{n-1}}{dP \wedge d\Omega^{n-2}}$ be considered as a $\mathcal{C}\{t\}$ -module; then clearly $tG \subset F$.

PROPOSITION 3.3. *We have the equality $tG = F$.*

Proof. Let us denote by $[\omega]$ the class in G of an element ω of Ω^n . We remark that the classes $[\omega_{\beta,i}]$, where $\omega_{\beta,i} = dP \wedge (x^\beta d^n x_i)$, generate F . A simple computation then leads to

$$t[\beta_i x^{\beta-I_i}] = \langle \beta, m \rangle + \sum_{\substack{j=1 \\ j \neq i}}^n m_j [\omega_{\beta,i}]. \quad \square$$

We can deduce from this proposition that $G/tG = G/F$. But then

$$G/F \simeq \Omega^n/dP \wedge \Omega^{n-1} \simeq \mathcal{O}/J_P.$$

The fact that the classes $[x^\alpha d^n x]$ ($\alpha \in A$) give a basis of G follows from Nakayama's lemma, so Theorem 3.0 is proved.

4. The non-isolated case (normal crossing)

How to extend the classification of couples functions-volume forms to the case where the function has a non-isolated singularity? Obviously, there are examples where the moduli space $\Omega^n/dP \wedge d\Omega^{n-2}$ is no longer a finitely generated $\mathcal{C}\{t\}$ -module.

But it is interesting to note first of all that we are in fact concerned with

$$G' = \Omega^n / \{d\eta \mid dP \wedge \eta = 0\},$$

and that this module is $C\{t\}$ finitely generated in the case where P is a normal crossing $P = x_1^{p_1} \dots x_n^{p_n}$. We write in this case $p_i = p'_i d$ ($i = 1, \dots, n$), where d is the g.c.d. of the p_i , and $\omega_j = \prod_{i=1}^n x_i^{j p_i - 1} d^n x$ for $j = 1, \dots, d$.

Given a couple $P = x_1^{p_1} \dots x_n^{p_n}$ and $\omega = a(x) d^n x$ with $a(0) = 1$, we have:

THEOREM 4.1. *There is a diffeomorphism φ which preserves P and transforms ω into the model*

$$\varphi^* \omega = dx_1 \wedge \dots \wedge dx_n + \sum_{j=1}^n \psi_j(P) \omega_j.$$

Let $\overline{\Omega^n}$ be the set of n -forms $d\eta$ such that $dP \wedge \eta = 0$. The same proof as in Theorem 2.1 gives the fact that if two volume forms ω and ω' are such that $\omega(0) = \omega'(0) = dx_1 \wedge \dots \wedge dx_n$ and $\omega - \omega' \in \overline{\Omega^n}$, then there is a diffeomorphism φ so that $\varphi^* P = P$ and $\varphi^* \omega = \omega'$.

The above theorem is a consequence of

PROPOSITION 4.1. *The $C\{t\}$ -module $G' = \Omega^n / \overline{\Omega^n}$ is free of rank d and the classes of the ω_j modulo $\overline{\Omega^n}$ give a basis of G' as a $C\{t\}$ -module.*

Proof. Given $f \in \mathcal{O}$ and $\omega = f dx_1 \wedge \dots \wedge dx_n$, we write $f = \sum_{\alpha} f_{\alpha} x^{\alpha}$ and we try to decompose ω into

$$f d^n x = d\eta + \sum_{j=1}^d \psi_j(P) \omega_j$$

with $d\eta \in \overline{\Omega^n}$.

We can treat each monomial separately, and so given a monomial $x^{\alpha} = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, we need to know for which values of $\alpha = (\alpha_1, \dots, \alpha_n)$ we can be sure to find η_1, \dots, η_n such that

$$\begin{aligned} x^{\alpha} &= \frac{\partial \eta_1}{\partial x_1} + \dots + \frac{\partial \eta_n}{\partial x_n}, \\ 0 &= \frac{\partial x^p}{\partial x_1} \eta_1 + \dots + \frac{\partial x^p}{\partial x_n} \eta_n. \end{aligned}$$

We can look for η_1, \dots, η_n of the type

$$\eta_1 = A_1 x_1^{\alpha_1 + 1} \dots x_n^{\alpha_n}, \quad \dots, \quad \eta_n = A_n x_1^{\alpha_1} \dots x_n^{\alpha_n + 1}.$$

We then have to solve the system

$$(\alpha_1 + 1) A_1 + \dots + (\alpha_n + 1) A_n = 1$$

$$p_1 A_1 + \dots + p_n A_n = 0.$$

Such a system has a solution if and only if $(\alpha_1 + 1, \dots, \alpha_n + 1)$ is not a multiple of (p_1, \dots, p_n) . This case corresponds to

$$\alpha_1 + 1 = \lambda p_1 = (\lambda d) p'_1,$$

$$\dots \dots \dots$$

$$\alpha_n + 1 = \lambda p_n = (\lambda d) p'_n.$$

λd is necessarily an integer and the different values $\lambda = j/d$, when j varies from 1 to d , correspond to the different monomials which give the d series $\psi_j(P) \omega_j$. \square

In [F₃], I have considered the question of C^∞ conjugacy for the real normal crossing.

The problem of C^∞ -conjugacy has been considered by Colin de Verdière and J. Vey ([C]) and V. Guillemin ([G]) for the Morse functions, and by C. Roche in general in his thesis ([R]).

References

- [B] E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. 2 (1970), 103–161.
- [C] Y. Colin de Verdière et J. Vey, *Le lemme de Morse isochore*, Topology 18 (1979), 283–293.
- [D] G. De Rham, *Sur la division des formes et des courants par une forme linéaire*, Comment. Math. Helv. 28 (1954), 346–352.
- [F₁] J.-P. Francoise, *Modèle local simultané d'une fonction et d'une forme de volume*, Astérisque 59–60 (1978), 119–130.
- [F₂] —, *Le théorème de M. Sebastiani pour une singularité quasi homogène isolée*, Ann. Inst. Fourier (Grenoble) 29 (1979), 247–254.
- [F₃] —, *Réduction simultanée d'un croisement normal et d'un volume*, Bol. Soc. Brasil. Mat. 13 (1982), 79–83.
- [G] V. Guillemin, *Band asymptotics in two dimensions*, Adv. in Math. 42 (1981), 248–282.
- [Ma₁] B. Malgrange, *Frobenius à singularités 1: codimension un*, Publ. Math. Inst. Hautes Études Sci. 46 (1976), 163–173.
- [Ma₂] —, *Intégrales asymptotiques et monodromie*, Ann. École Nat. Sup. Méc. Nantes 4 (1974), 405–430.
- [Mo] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. 120–121 (1965), 286–294.
- [R] C. Roche, Thèse, Université de Grenoble, 1981.
- [Se] M. Sebastiani, *Preuve d'une conjecture de Brieskorn*, Manuscripta Math. 2 (1970), 301–308.

- [Si] C. L. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag, Berlin-Heidelberg 1971.
- [St] S. Sternberg, *Infinite Lie groups and formal aspects of dynamics*, J. Math. Mech. 10 (1961).
- [To] J.-C. Tougeron, *Idéaux de fonctions différentiables*, Ergeb. Math. Grenzgeb. 71, Springer-Verlag, Berlin 1970.
- [V] J. Vey, *Sur le lemme de Morse*, Invent. Math. 40 (1977), 1-9.

*Presented to the semester
Singularities
15 February-15 June, 1985*
