DIFFERING AVERAGED AND QUENCHED LARGE DEVIATIONS FOR RANDOM WALKS IN RANDOM ENVIRONMENTS IN DIMENSIONS TWO AND THREE

ATILLA YILMAZ AND OFER ZEITOUNI

Abstract. We consider the quenched and averaged (or annealed) large deviation rate functions I_q and I_a for space-time and (the usual) space-only RWRE on $\mathbb{Z}^d$. By Jensen's inequality, $I_a \leq I_q$.

In the space-time case, when $d \geq 3+1$, I_q and I_a are known to be equal on an open set containing the typical velocity ξ_o. When $d = 1+1$, we prove that I_q and I_a are equal only at ξ_o. Similarly, when $d = 2+1$, we show that $I_a < I_q$ on a punctured neighborhood of ξ_o.

In the space-only case, we provide a class of non-nestling walks on $\mathbb{Z}^d$ with $d = 2$ or 3, and prove that I_q and I_a are not identically equal on any open set containing ξ_o whenever the walk is in that class. This is very different from the known results for non-nestling walks on $\mathbb{Z}^d$ with $d \geq 4$.

1. Introduction

1.1. **The models.** Consider a discrete time Markov chain on the d-dimensional integer lattice $\mathbb{Z}^d$ with $d \geq 1$. For any $x, z \in \mathbb{Z}^d$, denote the transition probability from x to $x+z$ by $\pi(x, x+z)$. Refer to the transition vector $\omega_x := (\pi(x, x+z))_{z\in\mathbb{Z}^d}$ as the *environment* at x. If the environment $\omega := (\omega_x)_{x\in\mathbb{Z}^d}$ is sampled from a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, then this process is called *random walk in a random environment* (RWRE). Here, $\mathcal{B}$ is the Borel σ-algebra corresponding to the product topology.

For every $y \in \mathbb{Z}^d$, define the shift T_y on Ω by $(T_y\omega)_x := \omega_{x+y}$. In order to have some statistical homogeneity in the environment, $\mathbb{P}$ is generally assumed to be stationary and ergodic with respect to $(T_y)_{y\in\mathbb{Z}^d}$. In this paper, we will make the stronger assumption that $\mathbb{P}$ is a product measure with equal marginals. In other words, $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ is a collection of independent and identically distributed (i.i.d.) random vectors.

The set $\mathcal{R} := \{z \in \mathbb{Z}^d : \mathbb{P}(\pi(0,z) > 0) > 0\}$ of allowed steps of the walk is called the *range* of the walk. We will assume that there exists a $\kappa > 0$ such that $\mathbb{P}(\pi(0,z) \geq \kappa) = 1$ for every $z \in \mathcal{R}$. This condition is known as *uniform ellipticity.*

Let $(e_i)_{i=1}^d$ denote the canonical basis for $\mathbb{Z}^d$. The walk is said to be *space-time* if

$$\mathcal{R} = \mathcal{R}_{st} := \{(z_1, \ldots, z_d) \in \mathbb{Z}^d : |z_1| + \cdots + |z_{d-1}| = 1, z_d = e_d\}, \tag{1.1}$$

and it is said to be *space-only* if

$$\mathcal{R} = \mathcal{R}_{so} := \{\pm e_i\}_{i=1}^d. \tag{1.2}$$

Space-time is a natural term since, under (1.1), the walk decomposes into two parts. Its projection on the e_d-axis is deterministic and can be identified with time. The motion in the span of $(e_i)_{i=1}^{d-1}$ can be thought of as a variation of space-only RWRE where the environment is freshly sampled at each time step. To emphasize this decomposition, we will write the dimension as $d = (d-1)+1$. For example, when $d = 3$, we will say that the dimension is $2+1$.

For every $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, the Markov chain with environment ω induces a probability measure P_x^ω on the space of paths starting at x. Statements about P_x^ω that hold for $\mathbb{P}$-a.e. ω are referred to as *quenched*. Statements about the semi-direct product $P_x := \mathbb{P} \times P_x^\omega$ are referred to as *averaged* (or *annealed*). Expectations under $\mathbb{P}, P_x^\omega$ and P_x are denoted by $\mathbb{E}, E_x^\omega$ and E_x, respectively.

See [23] for a survey of results and open problems on RWRE.

It is clear that no model satisfies both (1.1) and (1.2). Nevertheless, it turns out that many of the results that hold for space-only RWRE are valid under also the space-time assumption, and it is fair to say that space-time RWRE is easier to analyze than space-only RWRE because (1.1) ensures that the walk never visits the same point more than once.

Date: September 28, 2009.

2000 *Mathematics Subject Classification.* 60K37, 60F10, 82C44.

Key words and phrases. Random walks, large deviations, disordered media, fractional moment.

1.2. **Regeneration times.** In the next subsection, we will give a brief survey of the previous results on large deviations for RWRE in order to put the present work in context. Some of these results involve certain random times which are introduced below for convenience.

Let $(X_n)_{n\geq 0}$ denote the path of a space-only RWRE. Consider a unit vector $\hat{u}\in\mathcal{S}^{d-1}$. Define a sequence $(\tau_m)_{m\geq 0}$ of random times, which are referred to as *regeneration times* (relative to $\hat{u}$), by $\tau_o := 0$ and

$$\tau_m := \inf\left\{j > \tau_{m-1} : \langle X_i,\hat{u}\rangle < \langle X_j,\hat{u}\rangle \leq \langle X_k,\hat{u}\rangle \text{ for all } i,k \text{ with } i<j<k\right\}$$

for every $m\geq 1$. (Regeneration times were introduced in this context by Sznitman and Zerner, see [16].) Because we assumed the environment $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ to be an i.i.d. collection, if the walk is directionally transient relative to $\hat{u}$, i.e., if $P_o\left(\lim_{n\to\infty}\langle X_n,\hat{u}\rangle = \infty\right) = 1$, then $P_o\left(\tau_m < \infty\right) = 1$ for every $m\geq 1$. In this setup, as noted in [16], the significance of $(\tau_m)_{m\geq 1}$ is due to the fact that

$$\left(X_{\tau_m+1}-X_{\tau_m}, X_{\tau_m+2}-X_{\tau_m},\ldots,X_{\tau_{m+1}}-X_{\tau_m}\right)_{m\geq 1}$$

is an i.i.d. sequence under P_o.

The walk is said to satisfy Sznitman's transience condition (**T**) if

$$E_o\left[\sup_{1\leq i\leq\tau_1}\exp\left\{c_1\left|X_i\right|\right\}\right] < \infty \text{ for some } c_1 > 0.$$

When $d\geq 2$, Sznitman [15] proves that (1.2) and (**T**) imply a *ballistic* law of large numbers (LLN), an averaged central limit theorem and certain large deviation estimates.

Condition (**T**) holds as soon as the walk is *non-nestling* relative to $\hat{u}$, i.e., when the random drift vector

$$v(\omega) := \sum_{z\in\mathcal{R}}\pi(0,z)z \qquad \text{satisfies} \qquad \operatorname*{ess\,inf}_{\mathbb{P}}\langle v(\cdot),\hat{u}\rangle > 0. \tag{1.3}$$

The walk is said to be non-nestling if it is non-nestling relative to some unit vector. Otherwise, it is referred to as *nestling*. In the latter case, the convex hull of the support of the law of $v(\cdot)$ contains the origin.

In the case of space-time RWRE, regeneration times are defined naturally by taking $\hat{u} = e_d$ and $\tau_m = m$ for every $m\geq 1$.

1.3. **Previous results on large deviations for RWRE.** Recall that a sequence $(Q_n)_{n\geq 1}$ of probability measures on a topological space $\mathbb{X}$ satisfies the *large deviation principle* (LDP) with rate function $I:\mathbb{X}\to\mathbb{R}$ if I is non-negative, lower semicontinuous and for any measurable set G,

$$-\inf_{x\in G^o} I(x) \leq \liminf_{n\to\infty}\frac{1}{n}\log Q_n(G) \leq \limsup_{n\to\infty}\frac{1}{n}\log Q_n(G) \leq -\inf_{x\in\bar{G}} I(x).$$

Here, G^o is the interior of G, and $\bar{G}$ its closure. See [3] for general background regarding large deviations.

We will focus on the following large deviation principles.

Theorem 1 (Quenched LDP). *For $\mathbb{P}$-a.e. ω, $\left(P_o^\omega\left(\frac{X_n}{n}\in\cdot\right)\right)_{n\geq 1}$ satisfies the LDP with a deterministic and convex rate function I_q.*

Theorem 2 (Averaged LDP). *$\left(P_o\left(\frac{X_n}{n}\in\cdot\right)\right)_{n\geq 1}$ satisfies the LDP with a convex rate function I_a.*

There are many works on large deviations for space-only RWRE. We briefly mention them in chronological order. Greven and den Hollander [6] prove Theorem 1 for walks on $\mathbb{Z}$ under the i.i.d. environment assumption. They provide a formula for I_q and show that its graph typically has flat pieces. Zerner [24] establishes Theorem 1 for nestling walks on $\mathbb{Z}^d$ in i.i.d. environments. Comets, Gantert and Zeitouni [2] generalize the result of [6] to walks on $\mathbb{Z}$ in stationary and ergodic environments. Also, they prove Theorem 2 for walks on $\mathbb{Z}$ in i.i.d. environments and give a formula that links I_a to I_q. Varadhan [18] generalizes Zerner's result to stationary and ergodic environments without any nestling assumption. He also proves Theorem 2 for walks on $\mathbb{Z}^d$ in i.i.d. environments and gives a variational formula for I_a. Rassoul-Agha [10] generalizes the latter result of [18] to certain mixing environments. Rosenbluth [13] gives an alternative proof of Theorem 1 for walks on $\mathbb{Z}^d$ in stationary and ergodic environments, and provides a variational formula for I_q. Yilmaz [22] generalizes the result of [13] to a so-called level-2 LDP. Berger [1], Peterson and Zeitouni [9], and Yilmaz [19] obtain certain qualitative properties of I_a. Rassoul-Agha and Seppäläinen [12] generalize the result of [13] to a so-called level-3 LDP.

In the case of space-time RWRE, Rassoul-Agha and Seppäläinen [11] prove Theorem 1 by adapting the quenched argument in [18]. Theorem 2 does not require any work. Indeed, Assumption (1.1) implies that the walk under P_o is a sum of i.i.d. increments. The common distribution of these increments is $(q(z))_{z\in\mathcal{R}}$ where $q(z) := \mathbb{E}[\pi(0,z)]$ for every $z \in \mathcal{R}$. Therefore, Theorem 2 is simply Cramér's theorem, cf. [3].

In addition to the works mentioned in the last two paragraphs, there are two more results on large deviations for RWRE that are relevant to this paper. We state them in detail.

Theorem 3 (Yilmaz [20])**.** *Assume (1.1). If* $d \geq 3+1$, *then* $I_q = I_a$ *on a set* $\mathcal{A}_{st} \times \{e_d\}$ *containing the LLN velocity* ξ_o, *where* $\mathcal{A}_{st}$ *is an open subset of* $\mathbb{R}^{d-1}$.

Theorem 4 (Yilmaz [21])**.** *Assume (1.2),* $d \geq 4$, *and that Sznitman's* (***T***) *condition holds for some* $\hat{u} \in \mathcal{S}^{d-1}$.

- (a) *If the walk is non-nestling, then* $I_q = I_a$ *on an open set* $\mathcal{A}_{so}$ *containing the LLN velocity* ξ_o.
- (b) *If the walk is nestling, then*
  - (i) $I_q = I_a$ *on an open set* $\mathcal{A}_{so}^+$,
  - (ii) *there exists a* $(d-1)$*-dimensional smooth surface patch* $\mathcal{A}_{so}^b$ *such that* $\xi_o \in \mathcal{A}_{so}^b \subset \partial\mathcal{A}_{so}^+$,
  - (iii) *the unit vector* η_o *normal to* $\mathcal{A}_{so}^b$ *(and pointing inside* $\mathcal{A}_{so}^+$*) at* ξ_o *satisfies* $\langle \eta_o, \xi_o\rangle > 0$, *and*
  - (iv) $I_q(t\xi) = tI_q(\xi) = tI_a(\xi) = I_a(t\xi)$ *for every* $\xi \in \mathcal{A}_{so}^b$ *and* $t \in [0,1]$.

It is worthwhile to emphasize that the equality $I_q = I_a$ does not extend, in the setup of Theorems 3 and 4, to the whole space, see the discussion in [21].

1.4. **Our results.** For space-time RWRE, it is natural to ask whether Theorem 3 can be generalized to $d \geq 1+1$ or $2+1$. The answer turns out to be no.

Theorem 5. *Assume (1.1). If* $d = 1+1$, *then* $I_q(\xi) = I_a(\xi) < \infty$ *if and only if* $\xi = \xi_o$, *the LLN velocity.*

Theorem 6. *Assume (1.1). If* $d = 2+1$, *then* $I_a < I_q$ *on a set* $(\mathcal{G}_{st} \times \{e_d\}) \setminus \{\xi_o\}$, *where* $\mathcal{G}_{st} \subset \mathbb{R}^2$ *is open and* $\mathcal{G}_{st} \times \{e_d\}$ *contains* ξ_o.

In the case of space-only RWRE on $\mathbb{Z}$, Comets et al. [2] show that $I_q(\xi) = I_a(\xi) < \infty$ if and only if $\xi = 0$ or $I_a(\xi) = 0$. In particular, Theorem 4 cannot be generalized to $d \geq 1$. Our next result shows that its conclusion is false also for a class of space-only RWRE's in dimensions $d = 2,3$.

Definition 7. *Assume* $d \geq 2$, *and fix a triple* $p = (p^+, p^o, p^-)$ *of positive real numbers such that* $p^- < p^+$ *and* $p^+ + p^o + p^- = 1$. *For any* $\epsilon > 0$, *a probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{B})$ *is said to be in class* $\mathcal{M}_\epsilon(d,p)$ *if*

- (a) $\mathbb{P}$ *is a product measure with equal marginals,*
- (b) *(1.2) holds,*
- (c) $\mathbb{P}(\pi(0,e_d) = p^+,\ \pi(0,-e_d) = p^-) = 1$,
- (d) $\mathbb{P}(\epsilon/2 < |\pi(0,e_1) - \frac{p^o}{2(d-1)}| < \epsilon) = 1$, *and*
- (e) $\mathbb{P}$ *is invariant under the rotations of* $\mathbb{Z}^d$ *that preserve* e_d. *(We will refer to this as isotropy.)*

Theorem 8. *Assume* $d = 2$ *or* 3. *Fix a triple* $p = (p^+, p^o, p^-)$ *as in Definition 7. Then there exists an* $\epsilon_0 = \epsilon_0(p)$ *such that if* $\epsilon < \epsilon_0$ *and* $\mathbb{P}$ *is in class* $\mathcal{M}_\epsilon(d,p)$, *then the quenched and averaged rate functions* I_q *and* I_a *are not identically equal on any open set containing the LLN velocity* ξ_o.

The proofs of our results are based on a technique that combines the so-called *fractional moment* method with a certain change of measure (which we will refer to as *tilting the environment*). This technique has been developed for analyzing the so-called polymer pinning model, cf. [4, 17, 5], and it has been recently refined by Lacoin [8] for obtaining certain lower bounds for the free energy of directed polymers in random environments. Comparing with the polymer setup, an extra complication occurs in the RWRE model due to the dependence of the transition probabilities of the walk in the environment. (In the polymer model discussed above, the walk is a simple random walk, and the environment only appears in the evaluation of exponential moments with respect to the random walk.) The difficulty in the RWRE setup, and much of our work, lies in overcoming this dependency.

Here is how the rest of the paper is organized: In Section 2, we consider space-time RWRE and prove Theorems 5 and 6 by adapting the relevant arguments given in [8]. In Section 3, we focus on space-only walks that are non-nestling relative to e_d, and modify the previous proofs by making use of regeneration times. This way, we establish a result (see Theorem 18) analogous to Theorems 5 and 6. The only difference is that Theorem 18 is valid under a certain correlation condition, cf. (3.11). Finally, we prove Theorem 8 by checking that (3.11) holds whenever $\mathbb{P}$ is in class $\mathcal{M}_\epsilon(d,p)$ with some triple p (as in Definition 7) and a sufficiently small $\epsilon > 0$.

2. Inequality of the rate functions for space-time RWRE

2.1. **Reducing to a fractional moment estimate.** Assume $d \geq 1+1$. Recall (1.1). Consider a space-time random walk on $\mathbb{Z}^d$ in a uniformly elliptic and i.i.d. environment. For every $\theta \in \mathbb{R}^d$, define

$$\phi(\theta) := \sum_{z \in \mathcal{R}} \mathrm{e}^{\langle \theta, z \rangle} q(z)$$

where $q(z) := \mathbb{E}[\pi(0,z)]$. Since the walk visits every point at most once, $E_o\left[\exp\{\langle \theta, X_N \rangle\}\right] = \phi(\theta)^N$ for every $N \geq 1$.

Define the logarithmic moment generating functions

$$\Lambda_q(\theta) := \lim_{N\to\infty} \frac{1}{N} \log E_o^\omega\left[\exp\{\langle \theta, X_N\rangle\}\right] \quad \text{and} \quad \Lambda_a(\theta) := \lim_{N\to\infty} \frac{1}{N} \log E_o\left[\exp\{\langle \theta, X_N\rangle\}\right] = \log \phi(\theta).$$

By Varadhan's Lemma, cf. [3], $\Lambda_q(\theta) = \sup_{\xi \in \mathbb{R}^d}\left\{\langle \theta, \xi\rangle - I_q(\xi)\right\} = I_q^*(\theta)$, the convex conjugate of I_q at θ. Similarly, $\Lambda_a(\theta) = \log \phi(\theta) = I_a^*(\theta)$.

For every $N \geq 1$, $\theta \in \mathbb{R}^d$ and $\omega \in \Omega$, define

$$W_N(\theta, \omega) := E_o^\omega[\exp\{\langle \theta, X_N\rangle - N \log \phi(\theta)\}].$$

Given any $\alpha \in (0,1)$, Jensen's inequality and the bounded convergence theorem imply that

$$\begin{aligned} \Lambda_q(\theta) - \log\phi(\theta) &= \lim_{N\to\infty} \frac{1}{N} \log W_N(\theta,\cdot) = \mathbb{E}\left[\lim_{N\to\infty} \frac{1}{N}\log W_N(\theta,\cdot)\right] \\ &= \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\log W_N(\theta,\cdot)\right] = \lim_{N\to\infty} \frac{1}{N\alpha}\mathbb{E}\left[\log W_N(\theta,\cdot)^\alpha\right] \\ &\leq \limsup_{N\to\infty} \frac{1}{N\alpha} \log \mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right] \qquad (2.1) \\ &\leq \lim_{N\to\infty} \frac{1}{N\alpha} \log\left(\mathbb{E}\left[W_N(\theta,\cdot)\right]\right)^\alpha = 0. \end{aligned}$$

Lemma 9. *Assume (1.1). Fix any $\alpha \in (0,1)$. If $d = 1+1$, then*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right] < 0 \qquad (2.2)$$

whenever $\theta \notin sp\{e_2\}$, the one-dimensional vector space spanned by e_2.

Lemma 10. *Assume (1.1). Fix any $\alpha \in (0,1)$. If $d = 2+1$, then there exists a $\beta > 0$ such that (2.2) holds whenever* $\mathrm{dist}(\theta, sp\{e_3\}) \in (0,\beta)$.

Remark 11. *For every $\theta \in sp\{e_d\}$, (1.1) implies that $W_N(\theta,\cdot) = 1$ and $\Lambda_q(\theta) = \log\phi(\theta)$.*

When $d = 1+1$, it follows from (2.1) and Lemma 9 that $\Lambda_q(\cdot) < \log\phi(\cdot)$ on $\{\theta \in \mathbb{R}^2 : \theta \notin sp\{e_2\}\}$. By convex duality, $I_a < I_q$ on $\{\nabla \log\phi(\theta) : \theta \notin sp\{e_2\}\}$. It is easy to see that the latter set is equal to the domain of I_a minus the singleton $\{\xi_o\}$. This proves Theorem 5.

Similarly, when $d = 2+1$, Lemma 10 implies that $I_a < I_q$ on $\{\nabla \log\phi(\theta) : \mathrm{dist}(\theta, sp\{e_3\}) \in (0,\beta)\}$. One can check that this set is of the form $(\mathcal{G}_{st} \times \{e_d\}) \setminus \{\xi_o\}$ where $\mathcal{G}_{st} \subset \mathbb{R}^2$ is open and $\mathcal{G}_{st} \times \{e_d\}$ contains ξ_o. This proves Theorem 6.

The rest of this section is devoted to proving Lemmas 9 and 10.

2.2. **Decomposing into paths.** Assume $d = 1+1$ or $2+1$. Let $\mathbb{V}_d := \mathbb{Z}^{d-1} \times \{0\} \subset \mathbb{Z}^d$. Fix an n of the form k^2, with k an integer to be determined later (e.g., for $d = 1+1$, this n is chosen so that the conclusion of Lemma 12 below holds). When $d = 1+1$, let

$$J_y := [(y' - \tfrac{1}{2})\sqrt{n}, (y' + \tfrac{1}{2})\sqrt{n}) \times \{0\} \subset \mathbb{R}^2 \qquad (2.3)$$

for every $y = (y', 0) \in \mathbb{V}_2$. Similarly, when $d = 2+1$, let

$$J_y := [(y' - \tfrac{1}{2})\sqrt{n}, (y' + \tfrac{1}{2})\sqrt{n}) \times [(y'' - \tfrac{1}{2})\sqrt{n}, (y'' + \tfrac{1}{2})\sqrt{n}) \times \{0\} \subset \mathbb{R}^3$$

for every $y = (y', y'', 0) \in \mathbb{V}_3$.

Take $N = nm$ for some $m \geq 1$. For every $\theta \in \mathbb{R}^d$, $\omega \in \Omega$ and $Y = (y_1, \ldots, y_m) \in (\mathbb{V}_d)^m$, define

$$\bar{W}_N(\theta, \omega, Y) := E_o^\omega[\exp\{\langle \theta, X_N\rangle - N\log\phi(\theta)\}, X_{jn} - \lfloor jn\xi(\theta)\rfloor \in J_{y_j} \text{ for every } j \leq m] \qquad (2.4)$$

where $\xi(\theta) = \nabla \log \phi(\theta)$. (For $u \in \mathbb{R}^d$, $\lfloor u \rfloor$ denotes the closest element of $\mathbb{Z}^d$ to u. If there is more than one closest element, then take the one whose index is the smallest with respect to the lexicographic order.) Note that $\langle \xi(\theta), e_d \rangle = 1$ because $\langle z, e_d \rangle = 1$ for every $z \in \mathcal{R}_{st}$.

Since $\mathbb{V}_d$ is contained in the disjoint union $\cup_{y \in \mathbb{V}_d} J_y$, we see that $W_N(\theta, \omega) = \sum_Y \bar{W}_N(\theta, \omega, Y)$. Hence, $W_N(\theta, \omega)^\alpha \leq \sum_Y \bar{W}_N(\theta, \omega, Y)^\alpha$ by subadditivity, and

$$\mathbb{E}[W_N(\theta, \cdot)^\alpha] \leq \sum_Y \mathbb{E}\left[\bar{W}_N(\theta, \cdot, Y)^\alpha\right]. \tag{2.5}$$

In the rest of this section, we will treat the cases $d = 1 + 1$ and $d = 2 + 1$ separately.

2.3. **Tilting along a path ($d = 1 + 1$).** Given $m \geq 1$, $\theta \notin sp\{e_2\}$, $C_1 \geq 1$ and $Y = (y_1, \ldots, y_m) \in (\mathbb{V}_2)^m$, let

$$B_j := \{(s, i) \in \mathbb{Z}^2 : (j-1)n \leq i < jn, \|(s, i) - \lfloor i\xi(\theta) \rfloor - \sqrt{n} y_{j-1}\| \leq C_1 \sqrt{n}\} \tag{2.6}$$

for every $j \in \{1, \ldots, m\}$. Here, $y_o = (0, 0)$. Recall that $n = k^2$ for some integer k.

Fix a large K and a small δ_n, both to be determined later (depending on the choice of α, see (2.11), (2.12) and Lemma 12). Define $f_K(u) := -K \mathbb{1}_{u \geq e^{K^2}}$ and

$$g(\theta, \omega, Y) := \exp \sum_{j=1}^m f_K\left(\delta_n \sum_{(s,i) \in B_j} a(\theta, (s, i))\right) > 0, \tag{2.7}$$

where, for every $x \in \mathbb{Z}^2$, $a(\theta, x) := \langle \theta, v(T_x \omega) - \xi_o \rangle$ with the notation in (1.3). Note that $\mathbb{E}[a(\theta, x)] = 0$.

As before, take $N = nm$. By Hölder's inequality,

$$\begin{aligned} \mathbb{E}[\bar{W}_N(\theta, \cdot, Y)^\alpha] &= \mathbb{E}\left[(\bar{W}_N(\theta, \cdot, Y) g(\theta, \cdot, Y))^\alpha g(\theta, \cdot, Y)^{-\alpha}\right] \\ &\leq \mathbb{E}\left[\bar{W}_N(\theta, \cdot, Y) g(\theta, \cdot, Y)\right]^\alpha \mathbb{E}\left[g(\theta, \cdot, Y)^{-\frac{\alpha}{1-\alpha}}\right]^{1-\alpha}. \end{aligned} \tag{2.8}$$

Let us control the second term in (2.8). To abbreviate the notation, let

$$D(B_j) := \sum_{(s,i) \in B_j} a(\theta, (s, i)) \tag{2.9}$$

for every $j \in \{1, \ldots, m\}$. Since the environment is i.i.d.,

$$\begin{aligned} \mathbb{E}\left[g(\theta, \cdot, Y)^{-\frac{\alpha}{1-\alpha}}\right] &= \mathbb{E}\left[\exp\left(-\frac{\alpha}{1-\alpha} \sum_{j=1}^m f_K(\delta_n D(B_j))\right)\right] = \prod_{j=1}^m \mathbb{E}\left[\exp\left(-\frac{\alpha}{1-\alpha} f_K(\delta_n D(B_j))\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\frac{\alpha}{1-\alpha} f_K(\delta_n D(B_1))\right)\right]^m \leq \left(1 + e^{\frac{\alpha}{1-\alpha} K} \mathbb{P}\left(\delta_n D(B_1) \geq e^{K^2}\right)\right)^m. \end{aligned} \tag{2.10}$$

Note that, by Chebyshev's inequality,

$$\begin{aligned} \mathbb{P}\left(\delta_n D(B_1) \geq e^{K^2}\right) &\leq e^{-2K^2} \delta_n^2 \mathbb{E}\left[D(B_1)^2\right] = e^{-2K^2} \delta_n^2 \mathbb{E}\left[\sum_{(s,i) \in B_1} a(\theta, (s, i))^2\right] \\ &= e^{-2K^2} \delta_n^2 2C_1 n^{3/2} \mathbb{E}\left[a(\theta, (o, o))^2\right] \end{aligned}$$

since only the diagonal terms survive. Take

$$\delta_n = C_1^{-1/2} n^{-3/4}, \tag{2.11}$$

where C_1 is still to be defined (and will be chosen as in Lemma 12). Then, the RHS of (2.10) is bounded from above by

$$\left(1 + 2\mathbb{E}\left[a(\theta, (o, o))^2\right] e^{\frac{\alpha}{1-\alpha} K - 2K^2}\right)^m \leq \left(1 + 8e^{\frac{\alpha}{1-\alpha} K - 2K^2}\right)^m \leq 2^m$$

as soon as

$$8e^{\frac{\alpha}{1-\alpha} K - 2K^2} \leq 1. \tag{2.12}$$

Recalling (2.5) and (2.8), we see that

$$\mathbb{E}[W_N(\theta, \cdot)^\alpha] \leq 2^m \sum_Y \mathbb{E}\left[\bar{W}_N(\theta, \cdot, Y) g(\theta, \cdot, Y)\right]^\alpha. \tag{2.13}$$

2.4. **Estimating the expectation under the tilt ($d = 1+1$).** For every $m \geq 1$, $\theta \notin sp\{e_2\}$, $\omega \in \Omega$ and $Y \in (\mathbb{V}_2)^m$, let $N = nm$ as before. By the Markov property,

$$\begin{aligned}
\bar{W}_N(\theta,\omega,Y) &= \sum_{x_1,\ldots,x_m\in\mathbb{Z}^2} E_o^\omega[\exp\{\langle\theta,X_N\rangle - N\log\phi(\theta)\}, X_{jn} - \lfloor jn\xi(\theta)\rfloor = x_j \in J_{y_j}\ \forall j\leq m]\\
&= \sum_{x_1,\ldots,x_m\in\mathbb{Z}^2} E_o^\omega[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, X_n - \lfloor n\xi(\theta)\rfloor = x_1 \in J_{y_1}]\\
&\qquad\qquad \times E^\omega_{x_1+\lfloor n\xi(\theta)\rfloor}[\exp\{\langle\theta, X_n - (x_1 + \lfloor n\xi(\theta)\rfloor)\rangle - n\log\phi(\theta)\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad X_n - \lfloor 2n\xi(\theta)\rfloor = x_2 \in J_{y_2}]\\
&\qquad\qquad \times\cdots\\
&= \sum_{x_1,\ldots,x_m\in\mathbb{Z}^2} E_o^\omega[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, X_n - \lfloor n\xi(\theta)\rfloor = x_1 \in J_{y_1}]\\
&\qquad\qquad \times E^{T_{\lfloor n\xi(\theta)\rfloor+\sqrt{n}y_1}\omega}_{x_1-\sqrt{n}y_1}[\exp\{\langle\theta, X_n - (x_1 - \sqrt{n}y_1)\rangle - n\log\phi(\theta)\},\\
&\qquad\qquad\qquad\qquad X_n - \lfloor n\xi(\theta)\rfloor = x_2 - \sqrt{n}y_1 \in J_{y_2} - \sqrt{n}y_1]\\
&\qquad\qquad \times\cdots.
\end{aligned}$$

Recall (2.7) and (2.9). It follows from the i.i.d. environment assumption that

$$\begin{aligned}
&\mathbb{E}\left[\bar{W}_N(\theta,\cdot,Y)g(\theta,\cdot,Y)\right]\\
&\quad= \sum_{x_1,\ldots,x_m} \mathbb{E}\,[E_o^\omega[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor = x_1 \in J_{y_1}]\\
&\qquad\qquad \times E^{T_{\lfloor n\xi(\theta)\rfloor+\sqrt{n}y_1}\omega}_{x_1-\sqrt{n}y_1}\left[\exp\{\langle\theta, X_n - (x_1 - \sqrt{n}y_1)\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\},\right.\\
&\qquad\qquad\qquad\qquad \left. X_n - \lfloor n\xi(\theta)\rfloor = x_2 - \sqrt{n}y_1 \in J_{y_2} - \sqrt{n}y_1\right]\\
&\qquad\qquad \times\cdots]\\
&\quad= \sum_{x_1,\ldots,x_m} E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor = x_1 \in J_{y_1}]\\
&\qquad\qquad \times E_{x_1-\sqrt{n}y_1}\left[\exp\{\langle\theta, X_n - (x_1 - \sqrt{n}y_1)\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\},\right.\\
&\qquad\qquad\qquad\qquad \left. X_n - \lfloor n\xi(\theta)\rfloor = x_2 - \sqrt{n}y_1 \in J_{y_2} - \sqrt{n}y_1\right]\\
&\qquad\qquad \times\cdots\\
&\quad\leq E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_1}]\\
&\qquad \times \max_{x_1\in J_{y_1}} E_{x_1-\sqrt{n}y_1}\left[\exp\{\langle\theta, X_n - (x_1 - \sqrt{n}y_1)\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\},\right.\\
&\qquad\qquad\qquad\qquad \left. X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_2} - \sqrt{n}y_1\right]\\
&\qquad \times\cdots\\
&\quad= E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_1}]\\
&\qquad \times \max_{x_1\in J_o} E_{x_1}\left[\exp\{\langle\theta, X_n - x_1\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_2-y_1}\right]\\
&\qquad \times\cdots.
\end{aligned}$$

Plugging this in (2.13), we conclude that

$$\mathbb{E}[W_N(\theta,\cdot)^\alpha] \leq \left(2\sum_{y\in\mathbb{V}_2}\max_{x\in J_o} E_x\left[\exp\{\langle\theta,X_n - x\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_y\right]^\alpha\right)^m.$$

Since $N = nm$ and n was fixed, this proves Lemma 9 (and hence Theorem 5), provided that we have

Lemma 12. *Assume (1.1). If $d = 1+1$, $\alpha \in (0,1)$, $\theta \notin sp\{e_2\}$ and $\delta_n = C_1^{-1/2}n^{-3/4}$, then*

$$\sum_{y\in\mathbb{V}_2}\max_{x\in J_o} E_x\left[\exp\{\langle\theta,X_n - x\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_y\right]^\alpha < 1/2 \tag{2.14}$$

whenever n, K and C_1 are sufficiently large.

2.5. **Finishing the proof of Theorem 5.** It remains to give the

Proof of Lemma 12. We write the sum in (2.14) as

$$\sum_{y\in\mathbb{V}_2}\max_{x\in J_o}E_x\left[\cdots\right]=\sum_{\substack{y\in\mathbb{V}_2:\\ \|y\|>R}}\max_{x\in J_o}E_x\left[\cdots\right]+\sum_{\substack{y\in\mathbb{V}_2:\\ \|y\|\le R}}\max_{x\in J_o}E_x\left[\cdots\right]\tag{2.15}$$

with some large constant R. Since $f_K(u)=-K\mathbb{1}_{u\ge \mathrm{e}^{K^2}}\le 0$, the first sum on the RHS of (2.15) is bounded from above by

$$\begin{aligned}&\sum_{\substack{y\in\mathbb{V}_2:\\ \|y\|>R}}\max_{x\in J_o}E_x\left[\exp\{\langle\theta,X_n-x\rangle-n\log\phi(\theta)\},|X_n-\lfloor n\xi(\theta)\rfloor-\sqrt{n}y|\le\frac{\sqrt{n}}{2}\right]^\alpha\\ &\le\sum_{\substack{y\in\mathbb{V}_2:\\ \|y\|>R}}E_o\left[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)\},\left|\frac{X_n-\lfloor n\xi(\theta)\rfloor}{\sqrt{n}}-y\right|\le 1\right]^\alpha.\end{aligned}\tag{2.16}$$

Consider a tilted space-time walk on $\mathbb{Z}^2$ (in a deterministic environment) with transition probabilities $q^\theta(z):=q(z)\exp\{\langle\theta,z\rangle-\log\phi(\theta)\}$ for $z\in\mathcal{R}_{st}$. Let $\hat{P}_o^\theta$ denote the probability measure it induces on paths. Note that the LLN velocity under $\hat{P}_o^\theta$ is

$$\sum_{z\in\mathcal{R}_{st}}zq(z)\exp\{\langle\theta,z\rangle-\log\phi(\theta)\}=\nabla_\theta\log\phi(\theta)=\xi(\theta).$$

With this notation, (2.16) is equal to

$$\sum_{\substack{y\in\mathbb{V}_2:\\ \|y\|>R}}\hat{P}_o^\theta\left(\left|\frac{X_n-\lfloor n\xi(\theta)\rfloor}{\sqrt{n}}-y\right|\le 1\right)^\alpha\le\sum_{\substack{y\in\mathbb{V}_2:\\ \|y\|>R}}\hat{P}_o^\theta\left(\left|\frac{X_n-\lfloor n\xi(\theta)\rfloor}{\sqrt{n}}\right|\ge\|y\|-1\right)^\alpha$$

which, by Chebyshev's inequality, can be made arbitrarily small (uniformly in large n) by choosing R sufficiently large.

The second sum on the RHS of (2.15) is bounded from above by

$$2R\max_{x\in J_o}E_x\left[\exp\{\langle\theta,X_n-x\rangle-n\log\phi(\theta)+f_K(\delta_nD(B_1))\}\right]^\alpha.$$

Therefore, to conclude the proof of Lemma 12, it suffices to show that

$$E_o\left[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)+f_K(\delta_nD(B_1-x))\}\right]\le\left(\frac{1}{8R}\right)^{\alpha^{-1}}\tag{2.17}$$

for every $x\in J_o$.

Similar to B_1 defined in (2.6), introduce a new set

$$\bar{B}_1:=\{(s,i)\in\mathbb{Z}^2:0\le i<n,\|(s,i)-\lfloor i\xi(\theta)\rfloor\|\le(C_1-1/2)\sqrt{n}\}.$$

Note that $\bar{B}_1\subset B_1-x$ for every $x\in J_o$ since $\|x\|\le\sqrt{n}/2$.

$$\begin{aligned}&E_o\left[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)+f_K(\delta_nD(B_1-x))\}\right]\\ &\quad=\mathrm{e}^{-K}E_o\left[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)\},\delta_nD(B_1-x)\ge\mathrm{e}^{K^2}\right]\\ &\qquad+E_o\left[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)\},\{X_i:0\le i<n\}\not\subset\bar{B}_1,\delta_nD(B_1-x)<\mathrm{e}^{K^2}\right]\\ &\qquad+E_o\left[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)\},\{X_i:0\le i<n\}\subset\bar{B}_1,\delta_nD(B_1-x)<\mathrm{e}^{K^2}\right]\\ &\quad\le\mathrm{e}^{-K}+\hat{P}_o^\theta\left(\{X_i:0\le i<n\}\not\subset\bar{B}_1\right)\\ &\qquad+E_o\left[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)\},\{X_i:0\le i<n\}\subset\bar{B}_1,\delta_nD(B_1-x)<\mathrm{e}^{K^2}\right].\end{aligned}\tag{2.18}$$

The first term in (2.18) is small when K is large. Donsker's invariance principle ensures that the second term can be made arbitrarily small (uniformly in n) by choosing C_1 sufficiently large.

Let us focus on the third term in (2.18). For any sequence $(A_n)_{n\geq 1}$ of natural numbers,

$$
\begin{aligned}
&E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0 \leq i < n\} \subset \bar{B}_1, \delta_n D(B_1 - x) < \mathrm{e}^{K^2}] \\
&\quad\leq E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0 \leq i < n\} \subset \bar{B}_1, \delta_n \sum_{\substack{(s,i)\in B_1 - x \\ (s,i)\neq X_i}} a(\theta,(s,i)) < -A_n] \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0 \leq i < n\} \subset \bar{B}_1, \delta_n \sum_{i=0}^{n-1} a(\theta, X_i) < \mathrm{e}^{K^2} + A_n] \\
&\quad\leq \sum_{x_1,\ldots,x_{n-1}} \mathbb{E}[E_o^\omega\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, X_i = x_i\ \forall i < n\right], \delta_n \sum_{\substack{(s,i)\in B_1 - x \\ (s,i)\neq x_i}} a(\theta,(s,i)) < -A_n] \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta, X_i) < \mathrm{e}^{K^2} + A_n] \\
&\quad= \sum_{x_1,\ldots,x_{n-1}} E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, X_i = x_i\ \forall i < n\right] \times \mathbb{P}(\delta_n \sum_{\substack{(s,i)\in B_1 - x \\ (s,i)\neq x_i}} a(\theta,(s,i)) < -A_n) \qquad (2.19) \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta, X_i) < \mathrm{e}^{K^2} + A_n] \\
&\quad\leq \max_{x_1,\ldots,x_{n-1}} \mathbb{P}(\delta_n \sum_{\substack{(s,i)\in B_1 - x \\ (s,i)\neq x_i}} a(\theta,(s,i)) < -A_n) \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta, X_i) < \mathrm{e}^{K^2} + A_n] \\
&\quad\leq A_n^{-2}\delta_n^2 2C_1 n^{3/2}\mathbb{E}\left[a(\theta,(o,o))^2\right] \qquad (2.20) \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta, X_i) < \mathrm{e}^{K^2} + A_n].
\end{aligned}
$$

Here, (2.19) follows from the independence assumption on the environment, and (2.20) is an application of Chebyshev's inequality. Since $\delta_n = C_1^{-1/2} n^{-3/4}$, the first term in (2.20) goes to zero as $n \to \infty$ if $A_n \to \infty$.

Choose A_n such that $A_n \to \infty$ and $A_n = o(n^{1/4})$ as $n \to \infty$. For any $\mu \in \mathbb{R}^+$, the second term in (2.20) is equal to

$$
\begin{aligned}
&E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} (a(\theta, X_i) - \mu) < \mathrm{e}^{K^2} + A_n - \mu n\delta_n] \\
&\quad\leq M_n E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\} \sum_{i=0}^{n-1} (a(\theta, X_i) - \mu)^2] \qquad (2.21) \\
&\qquad + M_n E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\} \sum_{i\neq j} (a(\theta, X_i) - \mu)(a(\theta, X_j) - \mu)]
\end{aligned}
$$

by Chebyshev's inequality, where $M_n = \left(\frac{\delta_n}{\mu n\delta_n - A_n - \mathrm{e}^{K^2}}\right)^2 = O(n^{-2})$.

By the Harris inequality (cf. [7]),

$$
\begin{aligned}
E_o\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\} a(\theta,(o,o))\right] &= \mathbb{E}\left[E_o^\omega\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}\right] a(\theta,(o,o))\right] \\
&> \mathbb{E}\left[E_o^\omega\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}\right]\right] \mathbb{E}\left[a(\theta,(o,o))\right] = 0
\end{aligned}
\tag{2.22}
$$

since $E_o^\omega\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}\right]$ and $a(\theta,(o,o))$ are easily checked to be either both strictly increasing functions (when $\langle\theta, e_1\rangle > 0$) or both strictly decreasing functions (when $\langle\theta, e_1\rangle < 0$) of the random variable $\pi((o,o),(1,1))$. If we choose

$$
\mu = E_o\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\} a(\theta,(o,o))\right],
$$

then the second term in (2.21) vanishes by the independence assumption on the environment. Finally, observe that the first term in (2.21) is equal to

$$nM_nE_o\left[\exp\{\langle\theta,X_1\rangle-\log\phi(\theta)\}(a(\theta,(o,o))-\mu)^2\right]=O(n^{-1}).\quad\square$$

2.6. **Proof of Theorem 6.** Let us recall a few points regarding the arguments in Subsections 2.3 − 2.5. There, since $d=1+1$, the volume of B_1 (defined in (2.6)) is $O(n^{3/2})$. The variance of $D(B_1)$ (cf. (2.9)) scales like that volume. We take $\delta_n=O(n^{-3/4})$ so that the variance of $\delta_nD(B_1)$ is $O(1)$. With this choice, $n\delta_n\to\infty$ as $n\to\infty$. As we saw, this fact is crucial in the proof of Theorem 5.

In this subsection, we will assume that $d=2+1$. For every $m\geq 1$, $1\leq j\leq m$, $\theta\notin sp\{e_3\}$, $C_1\geq 1$ and $Y=(y_1,\ldots,y_m)\in(\mathbb{V}_3)^m$, we define

$$B_j:=\{(r,k):r\in\mathbb{Z}^2,(j-1)n\leq k<jn,\|(r,k)-\lfloor k\xi(\theta)\rfloor-\sqrt{n}y_{j-1}\|\leq C_1\sqrt{n}\},\tag{2.23}$$

similar to (2.6). Note that the volume of this new set is $O(n^2)$. If we were to define $D(B_1)$ analogously to (2.9), then we would have to take $\delta_n\leq O(n^{-1})$ in order to make the variance of $\delta_nD(B_1)$ not grow with n. Hence, the proof for $d=1+1$ does not directly carry over to the case $d=2+1$.

To resolve this issue, following [8], we will modify the proof by redefining $D(B_1)$ and δ_n. (We will continue using these names so that we can refer to the parts of Subsections 2.3 − 2.5 that carry over word by word.) The modification amounts essentially to using a tilting that is quadratic, instead of linear, in the local drift, as follows.

For every (r,k) and (s,l) with $r,s\in\mathbb{Z}^2$ and $1\leq k,l\leq n$, let

$$V((r,k),(s,l)):=\frac{1}{|k-l|}\mathbb{1}_{\{\|(s,l)-(r,k)-\lfloor(l-k)\xi(\theta)\rfloor\|<C_2\sqrt{|k-l|}\}}.\tag{2.24}$$

Here, the constant $C_2\geq 1$ will be determined later. Given any n integer and $x_1,\ldots,x_n\in\mathbb{Z}^3$ with $\langle x_k,e_3\rangle=k$, it follows easily from (2.24) that

$$\text{for any } s\in\mathbb{Z}^2,\ l\in\{1,\ldots,n\},\quad\sum_{k=1}^{n}V(x_k,(s,l))\leq 2\log n,$$

$$\sum_{k=1}^{n}\sum_{(s,l)\in B_1}V(x_k,(s,l))\leq 4C_2^2n^2,$$

$$\sum_{(s,l)\in B_1}\left(\sum_{k=1}^{n}V(x_k,(s,l))\right)^2\leq 8C_2^2n^2\log n,\quad\text{and}\tag{2.25}$$

$$\sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}}V((r,k),(s,l))^2\leq 32C_1^2C_2^2n^2\log n.\tag{2.26}$$

Recall the tilted law $\hat{P}_o^\theta$ introduced in the proof of Lemma 12.

Lemma 13. *For any $\delta>0$, there exists a $C_2\geq 1$ such that $\nu(n,X):=\sum_{1\leq i,j\leq n}V(X_i,X_j)$ satisfies*

$$\hat{P}_o^\theta(\nu(n,X)<n\log(n-1)/2)\leq\delta$$

for every $n\geq 2$.

Proof. For any realization of $X=(X_i)_{i\geq 1}$,

$$\nu(n,X)\leq\sum_{\substack{1\leq i,j\leq n\\ i\neq j}}\frac{1}{|i-j|}=:H(n).$$

Observe that

$$\hat{E}_o^\theta[\nu(n,X)]=\sum_{1\leq i,j\leq n}\hat{E}_o^\theta[V(X_i,X_j)]=\sum_{\substack{1\leq i,j\leq n\\ i\neq j}}\frac{1}{|i-j|}\hat{P}_o^\theta(\|(X_i-X_j-\lfloor(i-j)\xi(\theta)\rfloor\|<C_2\sqrt{|i-j|}).$$

When C_2 is sufficiently large, the CLT implies that

$$\hat{P}_o^\theta(\|(X_i-X_j-\lfloor(i-j)\xi(\theta)\rfloor\|<C_2\sqrt{|i-j|})\geq(1-\delta/2)$$

for any $i \neq j$. Therefore, $\hat{E}_o^\theta[\nu(n,X)] \geq (1-\delta/2)H(n)$. Applying Markov's inequality, we see that $\hat{P}_o^\theta(\nu(n,X) < H(n)/2) \leq \delta$. This implies the desired result since $H(n) \geq n\log(n-1)$. ☐

For any $\theta \in \mathbb{R}^3$ and $x \in \mathbb{Z}^3$, define $a(\theta,x) := \langle \theta, v(T_x\omega) - \xi_o\rangle$ as before, where $v(\omega) = \sum_{z\in\mathcal{R}} \pi(0,z)z$.

Lemma 14. *There exists a* $\beta > 0$ *such that*

$$\mu := E_o\left[\exp\{\langle \theta, X_1\rangle - \log\phi(\theta)\}a(\theta,o)\right] > 0$$

whenever $\mathrm{dist}(\theta, sp\{e_3\}) \in (0,\beta)$.

Proof. For every $\theta \notin sp\{e_3\}$, let

$$F(\theta) := \mathbb{E}\{E_o^\omega[\mathrm{e}^{\langle\theta,X_1\rangle}]E_o^\omega[\langle\theta,X_1\rangle]\} \quad \text{and} \quad G(\theta) := E_o[\mathrm{e}^{\langle\theta,X_1\rangle}]E_o[\langle\theta,X_1\rangle] = \phi(\theta)\langle\theta,\xi_o\rangle.$$

Our aim is to show that $F(\theta) > G(\theta)$.

Write $\theta = ce_3 + \theta'$ for some $c \in \mathbb{R}$ and $\theta' \in \mathbb{R}^3$ such that $\langle\theta', e_3\rangle = 0$. Then, $F(\theta) = \mathrm{e}^c F(\theta') + c\mathrm{e}^c\phi(\theta)$ and $G(\theta) = \mathrm{e}^c G(\theta') + c\mathrm{e}^c\phi(\theta)$. Therefore, it suffices to show that $F(\theta') > G(\theta')$.

Clearly, we have

$$\nabla F(\theta)\big|_{\theta=0} = \nabla G(\theta)\big|_{\theta=0} = E_o[X_1] = \xi_o.$$

Also, for any $u, u' \in \mathbb{R}^3$, with D^2F denoting the Hessian of F,

$$\langle u, D^2F(\theta)u'\rangle\big|_{\theta=0} = 2\mathbb{E}\{E_o^\omega[\langle X_1,u\rangle]E_o^\omega[\langle X_1,u'\rangle]\}$$

and

$$\langle u, D^2G(\theta)u'\rangle\big|_{\theta=0} = 2E_o[\langle X_1,u\rangle]E_o[\langle X_1,u'\rangle] = 2\langle\xi_o,u\rangle\langle\xi_o,u'\rangle.$$

By Schwarz' inequality (which is strict since the walk is uniform elliptic in the directions other than e_3),

$$\inf_{\substack{\|u\|=1\\ \langle u,e_3\rangle=0}} \left(\langle u, D^2F(\theta)u\rangle\big|_{\theta=0} - \langle u, D^2G(\theta)u\rangle\big|_{\theta=0}\right) > 0.$$

Finally, Taylor's theorem implies the existence of a $\beta > 0$ such that $F(\theta') > G(\theta')$ whenever $\|\theta'\| \in (0,\beta)$. ☐

Now, we are ready to give the new definition of $D(B_1)$ which is suitable for $d = 2+1$. For any $\theta \in \mathbb{R}^3$ such that $\mathrm{dist}(\theta, sp\{e_3\}) \in (0,\beta)$ (with β as in Lemma 14), let

$$D(B_1) := \sum_{\substack{(r,k)\in B_1,\\ (s,l)\in B_1}} V((r,k),(s,l))a(\theta,(r,k))a(\theta,(s,l)). \tag{2.27}$$

Note that $V((\cdot,k),(\cdot,k)) = 0$ for every $1 \leq k \leq n$. Since $\mathbb{E}[a(\theta,o)] = 0$, it follows from the independence of the environment that $\mathbb{E}[D(B_1)] = 0$. Also, $\mathbb{E}[D(B_1)^2] \leq 128|\theta|^2C_1^2C_2^2n^2\log n$ by (2.26) and the fact that $|a(\theta,o)| \leq 2|\theta|$.

If we choose

$$\delta_n := n\sqrt{\log n},$$

then the variance of $\delta_n D(B_1)$ is $O(1)$. Once we have this fact, the arguments in Subsections 2.3 – 2.5 carry over until (2.17). So, it suffices to show that $E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \delta_n D(B_1 - x) < \mathrm{e}^{K^2}]$ is small for all $x \in J_o$ when n and K are large. In the estimate below, we will (WLOG) take $x = 0$.

Let $\gamma = 1/2$, and observe that

$$\begin{aligned}
&E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \delta_n D(B_1) < \mathrm{e}^{K^2}]\\
&\quad\leq E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \nu(n,X) \geq \gamma n\log(n-1), \delta_n D(B_1) < \mathrm{e}^{K^2}]\\
&\qquad + E_o\left[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \nu(n,X) < \gamma n\log(n-1)\right]\\
&\quad= E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \nu(n,X) \geq \gamma n\log(n-1),\\
&\qquad\qquad \delta_n(D(B_1) - \mu^2\nu(n,X)) < \mathrm{e}^{K^2} - \mu^2\delta_n\nu(n,X)]\\
&\qquad + \hat{P}_o^\theta(\nu(n,X) < \gamma n\log(n-1))\\
&\quad\leq M_n E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \nu(n,X) \geq \gamma n\log(n-1), (D(B_1) - \mu^2\nu(n,X))^2] \qquad (2.28)\\
&\qquad + \hat{P}_o^\theta(\nu(n,X) < \gamma n\log(n-1))\\
&\quad\leq M_n E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}(D(B_1) - \mu^2\nu(n,X))^2] \qquad (2.29)\\
&\qquad + \hat{P}_o^\theta(\nu(n,X) < \gamma n\log(n-1)).
\end{aligned}$$

Here, (2.28) follows from the inequality $\mathbb{1}_{a<b} \le a^2/b^2$ with $a = \delta_n(D(B_1) - \mu^2\nu(n,X))$ and $b = e^{K^2} - \mu^2\delta_n\nu(n,X) < 0$, and

$$M_n = \left(\frac{\delta_n}{\mu^2\delta_n\gamma n\log(n-1) - \mathrm{e}^{K^2}}\right)^2. \tag{2.30}$$

Choose C_2 sufficiently large so that the second term in (2.29) is small for all n by Lemma 13.

It remains to control the first term in (2.29). Note that

$$\begin{aligned} &D(B_1) - \mu^2\nu(n,X)\\ &= 2\mu\sum_{k=1}^{n}\sum_{(s,l)\in B_1} V(X_k,(s,l))(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}})\\ &\quad + \sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}} V((r,k),(s,l))(a(\theta,(r,k)) - \mu\mathbb{1}_{\{X_k=(r,k)\}})(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}}),\end{aligned}$$

and

$$\begin{aligned} &(D(B_1) - \mu^2\nu(n,X))^2\\ &\le 8\mu^2\left(\sum_{k=1}^{n}\sum_{(s,l)\in B_1} V(X_k,(s,l))(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}})\right)^2\\ &\quad + 2\left(\sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}} V((r,k),(s,l))(a(\theta,(r,k)) - \mu\mathbb{1}_{\{X_k=(r,k)\}})(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}})\right)^2\end{aligned} \tag{2.31}$$

by the inequality $(a+b)^2 \le 2(a^2+b^2)$. When we substitute (2.31), the first term in (2.29) becomes a sum of two expectations. It is easy to see that the various cross terms in these expectations are equal to zero by independence and Lemma 14. Since $|a(\theta,\cdot)| \le 2|\theta|$, the first term in (2.29) is bounded from above by

$$\begin{aligned} &8\mu^2(2|\theta|+\mu)^2 M_n E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}\sum_{(s,l)\in B_1}(\sum_{k=1}^{n} V(X_k,(s,l)))^2]\\ &\quad + 2(2|\theta|+\mu)^2 M_n E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}\sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}} V((r,k),(s,l))^2]\\ &\le 64\mu^2C_2^2n^2\log n(2|\theta|+\mu)^2 M_n E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}]\\ &\quad\ + 64C_1^2C_2^2n^2\log n(2|\theta|+\mu)^2 M_n E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}]\\ &= O((\log n)^{-1})\end{aligned}$$

by (2.25), (2.26) and (2.30). We have proved Theorem 6.

3. Inequality of the rate functions for space-only RWRE

3.1. **Reducing to a fractional moment estimate.** Consider space-only RWRE on $\mathbb{Z}^d$ with $d \ge 1$. Assume that the walk is non-nestling relative to the canonical basis vector e_d. By Jensen's inequality, the quenched and averaged logarithmic moment generating functions

$$\Lambda_q(\theta) := \lim_{N\to\infty}\frac{1}{N}\log E_o^\omega\left[\exp\{\langle\theta,X_N\rangle\}\right] \quad\text{and}\quad \Lambda_a(\theta) := \lim_{N\to\infty}\frac{1}{N}\log E_o\left[\exp\{\langle\theta,X_N\rangle\}\right]$$

satisfy $\Lambda_q(\theta) \le \Lambda_a(\theta) \le |\theta|$ for every $\theta\in\mathbb{R}^d$.

Recall the definition of regeneration times $(\tau_n)_{n\ge0}$ (relative to e_d) given in Subsection 1.2. Let

$$\beta := \inf\{i\ge0 : \langle X_i,e_d\rangle < \langle X_o,e_d\rangle\} \in [1,\infty].$$

By the non-nestling assumption, there exist constants $c_2,c_3>0$ such that

$$\operatorname*{ess\,inf}_{\mathbb{P}} P_o^\omega(\beta=\infty) \ge c_2 \quad\text{and}\quad \operatorname*{ess\,sup}_{\mathbb{P}} P_o^\omega(\tau_1>n) \le \mathrm{e}^{-c_3 n} \tag{3.1}$$

for every $n \geq 1$, cf. [14]. These bounds clearly imply that

$$\operatorname*{ess\,sup}_{\mathbb{P}} E_o^\omega[\exp\{c\tau_1\} \,|\, \beta = \infty] \leq c_2^{-1} \operatorname*{ess\,sup}_{\mathbb{P}} E_o^\omega[\exp\{c\tau_1\}] =: H(c) < \infty \tag{3.2}$$

whenever $c < c_3$.

For every $c \in (0, c_3]$, introduce the set

$$\mathcal{C}(c) := \{\theta \in \mathbb{R}^d : 2|\theta| < c\}. \tag{3.3}$$

Lemma 15. *For every $\theta \in \mathcal{C}(c_3)$,*

$$E_o[\exp\{\langle \theta, X_{\tau_1} \rangle - \Lambda_a(\theta)\tau_1\} \,|\, \beta = \infty] = 1. \tag{3.4}$$

Λ_a is analytic on $\mathcal{C}(c_3)$. $\nabla\Lambda_a(0) = \xi_o$. The Hessian $\mathcal{H}_a$ of Λ_a is positive definite on $\mathcal{C}(c_3)$. For every $c < c_3$ and $\theta \in \mathcal{C}(c)$, the smallest eigenvalue of $\mathcal{H}_a(\theta)$ is bounded from below by a constant that depends only on c and the ellipticity constant κ of the walk.

Proof. See the proofs of Lemmas 6 and 12 of [19]. In particular, the desired lower bound for the smallest eigenvalue of $\mathcal{H}_a$ is evident from equation (2.10) of that paper. ☐

Given any $N \geq 1$, $\theta \in \mathcal{C}(c_3)$ and $\omega \in \Omega$, define

$$\hat{W}_N(\theta, \omega) := E_o^\omega[\exp\{\langle \theta, X_{\tau_N} \rangle - \Lambda_a(\theta)\tau_N\}].$$

Lemma 16. *For every $\theta \in \mathcal{C}(c_3)$, if*

$$\limsup_{N \to \infty} \frac{1}{N} \log \hat{W}_N(\theta, \omega) < 0$$

holds $\mathbb{P}$-a.s., then $\Lambda_q(\theta) < \Lambda_a(\theta)$.

Proof. Let $\theta \in \mathcal{C}(c_3)$. Then, $\theta \in \mathcal{C}(c)$ for some $c < c_3$. By hypothesis, for $\mathbb{P}$-a.e. ω, there exist $C_3 \geq 1$ and $c_4 > 0$ (both depending on ω) such that $\hat{W}_N(\theta, \omega) \leq C_3 \mathrm{e}^{-c_4 N}$ for every $N \geq 1$.

Given any $n \geq 1$ and $K \geq 1$, it follows from Chebyshev's inequality and (3.2) that

$$\begin{aligned}
&E_o^\omega[\exp\{\langle \theta, X_n \rangle - \Lambda_a(\theta)n\}] \\
&\quad = E_o^\omega[\exp\{\langle \theta, X_n \rangle - \Lambda_a(\theta)n\}, n < \tau_{\lfloor \frac{n}{K} \rfloor}] + \sum_{j=\lfloor \frac{n}{K} \rfloor}^{n} E_o^\omega[\exp\{\langle \theta, X_n \rangle - \Lambda_a(\theta)n\}, \tau_j \leq n < \tau_{j+1}] \\
&\quad \leq \mathrm{e}^{2|\theta|n} P_o^\omega(n < \tau_{\lfloor \frac{n}{K} \rfloor}) + \sum_{j=\lfloor \frac{n}{K} \rfloor}^{n} E_o^\omega[\exp\{\langle \theta, X_{\tau_j} \rangle - \Lambda_a(\theta)\tau_j\}] \operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{2|\theta|\tau_1\} \,|\, \beta = \infty] \\
&\quad \leq \mathrm{e}^{(2|\theta| - c)n} E_o^\omega[\exp\{c\tau_{\lfloor \frac{n}{K} \rfloor}\}] + \sum_{j=\lfloor \frac{n}{K} \rfloor}^{n} \hat{W}_j(\theta, \omega) \operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{c\tau_1\} \,|\, \beta = \infty] \\
&\quad \leq \mathrm{e}^{(2|\theta| - c)n} E_o^\omega[\exp\{c\tau_1\}] \left(\operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{c\tau_1\} \,|\, \beta = \infty] \right)^{\lfloor \frac{n}{K} \rfloor - 1} \\
&\qquad + \sum_{j=\lfloor \frac{n}{K} \rfloor}^{n} \hat{W}_j(\theta, \omega) \operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{c\tau_1\} \,|\, \beta = \infty] \\
&\quad \leq \mathrm{e}^{(2|\theta| - c)n} H(c)^{\lfloor \frac{n}{K} \rfloor} + H(c) \sum_{j=\lfloor \frac{n}{K} \rfloor}^{n} C_3 \mathrm{e}^{-c_4 j}.
\end{aligned}$$

Take K sufficiently large, and conclude that

$$\Lambda_q(\theta) - \Lambda_a(\theta) = \lim_{n \to \infty} \frac{1}{n} \log E_o^\omega[\exp\{\langle \theta, X_n \rangle - \Lambda_a(\theta)n\}] < 0. \quad ☐$$

For every $N \geq 1$, $\theta \in \mathcal{C}(c_3)$ and $\omega \in \Omega$, define

$$W_N(\theta, \omega) := E_o^\omega[\exp\{\langle \theta, X_{\tau_N} \rangle - \Lambda_a(\theta)\tau_N\} \,|\, \beta = \infty].$$

It follows from the renewal structure and (3.4) that

$$\begin{aligned}\mathbb{E}[P_o^\omega(\beta=\infty)W_N(\theta,\cdot)] &= P_o(\beta=\infty)E_o[\exp\{\langle\theta,X_{\tau_N}\rangle-\Lambda_a(\theta)\tau_N\}|\,\beta=\infty]\\ &= P_o(\beta=\infty)\left(E_o[\exp\{\langle\theta,X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\}|\,\beta=\infty]\right)^N\\ &= P_o(\beta=\infty).\end{aligned}$$

Given any $\alpha\in(0,1)$, by the same reasoning as in (2.1),

$$\limsup_{N\to\infty}\frac{1}{N}\log\hat{W}_N(\theta,\cdot)\le\limsup_{N\to\infty}\frac{1}{N\alpha}\log\mathbb{E}\left[\hat{W}_N(\theta,\cdot)^\alpha\right],\quad \mathbb{P}-a.s. \tag{3.5}$$

On the other hand, if $2|\theta|<c<c_3$, then we see by subadditivity, Chebyshev's inequality, and (3.2) that

$$\begin{aligned}\mathbb{E}\left[\hat{W}_{N+1}(\theta,\cdot)^\alpha\right] &= \mathbb{E}\left[\left(\sum_{x\in\mathbb{Z}^d}E_o^\omega[\exp\{\langle\theta,X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\},X_{\tau_1}=x]W_N(\theta,T_x\cdot)\right)^\alpha\right]\\ &\le \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d}\left(E_o^\omega[\exp\{\langle\theta,X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\},X_{\tau_1}=x]\right)^\alpha W_N(\theta,T_x\cdot)^\alpha\right]\\ &\le \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d}\left(E_o^\omega[\exp\{2|\theta|\tau_1\},\tau_1\ge|x|]\right)^\alpha W_N(\theta,T_x\cdot)^\alpha\right]\\ &\le \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d}\left(\mathrm{e}^{(2|\theta|-c)|x|}E_o^\omega[\exp\{c\tau_1\}]\right)^\alpha W_N(\theta,T_x\cdot)^\alpha\right]\\ &\le H(c)^\alpha\mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right]\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{(2|\theta|-c)\alpha|x|}.\end{aligned} \tag{3.6}$$

Lemma 17. *For every $\theta\in\mathcal{C}(c_3)$, if*

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right]<0 \tag{3.7}$$

for some $\alpha\in(0,1)$, then $\Lambda_q(\theta)<\Lambda_a(\theta)$. Hence, by convex duality, $I_a<I_q$ at $\xi=\nabla\Lambda_a(\theta)$.

Proof. This follows immediately from (3.5), (3.6) and Lemma 16. □

3.2. **The correlation condition.** In this subsection, we will consider space-only RWRE on $\mathbb{Z}^d$ with $d=2$ or 3, assume that the walk is non-nestling relative to e_d, and outline how one can modify the arguments given in Section 2 in order to reduce (3.7) to a simpler inequality.

Start with $d=2$. For every $n\ge1$ of the form k^2, and for every $y=(y',y'')\in\mathbb{Z}^2$, let

$$J_y:=[(y'-\frac{1}{2})\sqrt{n},(y'+\frac{1}{2})\sqrt{n})\times[(y''-\frac{1}{2})\sqrt{n},(y''+\frac{1}{2})\sqrt{n})\subset\mathbb{R}^2,$$

cf. (2.3). Take $N=nm$ for some $m\ge1$. For every $\theta\in\mathcal{C}(c_3)$, $\omega\in\Omega$ and $Y=(y_1,\ldots,y_m)\in(\mathbb{Z}^2)^m$, define

$$\bar{W}_N(\theta,\omega,Y):=E_o^\omega[\exp\{\langle\theta,X_{\tau_N}\rangle-\Lambda_a(\theta)\tau_N\},X_{\tau_{jn}}-\lfloor jn\zeta(\theta)\rfloor\in J_{y_j}\text{ for every }j\le m\big|\,\beta=\infty],$$

cf. (2.4), where $\zeta(\theta):=E_o[X_{\tau_1}\exp\{\langle\theta,X_1\rangle-\Lambda_a(\theta)\tau_1\}|\,\beta=\infty]$. By subadditivity,

$$\mathbb{E}[W_N(\theta,\cdot)^\alpha]\le\sum_Y\mathbb{E}\left[\bar{W}_N(\theta,\cdot,Y)^\alpha\right],$$

cf. (2.5). Finally, given any $C_1\ge1$, $Y=(y_1,\ldots,y_m)\in(\mathbb{Z}^2)^m$ and $j\in\{1,\ldots,m\}$, let

$$\begin{aligned}B_j:=\{(s,i)\in\mathbb{Z}^2: (j-1)n\langle\zeta(\theta),e_2\rangle+\sqrt{n}(y''_{j-1}+1/2)\le i<jn\langle\zeta(\theta),e_2\rangle+\sqrt{n}(y''_j-1/2),\\ \|(s-\sqrt{n}y'_{j-1})-\frac{\langle\zeta(\theta),e_1\rangle}{\langle\zeta(\theta),e_2\rangle}(i-\sqrt{n}y''_{j-1})\|\le C_1\sqrt{n}\},\end{aligned}$$

cf. (2.6). With these new definitions, the arguments in Subsections 2.3 – 2.5 carry over almost completely, once one replaces the i.i.d. random variables

$$E_o^{T_{X_1}\omega}[\exp\{\langle\theta,X_1\rangle-\log\phi(\theta)\}]$$

by the variables

$$E_o^{T_{X_{\tau_i}}\omega}[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}|\beta = \infty]\,.$$

(We leave the details to the reader.) However, there is one nontrivial difference. In the space-time case,

$$E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}\sum_{i=0}^{n-1} a(\theta, X_i)] \tag{3.8}$$

growing linearly in n was crucial, cf. (2.20) and (2.21), and it followed from an application of the Harris inequality, cf. (2.22). In the space-only case, one can check that (3.8) needs to be replaced by

$$E_o[\exp\{\langle\theta, X_{\tau_n}\rangle - \Lambda_a(\theta)\tau_n\}\sum_{x\in S(X,\tau_n)} a(\theta, x)|\beta = \infty] \tag{3.9}$$

growing linearly in n. Here, for any $j \geq 1$ and $x \in \mathbb{Z}^d$,

$$S(X, j) := \{X_i : 0 \leq i < j\}, \quad a(\theta, x) := \langle\theta, v(T_x\omega)\rangle - \mathbb{E}[\langle\theta, v(T_x\omega)\rangle], \quad \text{and} \quad v(\omega) = \sum_{z\in\mathcal{R}} \pi(0, z)z. \tag{3.10}$$

Note that a space-only walk can visit a point more than once, but such a point contributes only once to the sum in (3.9).

By the renewal structure, linear growth (in n) of (3.9) is equivalent to the following *correlation condition*:

$$E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}\sum_{x\in S(X,\tau_1)} a(\theta, x)|\beta = \infty] > 0. \tag{3.11}$$

For $d = 3$, one can apply exactly the same kind of modifications to the argument given in Subsection 2.6, and reduce (3.7) to (3.11). We omit the (routine) details.

Recall (3.3), and define

$$\mathcal{A}_{so} := \{\nabla\Lambda_a(\theta) : \theta \in \mathcal{C}(c_3)\}.$$

It follows easily from Lemma 15 that $\mathcal{A}_{so}$ is an open set containing ξ_o.

We have arrived at the following theorem.

Theorem 18. *Consider space-only RWRE on $\mathbb{Z}^d$ with $d = 2$ or 3. Assume that the walk is non-nestling relative to e_d. For every $\xi \in \mathcal{A}_{so}$, the strict inequality $I_a(\xi) < I_q(\xi)$ holds if (3.11) is satisfied at the unique $\theta \in \mathcal{C}(c_3)$ that solves $\xi = \nabla\Lambda_a(\theta)$.*

Proof. By Lemma 17, the desired result is implied by (3.7) which, in turn, follows from (3.11) as outlined above. □

3.3. **Proof of Theorem 8.** Consider space-only RWRE on $\mathbb{Z}^d$ with $d = 2$ or 3. Fix a triple $p = (p^+, p^o, p^-)$ of positive real numbers such that $p^- < p^+$ and $p^+ + p^o + p^- = 1$. Assume that $\mathbb{P}$ is in class $\mathcal{M}_\epsilon(d, p)$ for some small $\epsilon > 0$, cf. Definition 7. Assume that $\epsilon \leq \frac{p^o}{4(d-1)}$ so that the ellipticity constant κ of the walk satisfies

$$\kappa \geq \min(p^+, p^-, \frac{p^o}{4(d-1)}). \tag{3.12}$$

Lemma 19. *There exist $C_4 \geq 1$ and $c_5 > 0$ (depending only on p) such that $|\Lambda_a(\theta) - \langle\theta, \xi_o\rangle| \leq C_4|\theta|^2$ holds for every $\theta \in \mathcal{C}(c_5)$.*

Proof. Recall (3.1). Note that c_3 depends only on the law of the regeneration times which, in turn, is determined by the fixed triple p. Moreover, the ellipticity constant κ of the walk satisfies (3.12). Fix a $c_5 < c_3$. The desired result follows immediately from Lemma 15. □

Consider the set

$$\mathcal{C}_t(c_5) := \{\theta \in \mathcal{C}(c_5) : \langle\theta, e_d\rangle = 0\}.$$

(Here, the subscript stands for *transversal.*) Take any $\theta \in \mathcal{C}_t(c_5)$. Recall the notation in (3.10). Since $\mathbb{P}$ is in class $\mathcal{M}_\epsilon(d, p)$, it is easy to see that

$$\xi_o = (p^+ - p^-)e_d, \qquad \langle\theta, \xi_o\rangle = 0, \qquad \text{and} \qquad a(\theta, x) = \langle\theta, v(T_x\omega)\rangle \leq 2\epsilon(d-1)|\theta| \tag{3.13}$$

for every $x \in \mathbb{Z}^d$. Similarly, the isotropy assumption ensures that

$$Z(\theta) = Z(\theta, X, \tau_1, \omega) := \sum_{x \in S(X,\tau_1)} a(\theta, x)$$

satisfies

$$E_o[Z(\theta)|\beta = \infty] = E_o[\tau_1 Z(\theta)|\beta = \infty] = 0. \tag{3.14}$$

Our aim is to show that

$$E_o[\exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} Z(\theta)|\beta = \infty] > 0 \tag{3.15}$$

for certain choices of θ, to be determined later. Expanding the exponential on the LHS of (3.15), we see that

$$\begin{aligned} E_o[\exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} Z(\theta)|\beta = \infty] &\geq E_o[(1 + \langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1) Z(\theta)|\beta = \infty] - C_5|\theta|^3 \\ &= E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta)|\beta = \infty] - C_5|\theta|^3. \end{aligned} \tag{3.16}$$

Here, C_5 is some constant which depends only on p and c_5. The equality in (3.16) uses (3.14).

In order to estimate (3.16), we first provide a more convenient representation of RWRE. Let $(b_i)_{i\geq 0}$ be an i.i.d. sequence of random variables taking values in $\{e_d, 0, -e_d\}$, with

$$P(b_1 = e_d) = p^+, \quad P(b_1 = 0) = p^o, \quad \text{and} \quad P(b_1 = -e_d) = p^-.$$

Let $(f_i)_{i\geq 0}$ be another i.i.d. sequence of random variables (independent of $(b_i)_{i\geq 0}$) taking values in the set $\{\pm e_j : 1 \leq j < d\} \cup \{0\}$, with

$$P(f_1 = 0) = \frac{2\epsilon(d-1)}{p^o} \quad \text{and} \quad P(f_1 = \pm e_j) = \frac{1}{2(d-1)} - \frac{\epsilon}{p^o} \quad \text{if } 1 \leq j < d.$$

For any $\omega \in \Omega$, the walk $(X_i)_{i\geq 0}$ under P_o^ω can be constructed by setting

$$X_{i+1} - X_i := b_i + (1 - |b_i|) f_i + (1 - |b_i|)(1 - |f_i|) U_i\,,$$

where $(U_i)_{i\geq 0}$ is a sequence of independent random variables taking values in $\{\pm e_j : 1 \leq j < d\}$, with

$$P^\omega(U_i = \pm e_j | \mathcal{F}_i) = \frac{\pi(X_i, X_i \pm e_j) - (\frac{p^o}{2(d-1)} - \epsilon)}{2\epsilon(d-1)}.$$

Here, $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$. Note that the laws of the sequences $(b_i)_{i\geq 0}$ and $(f_i)_{i\geq 0}$ do not depend on the environment, and that τ_1 is a function of $(b_i)_{i\geq 0}$ only.

Let

$$N_i := \sum_{j=0}^{i-1} \mathbb{1}_{1 = (1-|b_i|)(1-|f_i|)}\,.$$

Introduce the events $L_0 := \{N_{\tau_1} = 0\}$, $L_1 := \{N_{\tau_1} = 1\}$, and $L_2 := \{N_{\tau_1} \geq 2\}$. Let $\mathcal{G} := \sigma((b_i, f_i)_{i\geq 0})$. Note that the events L_0, L_1 and L_2 are $\mathcal{G}$-measurable, and so is the event $\{\beta = \infty\}$. On the event L_0, we see that X_{τ_1} is $\mathcal{G}$-measurable. Also, note that $E_o[Z(\theta)|\mathcal{G}] = 0$ by isotropy. Therefore,

$$E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_0, \beta = \infty] = E_o[\langle \theta, X_{\tau_1}\rangle E_o[Z(\theta)|\mathcal{G}], L_0, \beta = \infty] = 0. \tag{3.17}$$

On the other hand, it is easy to check that $P_o(L_2) \leq c_6\epsilon^2$ for some $c_6 = c_6(p)$. By Hölder's inequality,

$$|E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_2, \beta = \infty]| \leq P_o(L_2)^{2/3} E_o[|\langle \theta, X_{\tau_1}\rangle Z(\theta)|^3, \beta = \infty]^{1/3} \leq c_7\epsilon^{7/3}|\theta|^2 \tag{3.18}$$

for some $c_7 = c_7(p) > 0$. (Recall that $a(\theta, \cdot) \leq 2\epsilon(d-1)|\theta|$, cf. (3.13).)

Finally, let $L_1^\ell = \{L_1, (1 - |b_\ell|)(1 - |f_\ell|) = 1, \ell < \tau_1\}$. Then,

$$E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_1, \beta = \infty] = \sum_{\ell=0}^{\infty} E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_1^\ell, \beta = \infty]. \tag{3.19}$$

For every $\ell \geq 0$,

$$\begin{aligned} E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_1^\ell, \beta = \infty] &= E_o[\langle \theta, X_\ell\rangle Z(\theta), L_1^\ell, \beta = \infty] \\ &\quad + E_o[\langle \theta, X_{\ell+1} - X_\ell\rangle Z(\theta), L_1^\ell, \beta = \infty] \\ &\quad + E_o[\langle \theta, X_{\tau_1} - X_{\ell+1}\rangle Z(\theta), L_1^\ell, \beta = \infty]. \end{aligned} \tag{3.20}$$

By computations similar to the one involving L_0, the first and the third terms on the RHS of (3.20) are zero. The second term is equal to

$$\begin{aligned} E_o[\langle\theta, X_{\ell+1} - X_\ell\rangle\langle\theta, v(T_{X_\ell}\omega)\rangle, L_1^\ell, \beta = \infty] &= P_o(L_1^\ell, \beta = \infty)\mathbb{E}[E^\omega[\langle\theta, U_o\rangle]\langle\theta, v(\omega)\rangle] \\ &= P_o(L_1^\ell, \beta = \infty)\mathbb{E}\left[\left(\sum_{z\neq\pm e_d} \frac{\pi(0,z) - (\frac{p^o}{2(d-1)} - \epsilon)}{2\epsilon(d-1)}\langle\theta, z\rangle\right)\langle\theta, v(\omega)\rangle\right] \\ &= \frac{P_o(L_1^\ell, \beta = \infty)}{2\epsilon(d-1)}\mathbb{E}\left[\langle\theta, v(\omega)\rangle^2\right]. \end{aligned}$$

Therefore, by (3.19),

$$E_o[\langle\theta, X_{\tau_1}\rangle Z(\theta), L_1, \beta = \infty] = \frac{P_o(L_1, \beta = \infty)}{2\epsilon(d-1)}\mathbb{E}\left[\langle\theta, v(\omega)\rangle^2\right].$$

It is easy to see that $P_o(L_1, \beta = \infty) \geq c_8\epsilon$ for some $c_8 = c_8(p) > 0$ if ϵ is small enough. Also, part (d) of Definition 7 ensures that $\mathbb{E}\left[\langle\theta, v(\omega)\rangle^2\right] \geq c_9\epsilon^2|\theta|^2$ for some $c_9 = c_9(p) > 0$. Hence,

$$E_o[\langle\theta, X_{\tau_1}\rangle Z(\theta), L_1, \beta = \infty] \geq c_{10}\epsilon^2|\theta|^2 \tag{3.21}$$

for some $c_{10} = c_{10}(p) > 0$. Combining (3.17), (3.18) and (3.21) gives

$$\begin{aligned} E_o[\langle\theta, X_{\tau_1}\rangle Z(\theta)|\beta = \infty] - C_5|\theta|^3 &\geq c_{10}\epsilon^2|\theta|^2 - c_7\epsilon^{7/3}|\theta|^2 - C_5|\theta|^3 \\ &= \left((c_{10} - c_7\epsilon^{1/3})\epsilon^2 - C_5|\theta|\right)|\theta|^2. \end{aligned} \tag{3.22}$$

If $\epsilon < (c_{10}/c_7)^3$, then, for every $\theta \in \mathcal{C}_t(c_5)$ such that $0 < |\theta| < (c_{10} - c_7\epsilon^{1/3})\epsilon^2/C_5$,

$$E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}Z(\theta)|\beta = \infty] > 0$$

by (3.16) and (3.22).

Finally, Theorem 18 implies that $I_a < I_q$ on the set

$$\{\nabla\Lambda_a(\theta) : \theta \in \mathcal{C}_t(c_5),\ 0 < |\theta| < (c_{10} - c_7\epsilon^{1/3})\epsilon^2/C_5\}$$

whose closure contains the LLN velocity $\xi_o = \nabla\Lambda_a(0)$. We have proved Theorem 8.

4. Open problems

Our technique of proof puts several restrictions on the class of models treated. The following are natural questions we have not addressed.

1. Does Theorem 8 extend to all space-only RWRE in dimension $d = 2, 3$, or at least to those satisfying Sznitman's condition (**T**)? Note that, for non-nestling walks, it suffices to show that the correlation condition (3.11) is satisfied on a sequence $(\theta_n)_{n\geq 1}$ that converges to zero, cf. Theorem 18.
2. In case $\sum\pi(o,z)\langle z, e\rangle$ is random for any $e \in \mathcal{R}_{so}$, is it true that $I_q(\xi) = I_a(\xi)$ only when $\xi = 0$ or $I_a(\xi) = 0$, as is the case in dimension $d = 1$?

Note that in our proof of Theorem 8, we used the isotropy assumption in order to get rid of a centering term under the (untilted) measure; this does not seem essential and probably, the lack of isotropy could be handled in the perturbative regime. However, getting rid of the perturbative restriction, or of the non-randomness in the e_d direction, requires additional arguments.

Acknowledgments

We thank Francis Comets for providing us with an update on polymer models and bringing the work of Lacoin [8] to our attention.

References

[1] Noam Berger. Limiting velocity of high-dimensional random walk in random environment. *Ann. Probab.*, 36(2):728–738, 2008.

[2] Francis Comets, Nina Gantert, and Ofer Zeitouni. Quenched, annealed and functional large deviations for one-dimensional random walk in random environment. *Probab. Theory Related Fields*, 118(1):65–114, 2000.

[3] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998.

[4] Bernard Derrida, Giambattista Giacomin, Hubert Lacoin, and Fabio Lucio Toninelli. Fractional moment bounds and disorder relevance for pinning models. *Comm. Math. Phys.*, 287(3):867–887, 2009.

[5] Giambattista Giacomin, Hubert Lacoin, and Fabio Lucio Toninelli. Marginal relevance of disorder for pinning models. arXiv:0811.0723, 2008.

[6] Andreas Greven and Frank den Hollander. Large deviations for a random walk in random environment. *Ann. Probab.*, 22(3):1381–1428, 1994.

[7] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[8] Hubert Lacoin. New bounds for the free energy of directed polymers in dimension $1+1$ and $1+2$. arXiv:0901.0699, 2009.

[9] Jonathon Peterson and Ofer Zeitouni. On the annealed large deviation rate function for a multi-dimensional random walk in random environment. To appear in *ALEA*, arXiv:0812.3619, 2008.

[10] Firas Rassoul-Agha. Large deviations for random walks in a mixing random environment and other (non-Markov) random walks. *Comm. Pure Appl. Math.*, 57(9):1178–1196, 2004.

[11] Firas Rassoul-Agha and Timo Seppäläinen. In preparation, 2009.

[12] Firas Rassoul-Agha and Timo Seppäläinen. Process-level quenched large deviations for random walk in random environment. Preprint, 2009.

[13] Jeffrey Rosenbluth. *Quenched large deviations for multidimensional random walk in random environment: a variational formula*. PhD thesis in Mathematics, New York University, 2006. arXiv:0804.1444.

[14] Alain-Sol Sznitman. Slowdown estimates and central limit theorem for random walks in random environment. *J. Eur. Math. Soc. (JEMS)*, 2(2):93–143, 2000.

[15] Alain-Sol Sznitman. On a class of transient random walks in random environment. *Ann. Probab.*, 29(2):724–765, 2001.

[16] Alain-Sol Sznitman and Martin Zerner. A law of large numbers for random walks in random environment. *Ann. Probab.*, 27(4):1851–1869, 1999.

[17] Fabio Lucio Toninelli. Coarse graining, fractional moments and the critical slope of random copolymers. *Electron. J. Probab.*, 14:no. 20, 531–547, 2009.

[18] S. R. S. Varadhan. Large deviations for random walks in a random environment. *Comm. Pure Appl. Math.*, 56(8):1222–1245, 2003. Dedicated to the memory of Jürgen K. Moser.

[19] Atilla Yilmaz. Averaged large deviations for random walk in a random environment. To appear in *Ann. Inst. H. Poincaré Probab. Statist.*, arXiv:0809.3467, 2008.

[20] Atilla Yilmaz. Large deviations for random walk in a space-time product environment. *Ann. Probab.*, 37(1):189–205, 2009.

[21] Atilla Yilmaz. On the equality of the quenched and averaged large deviation rate functions for high-dimensional ballistic random walk in a random environment. arXiv:0903.0410, 2009.

[22] Atilla Yilmaz. Quenched large deviations for random walk in a random environment. *Comm. Pure Appl. Math.*, 62(8):1033–1075, 2009.

[23] Ofer Zeitouni. Random walks in random environments. *J. Phys. A*, 39(40):R433–R464, 2006.

[24] Martin P. W. Zerner. Lyapounov exponents and quenched large deviations for multidimensional random walk in random environment. *Ann. Probab.*, 26(4):1446–1476, 1998.

Faculty of Mathematics, Weizmann Institute of Science
Current address: Department of Mathematics, University of California, Berkeley
E-mail address: `atilla@math.berkeley.edu`

Faculty of Mathematics, Weizmann Institute of Science and School of Mathematics, University of Minnesota
E-mail address: `zeitouni@math.umn.edu`