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Bijective Mappings of Meshes with Boundary and the Degree in Mesh Processing*

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Abstract. This paper introduces three sets of sufficient conditions for generating bijective simplicial mappings of manifold meshes. A necessary condition for a simplicial mapping of a mesh to be injective is that it either maintains the orientation of all elements or flips all the elements. However, this condition is known to be insufficient for injectivity of a simplicial map. In this paper we provide additional simple conditions that, together with the above-mentioned necessary condition, guarantee injectivity of the simplicial map. The first set of conditions generalizes classical global inversion theorems to the mesh (piecewise-linear) case. That is, it proves that in the case where the boundary simplicial map is bijective and the necessary condition holds, the map is injective and onto the target domain. The second set of conditions is concerned with mapping of a mesh to a polytope. It replaces the (often hard) requirement of a bijective boundary map with a collection of linear constraints and guarantees that the resulting map is injective over the interior of the mesh and onto. These linear conditions provide a practical tool for optimizing a map of the mesh onto a given polytope while allowing the boundary map to adjust freely and keeping the injectivity property in the interior of the mesh. Allowing more freedom in the boundary conditions is useful for two reasons: (a) it circumvents the hard task of providing a bijective boundary map, and (b) it allows one to optimize the boundary map *together* with the simplicial map to achieve lower energy levels. The third set of conditions adds to the second set the requirement that the boundary maps be orientation preserving as well (with a proper definition of boundary map orientation). This set of conditions guarantees that the map is injective on the boundary of the mesh as well as its interior. Several experiments using the sufficient conditions are shown for mapping triangular meshes. A secondary goal of this paper is to advocate and develop the tool of degree in the context of mesh processing.

Key words. meshes, simplicial maps, injectivity

AMS subject classifications. 65D18, 52, 57

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1. Introduction.

Triangular and tetrahedral meshes are prominent in representing surfaces and volumes in various fields such as computer graphics and vision, medical imaging, and engineering. Many of the algorithms and applications that use mappings of meshes require injectivity of the map to operate correctly. Nevertheless, injectivity in general is a hard constraint, and guaranteeing it poses a real challenge.

The main goal of this paper is to provide practical sufficient conditions that assure that a simplicial mapping $\Phi : \mathbf{M} \to \mathbb{R}^d$ is an injection taking a d-dimensional compact mesh $\mathbf{M}$ onto a prescribed polytope $\Omega \subset \mathbb{R}^d$. A secondary goal is to advocate and develop the tool of degree in the context of mesh processing.

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In classical analysis and elasticity theory, global inversion theorems provide sufficient conditions for injectivity, starting with the work of Hadamard, Darboux, and Stoilow and followed by the work of many others; see, for example, [16, 15, 4, 6, 14, 13, 7]. Nevertheless, most of the previous results deal with smooth mappings and/or require local injectivity of the map, both of which are unnatural for the mesh case (note that even if every element of a mesh is not inverted it is still not necessarily locally injective). More importantly, the sufficient conditions offered in previous works mostly assume that the mapping under consideration is bijective when restricted to the boundary of the domain. In practice, this condition is often too restrictive for two main (practical) reasons: first, constructing a bijective boundary mapping can be as challenging as the original problem, and second, in applications we are often required to find a mapping of a mesh that is minimizing a certain cost (or energy) and we do not know a priori what would be the optimal boundary map.

A necessary condition for a simplicial map Φ to be injective is that the map either maintains the orientations of all elements or inverts the orientation of all elements. Nevertheless, as is known (and shown later in this paper), this condition is not sufficient for injectivity. In this work we will prove three sets of conditions that, in addition to the necessary conditions, form sufficient conditions for injectivity.

The first set of conditions generalizes the global inversion theorems directly to meshes (albeit without the usual smoothness or local injectivity requirements). It shows that a sufficient condition for injectivity of a simplicial map Φ is that the boundary map $\Phi|_{\partial\mathbf{M}} : \partial\mathbf{M} \to \partial\mathbf{\Omega}$ is a bijection, together with the necessary condition of orientation consistency.

The second set of conditions aims at the problem of mapping the mesh $\mathbf{M}$ onto a polytope $\mathbf{\Omega}$ and allows weaker boundary conditions, where a particular boundary mapping is not required. The benefit of this second set is that it only adds a set of *linear* constraints (in addition to the necessary conditions) and therefore allows the building of algorithms that optimize over a *collection* of boundary maps while guaranteeing that the map Φ is injective over the interior of the mesh $\mathbf{M}$ and that it covers the target domain $\Phi(\mathbf{M}) = \mathbf{\Omega}$. Note that it does not in general guarantee the injectivity of the boundary mapping, as detailed later.

The third set of conditions also aims at mapping M to a polytope $\mathbf{\Omega}$ and adds to the second set of conditions the requirement that the boundary maps be also orientation preserving (to be defined precisely soon), therby guaranteeing the injectivity of the map over the entire $\mathbf{M}$ (including the boundary of $\mathbf{M}$).

A work related to ours is that of Floater [8] generalizing Tutte's paper on drawing a planar graph [21] that also provides a set of sufficient conditions for generating injective mappings of disk-type triangular meshes ($d = 2$ meshes) mapped onto a convex polygonal domain. More general sufficient conditions have been studied in [9]. Unfortunately, these constructions do not generalize to higher dimensions and/or nonconvex target domains. Other related works that studied local and global injective mappings include [3, 22, 19, 18]; however these works do not seem to overlap strongly with our goals. The current paper also provides full background and mathematical underpinning to the injectivity arguments from our previous papers [10, 1].

2. Preliminaries and main results.

Our object of interest is a d-dimensional compact manifold mesh $\mathbf{M} = (\mathbf{K})$, where $\mathbf{K} = \{\sigma\}$ is a d-dimensional finite simplicial complex, and σ denotes a simplex (we will also use other Greek letters such as τ, κ, α to denote simplices of $\mathbf{M}$). A *face* $\sigma \in \mathbf{K}$ will be our generic name for any simplex in $\mathbf{M}$. We will use the term ℓ-face to denote a simplex of dimension exactly ℓ. For example, 2-face is a triangle, and 3-face is a tetrahedron (tet). Each $\sigma \in \mathbf{K}$ has a fixed orientation: all $\sigma \in \mathbf{K}_d$ are assigned with consistent orientation, $\sigma \in \partial\mathbf{K}_{d-1}$ are assigned with the induced orientation, and all other faces are given some arbitrary but fixed orientation. We will use the notation $\mathbf{K}_\ell$ to denote the subset of ℓ-faces in $\mathbf{K}$, and so $\mathbf{K} = \mathbf{K}_0 \cup \mathbf{K}_1 \cup \cdots \cup \mathbf{K}_d$. For $d = 2$ (i.e., triangular mesh) we have $\mathbf{K} = \mathbf{K}_0 \cup \mathbf{K}_1 \cup \mathbf{K}_2$, where $\mathbf{K}_0 \subset \mathbb{R}^n$, $n \geq d$, is a collection of points, $\mathbf{K}_1$ is the set of edges, and $\mathbf{K}_2$ is the collection of triangles. For $d = 3$ (i.e., tetrahedral mesh), we have $\mathbf{K} = \mathbf{K}_0 \cup \mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3$, where $\mathbf{K}_3$ is the set of tetrahedra. Our simplices are considered by default to be closed sets (e.g., a point, a closed line segment, a triangle with its boundary). In this context the mesh $\mathbf{M}$ can be seen as the closed set of points in $\mathbb{R}^n$ which is constructed as the union of all the simplices in $\mathbf{K}$. We will restrict our attention to compact, orientable, connected meshes $\mathbf{M}$ with boundary $\partial\mathbf{M}$. We will mark by $\partial\mathbf{M} = (\partial\mathbf{K})$ the boundary mesh of $\mathbf{M}$ (e.g., a polygon for $d = 2$, and triangular mesh for $d = 3$) and by $\partial\mathbf{K}$ the set of all faces that are contained in $\partial\mathbf{M}$. $\partial\mathbf{K}_\ell$ will denote the subset of boundary ℓ-faces of $\mathbf{M}$, that is, all ℓ-faces $\sigma \in \mathbf{K}_\ell$ such that $\sigma \subset \partial\mathbf{M}$.

A simplicial map $\Phi : \mathbf{M} \to \mathbb{R}^d$ is a continuous map that is an affine map when restricted to each face σ of $\mathbf{M}$; we denote this affine map as $\Phi|_\sigma$. A simplicial map is uniquely determined by setting the image position of each point (0-face), that is, $u = \Phi(v) \in \mathbb{R}^d$ for all $v \in \mathbf{K}_0$, and extending linearly over all faces.

Our target domain $\boldsymbol{\Omega} \subset \mathbb{R}^d$ will be a (closed) polytope. Its boundary is denoted by $\partial\boldsymbol{\Omega}$, and, similarly to meshes, $\partial\boldsymbol{\Omega}_\ell$ will denote the collection of all boundary ℓ-faces (i.e., the polytope's faces of dimension ℓ contained in the boundary $\partial\boldsymbol{\Omega}$). For example, for $d = 2$, $\partial\boldsymbol{\Omega}$ is a polygonal line, while for $d = 3$, $\partial\boldsymbol{\Omega}$ is a polyhedral surface.

We are seeking conditions that guarantee that a simplicial mapping $\Phi : \mathbf{M} \to \boldsymbol{\Omega}$ is injective and onto. Let us start with a simple *necessary* condition: if Φ is injective, then it does not degenerate any d-face and either maintains the orientation of all d-faces or flips all orientations of d-faces.

Proposition 1 (necessary condition for injectivity of a simplicial map). *An injective simplicial map $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ of a d-dimensional compact mesh does not degenerate any d-face and satisfies exactly one of the following:* (1) *it maintains the orientation of all d-faces, or* (2) *it inverts the orientation of all d-faces.*

Indeed, if the affine map $\Phi|_\sigma$ is degenerate for some $\sigma \in \mathbf{K}_d$, then it is clearly not injective. So all d-faces are necessarily mapped to d-faces. Next, assume (in negation) that some of the d-faces' orientations are preserved and some are inverted; then by connectivity there have to be two adjacent (i.e., sharing a $(d-1)$-face) d-faces $\sigma_1, \sigma_2 \in \mathbf{K}_d$ such that $\Phi(\sigma_1)$ and $\Phi(\sigma_2)$ have opposite orientations. Since the two simplices $\Phi(\sigma_1), \Phi(\sigma_2)$ are sharing a $(d-1)$-face, the intersection of their interiors is not empty, leading to a contradiction with the assumption that Φ is injective.

A simplicial map that satisfies the above necessary condition is said to have the *consistent orientation* property. In this paper we will restrict our attention to *orientation preserving* simplicial maps. That is, all the affine maps $\Phi|_\sigma$, $\sigma \in \mathbf{K}_d$, are orientation preserving. The orientation reversing maps can be treated similarly.

Maintaining the orientation of the d-faces is not sufficient to guarantee injectivity; Figure 1

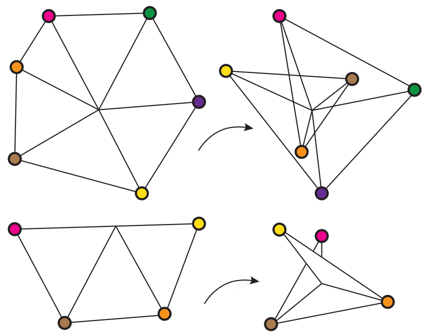

Figure 1. *Necessary condition for injectivity is not sufficient.*

shows two counterexamples for $d = 2$ (i.e., a triangular mesh). Similar and more elaborate examples can be given for the $d = 3$ case (i.e., a tetrahedral mesh). Our goal is to characterize additional "simple" conditions that will assure, along with the consistent orientation condition, injectivity of the simplicial map.

Before getting to the main results, in section 4 we will prove the following theorem, different versions of which appeared in [10, 1]. The current paper provides mathematical underpinning and generalizations to the arguments provided in these earlier works.

Theorem 1. *A nondegenerate orientation preserving simplicial map $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ of a d-dimensional compact mesh with boundary $\boldsymbol{M}$ is a bijection $\Phi : \boldsymbol{M} \to \boldsymbol{\Omega}$ if the boundary map $\Phi|_{\partial \boldsymbol{M}} : \partial \boldsymbol{M} \to \partial \boldsymbol{\Omega}$ is bijective.*

In practice, using this theorem to build an injective simplicial map $\Phi : \mathbf{M} \to \boldsymbol{\Omega}$ requires one to a priori generate a bijective boundary map $\Phi|_{\partial \mathbf{M}}$ and only later search for a simplicial map satisfying this boundary condition. In many common scenarios it is better to search for a map without restricting the boundary map to a fixed boundary map. For example, if seeking low distortion mappings, as is done in the parameterization problem, allowing the vertices at the boundary to move would help reach a lower distortion level (or even existence of a solution). Therefore, we will relax the above sufficient conditions and provide a more flexible setting to guarantee that a map is injective. These conditions, which are described next, will allow a certain freedom in the boundary maps while keeping the injectivity property. We start with some preparations.

As mentioned above, we would like to avoid prescribing a specific boundary map $\Phi|_{\partial \mathbf{M}}$ and allow each boundary face $\sigma \in \partial \mathbf{K}$ that is mapped onto $\partial \boldsymbol{\Omega}$ to "slide" on its target face of the polytope. To that end, every boundary $(d-1)$-face $\sigma \in \partial \mathbf{K}_{d-1}$ is assigned with a target boundary face of the polytope of the same dimension $\tau \in \partial \boldsymbol{\Omega}_{d-1}$. We will denote such an assignment by a function

$$\mathcal{A} : \partial \mathbf{K}_{d-1} \to \partial \boldsymbol{\Omega}_{d-1}. \tag{1}$$

The idea is that every face $\sigma \in \partial \mathbf{K}_{d-1}$ is mapped to somewhere on the planar boundary face $\tau = \mathcal{A}(\sigma) \in \partial \boldsymbol{\Omega}_{d-1}$, but its exact location in that affine space is unknown.

Unfortunately, although this condition is more flexible than fixing the boundary map, it can lead to rather complicated nonconvex constraints. For example, if one of the boundary

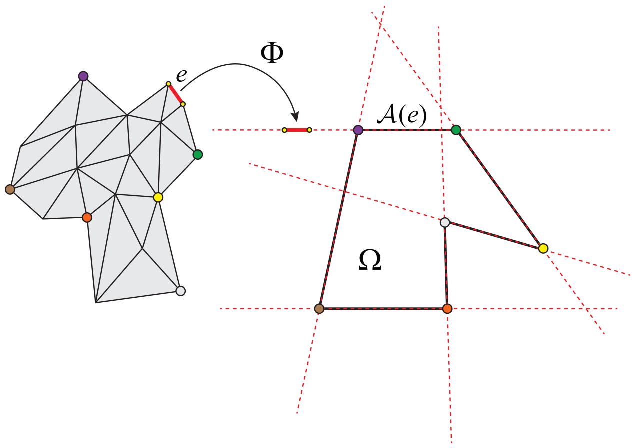

Figure 2. *Letting the boundary slide in the $d = 2$ case.*

faces of the polytope is nonconvex, this constraint will turn out to be nonconvex as well. See, for example, the polytope depicted in Figure 3. Hence, we will suggest a relaxation of this condition: we ask all $(d-1)$-faces $\sigma \in \partial\mathbf{K}_{d-1}$ to satisfy

$$\Phi(\sigma) \subset \operatorname{aff}(\mathcal{A}(\sigma)), \tag{2}$$

where $\operatorname{aff}(B)$ means the affine closure of a set B, namely, the smallest affine set containing B. In particular, these are *linear* equations when formulated in the unknowns of Φ (i.e., the target location of each vertex) and hence useful in practice. As noted above, in contrast to (2), the condition $\Phi(\sigma) \subset \mathcal{A}(\sigma)$ is not convex in the case when the face $\mathcal{A}(\sigma)$ is not convex, which is often the case. Even in the case when the face $\mathcal{A}(\sigma)$ is a convex face of the polytope's boundary, the linear condition in (2) would be still more efficient than the linear inequality constraint that is needed to realize the condition $\Phi(\sigma) \subset \mathcal{A}(\sigma)$.

It is important to note that (2) implies several necessary conditions as follows. Every face $\tau \in \partial\mathbf{K}$ that is in the intersection of several $(d-1)$-faces $\sigma_1, \dots, \sigma_k \in \partial\mathbf{K}_{d-1}$, that is, $\tau = \sigma_1 \cap \dots \cap \sigma_k$, must satisfy

$$\Phi(\tau) \subset \operatorname{aff}(\mathcal{A}(\sigma_1)) \cap \operatorname{aff}(\mathcal{A}(\sigma_2)) \cap \dots \cap \operatorname{aff}(\mathcal{A}(\sigma_k)) = \operatorname{aff}(\kappa), \tag{3}$$

where $\kappa = \mathcal{A}(\sigma_1) \cap \mathcal{A}(\sigma_2) \cap \dots \cap \mathcal{A}(\sigma_k)$, and we assume here that $\mathcal{A}$ is consistent in the sense that the intersection κ is a $(d-k)$-face and the normals to the $(d-1)$-faces $\mathcal{A}(\sigma_j)$ are linearly independent. Intuitively, $\Phi(\tau)$ is restricted to a lower dimensional affine space $\operatorname{aff}(\kappa)$. For later use, let us extend the function $\mathcal{A}$ to be defined over all boundary faces $\partial\mathbf{K}$ by defining $\mathcal{A}(\tau) = \kappa$.

To give a more concrete example, imagine a triangular mesh $\mathbf{M}$ (i.e., $d = 2$) mapped into a polygonal domain $\mathbf{\Omega} \subset \mathbb{R}^2$ in the plane (see Figure 2). Then (2) requires each boundary edge $e \in \partial\mathbf{K}_1$ to be mapped somewhere on the infinite line that supports the assigned edge $\mathcal{A}(e)$ in the target domain's boundary, $\partial\mathbf{\Omega}$. Furthermore, boundary vertices common to edges that are mapped to different (not colinear) polygon edges are restricted to the polygon's vertices. In Figure 2 these are shown with colored disks.

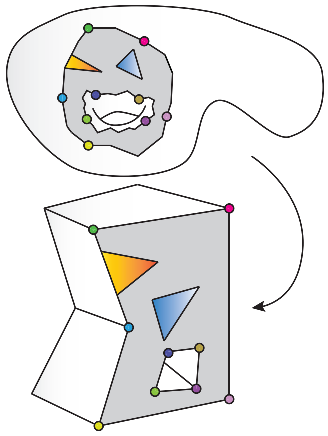

Figure 3. *The assignment for the $d = 3$ case.*

In the case of tetrahedral meshes (i.e., $d = 3$), boundary faces $f \in \partial \mathbf{K}_2$ are restricted to infinite planes supporting the relevant planar polygonal faces of the polytope; boundary edges that are adjacent to non-coplanar faces are mapped to infinite lines; and boundary vertices that are adjacent to three or more independent faces are mapped to fixed vertices of the polytope. Figure 3 illustrates an example: it shows in gray an area of the boundary of a tetrahedral mesh (top) that is mapped to a face of the polytope (bottom). Furthermore, it highlights the constrained vertices.

Naturally, we will need to assume that the assignment $\mathcal{A}$ is *topologically feasible*, namely, that there exists an orientation preserving boundary homeomorphism $\psi : \partial \mathbf{M} \to \partial \mathbf{\Omega}$ that satisfies the given assignment in some (arbitrary) way. Note that this homeomorphism need not be a simplicial map and that it can map the boundary faces of $\partial \mathbf{M}$ arbitrarily onto the boundary faces of $\partial \mathbf{\Omega}$, which is all we need to ensure that topologically the provided assignment $\mathcal{A}$ "makes sense." Furthermore, note that when computing the bijective simplicial mapping in practice we do not assume that we have such Ψ at hand or know it in any sense; only the knowledge of its *existence* is required. In the main result of this paper we will prove that any nondegenerate orientation preserving simplicial map Φ that satisfies (2) induced by some topologically feasible assignment $\mathcal{A}$ is an injection over the interior of $\mathbf{M}$ and onto $\mathbf{\Omega}$.

Theorem 2. *Let $\boldsymbol{M} \subset \mathbb{R}^n$ be a d-dimensional compact mesh with boundary embedded in $(n \geq d)$-dimensional Euclidean space, let $\mathbf{\Omega} \subset \mathbb{R}^d$ be a d-dimensional polytope, and let $\mathcal{A} : \partial \boldsymbol{K}_{d-1} \to \partial \mathbf{\Omega}_{d-1}$ be a topologically feasible assignment between their boundaries. Then, any nondegenerate orientation preserving $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ that satisfies the linear equation (2) for all $\sigma \in \partial \boldsymbol{K}_{d-1}$ satisfies the following:*

1. *Φ is injective over the interior of $\boldsymbol{M}$.*
2. *$\Phi(\boldsymbol{M}) = \mathbf{\Omega}$.*

Note that it is a delicate point, but the theorem above does not imply that Φ is injective on the *boundary* of $\mathbf{M}$. Although pretty rare in practice, it could happen that such a Φ is injective over Interior($\mathbf{M}$), $\Phi(\mathbf{M}) = \mathbf{\Omega}$, while $\Phi|_{\partial \mathbf{M}}$ is not injective.

For example, Figure 4 shows a mapping of an L-shaped triangular mesh, where each boundary edge is constrained to stay within its affine hull (as (2) requires); nevertheless, part

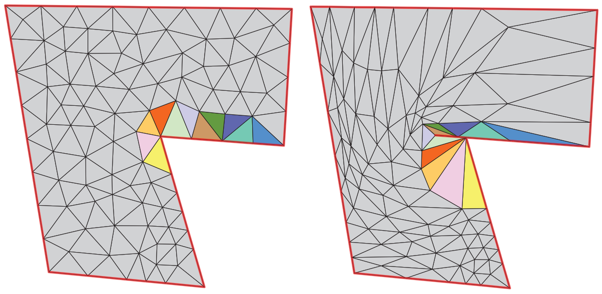

Figure 4. *A nondegenerate orientation preserving simplicial map of an L-shaped domain that is injective over the interior of the domain but not over its boundary (in red).*

of the boundary $\partial\mathbf{M}$ is mapped into the interior of $\mathbf{\Omega}$ (the entire boundary $\partial\mathbf{M}$ and its image $\Phi(\partial\mathbf{M})$ are highlighted in red). The colors indicate corresponding triangles. In a sense, what happened here is that the boundary of $\mathbf{\Omega}$ was extended into its interior. Note that in practice it is unlikely to generate such a map, and here it was artificially engineered. Nevertheless, in order to guarantee that the boundary is also mapped injectively, we can add the requirement that the boundary maps be orientation preserving in the following sense. First, for every boundary $(d-1)$-face $\tau \in \partial\mathbf{\Omega}_{d-1}$, all the faces $\mathcal{A}^{-1}(\tau)$ are mapped with their orientation preserved onto $\mathrm{aff}(\tau)$. Second, recursively, if $\partial\mathbf{\Omega}_{\ell} \ni \kappa \subset \tau \in \partial\mathbf{\Omega}_{\ell+1}$, then all the ℓ-faces $\mathcal{A}^{-1}(\kappa)$ should preserve their orientation, where the orientation of κ is taken to be induced by τ and the orientation of any $\sigma \in \mathcal{A}^{-1}(\kappa)$ is induced by a face $\alpha \in \mathcal{A}^{-1}(\tau)$ such that $\sigma \subset \alpha$. A map Φ that satisfies this condition is said to be *orientation preserving on the boundary.* This leads to our third and final set of sufficient conditions.

Theorem 3. *Let $\boldsymbol{M} \subset \mathbb{R}^n$ be a d-dimensional compact mesh with boundary embedded in $(n \geq d)$-dimensional Euclidean space, let $\mathbf{\Omega} \subset \mathbb{R}^d$ be a d-dimensional polytope, and let $\mathcal{A} : \partial\boldsymbol{K}_{d-1} \to \partial\mathbf{\Omega}_{d-1}$ be a topologically feasible assignment between their boundaries. Then, any nondegenerate orientation preserving $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ that satisfies the linear equation (2) for all $\sigma \in \partial\boldsymbol{K}_{d-1}$ and is orientation preserving on the boundary is a bijection between $\boldsymbol{M}$ and $\mathbf{\Omega}$.*

In the following section we will develop the tools that are used to prove the above theorems. The main tool will be a preimage counting argument that makes use of the power and elegance of the classical mapping degree. In short, the degree $\deg(\Phi_q, \partial\mathbf{M})$ is an integer that counts how many times the simplicial map Φ wraps the boundary of the mesh $\partial\mathbf{M}$ around the point q, and this number is equal almost everywhere to the number of preimages $\#\left\{\Phi^{-1}(q)\right\}$, as the following theorem states.

Theorem 4. *Let $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ be a nondegenerate orientation preserving simplicial map, and let $\boldsymbol{M}$ be a d-dimensional compact mesh with boundary. The number of preimages $\#\left\{\Phi^{-1}(q)\right\}$ of a point $q \in \mathbb{R}^d \setminus \Phi(\partial\boldsymbol{M})$ satisfies*

$$\#\left\{\Phi^{-1}(q)\right\} \leq \deg(\Phi_q, \partial\boldsymbol{M}).$$

If $q \in \mathbb{R}^d \setminus (\cup_{\tau \in \boldsymbol{K}_{d-1}} \Phi(\tau))$, the above inequality is replaced with an equality.

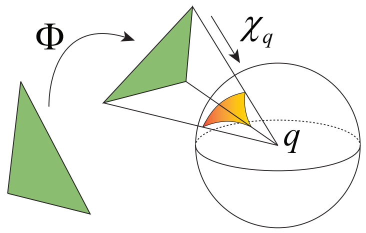

Figure 5. *Composition of a simplicial map and a projection onto a sphere.*

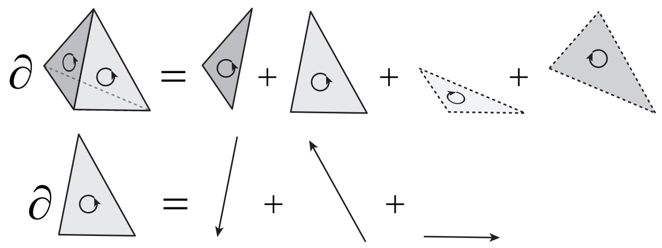

Figure 6. *The boundary operator.*

3. The cycle degree and preimage counting argument. Our goal in this section is to prove a useful preimage counting argument for orientation preserving simplicial maps. This is done by applying the mapping degree tool to simplicial maps restricted to cycles. This argument will be used in the subsequent section for proving the different sufficient conditions for injectivity of simplicial maps.

We will make use of the notion of degree, which in essence counts how many times a map between two closed manifold "wraps" the first manifold over the second one.

Let us define the projection onto a sphere $\chi_q : \mathbb{R}^d \setminus \{q\} \to \mathbb{S}_q$, where $\mathbb{S}_q$ is the unit $(d-1)$-sphere centered at q, that is, $\mathbb{S}_q = \left\{p \in \mathbb{R}^d | \, \|p - q\|_2 = 1\right\}$, by

$$\chi_q(p) = \frac{p - q}{\|p - q\|_2},$$

where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^d$. The key player in the upcoming theory is the composed map $\Phi_q = \chi_q \circ \Phi$. See Figure 5 for an illustration.

Before we define the notion of degree over cycles, let us recall some terminology. An ℓ-chain is a formal sum of ℓ-faces $c = \sum_i a_i \sigma_i$, $\sigma_i \in \mathbf{K}_\ell$, where $a_i \in \mathbb{Z}$. We denote the free abelian group of all ℓ-chains by the symbol $\mathcal{C}_\ell$. Let $\partial_\ell : \mathcal{C}_\ell \to \mathcal{C}_{\ell-1}$ be the boundary operator taking ℓ-chains to $(\ell-1)$-chains. Figure 6 shows an example of the boundary operator applied to $\sigma \in \mathbf{K}_3$ (tet, top) and to $\sigma \in \mathbf{K}_2$ (triangle, bottom). An ℓ-chain $c \in \mathcal{C}_\ell$ is called an ℓ-cycle if $\partial c = 0$. For example, $\partial\sigma$, where $\sigma \in \mathbf{K}_d$, is a $(d-1)$-cycle, as can be verified from Figure 6 for $d = 3$. The subgroup of ℓ-cycles is denoted as $\ker \partial_\ell$.

We will be interested in $(d-1)$-cycles. Namely, $c = \sum_i a_i \sigma_i \in \ker \partial_{d-1}$. The reason is that closed (possibly self-intersecting) $d-1$ subsurface meshes of $\mathbf{M}$ can be represented as $(d-1)$-cycles. For example, given a d-face $\sigma \in \mathbf{K}_d$, its boundary $c = \partial\sigma$ is an example of such a $d-1$ subsurface mesh and, as indicated above, is also a $(d-1)$-cycle. Furthermore, any (formal) sum of such elements, that is, $c = \sum_i \partial\sigma_i$, is also an example. In fact, this is the most general type of cycle we will need to consider. We denote by $I(c) = \{i | a_i \neq 0\}$ the index set of nonzero coefficients of the cycle $c = \sum_i a_i \sigma_i$.

Our goal is to define $\deg(\Phi_q, c)$, which is intuitively the degree of the map Φ_q restricted to the cycle c or, equivalently, how many times Φ_q wraps c over $\mathbb{S}_q$. Although the notion of degree is well established for mappings between manifolds and piecewise-linear manifolds (see [17] for a historic overview as well as a state-of-the-art report), we could not find in the literature a direct treatment of the degree of mapping restricted to cycles. Since this notion seems very natural for analyzing simplicial mappings of meshes, we develop it in detail here. We will adopt the so-called de Rham point of view using differential forms. The reason is that it seems to give a more efficient way to define the degree on cycles and to prove its properties. It might be useful to note, however, that in the rest of this paper we will use only the properties of the degree as summarized in Proposition 2 below.

That is, let ω be the normalized volume form on $\mathbb{S}_q$, i.e., $\int_{\mathbb{S}_q} \omega = 1$. Then we define the degree of Φ_q restricted to a $(d-1)$-cycle $c = \sum_i a_i \sigma_i$, $\sigma_i \in \mathbf{K}_{d-1}$, via

$$\deg(\Phi_q, c) = \int_c \Phi_q^* \omega = \sum_i a_i \int_{\sigma_i} \Phi_q^* \omega. \tag{4}$$

Note that $\int_{\sigma_i} \Phi_q^* \omega$ is simply the (normalized) signed area of $\Phi_q(\sigma_i)$. That is, the sign of $\int_{\sigma_i} \Phi_q^* \omega$ is $+1$ if Φ_q preserves the orientation of σ_i when mapping it to the sphere, and -1 if it inverts it.

In order to see that (4) actually well defines an integer and to give an alternative way of computing this number, we will adapt some arguments from classical degree theory of smooth mappings. For the sake of being self-contained we repeat some of the argumentation in our context. First, let $\Gamma^{d-1}(\mathbb{S}_q)$ denote the linear space of differential $(d-1)$-forms on the $(d-1)$-sphere $\mathbb{S}_q$. Given $\omega \in \Gamma^{d-1}(\mathbb{S}_q)$, we want to consider the relation between the two real scalars $\int_{\mathbb{S}_q} \omega$ and $\int_c \Phi_q^* \omega = \sum_i a_i \int_{\sigma_i} \Phi_q^* \omega$. Let us denote this relation by $T : \mathbb{R} \to \mathbb{R}$. Namely, we define T via $T(\int_{\mathbb{S}_q} \omega) = \int_c \Phi_q^* \omega$.

First, to see that this relation is well defined, take $\omega_1, \omega_2 \in \Gamma^{d-1}(\mathbb{S}_q)$ such that $\int_{\mathbb{S}_q} \omega_1 = \int_{\mathbb{S}_q} \omega_2$. Since $\int_{\mathbb{S}_q} \omega_1 - \omega_2 = 0$ and $\mathbb{S}_q$ is a connected, compact, and oriented manifold, there exists $\eta \in \Gamma^{d-2}(\mathbb{S}_q)$ such that $\omega_1 - \omega_2 = d\eta$ (the $d-1$ cohomology group in this case is $H^{d-1}(\mathbb{S}_q) \cong \mathbb{R}$; see, for example, pages 268–269 and Theorem 9 in [20]).

Hence,

$$\begin{aligned}
\int_c \Phi_q^* \omega_1 - \int_c \Phi_q^* \omega_2 &= \sum_i a_i \int_{\sigma_i} \Phi_q^* (\omega_1 - \omega_2) \\
&= \sum_i a_i \int_{\sigma_i} \Phi_q^* d\eta \\
&\overset{d \text{ commutes with pull-back}}{=} \sum_i a_i \int_{\sigma_i} d\Phi_q^* \eta
\end{aligned}$$

$$\overset{\text{Stokes}}{=} \sum_i a_i \int_{\partial\sigma_i} \Phi_q^* \eta$$
$$\overset{c \text{ is a cycle}}{=} 0,$$

where we used the fact that the operator d commutes with the pull-back operation, the Stokes theorem, and the fact that c is a cycle. The fact that $H^{d-1}(\mathbb{S}_q) \cong \mathbb{R}$ also implies that for every $\alpha \in \mathbb{R}$ there exists $\omega \in \Gamma^{d-1}(\mathbb{S}_q)$ such that $\int_{\mathbb{S}_q} \omega = \alpha$. That is, the domain of T is the whole real line $\mathbb{R}$, and it is well defined.

We have seen that $T : \mathbb{R} \to \mathbb{R}$ is well defined over the whole real line. Now we show that it is linear. Taking $\omega_1, \omega_2 \in \Gamma^{d-1}(\mathbb{S}_q)$ and scalars $\alpha, \beta \in \mathbb{R}$, we have for $\omega = \alpha\omega_1 + \beta\omega_2$ that

$$\begin{aligned} T\left(\alpha \int_{\mathbb{S}_q} \omega_1 + \beta \int_{\mathbb{S}_q} \omega_2\right) &= T\left(\int_{\mathbb{S}_q} \alpha\omega_1 + \beta\omega_2\right) \\ &= \int_c \Phi_q^* \left(\alpha\omega_1 + \beta\omega_2\right) \\ &= \alpha \int_c \Phi_q^* \omega_1 + \beta \int_c \Phi_q^* \omega_2 \\ &= \alpha T\left(\int_{\mathbb{S}_q} \omega_1\right) + \beta T\left(\int_{\mathbb{S}_q} \omega_2\right). \end{aligned}$$

This implies that $T : \mathbb{R} \to \mathbb{R}$ is linear, so it has to be of the form

$$\int_c \Phi_q^* \omega = T\left(\int_{\mathbb{S}_q} \omega\right) = d \int_{\mathbb{S}_q} \omega, \tag{5}$$

where d is some constant independent of the choice of $w \in \Gamma^d(\mathbb{S}_q)$. Since this is true for all forms in $\omega \in \Gamma^d(\mathbb{S}_q)$ we get that $d = \deg(\Phi_q, c)$, as defined in (4). It might seem that we have not accomplished anything with this definition of T; however, having at our disposal (5) (with constant $d = \deg(\Phi_q, c)$, regardless of the choice of ω) will help us easily prove all of the nice properties of the degree on cycles. In particular, we next prove that the degree (although still not clear from the above definition) is always an integer, coinciding with the classical Brouwer degree in the case when the cycle c represents a polyhedral surface, and show another useful way to calculate it.

Proposition 2. *Let $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ be a simplicial map of a d-dimensional compact mesh $\boldsymbol{M}$, let $\chi_q : \mathbb{R}^d \setminus \{q\} \to \mathbb{S}_q$ be a projection on the sphere centered at q, and let $\Phi_q = \chi_q \circ \Phi$ be their composition. Furthermore, let $c = \sum_i a_i\sigma_i \in \ker \partial_{d-1}$ be a $(d-1)$-cycle in $\boldsymbol{M}$. The number* $\deg(\Phi_q, c)$ *defined in* (4) *satisfies the following properties:*

1. *It is an integer.*
2. *In the case when the cycle c represents a $d-1$ piecewise-linear manifold (e.g., a polygon when $d = 2$, and a polyhedral surface when $d = 3$), this notion of degree coincides with the classical Brouwer degree.*

3. *For any* $p \in \mathbb{S}_q \setminus \bigcup_{i\in I(c)} \Phi_q(\partial\sigma_i)$,

$$\deg(\Phi_q, c) = \sum_{i\in I(c) \textit{ such that (s.t.) } p\in\Phi_q(\sigma_i)} a_i \operatorname{sign}_{\sigma_i}(\Phi_q),$$

where $\operatorname{sign}_{\sigma_i}(\Phi_q) = 1$ *if* Φ_q *preserves the orientation of* σ_i *as it maps* σ_i *onto the sphere* $\mathbb{S}_q$*, and* $\operatorname{sign}_{\sigma_i}(\Phi_q) = -1$ *if* Φ_q *inverts the orientation, and* $I(c) = \{i | a_i \neq 0\}$.

4. *Let* $c' = \sum_i a'_i\sigma_i \in \ker\partial_{d-1}$ *be another* $(d-1)$*-cycle in* $\boldsymbol{M}$*; then*

$$\deg(\Phi_q, c + c') = \deg(\Phi_q, c) + \deg(\Phi_q, c').$$

Proof. We start by proving property 3.

Take some $p \in \mathbb{S}_q \setminus \bigcup_{i\in I(c)} \Phi_q(\partial\sigma_i)$. Denote its "preimages" on c under Φ_q by $x_1, x_2, \ldots, x_n$. That is, there exists some σ_{i_j}, $i_j \in I(c)$, $j = 1, \ldots, n$, such that $p = \Phi_q(x_j)$ and $x_j \in \sigma_{i_j}$.

If there are no such preimages of p, then we can take a form $\omega \in \Gamma^{d-1}(\mathbb{S}_q)$ such that $\int_{\mathbb{S}_q} \omega > 0$ and $\omega = 0$ outside a small neighborhood V of p, where all points in V do not have preimages on c under Φ_q (existence of such V is implied by the continuity of Φ_q and the compactness of $\mathbf{M}$). Using ω in (5) shows that $d = \deg(\Phi_q, c) = 0$. This implies property 3 in this case.

We now assume that there exists at least one preimage, i.e., $n \geq 1$. There exist some neighborhood V of p, $p \in V \subset \mathbb{S}_q$, and disjoint neighborhoods U_j of x_j, $x_j \in U_j \subset \sigma_{i_j}$, such that $\Phi_q^{-1}(V) = \cup_{j=1}^n U_j$ and $\Phi_q(U_j) = V$. Take some $\omega \in \Gamma^{d-1}(\mathbb{S}_q)$ such that $\int_{\mathbb{S}_q} \omega > 0$ and $\omega = 0$ outside V. Then

$$\begin{aligned}\int_c \Phi_q^*\omega &= \sum_i a_i \int_{\sigma_i} \Phi_q^*\omega \\ &= \sum_{j=1}^n a_{i_j} \int_{U_j} \Phi_q^*\omega \\ &= \sum_{j=1}^n a_{i_j} \operatorname{sign}_{\sigma_{i_j}}(\Phi_q) \left(\int_V \omega\right).\end{aligned}$$

On the other hand (5) implies that $\int_c \Phi_q^*\omega = (\int_V \omega \deg(\Phi_q, c))$. Combining and dividing by $\int_V \omega$, we get property 3.

Property 1 follows from property 3 since all a_i and $\operatorname{sign}_\sigma(\Phi_q)$ are integers so the degree is an integer.

To prove property 4 we will simply use the definition in (4): let $c' = \sum_i a'_i\sigma_i$ be a $(d-1)$-cycle, and let ω be the normalized volume form on $\mathbb{S}_q$,

$$\begin{aligned}\deg(\Phi_q, c + c') &= \sum_i \left(a_i + a'_i\right) \int_{\sigma_i} \Phi_q^*\,\omega \\ &= \sum_i a_i \int_{\sigma_i} \Phi_q^*\,\omega + \sum_i a'_i \int_{\sigma_i} \Phi_q^*\,\omega \\ &= \deg(\Phi_q, c) + \deg(\Phi_q, c').\end{aligned}$$

Finally, property 2 can be proved by noting that property 3 coincides with the classical definition of the Brouwer degree in the case when c is a cycle representing a $(d-1)$-dimensional piecewise-linear manifold. ■

In the next lemma we compute the cycle degree for a simple $(d-1)$-cycle, that is, the boundary of a d-face.

Lemma 1. *Let $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ be a nondegenerate orientation preserving simplicial map of a d-dimensional compact mesh $\boldsymbol{M}$. For all $\sigma \in \boldsymbol{K}_d$ such that $q \notin \Phi(\partial\sigma)$ there exists*

$$\deg(\Phi_q, \partial\sigma) = \begin{cases} 1, & q \in \Phi(\sigma), \\ 0, & q \notin \Phi(\sigma). \end{cases}$$

Proof. If $q \notin \Phi(\sigma)$, there exists a hyperplane separating the point q and the convex set (tet) $\Phi(\sigma)$. This implies that at least half of the sphere $\mathbb{S}_q$ has no preimages by Φ_q. Property 3 of Proposition 2 then implies that $\deg(\Phi_q, \partial\sigma) = 0$.

In the case $q \in \text{Interior}(\Phi(\sigma))$, one can pick a point $p \in \mathbb{S}_q$ such that $p \in \text{Interior}(\Phi_q(\tau))$, where $\tau \subset \partial\sigma$. Since $\Phi(\sigma)$ is convex, it follows that τ is unique. Since $\Phi|_\sigma$ is orientation preserving and nondegenerate, $\text{sign}_\tau(\Phi_q) = 1$. Using property 3 of Proposition 2 again implies that $\deg(\Phi_q, \partial\sigma) = 1$. ■

Before we prove the main result of this section, namely the preimage counting argument, let us mention a useful property of orientation preserving simplicial maps, namely that they are *open* maps. That is, they map open sets to open sets. This property, which is proved in the appendix, is used later to extend injective mappings over zero-measure sets.

Lemma 2. *Let $\Phi : \boldsymbol{M} \subset \mathbb{R}^n \to \mathbb{R}^d$, $n \geq d$, be a nondegenerate orientation preserving simplicial map of a compact d-dimensional mesh $\boldsymbol{M}$ into $\mathbb{R}^d$. Then Φ is an open map.*

We now get to the main theorem of this section.

Theorem 4 (restated). *Let $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ be a nondegenerate orientation preserving simplicial map, and let $\boldsymbol{M}$ be a d-dimensional compact mesh with boundary. The number of preimages $\#\left\{\Phi^{-1}(q)\right\}$ of a point $q \in \mathbb{R}^d \setminus \Phi(\partial\boldsymbol{M})$ satisfies*

$$\#\left\{\Phi^{-1}(q)\right\} \leq \deg(\Phi_q, \partial\boldsymbol{M}).$$

If $q \in \mathbb{R}^d \setminus (\cup_{\tau\in\boldsymbol{K}_{d-1}} \Phi(\tau))$, the above inequality is replaced with an equality.

Before proving the theorem, note that the inequality can be strict. For example, consider the example in Figure 1: if we take q to be the central vertex in the top-right image in this example, then it has only one preimage, but its degree is $\deg(\Phi_q, \partial\mathbf{M}) = 2$, as this is precisely the winding number of the image boundary polygon w.r.t. the central vertex q.

Proof. Let us start with the second part of the theorem. Denote the set $Y = \cup_{\tau\in\mathbf{K}_{d-1}} \Phi(\tau)$. Let $q \in \mathbb{R}^d \setminus Y$. Lemma 1 and Proposition 2, property 4 imply that

$$\begin{aligned} \#\left\{\Phi^{-1}(q)\right\} &= \sum_{\sigma\in\mathbf{K}_d \text{ s.t. } q\in\Phi(\sigma)} 1 \\ &\overset{\text{Lemma 1}}{=} \sum_{\sigma\in\mathbf{K}_d} \deg(\Phi_q, \partial\sigma) \\ &\overset{\text{Prop. 2, property 4}}{=} \deg(\Phi_q, \partial\mathbf{M}). \end{aligned}$$

This proves the second claim of the theorem. Now we use Lemma 2 to prove the first claim. Assume that the claim is not true; that is, there exists $q \in \mathbb{R}^d \setminus \Phi(\partial\mathbf{M})$ such that $\#\left\{\Phi^{-1}(q)\right\} > \deg(\Phi_q, \partial\mathbf{M})$. By Lemma 2 the map Φ is open, so for every $x \in \left\{\Phi^{-1}(q)\right\}$ we can take an open neighborhood U_x (where all such neighborhoods U_x are pairwise disjoint), and $V = \cap_{x \in \{\Phi^{-1}(q)\}} \Phi(U_x)$ (note that there is a finite number of preimages, so this is a finite intersection) is an open neighborhood of q. Every point $q' \in V$ has strictly more preimages than $\deg(\Phi_q, \partial\mathbf{M})$. Since $q \cap \Phi(\partial\mathbf{M}) = \emptyset$, we can take V sufficiently small so that $V \cap \Phi(\partial\mathbf{M}) = \emptyset$. Since $\deg(\Phi_{q'}, \partial\mathbf{M})$ is an integer and continuous when $q' \in V$, it has to be constant in V. Since Y is of measure zero, we can find a point $q' \in V \setminus Y$. This point has more preimages than the integer $\deg(\Phi_q, \partial\mathbf{M}) = \deg(\Phi_{q'}, \partial\mathbf{M})$, in contradiction to the claim already proven above. This concludes the proof. ■

4. Sufficient conditions for injectivity.

Let us quickly recall our setting. We are interested in mapping a d-dimensional mesh embedded in $(n \geq d)$-dimensional Euclidean space, $\mathbf{M} \subset \mathbb{R}^n$, injectively and onto a target domain in the form of a d-dimensional polytope $\mathbf{\Omega} \subset \mathbb{R}^d$. In this section we prove three sets of sufficient conditions. Of course, all the sets of sufficient conditions ask the candidate simplicial map $\Phi : \mathbf{M} \to \mathbb{R}^d$ to satisfy the necessary condition for injectivity, namely, that it is nondegenerate and has consistent orientation (see Proposition 1). As stated above we will restrict our attention to orientation preserving simplicial maps while keeping in mind that the orientation reversing case is similar.

The first set of sufficient conditions generalizes the global inversion theorem from classical analysis to the piecewise-linear mesh case and simply states that in addition to the necessary nondegeneracy and orientation preserving condition, the boundary map should be bijective.

Theorem 1 (restated). *A nondegenerate orientation preserving simplicial map $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ of a d-dimensional compact mesh with boundary $\boldsymbol{M}$ is a bijection $\Phi : \boldsymbol{M} \to \boldsymbol{\Omega}$ if the boundary map $\Phi|_{\partial \boldsymbol{M}} : \partial\boldsymbol{M} \to \partial\boldsymbol{\Omega}$ is bijective.*

Proof. For an arbitrary point $q \in \mathrm{Interior}(\mathbf{\Omega})$, by assumption $q \notin \Phi(\partial\mathbf{M})$, and Theorem 4 implies that $\#\left\{\Phi^{-1}(q)\right\} \leq \deg(\Phi_q, \partial\mathbf{M}) = 1$. On the other hand, for q in the dense set $\mathrm{Interior}(\mathbf{\Omega}) \setminus \cup_{\tau \in \mathbf{K}_{d-1}} \Phi(\tau)$, we have $\#\left\{\Phi^{-1}(q)\right\} = 1$. By continuity of Φ and compactness of $\mathbf{M}$ this implies that $\#\left\{\Phi^{-1}(q)\right\} \geq 1$ for all $q \in \mathrm{Interior}(\mathbf{\Omega})$. Therefore, $\#\left\{\Phi^{-1}(q)\right\} = 1$ for all $q \in \mathrm{Interior}(\mathbf{\Omega})$. For arbitrary $q \in \mathbb{R}^d \setminus \mathbf{\Omega}$, Theorem 4 implies that $\#\left\{\Phi^{-1}(q)\right\} \leq \deg(\Phi_q, \partial\mathbf{M}) = 0$. This concludes the proof. ■

We now prove the second set of sufficient conditions, namely Theorem 2. In addition to the setting introduced in the beginning of this section, we are also given a topologically feasible assignment $\mathcal{A} : \partial\mathbf{K}_{d-1} \to \partial\mathbf{\Omega}_{d-1}$ of boundary faces of $\mathbf{M}$ to boundary faces of the target polytope domain $\mathbf{\Omega}$. We consider any nondegenerate orientation preserving $\Phi : \mathbf{M} \to \mathbb{R}^d$ that satisfies the linear equation (2) for all $\sigma \in \partial\mathbf{K}_{d-1}$, and we will prove that such a Φ is injective over $\mathrm{Interior}(\mathbf{M})$ and covers $\mathbf{\Omega}$, that is, $\Phi(\mathbf{M}) = \mathbf{\Omega}$. The main power in this formulation of sufficient conditions is that it allows us to consider a *collection* of different boundary mappings at the minor price of adding linear constraints to the problem. The idea of the proof is to reduce to the case of Theorem 1 using a homotopy argument. We start with calculating the degree $\deg(\Phi_q, \partial\mathbf{M})$ for *almost all* $q \in \mathbb{R}^d$.

Lemma 3. *Let $\boldsymbol{M} \subset \mathbb{R}^n$ be a d-dimensional mesh with boundary embedded in $(n \geq d)$-dimensional Euclidean space, let $\boldsymbol{\Omega} \subset \mathbb{R}^d$ be a d-dimensional polytope, and let $\mathcal{A} : \partial\boldsymbol{K}_{d-1} \to \partial\boldsymbol{\Omega}_{d-1}$ be a topologically feasible assignment. Then, for any nondegenerate orientation pre-*

serving $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ *that satisfies the linear equation* (2) *and for all* $\sigma \in \partial \boldsymbol{K}_{d-1}$, *there exists*

$$\deg(\Phi_q, \partial \boldsymbol{M}) = \begin{cases} 1, & q \in \boldsymbol{\Omega} \setminus Z, \\ 0, & q \in \mathbb{R}^d \setminus (\boldsymbol{\Omega} \cup Z), \end{cases}$$

where $Z = \bigcup_{\omega \in \partial \boldsymbol{\Omega}_{d-1}} \operatorname{aff}(\omega)$ *is the union of all affine hyperplanes supporting the boundary* $(d-1)$*-faces of* $\boldsymbol{\Omega}$.

A visualization of the set Z can be seen in Figure 2 as the union of all red dashed lines.

Proof. We will prove the lemma by employing a degree theory argument. By Proposition 2 our definition of the number $\deg(\Phi_q, \partial \mathbf{M})$ coincides with the well-known Brouwer degree of the continuous map $\Phi_q|_{\partial \mathbf{M}} : \partial \mathbf{M} \to \mathbb{S}_q$. Since we assume that $\mathcal{A}$ is topologically feasible, there exists an orientation preserving homeomorphism (not necessarily a simplicial map) $\Psi : \partial \mathbf{M} \to \partial \boldsymbol{\Omega}$ such that $\Psi(\sigma) \subset \mathcal{A}(\sigma)$ for all $\sigma \in \partial \mathbf{K}_{d-1}$. Let us consider the family of mappings $\varphi_q^t(\cdot)$ defined by

$$\varphi_q^t(x) = \chi_q\Big((1-t)\,\Phi(x) + t\Psi(x)\Big), \quad t \in [0,1], \ x \in \partial \mathbf{M}.$$

Let $Z = \bigcup_{\omega \in \partial \boldsymbol{\Omega}_{d-1}} \operatorname{aff}(\omega)$. We claim that, for all $q \in \mathbb{R}^d \setminus Z$, $\varphi_q^t : \partial \mathbf{M} \to \mathbb{S}_q$ is a homotopy. Indeed, fix any $q \in \mathbb{R}^d \setminus Z$ and consider an arbitrary $x \in \partial \mathbf{M}$. Since $x \in \sigma$ for some $\sigma \in \partial \mathbf{K}_{d-1}$, it means that $\Phi(x) \in \Phi(\sigma) \subset \operatorname{aff}(\mathcal{A}(\sigma))$ and $\Psi(x) \in \Psi(\sigma) \subset \mathcal{A}(\sigma) \subset \operatorname{aff}(\mathcal{A}(\sigma))$. This implies that

$$(1-t)\,\Phi(x) + t\Psi(x) \in \operatorname{aff}(\mathcal{A}(\sigma)) \subset Z.$$

Since Z is closed, q has some positive distance to Z, and we conclude that φ_q^t is continuous in t and x and hence is a homotopy. The invariance of the degree to homotopy now implies that, for all $q \in \mathbb{R}^d \setminus Z$,

$$\deg(\Phi_q, \partial \mathbf{M}) = \deg(\chi_q \circ \Psi, \partial \mathbf{M}),$$

and since $\Psi : \partial \mathbf{M} \to \partial \boldsymbol{\Omega}$ is an orientation preserving homeomorphism, for any interior point $q \in \boldsymbol{\Omega} \setminus Z$, $\deg(\chi_q \circ \Psi, \partial \mathbf{M}) = 1$, and for any $\mathbb{R}^d \setminus (\boldsymbol{\Omega} \cup Z)$ it equals 0, as required. ■

We now prove the second set of sufficient conditions.

Theorem 2 (restated). *Let* $\boldsymbol{M} \subset \mathbb{R}^n$ *be a d-dimensional compact mesh with boundary embedded in* $(n \geq d)$*-dimensional Euclidean space, let* $\boldsymbol{\Omega} \subset \mathbb{R}^d$ *be a d-dimensional polytope, and let* $\mathcal{A} : \partial \boldsymbol{K}_{d-1} \to \partial \boldsymbol{\Omega}_{d-1}$ *be a topologically feasible assignment between their boundaries. Then, any nondegenerate orientation preserving* $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ *that satisfies the linear equation* (2) *for all* $\sigma \in \partial \boldsymbol{K}_{d-1}$ *satisfies the following:*

1. Φ *is injective over the interior of* $\boldsymbol{M}$.
2. $\Phi(\boldsymbol{M}) = \boldsymbol{\Omega}$.

Proof. We start with proving statement 2.

Let $Y = \cup_{\tau \in \mathbf{K}_{d-1}} \Phi(\tau)$. Theorem 4 implies that $\#\left\{\Phi^{-1}(q)\right\} = \deg(\Phi_q, \partial \mathbf{M})$ for $q \in \mathbb{R}^d \setminus Y$. Let $Z = \bigcup_{\omega \in \partial \boldsymbol{\Omega}_{d-1}} \operatorname{aff}(\omega)$. Lemma 3 now implies that

$$\#\left\{\Phi^{-1}(q)\right\} = \begin{cases} 1, & q \in \boldsymbol{\Omega} \setminus (Z \cup Y), \\ 0, & q \in \mathbb{R}^d \setminus (\boldsymbol{\Omega} \cup Z \cup Y). \end{cases} \tag{6}$$

Denote $Q = Z \cup Y$. Note that Q is a set of measure zero. Take an arbitrary point $q \in \mathbb{R}^d \setminus \mathbf{\Omega}$. If q is outside Q, we already showed that it has no preimages in $\mathbf{M}$. Assume that $q \in Q$, and assume that q has a preimage $x \in \mathbf{M}$. $x \in \sigma$ for some $\sigma \in \mathbf{K}_d$, and since Φ is nondegenerate, the set $\Phi(\sigma)$ is a d-dimensional face, and therefore we can find $w \in \Phi(\sigma) \setminus (\mathbf{\Omega} \cup Q)$ with a preimage in σ, in contradiction with (6). Therefore $\#\left\{\Phi^{-1}(q)\right\} = 0$ for any $q \in \mathbb{R}^d \setminus \mathbf{\Omega}$. This implies that $\Phi(\mathbf{M}) \subset \mathbf{\Omega}$. Let us show that $\mathbf{\Omega} \subset \Phi(\mathbf{M})$. First, from (6) we have $\mathbf{\Omega} \setminus Q \subset \Phi(\mathbf{M})$. For any point $q \in \mathbf{\Omega} \cap Q$ we can take a series $q_n \to q$ such that $\#\left\{\Phi^{-1}(q_n)\right\} = 1$ (using (6)). Let $\{x_n\} \subset \mathbf{M}$ be their preimages, that is, $\Phi(x_n) = q_n$. Since $\mathbf{M}$ is compact, we can extract a convergent subsequence $x_n \to x \in \mathbf{M}$ (we abused notation and kept the original indexing). Now from continuity of Φ we have $\Phi(x) = \Phi(\lim(x_n)) = \lim \Phi(x_n) = \lim q_n = q$, and we have found a preimage of q, and therefore $q \in \Phi(\mathbf{M})$. Since this is true for all $q \in \mathbf{\Omega} \cap Q$, we have $\mathbf{\Omega} \subset \Phi(\mathbf{M})$, and therefore $\Phi(\mathbf{M}) = \mathbf{\Omega}$.

Let us prove the injectivity property (statement 1) next. Assume in negation that there exist two points $x, x' \in \mathrm{Interior}(\mathbf{M})$ such that $\Phi(x) = \Phi(x')$. Denote $q = \Phi(x)$. By the open map property of Φ (see Lemma 2) there exists a neighborhood U of q such that all $q' \in U$ have at least two preimages in $\mathbf{M}$. This contradicts (6) and concludes the proof. ■

As a side remark, we note that the open map property and (6) above also imply that $\Phi(\mathrm{Interior}(\mathbf{M})) \subset \mathrm{Interior}(\mathbf{\Omega})$.

As mentioned and demonstrated in section 2, Theorem 2 guarantees the injectivity of the map Φ in the interior of the mesh and that Φ is onto the domain $\mathbf{\Omega}$, but does not guarantee that the boundary is mapped bijectively. In the next and final set of sufficient conditions we show that adding the condition that Φ is orientation preserving on the boundary (as defined in section 2) leads to injectivity of Φ over the boundary of $\mathbf{M}$ as well.

Theorem 3 (restated). *Let $\boldsymbol{M} \subset \mathbb{R}^n$ be a d-dimensional compact mesh with boundary embedded in $(n \geq d)$-dimensional Euclidean space, let $\mathbf{\Omega} \subset \mathbb{R}^d$ be a d-dimensional polytope, and let $\mathcal{A} : \partial \boldsymbol{K}_{d-1} \to \partial \mathbf{\Omega}_{d-1}$ be a topologically feasible assignment between their boundaries. Then, any nondegenerate orientation preserving $\Phi : \boldsymbol{M} \to \mathbb{R}^d$ that satisfies the linear equation (2) for all $\sigma \in \partial \boldsymbol{K}_{d-1}$ and is orientation preserving on the boundary is a bijection between $\boldsymbol{M}$ and $\mathbf{\Omega}$.*

Proof. The proof is by induction on d.

Let us prove the theorem for $d = 1$. That is, $\mathbf{M}$ is a polygonal line defined by a finite series of ordered points $x_0, x_1, x_2, \ldots, x_L \in \mathbb{R}^n$, its boundary consists of the two end points $\partial \mathbf{M} = \{x_0, x_L\}$, and $\mathbf{\Omega}$ is a one-dimensional polytope in $\mathbb{R}$ homeomorphic to $\mathbf{M}$, namely a segment $[a, b] \subset \mathbb{R}$. Φ is nondegenerate and orientation preserving, which means that $\Phi(x_{i+1}) > \Phi(x_i)$ for all $i = 0, \ldots, L-1$. Furthermore, Φ satisfies (2), which in this case means that $\Phi(x_0) = a$ and $\Phi(x_L) = b$. Putting this together, we have $a = \Phi(x_0) < \Phi(x_1) < \cdots < \Phi(x_{L-1}) < \Phi(x_L) = b$, which implies that Φ is a bijection onto $[a, b]$.

Let us assume that we have proved the $d = \ell - 1$ case; now we prove the $d = \ell$ case. We have an ℓ-dimensional mesh $\mathbf{M}$ and a polytope $\mathbf{\Omega}$ with boundary $\partial \mathbf{\Omega}$ of dimension $\ell - 1$. Consider an arbitrary polytope's boundary $(\ell - 1)$-face $\tau \in \partial \mathbf{\Omega}_{\ell-1}$. Let $\mathbf{M}_\tau \subset \partial \mathbf{M}$ be the $(\ell - 1)$-dimensional submesh that is assigned to be mapped into $\mathrm{aff}(\tau)$. Remember that by our definition of the assignment function $\mathcal{A}$ we have that the set of $(\ell - 1)$-faces of $\mathbf{M}_\tau \subset \partial \mathbf{M}$ that is mapped into $\mathrm{aff}(\tau)$ is $\mathcal{A}^{-1}(\tau)$.

We know by assumption that Φ restricted to $\mathbf{M}_\tau$, that is, $\Phi|_{\mathbf{M}_\tau}$, does not degenerate and preserves the orientation of all faces in $\mathcal{A}^{-1}(\tau) \subset \partial \mathbf{K}_{\ell-1}$ as it maps them into $\mathrm{aff}(\tau)$. Then $\mathcal{A}$ restricted to $\partial \mathbf{M}_\tau$ is topologically feasible (since $\mathcal{A}$ is topologically feasible). Furthermore, by definition $\Phi|_{\mathbf{M}_\tau}$ is orientation preserving on the boundary $\partial \mathbf{M}_\tau$, and by (3) it satisfies (2) w.r.t. the submesh $\mathbf{M}_\tau$ and the restricted $\mathcal{A}$.

The induction assumption now implies that $\Phi|_{\mathbf{M}_\tau}$ is injective and onto τ. As $\tau \in \partial \mathbf{\Omega}_{\ell-1}$ was arbitrary, we have that $\Phi|_{\partial \mathbf{M}}$ is bijective, and since Φ is nondegenerate and orientation preserving, Theorem 1 implies that Φ is a bijection. ∎

5. Numerical experiments. Theorems 1–3 can be used to design algorithms that produce injective mappings of meshes onto polytopes. Of special interest is Theorem 2, as it adds simple linear conditions (i.e., (2)) in addition to the necessary orientation preserving constraints and allows working with weaker boundary conditions than prescribing a bijective boundary map, as required by Theorem 1. Namely, it requires only that we provide a topologically feasible assignment $\mathcal{A}$. In this section we demonstrate how this can be used for mapping triangular meshes, i.e., $d = 2$.

Let $\mathbf{M} = (\mathbf{K}_0 \cup \mathbf{K}_1 \cup \mathbf{K}_2)$ be a triangular mesh with boundary $\partial \mathbf{M}$, where $\mathbf{K}_0 = \{v_i\}$ is the set of vertices in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\mathbf{K}_1 = \{e_k\}$ is the set of edges, and $\mathbf{K}_2 = \{f_j\}$ is the set of triangles. Assume that we want to map $\mathbf{M}$ bijectively onto a planar polygonal domain $\Omega \subset \mathbb{R}^2$, that is, compute a bijective simplicial map $\Phi : \mathbf{M} \to \mathbf{\Omega}$. A simplicial map of $\mathbf{M}$ is uniquely described by setting the image of the vertices $\Phi(v_i)$, $v_i \in \mathbf{K}_0$. Then, the affine map of each triangle $f_j \in \mathbf{K}_2$ is the unique affine map that takes the corners v_1, v_2, v_3 of triangle f_j to $\Phi(v_1), \Phi(v_2), \Phi(v_3)$. Fixing a coordinate frame in triangle f_j and $\mathbb{R}^2$, we denote by $\boldsymbol{v}_i \in \mathbb{R}^2$, $i = 1, 2, 3$, the coordinate vector representing vertex v_i in the frame of triangle f_j, and by $\boldsymbol{u}_i \in \mathbb{R}^2$ the unknown vector representing $\Phi(v_i)$ in the global frame in the plane. Then, the affine map of face f_j, written in coordinates as $\mathcal{A}_j(\mathbf{x}) = A_j \mathbf{x} + \delta_j$, can be expressed as the unique solution of the linear system:

$$[A_j \ \delta_j] \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{bmatrix}.$$

That is, A_j, δ_j can be written as a constant linear combination of the variables $\boldsymbol{u}_i$.

To assure the necessary conditions for bijectivity, namely that each triangle's affine map A_j is orientation preserving, one can use the convex constraints introduced, for example, in [10, 11] or [5]. We will use the former mainly since the constraints in these works possess a maximality property. For completeness let us recap and reformulate the conditions here. For maintaining the orientation we ask that the determinant of the Jacobian be positive, namely, $\det(A_j) > 0$ for all $f_j \in \mathbf{K}_2$. By direct computation one can check that

$$2 \det(A_j) = \|B_j\|_F^2 - \|C_j\|_F^2 ,$$

where $B_j = \frac{A_j - A_j^T + \mathrm{tr}(A_j) I}{2}$ and $C_j = \frac{A_j + A_j^T - \mathrm{tr}(A_j) I}{2}$. Requiring that $\det(A_j) > 0$ is equivalent to requiring that

$$\|B_j\|_F > \|C_j\|_F .$$

As this is not a convex space we can, similarly to [10, 11], carve maximal convex subsets of this set via

$$\frac{\operatorname{tr}(R_j^T B_j)}{\sqrt{2}} > \|C_j\|_F \,, \tag{7}$$

where R_j is an arbitrary rotation matrix (we'll explain shortly how to choose it). This is a convex second-order cone constraint which can be optimized using standard second-order cone programming (SOCP) solvers such as MOSEK [2]. In fact, along with the positive determinant conditions, we found it to be numerically stable to bound also the condition number of A_j, where the condition number is defined as the ratio of the maximal to minimal singular values of A_j. Both can be enforced by a slight modification to (7):

$$\mu\frac{\operatorname{tr}(R_j^T B_j)}{\sqrt{2}} > \|C_j\|_F \,, \tag{8}$$

where $\mu = \frac{K-1}{K+1}$ and K is the desired bound on the condition number of A_j. (We used $K = 15$ for the planar mappings and $K = 5$ for the mesh parameterization example.) When working with planar meshes, that is, $\mathbf{K}_0 \subset \mathbb{R}^2$, we initialized $R_j = I$, and for the surface mesh we picked an arbitrary frame in each three-dimensional triangle, so R_j was initialized as an arbitrary 2×2 rotation matrix. We added the boundary conditions as required in Theorem 1 or Theorem 2; both are sets of linear constraints with the unknowns $\{\boldsymbol{u}_j\}$ and solve a feasibility problem to get an initial mapping. That is, we added to the left-hand side in (8) a new auxiliary variable t (the same t for all faces f_j) and minimized t. This is a convex problem. When a minimum is reached, if $t < 0$, we have found a feasible solution and hence a bijective mapping according to Theorem 1 or Theorem 2. If the minimal t is greater than zero, we reset the rotation R_j to be the rotation closest to B_j and repeat. As explained in [10], this procedure takes the largest symmetric convex space around the current map and hence allows further reduction of the functional, in this case t.

In applications one is often interested in mappings that possess some regularity. In this paper we chose to optimize a standard well-known regularity functional, namely, the Dirichlet energy. It is defined over triangular meshes as $E_{\mathrm{dir}}(\Phi) = \sum_{f_j \in \mathbf{K}_2} \|A_j\|_F^2 \operatorname{Area}(f_j)$, where $\operatorname{Area}(f_j)$ is the area of f_j. We optimized the Dirichlet energy with the convex constraints (8) using the rotation R_j achieved in the feasibility phase described above. Once converged, we reset the rotations and repeat until convergence (usually no more than 3–5 iterations are required). We demonstrate this algorithm in two scenarios: planar mesh mapping and surface mesh parameterization. The algorithm is implemented in the MATLAB environment using the MOSEK [2] and YALMIP [12] optimization packages.

Figure 7 depicts four examples of mapping a planar triangular mesh (the first column shows the source meshes) onto a polygonal domain. Each row shows the result of mapping the respective source mesh onto a polygonal domain by minimizing the Dirichlet energy of the map while constraining the map to be orientation preserving using the conic formulation described above. The second and third columns in Figure 7 show the result of minimizing the Dirichlet energy with the orientation preserving constraints while prescribing the boundary map to be the *fixed* uniform map, that is, mapping the boundary vertices of the source mesh to equally spaced locations along each edge of the target polygon. Since this boundary map is a

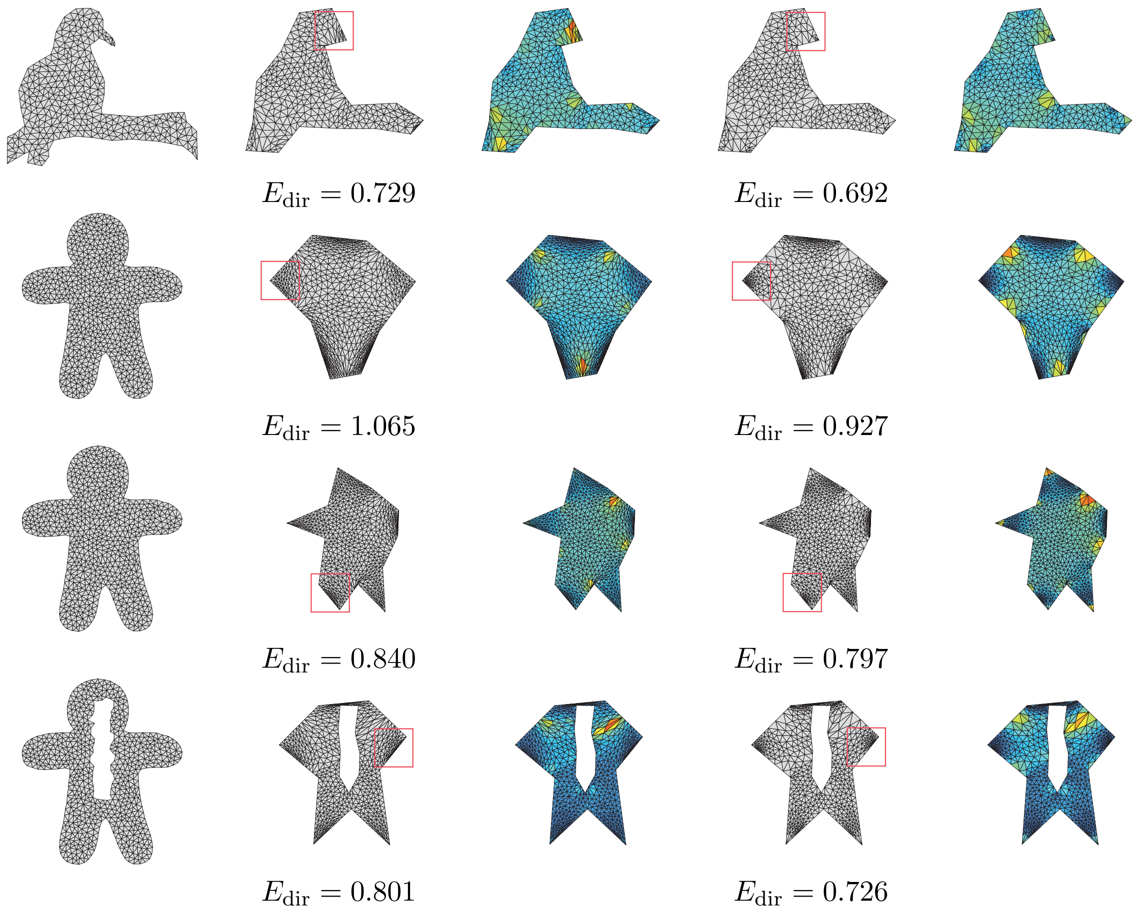

Figure 7. *Computing bijective simplicial mappings of planar meshes (source meshes are shown in first column) onto a polygonal domains using Dirichlet energy with orientation preserving constraints and a fixed uniform boundary map (second and third columns) and with Dirichlet energy with orientation preserving constraints and the linear boundary conditions of Theorem 2 (fourth and fifth columns). Note how the boundary map is optimized in the latter case to reduce the Dirichlet energy of the map (see E_{dir} value indicating the total Dirichlet energy). The third and fifth columns show the norm of the gradient of each map color coded (red indicates large gradients, and blue indicates small gradients).*

bijection, Theorem 1 implies that Φ is a bijection. The Dirichlet energy of the resulting maps (E_{dir}) are written below the second column images. The third column shows the gradient norm of the map (i.e., $\|A_j\|_F$) color coded. The fourth and fifth columns show the result of minimizing the Dirichlet energy with the orientation preserving constraints but this time with the boundary conditions of Theorem 2 (i.e., (2)). Theorem 2 guarantees the resulting map to cover the target domain exactly and to be injective over the interior of the source mesh. Note how the boundary map is optimized (several areas of interest are highlighted with red squares) to reduce the overall Dirichlet energy of the resulting map (compare the energies E_{dir} below each image in the second and fourth columns). The fifth column, similarly to the third column, shows the magnitudes of the gradients.

(a) (b)

Figure 8. *Injectively parameterizing a surface mesh* (a) *onto a polygonal domain in the plane* (b) *by optimizing the Dirichlet energy with orientation preserving constraints and the linear boundary conditions of Theorem* 2.

Figure 8 shows an example of mapping a triangular mesh embedded in three dimensions (mesh surface in (a)) onto a polygonal domain in the plane in (b). Here we have optimized the Dirichlet energy, together with the orientation preserving constraints, as above. The linear boundary conditions of Theorem 2 allowed the optimization to choose the boundary map that allowed low Dirichlet energy of this parameterization under the given assignment.

6. Conclusions.

This paper utilizes the concept of degree of simplicial maps over cycles to formulate and prove three sufficient conditions for guaranteeing that a simplicial map of a mesh with boundary onto a polytope is a bijection. The conditions are practical in the sense that they can be incorporated into algorithms that preserve orientations of simplices to produce bijective mappings. The conditions are appropriate for cases where a bijective mapping of a mesh with boundary onto a known polytope is sought and the boundary mapping is either supplied or unknown and needs to be optimized as well.

A limitation of the sufficient conditions developed in this paper is that they still do not allow complete freedom in optimizing the boundary map. For example, they do not allow boundary faces of the mesh to slide from one boundary face of the target polytope to another boundary face of the polytope. Ideally, such a property would allow optimizing the boundary map to reduce the energy further.

Finally, the paper focuses on the case of meshes with boundary, and a natural question is how to bijectively map manifold meshes without boundary. This question involves topological questions, and we leave this very interesting problem to future work.

7. Appendix.

In the appendix we prove the open map property of nondegenerate orientation preserving simplicial maps.

In the proof we will use the notion of 1-*ring*, which is defined for arbitrary $x \in \mathrm{Interior}(\mathbf{M})$ as $R_x = \bigcup_{\sigma \in \mathbf{K}_d, x \in \sigma} \sigma$. Figure 9 depicts an example of the 1-ring of a point (green) in the relative interior of an edge in a tetrahedral mesh ($d = 3$).

LEMMA 2. *Let $\Phi : \boldsymbol{M} \subset \mathbb{R}^n \to \mathbb{R}^d$, $n \geq d$, be a nondegenerate orientation preserving simplicial map of a compact d-dimensional mesh $\boldsymbol{M}$ into $\mathbb{R}^d$. Then Φ is an open map.*

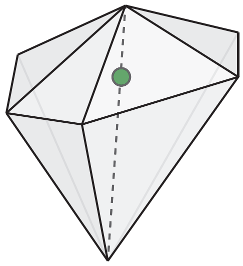

Figure 9. *An example of the* 1*-ring of an edge-interior point (in green).*

Proof. We need to show that, for every open set $U \subset \text{Interior}(\mathbf{M})$, the set $\Phi(U)$ is an open set. That is, any point $x \in U$ is mapped to an interior point $\Phi(x) \in \text{Interior}(\Phi(U))$. We will use the 1-ring of x, $R_x = \bigcup_{\sigma \in \mathbf{K}_d, x \in \sigma} \sigma$. Let $W = U \cap \text{Interior}(R_x)$. Since Φ is nondegenerate, $\Phi(x)$ has some positive distance to $Q := \Phi(\partial \text{Closure}(W))$. Let us consider a small open neighborhood V of $\Phi(x)$ such that all points in V are strictly closer to $\Phi(x)$ than Q.

We will compute the Brouwer degree (see Führer characterization on page 39 of [17]), $\deg(\Phi, W, p)$, for points $p \in V$. Let $Y = \cup_{\tau \in \mathbf{K}_{d-1}} \Phi(\tau)$. Y has measure zero. Since Φ is nondegenerate and continuous, we can find a point $q = \Phi(x') \in V \setminus Y$, where $x' \in W$. As Φ is orientation preserving, we have $\deg(\Phi, W, q) \geq 1$. By homotopy invariance, $\deg(\Phi, W, p)$ is constant for all $p \in V$. Hence, $\deg(\Phi, W, p) \geq 1$ for all $p \in V$. The existence of solution property of the degree now implies that any such $p \in V$ has a preimage in W, $p = \Phi(z)$, $z \in W \subset U$. Hence we proved $\Phi(x) \in V \subset \Phi(U)$, as required. ∎

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