Large deviations for zeros of random polynomials with i.i.d. exponential coefficients

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December 15, 2013

Dedicated to the memory of Wenbo Li

Abstract

We derive a large deviation principle for the empirical measure of zeros of the random polynomial $P_n(z) = \sum_{j=0}^{n} \xi_j z^j$, where the coefficients $\{\xi_j\}_{j\geq 0}$ form an i.i.d. sequence of exponential random variables.

1 Introduction

The study of the zero set $\{z_1, \dots, z_n\}$ of random polynomials

$$P_n(z) = \sum_{j=0}^{n} \xi_j z^j \tag{1}$$

with i.i.d. coefficients $\{\xi_j\}_{j\geq 0}$ has a long and rich history, which we will not review here; see [BRS86] for a classical account and [TV13] for the most recent results. Under mild conditions, the convergence of the empirical measure $L_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{z_i}$ of zeros of P_n. to the uniform measure on the unit circle goes back at least to [SS62] and [ET50]; scaled version of this convergence can be found in [SV95] (for the Gaussian case) and [IZ95] (for more general i.i.d. coefficients in the domain of attraction of stable laws).

We are interested in the large deviations for the empirical measure L_n. In the case of Gaussian coefficients, this has been studied before [ZZ10], [Be08], [Bl11], exploiting

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methods related to those used in the study of random matrices from the classical β-ensembles [BAG97, BAZ98, SS12]. Like in the case of random matrices, when one ventures away from the Gaussian setup (with i.i.d. coefficients), not much is known concerning large deviations.

Our goal in this paper is to exhibit a new class of coefficients, for which a large deviation principle for the empirical measure can be proved, namely the class of i.i.d. exponential coefficients, which for concreteness we normalize to have parameter 1. To our knowledge, the first to consider explicitly asymptotics for this class was Wenbo Li [Li11], who used general formulae of Zaporozhets [Za04] in order to compute the probability that all roots in such a polynomial are real. We relate our result to Li's computation in Theorem 1.2 below.

In order to state our results, we introduce some notation. In the rest of the paper, P_n denotes a random polynomial as in (1), with i.i.d. exponential (of parameter 1) coefficients $\{\xi_i\}$ and associated empirical measure of zeros L_n. For any Polish space X, let $\mathcal{M}_1(X)$ denote the space of probability measures on X, equipped with the topology of weak convergence. Let pol_+ denote the collection of polynomials (over $\mathbb{C}$) with coefficients that are real positive. For $p \in \mathrm{pol}_+$, let $\mu_p \in \mathcal{M}_1(\mathbb{C})$ denote the empirical measure of zeros of p. Note that μ_p depends on the set of zeros and not on a particular labeling of the zeros, that μ_p is symmetric with respect to the transformation $z \mapsto z^*$, and that $\mu_p(\mathbb{R}_+) = 0$. (Here and in the sequel, we use $\mathbb{R}_+$ to denote the interval $(0, \infty)$.)

We introduce the closure of the collection of empirical measures of polynomials with positive coefficients

$$\mathcal{P} = \overline{\{\mu_p : p \in \mathrm{pol}_+\}} \subset \mathcal{M}_1(\mathbb{C})\,. \tag{2}$$

Obviously, $L_n \in \mathcal{P}$.

Definition 1. *For any measure $\mu \in \mathcal{M}_1(\mathbb{C})$, define the logarithmic potential function to be*

$$\mathbb{L}_\mu(z) = \int \log|z-w| d\mu(w)$$

and the logarithmic energy to be

$$\Sigma(\mu) = \iint \log|z-w|\mu(z)\mu(w).$$

Definition 2. *Define the function $I : \mathcal{M}_1(\mathbb{C}) \to \mathbb{R}_+$ by*

$$I(\mu) = \begin{cases} \int \log|1-z| d\mu(z) - \frac{1}{2}\iint \log|z-w| d\mu(z) d\mu(w), & \text{if } \mu \in \mathcal{P}, \\ \infty, & \text{if } \mu \notin \mathcal{P} \end{cases}$$

We will see in Section 3.1 that I is well defined (for $\mu \in \mathcal{P}$, as the integral with respect to $\mu \times \mu$ of the function $f(z,w) = \log|1-z| + \log|1-w| - \log|z-w|$) and non-negative (the latter fact is immediate from the lower bound in Lemma 3.7)[1].

[1] A. Eremenko showed us a direct proof of the non-negativity of I, that bypasses the use of the lower bound from Lemma 3.7. Since we need the latter lemma for other reasons, we do not reproduce his proof here.

Our main result concerning large deviations of L_n is the following.

Theorem 1.1. *The random measures L_n satisfy a large deviation principle in the space $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function I. Explicitly, we have:*

(i) The function $I : \mathcal{M}_1(\mathbb{C}) \to [0,\infty]$ has compact level sets, i.e. the sets $\{\mu : I(\mu) \leq M\}$ are compact subsets of $\mathcal{M}_1(\mathbb{C})$ for each $M \in \mathbb{R}$.

(ii) For each open set $O \subset \mathcal{M}_1(\mathbb{C})$, we have

$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in O) \geq - \inf_{\mu\in O} I(\mu).$$

(iii) For each closed set $F \subset \mathcal{M}_1(\mathbb{C})$, we have

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in F) \leq - \inf_{\mu\in F} I(\mu).$$

Comparing the statement of Theorem 1.1 with the main results in [BAZ98] and [ZZ10], one sees that in spite of the fact that we are dealing with zeros of random polynomials, the rate function is closer to a random matrix theory rate function than to the one appearing in the Gaussian case. This is due to the expression for the joint distribution of zeros, see Section 2.1 below. We also note that because I is a good rate function, any minimizer μ of $I(\cdot)$ in $\mathcal{M}_1(\mathbb{C})$ must satisfy that $I(\mu) = 0$; in particular, the uniform measure on the unit circle is a minimizer, as one expects from the limit results in [ET50], [SS62]. The strict convexity of I (which follows from the same argument as in [BAG97]) shows that it is the unique minimizer.

As mentioned above, we tie our results to Li's computation in [Li11]. Toward this end, let $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+$ and define $\mu_R \in \mathcal{M}_1(\mathbb{R}_-) \subset \mathcal{M}_1(\mathbb{C})$ to be such that $I(\mu_R) = \inf_{\mu\in\mathcal{M}_1(\mathbb{R}_-)} I(\mu) =: I_R$. (Such a minimizer exists due to the lower semicontinuity of I.)

Theorem 1.2. *Conditioned on $L_n \in \mathcal{M}_1(\mathbb{R}_-)$, the sequence of random empirical measures L_n satisfy the large deviation principle in $\mathcal{M}_1(\mathbb{R}_-)$ with speed n^2 and rate function $I_R(\mu) = I(\mu) - I_R$. In particular, conditioned on $L_n \in M_1(\mathbb{R})$, the sequence L_n converges weakly to $\mu_R \in M_1(\mathbb{R}_-)$.*

A characterization of μ_R is given in the next theorem, due to J. Baik.

Theorem 1.3. *The minimizer μ_R has density with respect of Lebesgue measure on $\mathbb{R}_-$ equal to*

$$\phi(x) = \frac{1}{\pi(|x|+1)\sqrt{|x|}} \mathbf{1}_{\{x<0\}} \,. \tag{3}$$

An interesting feature of the minimizer μ_R is that it is not compactly supported. We discuss Theorems 1.2 and 1.3 in Section 6.

History and Acknowledgements: Our interest in this problem started when one of us (O.Z.) attended a talk by Wenbo Li on [Li11]; that talk suggested that an underlying large deviation principle should exist in the real case, and J. Baik computed its equilibrium measure, repeated here as Theorem 1.3. Wenbo Li's untimely death prompted S. G. and O. Z. to revisit the problem, and the important role of the class $\mathcal{P}$ in the complex case emerged. We posted the question concerning the characterization of $\mathcal{P}$ on MathOverflow [MO13], and the question was answered in [BES13].

We are indebted to J. Baik for allowing us to use his proof of Theorem 1.3, and to A. Eremenko for making [BES13] available to us as a preprint, for his patience in answering our questions, and for his comments on a preliminary draft of this paper.

2 Preliminaries

We discuss in this section several preliminaries. We first introduce the joint distribution of zeros and then we describe properties of $\mathcal{P}$.

2.1 The joint distribution of zeros

Let $p \in \mathrm{pol}_+$ be of degree n with $n-2k$ real zeros, $k = 0, 1, \ldots, \lfloor n/2 \rfloor$. We consider the zeros of p as a vector $(z_1, \cdots, z_n)$ with the convention that $z_1, \cdots, z_k$ are the non-real zeros with positive imaginary part, $z_{k+1} = \overline{z}_1, \cdots, z_{2k} = \overline{z}_k$ and $z_{2k+1}, \cdots, z_n$ denote the $n-2k$ real zeros. In this notation, for k fixed, a set of zeros is generically mapped to $k!(n-2k)!$ distinct points in $A^+_{n,k} = \mathbb{C}^k_+ \times \mathbb{C}^k_- \times \mathbb{R}^{n-2k}$, and $A^+_{n,k}$ is parametrized by $\mathbb{C}^k_+ \times \mathbb{R}^{n-2k}$.

Performing the change of variables from $(\xi_0, \ldots, \xi_n)$ to $(z_1, \ldots, z_k, z_{n-2k+1}, \ldots, z_n, \xi_n)$, counting multiplicities, using the form of the exponential density and integrating over the density of ξ_n (see [Za04] for a similar computation), one has that the random polynomial P_n induces the following measure on $\mathbb{C}^n$:

$$d\mathbb{P}_n(z_1, \cdots, z_n) = \tag{4}$$
$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} \frac{\prod_{1 \le i < j \le n} |z_i - z_j|}{\prod_{j=1}^n |1 - z_j|^{n+1}} 1_{B_{n,k}}(z_1, \cdots, z_n) d\mathcal{L}(z_1) \cdots d\mathcal{L}(z_k) d\ell(z_{2k+1}) \cdots d\ell(z_n).$$

Here $\mathcal{L}$ is the Lebesgue measure on $\mathbb{C}$, ℓ is the Lebesgue measure on $\mathbb{R}$, and $B_{n,k}$ consists of the n-tuples $(z_1, \ldots, z_n) \subset A_{n,k}$ that can be obtained as the zero set (with

$n-2k$ real zeros) of a polynomial of degree n with positive coefficients. In particular, letting $A_{n,k} = (\mathbb{C} \setminus \mathbb{R})^{2k} \times \mathbb{R}^{n-2k} \subset \mathbb{C}^n$, we see that the density of $\mathbb{P}_n$ on any fixed $A_{n,k}$ is

$$\frac{1}{\mathcal{Z}_{n,k}} 1_{B_{n,k}}(z_1, \cdots, z_n) \exp\left(\sum_{1 \le i < j \le n} \log|z_i - z_j| - (n+1) \sum_{j=1}^{n} \log|1 - z_j| \right). \tag{5}$$

where the constants $\mathcal{Z}_{n,k}$ satisfy that

$$\lim_{n\to\infty} \frac{1}{n^2} \log \max_{k=1}^{\lfloor n/2 \rfloor} \mathcal{Z}_{n,k} = \lim_{n\to\infty} \frac{1}{n^2} \log \min_{k=1}^{\lfloor n/2 \rfloor} \mathcal{Z}_{n,k} = 0\,. \tag{6}$$

The representation (5) with (6) is particularly suited for LDP analysis.

2.2 Properties of the class $\mathcal{P}$ of measures

Obviously, for any $p \in \mathrm{pol}_+$ with μ_p its empirical measure of zeros, we have that $\mu_p(\mathbb{R}_+) = 0$. However, that property is not preserved by weak convergence, and hence a-priori it is not clear that all measures in $\mathcal{P}$ satisfy it (although we will see, as a consequence of Obrechkoff's theorem below, that in fact they do). In this subsection, we discuss this and other properties of the class $\mathcal{P}$.

2.2.1 Obrechkoff's Theorem

A starting point for the description of $\mathcal{P}$ is the following classical theorem.

Theorem 2.1 (Obrechkoff)**.** *Let $p \in \mathrm{pol}_+$ and let*

$$C_\alpha = \{z \in \mathbb{C} : |\arg z| \le \alpha\}$$

denote the symmetric (around the positive real line) cone in $\mathbb{C}$ with apex at the origin and angle 2α. Then, $\mu_p(C_\alpha) \le 2\alpha/\pi$.

The proof, given in [Ob23], uses the argument principle. For our needs, note that Obrechkoff's Theorem implies that $\mu(C_\alpha) \le 2\alpha/\pi$ for any $\mu \in \mathcal{P}$. In particular, $\mu(\mathbb{R}_+) = 0$ for such μ.

Obrechkoff's Theorem leads to the following lemma on the integrability of the logarithm near 1 for $\mu \in \mathcal{P}$.

Lemma 2.2. *Let $M > 0$ and set $A_M = \{z : \log|1-z| \le -M\}$. Then there is a positive quantity $C(M)$ satisfying* $\lim_{M\to\infty} C(M) = 0$ *such that for any $\mu \in \mathcal{P}$,*

$$\max\{\mu(A_M), \int_{A_M} |\log|1-z||\, d\mu(z)\} \le C(M). \tag{7}$$

Proof. We first consider $p \in \mathrm{pol}_+$ of degree N. For $M > 0$, let Z_M consists of all zeros z_i of p such that $\log|1 - z_i| < -M$. Let $N(p, M)$ be the cardinality of Z_M and $S(p, M) = \sum_{i:z_i \in Z_M} \log|1 - z_i|$. By Lemma 2.1, there exists a constant M_0 independent of p or N such that for $M > M_0$, $|Z_M| < 2e^{-M}N$. Thus, with $B_j := \{z : -(j+1) < \log|1-z| \le -j : j \ge M\}$, we get

$$\frac{1}{N}|S(p, M)| \le \frac{1}{N}\sum_{j=M}^{\infty}\sum_{i:z_i \in B_j} |\log|1 - z_i|| \le \sum_{j=M}^{\infty} j \cdot 4e^{-j} =: c(M)\,,$$

with $c(M) \to_{M\to\infty} 0$. Since $|S(p, M)| \ge N(p, M)$ if M_0 is chosen large enough, we obtain the same inequality for $N(p, M)/N$. Thus, (7) holds for μ_p, uniformly in p, N.

To obtain the same inequality for $\mu \in \mathcal{P}$, take an approximating sequence $\mu_{p_n} \to \mu$, and use that $\mu(A_M) \le \limsup_{n\to\infty} \mu_{p_n}(A_{M-1})$ together with

$$\int_{A_M} |\log|1 - z| \vee -K| d\mu(z) \le \limsup_{n\to\infty} \int_{A_{M-1}} |\log|1 - z| \vee -K| d\mu_{p_n}(z) \le c(M - 1)\,,$$

and then apply monotone convergence over K. One concludes that (7) holds with $C(M) = c(M - 1)$. ∎

2.2.2 The Bergweiler-Eremenko-Sokal Theorem

For $\mu \in \mathcal{M}_1(\mathbb{C})$, let

$$\widehat{\mathbb{L}}_\mu(z) = \int_{|w|\le 1} \log(|z - w|)d\mu(w) + \int_{|w|>1} \log(|1 - \frac{z}{w}|)d\mu(w)\,.$$

Whenever $C_\mu := \int \log_+ |w|\mu(dw) < \infty$, it holds that

$$\widehat{\mathbb{L}}_\mu(z) = \mathbb{L}_\mu(z) - C_\mu\,.$$

In a recent work [BES13], Bergweiler, Eremenko and Sokal proved the following.

Theorem 2.3 (Bergweiler-Eremenko-Sokal)**.** *$\mu \in \mathcal{P}$ if and only if it is invariant with respect to conjugation and satisfies $\widehat{\mathbb{L}}_\mu(z) \le \widehat{\mathbb{L}}_\mu(|z|)$ for all $z \in \mathbb{C}$.*

In the proof of Theorem 1.1, we will exploit this result, and in addition, its proof.

3 Proof of Theorem 1.1

We prove in this section Theorem 1.1. The proof is divided into sections. We first study, in section 3.1, properties of I and establish that it is well defined and lower-semicontinuous. In Section 3.2, we prove the exponential tightness of $\{\mathbb{P}_n\}$. Section 3.3 is devoted to the proof of the upper bound. Finally, Section 3.4 states and proves Lemma 3.7, which is the lower bound; the proof of Lemma 3.7 uses some some technical approximation lemmas whose proofs are postponed to Sections 4 and 5.

3.1 I is well defined and has compact level sets.

We prove the following.

Lemma 3.1. *The function I is well defined on $\mathcal{M}_1(\mathbb{C})$ and it possesses compact level sets.*

Lemma 3.1 almost shows that I is a good rate function; what is missing is a proof that $I(\mu) \geq 0$ for $\mu \in \mathcal{M}_1(\mathbb{C})$. This fact is a consequence of Lemma 3.7 below.

Proof. Define $f(z,w) = \log|1-z| + \log|1-w| - \log|z-w|$. We first show that one can choose a function $K(L) \to_{L\to\infty} \infty$ so that the following inclusion holds for all L large:

$$\{(z,w) : |z| > L, |w| > L\} \subset \{f(z,w) \geq K(L)\}. \tag{8}$$

Indeed, setting $z' = 1-z$ and $w' = 1-w$, we get

$$f(z,w) = \log|z'| + \log|w'| - \log|z'-w'|.$$

But

$$\frac{|z'w'|}{|z'-w'|} \geq \frac{1}{\frac{1}{|z'|} + \frac{1}{|w'|}} \geq \frac{1}{2}\min\{|z'|, |w'|\}.$$

Clearly, this implies (8). Further, the last inequality also implies that, with $A = \{(z,w) \in \mathbb{C}^2 : |1-z| > 1/4, |1-w| > 1/4\}^{\complement}$,

$$\inf_{A^{\complement}} f(z,w) \geq \frac{1}{8}\,. \tag{9}$$

We next show that $I(\mu)$ is well defined. For that it is enough to consider $\mu \in \mathcal{P}$. Since $f(z,w) \geq c + \log\min(|z-1|, |w-1|)$ for some constant C, an application of Lemma 2.2 implies that the integral of f is well defined (and bounded below).

We next show that the level sets of I are precompact. Choose L large enough so that $K(L) > 1$. Then,

$$\begin{aligned}
\mu(|z| > L)^2 &= \mu \otimes \mu(|z| > L, |w| > L) \qquad (10)\\
&\leq \mu\left(\{f(z,w) \geq K(L)\} \cap \{|1-z| > 1/4, |1-w| > 1/4\}\right)\\
&\leq \frac{1}{K(L) - 1/8} \iint_{A^{\complement}} (f(z,w) - 1/8)\, d\mu(z) d\mu(w)\\
&\leq \frac{1}{K(L) - 1/8} \left(\iint f(z,w) d\mu(z) d\mu(w) - \iint_A f(z,w) d\mu(z) d\mu(w) \right),
\end{aligned}$$

where we used (9) in the second inequality.

Our next task is to show that

$$-\iint_A f(z,w)d\mu(z)d\mu(w) \le c\,, \tag{11}$$

for some constant c independent of $\mu \in \mathcal{P}$. To this end, we write $A = A_1 \cup A_2 \cup A_3 \cup A_4$ with

$$\begin{aligned} &A_1 := \{|1-z| \le 1/2, |1-w| \le 1/4\},\ A_2 := \{|1-z| > 1/2, |1-w| \le 1/4\},\\ &A_3 := \{|1-w| \in [1/4,1/2], |1-z| \le 1/4\},\ A_4 := \{|1-w| > 1/2, |1-z| \le 1/4\}. \end{aligned}$$

Since $|z-w| \le 3/4$ for $(z,w) \in A_1$, we have

$$\begin{aligned} -\iint_{A_1} f(z,w)d\mu(z)d\mu(w) &\le \iint_{A_1} \log|z-w|d\mu(z)d\mu(w) - 2\int_{\{z:|1-z|\le 1/2\}} \log|1-z|d\mu(z)\\ &\le -2\int_{\{z:|1-z|\le 1/2\}} \log|1-z|d\mu(z) \le C(\log 2)\,, \end{aligned} \tag{12}$$

where $C(\log 2)$ is given by Lemma 2.2. With the same argument, we also have

$$-\iint_{A_3} f(z,w)d\mu(z)d\mu(w) \le C(\log 2). \tag{13}$$

For the integral over the set A_2, we note that $|1-(1-w)/(1-z)| \in (1/2, 3/2)$ for $(z,w) \in A_2$, and therefore

$$\begin{aligned} -\iint_{A_2} f(z,w)d\mu(z)d\mu(w) &= -\iint_{A_2} \log\frac{|1-w|}{|1-\frac{1-w}{1-z}|}d\mu(z)d\mu(w) \\ \le\ \log(3/2) - \int_{\{w:|1-w|\le 1/4\}} &\log|1-w|d\mu(w) \le \log(3/2) + C(\log 4)\,, \end{aligned} \tag{14}$$

where $C(\log 4)$ is again given by Lemma 2.2.

Since $\iint_{A_4} f(z,w)d\mu(z)d\mu(w) = \iint_{A_2} f(z,w)d\mu(z)d\mu(w)$, we obtain by combining (12), (13) and (14) that (11) holds.

From (11) we obtain that for any $M > 0$,

$$\sup_{\{\mu: I(\mu)\le M\}} \mu(|z| > L) \to_{L\to\infty} 0\,,$$

which yields the pre-compactness of the level sets of I by an application of Prohorov's criterion.

It remains to show that I is lower semicontinuous. Since $\mathcal{P}$ is closed in $\mathcal{M}_1(\mathbb{C})$, it is enough to check the lower semicontinuity in $\mathcal{P}$. Toward this end, for $\epsilon, M > 0$ define

$$f^{\epsilon,M}(z,w) = \left[\left(\log|1-z| \vee (-\frac{1}{\epsilon})\right) + \left(\log|1-w| \vee (-\frac{1}{\epsilon})\right) - (\log|z-w| \vee (-M))\right] \wedge M$$

and

$$f^\epsilon(z,w) = \log|1-z| \vee (-\frac{1}{\epsilon}) + \log|1-w| \vee (-\frac{1}{\epsilon}) - \log|z-w|.$$

Set

$$I^{\epsilon,M} := \frac{1}{2}\iint f^{\epsilon,M}(z,w)d\mu(z)d\mu(w)$$

and

$$I^\epsilon := \frac{1}{2}\iint f^\epsilon(z,w)d\mu(z)d\mu(w).$$

Note that by monotone convergence, $I^\epsilon = \sup_{M>0} I^{\epsilon,M}$, and since $I^{\epsilon,M} : \mathcal{M}_1(\mathbb{C}) \to \mathbb{R}$ is continuous, we have that I^ϵ is lower semicontinuous on $\mathcal{M}_1(\mathbb{C})$, and therefore on $\mathcal{P}$. On the other hand, I^ϵ converges uniformly to I on $\mathcal{P}$ by Lemma 2.2. It follows that I is also lower semicontinuous on $\mathcal{P}$, completing the proof of the lemma. ■

3.2 Exponential Tightness of $\{\mathbb{P}_n\}$.

We prove in this subsection the exponential tightness of the family $\{\mathbb{P}_n\}$.

Lemma 3.2. *The family $\{\mathbb{P}_n\}$ is exponentially tight. That is, with $T > 0$ there exist compact sets $K_T \subset \mathcal{P}$ so that*

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P(L_n \in K_T^\complement) \leq -T.$$

Proof. Introduce the function $g(z,w) = \log|1-z| + \log|1-w| - \log_+(|z-w|)$, and define the function J on $\mathcal{P}$ by

$$J(\mu) = \iint g(z,w)d\mu(z)d\mu(w)\,.$$

Using Lemma 2.2 and arguing as in subsection 3.1, one sees that the sets

$$K_B := \{\mu \in \mathcal{P} : J(\mu) \leq 5B\}$$

are compact in $\mathcal{M}_1(\mathbb{C})$ for B large.

We need thus to estimate $P(L_n \in K_B^\complement)$. Introduce the random variables

$$X_n = \frac{1}{n^2}\sum_{i=1}^n \log|1-z_i| = \frac{1}{n^2}\log\frac{P_n(1)}{\xi_n} = \frac{1}{n^2}\log\frac{\xi_0+\cdots+\xi_n}{\xi_n}$$

and

$$Y_n = \frac{1}{n}\sum_{\{i:|1-z_i|<1\}} |\log|1-z_i||\,.$$

We need the following estimate, whose proof is postponed to the end of the subsection.

Lemma 3.3. *There exists a constant $c > 0$ such that for all n large,*

$$\mathbb{P}_n(|X_n| > B) \leq 20ne^{-Bn^2} \tag{15}$$

and

$$\mathbb{P}_n(Y_n > c) = 0\,. \tag{16}$$

Continuing with the proof of Lemma 3.2, we have

$$\begin{aligned} \mathbb{P}_n(L_n \in K_B^{\complement}) &\leq \mathbb{P}_n\left(\{L_n \in K_B^{\complement}\} \cap \{|X_n| \leq B\}\right) + \mathbb{P}_n\left(\{|X_n| > B\}\right) \\ &\leq \sum_{k=0}^{\lfloor n/2 \rfloor} \mathbb{P}_n\left(\{L_n \in K_B^{\complement}\} \cap \{|X_n| \leq B\} \cap A_{n,k}\right) + 20ne^{-Bn^2}\,, \end{aligned} \tag{17}$$

see (5) for the definition of $A_{n,k}$.

We next consider the density of $\mathbb{P}_n$ on $A_{n,k}$, see (5), which we write as

$$\begin{aligned} &f_{k,n}(z_1, \cdots, z_N) = \\ &\frac{1}{\mathcal{Z}_{n,k}} \exp\left(\frac{n^2}{2}(\frac{1}{n^2}\sum_{i\neq j}\log|z_i - z_j| - \frac{2}{n}\sum_i \log|1 - z_i| + \frac{4}{n^2}\sum_i \log|1 - z_i|)\right) \\ &\times \exp\left(-3\sum_i \log|1 - z_i|\right) \mathbf{1}_{B_{n,k}}(z_1, \ldots, z_n). \end{aligned}$$

Note that

$$\frac{1}{n^2}\sum_{i\neq j}\log|z_i - z_j| \leq \frac{1}{n^2}\sum_{i\neq j}(\log_+|z_i - z_j|) = \iint \log_+|z - w| dL_n(z)dL_n(w).$$

Thus, on the event $\{L_n \in K_B^{\complement}\} \cap \{|X_n| \leq B\} \cap A_{n,k}$, we have that

$$\frac{1}{n^2}\sum_{i\neq j}\log|z_i - z_j| - \frac{2}{n}\sum_i \log|1 - z_i| + \frac{4}{n^2}\sum_i \log|1 - z_i| \leq -5B + 4B = -B$$

and therefore on this event,

$$f_{k,n}(z_1, \cdots, z_N) \leq \frac{1}{\mathcal{Z}_{n,k}} e^{-n^2B/2} \exp\left(-3\sum_i \log|1 - z_i|\right) \mathbf{1}_{B_{n,k}}(z_1, \ldots, z_n).$$

Thus, using (16) and the constant c in the statement of the lemma,

$$\begin{aligned} &\mathbb{P}_n\left(\{L_n \in K_B^{\complement}\} \cap \{|X_n| \leq B\} \cap A_{n,k}\right) \\ \leq\ & \frac{1}{\mathcal{Z}_{n,k}} e^{-n^2B/2} \int \cdots \int \left(\left[\prod_{i=0}^{n} \frac{1}{|1 - z_i|^3}\right] \wedge e^{3cn}\right) d\mathcal{L}(z_1) \cdots d\mathcal{L}(z_k) d\ell(z_{2k+1}) \cdots d\ell(z_n). \end{aligned}$$

Lemma 3.2 follows from substituting the last display in (17) and performing the integration. ■

Proof of Lemma 3.3. By the argument in the proof of Lemma 2.2, we have that for any j non-negative integer,

$$\frac{1}{n}\sum_{z_i:|1-z_i|\in[2^{-j+1},2^{-j}]} |\log|1-z_i|| \le j\cdot 2^{-j}\,.$$

Thus,

$$Y_n = \frac{1}{n}\sum_{z_i:|1-z_i|\le 1} |\log|1-z_i|| \le \sum_{j=0}^{\infty} j2^{-j}\,. \tag{18}$$

In particular, for all n large,

$$\mathbb{P}_n(X_n \le -1) = 0\,. \tag{19}$$

Next, we control the upper tail of X_n. We have

$$\begin{aligned} P(X_n > B) &= P(\sum_{i=0}^{n-1}\xi_i > (e^{Bn^2}-1)\xi_n) \le P(\sum_{i=1}^{n-1}\xi_i > \frac{1}{2}e^{Bn^2}\xi_n) \\ &= \int_0^\infty e^{-x}P(\sum_{i=1}^{n-1}\xi_i > \frac{1}{2}e^{Bn^2}x)dx\,. \end{aligned} \tag{20}$$

Using Chebycheff's inequality, we have

$$P(\sum_{i=1}^{n-1}\xi_i > \frac{1}{2}e^{Bn^2}x) \le e^{-\lambda e^{Bn^2}x/2}E(e^{\lambda\xi_1})^n \le \frac{e^{-\lambda e^{Bn^2}x/2}}{(1-\lambda)^n}\,.$$

Choosing $\lambda = 1/n$ and substituting in (20) gives

$$P(X_n > B) \le e\cdot\int_0^\infty e^{-x(1+e^{Bn^2}/2n)}dx \le 4e\cdot ne^{-Bn^2}\,.$$

Combining the last display with (19) completes the proof. ■

3.3 The Upper Bound

Recall the notation $f(z,w) = \log|1-z| + \log|1-w| - \log|z-w|$. We prove in this subsection the following.

Lemma 3.4. *For any $\mu \in \mathcal{P}$,*

$$\lim_{\epsilon\to 0}\overline{\lim_{n\to\infty}}\frac{1}{n^2}\log\mathbb{P}_n\left(d(L_n,\mu)\le\epsilon\right) \le -\frac{1}{2}\iint f(z,w)d\mu(z)d\mu(w)\,. \tag{21}$$

Here, $d(\cdot,\cdot)$ is an arbitrary metric on $\mathcal{M}_1(\mathbb{C})$ which is compatible with the weak topology, e.g. the Lévy metric.

Proof. Define the set of measures

$$E_n := \{\nu \in \mathcal{P} : \frac{1}{n}\int \log|1-z|d\nu(z) \geq \frac{1}{2}\iint f(z,w)d\nu(z)d\nu(w)\}.$$

The set E_n corresponds to a subset of $\cup_{k=0}^{\lfloor n/2\rfloor} A_{n,k}$ which gives rise to empirical measures ν as described in Section 1. By abuse of notation, we denote this set by E_n as well.

An application of (15) of Lemma 3.3 gives the following.

Proposition 3.5. *With notation as above,* $\mathbb{P}_n(E_n) \leq 20n \exp\left(-\frac{1}{2}n^2 \int\int f(z,w)d\mu(z)d\mu(w)\right)$.

Now,

$$\mathbb{P}_n\left(d(L_n,\mu) \leq \epsilon\right) \leq \mathbb{P}_n\left(E_n^{\complement} \cap \{d(L_n,\mu) \leq \epsilon\}\right) + \mathbb{P}_n(E_n).$$

Therefore,

$$\overline{\lim_{n\to\infty}} \frac{1}{n^2}\mathbb{P}_n\left(d(L_n,\mu)\leq\epsilon\right) = \text{Max}\{\overline{\lim_{n\to\infty}}\frac{1}{n^2}\mathbb{P}_n(E_n), \overline{\lim_{n\to\infty}}\frac{1}{n^2}\mathbb{P}_n\left(E_n^{\complement}\cap\{d(L_n,\mu)\leq\epsilon\}\right)\}.$$

Since $\overline{\lim}_{n\to\infty}\frac{1}{n^2}\log\mathbb{P}_n(E_n)$ is bounded above by the desired upper bound, it remains to deal with $\overline{\lim}_{n\to\infty}\frac{1}{n^2}\mathbb{P}_n\left(E_n^{\complement}\cap\{d(L_n,\mu)\leq\epsilon\}\right)$.

We begin with

$$\mathbb{P}_n(E_n^{\complement}\cap\{d(L_n,\mu)<\epsilon\}) = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{\mathcal{Z}_{n,k}}I_{k,n}^{\epsilon},$$

where

$$I_{k,n}^{\epsilon} = \int_{\{E_n^{\complement}\cap A_{n,k}\cap B_{n,k}\cap\{d(W_n,\mu)\leq\epsilon\}\}} \exp\left(\sum_{1\leq i<j\leq n}\log|z_i-z_j| - (n+1)\sum_{j=1}^{n}\log|1-z_j|\right)$$

$$d\mathcal{L}(z_1)\cdots d\mathcal{L}(z_k)d\ell(z_{2k+1})\cdots d\ell(z_n)$$

where $W_n(z_1,\cdots,z_n)$ is the empirical measure $\frac{1}{n}\sum_{i=1}^{n}\delta_{z_i}$.

We will upper bound $\overline{\lim}_{n\to\infty}\frac{1}{n^2}\log I_{k,n}^{\epsilon}$ for each $0\leq k\leq\lfloor n/2\rfloor$, uniformly in k; by summing over k, this (together with (6)) will be sufficient for the overall upper bound on $\mathbb{P}_n\left(E_n^{\complement}\cap\{d(L_n,\mu)\leq\epsilon\}\right)$.

For reasons similar to those encountered in the proof of exponential tightness, we write the integrand in $I_{k,N}^{\epsilon}$ as

$$\exp\left(\sum_{1\leq i<j\leq n}\log|z_i-z_j| - (n-1)\sum_{j=1}^{n}\log|1-z_j| + \epsilon\sum_{j=1}^{n}\log|1-z_j|\right)\prod_{j=0}^{n}\frac{1}{|1-z_j|^{2+\epsilon}}. \tag{22}$$

Note that to upper bound the exponent in (22), it suffices to truncate $\log|z_i - z_j|$ from below and $\log|1 - z_i|$ from above. To this end, we fix a big positive number M and define the truncated function

$$f_M(z, w) = f(z, w) \wedge M.$$

The exponent in (22) is

$$\mathcal{E}_n(z_1, \cdots, z_N) \leq \frac{n^2}{2}\left(-\iint_{z\neq w} f_M(z,w)dL_n(z)dL_n(w) - \frac{2\epsilon}{n}\int \log|1-z|dL_n(z)\right) \tag{23}$$

and $\exp(\mathcal{E}_n(z_1, \cdots, z_n))$ is integrated, for each fixed k, with respect to the measure

$$\prod_{j=0}^{n} \frac{1}{|1-z_j|^{2+\epsilon}} \mathbf{1}_{B_{n,k}}(z_1, \ldots, z_n) d\mathcal{L}(z_1)\cdots d\mathcal{L}(z_k) d\ell(z_{2k+1})\cdots d\ell(z_n). \tag{24}$$

But

$$\iint_{z\neq w} f_M(z,w)dL_n(z)dL_n(w) = \iint f_M(z,w)dL_n(z)dL_n(w) - M/n$$

In the above equality, the M/n term comes from the diagonal terms in the discrete sum $\iint f_M(z,w)dL_n(z)dL_n(w)$.

We handle (23) with the following proposition, whose proof is defered to the end of this section.

Proposition 3.6. *There exist $\delta_M(\epsilon) > 0$ and $c(M) > 0$ such that for all $\nu \in \mathcal{P}$ such that $d(\nu, \mu) < \epsilon$ we have*

$$\left|\iint f_M(z,w)d\nu(z)d\nu(w) - \iint f_M(z,w)d\mu(z)d\mu(w)\right| < \delta_M(\epsilon) + c(M).$$

where $\delta_M(\epsilon) \to 0$ as $\epsilon \to 0$ for each fixed M (bigger than some universal constant) and $c(M) \to 0$ uniformly in $\nu \in \mathcal{P}$ and ϵ.

Continuing with the proof of the upper bound, we use Proposition 3.6 in (23) together with (16) to write

$$\begin{aligned} I_{k,n}^{\epsilon} \quad &\leq \quad Cn \exp\left\{4n^2(\delta_M(\epsilon) + c(M) + n^{-1}M)\right\} \\ &\times \exp\left\{-\frac{n^2}{2}\left(\iint f(z,w)d\mu(z)d\mu(w) - \epsilon\iint f(z,w)d\mu(z)d\mu(w)\right)\right\} \\ &\int\cdots\int\left(\prod_{i=0}^{n}\frac{1}{|1-z_i|^{2+\epsilon}} \wedge c^{(2+\epsilon)n}\right) d\mathcal{L}(z_1)\cdots d\mathcal{L}(z_k)d\ell(z_{2k+1})\cdots d\ell(z_n). \end{aligned}$$

The last integral is dominated by $e^{C(\epsilon)n}$ for appropriate $C(\epsilon)$. Taking logarithm, dividing by n^2 and letting $n \to \infty$, $\epsilon \to 0$ and $M \to \infty$ (in that order) we get the desired upper bound (21). ∎

Proof of Proposition 3.6. The statement would follow immediately from the definitions if f_M was a bounded continuous functions. Although f_M is not a bounded continuous function, f_M is clearly bounded above. We introduce $g_M = f_M \vee (-M)$. Note that switching with $z' = 1-z, w' = 1-w$, we have $g_M(1-z', 1-w') = (-M) \vee M \wedge (-\log|\frac{1}{z'} - \frac{1}{w'}|)$.

Let A_M be the set

$$A_M := \{(z,w) : f_M(z,w) < -M\}.$$

Clearly,

$$\iint f_M(z,w)d\nu(z)d\nu(w) = \iint g_M(z,w)d\nu(z)d\nu(w) + \iint_{A_M} (f_M(z,w)+M)d\nu(z)d\nu(w).$$

We consider integration over the domain $|w'| \le |z'|$; by symmetry, the complementary domain can be handled similarly. Then on the set A_M, we have $f_M(z,w) = -\log|1 - \frac{w'}{z'}| + \log|w'|$. But since $|w'| \le |z'|$, we have $-\log|1-\frac{w'}{z'}| \ge -\log 2$ and therefore on the set A_M we have

$$\log|w'| \le -M + \log 2$$

and

$$-\log 2 + \log|w'| \le f_M(z,w) \le -M. \tag{25}$$

Let $B_{M,\nu}$ be the event that for two i.i.d. variables (X,Y) sampled from ν, the minimum satisfies

$$\log(\min(|1-X|, |1-Y|)) \le -M + 2.$$

Clearly, $A_M \subset B_{M,\nu}$. From Lemma 2.2, we deduce that

$$\nu(B_{M,\nu}) < c_1(M) \tag{26}$$

where $c_1(M) \to 0$ as $M \to \infty$ uniformly in ν. Furthermore, the same lemma implies that

$$\left| \int_{B_{M,\nu}} \log(\min(|1-z|, |1-w|))\, d\nu(z)d\nu(w) \right| < c_2(M) \tag{27}$$

where $c_2(M) \to 0$ as $M \to \infty$ uniformly in ν.

Combining (25), (26) and (27) we get

$$\iint_{A_M} (f_M(z,w) + M)d\nu(z)d\nu(w) = c_3(M, \nu)$$

where $c_3(M,\nu) \to 0$ as $M \to \infty$ uniformly in $\nu \in \mathcal{P}$. In other words, we have

$$\left| \iint f_M(z,w)d\nu(z)d\nu(w) - \iint g_M(z,w)d\nu(z)d\nu(w) \right| \to 0$$

as $M \to \infty$, uniformly in $\nu \in \mathcal{P}$.

It remains to show that $\iint g_M d\nu \otimes d\nu \to \iint g_M d\mu \otimes d\mu$ as $\nu \to \mu$ for a fixed M. But this is true by definition since g_M is a bounded continuous function. ■

3.4 The Lower Bound

Our goal in this subsection is to prove the following.

Lemma 3.7. *For any $\mu \in \mathcal{P}$ with $I(\mu) < \infty$,*

$$\lim_{\epsilon \to 0} \varliminf_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu) \leq \epsilon \right) \geq -I(\mu) = -\frac{1}{2} \iint f(z,w) d\mu(z) d\mu(w). \tag{28}$$

(Recall that $f(z,w) = \log|1-z| + \log|1-w| - \log|z-w|$.)

The proof of Lemma 3.7 proceeds by several approximation steps. Those are detailed in the rest of the subsection, with several technical propositions deferred to Sections 4 and 5.

3.4.1 A dense subclass $\mathcal{D} \subset \mathcal{P}$

We introduce a dense (in the metric of $\mathcal{M}_1(\mathbb{C})$) subset $\mathcal{D} \subset \mathcal{P}$ such that for any measure $\mu \in \mathcal{P}$ there is a sequence $\{\mu_m\}_{m=1}^{\infty}$ from $\mathcal{D}$ such that

(i) $\mu_m \to \mu$ as $m \to \infty$ (convergence in the weak topology of $\mathcal{M}_1(\mathbb{C})$),

(ii) $I(\mu_m) \to I(\mu)$ as $m \to \infty$,

(iii) For any $\nu \in \mathcal{D}$, the estimate (28) holds.

Once such a subset $\mathcal{D}$ is constructed, Lemma 3.7 follows at once.

Such a desirable dense subset $\mathcal{D} \subset \mathcal{P}$ will be obtained from the proof of the characterization Theorem 2.3 for $\mathcal{P}$. In [BES13, Proof of Theorem 2], the authors begin with any fixed measure $\mu \in \mathcal{P}$, and perform a sequence of approximation steps (steps 1-5 in their paper) to obtain a measure $\mu_\epsilon \in \mathcal{P}$, which has (roughly) the following support properties:

(a) $\mathrm{supp}(\mu_\epsilon)$ is contained in a compact annulus centred at the origin.

(b) $\mathrm{supp}(\mu_\epsilon)$ is disjoint from a cone with apex at the origin and axis the positive ray $\mathbb{R}_+$.

(c) $\widehat{\mathbb{L}}_{\mu_\epsilon}(z) < \widehat{\mathbb{L}}_{\mu_\epsilon}(|z|)$ for each $z \in \mathbb{C} \setminus \bar{\mathbb{R}}_+$, while $\widehat{\mathbb{L}}_{\mu_\epsilon}(z) = \widehat{\mathbb{L}}_{\mu_\epsilon}(0) + a\Re z + O(|z|^2)$ as $|z| \to 0$ and $\widehat{\mathbb{L}}_{\mu_\epsilon}(z) = \log|z| + b/\Re(z) + O(1/|z|^2)$ as $|z| \to \infty$, with both a and b being positive.

In addition, $\mu_\epsilon \to_{\epsilon\to 0} \mu$ (again, in the weak topology of $\mathcal{M}_1(\mathbb{C})$.

We define $\mathcal{D}$ to be the subset of $\mathcal{M}_1(\mathbb{C})$ consisting of probability measures μ_ϵ satisfying (a)–(c). We will use the [BES13] construction (with a slight modification of their step 5), in order to construct, for any $\mu \in \mathcal{P}$ with $I(\mu) < \infty$, a sequence $\mu_\epsilon \in \mathcal{D}$ with $\mu_\epsilon \to \mu$ and in addition $I(\mu_\epsilon) \to_{\epsilon\to 0} I(\mu)$. For the sake of completeness, we give a complete account of the construction, its modifications and approximation properties in Section 5. That is, we prove in Section 5 the following proposition.

Proposition 3.8. *For any $\mu \in \mathcal{P}$ with $I(\mu) < \infty$, there exists a sequence $\mu_\epsilon \in \mathcal{D}$ so that $\mu_\epsilon \to_{\epsilon\to 0} \mu$ and $I(\mu_\epsilon) \to_{\epsilon\to 0} I(\mu)$.*

Equipped with Proposition 3.8 and in view of properties (i)–(iii) above, Lemma 3.7 is an immediate consequence of the following proposition and the local nature of the large deviations lower bound.

Proposition 3.9. *The lower bound* (28) *holds for any $\mu \in \mathcal{D}$.*

The rest of this section is devoted to the proof of Proposition 3.9.

3.4.2 Proof of Proposition 3.9.

The proof proceeds in several approximation steps. We fix throughout a $\mu \in \mathcal{D}$. We first construct in Proposition 3.10 a sequence of polynomials with positive coefficients whose empirical measure of zeros approximates μ and so that their (discrete) logarithmic energies approximate the logarithmic energy of μ. We then show in Proposition 3.11 that it is enough to prove the lower bound for balls centered at the empirical measure μ_k of zeros of these approximating polynomials. The proof of the later lower bound is then obtained by first constructing appropriate neighborhoods of μ_k, and then lower bounding the probability by lower bounding the density of $\mathbb{P}_n$ on these neighborhoods.

Stage I: Reduction to atomic measures
We introduce the discrete version of logarithmic energy for atomic measures with distinct atoms, as follows.

Definition 3. *For an atomic measure μ with equal mass $\frac{1}{k}$ at the k distinct atoms $\{z_i\}_{i=1}^k$, let*

$$\Sigma_{\mathrm{a}}(\mu) = \frac{1}{k^2} \sum_{i \neq j} \log |z_i - z_j|$$

denote the modified logarithmic energy.

By an abuse of notation, we will also use the same notation $\Sigma_{\mathrm{a}}(P)$ where P is a polynomial (with distinct zeros); in that case, the atomic measure being considered is the empirical measure of the zeros of P. We begin with the following proposition.

Proposition 3.10. *For each $\mu \in \mathcal{D}$ one may find a sequence of monic polynomials $\{P_k\}$, with empirical measure of zeros $\{\mu_k\}$, satisfying the following properties.*

(i) *P_k has positive coefficients.*

(ii) *$\mu_k \in \mathcal{P}$, $\mu_k \to \mu$ in $\mathcal{M}_1(\mathbb{C})$ and $\Sigma_{\mathrm{a}}(\mu_k) \to \Sigma(\mu)$ as $k \to \infty$.*

(iii) *μ_k has $a(k)$ distinct atoms, none of which is real, and μ_k puts equal mass of $1/a(k)$ on each atom, with $a(k) \to \infty$ as $k \to \infty$. In particular, $a(k)$ is even.*

(iv) *P_k is of degree $d(k) = a(k)n(k)$ for some positive integer $n(k)$.*

The proof of Proposition 3.10 is given in Section 4.

The importance of the polynomials in Proposition 3.10 lies in the following proposition.

Proposition 3.11. *To obtain (28) for some $\mu \in \mathcal{D}$, it suffices to prove that for all large enough k, and with μ_k as in Proposition 3.10, we have*

$$\varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu_k) \le \epsilon \right) \ge -\frac{1}{2} \left(\int \log|1-z| d\mu_k(z) + \int \log|1-z| d\mu_k(w) - \Sigma_{\mathrm{a}}(\mu_k) \right). \tag{29}$$

Proof of Proposition 3.11. Given $\epsilon > 0$, we have for all k large enough the inclusion of sets

$$\{\nu : d(\nu, \mu_k) < \epsilon/2\} \subset \{\nu : d(\nu, \mu) < \epsilon\}.$$

This implies that given $\epsilon > 0$, we have for all large enough k

$$\begin{aligned} \varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu) \le \epsilon \right) &\ge \varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu_k) \le \epsilon/2 \right) \\ &\ge -\frac{1}{2} \left(\int \log|1-z| d\mu_k(z) + \int \log|1-z| d\mu_k(w) - \Sigma_{\mathrm{a}}(\mu_k) \right). \end{aligned} \tag{30}$$

Now, the support of each μ_k is contained in the $1/a(k)$ thickening of the support of μ which is a compact set bounded away from 1. Hence the function $\log|1-z|$ is continuous on a closed neighborhood of the support of μ, and therefore $\mu_k \to \mu$ in $\mathcal{M}_1(\mathbb{C})$ implies that $\int \log|1-z| d\mu_k(z) \to \int \log|1-z| d\mu(z)$. Moreover, by property (ii) of μ_k we have that $\Sigma_{\mathrm{a}}(\mu_k) \to \Sigma(\mu)$ as $k \to \infty$. Letting now $k \to \infty$ first and then $\epsilon \to 0$ in (30) yields (28). ■

Stage II: The Neighbourhoods $\mathcal{N}_n(\epsilon, \delta)$ of μ_k

In this stage we will fix k and consider a positive number $0 < \delta < 1$ (we will eventually let $\delta \to 0$). We will define suitable n-dependent neighbourhoods $\mathcal{N}_n(\epsilon, \delta)$ of μ_k in $\mathcal{M}_1(\mathbb{C})$ (depending on N), which are contained in the set $\{\nu : d(\nu, \mu_k) < \epsilon\}$.

We begin with several definitions. If $\{w_i\}_{i=1}^p$ is a collection of complex numbers (multiplicities allowed), we say that the collection of complex numbers $\{\widetilde{w}_i\}_{i=1}^p$ is *ρ-compatible* with $\{w_i\}_{i=1}^p$ if $|w_i - \widetilde{w}_i| < \rho$ for all i and $\widetilde{w}_i$ is real whenever w_i is real.

Definition 4. *Let P be a polynomial of degree p with positive coefficients and zero set $\{w_i\}_{i=1}^p$ (multiplicities allowed). Then ρ_P is defined as the largest $\rho > 0$ satisfying that for any $i \neq j$, either $w_i = w_j$ or $B(w_i, \rho) \cap B(w_j, \rho) = \emptyset$, and in addition any collection $\{\widetilde{w}_i\}_{i=1}^p$ which is symmetric under conjugation and ρ-compatible with $\{w_i\}_{i=1}^p$ is the zero set of a polynomial with positive coefficients.*

For any $\rho < \rho_P$, we denote by $S(P, \rho)$ the set of all empirical measures corresponding to such collections $\{\widetilde{w}_i\}$.

Let P_k be as in Proposition 3.10, with corresponding empirical measure μ_k and set of atoms $\{z_i\}_{i=1}^{a(k)}$. Recall that none of the zeros of P_k is real. We will consider the set of atomic measures $S(P_k, \rho(\delta))$ (where δ is a small parameter to be sent to 0 eventually) and choose $\rho(\delta) \to 0$ as $\delta \to 0$ depending on P_k so that:

- $d(\mu_k, \nu) < \epsilon/2$ for any $\nu \in S(P_k, \rho(\delta))$.
- $\left|\int \log|1-z| d\mu_k - \int \log|1-z| d\nu(z)\right| < \delta$ for each $\nu \in S(P_k, \rho(\delta))$.
- $\rho(\delta) < \delta \cdot \min_{i \neq j} |z_i - z_j|$.

Given a positive integer n, set $m = \lfloor n/d(k) \rfloor$ (where we recall that $d(k) = a(k)n(k)$ is the degree of P_k). Define the set of atomic measures

$$S^{(m)}(P_k, \rho(\delta)) := \{\nu : \frac{1}{m}\sum_{i=1}^{m} \nu_i;\ \nu_i \in S(P_k, \rho(\delta)) \text{ and } \nu \text{ has } md(k) \text{ distinct atoms}\}.$$

Each $\nu \in S^{(m)}(P_k, \rho(\delta))$ has the following structure: it has $md(k)$ distinct atoms with equal mass at each atom, and the atoms can be grouped into $a(k)$ sets according to their nearest z_i; by the definition of $S(P_k, \rho(\delta))$, each atom is at distance at most $\rho(\delta)$ from the corresponding z_i. In particular, none of the atoms is real.

Fix a bounded interval $\mathcal{I}$ of length < 1 on the negative real line such that $\mathcal{I}$ is also bounded away from the support of μ_k by a distance ≥ 2. Define the set of measures

$$\Upsilon^{(m)} := \left\{ \frac{1}{n - md(k)} \sum_{i=1}^{n-md(k)} \delta_{\theta_i} : \theta_i \text{ are distinct numbers } \in \mathcal{I} \right\}.$$

Finally, define $\mathcal{N}_n(\epsilon, \delta)$ to be the set of measures

$$\mathcal{N}_n(\epsilon, \delta) := \left\{ \frac{md(k)}{n}\nu_1 + \left(1 - \frac{md(k)}{n}\right)\nu_2 : \nu_1 \in S^{(m)}(P_k, \rho(\delta)), \nu_2 \in \Upsilon^{(m)} \right\}.$$

Note that all measures in $\mathcal{N}_n(\epsilon,\delta)$ possess precisely $(n-md(k))$ real zeros. Since $0 \le n-md(k) \le d(k)$ (which is fixed since we are considering k to be fixed), for large enough n we have $d(\mu_k,\nu)<\epsilon$ for all measures $\nu \in \mathcal{N}_n(\epsilon,\delta)$. Therefore,

$$\varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n,\mu_k) \le \epsilon \right) \ge \varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(\mathcal{N}_n(\epsilon,\delta) \right). \tag{31}$$

Remark 3.1. *Each $\nu \in \mathcal{N}_n(\epsilon,\delta)$ has the following structure:*

- *ν has n distinct atoms, each having equal mass $1/n$.*
- *The atoms of ν are the disjoint union of $(a(k)+1)$ subsets as follows:*
  - *$\Lambda_i(\nu) := \{w : w$ an atom of $\nu, |z_i - w| \le \rho(\delta)\}, 1 \le i \le a(k)$, with $|\Lambda_i(\nu)| = d(k)m/a(k)$ for each $i \in \{1,\cdots,a(k)\}$.*
  - *$\Lambda_0(\nu) := \{w : w$ an atom of $\nu, w \in \mathcal{I}\}$ with $|\Lambda_0(\nu)| = n - d(k)m$.*

Conversely, every collection of n points satisfying the above structure has the property that the corresponding empirical measure is in $\mathcal{N}_n(\epsilon,\delta)$.

For each $\nu \in \mathcal{N}_n(\epsilon,\delta)$, we define the atomic measure

$$\nu(i) := \frac{1}{|\Lambda_i(\nu)|} \sum_{w \in \Lambda_i(\nu)} \delta_w$$

for $0 \le i \le a(k)$.

Stage III: A good subset $\widetilde{\mathcal{N}}_n(\epsilon,\delta) \subset \mathcal{N}_n(\epsilon,\delta)$ and completion of the proof of Proposition 3.9.

We introduce a subset $\widetilde{\mathcal{N}}_n(\epsilon,\delta) \subset \mathcal{N}_n(\epsilon,\delta)$, and estimate its volume in Proposition 3.12. We then use the estimate to complete the proof of Proposition 3.9.

Throughout, we fix $q = q(n) = md(k)/2$, recalling that $m = \lfloor n/d(k) \rfloor$ and that by construction, $d(k)$ is even.

Definition 5. *For $1 \le i \le a(k)$, define U_i to be the ball of radius $\rho(\delta)$ centered at z_i. Define U_0 to be the interval $\mathcal{I}$.*

Define the set of atomic measures $\widetilde{\mathcal{N}}_n(\epsilon,\delta) \subset \mathcal{N}_n(\epsilon,\delta)$ as follows:

$$\widetilde{\mathcal{N}}_n(\epsilon,\delta) := \{\nu \in \mathcal{N}_n(\epsilon,\delta) : \Sigma_{\mathrm{a}}(\nu(i)) > 2\Sigma(U_i) \text{ for each } 0 \le i \le a(k)\}.$$

By an abuse of notation, we also denote by $\widetilde{\mathcal{N}}_n(\epsilon,\delta)$ the subset of $\mathbb{C}^q \times \mathbb{R}^{n-2q}$ induced by the atoms of measures $\nu \in \widetilde{\mathcal{N}}_n(\epsilon,\delta)$ in the manner described in Section 1.

Note that by the choice of $\rho(\delta)$, $U_i \cap \mathbb{R} = \emptyset$ for $i = 1, \dots, a(k)$.

With this definition, we have the following proposition, whose proof is given at the end of this subsection.

Proposition 3.12. *With notation as above, we have*

(i) *Volume*$(\widetilde{\mathcal{N}}_n(\epsilon, \delta)) \geq \frac{1}{2^{a(k)}} \rho(\delta)^{d(k)m} |\mathcal{I}|^{d(k)}$.

(ii) *For each* $\nu \in \widetilde{\mathcal{N}}_n(\epsilon, \delta)$, *we have*

$$\Sigma_{\mathrm{a}}(\nu) \geq \Sigma_{\mathrm{a}}(\mu_k) + \log|1 - 2\delta| + \frac{2}{a(k)} A(\delta) + \frac{C(k)}{n^2}$$

where $A(\delta)$ *and* $C(k)$ *are positive quantities such that* $A(\delta) \to 0$ *as* $\delta \to 0$ *with* k *fixed.*

(iii) $\left| \int \log|1 - z| d\mu_k - \int \log|1 - z| d\nu(z) \right| < \delta$ *for each* $\nu \in \widetilde{\mathcal{N}}_n(\epsilon, \delta)$.

In the proposition, by Volume we mean the Euclidean volume.

We can now complete the following proof.

Proof of Proposition 3.9. In what follows, we will use the notation $\widetilde{\mathcal{N}}_n$ to denote $\widetilde{\mathcal{N}}_n(\epsilon, \delta)$, unless explicitly mentioned otherwise. We have

$$P_n\left(\widetilde{\mathcal{N}}_n\right) \geq \frac{1}{\mathcal{Z}_{n,q}} \times$$
$$\int_{\widetilde{\mathcal{N}}_n} \exp\left\{\frac{n^2}{2}\left(\Sigma_{\mathrm{a}}(\mathcal{W}) - 2\frac{n+1}{n^2}\sum_{i=1}^{n} \log|1 - w_i|\right)\right\} \prod_{i=1}^{q} d\mathcal{L}(w_i) \prod_{j=0}^{n-2q-1} d\ell(w_{N-j}),$$

where we recall that each measure in $\widetilde{\mathcal{N}}_N$ has $n - 2q$ real atoms, and $\mathcal{W}$ is the empirical measure corresponding to the set of atoms $\{w_i\}_{i=1}^n$.

By Proposition 3.12, the exponent in the last integral is lower bounded (uniformly over all measures in $\widetilde{\mathcal{N}}_n$) by

$$\mathcal{E}_n = \frac{n^2}{2}\left(\Sigma_{\mathrm{a}}(\mu_k) + \log|1 - 2\delta| + \frac{2}{a(k)} A(\delta) + \frac{C(k)}{n^2} - (1 + \frac{1}{n}) \cdot 2 \int \log|1 - z| d\mu_k(z) - 2(1 + \frac{1}{n})\delta\right).$$

Hence, using also (6),

$$\mathbb{P}_n(\widetilde{\mathcal{N}}_n) \geq \exp\left(o(n^2) + \mathcal{E}_n\right) \mathrm{Volume}(\widetilde{\mathcal{N}}_n).$$

From Proposition 3.12 it follows that $\frac{1}{n^2} \log \mathrm{Volume}(\widetilde{\mathcal{N}}_n) \to 0$ as $n \to \infty$.

Therefore,

$$\varliminf_{n\to\infty} \frac{1}{n^2}\mathbb{P}_n(\widetilde{\mathcal{N}}_n(\epsilon,\delta)) \geq \frac{1}{2}\left(\Sigma_{\mathrm{a}}(\mu_k) + \log|1-2\delta| + \frac{2}{a(k)}A(\delta) - 2\int \log|1-z|d\mu_k(z) - 2\delta\right).$$

Letting $\delta \to 0$, holding k fixed, we obtain (29). Proposition 3.11 now implies that the proof of (28) is complete. ■

Proof of Proposition 3.12. We begin with part (i). Note that for $i = 1, \ldots, a(k)$, U_i does not intersect the real axis. Recall also that $|\Lambda_i(\nu)| = d(k)m/a(k) =: p$.

Let a p-tuple $\gamma = (\gamma_1, \cdots, \gamma_p)$ be sampled with each co-ordinate drawn independently from the uniform distribution on U_i; by abuse of notation, we still use γ to denote the atomic measure $\frac{1}{p}\sum_j \delta_{\gamma_j}$. Then, $\mathbb{E}(\Sigma_{\mathrm{a}}(\gamma)) = (1-\frac{1}{p})\Sigma(U_i)$. Notice here that since the radius of U_i is less than 1, $\Sigma_{\mathrm{a}}(\gamma)$ and $\Sigma(U_i)$ both have negative signs. Therefore, by a first moment bound,

$$\mathrm{Volume}\{\gamma \in U_i^p : |\Sigma_{\mathrm{a}}(\gamma)| = -\Sigma_{\mathrm{a}}(\gamma) \geq 2|\Sigma(U_i)| = -2\Sigma(U_i)\} \leq \frac{1}{2}\mathrm{Volume}(U_i^p).$$

This implies that

$$\mathrm{Volume}\{\gamma \in U_i^p : \Sigma_{\mathrm{a}}(\gamma) > 2\Sigma(U_i)\} > \frac{1}{2}\mathrm{Volume}(U_i^p).$$

Note that the quantity $\Sigma(\mu)$ is invariant under translations of the measure μ, hence any $\Sigma(U_i)$ with $i \geq 1$ is equal to $\Sigma(U)$, where U is the disk of radius $\rho(\delta)$ centered at the origin. Note that $\Sigma(U) \to 0$ as $\delta \to 0$.

A similar argument with the atoms in $\Lambda_0(\nu)$ for $\nu \in \mathcal{N}_n$ implies that

$$\mathrm{Volume}\{\gamma \in \mathcal{I}^{n-d(k)m} : \Sigma_{\mathrm{a}}(\gamma) > 2\Sigma(\mathcal{I})\} > \frac{1}{2}\mathrm{Volume}(|\mathcal{I}|^{n-d(k)m}).$$

Definition 5 and Remark 3.1 now imply part (i) of Proposition 3.12.

We next turn to the proof of part (ii). For $\nu \in \mathcal{N}_n(\epsilon,\delta)$ we have

$$\Sigma_{\mathrm{a}}(\nu) = \sum_{i=1}^{a(k)} \frac{d(k)^2m^2}{a(k)^2n^2}\Sigma_{\mathrm{a}}(\nu(i)) + \frac{(n-d(k)m)^2}{n^2}\Sigma_{\mathrm{a}}(\nu(0)) + \frac{1}{n^2}\sum_{w_i,w_j \text{ not in same } \Lambda_l} \log|w_i - w_j|. \tag{32}$$

Recall that $\rho(\delta) < \delta \cdot \min_{i\neq j}|z_i - z_j|$. This implies that for w_i, w_j from $\Lambda_\alpha, \Lambda_\beta$ respectively with $\alpha \neq \beta \neq 0$, we have

$$(1-2\delta)|z_\alpha - z_\beta| \leq |w_i - w_j| \leq (1+2\delta)|z_\alpha - z_\beta|.$$

On the other hand, if $w_i \in \Lambda_0$ and $w_j \notin \Lambda_0$, then

$$0 \leq \log|w_i - w_j| \leq \log|1 + |\mathcal{I}| + D + \rho(\delta)|,$$

where D is the diameter of the support of μ_k. Hence

$$\left|\frac{1}{n^2}\sum_{\text{Exactly one of } w_i,w_j\in\Lambda_0}\log|w_i-w_j|\right|\le\frac{d(k)(n-d(k)m)}{n^2}\log|1+|\mathcal{I}|+D+\rho(\delta)|\le C(k)/n^2$$

for some function $C(k)$. Hence we have

$$\log|1-2\delta|-\frac{C(k)}{n^2}\le\left|\frac{1}{n^2}\sum_{w_i,w_j\text{ not in same }\Lambda_l}\log|w_i-w_j|-\Sigma_{\mathrm{a}}(\mu_k)\right|\le\log|1+2\delta|+\frac{C(k)}{n^2}. \tag{33}$$

This is true for every $\nu\in\mathcal{N}_n(\epsilon,\delta)$, and therefore for every $\nu\in\widetilde{\mathcal{N}}_n(\epsilon,\delta)$. By definiton of $\widetilde{\mathcal{N}}_n$, we have $\Sigma_{\mathrm{a}}(\nu(i))\ge 2\Sigma(U_i)\ge 2B(\delta)$ for each $\nu\in\widetilde{\mathcal{N}}_n$ and each $0\le i\le a(k)$, where the function $B(\delta)=\max(\Sigma(U),\Sigma(I))$. Since $m=\lfloor n/d(k)\rfloor$, we have

$$\sum_{i=1}^{a(k)}\frac{d(k)^2m^2}{a(k)^2n^2}\Sigma_{\mathrm{a}}(\nu(i))+\frac{(n-d(k)m)^2}{n^2}\Sigma_{\mathrm{a}}(\nu(0))\ge\frac{A(\delta)}{a(k)}$$

for some $A(\delta)\to 0$ as $\delta\to 0$. This completes the proof of part (ii) of the proposition.

Part (iii) of the proposition is immediate as the statement holds for all measures $\nu\in\mathcal{N}_n(\epsilon,\delta)$ and $\widetilde{\mathcal{N}}_n(\epsilon,\delta)\subset\mathcal{N}_n(\epsilon,\delta)$. ■

4 Proof of Proposition 3.10

We begin with a general approximation result.

Lemma 4.1. *Let $\mu\in\mathcal{M}_1(\mathbb{C})$ be of compact support and such that μ is symmetric under conjugation, $\Sigma(\mu)<\infty$, and μ does not possess atoms. Then there exists a sequence of point configurations with distinct points and empirical measures ν_k such that*

(1) ν_k is symmetric under conjugation, the support of ν_k is contained inside the $1/k$ thickening of the support of μ, and ν_k does not charge the real line.

(2) $\nu_k\to\mu$ as $k\to\infty$.

(3) $\Sigma_{\mathrm{a}}(\nu_k)\to\Sigma(\mu)$ as $k\to\infty$.

Proof. For an discrete measure ν having n distinct atoms and equal mass $1/n$ on each atom, and a function f, define $\Sigma^f_{\mathrm{a}}(\nu)=\frac{1}{n^2}\sum_{i\ne j}f(x_i,x_j)$ where $\{x_i\}_{i=1}^n$ are the atoms of ν. Thus, for $f(z,w)=\log|z-w|$, we have $\Sigma^f_{\mathrm{a}}(\nu)=\Sigma_{\mathrm{a}}(\nu)$ as defined earlier.

Recall that the measure μ is compactly supported and symmetric under conjugation. Let μ_R be μ restricted to $\mathbb{R}$ and let μ_u be μ restricted to the upper half plane. Define the measures $\mu_1 = \frac{1}{2}\mu_R + \mu_u$ and $\mu_2 = \frac{1}{2}\mu_R + \overline{\mu_u}$ where $\overline{\mu_u}$ is the measure supported on the lower half plane and defined by $\overline{\mu_u}(A) = \mu_u(\overline{A})$. Obviously, $\mu = \mu_1 + \mu_2$.

For each n, we will obtain conjugation symmetric point sets of size $2n$ whose empirical measures will approximate μ in the following way. First, consider n i.i.d. random samples $\{x_1, \cdots, x_n\}$ from μ_1 (to obtain random samples we consider the probability measure obtained by appropriately normalizing μ_1). Consider the point set $Y := \{y_1, \cdots, y_n\}$ where $y_j = x_j + \frac{i}{n}$ and i is the imaginary unit. Consider the point set $Z := Y \cup \overline{Y}$, and let $L_n = \frac{1}{2n}\sum_{z\in Z}\delta_z$ denote the (random) empirical measure associated with Z; one has that $L_n \to \mu$ in distribution (for example, by an application of Sanov's theorem).

Fix a positive number M (to be thought of as large). Let $K > 1$ be a bound on the diameter of the support of M. Set $f(z, w) = \log|z - w|$. Define $f_M(z, w) = f(z, w) \wedge M \vee (-M)$ and $g_M = f - f_M$. Because $\Sigma(\mu) < \infty$ we have that

$$\alpha(M) = \iint |g_M(z, w)| d\mu(z) d\mu(w) \to 0$$

as $M \to \infty$.

We have that $\frac{1}{n} \le |Y_i - \overline{Y}_i| \le K$. Therefore, for $n > n_0(K)$,

$$\frac{1}{n^2}\left|\sum_{i=1}^{n} \log|Y_i - \overline{Y}_i|\right| \le \log n/n.$$

On the other hand, for $i \neq j$ we have $\mathbb{E}[|g_M(Y_i, Y_j)|] = \iint |g_M(z, w)| d\mu_1(z) d\mu_1(w)$ and $\mathbb{E}[|g_M(\overline{Y}_i, \overline{Y}_j)|] = \iint |g_M(z, w)| d\mu_2(z) d\mu_2(w)$.

We next claim that, for $M > \log K$, we have $\mathbb{E}[|g_M(Y_i, \overline{Y}_j)|] \le \iint |g_M(z, w)| d\mu_1(z) d\mu_2(w)$. To see this, note that

$$g_M(z, w) = \begin{cases} 0, & \text{if } -M \le \log|z - w| \le M, \\ \log|z - w| - M, & \text{if } \log|z - w| \ge M, \\ \log|z - w| + M & \text{if } \log|z - w| \le -M. \end{cases}$$

The case $\log|z - w| \ge M$ does not arise when we consider $g_M(Y_j, \overline{Y}_j)$ because $M > \log K$. On the other hand,

$$|Y_k - \overline{Y}_j| = |X_k - \overline{X}_j + \frac{2i}{n}| \ge |X_k - \overline{X}_j|$$

because the X_j-s belong to the upper half plane. Thus, $\log|Y_k - \overline{Y}_j| \le -M$ implies $\log|X_k - \overline{X}_j| \le -M$ and on this event we have $\log|X_k - \overline{X}_j| + M \le \log|Y_k - \overline{Y}_j| + M$.

Since these quantities are negative, this is equivalent to $|\log|Y_k - \overline{Y}_j| + M| \le |\log|X_k - \overline{X}_j| + M|$. This implies that

$$\mathbb{E}[|g_M(Y_i, \overline{Y}_j)|] \le \mathbb{E}[|g_M(X_i, \overline{X}_j)|] = \iint |g_M(z,w)| d\mu_1(z) d\mu_2(w),$$

as claimed.

We now have that

$$\begin{aligned} \mathbb{E}[|\Sigma_{\mathrm{a}}^{g_M}(L_n)|] &\le \frac{1}{4n^2} \sum_{i \ne j} \mathbb{E}[|g_M(Z_i, Z_j)|] \\ = \quad & \frac{1}{4n^2} \sum_{i \ne j} \mathbb{E}[|g_M(Y_i, Y_j)|] + \frac{1}{4n^2} \sum_{i \ne j} \mathbb{E}[|g_M(Y_i, \overline{Y}_j)|] \\ & + \frac{1}{4n^2} \sum_{i \ne j} \mathbb{E}[|g_M(\overline{Y}_i, \overline{Y}_j)|] + \frac{1}{4n^2} \sum_{i=1}^{n} \mathbb{E}[|g_M(Y_i, \overline{Y}_i)|]. \end{aligned}$$

Thus, from the previous computations, we can choose $n_1 = n_1(M)$ so that for all $n > n_1(M)$,

$$\mathbb{E}[|\Sigma_{\mathrm{a}}^{g_M}(L_n)|] \le 2\alpha(M)\,.$$

In particular, for fixed $\delta > 0$ there exists $n_2(M) = n_2(M, \delta)$ so that for $n > n_2(M)$, there exists a realization ν_n of L_n such that $d(\nu_n, \mu) \le \delta$ and $|\Sigma_{\mathrm{a}}^{g_M}(\nu_n)| \le 2\alpha(M)$. Applying a diagonalization argument (with $\delta \to 0$ while M is kept fixed), we find a sequence (with some abuse of notation, denoted ν_k), so that such that $d(\nu_k, \mu) \le 1/k$ and $|\Sigma_{\mathrm{a}}^{g_M}(\nu_k)| \le 2\alpha(M)$.

Now, $\Sigma_{\mathrm{a}}(\nu_k) = \Sigma_{\mathrm{a}}^{f_M}(\nu_k) + \Sigma_{\mathrm{a}}^{g_M}(\nu_k)$. Since f_M is bounded and continuous, we have $\Sigma_{\mathrm{a}}^{f_M}(\nu_k) \to \iint f_M(z,w) d\mu(z) d\mu(w)$ as $k \to \infty$. This implies that

$$\overline{\lim_{k \to \infty}} |\Sigma_{\mathrm{a}}^{f}(\nu_k) - \iint f(z,w) d\mu(z) d\mu(w)| \le 3\alpha(M).$$

Applying again a diagonalization argument (this time over M), one obtains the lemma. ■

Proof of Proposition 3.10. Let ν_k be the sequence of atomic measures constructed in Lemma 4.1. Note that each on the ν_ks has $a(k)$ distinct (non real) atoms, is supported within the $1/a(k)$ thickening of the support of μ, and satisfies $\Sigma_{\mathrm{a}}(\nu_k) \to \Sigma(\nu_k)$. Each ν_k gives rise to a monic polynomial Q_k with distinct zeros (so that the zeros of Q_k are precisely the atoms of ν_k). Since $\mu \in \mathcal{P}$, Step 6 in [BES13, Proof of Theorem 2] shows that for some $n(k)$ large enough, the polynomial $P_k = Q_k^{n(k)}$ has all coefficients real and positive. Note that the empirical measure of zeros of P_k coincides with the empirical measure of zeros of Q_k. This completes the proof. ■

5 The Bergweiler-Eremenko-Sokal approximation and proof of Proposition 3.8.

We begin with a fixed measure $\mu \in \mathcal{P}$ satisfying $I(\mu) < \infty$. Let

$$\widehat{\mathbb{L}}_\mu(z) = \int_{|w|\le 1} \log(|z-w|)d\mu(w) + \int_{|w|>1} \log(|1-\frac{z}{w}|)d\mu(w)\,,$$

and denote by $u(z) = \mathbb{L}_\mu(z) = \int \log|z-w|d\mu(w)$ its logarithmic potential. Because $I(\mu) < \infty$, the Bergweiler-Eremenko-Sokal condition can be written as $u(z) \le u(|z|)$ for all $z \in \mathbb{C} \setminus \mathbb{R}_+$.

The Bergweiler-Eremenko-Sokal approximation proceeds in five steps to construct a sequence of approximations $\mu_i = \mu_i^\epsilon$, $i = 1, \ldots, 5$, with $\mu_5 \in \mathcal{D}$. In the i-th step, one starts with a measure μ_{i-1} (with $\mu_0 = \mu$ and $u_0 = u$) and constructs measures $\mu_i \in \mathcal{P}$ (depending on a small parameter $\epsilon > 0$) such that $\mu_i \to \mu_{i-1}$ weakly as $\epsilon \to 0$. The measures μ_i are defined via subharmonic functions u_i, such that $\mu_i = (2\pi)^{-1}\Delta u_i$ in the sense of distributions. One shows, see Section 4 of [BES13]) that in each of the 5 steps, one has $L_{\mu_i}(z) = u_i(z) + k_i^\epsilon$ where k_i^ϵ is a constant (as a function of z) depending on ϵ, and that $u_i(z) \to u_{i-1}(z)$ for each $z \in \mathbb{R}_+$, while in some of the steps the above convergence will occur pointwise in $\mathbb{C}$. It is a consequence of Proposition 5.1 below that in each of the Steps 1-5, $k_i^\epsilon \to 0$ as $\epsilon \to 0$.

5.1 Preliminaries

We begin with several preliminary proerties concerning the convergence of subharmonic functions.

Proposition 5.1. *Let u_ϵ be a sequence of subharmonic functions converging in the sense of distributions to a subharmonic function u_0 as $\epsilon \to 0$. Assume further that u_ϵ converges pointwise to u_0 on $\mathbb{R}_+$. Let $\mu_\epsilon, \epsilon \ge 0$ denote the Riesz measure of u_ϵ and assume that $u_\epsilon(z) = \mathbb{L}_{\mu_\epsilon}(z) + k_\epsilon$ where k_ϵ is independent of z. Then $\mu_\epsilon \to \mu_0$ weakly and $k_\epsilon \to k_0$.*

Proof. By standard results, see [BES13, Appendix A], one gets the weak convergence $\mu_\epsilon \to \mu_0$ and the a.e. (with respect to Lebesgue measure on $\mathbb{R}_+$) convergence of $\mathbb{L}_{\mu_\epsilon}$ to $\mathbb{L}_{\mu_0}$. In particular, the convergence $L_{\mu_\epsilon}(z) \to L_{\mu_0}(z)$ occurs at a point $z \in \mathbb{R}_+$. This yields the convergence $k_\epsilon \to k_0$. ■

We next show that pointwise monotone convergence of subharmonic functions implies the convergence of the associated logarithmic energies. Results of this nature are known in the literature; here we follow an approach based on the proof of a related result in [D84].

Proposition 5.2. *Suppose $\{u_n\}$ is a sequence of subharmonic functions decreasing pointwise to a subharmonic function u_0, as $n \to \infty$. Let μ_n be the Riesz measure of u_n, and assume $u_n(z) = \mathbb{L}_{\mu_n}(z) + k_n$, with $k_n \to k_0$ as $n \to \infty$. Then $\Sigma(\mu_n) \to \Sigma(\mu_0)$ as $n \to \infty$.*

Proof. By subtracting off k_0, we can assume $u_0 = \mathbb{L}_{\mu_0}$ and $k_\epsilon \to 0$. Introduce the notation $[\mu, \nu] := \iint \log|z - w| d\mu(z) d\nu(w)$.

The hypotheses imply in particular that $\mu_n \to \mu_0$ weakly. The lower semicontinuity of $-\Sigma(\cdot)$ then implies that $\limsup \Sigma(\mu_n) \leq \Sigma(\mu_0)$. To see the other direction, note that if either $n = 0$ or $n > m$ we have

$$\begin{aligned}
\Sigma(\mu_n) &= [\mu_n, \mu_n] = \int (u_n(z) - k_n)\, d\mu_n(z) = \int u_n(z) d\mu_n(z) - k_n \\
&\leq \int u_m(z) d\mu_n(z) - k_n = [\mu_m, \mu_n] + k_m - k_n \leq [\mu_m, \mu_m] + 2(k_m - k_n) \\
&= \Sigma(\mu_m) + 2(k_m - k_n),
\end{aligned}$$

where the monotonicity of the sequence $\{u_n\}$ was used in the inequalities. We conclude that $\liminf_{m\to\infty} \Sigma(\mu_n) \geq \Sigma(\mu_0)$, completing the proof. ■

5.2 Approximation steps

We describe each of the approximation steps $\mu_i^\epsilon \to \mu_{i-1}$, $i = 1, \ldots, 5$, and show that for each, both

$$\mathbb{L}_{\mu_i^\epsilon}(1) \to \mathbb{L}_{\mu_{i-1}}(1)\,, \quad \Sigma(\mu_i^\epsilon) \to \Sigma(\mu_{i-1}). \tag{34}$$

In the sequel, we omit the subscript ϵ when it is clear from the context.

5.2.1 Step 1:

Given $\epsilon > 0$, define

$$u_1(z) = \max\{u(ze^{i\alpha}) : |\alpha| \leq \epsilon\}.$$

Note that $u_1 \geq u$ pointwise, and that u_1 decreases in ϵ. It is proved in [BES13, Section 4] that $u_1 \to_{\epsilon\to 0} u$ weakly; however, due to upper semicontinuity of u, this implies also the pointwise convergence $u_1 \searrow_{\epsilon\to 0} u$. An application of Propositions 5.1 and 5.2 yields (34) for $i = 1$.

5.2.2 Step 2:

For $\epsilon_1 \in (0, \epsilon)$ where ϵ is as chosen in Step 1, define $D_{\epsilon_1} = \{z : |\arg(z)| \leq \epsilon_1\}$. Let v denote the solution to the Dirichlet problem in D_{ϵ_1} with boundary conditions $u_1(z)$ and $v(z) = O(\log|z|)$ as $z \to \infty$. Define $u_2 = u_2(\epsilon_1)$ to be the function obtained by "balayage" (i.e., sweeping out of the Riesz measure) from the domain D_{ϵ_1}. In other words, $u_2(z) = v(z)$ if $z \in D_{\epsilon_1}$, and $= u_1(z)$ otherwise. In this step too, it follows from [BES13, Section 4] and the upper semicontinuity of u_1 that $u_2 \searrow_{\epsilon_1 \to 0} u_1$ pointwise. An application of Propositions 5.1 and 5.2 yields (34) for $i = 2$.

5.3 Step 3:

For $\epsilon > 0$, define $u_3(z) = u_2(z + \epsilon)$. We have $d\mu_3(z) = d\mu_2(z + \epsilon)$. We have that $u_3 \to u_2$ pointwise on $\mathbb{R}_+$ and hence Proposition 5.1 yields that $\mathbb{L}_{\mu_3}(1) \to \mathbb{L}_{\mu_2}(1)$. Since $\Sigma(\mu_3) = \Sigma(\mu_2)$, we conclude that (34) holds for $i = 3$.

5.4 Step 4:

For $\epsilon > 0$, define $v(z) = u_3(1/z) + \log|z|$ for $z \neq 0$, and extend $v(\cdot)$ to $\mathbb{C}$ by defining $v(0) = \limsup_{z\to 0} v(z)$. This definition preserves the sub-harmoniticity of v, and in fact

$$v(0) = \lim_{r \searrow 0} v(r) \tag{35}$$

because $u_3(z) \leq u_3(|z|)$, hence the limsup is attained as $r \searrow 0$, and the convexity of $v(r)$ in $\log r$ then yields the existence of the limit. We claim that in fact, $v(0) > -\infty$ (and therefore, by sub-harmoniticity, is finite). Indeed, $\mathbb{L}_{\mu_v}(0) = -\mathbb{L}_{\mu_3}(0)$; by construction, u_3 is harmonic in a neighborhood of 0, see [BES13]. Hence, $\mathbb{L}_{\mu_v}(0) = -\mathbb{L}_{\mu_3}(0)$ is finite, as claimed. We note in passing that $\mathbb{L}_{\mu_v}(1) = \mathbb{L}_{\mu_3}(1)$.

Next, define $w(z) = v(z + \epsilon)$ and, finally, $u_4(z) = w(1/z) + \log|z|$. As in the previous step, we have that $\mathbb{L}_{\mu_4}(1) = \mathbb{L}_{\mu_w}(1) \to_{\epsilon\to 0} \mathbb{L}_{\mu_v}(1)$. So it only remains to check the convergence of the logarithmic energy. To that end, note that under the transformation $v(z) = u(1/z) + \log|z|$, one has

$$\begin{aligned}
\Sigma(\mu_v) &= \iint \log|z - w| d\mu_v(z) d\mu_v(w) = \iint \log\left|\frac{1}{z} - \frac{1}{w}\right| d\mu_u(z) d\mu_u(w) \\
&= \iint \log|z - w| d\mu_u(z) d\mu_u(w) - 2\int \log|z| d\mu_u(z) = \Sigma(\mu_u) - 2\mathbb{L}_{\mu_u}(0).
\end{aligned}$$

With this computation in hand, we trace the changes in the logarithmic energy Σ in Step 4 as follows:

$$\begin{aligned}
\Sigma(\mu_v) &= \Sigma(\mu_{u_3}) - 2\mathbb{L}_{\mu_3}, \Sigma(\mu_w) = \Sigma(\mu_v), \mathbb{L}_{\mu_w}(0) = \mathbb{L}_{\mu_v}(\epsilon) \\
\Sigma(\mu_{u_4}) &= \Sigma(\mu_w) - 2\mathbb{L}_{\mu_w}(0) = \Sigma(\mu_v) - 2\mathbb{L}_{\mu_v}(\epsilon) = \Sigma(\mu_{u_3}) - 2\mathbb{L}_{\mu_{u_3}}(0) - 2\mathbb{L}_{\mu_v}(\epsilon) \\
&= \Sigma(\mu_{u_3}) + 2\mathbb{L}_{\mu_v}(0) - 2\mathbb{L}_{\mu_v}(\epsilon).
\end{aligned}$$

But

$$\mathbb{L}_{\mu_v}(\epsilon) - \mathbb{L}_{\mu_v}(0) = v(\epsilon) - v(0) \to 0\,,$$

due to (35). This completes the proof of (34) for Step 4.

5.5 Step 5:

In this step, we slightly differ from the recipe of [BES13]. Let I denote an interval of length θ (which is assumed to be small but fixed) centered at -1. Let α be the normalized Lebesgue measure supported on the set I. For $\epsilon > 0$, define the measure

$$\mu_5 := (1-\epsilon)\mu_4 + \epsilon\alpha.$$

Define $u_5 = \mathbb{L}_{\mu_5}$. Via a series expansion of $\log|1+z|$ for small and large z, one checks that this has the intended effect of making the coefficients b and c in Step 5 of the [BES13] argument positive, while preserving the inequality $u_5(z) < u_5(|z|)$ for $z \notin \bar{\mathbb{R}}_+$. Thus, $\mu_5 \in \mathcal{D}$.

To see (34) for μ_5, note that $\mathbb{L}_{\mu_5} = (1-\epsilon)\mathbb{L}_{\mu_4} + \epsilon\mathbb{L}_\alpha$ and

$$\Sigma(\mu_5) = (1-\epsilon)^2\Sigma(\mu_4) + \epsilon^2\Sigma(\alpha) + 2\epsilon(1-\epsilon)\int \mathbb{L}_\alpha(z) d\mu_4(z).$$

Since I is a bounded interval, one has that $\mathbb{L}_\alpha(z)$ is uniformly bounded in z on any compact set, in particular, on the support of μ_4. Letting $\epsilon \to 0$, one gets (34) for μ_5.

This completes the proof of Proposition 3.8. ■

6 Conditioning on all zeros being real

One notes from the expression for the density (5) in case $k = 0$ that $P(L_n \in \mathcal{M}_1(\mathbb{R}_-)) > 0$. One also notes that $\{\mu \in \mathcal{P} : \text{supp}(\mu) = \mathbb{R}_-\} = \mathcal{M}_1(\mathbb{R}_-)$. Thus, one can rerun the proof of the lower bound in Theorem 1.1 replacing throughout $\mathcal{P}$ by $\mathcal{M}_1(\mathbb{R}_-)$ as a particular case of the proof in [BAG97]. One obtains that

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in \mathcal{M}_1(\mathbb{R}_-)) = -I_R\,,$$

and one immediately deduces Theorem 1.2 by noting that the minimizer μ_R is unique due to the strict convexity of I (applied on $\mathcal{M}_1(\mathbb{R}_-)$).

To see Theorem 1.3, we can make the transformation $x \mapsto -x$ to see that we are interested in solving the variational problem

$$\inf_{\mu\in M_1([1,\infty))} \{ \int_0^\infty \log(x+1) d\mu(x) - \gamma \int_0^\infty \int_0^\infty \log|x-y| d\mu(x) d\mu(y)\,, \tag{36}$$

with $\gamma = 1/2$.

A standard application of calculus of variation methods (as e.g. in [AGZ10, Lemma 2.6.2]) shows that the minimizer $\bar{\mu}$ of (36) is characterized as the unique solution, for some constant C, of

$$2\gamma \mathbb{L}_{\bar{\mu}}(x) \begin{cases} = \log(x+1) + C, & \bar{\mu} - a.e. \\ > \log|x+1| + C, & x \in \mathbb{R}_+ \setminus \mathrm{supp}(\bar{\mu}). \end{cases} \tag{37}$$

One can proceed by first guessing the form of the minimizer and then verifying that it satisfies indeed (37). For $\gamma > 1/2$, this is can be achieved solving, in a compact interval, the associated Riemann-Hilbert problem, and then taking the limit $\gamma \to 1/2$. We do not detail these computations, instead presenting the *ansatz* that the minimizer in (36) has density with respect to Lebesgue measure on $[0, \infty)$ of the form

$$\psi(x) = \frac{1}{\pi(x+1)\sqrt{x}}\,. \tag{38}$$

We need to verify that $\psi(x)dx$ satisfies (37). Making the change of variables $w = \sqrt{x}$, we have

$$\mathbb{L}_{\bar{\mu}}(x) = \frac{2}{\pi} \int_0^\infty \frac{\log|x - w^2|}{w^2+1} dw\,.$$

Choosing the contour of integration $\mathcal{C} := \{r\}_{r=-R}^{R} \cup \{Re^{i\theta}\}_{\theta=0}^{\pi}$ for R large, and noting the pole at i, one obtains from a residue computation that $\mathbb{L}_{\bar{\mu}}(x) = \log(x+1)$ for $x \in \mathbb{R}_+$, i.e. that (37) holds with density ψ. This completes the proof of Theorem 1.3.

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