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The Nonlinear Riemann–Hilbert Problem[1]

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INTRODUCTION

1. Statement of the problem. The nonlinear Riemann–Hilbert problem is a natural analog of the classical linear problem. The latter is formulated as follows.

Problem. *Given the linear operators* $G_j: \mathbb{C}^n \to \mathbb{C}^n$, $j = 1, \ldots, m$, *find the Fuchsian system*

$$\dot{z} = \sum \frac{A_j}{t - a_j} z, \tag{1}$$

whose monodromy transformation corresponding to the standard circuits of the singular point a_j *is equal to* G_j.

The standard circuits are defined as follows. We take an arbitrary base point on the Riemann sphere such that the line segments which connect it with the singular points have no points in common (except for the origin). The standard circuit of the point a_j consists of a part of the corresponding segment, a positively oriented small circle that circuits a_j, and the same segment traversed in the opposite direction. We enumerate the singular points so that the product of their standard circuits in the order of numbering is contractible on the Riemann sphere with deleted singular points. In what follows, we understand the monodromy transformation corresponding to the point a_j to be a transformation corresponding to the standard circuit.

The nonlinear problem refers to equations in the direct product of a small n-dimensional ball by the Riemann sphere with a finite number of deleted points. Suppose that we are given an equation

$$\dot{z} = v(t, z), \qquad v(t, 0) = 0, \tag{2}$$

where the field v is holomorphic outside of the finite number of singular points a_j and the field $(t - a_j)v$ is holomorphic at the singular point a_j. Let the singular points of (2) be nondegenerate and the point ∞ be regular, i.e., $t^2 v = O(1)$ as $t \to \infty$. Then we call this equation an equation of the class NF (from "nonlinear Fuchsian").

The circuit about every singular point is associated with a germ of the nonlinear mapping

$$\Delta_j: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), \qquad \prod \Delta_j = id. \tag{3}$$

The nonlinear Riemann–Hilbert problem is an inverse problem, namely, in the direct product

$$\hat{\mathbb{C}} \times (\mathbb{C}^n, 0)$$

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we must find the differential equation (2) with given nondegenerate singular points a_j on the Riemann sphere whose monodromy transformations corresponding to the standard circuits of singular points are equal to the specified germs of the holomorphic mappings (3).

This problem has a natural linearization, namely, equation (2) is replaced by a variation equation with respect to the initial condition which varies in the plane z:

$$\dot{z} = (\partial v/\partial z(t,0))z.$$

By virtue of the assumption of the holomorphy of the field $(t-a_j)v$ at the point $(a_j, 0)$, this linear system turns out to be Fuchsian.

The Riemann–Hilbert problem for Fuchsian systems has a dramatic history. Riemann stated it not long before he died and only outlined its solution. Poincaré repeatedly attempted to solve this problem, but without success. Hilbert included it into his famous list with the number 21. In 1908, Hilbert's disciple Plemelj published the solution of the problem [1], which for long was considered to be final. Indeed, it holds if, in the preceding statement, the Fuchsian systems are replaced by the so-called regular systems, or if, in the statement given above, one of the matrices G_j is equivalent to a diagonal matrix [2]. In 1970s, the author noticed that Plemelj's solution did not hold for arbitrary matrices of monodromy and Fuchsian systems [3]. In 1989 Bolibrukh found out that in the general form the Riemann–Hilbert problem was unsolvable [4–7].

Obviously, the solvability of the linearized problem is necessary for the solvability of the nonlinearized one. The question arises as to whether there exist other, essentially nonlinear, obstacles for the solvability of the nonlinear Riemann–Hilbert problem? Unexpectedly, the answer turns out to be positive. Some obstacles arise already at $n = 1$.

2. The necessary conditions for solvability. The one-dimensional linear Riemann–Hilbert problem is trivially solvable. We give the following definition to describe the obstacles in the way of solving the nonlinear problem.

Definition 1. The germ of the mapping $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is said to be *wild* if it has a nonresonance multiplicator (the argument of its derivative at zero is irrational relative to a complete rotation) and is analytically nonequivalent to the linear one. The germ is *resonant* if its multiplicator is the root of unity.

Note that the multiplicator of a wild mapping is modulo 1.

Theorem 1. *A collection consisting of only wild mappings cannot be realized as a collection of monodromy transformations of equation* (2) *for* $n = 1$.

Proof. Let us assume the contrary. Let (2) be the required equation. For $n = 1$, in the neighborhood of the j-th singular point transferred to zero, this equation is equivalent to the system

$$\dot{z} = \lambda_j z + O(|z|^2 + |t|^2), \qquad \dot{t} = t. \tag{4}$$

The linear part of the monodromy is

$$z \mapsto \nu_j z, \qquad \nu_j = \exp 2\pi i \lambda_j.$$

The number λ_j is real since the monodromy transformation is wild. As above, we assume that the infinite point is nonsingular. According to the theorem of the residues applied to the function $\partial v/\partial z\,(t, 0)$ we have

$$\sum \lambda_j = 0. \tag{5}$$

Therefore, at least one of the eigenvalues of λ_j is positive. By the definition of wild singular points, it is irrational. According to Poincaré theorem, in the neighborhood of the singular point $(0, 0)$ equation (4) is analytically equivalent to its linear part. Consequently, its monodromy transformation is analytically equivalent to the linear one and is not wild. We have obtained a contradiction.

Other obstacles, which give the necessary and sufficient conditions for solvability of the one-dimensional nonlinear Riemann–Hilbert problem, are described in Section 1.

The general theorem of unsolvability, whose proof is very close to the proof of the preceding theorem, can be formulated as follows.

Theorem 2. *Let $n > 1$. The collection of germs of mapping* (3), *none of which has a nontrivial (distinct from a fixed point and its full neighborhood) analytical invariant manifold, cannot be realized as a collection of generators of the monodromy group of equation* (2).

It should be pointed out that the necessary conditions for solvability of the nonlinear Riemann–Hilbert problem are rather simple. However, this is not true for the sufficient conditions. The main part of the article is devoted to the proof of the next theorem.

Theorem 3. *Supose that the points a_j are marked on the Riemann sphere, the germs of the biholomorphic mappings* (3) *are given, and let the classical Riemann–Hilbert problem with $d\Delta_j(0)$ be solvable. Let at least one of the mappings Δ_j be analytically equivalent to the linear one. Then the germs Δ_j can be realized as the monodromy transformation of equation* (2).

3. The idea of the proof. The proof of the solvability of the linear Riemann–Hilbert problem (in the case where it can be carried out) can be constructed according to the following plan. First, on the direct product of a sphere with deleted disks, whose centers are at singular points, by $\mathbb{C}^n$, we construct a foliation with a specified monodromy group. This construction is standard.

Second, for every singular point we construct a foliation on a cylinder (the product of the neighborhood of the closure of the deleted disk by $\mathbb{C}^n$) which has (one) specified monodromy transformation. This construction is simpler still and is defined by explicit relations.

Third, we glue the holes obtained at the first stage by the cylinders constructed at the second stage so that the glued foliations form one new foliation. As a result we obtain a linear foliation on the skew product of the Riemann sphere by $\mathbb{C}^n$. The leaves can be regarded as the integral curves of the original equation.

The only difficulty that we must overcome is to prove that the skew product $\mathbb{C} \times \mathbb{C}^n$ that we have constructed is actually direct. Then the foliation mentioned above is defined by equation (2). However, without a special selection of foliations, constructed at the second stage, the corresponding skew product will not be direct. This selection is precisely the main difficulty in the proof of the solvability in the linear case.

The nonlinear Riemann–Hilbert problem can be solved in parallel. The first stage in the nonlinear case is as standard as in the linear one. The second stage, in contrast to the standard linear case, presents the main difficulty. This is the so-called *realization theorem*, namely, the statement that any biholomorphic germ can be represented as the monodromy transformation of a singular point of a holomorphic vector field.

Definition 2. A singular point of a holomorphic vector field is said to be 1-hyperbolic if one of its eigenvalues is positive and the other eigenvalues lie in the left half-plane.

According to the Hadamard–Perron theorem, the positive eigenvalue is associated with a one-dimensional holomorphic invariant manifold. Its monodromy transformation is called the monodromy transformation of the singular point itself.

Theorem 4. *Every germ of the biholomorphic mapping $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ can be realised as the monodromy transformation of a 1-hyperbolic singular point of a holomorphic vector field.*

It turns out that even the formal analog of this theorem is nontrivial. When we obtain this analog in Section 4 below, we carry out the proof according to the plan used in [8] for a one-dimensional case.

In order to use the realization theorem in the proof of Theorem 3, we must know how to construct a 1-hyperbolic singular point with a defined monodromy, for which the variational equation along a one-dimensional unstable manifold in the sense defined below coincides almost completely with the given Fuchsian system. With this aim in view, we prove the following theorem.

Theorem 5. *For every Fuchsian system*

$$\dot{z} = \frac{A(t)}{t} z, \tag{6}$$

defined in the neighborhood of zero, for every biholomorphic germ $\Delta\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ whose derivative at zero coincides with the monodromy transformation of the system, and for any sufficiently large natural l, there exists an equation

$$\dot{z} = \frac{(A(t) - lE)z + f(t,z)}{t}, \qquad f(t,z) = O(|z|^2), \tag{7}$$

whose monodromy transformation, which corresponds to the leaf $z = 0$, coincides with Δ.

Theorem 4 obviously follows from Theorem 5.

The article is organized according to the following plan. The first section contains the full solution of a one-dimensional problem. In the second section we discuss the necessary conditions for solvability for the arbitrary n. In the third section, Theorem 3 of solvability is reduced to Theorem 5 of realization. In the fourth section we prove the formal analog of Theorem 5, and in the fifth section we prove the theorem itself.

1. THE NECESSARY AND SUFFICIENT CONDITIONS FOR SOLVABILITY OF A ONE-DIMENSIONAL PROBLEM. VANISHING HOLONOMY

In this section, the condition indicated in the title is proved modulo the realization theorem and trivialization lemma (Theorem 6 in 1.1 and Lemma 3.1 in Section 3 below). A full description is given for groups of vanishing holonomy for germs of equations in a complex plane whose nice blowing up can be obtained in one step.

1.1. Realization: a one-dimensional case.

Theorem 6. *Let $\alpha \in (0,1)$ be an arbitrary number. Then, for any germ of the conformal mapping*

$$\Delta\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0),$$

for which $\Delta'(0) = \exp 2\pi i\alpha$ and for any natural l, there exists a germ of a holomorphic vector field with the ratio of eigenvalues equal to $\lambda_2/\lambda_1 = \alpha - l$, such that the monodromy transformation for the separatrix, which corresponds to λ_1, is equal to Δ.

In [8] this theorem is proved for an irrational α instead of an arbitrary one. For the arbitrary α it is proved in the same way as Theorem 5 (and at the same time as Theorem 5), but the proof is simpler. Note that in Theorem 5 the natural l is chosen whereas in Theorem 6 it is arbitrary.

The only difference between the proofs of Theorems 5 and 6 is indicated in the process of proving Theorem 5.

For $n = 1$ the ratio of the eigenvalues of the singular point $(0, a_j)$ of equation (2) (the eigenvalue in the denominator corresponds to the vector tangent to the t-axis) is called a *characteristic number* of this point and is denoted by λ_j.

1.2. One-dimensional problem. An equation (in the plane) of the Poincaré type with a nonlinearizable monodromy transformation is a *resonant node.* The ratio of its eigenvalues is a natural number or the inverse of a natural number. A resonant node has a unique holomorphic invariant manifold which touches, at a singular point, a vector with an eigenvalue which is the least in modulus. For equations of the NF class this manifold is the straight line $z = 0$. Consequently, the characteristic numbers of resonant nodes of equations of the NF class are inverse to natural numbers.

The monodromy transformation of a resonant node corresponding to the solution $z = 0$ is called a *Dulac transformation.*

Theorem 1′. *Theorem* 1 *holds true if "a collection consisting of only wild mappings" in its statement should be replaced by "a collection consisting of only wild and resonant nonDulac mappings."*

The proof is the same as for Theorem 1 above.

By definition, for every Dulac transformation there exists a resonant node whose monodromy transformation coincides with this transformation. However, according to Theorem 6, the Dulac transformation can also be realized as the monodromy transformation of a complex saddle.

The number

$$\ln_0 \nu = \frac{\ln \nu}{2\pi i}, \qquad \operatorname{Re}(\ln_0 \nu) \in [-1, 0)$$

is called the *normed logarithm* of the number ν.

Theorem 7. *Let $n = 1$, and suppose that we are given mappings* (3) *and that*

$$\nu_j = \Delta_j'(0), \qquad \prod \nu_j = 1.$$

Collection (3) *can be realized as a collection of monodromy transformations of equation* (2) *which corresponds to standard circuits if and only if one of the following two conditions is fulfilled:*

(1) *at least one of mappings* (3) *is analytically equivalent to a linear mapping;*

(2) *among mappings* (3) *there are k Dulac mappings, the other mappings being wild or resonant. In this case*

$$k + \sum \ln_0 \nu_j \geq 0.$$

Proof. *Necessity.* Supose that none of conditions 1 and 2 is fulfilled but collection (3) is nevertheless realized. Suppose that (2) is the corresponding equation and

$$\mu_j = 1/n_j$$

are characteristic numbers corresponding to its resonant nodes, λ_l are the other characteristic numbers. Let m be the number of resonant nodes. Obviously, $m \leq k$.

Since condition 1 is not fulfilled, all mappings which are not Dulac are wild, or resonant. For them $\lambda_l < 0$.

The Dulac mappings are included in collection (3) according to Theorem 1′. Wild or resonant mappings also belong to this collection since, otherwise, the sum of all characteristic numbers would be positive.

Let α_j, β_l be normed logarithms of the multiplicators of the monodromy transformation of resonant nodes and the other mappings (3) respectively. Then $\mu_j = \alpha_j + 1$, $\lambda_l \le \beta_l$. The sum of all characteristic numbers of equation (2) is zero. Consequently,

$$0 = \sum \mu_j + \sum \lambda_l \le m + \sum \alpha_j + \sum \beta_l \le k + \sum \ln_0 \nu_j.$$

Hence follows condition 2, a contradiction.

Sufficiency. Let us derive the following requirement from conditions 1 and 2.

$*$. *The mappings Δ_j are realized as monodromy transformations of the vector fields v_j with characteristic numbers λ_j which satisfy condition* (5)*: $\sum \lambda_j = 0$.*

In Section 3, we shall use this statement to derive the solvability of a nonlinear problem, i.e., we shall prove the sufficiency in Theorem 7. Here we shall prove requirement $*$.

Suppose that condition 1 is fulfilled, i.e., one of the given mappings, say, Δ_1, is analytically equivalent to a linear one. We can assume, without loss of generality, that it is linear. By definition, all Dulac mappings can be realized as monodromy transformations of resonant nodes. All the other mappings, except for Δ_1, can be realized, according to Theorem 6, as monodromy transformations of a singular point with a characteristic number equal to the normed logarithm of the monodromy multiplicator. The linear mapping Δ_1 can be realized as the monodromy transformation of a germ with a characteristic number equal to any preassigned logarithm of the monodromy multiplicator divided by $2\pi i$. Equality (5) can be ensured by the choice of this number. We have proved requirement $*$ under condition 1.

Suppose now that condition 1 is not fulfilled. Then condition 2 is fulfilled. Consequently, there exists a natural $m \le k$ such that $m + \sum \ln_0 \nu_j = 0$. Let us realize any m Dulac mappings as monodromy transformation of resonant nodes. The other mappings can be realized according to Theorem 6 as monodromy transformation of complex saddles with characteristic numbers equal to $\ln_0 \nu_j$. This implies the fulfillment of requirement $*$.

1.3. Vanishing holonomy. We can prove the analog of the realization theorem for a vanishing holonomy. Let us consider a class of germs of analytic vector fields on $(\mathbb{C}^2, 0)$ with a zero linear part. Moreover, we assume that these germs have a zero $(m-1)$-jet and a nonzero m-jet. We denote this class of germs by $\mathcal{V}_m$. After one blow-up, the typical germs of the class $\mathcal{V}_m$ pass into germs of fields of lines on the glued the Riemann sphere $\widehat{\mathbb{C}}$ with $m+1$ nondegenerate singular points and the sphere itself, with the singular points deleted, is a leaf of a foliation. To the germ v of the class $\mathcal{V}_m$ there corresponds the germ resulting from the blow-up and denoted by $\tilde{v}$. Let a_j be singular points of the germ $\tilde{v}$ on the glued the Riemann sphere. The standard circuits of these points give rise to the germs of monodromy transformations $\Delta_j\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. The group generated by them is called a *group of vanishing holonomy* (or simply a group of monodromy) and is an invariant of the analytic classification of germs of the class $\mathcal{V}_m$. A natural inverse problem arises, namely, can the given germs Δ_j be realized as the generators of the group of vanishing holonomy of an equation of the class $\mathcal{V}_m$? The following theorem provides an answer.

Theorem 8. *Let $n = 1$ and mappings* (3) *be defined. Collection* (3) *can be realized as a collection of generators, corresponding to standard circuits, of the group of a vanishing holonomy* *of a certain germ of the class $\mathcal{V}_m$ if and only if one of the following two conditions is fulfilled:*

(1) *at least one of mappings* (3) *is analytically equivalent to a linear mapping;*

(2) *among mappings* (3) *there are k Dulac mappings, the other mappings being wild or resonant. In this case,*

$$k + \sum \ln_0 \nu_j \geq -1.$$

The proof of Theorem 8 is completely similar to the preceding proof with the only difference that condition (5) imposed on the characteristic numbers is replaced by the condition

$$\sum \lambda_j = -1. \tag{5'}$$

Condition (5′) is a special case of the Camacho–Sed theorem [9]. It can be explained by the fact that the self-intersection index of the glued in the Riemann sphere is -1. This motivates the difference between conditions (5) and (5′). Otherwise, the proof repeats the preceding one.

Besides the vanishing holonomy, the invariant of the orbital analytical classification of germs of the class $\mathcal{V}_m$ is a finite jet of the so-called separatrix set. The description of this invariant and the theorem on its realization can be found in the next article in this collection [10].

2. THE NECESSARY CONDITION FOR SOLVABILITY OF THE MULTIDIMENSIONAL NONLINEAR RIEMANN–HILBERT PROBLEM

In this section we prove Theorem 2 from Introduction.

Let us assume that the theorem is not valid. Let us consider the collection of germs (3) and assume that none of them has a nontrivial analytical invariant manifold. Let us also assume that these germs can be realized as monodromy transformations of equation (2) of the NF class. Let us reduce these assumptions to a contradiction. Consider the following linearized system (1):

$$\dot{z} = A(t)z, \qquad A(t) = \frac{\partial v}{\partial z}(t, 0).$$

This is a Fuchsian system since the fields $(t - a_j)v$ are analytic at the points a_j. Its residue matrices are nondegenerate since the singular points of the equations of the NF class are nondegenerate. The point ∞ is regular for it by virtue of the assumption that $t^2 v \to 0$ as $t \to \infty$. Consequently, $\sum A_j = 0$. Therefore the sum of the traces of the matrices A_j is zero. Consequently, at least one of these matrices, say, A_1, has at least one eigenvalue in the right half-plane. Let us transfer zero to the point a_1. In the neighborhood of this point system (2) takes the form

$$\dot{z} = A_1(t)z + \dots, \qquad \dot{t} = t. \tag{2.1}$$

The dots denote terms which are nonlinear with respect to t, z. Let us consider two cases.

Case 1. Not all eigenvalues of the matrix A_1 lie in the right half-plane. Then the matrix A_1 has a nonempty set of eigenvalues both in the right and the left half-plane. In this case, the linearization of equation (3.1) has an unstable invariant subspace of dimension exceeding 1, i.e., the eigenvalue 1 corresponding to the variable t is added to the unstable eigenvalues of the matrix A_1. This manifold does not coincide with the whole space (t, z) by the assumption of Case 1. According to the Hadamard–Perron theorem, equation (3.1) has an unstable analytical invariant manifold W^u. Since $1 < \dim W^u < n+1$, it intersects the transversal $t = 1$ along the nontrivial analytical submanifold. This submanifold is invariant under the monodromy transformation Δ_1 since the manifold W^u is invariant. The contradiction obtained proves the theorem in Case 1.

Case 2. All eigenvalues of the matrix A_1 lie in the right half-plane. In this case, the equation is analytically equivalent to its resonance normal form (Dulac's theorem). Suppose that the matrix A_1 has been reduced to the Jordan normal form and λ_1 is its eigenvalue which has the smallest real part (if there are several values of this kind, then we take any one of them). Let z_1 be a variable which corresponds, in a linearized system to the equation $\dot{z}_1 = \lambda_1 z_1$. Then, in the resonance normal form, the equation for z_1 does not have nonlinear terms by virtue of the absence of a resonance of the form $\lambda_1 = (\lambda, k)$. Consequently, system (3.1) is equivalent to the system

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{Z} = \ldots, \qquad \dot{t} = t,$$

where $Z = (z_2, \ldots, z_n)$. Therefore the z_1-axis is an invariant manifold of the monodromy transformation Δ_1. The contradiction obtained proves Theorem 2.

3. THE SUFFICIENT CONDITION FOR THE SOLVABILITY MODULO THE REALIZATION THEOREM

In this section we prove Theorem 3. The proof will be carried out according to the plan outlined in item 3 of Introduction.

3.1. The solution of a nonlinear problem on a sphere with holes. Suppose, as before, that a_j are deleted points on a Riemann sphere, a is a base point, γ_j are standard circuits of the deleted points. We denote by β_j small circles belonging to the standard loops γ_j. Let D_j be disks with centers a_j lying entirely inside the circles β_j. We denote by $\widehat{U}$ the universal covering over the domain $U = \widehat{\mathbb{C}} \setminus \cup D_j$ with the base point a. Let T_j be the shift transformation of the universal covering $\widehat{U}$ corresponding to the loop γ_j. In the space $\widehat{U} \times \mathbb{C}^n$ we shall consider germs on the curve $\widehat{U} \times 0$ of the mappings

$$\widehat{\Delta}_j \colon (z, \hat{t}) \mapsto (\Delta_j^{-1}(z), T_j(\hat{t})). \tag{3.1}$$

Let $\mathbf{G} = \langle \widehat{\Delta}_j \rangle$ be a subgroup of the group of germs on the curve $\widehat{U} \times 0$ of the mappings with the superposition operation. There exists a neighborhood $\widehat{V}$ of this curve which possesses the following property: the mapping $\widehat{\Delta}_j$ is defined on $\widehat{V}$; the quotient space $V = \widehat{V}/\mathbf{G}$ is a complex manifold. The factorization means the identification of points and their images under the action of the mappings $\widehat{\Delta}_j$.

Let us consider the foliation of the domain $\widehat{V}$ into horizontal straight lines (with holes) $\{z\} \times \widehat{U}$. Mappings (3.1) respect this foliation. Therefore, the factorization gives rise to new foliation. We denote it by $\mathcal{F}_U$. By construction, the monodromy transformations of this foliation, corresponding to the curves γ_j, are equal to Δ_j. In order to define this foliation by a differential equation, we must introduce the structure of a direct product in a certain neighborhood of U in V.

The curve U is embedded into V in a natural way. Recall that two embeddings i_1, i_2 of a holomorphic curve into a complex manifold are equivalent if the embedded curves have biholomorphically equivalent neighborhoods and the holomorphic diffeomorphism h of one neighborhood into the other is permutable with the mappings of the embeddings in the sense $h \circ i_1 = i_2$.

Mappings (3.1) commute with the projection $(z,t) \mapsto t$ and therefore the projection $\pi: V \to U$ is defined on the quotient space V, the projection being regular everywhere and identical on U. The tangent space to the fibers form a *normal bundle* N which corresponds to the embedding $U \to V$. Note that the curve U is a Stein manifold. Therefore the following general theorem is applicable.

Theorem (Siu) [11]. *The embedding of Stein's manifold into an arbitrary complex manifold is equivalent to the embedding of the first manifold, as a zero section, into the space of the corresponding normal bundle.*

Any vector bundle over a proper subdomain of the Riemann sphere is trivial, and therefore the embedding $U \to V$ is equivalent to the embedding $U \to \mathbb{C}^n \times U$ as a zero section. This defines the required structure and makes it possible to define the foliation $\mathcal{F}_U$ on a certain neighborhood $V' \in V$ of the curve U by the differential equation $\dot{z} = v(z,t)$. This completes the first step of the proof.

3.2. Pasting up the holes. At the second stage we construct the foliation $\mathcal{F}$ with a given monodromy and nondegenerate singular points on the neighborhood of the Riemann sphere in an abstract $(n+1)$-dimensional manifold. At the third stage, this neighborhood is embedded biholomorphically into the direct product $\mathrm{C}^n \times \mathrm{C}$ and the foliation $\mathcal{F}$ turns into equation (2) whose linear part with respect to z is system (1). This completes the proof of Theorem 3.

Let Δ_j be the same collection as in the hypothesis of Theorem 3. Suppose that equation (1) gives the solution to the classical Riemann–Hilbert problem with the given $T_j = d\Delta_j(0)$.

Let α_j be the arc of the curve γ_j from the initial point a to the point b_j on the circle β_j. Suppose that $\mathbf{G}_j: (\mathbb{C}^n, 0) \times \{a\} \to (\mathbb{C}^n, 0) \times \{b_j\}$ is a germ of the mapping along the leaves of the foliation $\mathcal{F}_U$. We denote by $\mathbf{D}_j$ the germ $\mathbf{G}_j \circ \Delta_j \circ \mathbf{G}_j^{-1}$.

Now we apply Theorem 5 to every germ $\mathbf{D}_j$ and to equation (6) obtained from (1) by the shift $t \mapsto t - a_j$. Let (5.1) be an equation given by this theorem and l be a natural number in relation (5.1) which now depends on j and is denoted by l_j. The system

$$\dot{z} = \sum \frac{A_j - l_j E}{t - a_j} z \tag{3.2}$$

has the same monodromy as system (1). Indeed, if we add a scalar matrix with integer residues to the matrix of the system, the monodromy will not be changed since the scalar matrix commutes with any matrix and the integral power of t is a one-valued function.

However, the sum of the residue matrices of system (3.2) at the points a_j is no longer equal to zero since infinity is a new singular point. However, by the hypothesis it must be a regular point of the system which is to be constructed. In order to regularize the system at infinity, we recall that one of the given germs of the mappings is equivalent to a linear one. We assume that this germ is $\mathbf{D}_1$. Let $l' = -\sum_{j>1} l_j$. Let us consider the system

$$\dot{w} = \sum \frac{B_j}{t - a_j} w, \quad B_j = A_j - l_j E, \quad j \geq 2; \qquad B_1 = A_1 - l' E. \tag{3.3}$$

System (3.3) has the same monodromy group as (3.2), but the infinite point is already regular for it: $\sum B_j = 0$. According to Theorem 5, in the neighborhood of every point a_j, $j \geq 2$, we can add nonlinear terms to the numerator of equation (3.2) so that the monodromy transformation of the resulting equation will be equal to $\mathbf{D}_j$.

For the negative $l = l'$, Theorem 5 is inapplicable. However, by the hypothesis, the germ $\mathbf{D}_1$ can be linearized. This means that there exists a substutution

$$z = h(w), \tag{3.4}$$

which transforms the germs $\mathbf{D}_1$ the monodromy map $T_1 = h^{-1} \circ \mathbf{D}_1 \circ h$ of equation (3.3) corresponding to the point a_1. The substitution inverse to (3.4) takes equation (3.3) to a nonlinear equation whose monodromy of the point a_1 is equal to $\mathbf{D}_1$. This gives a nonlinear equation of the form (7) with the monodromy $\mathbf{D}_1$ in the neighborhood of the point a_1.

Thus, let $A(t)$ be a matrix of system (3.3). In the neighborhood of every one of the points A_j we have got a system

$$\dot{z} = \frac{A(t)z + f_j(t,z)}{t - a_j}, \qquad f_j = O(|z|^2),$$

whose monodromy transformation, corresponding to the circle β_j, is equal to $\mathbf{D}_j$. We denote the corresponding foliation by $\mathcal{F}_j$. Let R_j be a ring adjoining from the outside the boundary of the disk D_j and let $V_j = (\mathbb{C}^n, 0) \times (D_j \cup R_j)$. We glue this domain to the domain V so that the following requirements are met: the glueing is biholomorphic, commutes with the projection $(z,t) \mapsto t$, and transfers the leaves of the foliation $\mathcal{F}_j$ into those of the foliation $\mathcal{F}_U$. This glueing is uniquely defined by its restriction F_j to the fiber $t = b_j$ and is correctly defined since the monodromy transformation of the foliation to be glued, corresponding to the circle β_j, coincide.

It is convenient to subject the constructed glueing to an additional requirement. Note that the restriction F_j can be arbitrary. We assume that it has an identical linear part. *Then everywhere on R_j, the part of the glueing which is linear with respect to z, is identical* since the glueing is carried on along the phase curves of the differential equations which have a common linear part with respect to z.

We have obtained an $(n+1)$-dimensional manifold M, which contains a Riemann sphere, and a foliation on it with given singular points and a given monodromy. Note that the holomorphic retraction $\pi: M \to \widehat{\mathbb{C}}$ is defined.

It remains to introduce, on the glued manifold M, the structure of a direct product, to be more precise, define the chart z on the fibers of the projection π which is zero on $\widehat{\mathbb{C}}$.

3.3. Trivialization. If we consider a circle instead of a Riemann sphere, then the structure which was discussed in the preceding subsection exists. By a linear-fractional transformation we transfer one of the singular points a_j, say, a_1, to infinity. Let us consider two disks, L_+ and L_-, with centers 0 and ∞ respectively, which cover the Riemann sphere and are such that the disk L_+ contains all singular points a_j except for ∞. They have neighborhoods U_+ and U_- in M with the charts z_+, z_- on the fibers. In these charts the foliation $\mathcal{F}$ is defined by the equations

$$\dot{z} = v_+(z,t), \qquad \dot{z} = v_-(z,t).$$

The fields v_+ and v_- are defined in U_+ and U_-, respectively, and are holomorphic outside of the planes $t = a_j$, $t = \infty$, and in the neighborhood of these planes the vector-functions $(t - a_j)v_+$ and $t^{-1}v_-$ are holomorphic.

The manifold M is obtained from U_+ and U_- by a holomorphic glueing defined in the neighborhood of the ring $R = U_+ \cap U_-$. Just as in the preceding subsection, we can assume that it preserves t and its linear part is identical with respect to z. Thus the glueing has the form

$$\mathbf{F}(z,t) = (z + f(z,t), t), \qquad f(z,t) = O(|z|^2).$$

A glueing of this kind can always be trivialized. We can say that the nonlinear terms do not introduce any new obstacles into the trivialization of the neighborhood of the embedded sphere when the linear terms define a direct product. The statement concerning the trivialization of a glueing is formalized in the following lemma.

Suppose that U is an arbitrary domain on a complex straight line and $\mathcal{U}$ is the direct product $\mathbb{C}^n \times U$. We denote by $\mathcal{A}(U)$ the space of germs on U of the mappings into $\mathbb{C}^n$ which are holomorphic in a certain domain of the space $\mathcal{U}$ that includes the product of a ball in a fiber by a base, and are such that the 1-jet of the germs on U is zero. For every germ $f \in \mathcal{A}(U)$ the family of norms

$$\|f\|_\rho = \max_{U_\rho} |f|, \qquad U_\rho = \{|z| \le \rho\},$$

is defined for all sufficiently small ρ which run over the interval dependent of the germ.

Let $U_0 = U_+ \cap U_-$. Suppose that (z_+, t), (z_-, t) are two systems of coordinates on U_+ and U_-, respectively, $(z_-, t) = \mathbf{F}(z_+, t) = (z_+ + f(z_+, t), t)$. In this case, $f \in \mathcal{A}(U_0)$.

Lemma 3.1. *For every holomorphic vector-function $f \in \mathcal{A}(U_0)$ there exist neighborhoods U_+ and U_- of the disks $L+$ and L_- and the mappings $\mathbf{H}_+$, $\mathbf{H}_-$ which map biholomorphically U_+ and U_- onto its image, have an identical 1-jet at $z = 0$, and are such that the equation*

$$\mathbf{F} \circ \mathbf{H}_+ = \mathbf{H}_-, \qquad \mathbf{H}_+(z,t) = (z + h_+(z,t), t), \qquad \mathbf{H}_-(z,t) = (z + h_-(z,t) \tag{3.5}$$

is solvable for the germs $h_+(z,t)$, $h_-(z,t) \in \mathcal{A}(U_0)$.

Proof. Simple calculations turn equation (3.5) into the *functional equation*

$$h_- - h_+ = f \circ (z + h_+, t). \tag{3.6}$$

We shall first investigate the *homological equation*

$$h_- - h_+ = f. \tag{3.7}$$

Its solution is described in the following proposition.

Proposition 3.1. *Let $\mathcal{L}$ be an operator which brings the vector-function, which is holomorphic inside a ring and continuous on its closure $R_- < |t| < R_+$, into the decomposition to its Taylor and Laurent parts: two vector-functions, one of which is holomorphic in the disk containing 0, and the other is holomorphic in the disk containing infinity (and is zero at infinity) and whose difference is equal to the given function. Then the operator $\mathcal{L}$ is restricted in the norm of the space C.*

Proof. The operator $\mathcal{L}$ is given by the relation

$$\mathcal{L}f = (h_+, h_-), \qquad h_\pm(t) = -\frac{1}{2\pi i}\int_{\gamma_\pm} \frac{f(\zeta)\,d\zeta}{\zeta - t},$$

where $\gamma_\pm$ is a positively oriented circle $|t| = R_\pm$. The equality $h_- - h_+ = f$ follows from the Cauchy integral formula. Let γ_0 be a circle $|t| = R_0 = (R_+ + R_-)/2$ and L_+, L_-, K_+, K_- be disks

$|t| \leq R_+$, $|t| \geq R_-$, $|t| \leq R_0$, $|t| \geq R_0$. Then there exists c, which depends on R_+, R_- and such that

$$\max_{K_+} |h_+| \leq c \max_{L_+ \cap L_-} |f|; \qquad \max_{K_-} |h_-| \leq c \max_{L_+ \cap L_-} |f|.$$

However, the relations $h_+ = h_- - f$, $h_- = h_+ + f$ hold in the rings $L_+ \setminus K_+$, $L_- \setminus K_-$, and, consequently,

$$\max_{L_\pm} |h_\pm| \leq (c+1) \max_{L_+ \cap L_-} |f|.$$

This proves the proposition.

Let us return to the proof of the lemma. The operator $\mathcal{L}f$ used fiberwise on the straight lines $z = \text{const}$ gives the solution of the homological equation. Applying it to both sides of the functional equation, we get the problem on the fixed point: $(h_+, h_-) = \mathcal{L}f(z + h_+, t)$. The operator $\Phi: h = (h_+, h_-) \mapsto f(z + h_+, t)$ is a strongly contracting in the following sense: there exists c such that $\|\Phi(0)\|_\rho \leq c\rho^2$, $\operatorname{Lip} \Phi \leq c\rho$ for all sufficiently small ρ. This definition is given in [12] and the same work contains the proof of the following (obvious) statement: the product of a restricted operator by a strongly contracting operator is a contracting operator in the ball $\|h\|_\rho \leq \rho$ for all sufficiently small ρ.

It follows from this statement that the functional equation (3.6) is solvable for $h_\pm \in A(U_0)$. This proves Lemma 3.1 and, together with it, Theorem 3, modulo Theorem 5. Theorem 5 is proved in the next two sections.

Remark. The proof of Lemma 3.1 for the case of $n = 1$ was first obtained by Savel'ev with the aid of the Newton–Kolmogorov method [13]. The proof of Proposition 3.1 presented here follows that of Birkhoff. A. Bolibrukh attracted the attention of the author to the fact that this reasoning reduces the proof of Lemma 3.1 to the application of the principle of contraction mappings. The investigation of the neighborhood of an embedded sphere is carried out here according to the same plan that is used for the investigation of the neighborhood of an embedded torus [14]; by the way, the case of a torus is considerably more difficult.

4. FORMAL THEOREM ON THE REALIZATION OF MONODROMY

In this section we make the first step in the proof of Theorem 5 on realization. For $n = 1$, in the case of resonant multiplicators, this theorem was proved by Martinet and Ramis [15], and, independently, by Voronin and the author (not published). The theorem on the realization of wild one-dimensional mappings was proved, for the first time, by Peretz-Marco and Yoccoz [16]. After the talk with Peretz-Marco at the conference in Riŏ de Janéirò (January 1992), where this proof was presented, the author obtained the proof of the same theorem [8]. It served as the basis for the proof of Theorem 5 given below.

4.1. Formulation of the fundamental lemma.

Fundamental lemma. *For every Fuchsian system*

$$\dot{z} = \frac{A(t)}{t} z, \tag{4.1}$$

defined in the neighborhood of zero, for every natural N, and every biholomorphic germ $\Delta: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, whose derivative at zero coincides with the monodromy transformation of the system, there *exists a natural l, which depends only on the spectrum λ of the matrix $A(0)$, and there also exists an equation*

$$\dot{z} = \frac{(A(t) - lE)z + f(t,z)}{t}, \qquad f(t,z) = O(|z|^2), \tag{4.2}$$

which depends on l and on N, whose monodromy transformation corresponding to the fiber $z = 0$ coincides with Δ with an accuracy to within $O(|z|^N)$.

It is natural to call this lemma a formal analog of Theorem 5. The theorem will be derived from the lemma in the next section, and in this section the lemma is proved.

4.2. The Poincaré–Dulac theorem and Level's normal form for Fuchsian systems. We can assume without loss of generality that the residue matrix $A(0)$ of the Fuchsian system (4.1) is reduced to the Jordan normal form. In addition, its eigenvalues, which differ from one another by an integer (i.e., which are resonant), are arranged on the diagonal in the order of decrease of the real parts. Let us consider the autonomous system

$$\dot{z} = A(t)z, \qquad \dot{t} = t, \tag{4.1$'$}$$

corresponding to the nonautonomous system (4.1). By the formal linear substitution

$$w = H(t)z \tag{4.3}$$

with the matrix $H(t)$ which is holomorphic and invertible in the neighborhood of zero, we can reduce system (4.1) to the Poincaré–Dulac normal form. The resonant terms for the original system are linear with respect to w and have the form $w_j t^k \partial/\partial w_i$. Only these terms remain in normal form. The corresponding resonant relations have the form $\lambda_i = \lambda_j + k$. Thus, system (4.1$'$) is formally equivalent to the system

$$\dot{w} = B(t)w, \qquad \dot{t} = t, \tag{4.2$'$}$$

in which

$$b_{ij}(t) = b_{ij}t^{\lambda_i - \lambda_j}, \qquad b_{ij} \in \mathbb{C} \tag{4.4}$$

under the condition that the exponent is a positive integer; in addition, terms of zero power with respect to t have the form $B(0)w$, where $B(0) = A(0)$ is a matrix written in Jordan normal form. The matrix $B(t)$ is upper-triangular since the eigenvalues of the matrix $B(0)$ are ordered so that the relation $\lambda_i - \lambda_j = k \in \mathbb{N}$ implies $i < j$. Note that $b_{ii}(t) = b_{ii} = \text{const}$. The numbers $\mu_i = b_{ii}$ are eigenvalues of the residue matrix $B(0)$.

Consequently, system (4.1) is formally equivalent to the system

$$\dot{w} = \frac{B(t)}{t}w. \tag{4.5}$$

It turns out that in this way we have also described the analytic normal form of system (4.1). The matter is that the formal equivalence of Fuchsian system implies their analytic equivalence [17]. Therefore the formal coordinate change H is, in fact, convergent.

It is sufficient to prove the fundamental lemma if system (4.1) in its formulation is replaced by its normal form (4.4), (4.5) and the monodromy transformation Δ is replaced by $\Phi = H^{-1}(1)\Delta H(1)$.

By the hypothesis of the lemma, the monodromy transformation of system (4.4), (4.5) is equal to $d\Phi(0)$. For brevity, we shall call the monodromy matrix and the residue matrix of the Fuchsian system by its monodromy and residue. For system (4.5) we denote these matrices by Q and $B(0)$ respectively.

Proposition 4.1. *Two systems* (4.4), (4.5) *with the same monodromy and residue coincide.*

Proof. We first construct the fundamental matrix W of system (4.4), (4.5) normalized by the condition that one of the values $W(1)$ is equal to E and then prove that W is uniquely defined by the monodromy and residue of this system.

Let $B(t)$ be the matrix of system (4.5). We set

$$B(1) = B_0 + B_1, \qquad B_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

The matrix B_1 is upper-triangular nilpotent since $B(t)$ is upper-triangular with the same diagonal as B_0.

The normalized system (4.4), (4.5) is integrable and has a fundamental matrix of the form

$$W = t^{B_0} t^{B_1}. \tag{4.6}$$

To verify this fact, it suffices to verify the relation

$$t\dot{W}W^{-1} = B(t). \tag{4.7}$$

We have

$$t\dot{W}W^{-1} = B_0 + t^{B_0} B_1 t^{-B_0}.$$

The second term on the right-hand side is an upper-triangular matrix. We denote it by $C(t)$ and its elements by $c_{ij}(t)$, and the elements of the matrix B_1 by b_{ij}. Then

$$c_{ij}(t) = b_{ij} t^{\lambda_i - \lambda_j}.$$

Relation (4.4) implies $B_0 + C(t) = B(t)$. This proves (4.7).

The matrix $B(t)$ of system (4.5) may be split into blocks corresponding to the eigenvalues of the matrix $B(0)$ which are resonant. The monodromy matrix may be split into similar blocks. They are known as *resonance blocks*. Let us prove that the monodromy matrix of system (4.5) is equal to

$$G = \exp 2\pi i B_0 \exp 2\pi i B_1 \tag{4.8}$$

and has eigenvalues

$$\nu_j = \exp 2\pi i \lambda_j, \qquad j = 1, \dots, n. \tag{4.9}$$

Indeed,

$$W(t \exp 2\pi i) = t^{B_0} \exp 2\pi i B_0 t^{B_1} \exp 2\pi i B_1 = W(t) \exp 2\pi i B_0 \exp 2\pi i B_1.$$

However, for $t = 1$ we have $t^{B_0} = t^{B_1} = E$. This proves relation (4.8). It immediately implies (4.9).

Proposition 4.1 follows from relation (4.8). Indeed, the matrix B_1 can be uniquely restored from the matrix G. The matrix B_0 can be uniquely restored from the matrix $B(0)$. In turn, the matrix $B(t)$ can be restored from the matrix $B(1) = B_0 + B_1$. This proves the proposition.

4.3. Extension of the normalized monodromy transformation to an extended phase space. We denote by $\mathbf{A}$ the linear vector field

$$\mathbf{A} = \sum \mu_j w_j \frac{\partial}{\partial w_j} + t\frac{\partial}{\partial t}, \qquad \mu_j = \lambda_j - l,$$

and choose the natural l below.

The extension indicated in the title is carried out so that the extended mapping would commute with the field $\mathbf{A}$.

Remark. The coordinates w, t are introduced in the space $\mathbb{C}^{n+1}$. This makes it possible to write the vector fields and mappings by the same formulas, interpreting the vector fields and mappings as vector-functions. The mapping

$$H = \sum h_{kmj} w^k t^m \frac{\partial}{\partial w_j} + t\frac{\partial}{\partial t}$$

commutes with $\mathbf{A}$:

$$(dH)\mathbf{A} = \mathbf{A} \circ H,$$

if and only if the relation

$$\mu_j = (k, \mu) + m$$

is satisfied for some integer m and for all monomials in the decomposition of H.

This remark motivates the definition of resonances.

The shortened normalized transformation Φ consists only of resonance monomials

$$\Phi(w) = \sum g_{ij} w_j \frac{\partial}{\partial w_i} + \sum_{|k|=2}^{N} a_{kj} w^k \frac{\partial}{\partial w_j},$$

with

$$a_{kj} \neq 0 \Rightarrow \nu_j = \nu^k, \qquad g_{ij} \neq 0 \Rightarrow \nu_i = \nu_j, \qquad i < j. \tag{4.10}$$

It follows from (4.10) that for $a_{kj} \neq 0$ we have $\lambda_j = (\lambda, k) + m_{jk}$, where m_{jk} are certain integers, not necessarily nonnegative. We replace λ_j by $\mu_j + l$ with a natural l. The number l, dependent only on λ, can be chosen so large that for the same j, k

$$\mu_j = (\mu, k) + n_{j,k}$$

with a natural $n_{j,k}$.

We extend the mapping Φ to the mapping

$$\mathbf{F}\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0), \qquad \mathbf{F}(w, t) = (F_t(w), t)$$

so that the extended mapping would commute with $\mathbf{A}$, namely, we set

$$F_t(w) = \sum g_{ij} w_i t^{\lambda_i - \lambda_j} \frac{\partial}{\partial w_j} + \sum_{|k|=2}^{N} a_{kj} w^k t^{n_{k,j}} \frac{\partial}{\partial w_j}.$$

We have constructed the required extension.

4.4. Completion of the proof of the fundamental lemma. Here we construct a quasi-generator $\mathbf{V} = (v, t)$ of the mapping $\mathbf{F}$ whose linear part with respect to w, are equal to $(B(t)w, t)$. The phase flow transformation g of such a field corresponding to the time $2\pi i$, restricted to the plane $t = 1$, coincides with the monodromy transformation Δ of the equation

$$\dot{w} = \frac{v}{t}, \tag{4.11}$$

and the equation itself, after substitution (4.3), passes into (4.2). In turn, the transformation g coincides with $\mathbf{F}$ with an accuracy up to $o(|w|^N)$. Consequently, the same holds true for the mappings $\Phi = \mathbf{F}|_{t=1}$ and Δ.

Thus, it remains to construct a quasi-generator $\mathbf{V}$. Let $\mathbf{F}_0$ be a diagonal linear operator such that the mapping $d\mathbf{F}(0)\mathbf{F}_0^{-1}$ is unipotent.

We set

$$\mathbf{F} = \mathbf{F}_0 \circ \mathbf{F}_1.$$

The mapping $\mathbf{F}_1$ has a unipotent linear part and commutes with the vector field $\mathbf{A}$.

Proposition 4.2. *Suppose that a mapping with a unipotent linear part commutes with the diagonal linear vector field* $\mathbf{A}$. *Then it has a quasi-generator* $\mathbf{v}$ *with a nilpotent linear part commuting with* $\mathbf{A}$.

This proposition will be proved later. Let us derive from it the fundamental lemma.

Obviously, the field $\mathbf{A}$ is a generator of the mapping $\mathbf{F}_0$. Let $\mathbf{v}$ be an almost-generator of the mapping $\mathbf{F}_1$ given by Proposition 4.2. Then the field $\mathbf{A} + \mathbf{v}$ is an almost-generator of the mapping $\mathbf{F} = \mathbf{F}_0 \circ \mathbf{F}_1$. Indeed, the transformations of the phase flows of the fields $\mathbf{A}$ and $\mathbf{v}$ commute. Therefore

$$g^{2\pi i}_{\mathbf{A}+\mathbf{v}} = g^{2\pi i}_{\mathbf{A}} \circ g^{2\pi i}_{\mathbf{v}} = \mathbf{F}_0 \circ (\mathbf{F}_1 + O(|w|^N)).$$

Note that the linearization of system (4.11) with respect to w is a Fuchsian system. The corresponding autonomous system (4.5) commutes with the field $\mathbf{A}$ and, consequently, is a resonant system. Its monodromy transformation is equal to $G = d\Phi(0)$ and the diagonal of the residue matrix coincides with μ. According to Proposition 4.1, this system is obtained from the z-linear part of (4.2), by means of substitution (4.3).

We have identified the linear part of system (4.11). The fundamental lemma is proved modulo Proposition 4.2.

Proof of Proposition 4.2. It is well known that the germ of a biholomorphic mapping with a unipotent linear part is included into the flow with an accuracy to within the terms of any power, and the generator is a field with a nilpotent linear part. Let us recall the proof of this fact. We fix the arbitrary N and consider the group $\mathbf{G}$ of N-jets of the mappings $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ with a unipotent linear part. We construct a sharp linear representation of this group in the space J of all N-jets of the mappings $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$. The group $\mathbf{G}$ acts on J by right shifts. To be more precise,

$$T_F \colon J \mapsto J, \qquad \phi \mapsto j^N(\phi \circ F^{-1}).$$

This is obviously a homomorphism and even monomorphism: the germ F can be restored from its action on the identity mapping ϕ.

The operators T_F are unipotent. The Lie group of all unipotent operators is exponential; its Lie algebra consists of nilpotent operators. Let L_F be a nilpotent logarithm of T_F. Then $-L_F$ corresponds to the almost-generator of the germ F. The components of the vector field F are components of the germ $L_F\phi$ for $\phi = id$.

The proof of Proposition 4.2 can be obtained from this reasoning by a simple modification. Let $\mathbf{G_A}$ and $J_\mathbf{A}$ be a subgroup of the group $\mathbf{G}$ and a subspace of the space J consisting of jets that commute with $\mathbf{A}$. Obviously, as before, $\mathbf{G_A}$ transfers $J_\mathbf{A}$ into itself acting by right shifts. All mappings from $J_\mathbf{A}$ may be decomposed into the sums of resonant monomials (commuting with $\mathbf{A}$). Consequently, the almost-generator of the jet $F \in \mathbf{G_A}$ has the same property.

This proves Proposition 4.2 and, together with it, the fundamental lemma.

5. GEOMETRIC THEOREM ON THE REALIZATION OF MONODROMY TRANSFORMATION

Here we complete the proof of the strong theorem on the realization of monodromy (Theorem 5). Together with the theorem from Section 3, it proves the sufficient condition of solvability of the nonlinear Riemann–Hilbert problem.

5.1. Annihilation of a small residual. The following lemma constitutes the main result of this subsection.

Geometric lemma. *For every biholomorphic germ* $\Delta: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *and every Fuchsian system* (4.1) *there exist numbers* N *and* l *which possess the following property. Suppose that the monodromy transformation of the equation*

$$\dot z = \frac{(A(t) - lE)z + f(t,z)}{t}, \qquad f(t,z) = O(|z|^2), \tag{4.2}$$

corresponding to the fiber $z = 0$, *coincides with* Δ *with an accuracy* $O(|z|^N)$. *Then there exists an equation*

$$\dot z = \frac{(A(t) - lE)z + g(t,z)}{t}, \qquad g(t,z) = O(|z|^2), \tag{5.1}$$

whose monodromy transformation, which corresponds to the same fiber, exactly coincides with Δ.

The existence of equation (4.2), whose monodromy is a good approximation of the germ is included into the hypothesis of the geometric lemma. The existence of such an equation is proved by the fundamental lemma. Therefore Theorem 5 follows from the fundamental and geometric lemmas.

The proof of the geometric lemma is based on the idea which is a key idea for the whole investigation of the nonlinear Stokes phenomenon. The equation with the given monodromy transformation can be constructed "by hands." In this case, its phase space turns out to be an abstract manifold a priori not embedded into any numerical space, and the equation on it is defined not by a relation of the type of (5.1), but simply by a holomorphic field of lines which defines the foliation with one-dimensional leaves. The main part of the proof consists in *identifying* the constructed phase space and foliation as the domain in $\mathbb{C}^{n+1}$ and equation (5.1).

5.2. Foliation with a given monodromy on a glued manifold. Here we construct a manifold, which is diffeomorphic to the product of a punctured circle by an n-dimensional ball, and a foliation on it, which has the given monodromy. The construction is carried out by means of glueing from the following data.

Let N be a natural number as large as required for the sequel. Suppose that Δ is a germ mentioned in the geometric lemma, Δ_0 is a germ of the monodromy transformation of equation (4.2) (everywhere in what follows the monodromy corresponds to the fiber $z = 0$), which approximates the initial germ with an accuracy $O(|z|^N)$:

$$\Delta = (id + h) \circ \Delta_0, \qquad h = O(|z|^N). \tag{5.2}$$

Suppose that the eigenvalues of the matrix $A(0) - lE$ are negative and $\mathcal{F}_0$ is a foliation defined by equation (4.2). Let v_0 be the corresponding vector field

$$v_0 = (t, (A(t) - lE)z + f(t, z)),$$

and λ be the spectrum of its linearization at zero

$$\lambda = (1, \lambda_1, \dots, \lambda_n).$$

By the hypothesis of the lemma, $\operatorname{Re} \lambda_j < 0$. According to the Hadamard–Perron theorem, the stable manifold of the germ v_0 is holomorphic; we can assume that it has the form $t = 0$.

We denote by K a punctured one-dimensional disk and by B an n-dimensional ball (everything is complex), and set $M_0 = K \times B$. Let t be a chart on K such that $t = 0$ at the deleted point.

Let us consider the sector $\widetilde{K}$ on the universal covering over K:

$$\widetilde{K} = \{\tilde{t} = re^{i\theta},\ \theta \in (-\varepsilon, 2\pi + \varepsilon)\}.$$

The natural projection $\widetilde{K} \to K$ is defined. We set

$$\widetilde{M} = \widetilde{K} \times B, \qquad S = \{re^{i\theta} |\ \theta \in (-\varepsilon, \varepsilon)\}, \qquad S' = \{re^{i\theta} |\ \theta \in (2\pi - \varepsilon, 2\pi + \varepsilon)\}.$$

We denote the foliation $\mathcal{F}_0$, lifted to $\widetilde{M}$, by $\widetilde{\mathcal{F}}$. Let $t \in S$, and then $t' \in S'$ is a point with the same projection as t.

Let us construct a biholomorphic glueing

$$\Phi: S' \times B \to S \times \mathbb{C}^n$$

with the following properties:

Φ preserves the function t;

Φ respects the foliation $\widetilde{\mathcal{F}}$, i.e., transfers the leaves to the leaves.

The third, principal, property is based on the first two and on the following remark. The glueing Φ turns $\widetilde{M}$ into an analytical manifold which, as is shown below, is diffeomorphic to M_0. Under the action of the glueing, the foliation $\widetilde{\mathcal{F}}$ turns into the foliation $\mathcal{F}$ over M.

The third requirement to the glueing Φ is that the foliation $\mathcal{F}$ should have the monodromy Δ.

Let us prove that such a glueing does exist. We shall seek it in the form

$$\Phi(t, z) = (t, \Phi_t(z)).$$

This ensures the first requirement. Let h be the same function as in (5.2).

We shall glue together the points in the plane $t = 1$:

$$\Phi: (1', z) \mapsto (1, \Phi_1(z)), \qquad \Phi_1(z) = z + h(z). \tag{5.3}$$

We extend this mapping along the leaves of the foliation $\mathcal{F}_0$. To this end, we consider the mapping

$$G^t: 1 \times (\mathbb{C}^n, 0) \to t \times (\mathbb{C}^n, 0)$$

along the solutions of equation (4.2). By changing the scale, we can obtain $1 \in S$. It follows from the second requirement (the preservation of the foliation $\mathcal{F}_0$) that

$$\Phi(t', z) = (t, G^t \circ \Phi_1 \circ (G^t)^{-1}(z)). \tag{5.4}$$

This glueing map biholomorphically transforms $S' \times B$ onto its image, as follows from Proposition 5.1 proved below. The glueing map Φ transforms the leaf φ_p into the leaf φ_q, where $p = (1', z)$, $q = (1, \Phi_1(z))$. Finally, the foliation $\mathcal{F}$ has the monodromy Δ. Indeed, upon the extension over the circle $\{t = e^{i\theta},\ \theta \in [0, 2\pi]\}$, the point $(1, z)$ passes into the point $(1', \Delta_0(z))$, and upon the glueing Φ into the point $(1, \Delta(z))$. Note that the mapping Φ can be holomorphically extended to a certain neighborhood of the curve $z = 0$ in $\widetilde{M}$. We denote the extended mapping by the same symbol. We have constructed the required manifold and foliation. It only remains to identify them. However, we shall first prove that the glueing Φ rapidly tends to an identity mapping as the argument tends to the plane $t = 0$.

5.3. Investigating the glueing map Φ.

Proposition 5.1. *Let S' be an arbitrary sector with vertex at zero, containing S and contained in $\widetilde{K}$. For every $k > 0$, there exists $N > 0$ such that the glueing map* (5.3), (5.4) *possesses the property*

$$|\Phi_t(z) - z| = o(t^k) \quad \textit{in} \quad S' \times B$$

as $z \to 0$, if relation (5.2) *is satisfied.*

Proof. The foliation $\mathcal{F}_0$, which is used to construct the glueing map, is defined by equation (4.2). The principal part of the numerator is equal to Cz, where

$$C = A(0) - lE.$$

We choose the number l such that the eigenvalues μ_j of the matrix C lie in the left half-plane. Let us consider the mapping

$$G^t: 1 \times (\mathbb{C}^n, 0) \to t \times (\mathbb{C}^n, 0)$$

along the solution of equation (4.2). By definition, the glueing map Φ has the form (5.4). It follows from the inequality $\operatorname{Re} \mu_j < 0$ that G^{-t} is a strong contraction for small t lying in the sector S. For any mapping $\Psi: z \mapsto z + o(|z|^N)$ the operator

$$R^t: \Psi \mapsto G^t \circ \Psi \circ (G^t)^{-1}$$

diminishes the discrepancy the stronger the larger N is (R from recalling).

Before proving this statement in full generality, we shall illustrate it by two particular cases.

Case 1. Suppose, in equation (4.2), $n = 1$, $A(t) \equiv A(0)$, $f \equiv 0$. Then equation (4.2) takes the form

$$\dot{z} = \frac{C}{t}z, \quad \text{where} \quad z \in \mathbb{C}^1.$$

For such an equation, $G^t = t^C z$, $\operatorname{Re} C < 0$. We fix a small t and set $\varepsilon = t^C$, $|\varepsilon| \ll 1$. The operator R^t acts according to the relation

$$R^t\colon z + \sum_N^\infty a_k z^k \mapsto z + \sum_N^\infty \varepsilon^{k-1} a_k z^k,$$

whence immediately follows case 1 of the proposition.

Case 2. Suppose, in equation (4.2) n is arbitrary, $A(t) \equiv A(0)$, $f \equiv 0$. Equation (4.2) has the same form as in Case 1, with the only difference that $z \in \mathbb{C}^n$. By a linear substitution in the space $\mathbb{C}^n$ we can achieve a situation where the operator C will be almost diagonal: its off-diagonal terms will be small. The basis in which the operator can be written by this matrix will be regarded as orthonormal in $\mathbb{C}^n$. We set

$$\alpha = \min(-\operatorname{Re}\mu_j), \qquad \beta = \max(-\operatorname{Re}\mu_j). \tag{5.5}$$

Then, for small $t \in S$ and small ε,

$$\|t^{-C}\| \le |t|^{\alpha/2}, \qquad \|t^C\| \le |t|^{-2\beta}. \tag{5.6}$$

We can now repeat the relation for the action of R^t almost verbatim, with the only difference that z is a vector and k is a multiindex:

$$R^t\colon z + \sum_{|k|=N}^\infty a_k z^k \mapsto z + \sum_{|k|=N}^\infty t^C a_k (t^{-C} z)^k.$$

Consequently, if $\psi(z) = z + O(z^N)$, then, for a certain $L = L(\psi)$, we have

$$|R^t\psi(z) - z| \le L|t|^{c(N)}, \qquad c(N) = N\alpha/2 - 2\beta. \tag{5.7}$$

General case. In the general case, the analog of inequalities (5.6) holds true, i.e., for small $t \in S$, we have

$$|(G^t)^{-1}(z)| \le |t|^{\alpha/2}, \ |G^t(z)| \le |t|^{-2\beta}. \tag{5.8}$$

These inequalities will be proved below.

Proposition 5.1 can be derived from them in the same way as in Case 2:

$$R^t\colon z + \sum_N^\infty a_k z^k \mapsto G^t\Big[(G^t)^{-1}z + \sum_N^\infty a_k((G^t)^{-1}z)^k\Big].$$

According to the Cauchy inequality, for all t lying in a sector, which is narrower than S,

$$\left\|\frac{dG^t}{dz}\right\| \le \gamma t^{-(2\beta+1)}$$

for a certain $\gamma > 0$. Consequently, for the same ψ as in Case 2, estimate (5.7) in which $c(N) = N\alpha/2 - 2\beta - 1$ holds true. This proves the proposition modulo (5.8).

Inequalities (5.8) can be proved by means of standard estimates used, for instance, in the proof of Lyapunov's stability theorem. Namely, we set $t = re^{i\varphi}$. Then, for $t \in S$, we have

$$\frac{\partial z}{\partial r} = \dot{z}e^{i\varphi} = \frac{Cz + (A(t) - A(0))z + f(t,z)}{r}.$$

Consequently,

$$\frac{\partial |z|^2}{\partial r} = \operatorname{Re}\frac{(Cz, \bar{z})}{r} + O(1)|z|^2.$$

Inequalities (5.8) now follow from the almost-diagonality of C and definitions (5.5).

This completes the proof of Proposition 5.1.

5.4. Identifying a phase space.

Proposition 5.2. *The manifold M is diffeomorphic to M_0.*

Proof. We construct a mapping $\widetilde{G}: \widetilde{M} \to M_0$ which is lowered to the mapping $G: M \to M_0$. For our purpose, the relation

$$\widetilde{G}|_{S' \times B} = \widetilde{G} \circ \Phi|_{S \times B} \tag{5.9}$$

is sufficient. On the manifold $\widetilde{M}$ we construct a function y such that the mapping $\widetilde{G}: (\tilde{t}, z) \mapsto (\tilde{t}, y)$ satisfies relation (5.9).

A wide arbitrariness is possible in the construction. We set

$$y|_{S \times B} = z.$$

Then, by virtue of (5.9),

$$y|_{S' \times B}(t,z) = \Phi_t(z).$$

Suppose that χ is a smooth function on $(-\varepsilon, 2\pi + \varepsilon)$ equal to zero on the interval $(-\varepsilon, \varepsilon)$ and to 1 on $(2\pi - \varepsilon, 2\pi + \varepsilon)$.

Let $\theta(\tilde{t}) = \arg \tilde{t}$, $\tilde{\chi}(\tilde{t}) = \chi(\theta(\tilde{t}))$. Suppose that π is a natural projection $\widetilde{M} \to M_0$, $(z, \tilde{t}) \mapsto (z, t)$. We set

$$\widetilde{G}(\tilde{t}, z) = (y(\tilde{t}, z), t), \qquad y(\tilde{t}, z) = z + \tilde{\chi}(\tilde{t})(\Phi_{\tilde{t}}(z) - z). \tag{5.10}$$

We define the diffeomorphism G by the relation

$$\widetilde{G} = G \circ \pi. \tag{5.11}$$

This definition is correct since $\widetilde{G}$ satisfies relation (5.9). It follows from Proposition 5.1 that G is a diffeomorphism, at least on the manifold M^ε, obtained from

$$\widetilde{M}^\varepsilon = \{(t,z) \mid t \in \widetilde{K},\ |z| < \varepsilon\}$$

by the glueing Φ.

Lemma 5.1. *For a sufficiently small ε the manifold M^ε is biholomorphically equivalent to the neighborhood of zero in $(\mathbb{C}^{n+1}, 0,)$, from which the plane $t = 0$ is deleted.*

Idea of the proof. The mapping G transfers the complex structure defined on M into the almost-complex structure on M_0. This structure turns out to be integrable and, what is most essential, can be extended up to a smooth integrable, almost-complex structure to the neighborhood of zero in $\mathbb{C}^{n+1}$ enveloping M_0 (see Proposition 5.3 below). According to the Nirenberg–Newlandler theorem, there exists a mapping H of a certain neighborhood of zero in $\mathbb{C}^{n+1}$ into a similar neighborhood which is biholomorphic in the sense of this structure. Without loss of generality, $dH(0) = \mathrm{id}$. The mapping $F = H \circ G$ is biholomorphic and transfers the manifold M_ε into the domain which contains the neighborhood of zero in the space $\mathbb{C}^{n+1}$. This completes the identification of the manifold M.

Proposition 5.3. *Let $G: M \to M_0$ be the diffeomorphism* (5.11), (5.10) *and σ be almost-complex structure on M_0 transferred from M by the mapping G. Then for a sufficiently large N in relation* (5.2), *the structure σ can be C^{n+2}-smoothly extended up to the integrable, almost-complex structure on* $(\mathbb{C}^{n+1}, 0)$.

Proof. An almost-complex structure is defined by the subbundle of the cotangent bundle consisting of the forms called forms of the type of $(1, 0)$.

We describe the complex structure on M. Let us consider on $\widetilde{M}$ two collections of $n+1$ holomorphic 1-forms

$$\{dt,\ dz = (dz_1, \dots, dz_n)\} \quad \text{and} \quad \{dt, d\Phi_t(z)\}.$$

Any collection of the forms

$$dt, \quad \zeta, \quad \zeta = \alpha\, dz + \beta\, d\Phi_t(z)$$

with nonzero α, β generates the same complex structure on $\widetilde{M}$. Let us construct a collection ζ of n 1-forms such that it is a collection of forms on M. Let

$$\zeta|_{S\times B} = dz.$$

Then

$$\zeta|_{S'\times B} = d\Phi_t(z).$$

Suppose that the function χ is the same as above. We set

$$\zeta = dz + \tilde{\chi}\, d(\Phi_t(z) - z).$$

This is a correctly defined collection of forms on M. On the manifold M_0, there arises an almost-complex structure transferred from M by the mapping G and defined by the forms

$$dt, \quad \omega = (G^{-1})^* \zeta.$$

If the number N in relation (5.2) is sufficiently large, then

$$j^{n+2}(\omega - dz) \to 0 \tag{5.12}$$

as $t \to 0$ in the sector with vertex 0 whose closure contains the support of the truncating function $\tilde{\chi}$. This follows from Proposition 5.1. Let us extend the form ω to the plane $t = 0$ by the form dz. It follows from (5.12) that this extension is C^{n+2}-smooth. Thus, the almost-complex structure dt, ω

is smoothly extended to the neighborhood of zero in the space $\mathbb{C}^{n+1}$. The condition of integrability of this structure is fulfilled for $t \neq 0$ since the structure is transferred to M_0 from M by means of the diffeomorphism. In the plane $t = 0$, the integrability condition is fulfilled by continuity.

This proves Proposition 5.3.

Proposition 5.3 yields Lemma 5.1 as was explained at the beginning of the proof of the lemma.

5.6. Identification of a foliation. It remains to prove that the foliation $F_*\mathcal{F}$ is defined by equation (5.1) in a suitable coordinate system.

Proposition 5.4. *There exists a coordinate system in the space $\mathbb{C}^{n+1}$ near zero in which the foliation $F_*\mathcal{F}$ is defined by an equation of the form* (5.1) *with the monodromy Δ.*

Proof. The foliation $F_*\mathcal{F}$ coincides with $\tilde{F}_*\mathcal{F}_0$. The mapping $\tilde{F} = H \circ \tilde{G}$ transforms some a certain neighborhood of the sector $\widetilde{K}' \subset \widetilde{K} \subset \widetilde{M}$, with the same radius as $\widetilde{K}$ has and a smaller angle into the product $K' \times B'$. Here $B' \subset B$ is a smaller ball with the same center and K' is a punctured circle covered by the sector $\widetilde{K}'$. Let us investigate the mapping $\tilde{F}$.

Proposition 5.5. *The mapping $F_1 = \tilde{F}|_{\tilde{t}=1'}$ is holomorphic; the mapping $L(t) = \frac{\partial \tilde{F}}{\partial z}|_{z=0}$ is correctly defined on the circle K', is holomorphic and invertible.*

Proof. Over the sector S the almost-complex structure defined by the forms dt, ω coincides with the initial complex structure. Therefore the mapping H is holomorphic over S. The mapping $\tilde{G}$ has the same property, whence follows the first statement of the proposition.

Next, the mapping $\frac{\partial \tilde{F}}{\partial z}|_{z=0}$ is also holomorphic and invertible on $\widetilde{K}'$. However, by construction of the mappings $\tilde{G}$, $\frac{\partial \tilde{G}}{\partial z}|_{z=0} = E$. Consequently, L is correctly defined and holomorphic on K'. However, L is bounded on the disk $D = K + \{0\}$. According to the removable singularity theorem, L is holomorphic on D.

The foliation $F_*\mathcal{F} = \tilde{F}_*\mathcal{F}_0$ is defined by an equation whose similarity to (4.2) will be established below. We write equation (4.2) in the form

$$\dot{w} = \frac{v(t, w)}{t}. \tag{5.13}$$

The vector-function v is holomorphic in $(\mathbb{C}^{n+1}, 0)$. According to Proposition 5.3, the coordinate change

$$U: (z, t) \mapsto (u(z, t), t), \qquad u(z, t) = L^{-1}(t)L(1)F_1(z)$$

is holomorphic in $(\mathbb{C}^{n+1}, 0)$. The change $U \circ \tilde{F}$ transforms (5.13) into the equation

$$\dot{u} = \frac{\tilde{v}(u, t)}{t}. \tag{5.14}$$

However, the last change is identical in the plane $\tilde{t} = 1'$ and has an identical linear part with respect to z for $z = 0$ on the punctured circle K'. In addition, the derivative $d\tilde{F}$ is bounded. Consequently, the field $\tilde{v}$ can be holomorphically extended to the plane $t = 0$, and its linear part coincides with the linear part of the field v.

By the construction of the glueing Φ, equation (5.14) has monodromy Δ. This proves Proposition 5.4.

We have thus identified the foliation and completely proved the geometric lemma. It yields Theorem 5, as was proved in 5.1.

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