Convergence in law of the maximum of nonlattice branching random walk

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Abstract

Let η_n^* denote the maximum, at time n, of a nonlattice one-dimensional branching random walk η_n possessing (enough) exponential moments. In a seminal paper, Aïdekon [2] demonstrated convergence of η_n^* in law, after recentering, and gave a representation of the limit. We give here a shorter proof of this convergence by employing reasoning motivated by Bramson, Ding and Zeitouni [5]. Instead of spine methods and a careful analysis of the renewal measure for killed random walks, our approach employs a modified version of the second moment method that may be of independent interest.

1 Introduction

We consider nonlattice one-dimensional branching random walk (BRW), $\{\eta_n\}_{n=0,1,2,\ldots}$, with offspring distribution $\{p_i\}_{i=1,2,\ldots}$ and random walk increments $\{w(dy)\}_{y\in\mathbb{R}}$. The BRW is constructed in the usual inductive manner using $p.$ and $w(\cdot)$, with individuals of the nth generation moving independently of each other according to $w(\cdot)$, from the site of their parent in the $(n-1)$st generation. We denote by ρ the mean of $p.$, by γ_0 the mean of $w(\cdot)$, assume that $p.$ has finite second moment, and that $w(\cdot)$ is nonlattice and has exponential moments in an appropriate interval (which will be specified shortly), using the notation

$$K = \sum_i i^2 p_i, \quad \phi(\theta) = \int e^{\theta y} w(dy)\,. \tag{1}$$

(Here, nonlattice means that the support of $w(\cdot) + y$ is not contained in any discrete subgroup of $\mathbb{R}$ for any y.) We denote by V_n the set of nth generational offspring, with $\eta_{v,n}, v \in V_n$, being the positions of these offspring, and set $\eta_n^* = \max_{v\in V_n} \eta_{v,n}$.

The limiting behavior of η_n^*, as $n \to \infty$, has been studied since the early 1970s. A strong law of large numbers for η_n^*/n was first given in Kingman [11]; see Aïdekon [2] for general literature on the subject of branching random walk. In his recent seminal paper, Aïdekon [2] has shown the

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sharp result that $\eta_n^* - (c_1 n - c_2 \log n)$ converges in distribution for appropriate c_1, c_2, which depend on $p.$ and $w(\cdot)$; he also identified the limit as a Gumbel distribution shifted by a particular random variable, the limit of the derivative martingale of the branching random walk.

The behavior of η_n^* is related to the limiting behavior of the maximum of branching Brownian motion. The latter problem traces its roots back to Kolmogorov, Petrovsky, and Piscounov [12] and Fisher [10]; sharp results were obtained in Bramson [4], and an identification of the limit as a Gumbel distribution shifted by the derivative martingale was obtained by Lalley and Sellke [13]. Results comparable to those in [4] were obtained in the context of the two-dimensional discrete Gaussian free field in Bramson, Ding, and Zeitouni [5]. Here, we employ reasoning related to that in the last paper to show convergence in distribution of η_n^* after recentering, and to identify the limit.

To state our main result, Theorem 1.1, we first introduce the following terminology. Let $I(\cdot)$ denote the rate function for $w(\cdot)$, that is, for $\lambda > \gamma_0$,

$$I(\lambda) = \sup_{\theta > 0}[\theta\lambda - \log \phi(\theta)] \,. \tag{2}$$

Assume that

$$\log \rho \in \{I(\cdot)\}^\circ \,, \quad c_1 \in \left\{(\log \phi)'(\cdot)\right\}^\circ, \tag{3}$$

where c_1 satisfies $I(c_1) = \log \rho$ (and G° denotes the interior of G). Then, $I(\cdot)$ is convex and differentiable in a neighborhood of c_1. Denote by $\bar\theta$ the value of θ at which the supremum in (2) is taken for $\lambda = c_1$, and set $c_2 = 3/2\bar\theta$. We then set $m_n = c_1 n - c_2 \log n$. Also, set

$$Z_k = \sum_{v \in V_k} (c_1 k - \eta_{v,k}) \mathrm{e}^{-\bar\theta(c_1 k - \eta_{v,k})} \,,$$

and denote by $\mathcal{F}_k$ the σ-algebra generated by the BRW up through time k.

Our main result is the following theorem.

Theorem 1.1. *Assume that η_n is a nonlattice branching random walk satisfying (3), with $K < \infty$. Then, $\eta_n^* - m_n$ converges in law as $n \to \infty$. Moreover, $Z = \lim_{k\to\infty} Z_k$ exists and is finite and positive with probability 1, and there exists a constant $\alpha^* > 0$ so that, for each $z \in \mathbb{R}$,*

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathbb{P}(\eta_n^* \leqslant m_n + z | \, \mathcal{F}_k) = \exp\{-\alpha^* Z \mathrm{e}^{-\bar\theta z}\} \quad a.s. \tag{4}$$

Remarks.

1. Theorem 1.1 is the analog of Theorem 1.1 of Aïdekon [2]. The latter paper has nearly optimal conditions on the branching and random walk distributions, which we have not tried to duplicate here.

2. Our proof of Theorem 1.1 is, we believe, shorter and more elementary than that in [2], employing techniques developed in Ding and Zeitouni [8] and Bramson, Ding, and Zeitouni [5]. In particular, we do not use the convergence of the derivative martingale in the convergence in law proof, we do not use renewal theory (except to the extent that certain estimates from random walk, developed in [6], are used), and we do not work with the spine representation. Instead, we employ a variant of the second moment method that is tailored toward deriving tail estimates and involves a truncation that keeps only the leading particle in each subtree of depth k rooted at a vertex in V_{n-k}.

3. The result (4) for branching Brownian motion dates back to Lalley and Sellke [13] and states that the limit can be written as a random shift (by the limit of the so called derivative martingale) of the Gumbel distribution.

4. When the first part of (3) does not hold (which is only possible if the support of $w(\cdot)$ has a finite upper bound), non-standard centering and limit behavior is possible for η_n^* (see, for example, Bramson [3]). The second part of (3) ensures that, after an exponential change of measure that recenters the measure at c_1, the resulting measure still possesses exponential moments.

5. We believe that the approach discussed in this paper allows one to also handle the case of lattice BRWs. We discuss this extension in Section 5.

An important part of the demonstration of Theorem 1.1 involves showing that $\mathbb{P}(\eta_n^* - m_n > z) \sim \alpha^* z e^{-\bar{\theta} z}$ for large z, which is done in Proposition 3.1. (Here and later, we write $a_n(z) \sim b_n(z)$ if $\lim_{z\to\infty} \limsup_{n\to\infty} a_n(z)/b_n(z) = \lim_{z\to\infty} \liminf_{n\to\infty} a_n(z)/b_n(z) = 1$.) The long Section 3 is devoted to showing this proposition, with the two main steps being Propositions 3.2 and 3.5. Proposition 3.2 compares $\mathbb{P}(\eta_n^* - m_n > z)$ with an appropriate expectation corresponding to the number of particles present at a time $n - \ell$, $\ell \ll n$, that lie below a given boundary until then and that have at least one offspring above $m_n + z$ at time n; related estimates are also present in [2]. The second moment estimates used here (in Proposition 3.4) are a more refined version of those used elsewhere in the branching literature. Proposition 3.5 then shows that this expectation is approximated by $\alpha^* z e^{-\bar{\theta} z}$.

In the proof of Theorem 1.1, we divide the evolution of $\{\eta_j\}_{0\leqslant j\leqslant n}$ into two time intervals, $[0, k]$ and $[k, n]$, first letting $n \to \infty$ and then $k \to \infty$. At time k, we decompose the process $\{\eta_j\}_{j=0,1,2,\ldots}$ into $|V_k|$ processes, each given by a BRW $\{\eta_{v,j}^{v'}\}_{j=0,1,2,\ldots}$ descending from $v' \in V_k$ and restarted at position 0, i.e.,

$$\eta_{v,j}^{v'} = \eta_{v,j+k} - \eta_{v',k} \quad \text{for } v \in V_j^{v'}, \tag{5}$$

where $V_j^{v'} = V_j^{v',k}$ denotes the set of jth generation descendents of v' in the $(j+k)$th generation of the BRW; the processes $\{\eta_j^{v'}\}_{j=0,\ldots,n'}$ will be independent copies of $\{\eta_j\}_{j=0,\ldots,n'}$.

The first part of Theorem 1.1 follows quickly from Proposition 3.1 together with the decomposition in (5). The limit (4) of Theorem 1.1 employs reasoning similar to that for Theorem 1 of Lalley and Sellke [13]. Both results are proved in Section 4.

In Section 2, various technical results are demonstrated that will be needed in Sections 3 and 4. Basic tools for these results are the crossing probabilities for random walks of certain curves, which are given in Lemmas 2.1 – 2.3, with the proof of the last two being deferred to the appendix.

Notation. For functions $F(\cdot)$ and $G(\cdot)$, we write $F \lesssim G$ or $F = O(G)$ if there exists an absolute constant $C > 0$ such that $F \leqslant CG$ everywhere in the domain, and $F \asymp G$ if $F \lesssim G$ and $G \lesssim F$. We sometimes abbreviate $F(x) = o_x(1)$ if $F(x) \to_{x\to\infty} 0$. For functions F, G of a real or integer variable, we write $F \sim G$ if F/G converges to 1 as the variable tends to infinity. Finally, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer not greater than x.

2 Preliminaries

2.1 Some random walk inequalities

In this subsection, we state two lemmas, Lemma 2.2 and Lemma 2.3, that give bounds on the probability of mean zero random walks not crossing specified curves. These lemmas will be applied repeatedly in this section and the next.

For both lemmas, we will employ a version of the ballot theorem that is a slight modification of that given in Theorem 1 of Addario-Berry and Reed [1]. Here, $\{X_k\}_{k=1,2,\dots}$ denote independent copies of mean zero random variables X, and $S_n = \sum_{k=1}^n X_k$; X will also be assumed to be nonlattice.

Lemma 2.1. *In addition to the above assumptions, assume that X has finite variance. For all n and all $a, y \geqslant 0$, $b > a$, there is a $C > 0$, depending only on the law of X and on $b - a$, such that*

$$\mathbb{P}(S_n \in (a,b), S_k > -y \text{ for all } 0 < k < n) \leqslant \frac{C(y \vee 1)((y+a) \vee 1)}{n^{3/2}} \tag{6}$$

and such that, for all a with $0 \leqslant a \leqslant \sqrt{n}$,

$$\mathbb{P}(S_n \in (a,b), S_k > 0 \text{ for all } 0 < k < n) \geqslant \frac{(a \vee 1)}{Cn^{3/2}}. \tag{7}$$

Lemma 2.1 differs from Theorem 1 of Addario-Berry and Reed [1] only in that (6) is phrased here for general $y > 0$, rather than just for $y = 0$ as in the paper. The proof of (6) remains essentially the same as in [1]: The time interval $[0, n]$ is divided into three parts, $[0, n/4]$, $[n/4, 3n/4]$, and $[3n/4, n]$. For both the first and third subintervals, Lemma 3 (iii) of [1], which gives an upper bound on the first time at which $S_k < -y'$, $y' \geqslant 0$, is applied in its general form, rather than being restricted to $y' = 0$ for the first subinterval as in the paper. As in [1], for the middle term, one employs an upper bound on the density of $S_{3n/4} - S_{n/4}$. The three upper bounds are then multiplied together to give (6).

In our applications, the above random walk $\{S_k\}_{k=0,\dots,n}$ will correspond to the random walk obtained by first subtracting $c_1 k$ from the random walk underlying our BRW, and then tilting the corresponding measure so that the mean of the random walk associated with the tilting is 0. Since Theorem 1.1 instead requires the nonlinear centering m_n at time n, which differs from $c_1 n$ by $c_2 \log n$, we will in practice apply the following perturbation of Lemma 2.1, which instead bounds the random walk $\{S_k^{(n)}\}_{k=0,\dots,n}$ defined below. Note that, for $d^{(n)} = 0$, $S_k^{(n)} \stackrel{d}{=} S_k$.

Lemma 2.2. *Let X and S_n be as above, and in addition assume that $\mathbb{E}(\mathrm{e}^{\theta X}) < \infty$, for $|\theta| \leqslant \theta_0$ for some $\theta_0 > 0$. Set $S_k^{(n)} = \sum_{i=1}^k X_i^{(n)}$, where $X_i^{(n)} = X_i + d^{(n)}$. Assume that either $d^{(n)} > 0$ for all n, that $d^{(n)} < 0$ for all n, or that $d^{(n)} \equiv 0$, with in each case $|d^{(n)}| \leqslant c(\log n)/n$ for some $c > 0$. Define the probability measure $\mathbb{P}^{(n)}$, on paths in $[0, n]$, by*

$$\frac{d\mathbb{P}^{(n)}}{d\mathbb{P}} = \frac{\mathrm{e}^{-\theta^{(n)} S_n}}{\mathbb{E}(\mathrm{e}^{-\theta^{(n)} S_n})}, \tag{8}$$

with $\theta^{(n)}$ being chosen so that $\mathbb{E}^{(n)}(X_1^{(n)}) = 0$. Then $S_k^{(n)}$ satisfies the analogs of (6) and (7), with $\mathbb{P}^{(n)}$ replacing $\mathbb{P}$ and the constants C depending on c. (We will refer to these inequalities as (6) and (7) as well.)

Lemma 2.2, together with the following lemma, will be proved in the appendix. Here, $h(\cdot)$ is a non-negative function such that $h(0)=0$, and $h(n)\leqslant C'\log(n+1)$ for a given constant $C'>0$ and all $n\in\mathbb{Z}_+$.

Lemma 2.3. *Let X, S_n, $S_k^{(n)}$, $\mathbb{P}^{(n)}$, $d^{(n)}$, and c be as in Lemma 2.2. For any $y\geqslant 1$ and $-y+1\leqslant a<b<\infty$, there exists $\beta_{y,a,b}>0$ such that*

$$\lim_{n\to\infty} n^{3/2}\mathbb{P}^{(n)}(S_n^{(n)}\in(a,b), S_k^{(n)}\geqslant -y \text{ for all } 0<k<n)=\beta_{y,a,b} \tag{9}$$

with, for some $\beta^*>0$,

$$\lim_{y,y+a\to\infty}\beta_{y,a,b}/(b-a)y(y+a)=\beta^* \tag{10}$$

if $b-a>0$ is fixed as $y,a\to\infty$; $\beta_{y,a,b}$ is continuous in a and b and right continuous in y. Furthermore, there exists $\delta_{\bar{y}}$, with $\delta_{\bar{y}}\searrow 0$ as $\bar{y}:=y\wedge(y+a)\nearrow\infty$, such that, for $-y+1\leqslant a<b<\infty$,

$$\limsup_{n\to\infty} n^{3/2}\mathbb{P}^{(n)}(S_n^{(n)}\in(a,b), S_k^{(n)}\geqslant -y-y^{1/10}-h(k\wedge(n-k)) \text{ for all } 0<k<n)\leqslant\beta_{y,a,b}(1+\delta_{\bar{y}})\,. \tag{11}$$

If, in addition, $h(\cdot)$ is increasing and concave, then, for fixed $\varepsilon=b-a>0$, there exist $C>0$ and $n_\varepsilon\in\mathbb{Z}_+$ such that, for $n\geqslant n_\varepsilon$, $n/2\leqslant j\leqslant n$, $y\geqslant 1$, and $-y-h(n-j)+1\leqslant a<b<\infty$,

$$n^{3/2}\mathbb{P}^{(n)}(S_j^{(n)}\in(a,b), S_k^{(n)}\geqslant -y-h(k\wedge(n-k)) \text{ for all } 0<k<j)\leqslant Cy(y+a+h(n-j))\,. \tag{12}$$

If $S_k^{(n)}\geqslant -y$ is replaced by the strict inequality $S_k^{(n)}>-y$ in (9), and the analogous change is made in (11), then the analogs of (9) and (11) continue to hold for appropriate $\beta^o_{y,a,b}$, which is continuous in a and b, and left continuous in y. None of the terms $\beta_{y,a,b}$, $\beta^o_{y,a,b}$, $\delta_{\bar{y}}$, C and n_ε depends on $d^{(n)}$, for fixed c.

It follows from (9) and (11) that the ratio of the probabilities in these two displays lies within $[1,1+2\delta_y]$ for given y, $a\geqslant 0$, and large enough n. Since both probabilities are increasing in y, this ratio also holds uniformly for $y'\in[y,y+M]$ and fixed $M>0$. A similar observation holds, as n increases, for $\beta_{y,a,b}/y(y+a)$ with large (but bounded) values of y and a, if $b-a$ is fixed. Note that the limits $\beta_{y,a,b}$ and $\beta^o_{y,a,b}$ may depend on the sign of $d^{(n)}$, although β^* will not.

2.2 Preliminary bounds on the right tail of the maximum of BRW

In this subsection, we give preliminary upper (Corollary 2.5) and lower (Lemma 2.7) bounds on the right tail of the maximum of BRW. We first introduce some terminology.

Throughout the paper, we will write $\{\eta_{v,n}(k)\}_{k=0,1,\ldots,n}$ for the random walk where $\eta_{v,n}(k)$ is the position of the kth generation individual in the family tree of individuals leading to $v\in V_n$; recall that $\eta_{v,n}(k+1)-\eta_{v,n}(k)$, $k=0,\ldots,n-1$, each have law $w(\cdot)$. We also set $\bar{\eta}_{v,n}(k)=\eta_{v,n}(k)-km_n/n$.

For $\beta>0$, set

$$G_{n,\beta}=\bigcup_{v\in V_n}\bigcup_{0\leqslant k\leqslant n}\{\bar{\eta}_{v,n}(k)\geqslant\beta+(4/\bar{\theta})(\log(k\wedge(n-k)))_+\}\,, \tag{13}$$

where $\bar{\theta}$ is defined below (3). We also set $g_{n,\delta}(i)=\exp\{-\delta|i|(\frac{|i|}{n\log n}\wedge 1)\}$, where $\delta>0$ is a constant that will be specified shortly.

In order to show Corollary 2.5, we first obtain, in Lemma 2.4, an upper bound on the probability that BRW takes atypically large values over $[0,n]$. Lemma 2.4 will also be applied in Section 3.

Lemma 2.4. *There exists a constant $\delta > 0$ such that $\mathbb{P}(G_{n,\beta}) \lesssim \beta \mathrm{e}^{-\bar\theta\beta} g_{n,\delta}(\beta)$ for all $n \geqslant 2$ and $\beta \geqslant 1$.*

For many of the applications in the next section, the weaker bound $\mathbb{P}(G_n(\beta)) \lesssim \beta \mathrm{e}^{-\bar\theta\beta}$ will suffice. We remark that one can show the bound in Lemma 2.4 still holds if the denominator $n \log n$ in $g_{n,\delta}(\cdot)$ is replaced by n (by using the Skorokhod embedding), although we have not done so here.

Proof of Lemma 2.4. For given $v \in V_n$, we define the probability measure $\mathbb{Q}^{(n)}$, on paths in $[0,n]$, by

$$\frac{d\mathbb{P}}{d\mathbb{Q}^{(n)}} := \mathrm{e}^{-\theta_n \bar\eta_{v,n}(n) - nI(m_n/n)} = (1 + O(\frac{1}{n}\log^2 n))\, n^{3/2}\rho^{-n}\mathrm{e}^{-\theta_n\bar\eta_{v,n}(n)}\,, \tag{14}$$

where $I(\lambda)$ is the rate function in (2) and $\theta_n = \theta_n(m_n/n)$ is the value of θ at which the supremum in (2) is taken when $\lambda = m_n/n$. The second equality is a consequence of the definition of m_n and the differentiability of $I(\cdot)$, which imply that

$$0 \leqslant \bar\theta - \theta_n \lesssim \frac{1}{n}\log n\,, \tag{15}$$

and of $I'(c_1) = \bar\theta$.

For $0 \leqslant k \leqslant n$, write $\psi_{n,\beta}(k) = \beta + (4/\bar\theta)(\log(k \wedge (n-k)))_+$ and set

$$\begin{aligned}
\chi^{\mathbb{P}}_{n,j}(i) &= \mathbb{P}(\bar\eta_{v,n}(k) \leqslant \psi_{n,\beta}(k) \text{ for all } k \leqslant j,\, \bar\eta_{v,n}(j) \in [i-1,i))\,,\\
\chi^{\mathbb{Q}^{(n)}}_{n,j}(i) &= \mathbb{Q}^{(n)}(\bar\eta_{v,n}(k) \leqslant \psi_{n,\beta}(k) \text{ for all } k \leqslant j,\, \bar\eta_{v,n}(j) \in [i-1,i))\,.
\end{aligned}$$

By an elementary union bound,

$$\mathbb{P}(G_{n,\beta}) \leqslant \sum_{j=1}^{n-1} \rho^{j+1} \sum_{i=-\infty}^{\lfloor \psi_{n,\beta}(j)+1 \rfloor} \chi^{\mathbb{P}}_{n,j}(i)\mathbb{P}(i + \bar\eta_{v,n}(j+1) - \bar\eta_{v,n}(j) \geqslant \psi_{n,\beta}(j+1))\,. \tag{16}$$

We will obtain upper bounds for each of the two factors inside the inner sum in (16); the bound in (21) for the first factor $\chi^{\mathbb{P}}_{n,j}(i)$ requires most of the work.

For $\chi^{\mathbb{P}}_{n,j}(i)$, we will need an upper bound on $\chi^{\mathbb{Q}^{(n)}}_{n,j}(i)$, for which we consider the probability measure $\mathbb{Q}$ on paths in $[0,n]$ defined by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \mathrm{e}^{-\bar\theta(\eta_{v,n}(n) - c_1 n) - nI(c_1)},$$

for given $v \in V_n$. Note that, under $\mathbb{Q}$, $\{\eta_{v,n}(k) - c_1 k\}_{k=0,1,\ldots,n}$ is a mean zero random walk that satisfies the moment assumptions of Lemma 2.2, and, under $\mathbb{Q}^{(n)}$, $\{\bar\eta_{v,n}(k)\}_{k=0,1\ldots,n}$ is also a mean zero random walk. Setting $S_k = \eta_{v,n}(k) - c_1 k$ and $S^{(n)}_k = \bar\eta_{v,n}(k)$, one has $S^{(n)}_k = S_k + c_2 k(\log n)/n$, and $\mathbb{Q}$ and $\mathbb{Q}^{(n)}$ satisfy the analog of (8) for $\theta^{(n)}$ chosen as in the lemma. The assumptions of Lemma 2.3 are therefore satisfied for S_k and $S^{(n)}_k$, and consequently, by (12) of Lemma 2.3,

$$\chi^{\mathbb{Q}^{(n)}}_{n,j}(i) \lesssim j^{-3/2}\psi_{n,\beta}(0)(\psi_{n,\beta}(j) - i + 2) \tag{17}$$

for $i \leqslant \psi_{n,\beta}(j) + 1$ and $n/2 \leqslant j \leqslant n$.

Since $\chi^{\mathbb{P}}_{n,j}(i) \lesssim \frac{d\mathbb{P}}{d\mathbb{Q}^{(n)}}|_j([i-1,i))\chi^{\mathbb{Q}^{(n)}}_{n,j}(i)$, (17), together with (14), (15), and $\frac{1}{n}\log n \leqslant \frac{1}{j}\log j$, implies that, for given $C>0$,

$$\chi^{\mathbb{P}}_{n,j}(i) \lesssim \psi_{n,\beta}(0)(\psi_{n,\beta}(j)-i+2)\rho^{-j}\mathrm{e}^{-\theta_n i} \lesssim \beta(\psi_{n,\beta}(j)-i+2)\rho^{-j}\mathrm{e}^{-\bar{\theta} i} \tag{18}$$

for $\beta - C\sqrt{n\log n} \leqslant i \leqslant \psi_{n,\beta}(j)+1$ and $n/2 \leqslant j \leqslant n$. When $0 \leqslant j < n/2$, instead of $n/2 \leqslant j \leqslant n$, is assumed and the same range of i is kept, one obtains from (14) the simpler

$$\chi^{\mathbb{P}}_{n,j}(i) \leqslant \mathbb{P}(\bar{\eta}_{v,n}(j) \in [i-1,i)) \lesssim j\rho^{-j}\mathrm{e}^{-\bar{\theta} i}\,; \tag{19}$$

we will denote this collection of pairs (i,j) by A_n. (Later on, the term ρ^{-j} in (18) and (19) will cancel with the corresponding prefactor in (16), and $\mathrm{e}^{-\bar{\theta} i}$ will cancel with the corresponding term in (22).) The bound on the right hand sides of (18) and (19) still holds after multiplication by $g_{n,\delta}(i)$ on the right (because of the above lower bound on i). On the other hand, since the distribution of $w(\cdot)$ has exponential moments in a neighborhood of $\bar{\theta}$, it will follow from a moderate deviation estimate (using Markov's inequality) and (15) that, for $i < \beta - C\sqrt{n\log n}$,

$$\chi^{\mathbb{P}}_{n,j}(i) \leqslant \mathbb{P}(\bar{\eta}_{v,n}(j) \in [i-1,i)) \lesssim j\rho^{-j}\mathrm{e}^{-\theta_n i}\exp\{-\delta i(\frac{i}{n}\wedge 1)\} \leqslant \rho^{-j}\mathrm{e}^{-\bar{\theta} i}g_{n,\delta}(i) \tag{20}$$

for small enough $\delta>0$ and large enough C. (See, e.g., Dembo and Zeitouni [7, Theorem 3.7.1] for the moderate deviation estimate.) Grouping (18) and (20) together, one obtains

$$\chi^{\mathbb{P}}_{n,j}(i) \lesssim \begin{cases} \beta(\psi_{n,\beta}(j)-i+2)\rho^{-j}\mathrm{e}^{-\bar{\theta} i}g_{n,\delta}(i) & \text{for } (i,j)\notin A_n\,, \\ j\rho^{-j}\mathrm{e}^{-\bar{\theta} i}g_{n,\delta}(i) & \text{for } (i,j)\in A_n\,. \end{cases} \tag{21}$$

For the upper bound of the second factor in (16), note that, since $w(\cdot)$ has exponential moments in a neighborhood of $\bar{\theta}$,

$$\mathbb{P}(i+\bar{\eta}_{v,N}(j+1)-\bar{\eta}_{v,N}(j) \geqslant \psi_{N,\beta}(j+1)) \lesssim \exp\{-(\bar{\theta}+\delta')(\psi_{n,\beta}(j+1)-i)\} \tag{22}$$

for some $\delta'>0$.

Plugging (21) and (22) into (16) and summing over the inner sum implies that

$$\mathbb{P}(G_{n,\beta}) \lesssim \sum_{j=1}^{n}\beta(j\wedge(n+1-j))^{-2}\mathrm{e}^{-\bar{\theta}\beta}g_{n,\delta}(\beta) \lesssim \beta\mathrm{e}^{-\bar{\theta}\beta}g_{n,\delta}(\beta) \tag{23}$$

for small enough $\delta>0$ and all $\beta\geqslant 1$, where the power -2 in $(j\wedge(n+1-j))^{-2}$ is obtained from the term $(4/\bar{\theta})(\log(j\wedge(n-j)))_+$. This completes the proof of the lemma. □

Our main application of Lemma 2.4 in this section is the following upper bound on $\mathbb{P}(\eta^*_n > m_n+z)$. Let $\theta^*_n := \bar{\theta}$ for $z\leqslant n$ and $\theta^*_n := \bar{\theta}+\delta$ for $z>n$. The upper bound involving θ^*_n, in Corollary 2.5, will suffice except in two places ((46) and (50) of Lemma 3.4).

Corollary 2.5. *For appropriate $\delta>0$ and all $n,z\geqslant 2$,*

$$\mathbb{P}(\eta^*_n > m_n+z) \lesssim z\mathrm{e}^{-\bar{\theta} z}g_{n,\delta}(z)\,. \tag{24}$$

In particular, $\mathbb{P}(\eta^*_n > m_n+z) \lesssim z\mathrm{e}^{-\theta^*_n z}$ for all $n,z\geqslant 2$.

Proof. Since $\{\eta_n^* > m_n + z\} \subseteq G_{n,z}$, the bound in (24) follows immediately from Lemma 2.4. $\square$

The following result is a quick consequence of Corollary 2.5 and the definition of m_n.

Corollary 2.6. *For appropriate $\delta > 0$, and all $2 \leqslant \ell \leqslant \sqrt{n}$ and $z \geqslant -\log \ell + 1$,*

$$\mathbb{P}(\eta_\ell^* > \ell m_n/n + z) \lesssim \ell^{-3/2}(z + \log \ell)\mathrm{e}^{-\bar{\theta} z} g_{\ell,\delta}(z)\,. \tag{25}$$

In particular, $\mathbb{P}(\eta_\ell^* > \ell m_n/n + z) \lesssim \ell^{-3/2}(z + \log \ell)\mathrm{e}^{-\theta_\ell^* z}$.

The last result of the section gives a lower bound on the right tail of the maximum of BRW. For $v, w \in V_n$, we say that v and w *split* at time $j_s = n - s$, denoted by $v \sim_s w$, if s is the maximal integer such that $\{\eta_{v,n}(j) - \eta_{v,n}(j_s) : j_s \leqslant j \leqslant n\}$ is independent of $\{\eta_{w,n}(j) - \eta_{w,n}(j_s) : j_s \leqslant j \leqslant n\}$, i.e., the last common ancestor of v and w occurs at time j_s.

Lemma 2.7. *For all n and z satisfying $z \leqslant \sqrt{n}$,*

$$\mathbb{P}(\eta_n^* > m_n + z) \gtrsim z\mathrm{e}^{-\bar{\theta} z}\,. \tag{26}$$

The argument for Lemma 2.7 involves well-known second moment estimates. More precise second moment estimates will be shown in Proposition 3.1.

Proof of Lemma 2.7. For $v \in V_n$, set

$$H_{v,n}(z) = \{\bar{\eta}_{v,n}(k) \leqslant z \text{ for all } k \leqslant n-1,\ \bar{\eta}_{v,n}(n) \in (z, z+1]\}$$

and $\Delta_{n,z} = \sum_{v \in V_n} \mathbf{1}_{H_{v,n}(z)}$. We will apply the elementary bound

$$\mathbb{P}(\eta_n^* > m_n + z) \geqslant (\mathbb{E}\Delta_{n,z})^2/\mathbb{E}(\Delta_{n,z})^2\,, \tag{27}$$

which is a consequence of Jensen's inequality.

We obtain a lower bound on $\mathbb{E}\Delta_{n,z}$ by employing the change of measure in (14) as was done immediately below (17), but reversing the inequalities there and applying (7) instead of (12) for the lower bound corresponding to (17) (and with z in place of $\psi_{n,\beta}(\cdot)$). Multiplying this bound by ρ^n, we obtain

$$\mathbb{E}\Delta_{n,z} \gtrsim z\mathrm{e}^{-\theta_n z} \geqslant z\mathrm{e}^{-\bar{\theta} z}\,, \tag{28}$$

with the first inequality holding for $z \leqslant \sqrt{n}$.

For the upper bound on $\mathbb{E}(\Delta_{n,z})^2$, we employ the decomposition

$$\mathbb{E}(\Delta_{n,z})^2 = \mathbb{E}\Delta_{n,z} + K^*\rho^{n-2}\sum_{s=1}^{n} \rho^s\, \mathbb{P}(H_{v,n}(z) \cap H_{w,n}(z) \text{ for } v \sim_s w)\,, \tag{29}$$

where $K^* = K - \rho$, and K is defined in (1). Set $J_i = z + (-i-1, -i]$. Conditioning on the value at $\bar{\eta}_{v,n}(n-s)$, one has, for $v \sim_s w$,

$$\begin{aligned}\mathbb{P}(H_{v,n}(z) \cap H_{w,n}(z)) &\leqslant \sum_{i=0}^{\infty} \mathbb{P}(\bar{\eta}_{v,n}(k) \leqslant z \text{ for } k < n-s;\ \bar{\eta}_{v,n}(n-s) \in J_i) \\ &\times \big(\sup_{y \in J_i}(\mathbb{P}(\bar{\eta}_{v,n}(n-s+j) \leqslant z \text{ for } j < s;\ \bar{\eta}_{v,n}(n) \in (z, z+1]) \,|\, \bar{\eta}_{v,n}(n-s) = y)\big)^2\,.\end{aligned}$$

By the same reasoning as in (14)–(18) for $\chi^{\mathbb{P}}_{n,j}(\cdot)$, one obtains upper bounds (up to multiplicative constants) for the probabilities on the right hand side of the above display: two applications of (12) yield the bound

$$\frac{zi}{((n-s)\vee 1)^{3/2}}\cdot\left(\frac{i}{s^{3/2}}\right)^2$$

and two applications of (14) yield

$$\mathrm{e}^{-\theta_n(z+i)}\rho^{-(n+s)}n^{3/2}\exp\{((3/2)s\log n)/n\}\,,$$

with the summands in the above display being bounded by the product of these two quantities. For $z\leqslant\sqrt{n}$, substituting these bounds into the above display and factoring out the terms not involving i gives the sum, $\sum_{i=0}^{\infty} i^3\mathrm{e}^{-\theta_n i}\leqslant\sum_{i=0}^{\infty} i^3\mathrm{e}^{-\bar\theta i/2}<\infty$. Consequently,

$$\begin{aligned}K^*\rho^{n-2}\sum_{s=1}^{n}\rho^s\,\mathbb{P}(H_{v,n}(z)\cap H_{w,n}(z)\text{ for }v\sim_s w)&\lesssim z\mathrm{e}^{-\bar\theta z}\sum_{s=1}^{n}\frac{n^{3/2}\exp\{(3/2)(s/n)\log n\}}{((n-s)\vee 1)^{3/2}s^3}\\&\lesssim z\mathrm{e}^{-\bar\theta z}\sum_{s=1}^{\infty}s^{-3/2}\lesssim z\mathrm{e}^{-\bar\theta z}\lesssim\mathbb{E}\Delta_{n,z}\,,\end{aligned}$$

where the second inequality uses $(s/n)\log n\leqslant\log s$ and $(n-s)\vee s\geqslant n/2$, and the last inequality follows from (28). It therefore follows from (29) that $\mathbb{E}(\Delta_{n,z})^2\lesssim\mathbb{E}\Delta_{n,z}$. This, together with (27), (28) and (29), implies (26) and completes the proof of the lemma. □

3 The limiting right tail of the maximum of BRW

The main result of this section is the following proposition.

Proposition 3.1. *There exists a constant $\alpha^*>0$ such that*

$$\lim_{z\to\infty}\limsup_{n\to\infty}|z^{-1}\mathrm{e}^{\bar\theta z}\mathbb{P}(\eta_n^*>m_n+z)-\alpha^*|=0\,. \tag{30}$$

The section consists of two parts. In Subsection 3.1, our main result is Proposition 3.2, which compares $\mathbb{P}(\eta_n^*>m_n+z)$ with $E\Lambda_{n,z}$, which is defined in (32). The main result in Subsection 3.2, Proposition 3.5, shows that $E\Lambda_{n,z}\sim\alpha^* ze^{-\bar\theta z}$ for large z.

3.1 Expectation bounds for $\mathbb{P}(\eta_n^*>m_n+z)$

In order to prove Proposition 3.1, we will study the BRW at intermediate times $n-\ell\in(0,n)$, where $\ell=\ell(z)$ is an integer function of z, not depending on n, and which satisfies

$$\ell(z)\leqslant z,\quad \ell(z)\to_{z\to\infty}\infty\,. \tag{31}$$

When taking multiple limits, we will let $n\to\infty$ before $z\to\infty$. (The restriction $\ell(z)\leqslant z$ is employed, for example, in (54), but is needed because of the term $\log n$ in $g_{n,\delta}(\cdot)$.)

For $v' \in V_{n-\ell}$, define

$$
\begin{aligned}
E_{v',n}(z) &= \{\eta_{v',n-\ell}(j) \leqslant jm_n/n + z \text{ for all } 0 \leqslant j \leqslant n-\ell, \text{ and } \max_{v \in V_\ell^{v'}} \eta_{v,n} > m_n + z\}\,, \\
F_{v',n}(z) &= \{\eta_{v',n-\ell}(j) \leqslant jm_n/n + z + \frac{1}{2}\log \ell + \frac{4}{\theta}(\log[j \wedge (n-\ell-j)])_+ \\
&\qquad\qquad \text{for all } 0 \leqslant j \leqslant n-\ell, \text{ and } \max_{v \in V_\ell^{v'}} \eta_{v,n} > m_n + z\}\,, \\
G_n(z) &= \bigcup_{v' \in V_{n-\ell}} \bigcup_{0 \leqslant j \leqslant n-\ell} \{\eta_{v',n-\ell}(j) > jm_n/n + z + \frac{1}{2}\log \ell + \frac{4}{\theta}(\log[j \wedge (n-\ell-j)])_+\}\,.
\end{aligned}
\tag{32}
$$

Also define

$$
\Lambda_{n,z} = \sum_{v' \in V_{n-\ell}} \mathbf{1}_{E_{v',n}(z)}\,, \quad \Gamma_{n,z} = \sum_{v' \in V_{n-\ell}} \mathbf{1}_{F_{v',n}(z)}\,.
$$

In words, the random variable $\Lambda_{n,z}$ counts the number of $(n-\ell)$th generation individuals v' for which (1) over $j \in [0, n-\ell]$, $\eta_{v',n}(\cdot)$, stays below the line connecting $(0,z)$ to $(n, m_n + z)$ and (2) at least one of its descendents at time n has value greater than $m_n + z$. The random variable $\Gamma_{n,z}$ counts the number of individuals v' whose ancestors are instead constrained to stay below a higher, slightly concave curve. (Here and later on, we will often suppress ℓ from the notation.)

The main result of this subsection is the following proposition.

Proposition 3.2. *For $\Lambda_{n,z}$ defined as above,*

$$
\lim_{z\to\infty} \limsup_{n\to\infty} \frac{\mathbb{P}(\eta_n^* > m_n + z)}{\mathbb{E}\Lambda_{n,z}} = \lim_{z\to\infty} \liminf_{n\to\infty} \frac{\mathbb{P}(\eta_n^* > m_n + z)}{\mathbb{E}\Lambda_{n,z}} = 1\,. \tag{33}
$$

In order to demonstrate Proposition 3.2, we separately derive lower and upper bounds on truncations of the BRW at time $n-\ell$ in terms of the curves in the definitions of $\Lambda_{n,z}$ and $\Gamma_{n,z}$. The following two requirements motivate our choices of $\Lambda_{n,z}$ and $\Gamma_{n,z}$:

(1) The truncations corresponding to $\Lambda_{n,z}$ and $\Gamma_{n,z}$ should result asymptotically in the same expectation; this will be shown in Lemma 3.3.

(2) After truncation with respect to the curve corresponding to $\Lambda_{n,z}$, the resulting second moment of the number of curves should be asymptotically the same as the corresponding expectation; this will be shown in Lemma 3.4.

We first compare $\mathbb{E}\Lambda_{n,z}$ and $\mathbb{E}\Gamma_{n,z}$. Note that $\mathbb{E}\Lambda_{n,z} \leqslant \mathbb{E}\Gamma_{n,z}$.

Lemma 3.3. *For $\Lambda_{n,z}$ and $\Gamma_{n,z}$ as above,*

$$
\lim_{z\to\infty} \liminf_{n\to\infty} \frac{\mathbb{E}\Lambda_{n,z}}{\mathbb{E}\Gamma_{n,z}} = 1\,. \tag{34}
$$

Proof. For $v' \in V_{n-\ell}$, we write $\hat{\eta}_{v',n-\ell}(k) = \eta_{v',n-\ell}(k) - km_n/n$, and define the probability measures $\mathbb{Q}^{(n)}$, on paths in $[0, n-\ell]$, by

$$
\frac{d\mathbb{P}}{d\mathbb{Q}^{(n)}} = \mathrm{e}^{-\theta_n \hat{\eta}_{v',n}(n-\ell) - (n-\ell)I(m_n/n)}\,, \tag{35}
$$

where θ_n is defined below (14). Under $\mathbb{Q}^{(n)}$, $\hat{\eta}_{v',n-\ell}(\cdot)$ is a random walk with increments whose law depends on n but possess a uniformly bounded variance.

For $\psi_{n,\ell,z}(k) = z + \frac{1}{2}\log\ell + \frac{4}{\theta}(\log[k\wedge(n-\ell-k)])_+$ and $J_i = z + \frac{1}{2}\log\ell + (-i-1,-i]$, set

$$\varphi^U_{n,\ell,z}(i) = \mathbb{P}(\hat{\eta}_{v',n-\ell}(k) \leqslant \psi_{n,z}(k) \text{ for all } k \leqslant n-\ell,\ \hat{\eta}_{v',n-\ell}(n-\ell) \in J_i)\,,$$

$$\varphi^L_{n,\ell,z}(i) = \mathbb{P}(\hat{\eta}_{v',n-\ell}(k) \leqslant z \text{ for all } k \leqslant n-\ell,\ \hat{\eta}_{v',n-\ell}(n-\ell) \in J_i)\,.$$

One then has the upper bound

$$\begin{aligned}\mathbb{P}(F_{v',n}(z)\setminus E_{v',n}(z)) \leqslant & \sum_{i\in A_1\cup A_3} \varphi^U_{n,\ell,z}(i)\mathbb{P}(\eta^*_\ell > \ell m_n/n + i)\\ &+ \sum_{i\in A_2}(\varphi^U_{n,\ell,z}(i) - \varphi^L_{n,\ell,z}(i))\mathbb{P}(\eta^*_\ell > \ell m_n/n + i)\,,\end{aligned}$$

where A_1, A_2, and A_3 are the integers restricted to $[0,\, \ell^{1/3} + \frac{1}{2}\log\ell]$, $(\ell^{1/3} + \frac{1}{2}\log\ell\,, \ell + z]$, and $(\ell + z, \infty)$, respectively. One can bound the first sum over A_1, respectively, over A_3, by using the analog of (18), respectively, (20), to bound $\varphi^U_{n,\ell,z}(i)$, and by using Corollary 2.6 to bound the second term, from which one obtains the upper bounds of the sums over A_1 and A_3,

$$C\rho^{-(n-\ell)}\ell^{-1/2}z\mathrm{e}^{-\bar{\theta}z} \quad \text{and} \quad C\rho^{-(n-\ell)}z^3\mathrm{e}^{-(\bar{\theta}+\delta)z}\,,$$

for an appropriate constant C and large enough n, where $\delta > 0$ is as in Corollary 2.6. It moreover follows from the comment after Lemma 2.3 that the sum over A_2 is at most $\delta_{\frac{1}{2}\ell^{1/3}}\mathbb{P}(E_{v,n}(z))$, where $\delta_y \to_{y\to\infty} 0$ and δ_y is as in Lemma 2.3. Summation over $v' \in V_{n-\ell}$ therefore implies that

$$\mathbb{E}\Gamma_{n,z} - \mathbb{E}\Lambda_{n,z} \lesssim \ell^{-1/2}z\mathrm{e}^{-\bar{\theta}z} + \delta_{\frac{1}{2}\ell^{1/3}}\mathbb{E}\Lambda_{n,z}\,. \tag{36}$$

By Lemmas 2.4 and 2.7 (applied with $\beta = z + \frac{1}{2}\log\ell$), for all $\ell, n \geqslant 2$ and $z \leqslant \sqrt{n}$,

$$\mathbb{E}\Gamma_{n,z} \gtrsim z\mathrm{e}^{-\bar{\theta}z}\,. \tag{37}$$

Together, (36) and (37) imply (34), which completes the proof of the lemma. □

We next provide a precise estimate of the second moment of $\Lambda_{n,z}$ in terms of its first moment.

Lemma 3.4. *For $\Lambda_{n,z}$ as above,*

$$\lim_{z\to\infty}\limsup_{n\to\infty}\frac{\mathbb{E}(\Lambda_{n,z})^2}{\mathbb{E}\Lambda_{n,z}} = 1\,. \tag{38}$$

Proof. By Lemma 3.3 and (37),

$$\lim_{z\to\infty}\liminf_{n\to\infty}(\mathbb{E}\Lambda_{n,z}/z\mathrm{e}^{-\bar{\theta}z}) \gtrsim 1\,. \tag{39}$$

We need to estimate the above second moment, which we rewrite as

$$E(\Lambda_{n,z})^2 = E\Lambda_{n,z} + \sum_{v,w\in V_{n-\ell}, v\neq w}\mathbb{P}(E_{v,n}(z)\cap E_{w,n}(z))\,. \tag{40}$$

In analogy with the previous section, for $v, w \in V_{n-\ell}$, we will write $v \sim_s w$ when v and w split at time $j_s = n - \ell - s$.

As in (29) of Lemma 2.7, we will show that, for large z, the sum in (40) is small in comparison with the first term on the right hand side. For $v \sim_s w$, with given s, we will employ the upper bound

$$\begin{aligned}
&\mathbb{P}(E_{v,n}(z) \cap E_{w,n}(z)) \\
=&\mathbb{P}(\bar{\eta}_{v,n}(j), \bar{\eta}_{w,n}(j) \leqslant z \text{ for all } j \in [0, n-\ell];\ \max_{u \in V_\ell^v} \bar{\eta}_{u,n}, \max_{u \in V_\ell^w} \bar{\eta}_{u,n} > z) \\
=&\sum_{i=0}^{\infty} \mathbb{P}(\bar{\eta}_{v,n}(j), \bar{\eta}_{w,n}(j) \leqslant z \text{ for all } j \in [0, n-\ell];\ \max_{u \in V_\ell^v} \bar{\eta}_{u,n}, \max_{u \in V_\ell^w} \bar{\eta}_{u,n} > z; \bar{\eta}_{v,n}(j_s) \in J_i) \\
\leqslant&\sum_{i=0}^{\infty} \mathbb{P}(\bar{\eta}_{v,n}(j) \leqslant z \text{ for all } j \in [0, j_s];\ \bar{\eta}_{v,n}(j_s) \in J_i)(\Gamma_{v,i,z,s})^2\,, \qquad (41)
\end{aligned}$$

where, in the above sums, $J_i = z + (-i-1, -i]$, and

$$\Gamma_{v,i,z,s} = \sup_{\bar{\eta}_{v,n}(j_s) \in J_i} \mathbb{P}(\bar{\eta}_{v,n}(j) \leqslant z \text{ for all } j_s < j \leqslant n-\ell, \max_{u \in V_\ell^v} \bar{\eta}_{u,n} > z \mid \bar{\eta}_{v,n}(j_s)).$$

We decompose the range for s into three regions, given by $[0, \lfloor \ell^{1/3} \rfloor)$, $[\lfloor \ell^{1/3} \rfloor, n-\ell-\lfloor \ell^{1/3} \rfloor)$ and $[n-\ell-\lfloor \ell^{1/3} \rfloor, n-\ell]$; the arguments for each part are similar, with minor differences. We first handle the main interval, consisting of $s \in [\lfloor \ell^{1/3} \rfloor, n-\ell-\lfloor \ell^{1/3} \rfloor)$.

Set $\gamma_{h,\ell} = \mathbb{P}(\eta_\ell^* > \ell m_n/n + h)$. Restarting the BRW at time j_s, then applying the change of measure (35), (6) of the ballot theorem, and reasoning similarly to the upper bound for $\chi_{n,j}^{\mathbb{P}}(\cdot)$ in (18)–(21), one obtains

$$\begin{aligned}
\Gamma_{v,i,z,s} &\leqslant \sum_{h=-1}^{\infty} \mathbb{P}(\bar{\eta}_{v,s+\ell}(j) \leqslant i+1 \text{ for all } j \in [0,s], \bar{\eta}_{v,s+\ell}(s) \in (i-h-1, i-h])\gamma_{h,\ell} \\
&\lesssim \sum_{h=-1}^{\infty} (h+2)(i+1)\mathrm{e}^{-\bar{\theta}(i-h)} g_{s,\delta}(i-h)\mathrm{e}^{-sI(m_n/n)}\gamma_{h,\ell}/s^{3/2} \qquad (42)
\end{aligned}$$

for appropriate $\delta > 0$; similarly,

$$\mathbb{P}(\bar{\eta}_{v,n}(j) \leqslant z \text{ for all } j \in [0, j_s];\ \bar{\eta}_{v,n}(j_s) \in J_i) \lesssim \frac{z(i+1)}{j_s^{3/2}}\mathrm{e}^{-\bar{\theta}(z-i)}\mathrm{e}^{-j_s I(m_n/n)}\,. \qquad (43)$$

Substitution of (42) and (43) into (41) implies that

$$\mathbb{P}(E_{v,n}(z) \cap E_{w,n}(z)) \lesssim z\mathrm{e}^{-\bar{\theta}z}[\mathrm{e}^{-(n-\ell+s)I(m_n/n)}/(s^3 j_s^{3/2})][\sum_{i=0}^{\infty}(i+1)^3\mathrm{e}^{-\bar{\theta}i}][\sum_{h=-1}^{\infty}(h+2)\mathrm{e}^{\bar{\theta}h}\gamma_{h,\ell}]^2\,. \qquad (44)$$

(Here, we have used $g_{s,\delta}(i-h) \leqslant 1$; the term $g_{s,\delta}(i-h)$ will be needed when applying (42) for $s \in [1, \lfloor \ell^{1/3} \rfloor)$.) Using the definitions of c_1 and m_n, one has the upper bound, for the quantity in the first brackets,

$$\rho^{-(n-\ell+s)}\frac{n^{3/2}}{(sj_s)^{3/2}}\frac{\exp\{(3/2)(s/n)\log n\}}{s^{3/2}} \lesssim \rho^{-(n-\ell+s)}\frac{n^{3/2}}{(sj_s)^{3/2}}\,, \qquad (45)$$

whereas, plugging in Corollary 2.6, one has, for $n \geqslant \ell^2$, the upper bound for the quantity in the last brackets,

$$\ell^{-3/2} \sum_{h=-1}^{\infty} (h+2)(h+\log \ell) g_{\ell,\delta}(h) \lesssim (\log \ell)^{3/2}\,, \tag{46}$$

since the sum is of order $\sqrt{\ell \log \ell}$ times the variance $\ell \log \ell$ of the corresponding normal. Together with (44), these two bounds imply that, for $s \in [\lfloor \ell^{1/3} \rfloor, n-\ell-\lfloor \ell^{1/3} \rfloor)$,

$$\mathbb{P}(E_{v,n}(z) \cap E_{w,n}(z)) \lesssim z\mathrm{e}^{-\bar{\theta} z} \rho^{-(n-\ell+s)} \frac{(n \log \ell)^{3/2}}{(sj_s)^{3/2}}\,. \tag{47}$$

The argument for $s \in [n-\ell-\lfloor \ell^{1/3} \rfloor, n-\ell]$ is essentially the same as the previous argument, but, instead of (43), which employs the ballot theorem, we use the simpler upper bound

$$\mathbb{P}(\bar{\eta}_{v,n}(j_s) \in J_i) \lesssim \frac{1}{(j_s \vee 1)^{1/2}} \mathrm{e}^{-\bar{\theta}(z-i)} \mathrm{e}^{-j_s I(m_n/n)}, \tag{48}$$

which avoids the coefficient z in the numerator. Continuing as above, instead of (47), one obtains that, for $s \in [n-\ell-\lfloor \ell^{1/3} \rfloor, n-\ell]$,

$$\mathbb{P}(E_{v,n}(z) \cap E_{w,n}(z)) \lesssim \mathrm{e}^{-\bar{\theta} z} \rho^{-(n-\ell+s)} \frac{(n \log \ell)^{3/2}}{s^{3/2}(j_s \vee 1)^{1/2}}\,. \tag{49}$$

The argument for $s \in [0, \lfloor \ell^{1/3} \rfloor)$ is also similar to that for $s \in [\lfloor \ell^{1/3} \rfloor, n-\ell-\lfloor \ell^{1/3} \rfloor)$, but one retains the term $g_{s,\delta}(i-h)$ in (42), and therefore replaces the double sum in (44) by

$$\sum_{i=0}^{\infty} (i+1)^3 \mathrm{e}^{-\bar{\theta} i} [\sum_{h=-1}^{\infty} (h+2) \mathrm{e}^{\bar{\theta} h} g_{s,\delta}(i-h) \gamma_{h,\ell}]^2 = \sum_{i=0}^{\infty} (i+1)^3 \mathrm{e}^{-\bar{\theta} i/3} [\sum_{h=-1}^{\infty} (h+2) \mathrm{e}^{\bar{\theta} h} \mathrm{e}^{-\bar{\theta} i/3} g_{s,\delta}(i-h) \gamma_{h,\ell}]^2\,. \tag{50}$$

Bounding $\gamma_{h,\ell}$ as before, this is

$$\lesssim \sum_{i=0}^{\infty} (i+1)^3 \mathrm{e}^{-\bar{\theta} i/3} [\sum_{h=-1}^{\infty} \ell^{-3/2} (h+2)(h+\log \ell) \mathrm{e}^{-\bar{\theta} i/3} g_{s,\delta}(i-h)]^2\,. \tag{51}$$

By completing the square, one can show that, for appropriate $\varepsilon = \varepsilon_{\bar{\theta},\delta} > 0$,

$$\mathrm{e}^{-\bar{\theta} i/3} g_{s,\delta}(i-h) \lesssim g_{s,\varepsilon}(h)$$

for all s, h, and i. Employing (46), but with $g_{s,\varepsilon}(h)$ in place of $g_{\ell,\delta}(h)$, it follows that (51), and hence (50), is at most (up to a constant multiple)

$$\{\ell^{-3/2}[(s \log s)^{3/2} + (s \log s)^2 \log \ell]\}^2 \lesssim \ell^{-1/2}\,,$$

on accout of $s \leqslant \ell^{1/3}$. Plugging in this bound for the product of the last two bracketed quantities in (44), and employing (45) for the first bracketed quantity, this implies that

$$\mathbb{P}(E_{v,n}(z) \cap E_{w,n}(z)) \lesssim z\mathrm{e}^{-\bar{\theta} z} \rho^{-(n-\ell+s)} \frac{n^{3/2} \ell^{-1/2}}{((s \vee 1) j_s)^{3/2}}\,. \tag{52}$$

Using $\mathbb{E}(\sum_{v,w\in V_{n-\ell},v\neq w}\mathbf{1}_{v\sim_s w})=K^*\rho^{n-\ell+s-2}$, the bounds in (47), (49), and (52) together show that the sum on the right hand side of (40) is at most (up to a constant multiple)

$$ze^{-\bar{\theta}z}\{\sum_{s=0}^{\lfloor\ell^{1/3}\rfloor}\frac{n^{3/2}\ell^{-1/2}}{((s\vee 1)j_s)^{3/2}}+\sum_{s=\lfloor\ell^{1/3}\rfloor+1}^{n-\ell-\lfloor\ell^{1/3}\rfloor}\frac{(n\log\ell)^{3/2}}{(sj_s)^{3/2}}+\sum_{s=n-\ell-\lfloor\ell^{1/3}\rfloor+1}^{n-\ell}\frac{(n\log\ell)^{3/2}\ell^{1/3}}{(s(j_s\vee 1))^{3/2}z}\}\,. \tag{53}$$

Since $\sum_{s=k}^{\infty}1/s^{3/2}\lesssim 1/k^{1/2}$, this is, for given z and large n,

$$\lesssim ze^{-\bar{\theta}z}\{\ell^{-1/2}+(\log\ell)^{3/2}\ell^{-1/6}+(\log\ell)^{3/2}z^{-2/3}\}\lesssim\ell^{-1/8}ze^{-\bar{\theta}z}\lesssim\ell^{-1/8}\,\mathbb{E}\Lambda_{n,z}\,, \tag{54}$$

where the first two inequalities use (31) and the third inequality uses (39). The coefficient $\ell^{-1/8}$ of $\mathbb{E}\Lambda_{n,z}$ goes to 0 as $z\to\infty$. This shows that, for large z, the sum in (40) is small in comparison with the preceding term in (40), which completes the proof of the lemma. □

We now complete the demonstration of Proposition 3.2.

Proof of Proposition 3.2, By a simpler version of the argument in Lemma 2.4,

$$\mathbb{P}(G_n(z))\lesssim\ell^{-\bar{\theta}/2}ze^{-\bar{\theta}z} \tag{55}$$

for $\ell\leqslant\sqrt{n}$; the factor $\frac{1}{2}\log\ell$ in the definition of $G_n(z)$ has been employed here. (The analog of (18) (rather than (21)) suffices, for which one employs the change of measure (35) (rather than (14)).) Together, (55), Lemma 3.3, (37), and the trivial estimate

$$\mathbb{P}(G_n(z))+\mathbb{E}\Gamma_{n,z}\geqslant\mathbb{P}(\eta_n^*>m_n+z)$$

imply the upper bound

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\frac{\mathbb{P}(\eta_n^*>m_n+z)}{\mathbb{E}\Lambda_{n,z}}\leqslant 1\,. \tag{56}$$

On the other hand, the lower bound

$$\liminf_{z\to\infty}\liminf_{n\to\infty}\frac{\mathbb{P}(\eta_n^*>m_n+z)}{\mathbb{E}\Lambda_{n,z}}\geqslant 1 \tag{57}$$

is an immediate consequence of Lemma 3.4 and the inequalities

$$\mathbb{P}(\eta_n^*>m_n+z)\geqslant\mathbb{P}(\bigcup_{v'\in V_{n-\ell}}E_{v',n}(z))\geqslant\frac{(\mathbb{E}\Lambda_{n,z})^2}{\mathbb{E}(\Lambda_{n,z})^2}$$

(with the latter following from Jensen's inequality). Together, (56) and (57) imply (33). □

3.2 Asymptotics for $\mathbb{E}\Lambda_{n,z}$

This subsection is devoted to demonstrating Proposition 3.5, which gives the asymptotic behavior of $E\Lambda_{n,z}$ for large n and z.

Proposition 3.5. *There exists a constant $\alpha^*>0$ such that*

$$\lim_{z\to\infty}\limsup_{n\to\infty}\frac{\mathbb{E}\Lambda_{n,z}}{\alpha^*ze^{-\bar{\theta}z}}=\lim_{z\to\infty}\liminf_{n\to\infty}\frac{\mathbb{E}\Lambda_{n,z}}{\alpha^*ze^{-\bar{\theta}z}}=1\,. \tag{58}$$

Together, Propositions 3.2 and 3.5 imply Proposition 3.1.

For $v \in V_{n-\ell}$, denote by $\nu_{n,z}(\cdot)$ the measure satisfying

$$\nu_{n,z}(I) = \mathbb{P}(\bar{\eta}_{v,n}(j) \leqslant z \text{ for all } 0 \leqslant j \leqslant n-\ell;\ \bar{\eta}_{v,n}(n-\ell) \in z+I)$$

for all intervals $I \in \mathbb{R}$. Also, set $\gamma_\ell(y) = \mathbb{P}(\eta_\ell^* > \ell m_n/n + y)$. From the definition of $E_{v,n}(z)$ in (32) and $J_i := (-i-1, -i]$, $i = 0, 1, \ldots$, one has

$$\mathbb{P}(E_{v,n}(z);\ \bar{\eta}_{v,n}(n-\ell) \in z+I) = \int_{I\cap(-\infty,0]} \gamma_\ell(-y)\nu_{n,z}(dy) \leqslant \sum_{i=0}^{\infty} \gamma_\ell(i)\nu_{n,z}(I \cap J_i)\,. \tag{59}$$

We will denote by $\Lambda_{n,z,I}$ the analog of $\Lambda_{n,z}$, but with the added restriction $\bar{\eta}_{v,n}(n-\ell) \in z+I$; then $\mathbb{E}\Lambda_{n,z,I} = \rho^{n-\ell}\mathbb{P}(E_{v,n}(z);\ \bar{\eta}_{v,n}(n-\ell) \in z+I)$, for any $v \in V_{n-\ell}$.

Set $L_\ell = (-\ell, -\ell^{2/5}]$; the following lemma shows that the main contribution to $\mathbb{E}\Lambda_{n,z}$ is from values $y \in I := L_\ell$, as in (59). (The choice of the exponent $2/5$ here is somewhat arbitrary; only $0 < 2/5 < 1/2$ is used.) In the lemma, we will treat z and ℓ as independent variables, and will only employ the relationship (31) at the end of the subsection.

Lemma 3.6. *For Λ_{n,z,L_ℓ} defined as above,*

$$\lim_{z,\ell\to\infty} \liminf_{n\to\infty} \frac{\mathbb{E}\Lambda_{n,z,L_\ell}}{\mathbb{E}\Lambda_{n,z}} = 1\,.$$

Proof. Using (59), it suffices to show that

$$\lim_{\ell\to\infty} \sup_{z\geqslant 1} \limsup_{n\to\infty} \rho^{n-\ell}\Big(\sum_{i=0}^{\infty} \gamma_\ell(i)\nu_{n,z}(I_h \cap J_i)\Big)\Big/\mathbb{E}\Lambda_{n,z} = 0 \tag{60}$$

for $h = 1, 2$, with $I_1 = (-\infty, -\ell]$ and $I_2 = (-\ell^{2/5}, 0]$.

As in previous applications, (6) and the change of measure (35), together with (15), imply that, for $z, \ell \leqslant \sqrt{n}$,

$$\nu_{n,z}(J_i) \lesssim (i+1)ze^{-\bar{\theta}(z-i)}\rho^{-(n-\ell)} \tag{61}$$

for $i \geqslant 0$. Also, by Corollary 2.6,

$$\gamma_\ell(i) \lesssim (i+1+\log\ell)\ell^{-3/2}\mathrm{e}^{-\theta_\ell^* i}\,. \tag{62}$$

Combining (61) and (62), it follows that

$$\sum_{i=\lfloor\ell\rfloor}^{\infty} \gamma_\ell(i)\nu_{n,z}(J_i) \lesssim \rho^{-(n-\ell)}z\mathrm{e}^{-\bar{\theta}z} \sum_{i=\lfloor\ell\rfloor}^{\infty} (i+1+\log\ell)^2\ell^{-3/2}\mathrm{e}^{-\delta i} \lesssim \mathrm{e}^{-\delta\ell/2}\rho^{-(n-\ell)}z\mathrm{e}^{-\bar{\theta}z}\,,$$

where $\delta > 0$ is as in Corollary 2.6, and

$$\sum_{i=0}^{\lfloor\ell^{2/5}\rfloor} \gamma_\ell(i)\nu_{n,z}(J_i) \lesssim \rho^{-(n-\ell)}z\mathrm{e}^{-\bar{\theta}z} \sum_{i=0}^{\lfloor\ell^{2/5}\rfloor} (i+1+\log\ell)^2\ell^{-3/2} \lesssim \ell^{-3/10}\rho^{-(n-\ell)}z\mathrm{e}^{-\bar{\theta}z}\,.$$

These bounds, together with (34) and (37), imply (60). $\square$

We employ the previous lemma, together with (10) to demonstrate Proposition 3.5.

Proof of Proposition 3.5. Write $x_n = \ell m_n/n - c_1\ell$, set $J_i^N = (-(i-1)/N + x_n, -i/N + x_n]$, $i = 0, 1, \ldots$, for given $N \in \mathbb{Z}_+$. Note that, for fixed ℓ, $x_n \to_{n\to\infty} 0$. Similar reasoning to that leading to (61), but with the sharper (10) in place of (6), implies that, for fixed N,

$$\lim_{z,i\to\infty} \overline{\underline{\lim}}_{n\to\infty} \rho^{n-\ell}\nu_{n,z}(J_i^N)\Big/\Big(\beta^*((i/N - x_n)/N)ze^{-\bar{\theta}(z-i/N+x_n)}\Big) = 1\,, \tag{63}$$

where $\overline{\underline{\lim}}_{n\to\infty} f(n)$ is shorthand for the bounds given by both $\limsup_{n\to\infty} f(n)$ and $\liminf_{n\to\infty} f(n)$. For the moment treating z and ℓ as independent variables, it follows from (63) that

$$\lim_{z,\ell\to\infty} \overline{\underline{\lim}}_{n\to\infty} \rho^{n-\ell} \sum_{i/N\in -L_\ell} \gamma_\ell(i/N)\nu_{n,z}(J_i^N)\Big/\Big(\beta^* \sum_{i/N\in -L_\ell} \gamma_\ell(i/N)((i/N - x_n)/N)ze^{-\bar{\theta}(z-i/N+x_n)}\Big) = 1$$

for fixed N. Application of bounded convergence to the denominator, as $n \to \infty$, therefore implies

$$\lim_{z,\ell\to\infty} \overline{\underline{\lim}}_{n\to\infty} \rho^{n-\ell} \sum_{i/N\in -L_\ell} \gamma_\ell(i/N)\nu_{n,z}(J_i^N)\Big/\Big(\beta^* \sum_{i/N\in -L_\ell} \gamma_\ell(i/N)(i/N^2)ze^{-\bar{\theta}(z-i/N)}\Big) = 1\,. \tag{64}$$

On the other hand, because of the monotonicity of $\gamma_\ell(\cdot)$ and $\nu_{n,z}(J_i^N)$ on $-L_\ell$, for given N,

$$1 - 2\bar{\theta}/N \leqslant \lim_{z,\ell\to\infty} \overline{\underline{\lim}}_{n\to\infty} \mathbb{E}\Lambda_{n,z,L_\ell}\Big/\Big(\rho^{n-\ell} \sum_{i/N\in -L_\ell} \gamma_\ell(i/N)\nu_{n,z}(J_i^N)\Big) \leqslant 1 + 2\bar{\theta}/N\,,$$

where one uses (63) to bound $\nu_{n,z}(J_{i-1}^N)/\nu_{n,z}(J_i^N)$. Letting $N \to \infty$, it follows from this and (64) that

$$\lim_{z,\ell\to\infty} \overline{\underline{\lim}}_{n\to\infty} \mathbb{E}\Lambda_{n,z,L_\ell}\Big/\Big(\beta^* ze^{-\bar{\theta}z} \int_{-L_\ell} ye^{\bar{\theta}y}\gamma_\ell(y)\,dy\Big) = 1\,. \tag{65}$$

It follows from Lemma 3.6 that the analog of (65) also holds, with $\mathbb{E}\Lambda_{n,z}$ in place of $\mathbb{E}\Lambda_{n,z,L_\ell}$, and so

$$\lim_{z,\ell\to\infty} \overline{\underline{\lim}}_{n\to\infty} \mathbb{E}\Lambda_{n,z}\Big/\Big(\beta^* ze^{-\bar{\theta}z} \int_{-L_\ell} ye^{\bar{\theta}y}\gamma_\ell(y)\,dy\Big) = 1\,.$$

(Using the same reasoning as in Lemma 3.6, one could also replace the region of integration $-L_\ell$ by $[0, \infty)$.) Consequently, for given $\varepsilon > 0$ and $\ell \geqslant \ell_0$, with ℓ_0 large but fixed, the limiting ratio will lie within $(1-\varepsilon, 1+\varepsilon)$ as $z \to \infty$. The limits in (58) follow from this for any choice of $\ell = \ell(z)$ with $\ell(z) \to_{z\to\infty} \infty$. □

4 Proof of Theorem 1.1

We first show that, under the assumptions of Theorem 1.1, $\eta_n^* - m_n$ converges in distribution. One obtains, by decomposing the BRW over the time intervals $[0, k]$ and $[k, n]$,

$$\begin{aligned} \mathbb{P}(\eta_n^* - m_n \leqslant z \,|\, \mathcal{F}_k) &= \prod_{v'\in V_k} \mathbb{P}(\eta_{v',k} + \tilde{\eta}_{n-k}^* - m_n \leqslant +z) \\ &= \prod_{v'\in V_k} \mathbb{P}(\tilde{\eta}_{n-k}^* - m_{n-k} \leqslant m_n - m_{n-k} - \eta_{v',k} + z)\,, \end{aligned} \tag{66}$$

where $\eta_{v',k}$ are the positions of the particles for the process $\eta_\cdot$ at time k, and $\tilde{\eta}_\cdot$ is an independent BRW. Note that, from the definition of m_n, $m_n - m_{n-k} = c_1 k + \delta_k(n)$, with $\delta_k(n) \to_{n\to\infty} 0$ for fixed k, and that, by Proposition 3.1, the distribution of $\eta^*_{n-k} - m_{n-k}$ has right tail $\alpha^* z \mathrm{e}^{-\bar{\theta} z}$ for large $n-k$. Since

$$\eta^*_k - c_1 k \to_{k\to\infty} -\infty \tag{67}$$

in probability, it follows that, for large k and much larger n, the logarithm of the right hand side of (66) is asymptotically

$$\sum_{v'\in V_k} \log[1 - \alpha^* z_{v',k} \mathrm{e}^{-\bar{\theta} z_{v',k}}] \sim -\alpha^* \sum_{v'\in V_k} z_{v',k} \mathrm{e}^{-\bar{\theta} z_{v',k}}, \tag{68}$$

for fixed z, where $z_{v',k} = c_1 k - \eta_{v',k} + z$. Setting

$$Z_k = \sum_{v'\in V_k} (c_1 k - \eta_{v',k}) \mathrm{e}^{-\bar{\theta}(c_1 k - \eta_{v',k})}, \quad Y_k = \sum_{v'\in V_k} \mathrm{e}^{-\bar{\theta}(c_1 k - \eta_{v',k})},$$

this equals $-\alpha^* \mathrm{e}^{-\bar{\theta} z}(Z_k + zY_k) \sim -\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k$, since (67) implies that $Y_k/Z_k \to 0$ in probability.

It follows from the last paragraph that, for appropriate $A_k \in \mathcal{F}_k$ and $\varepsilon_k > 0$, with $Z_k > 0$ on A_k and $\mathbb{P}(A_k^c) \leqslant \varepsilon_k \to_{k\to\infty} 0$,

$$1 - \varepsilon_k \leqslant \log \mathbb{P}(\eta^*_n - m_n \leqslant z \,|\, \mathcal{F}_k) / \big(-\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k\big) \leqslant 1 + \varepsilon_k \quad \text{on } A_k, \tag{69}$$

for $n \geqslant n_k$ and appropriate n_k, with $n_k \to_{k\to\infty} \infty$, and hence

$$\mathbb{E}[\exp\{-(1+\varepsilon_k)\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k\}; A_k] \leqslant \mathbb{P}(\eta^*_n - m_n \leqslant z) \leqslant \mathbb{E}[\exp\{-(1-\varepsilon_k)\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k\}; A_k] + \varepsilon_k\,. \tag{70}$$

As $k \to \infty$, both the left and right hand sides of (70) converge to $\mathbb{E}[\exp\{-\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k\}]$. (To see this, note that $\exp\{-\mathrm{e}^{-x}\}$ is uniformly continuous in x.) Consequently, for fixed z,

$$\lim_{k\to\infty} \limsup_{n\to\infty} |\mathbb{P}(\eta^*_n - m_n \leqslant z) - \mathbb{E}[\exp\{-\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k\}; A_k]| = 0\,. \tag{71}$$

It follows from this that, for some function $w(\cdot)$,

$$\mathbb{P}(\eta^*_n - m_n \leqslant z) \to_{n\to\infty} w(z)\,. \tag{72}$$

One can check that $\lim_{z\to-\infty} w(z) = 0$ (since $\max_v \eta^{v'}_{v,n} - m_n$ are independent for different $v' \in V_1$, but the limits of their distributions must share this same limit as first $n \to \infty$ and then $z \to -\infty$). So $\eta^*_n - m_n$ is in fact tight, and hence $w(\cdot)$ is a distribution function. This completes the proof of the first part of Theorem 1.1.

We still need to verify (4). To show this, we will first show that $\lim_{n\to} Z_n$ exists a.s. (In probability convergence of Z_n is automatic from (71), although we do not use this.) Because of (72), $m_{n+1} - m_n \to_{n\to\infty} c_1$, and the BRW property,

$$w(0) = \mathbb{E}[\prod_{v\in V_1} w(c_1 - \eta_{v,1})].$$

Consequently, $W_n := \prod_{v\in V_n} w(c_1 n - \eta_{v,n})$ is a martingale with respect to $\mathcal{F}_n$. Since W_n is non-negative, $W_n \to_{n\to\infty} W$ exists a.s. by the martingale convergence theorem. On the other hand, it follows from the definition of c_1 that

$$\mathbb{E}[\sum_{v\in V_1} \mathrm{e}^{-\bar{\theta}(c_1 - \eta_{v,1})}] = 1\,,$$

and hence Y_n is also a nonnegative martingale. Another application of the martingale convergence theorem implies that $Y_n \to_{n\to\infty} Y$ exists a.s. (In fact, $Y = 0$ since $Y_k/Z_k \to 0$ in probability.) This implies, in (67), the stronger a.s. convergence in fact holds.

It follows from Proposition 3.1 and (72) that $1 - w(z) \sim \alpha^* z \mathrm{e}^{-\bar{\theta} z}$ as $z \to \infty$. This same property was exploited for $\mathbb{P}(\eta_n^* - m_n > z) \mid \mathcal{F}_k)$ in the first part of the proof, which we now also use for W_n. Proceeding as in (66)–(68), but employing a.s. rather than in probability convergence, one obtains the following a.s. analog of (69),

$$(\log W)\big/ \lim_{k\to\infty} (-\alpha^* Z_k) = \lim_{k\to\infty} (\log W_k)\big/(-\alpha^* Z_k) = 1 \quad \text{a.s.}$$

Therefore, $Z = \lim_{k\to\infty} Z_k$ exists a.s., with $Z = -(\alpha^*)^{-1} \log W$.

The reasoning leading up to (69) also implies the a.s. version,

$$\begin{aligned}\lim_{k\to\infty} \liminf_{n\to\infty} &(\log \mathbb{P}(\eta_n^* - m_n \leqslant z \mid \mathcal{F}_k))\big/(-\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k) \\ &= \lim_{k\to\infty} \limsup_{n\to\infty} (\log \mathbb{P}(\eta_n^* - m_n \leqslant z \mid \mathcal{F}_k))\big/(-\alpha^* \mathrm{e}^{-\bar{\theta} z} Z_k) = 1 \quad \text{a.s.}\end{aligned}$$

Plugging in $Z = \lim_{k\to\infty} Z_k$ reduces the above limits to (4). Taking expectations on both sides of (4), and applying bounded convergence to the limit as $z \to -\infty$, together with $w(-\infty) = 0$, implies that Z is a.s. strictly positive. This completes the proof of the second part of Theorem 1.1.

5 The lattice case

The methods in this paper allow one to also handle the lattice case, which is mentioned as an open problem in Aidekon [2]. Unfortunately, in many places both the statements and some steps in the proof need to be modified, albeit in a minor way. To avoid repetitions or burdening the main text with extra details geared toward the lattice case, we decided to only summarize in this short section the result and the needed adaptations, and to leave the actual proof for either future work or the interested reader.

Recall that a random walk with increments distribution $w(\cdot)$ is called *lattice* if for some y, the support of $w(\cdot) + y$ is contained in a discrete subgroup of $\mathbb{R}$. By rescaling and shifting, we can and will assume that $y = 0$ and that the discrete subgroup is $\mathbb{Z}$. We define m_n and Z_k as in the nonlattice case, and set $A_n = \mathbb{Z} - m_n$.

The analog of Theorem 1.1 is the following.

Theorem 5.1. *Assume that η_n is a lattice branching random walk satisfying (3), with $K < \infty$. Then, $Z = \lim_{k\to\infty} Z_k$ exists and is finite and positive with probability 1, and there exists a constant $\alpha^* > 0$ so that, for each $z \in \mathbb{R}$,*

$$\lim_{k\to\infty} \lim_{n\to\infty} \sup_{z\in A_n} |\mathbb{P}(\eta_n^* \leqslant m_n + z | \mathcal{F}_k) - \exp\{-\alpha^* Z \mathrm{e}^{-\bar{\theta} z}\}| = 0 \quad \textit{a.s.} \tag{73}$$

Note that, by taking expectations in (73), one obtains

$$\lim_{n\to\infty} \sup_{z\in A_n} |\mathbb{P}(\eta_n^* \leqslant m_n + z) - \mathbb{E} \exp\{-\alpha^* Z \mathrm{e}^{-\bar{\theta} z}\}| = 0\,. \tag{74}$$

If one wishes, one can also rephrase (73) and (74) as convergence results over all $\mathbb{R}$ (rather than over A_n) by appropriate interpolation of $\mathbb{P}(\cdot)$ within the intervals between lattice points.

We indicate the main modifications in the argument for obtaining Theorem 5.1 in place of Theorem 1.1, while including some details for the curious reader. Most importantly, we need a modified version of the technical estimates in Subsection 2.1, with the main modifications being in (9)–(11) of Lemma 2.3. In the lemma, we replace the interval (a, b) by the half open interval $(a, b]$, with $b - a = 1$, which contains exactly one lattice point. Also, we replace the term y, $y \geqslant 1$, given in the definition of the boundary, by an appropriately chosen y' with $|y' - y| < 1$.

In the proofs in the appendix of the different parts of Lemma 2.3, various details for the lattice case differ from those for the nonlattice case. One needs to replace the conditional local CLT of [6, Theorem 1] and the asymptotic expansion of [9, Theorem 16.4.1] by their lattice analogs [6, Theorem 2] and [9, Theorem 16.4.2]. The inequality (75) obtained by reversing time can be replaced by a corresponding equality if the interval (a, b) is replaced by a single point; this simplifies somewhat the proofs of (9) and (10). In (98) of the important Lemma A.3, one sets $\delta = 0$, since expanding the interval by $\delta > 0$ no longer ensures the inequality (109) needed for step (C) of the proof; rather, in (98), one instead replaces the term y by an appropriate y' satisfying $|y' - y| < 1$. (The existence of such a y' can be shown by noting that the maximum separation at any time between the two boundaries in (109) is of order $(\log n)/n^{3/4}$, and considering the probability that the random walk visits this region at some time, for y' is chosen randomly in $[0, 1) + y$. Since the time interval has length $\lfloor n^{1/4} \rfloor$, application of the union bound over these times shows that the probability of this event occurring is of smaller order than 1.)

Once the estimates in Subsection 2.1 are obtained, the rest of the proof in Sections 2– 4 proceeds for the most part as in the nonlattice case, with the change that, in Section 3, all limits in z are taken over lattice points, which is reflected in the argument in Section 4. In particular, (30) of Proposition 3.1 and (58) of Proposition 3.5 need to be modified in a manner analogous to (73) of Theorem 5.1, with z restricted to A_n at a given time n. (In the proof of Proposition 3.5 at the end of the subsection, one also sets $N = 1$ rather than letting $N \to \infty$.) One also needs to modify slightly the definitions of the terms $E_{v',n}(z)$, $F_{v',n}(z)$, and $G_n(z)$, given in (32), because of the double role played by z there, which appears in both the definition of the boundary and the value of the trajectory at time n; in the first instance, one instead employs z' with $|z' - z| < 1$. (This modification does not affect the bounds that follow because of the upper bound in (55).) The reasoning in Section 4 is analogous to that for the nonlattice case, although one needs to put more effort into constructing the function $w(\cdot)$ in (72), because of the restraint $z \in A_n$, at time n, on the analog of the left hand side of (72). (To get around this, one can employ the limit $z_{n_i} \to_{i\to\infty} z$, for any fixed z, which will hold at appropriate times $n_i \to_{i\to\infty} \infty$, and points z_{n_i} with $z_{n_i} \in A_{n_i}$.)

Appendix A

The appendix is devoted to the demonstration of Lemma 2.2 and Lemma 2.3. The special case of Lemma 2.2 with $S_k^{(n)} = S_k$, i.e., $d^{(n)} \equiv 0$, was given in Lemma 2.1. In the first subsection of the appendix, we demonstrate the special case of Lemma 2.3 with $d^{(n)} \equiv 0$ and, for (12), when $j = n$ is also assumed. In the second subsection, we then show, in Corollary A.4, that essentially the same asymptotic behavior holds for $\{S_k\}_{k=0,\ldots,n}$ and $\{S_k^{(n)}\}_{k=0,\ldots,n}$ as $n \to \infty$, when $d^{(n)} = O((\log n)/n)$. Lemmas 2.2 and 2.3 will follow from Lemma 2.1, the special case of Lemma 2.2, and Corollary A.4.

Variants of the following inequality will be used several times in the appendix. Let $\tilde{S}_k$ be the random walk obtained by reversing time, i.e., $\tilde{S}_k = \sum_{i=1}^k (-X_i)$, and define $\tilde{S}_k^{(n)}$ by $\tilde{S}_k^{(n)} = \sum_{i=1}^k -(X_i + d^{(n)})$ for $k \leqslant n$. It is easy to check that, for a given choice of a, b, and y, with $a < b$

and $y \geqslant 0$,

$$\begin{aligned}
&\mathbb{P}(\tilde{S}_n^{(n)} \in (-b,-a), \tilde{S}_k^{(n)} \geqslant -y - h(k \wedge (n-k)) - a \text{ for all } 0 < k < n) \\
&\quad \leqslant \mathbb{P}(S_n^{(n)} \in (a,b), S_k^{(n)} \geqslant -y - h(k \wedge (n-k)) \text{ for all } 0 < k < n) \\
&\quad \leqslant \mathbb{P}(\tilde{S}_n^{(n)} \in (-b,-a), \tilde{S}_k^{(n)} \geqslant -y - h(k \wedge (n-k)) - b \text{ for all } 0 < k < n).
\end{aligned} \tag{75}$$

By first partioning $(-b,-a)$ into smaller intervals and then applying the analog of (75) to each subinterval, one obtains more accurate bounds for the middle probability in (75).

A.1 Demonstration of Lemma 2.3, with $d^{(n)} = 0$

Here, we demonstrate Lemma 2.3 with $d^{(n)} = 0$. We also restrict (12) to the case $j = n$.

For the first two parts of Lemma 2.3, we will employ the following conditioned local central limit theorem, which follows from [Caravenna [6], Theorem 1]. As at the beginning of Section 2, $\{X_k\}_{k=1,2,\ldots}$ denotes independent copies of a mean zero random variable X and $S_n = \sum_{k=1}^n X_k$; X is also assumed to be nonlattice. We denote by $\mathcal{C}_n$ the set where $S_1, S_2, \ldots, S_n > 0$.

Proposition A.2. *Let X be as above, with variance 1. Then,*

$$\sup_{x \in \mathbb{R}_+} |n^{1/2}\, \mathbb{P}(S_n \in [x, x+q) \,|\, \mathcal{C}_n) - qx\mathrm{e}^{-x^2/2n}/n^{1/2}| \to_{n\to\infty} 0 \tag{76}$$

uniformly over q in compact sets in $\mathbb{R}_+$.

Also note that, by [Caravenna [6], (2.6)] (or from older references),

$$\mathbb{P}(\mathcal{C}_n) \sim C/\sqrt{n} \quad \text{for some } C > 0\,. \tag{77}$$

In order to demonstrate (9) and (10) of Lemma 2.3, with $S_k^{(n)} = S_k$, we first demonstrate the limits (78) and (79) below, which are extensions of (76), with the restriction $S_1, S_2, \ldots, S_m \geqslant -y$, for fixed $y \geqslant 1$, replacing that of $S_1, S_2, \ldots, S_m > 0$, for $m = \lfloor n/2 \rfloor$. We then express S_n by "gluing together" independent copies of S_m and $-S_{n-m}$, and apply (78) and (79) to each half to obtain (9) and (10).

Proof of (9) and (10) of Lemma 2.3 for S_k. We may assume WLOG that $\sigma^2(X) = 1$ by rescaling the random walk. In order to demonstrate (9) and (10), we first show that, for appropriate $C > 0$,

$$\sup_{x \geqslant -y} |m\, \mathbb{P}(S_m \in [x, x+q); S_k \geqslant -y \text{ for all } 1 \leqslant k \leqslant m) - \bar{\beta}_y qx\mathrm{e}^{-x^2/2m}/m^{1/2}| \to_{m\to\infty} 0\,, \tag{78}$$

for fixed $y, q > 0$, where $\bar{\beta}_y$ is right continuous and monotone in y, and

$$\bar{\beta}_y/y \to_{y\to\infty} \bar{\beta}^* > 0\,. \tag{79}$$

(Although the latter term in (78) is negative for $x < 0$, it will be negligible for large m and will not affect the limit.)

To show (78) and (79), we introduce the following terminology. Let τ be the first time at which $\{S_k : k \leqslant n\}$ takes its minimum. Abbreviate the event on the left hand side of (78) by

$\Psi^y_m([x,x+q))$, and partition $\Psi^y_m([x,x+q))$ by $\{\Psi^{y,1}_{m,M}([x,x+q)),\Psi^{y,2}_{m,M}([x,x+q))\}$, where $\Psi^{y,1}_{m,M}(\cdot)$ is given by the restriction $\tau\leqslant\lfloor M^2y^2\rfloor$ and $\Psi^{y,2}_{m,M}(\cdot)$ by the restriction $\tau\in[\lfloor M^2y^2\rfloor+1,m-1]$.

Subdividing the interval $(\lfloor M^2y^2\rfloor,m-1]$ into $[\lfloor M^2y^2\rfloor+1,\lfloor m/2\rfloor]$, $[\lfloor m/2\rfloor+1,\lfloor m(1-\varepsilon_{M,y})\rfloor]$, and $[\lfloor m(1-\varepsilon_{M,y})\rfloor+1,m-1]$, with $\varepsilon_{M,y}=1/M^2y^2$, one obtains

$$
\begin{aligned}
\mathbb{P}(\Psi^{y,2}_{m,M}([x,x+q))) &\lesssim \sum_{k=\lfloor M^2y^2\rfloor+1}^{\lfloor m/2\rfloor}\frac{qy^2}{k^{3/2}m}\Big(\frac{x}{m^{1/2}}\mathrm{e}^{-x^2/2m}+o_m(1)\Big)\\
&+\sum_{k=\lfloor m/2\rfloor+1}^{\lfloor m(1-\varepsilon_{M,y})\rfloor}\frac{qy^2}{m^{3/2}(m-k)}\Big(\frac{x}{(m-k)^{1/2}}\mathrm{e}^{-x^2/2(m-k)}+o_{m-k}(1)\Big)+\sum_{k=\lfloor m(1-\varepsilon_{M,y})\rfloor+1}^{m-1}\frac{y^2}{m^{3/2}(m-k)^{1/2}}\\
&\lesssim \frac{qy}{Mm}\Big[\Big(\frac{x}{m^{1/2}}\mathrm{e}^{-x^2/2m}\Big)(1+1/\varepsilon_{M,y}m^{1/2})+1/q+o_m(1)+\frac{M^3y^3}{m^{1/2}}o_{\varepsilon_{M,y}m}(1)\Big]\\
&\lesssim \frac{qy}{Mm}\Big[\Big(\frac{x}{m^{1/2}}\mathrm{e}^{-x^2/2m}\Big)(1+o_m(1))+2/q\Big]\,,
\end{aligned}
$$

with $o_m(1)$ being uniform in $x\geqslant -y$. The first inequality uses (6) of Lemma 2.1 together with the reversed random walk $\tilde{S}_k$ having increments $-X_k$, and (76) and (77).

So, to demonstrate (78), it suffices to show

$$
\lim_{M\to\infty}\limsup_{m\to\infty}\sup_x\Big|m\,\mathbb{P}(\Psi^{y,1}_{m,M}([x,x+q)))-\frac{\bar\beta_y qx}{m^{1/2}}\mathrm{e}^{-x^2/2m}\Big|=0. \tag{80}
$$

Denoting by $\mu_k(\cdot)$ the subprobability measure

$$
\mu_k([w_1,w_2))=\mathbb{P}(S_k<\min_{j\leqslant k-1}S_j;S_k\in[w_1,w_2))\,,
$$

one has, again using (76), (77), and the reversed random walk $\tilde{S}_k$,

$$
\begin{aligned}
m\mathbb{P}(\Psi^{y,1}_{m,M}([x,x+q))) &= m\sum_{k=0}^{\lfloor M^2y^2\rfloor}\int_{-y}^{0}\Psi^0_{m-k}([x-w,x-w+q))\mu_k(dw)\\
&= Cq\Big(\frac{x}{m^{1/2}}\mathrm{e}^{-x^2/2m}+o_m(1)\Big)\sum_{k=0}^{\lfloor M^2y^2\rfloor}\mu_k([-y,0])\,,
\end{aligned} \tag{81}
$$

for $C>0$ as in (77), where $o_m(1)$ is uniform in $x\in\mathbb{R}_+$. Hence, (80) holds, with $\bar\beta_y=C\sum_{k=0}^{\infty}\mu_k([-y,0])$ (which is $\lesssim\sum_{k=1}^{\infty}y^2/k^{3/2}<\infty$), and $\bar\beta_y$ is obviously monotone and right continuous in y.

In order to demonstrate (79), we further partition $\Psi^{y,1}_{m,M}(\cdot)$ into $\Psi^{y,1a}_{m,M}(\cdot)$ and $\Psi^{y,1b}_{m,M}(\cdot)$, with $\Psi^{y,1a}_{m,M}(\cdot)$ denoting the restriction of $\Psi^{y,1}_{m,M}(\cdot)$ to $\tau<\lfloor y^2/M^2\rfloor$ and $\Psi^{y,1b}_{m,M}(\cdot)$ the restriction to $\tau\in[\lfloor y^2/M^2\rfloor,[\lfloor M^2y^2\rfloor]$. We note that, using (76) and (77),

$$
y^{-1}m\,\mathbb{P}(\Psi^{y,1a}_{m,M}([x,x+q)))\lesssim\frac{q}{M}\Big(\frac{x}{m^{1/2}}\mathrm{e}^{-x^2/2m}+o_m(1)\Big)\,, \tag{82}
$$

where $o_m(1)$ is uniform in $x\geqslant -y$ and $y\in\mathbb{R}_+$, and hence

$$
\lim_{M\to\infty}\limsup_{m\to\infty}\sup_{x,y\in\mathbb{R}+}y^{-1}m\,\mathbb{P}(\Psi^{y,1a}_{m,M}([x,x+q)))=0\,.
$$

So, for (79), it suffices to demonstrate

$$\lim_{M\to\infty}\limsup_{y\to\infty}\limsup_{m\to\infty}|y^{-1}m\mathbb{P}(\Psi^{y,1b}_{m,M}([x,x+q)))-\bar\beta^* q\frac{x}{m^{1/2}}\mathrm{e}^{-x^2/2m}|=0 \tag{83}$$

for some $\bar\beta^*>0$ and each x.

One proceeds as in (81), but also applying (76) to $\mu_k(\cdot)$, to obtain

$$\begin{aligned} y^{-1}m\,\mathbb{P}(\Psi^{y,1b}_{m,M}([x,x+q))) &= y^{-1}m\sum_{k=\lfloor y^2/M^2\rfloor}^{\lfloor M^2y^2\rfloor}\int_{-y}^{0}\Psi^0_{m-k}([x-w,x-w+q))\mu_k(dw) \\ &= C^2q(\frac{x}{m^{1/2}}\mathrm{e}^{-x^2/2m}+o_{m,1}(1))[y^{-1}\sum_{k=\lfloor y^2/M^2\rfloor}^{\lfloor M^2y^2\rfloor}k^{-1/2}(1-\mathrm{e}^{-y^2/2k})+o_{k,2}(1)]\,, \end{aligned} \tag{84}$$

where $o_{m,1}(1)$ and $o_{k,2}(1)$ are uniform in $x\geqslant -y$ and $y\in\mathbb{R}_+$. (The bound $o_{k,2}(1)$ requires some estimation: for $k<<y^2$ one employs (77) as an upper bound for $\mu_k([-y,0])$; for $k>>y^2$, one employs the upper bound $C'y^2/k^{3/2}$, for some $C'>0$.) Setting $\ell=k/y^2$, one has

$$y^{-1}\sum_{k=\lfloor y^2/M^2\rfloor}^{\lfloor M^2y^2\rfloor}k^{-1/2}(1-\mathrm{e}^{-y^2/2k})\to\int_0^\infty(1-\mathrm{e}^{-1/2\ell})/\sqrt{\ell}\,d\ell$$

as first $y\to\infty$ and then $M\to\infty$. Together with (84), this implies (83) with $\bar\beta^*=C^2\int_0^\infty(1-\mathrm{e}^{-1/2\ell})/\sqrt{\ell}\,d\ell$.

We now show that (9) and (10) follow from (78) and (79). Denote by S^x_k and $\tilde S^x_k$ the random walks with increments X_k and $-X_k$ started at x, and abbreviate by setting $A^x_{n,y}=\{S^x_k\geqslant -y \text{ for all } 0<k<n\}$, with $A_{n,y}=A^0_{n,y}$. We assume n is even; the argument for odd n is the same.

Setting $m=n/2$, one has, for given $[a_i,b_i)$, with $a_i<b_i$, and $N\in\mathbb{Z}_+$, the decomposition

$$\begin{aligned} &n^{3/2}\,\mathbb{P}(S_n\in[a_i,b_i);A_{n,y}) \\ &\quad= n^{3/2}(\sum_{x\in -y+\mathbb{Z}_+/N}\mathbb{P}(S_m\in[x,x+1/N);A_{m,y})\int_{[x,x+1/N)}\mathbb{P}(S^z_m\in[a_i,b_i);A^z_{m,y})\nu^x_m(dz))\,, \end{aligned} \tag{85}$$

where $\nu^x_m(\cdot)$ is some probability measure over $[x,x+1/N)$. (We will shortly choose $[a_i,b_i)$ to be a small subinterval of $[a,b)$.) As in (75),

$$\mathbb{P}(S^z_m\in[a_i,b_i);A^z_{m,y})\in[\mathbb{P}(\tilde S^{a_i}_m\in(z+a_i-b_i,z];A^{a_i}_{m,y}),\,\mathbb{P}(\tilde S^{b_i}_m\in(z,z+b_i-a_i];A^{b_i}_{m,y})]\,. \tag{86}$$

Plugging (86) into (85), employing (77) and (78), and letting $N\to\infty$, one obtains that the right hand side of (85) equals

$$2^{3/2}(b_i-a_i)\bar\beta_y\hat\beta_{i,y}(\int_0^\infty u^2\mathrm{e}^{-u^2}du+o_m(1))=(1+o_m(1))2^{1/2}(b_i-a_i)\bar\beta_y\hat\beta_{i,y}, \tag{87}$$

where $\hat\beta_{i,y}\in[\bar\beta_{y+a_i},\bar\beta_{y+b_i}]$ and $o_m(1)$ is uniform in i.

Now, partition $[a,b)$ by $\{[a_i,b_i)\}_{i=1,\ldots,I}$ so that $b_i-a_i\leqslant\varepsilon$, for given $\varepsilon>0$. By (85)–(87),

$$n^{3/2}\mathbb{P}(S_m\in[a,b);A_{m,y})=(1+o_m(1))2^{1/2}\bar\beta_y\sum_i(b_i-a_i)\hat\beta_{i,y}\,,$$

which converges to $2^{1/2}\bar{\beta}_y \int_a^b \bar{\beta}_{y+v} dv =: \beta_{y,a,b}$ as $m \to \infty$ and then $\varepsilon \to 0$. This implies (9). Clearly, $\beta_{y,a,b}$ is continuous in a and b, and, since $\bar{\beta}_y$ is right continuous in y, so is $\beta_{y,a,b}$. The limit (10) follows immediately from this and (79), with $\beta^* = 2^{1/2}(\bar{\beta}^*)^2$.

The argument for the analog of (9), but with the restriction $S_k > -y$ instead of $S_k \geqslant -y$, is the same as that for (9), but with $\bar{\beta}_y$ replaced by $\bar{\beta}_y^o = C\sum_{k=0}^{\infty} \mu_k((-y,0])$ and $\beta_{y,a,b}^o := 2^{1/2}\bar{\beta}_y^o \int_a^b \bar{\beta}_{y+v}^o dv$. $\square$

We now demonstrate (11) and (12) of Lemma 2.3 in the case where $S_k^{(n)} = S_k$. Nearly all of the work is devoted to (11); the proof of (12) will follow quickly by re-applying some of the steps for (11). To show (11), we partition time into three intervals, corresponding to the value of τ (as defined above, the first time at which S_k takes its minimum). Define the sets $\Omega_{n,y}$ and $\Omega'_{n,y}$ as in (88) below. We will show that, over each of the three intervals, either $\mathbb{P}(\Omega_{n,y} \cap \{\tau = k\})$ and $\mathbb{P}(\Omega'_{n,y} \cap \{\tau = k\})$ are close, or $\mathbb{P}(\Omega'_{n,y} \cap \{\tau = k\})$ is insignificant. Hence, $\mathbb{P}(\Omega_{n,y})$ and $\mathbb{P}(\Omega'_{n,y})$ will be close.

Proof of (11) and (12) of Lemma 2.3 for S_k. For both (11) and (12), we first restrict ourselves to the case $a \geqslant -1/2$, and afterwards show $a < -1/2$ by reversing the random walk. We first demonstrate (11) and note that the reasoning for the analog of (11), but with "$S_k \geqslant$" replaced by the strict inequality "$S_k >$", will be identical.

Let τ be defined as above. Set $I_y = \mathbb{Z}_+ \cap (y^7, n - y^7)$, $h_n(x) = y^{1/10} + h(x \wedge (n-x))$, and

$$\begin{aligned} \Omega_{n,y} &= \{S_n \in (a,b), S_k \geqslant -y \text{ for all } 1 \leqslant k \leqslant n\}, \\ \Omega'_{n,y} &= \{S_n \in (a,b), S_k \geqslant -y - h_n(k) \text{ for all } 1 \leqslant k \leqslant n\}. \end{aligned} \tag{88}$$

Applying the first part of Lemma 2.1, with $y = 0$, we obtain

$$\begin{aligned} \mathbb{P}(\tau \in I_y; \Omega'_{n,y}) &\leqslant C^2 \sum_{k \in I_y} \sum_{j=0}^{y+h_n(k)} \frac{j(j+b)}{k^{3/2}(n-k)^{3/2}} \\ &\leqslant C^2 b_1 \sum_{k \in I_y} \frac{(y+h_n(k))^3}{k^{3/2}(n-k)^{3/2}} \leqslant 2C_1 C^2 b_1 y^{-1/2} n^{-3/2}, \end{aligned} \tag{89}$$

where $b_1 := b \vee 1$, C is as in Lemma 2.1, and $C_1 > 0$. To prove (11), we will employ (89), together with suitably small upper bounds on $\mathbb{P}(\tau = k; \Omega'_{n,y} \setminus \Omega_{n,y})$ relative to $\mathbb{P}(\tau = k; \Omega_{n,y})$, for every $k \notin I_y$. Using the symmetry of $h_n(\cdot)$, we only consider $k < y^7$, which we break into the subcases $k \leqslant y^{19/10}$ and $k \in (y^{19/10}, y^7)$.

For $k \leqslant y^{19/10}$,

$$\begin{aligned} &\mathbb{P}(\tau = k; \Omega'_{n,y} \setminus \Omega_{n,y}) \\ \leqslant &\mathbb{P}(-y - h_n(k) \leqslant S_k \leqslant -y) \max_{-y^* - h_n(k) \leqslant x \leqslant -y} \mathbb{P}(S_j \geqslant 0 \text{ for all } 1 \leqslant j \leqslant n-k; S_{n-k} \in (a-x, b-x)) \\ \leqslant &C_2 C y^{-2} \cdot (b+y) n^{-3/2} \leqslant 2C_2 C b_1 y^{-1} n^{-3/2}, \end{aligned} \tag{90}$$

where $C_2 > 0$. The second term of the first inequality, on the third line, follows from (6) of Lemma 2.1 with $y = 0$, and the first term follows from the assumption of exponential moments on the random walk (40 moments suffices) together with a moment estimate. On the other hand, by (7) of Lemma 2.1 (for large enough n),

$$\mathbb{P}(\tau = k;\, \Omega_{n,y}) \gtrsim n^{-3/2} \sum_{i=1}^{\sqrt{k}} \frac{i(i+a)}{k^{3/2}} \asymp (1 + b_1/\sqrt{k}) n^{-3/2} \geqslant (1 + b_1/y) n^{-3/2} \,.$$

Combined with (90), it follows that, for $k \leqslant y^{19/10}$,

$$\mathbb{P}(\tau = k;\, \Omega'_{n,y} \setminus \Omega_{n,y}) \lesssim y^{-1} \mathbb{P}(\tau = k;\, \Omega_{n,y}) \,. \tag{91}$$

We next consider $k \in (y^{19/10}, y^7)$. Since, for appropriate C'', the boundary for $\Omega'_{n,y}$ lies at most $C'' y^{1/10}$ below that for $\Omega_{n,y}$ at any time k, another application of (6) implies that

$$\begin{aligned} \mathbb{P}(\tau = k; \Omega'_{n,y} \setminus \Omega_{n,y}) &\leqslant C''' y^{1/10} \cdot k^{-3/2} (y + \log k)(y + \log k + b) n^{-3/2} \\ &\leqslant 4C''' k^{-3/2} y^{11/10} (y + b) n^{-3/2} \,, \end{aligned} \tag{92}$$

for appropriate C'''. Another application of (7), after summation, implies that

$$\mathbb{P}(\tau = k;\, \Omega_{n,y}) \gtrsim (y \wedge \sqrt{k})^2 (y \wedge \sqrt{k} + b) k^{-3/2} n^{-3/2} \,. \tag{93}$$

Combining the last two displays, it follows that, for $k \in (y^{19/10}, y^7)$,

$$\mathbb{P}(\tau = k;\, \Omega'_{n,y} \setminus \Omega_{n,y}) \lesssim y^{-7/10} \mathbb{P}(\tau = k;\, \Omega_{n,y}) \,. \tag{94}$$

Also note that summing (93) over $k \in [y^2, 2y^2]$ implies

$$\mathbb{P}(\Omega_{n,y}) \gtrsim y(y + b) n^{-3/2}, \tag{95}$$

which dominates the bound in (89) for large y.

Together, (89), (91), and (94) imply that

$$\mathbb{P}(\Omega'_{n,y}) \lesssim (1 + y^{-7/10}) \mathbb{P}(\Omega_{n,y}) + b_1 y^{-1/2} n^{-3/2} \,. \tag{96}$$

Along with (95) and (9), this completes the proof of (11).

The inequality (12), for $j = n$ and $a \geqslant -1/2$, follows by applying the first part of Lemma 2.1 (with general y) to bound $\mathbb{P}(\Omega_{n,y})$ from above, and then combining this with the upper bounds (89), (90), and (92), for $\mathbb{P}(\tau = k; \Omega'_{n,y} \setminus \Omega_{n,y})$ over the three ranges of k. Specifically, the bound in (12) is of the same order as that in (6) and, in place of the y coefficients in (6), the bound in (89) contributes the coefficient $b_1 y^{-1/2}$, the bound in (90) contributes the coefficient $b_1 y^{9/10}$ (after summing over the region $k \leqslant y^{19/10}$), and the bound in (92) contributes the coefficient $y^{3/20}(y + b)$ (after summing over the region $k \in (y^{19/10}, y^7)$).

In order to show (11) and (12) for $a < -1/2$, it suffices to also assume that $b \leqslant 0$ and $b - a \leqslant 1/2$. Denoting by $\tilde{\Omega}_{n,y}$ and $\tilde{\Omega}'_{n,y}$ the analogs of the sets $\Omega_{n,y}$ and $\Omega'_{n,y}$, but for the reversed random walk $\tilde{S}_k$ rather than S_k, and using (75), for (11) it is enough to show that

$$\mathbb{P}(\tilde{\Omega}'_{n,y+1/2} \setminus \tilde{\Omega}_{n,y-1/2}) \leqslant \delta_y \mathbb{P}(\tilde{\Omega}_{n,y-1/2}) \tag{97}$$

for the interval $(\tilde{a}, \tilde{b})$, with $\tilde{a} = -b \geqslant 0$ and $\tilde{b} = -a$, where $\delta_y \to 0$ as $y \to \infty$. One can employ the same reasoning as for the case $a \geqslant -1/2$, with the bounds in (89) –(92) holding for different constants in front. (The term $h(k)$ there needs to be increased by 1.) The bound (97) follows after combining these inequalities as in (96). For (12) with $j = n$ and $a < -1/2$, one employs the same reasoning as in the previous paragraph, but for $\tilde{S}_k$ instead of S_k. $\square$

A.2 Demonstration of Lemmas 2.2 and 2.3 for general $d^{(n)}$

Let the random walks $\{S_k\}_{k=0,\dots,n}$ and $\{S_k^{(n)}\}_{k=0,\dots,n}$ be as in Lemmas 2.2 and 2.3. In this subsection, we show that the non-crossing probabilities of the curves there are asymptotically the same as $n \to \infty$, which enables us to show Lemmas 2.2 and 2.3 for general $d^{(n)}$. We will find it convenient to consider two choices of translation terms $d^{(n,i)}$ satisfying $d^{(n,1)} < d^{(n,2)}$ and $|d^{(n,i)}| \leqslant c(\log n)/n$. Also, set $h^{(n,i)}(k) = h(k \wedge (n-k)) - d^{(n,i)}k$, $a^{(n,i)} = a + d^{(n,i)}n$ and $b^{(n,i)} = b + d^{(n,i)}n$.

Lemma A.3. *Let S_k, c, and $d^{(n,i)}$, $i = 1, 2$, be as above. Then, for fixed $\varepsilon > 0$, $\delta > 0$, and appropriate $C > 0$,*

$$\begin{aligned} &\mathbb{P}(S_n \in (a^{(n,2)}, b^{(n,2)}), S_k \geqslant -y - h^{(n,2)}(k) \text{ for all } 0 < k < n) \\ &\quad \leqslant (1+\delta)\mathbb{P}(S_n \in (a^{(n,1)} - \delta, b^{(n,1)} + \delta), S_k > -y - h^{(n,1)}(k) \text{ for all } 0 < k < n) \\ &\qquad + C(y \vee 1)((y+a) \vee 1)/n^{25/16} \end{aligned} \tag{98}$$

for all n, $y \geqslant 0$, and $-y \leqslant a < b < \infty$ with $b - a = \varepsilon$.

We will demonstrate Lemma A.3 at the end of this subsection. In the following corollary, $h^{(n)}(k)$, $a^{(n)}$, and $b^{(n)}$ are the analogs of $h^{(n,i)}(k)$, $a_i^{(n)}$, and $b_i^{(n)}$ with a given $d^{(n)}$.

Corollary A.4. *Suppose that $d^{(n)}$ is as above and $d^{(n)} > 0$ for all n. For any $y \geqslant 1$ and $1 - y \leqslant a < b < \infty$,*

$$\lim_{n\to\infty} \frac{\mathbb{P}(S_n \in (a^{(n)}, b^{(n)}), S_k \geqslant -y - h^{(n)}(k) \text{ for all } 0 < k < n)}{\mathbb{P}(S_n \in (a,b), S_k > -y - h(k \wedge (n-k)) \text{ for all } 0 < k < n)} = 1\,, \tag{99}$$

with the rate of convergence being uniform over all sequences $d^{(n)}$ satisfying $|d^{(n)}| \leqslant c(\log n)/n$ for given $c > 0$. If, instead, $d^{(n)} < 0$ is assumed for all n, then the analog of (99) holds, but with "$S_k >$" replaced by "$S_k \geqslant$" in the denominator. The same limits hold, in each case, if "$S_k \geqslant$" is replaced by "$S_k >$" in the numerator.

Proof. Since the proofs of all four statements are similar, we prove just (99). The upper bound 1 for the limit on the left hand side of (99) is obtained by setting $d^{(n,1)} = 0$ and $d^{(n,2)} = d^{(n)}$ in (98), and then employing the continuity of $\beta_{y,a,b}$ in a and b, together with the lower bound given by (9) for $d^{(n,1)} = 0$, which decays more slowly than $n^{-25/16}$.

The lower bound 1 is obtained by applying (98) to $\tilde{S}_k$ and reversing the roles of $d^{(n,1)}$ and $d^{(n,2)}$. After partitioning (a, b) into subintervals with endpoints a_i and b_i satisfying $0 \leqslant b_i - a_i \leqslant 1/N$, for given $N \in \mathbb{Z}_+$, the analog of the lower bound in (75) implies that

$$\begin{aligned} &\mathbb{P}(\tilde{S}_n \in [-b_i^{(n)}, -a_i^{(n)}), \tilde{S}_k > -y - \hat{h}^{(n)}(k) - a_i \text{ for all } 0 < k < n) \\ &\quad \leqslant \mathbb{P}(S_n \in (a_i^{(n)}, b_i^{(n)}], S_k \geqslant -y - h^{(n)}(k) \text{ for all } 0 < k < n)\,, \end{aligned} \tag{100}$$

where $\hat{h}^{(n)}(k)$ is the analog of $h^{(n)}(k)$, but for the translation $-d^{(n)}$ instead of $d^{(n)}$. (For the terminal subinterval with $b_i = b$, we instead use $(a_i^{(n)}, b_i^{(n)})$.) On the other hand, by (98), for any $\delta > 0$,

$$\liminf_{n\to\infty} \frac{\mathbb{P}(\tilde{S}_n \in [-b_i^{(n)}, -a_i^{(n)}), \tilde{S}_k > -y - \hat{h}^{(n)}(k) - a_i \text{ for all } 0 < k < n)}{\mathbb{P}(\tilde{S}_n \in [-b_i + \delta, -a_i - \delta), \tilde{S}_k \geqslant -y - h(k \wedge (n-k)) - a_i \text{ for all } 0 < k < n)} \geqslant 1\,, \tag{101}$$

with convergence being uniform over all sequences satisfying $|d^{(n)}| \leqslant c(\log n)/n$. Application of the analog of the upper bound in (75) implies that the denominator in (101) is at least the denominator of (99), but with the interval (a, b) there replaced by $(a_i + \delta, b_i - \delta]$ and the term $h(k \wedge (n-k))$ replaced by $h(k \wedge (n-k)) - 1/N$. Together with (100), (101), and the continuity of $\beta^o_{y,a,b}$ in a and b, this implies

$$\liminf_{n\to\infty} \frac{\mathbb{P}(S_n \in (a_i^{(n)}, b_i^{(n)}], S_k \geqslant -y - h^{(n)}(k) \text{ for all } 0 < k < n)}{\mathbb{P}(S_n \in (a_i, b_i], S_k > -y - h(k \wedge (n-k)) + 1/N \text{ for all } 0 < k < n)} \geqslant 1 \tag{102}$$

uniformly in $d^{(n)}$. Summation of the probabilities over all i in the numerator and in the denominator of (102), and then letting $N \to \infty$ gives the lower bound in (99), since $\beta^o_{y,a,b}$ is left continuous in y. $\square$

We next show that both Lemma 2.2 and Lemma 2.3 follow from Corollary A.4 and the restricted version of Lemma 2.3 with $d^{(n)} \equiv 0$. Recall that the random walk $\{S_k^{(n)}\}_{k=0,\ldots,n}$ is defined in Lemma 2.2 and satisfies $\mathbb{E}^{(n)}(S_k^{(n)}) = 0$.

Proof of Lemmas 2.2 and 2.3. Let $\hat{a}^{(n)}$, $\hat{b}^{(n)}$, and $\hat{h}^{(n)}(k)$ be the analogs of $a^{(n)}$, $b^{(n)}$, and $h^{(n)}(k)$, but for the translation $-d^{(n)}$ instead of $d^{(n)}$. In order to show (9) –(11) of Lemma 2.3, we note that

$$\begin{aligned}
&\mathbb{P}^{(n)}(S_n^{(n)} \in (a,b), S_k^{(n)} \geqslant -y - h(k \wedge (n-k)) \text{ for all } 0 < k < n)\\
&\quad = \mathbb{P}^{(n)}(S_n \in (\hat{a}^{(n)}, \hat{b}^{(n)}), S_k \geqslant -y - \hat{h}^{(n)}(k) \text{ for all } 0 < k < n)\\
&\quad = \gamma^{(n)}_{a,b,d,y,h} \mathbb{P}(S_n \in (\hat{a}^{(n)}, \hat{b}^{(n)}), S_k \geqslant -y - \hat{h}^{(n)}(k) \text{ for all } 0 < k < n),
\end{aligned} \tag{103}$$

where $\gamma^{(n)}_{a,b,d,y,h}$ is bounded above by $\mathrm{e}^{-\theta^{(n)}\hat{a}^{(n)}}$ and bounded below by $\mathrm{e}^{-\theta^{(n)}\hat{b}^{(n)}}/\mathbb{E}(\mathrm{e}^{-\theta^{(n)}S_n})$, and $\theta^{(n)}$ is as in (8). Since $|d^{(n)}| \leqslant c(\log n)/n$, one can check that $|\theta^{(n)}| \leqslant c'(\log n)/n$ for some $c' > 0$. Since a and b are fixed, $\mathbb{E}(S_n) = 0$, and S_n has exponential moments, it follows that both bounds converge to 1 as $n \to \infty$ uniformly in $d^{(n)}$. Together with (99) (with $-d^{(n)}$ in place of $d^{(n)}$) and the restricted versions of (9) –(11) for $d^{(n)} = 0$, this implies (9) –(11) for general $d^{(n)}$.

We next show (12) of Lemma 2.3, first restricting ourselves to the case where $j = n$. Using the above bound on $|\theta^{(n)}|$, when $a > n/\log n$, it is not difficult to show (12) by ignoring the boundary and using the same moderate deviation inequality as in (20). When $a \leqslant n/\log n$, we first consider the case where $d^{(n)} < 0$. Then (103) again holds and, using the above bound on $|\theta^{(n)}|$,

$$\begin{aligned}
&\mathbb{P}^{(n)}(S_n^{(n)} \in (a,b), S_k^{(n)} \geqslant -y - h(k \wedge (n-k)) \text{ for all } 0 < k < n)\\
&\quad \lesssim \mathbb{P}(S_n \in (\hat{a}^{(n)}, \hat{b}^{(n)}), S_k \geqslant -y - \hat{h}^{(n)}(k) \text{ for all } 0 < k < n).
\end{aligned} \tag{104}$$

Together with (98) (with $-d^{(n)}$ in place of $d^{(n)}$) and the restricted version of (12), this implies the general version of (12) for $d^{(n)} < 0$. When $d^{(n)} > 0$, we employ the reversed random walk $\tilde{S}_k$ and note that, as in (75),

$$\begin{aligned}
&\mathbb{P}^{(n)}(S_n^{(n)} \in (a,b), S_k^{(n)} \geqslant -y - h(k \wedge (n-k)) \text{ for all } 0 < k < n)\\
&\quad \leqslant \mathbb{P}^{(n)}(\tilde{S}_n^{(n)} \in (-b,-a), \tilde{S}_k^{(n)} \geqslant -y - h(k \wedge (n-k)) - b \text{ for all } 0 < k < n)
\end{aligned} \tag{105}$$

for any a, b, and y. Since $-d^{(n)} < 0$, one can apply (104), with $\tilde{S}_k^{(n)}$ in place of $S_k^{(n)}$, and then reason as above.

The demonstration of (12) for general $j \geqslant n/2$ requires just a slight modification of the argument in the previous paragraph. Since $h(\cdot) \geqslant 0$ is increasing and concave, we have

$$h(k \wedge (n-k)) \leqslant 2h(k \wedge (j-k)) + (k/j)h(n-j) \quad \text{for all } 0 \leqslant k \leqslant j\,. \tag{106}$$

(This is the only point in the proof of (12) where the monotonicity and concavity of $h(\cdot)$ are used.) Indeed, the inequality in (106) holds trivially when $k \leqslant j/2$ and it holds with equality when $k = j$; for $j/2 < k < j$, it follows from the inequalities

$$h(k \wedge (n-k)) - h(n-j) \leqslant h(n-k) - h(n-j) \leqslant h(j-k)$$

and

$$\frac{h(j-k)}{j-k} \geqslant \frac{h(j)}{j} \geqslant \frac{h(n-j)}{j}\,,$$

where $j \geqslant n/2$ was used in the last inequality (multiply the latter display by $j-k$ and add $h(j-k)$ to both sides, before applying it to the preceding display). Setting $d_2^{(n)} = 2C'(\log\lfloor n/2 \rfloor)/n$ and $h_2^{(n,j)}(k) = 2h(k \wedge (j-k)) + d_2^{(n)}k$, it follows that the probability on the left hand side of (12) is, for any $j \in [n/2, n]$, at most

$$\mathbb{P}^{(n)}(S_j^{(n)} \in (a,b), S_k^{(n)} \geqslant -y - h_2^{(n,j)}(k) \text{ for all } 0 < k < j)\,. \tag{107}$$

One can now apply the same reasoning as for the left hand side of (105), but stopping the process $S_k^{(n)}$ at time j instead of n and applying the additional tilting induced by $d_2^{(n)}$. The bound here, as above, is uniform up to the choice of c. This concludes the proof of Lemma 2.3.

The equation (6) of Lemma 2.2 is a special case of (12), with $h(k) \equiv 0$. So, to prove Lemma 2.2, we need only still to show (7). Supposing that $d^{(n)} > 0$, we again apply (103). Since $a \leqslant \sqrt{n}$, $\gamma_{a,b,d,0,0} \geqslant C'$ for some constant $C' > 0$. Using (98), with $-d^{(n)}$ in place of $d^{(n,1)}$ and $d^{(n,2)} \equiv 0$, (7) follows from this and the special case of (7) with $d^{(n)} = 0$. For $d^{(n)} < 0$, similar reasoning holds after employing the reversed random walk $\tilde{S}_k^n$ and (75). $\square$

We now demonstrate Lemma A.3.

Proof of Lemma A.3. In order to show (98), we decompose the interval $[0, n]$ into three parts, $[0, e_1]$, $[e_1, e_2]$, and $[e_2, n]$, where $e_1 = n - \lfloor n^{1/4} \rfloor - \lfloor n^{1/12} \rfloor$ and $e_2 = n - \lfloor n^{1/4} \rfloor$. (We could choose larger powers of n, e.g., $n^{3/4}$ instead of $n^{1/4}$ and $n^{1/4}$ instead of $n^{1/12}$. However, these smaller powers of n are required in the lattice setting of Section 5, and so we also employ them here for conformity.) To compare the probabilities on the left and right hand sides of (98), we will proceed in essence as follows:

- (A) Over the first interval $[0, e_1]$, compare the probabilities that the same path on each of the two sides always lies above the corresponding boundary. Since $d^{(n,2)} \geqslant d^{(n,1)}$, the boundary on the left hand side is higher than that on the right hand side, and so the inequality is automatic on this interval.
- (B) Over the middle interval $[e_1, e_2]$, compare a path on the right hand side with the path on the left hand side that at time e_1 takes the same value, but at time e_2 is larger by $\ell_{1,2}^{(n)} := (d^{(n,2)} - d^{(n,1)})e_2$. We will then employ a version of the local central limit theorem at time e_2 to compare probabilities for corresponding paths; since $d_{1,2}^{(n)} := d^{(n,2)} - d^{(n,1)}$ is of order $(\log n)/n$, the probabilities will be approximately the same.

• (C) Over the last interval $[e_2, n]$, compare paths that are identical over the interval, except for the translation $\ell_{1,2}^{(n)}$ inherited from the middle interval. After also translating the boundaries by $\ell_{1,2}^{(n)}$, the boundary on the left hand side lies above the boundary on the right hand side. If the value taken at time n by the path on the left hand side lies in $(a^{(n,2)}, b^{(n,2)})$, the value on the right hand side lies in $(a^{(n,1)}, b^{(n,1)} + \delta)$ if $\delta \geqslant d_{1,2}^{(n)}(n - e_2) = d_{1,2}^{(n)}\lfloor n^{1/4}\rfloor = o(1)$.
• The inequality (98) will then follow by combining (A)–(C) using the Markov property of random walk.

As indicated above, the inequality that is employed for (A) is immediate: for $a_1 \leqslant b_1$,

$$\begin{aligned} &\mathbb{P}(S_{e_1} \in (a_1, b_1), S_k \geqslant -y - h^{(n,2)}(k) \text{ for all } 0 < k \leqslant e_1) \\ &\qquad \leqslant \mathbb{P}(S_{e_1} \in (a_1, b_1), S_k > -y - h^{(n,1)}(k) \text{ for all } 0 < k \leqslant e_1)\,. \end{aligned} \tag{108}$$

The inequality that is employed for (C) is also immediate: for fixed $\delta \geqslant d_{1,2}^{(n)}\lfloor n^{1/4}\rfloor$ and any $a \leqslant b$ and x_2,

$$\begin{aligned} &\mathbb{P}(S_{\lfloor n^{1/4}\rfloor} \in (a^{(n,2)} - \ell_{1,2}^{(n)}, b^{(n,2)} - \ell_{1,2}^{(n)}), S_k \geqslant -y - h^{(n,2)}(k') - x_2 - \ell_{1,2}^{(n)},\ 0 < k \leqslant \lfloor n^{1/4}\rfloor) \\ &\qquad \leqslant \mathbb{P}(S_{\lfloor n^{1/4}\rfloor} \in (a^{(n,1)}, b^{(n,1)} + \delta), S_k > -y - h^{(n,1)}(k') - x_2\,,\ 0 < k \leqslant \lfloor n^{1/4}\rfloor), \end{aligned} \tag{109}$$

with $k' := \lfloor n^{1/4}\rfloor - k$, since $\ell_{1,2}^{(n)} = d_{1,2}^{(n)} n - d_{1,2}^{(n)}\lfloor n^{1/4}\rfloor$ with $h^{(n,2)}(k') + \ell_{1,2}^{(n)} \leqslant h^{(n,1)}(k')$ on the interval.

We still need to obtain bounds corresponding to (B), for which we first need to restrict the range of the values at the initial and terminal points of the interval and to obtain several related bounds. We will show that, for given $\varepsilon > 0$, $\delta > 0$, and large enough n,

$$\mathbb{P}(S_{\lfloor n^{1/12}\rfloor} \in (a_2 + \ell_{1,2}^{(n)}, b_2 + \ell_{1,2}^{(n)})) \leqslant (1+\delta)\mathbb{P}(S_{\lfloor n^{1/12}\rfloor} \in (a_2, b_2)) \tag{110}$$

for all a_2 and b_2 satisfying $\varepsilon = b_2 - a_2$ and $|a_2| \leqslant n^{1/16}$. On the other hand, it follows from a moderate deviation estimate that, for any M and large enough C,

$$\mathbb{P}(|S_{k^{1/12}}| > \tfrac{1}{2} n^{1/16} \text{ for some } 0 < k \leqslant n^{1/12}) \leqslant C n^{-M} \tag{111}$$

for all n. By applying (12) with $j = n$ in the case $d^{(n)} \equiv 0$, over both $[0, e_1]$ and $[e_1, n]$, it moreover follows that

$$\begin{aligned} &\mathbb{P}(S_n \in (a^{(n,2)}, b^{(n,2)}), S_{e_1} \leqslant n^{1/12}, S_k \geqslant -y - h^{(n,2)}(k) \text{ for all } 0 < k < n) \\ &\qquad \leqslant C_1(y \vee 1)((y + a^{(n,2)}) \vee 1) n^{-13/8} \leqslant C_1'(y \vee 1)((y + a) \vee 1) n^{-25/16} \end{aligned} \tag{112}$$

for large enough C_1 and C_1' depending on c, and any n, $y \geqslant 0$, and a and b with $b - a > 0$ fixed.

The inequality (98) follows from the inequalities (108)–(112): Dividing $[0, n]$ into the three subintervals $[0, e_1]$, $[e_1, e_2]$, and $[e_2, n]$ defined earlier, we will apply the Markov property to S_k, letting $S_{e_1} \in [x_1, x_1 + dx)$ and $S_{e_2} \in [x_2, x_2 + dx)$, for given x_1 and x_2, and then integrating over x_1 and x_2. On account of (112), we may restrict attention to $x_1 \geqslant n^{1/12}$. Restarting the process S_k at time e_1 and applying (111), we may also disregard paths that cross the boundary over $[e_1, e_2]$, which enables us to apply (110). Together with (108)–(109), this implies (98).

To complete the proof of (98), we still need to verify (110). We introduce two variables, $S^{(n,1)}$ and $S^{(n,2)}$, with $S^{(n,1)} = S_{\lfloor n^{1/12}\rfloor} - a_2$ and $S^{(n,2)} = S_{\lfloor n^{1/12}\rfloor} - a_2 - \ell_{1,2}^{(n)}$, and two measures $\mathbb{P}^{(n,1)}$

and $\mathbb{P}^{(n,2)}$, defined as in (8) by tilting the measure $\mathbb{P}$ with appropriate $\theta^{(n,1)}$ and $\theta^{(n,2)}$, so that the corresponding expectations satisfy $\mathbb{E}^{(n,2)}(S^{(n,2)}) = \mathbb{E}^{(n,1)}(S^{(n,1)}) = 0$. Reasoning as in (103), one can show that

$$\begin{aligned}\mathbb{P}(S_{\lfloor n^{1/12}\rfloor} \in (a_2 + \ell_{1,2}^{(n)}, b_2 + \ell_{1,2}^{(n)}) &= \gamma_{a_2,b_2,d}^{(n)} \mathbb{P}^{(n,2)}(S^{(n,2)} \in (0,\varepsilon)),\\ \mathbb{P}(S_{\lfloor n^{1/12}\rfloor} \in (a_2, b_2) &= \gamma_{a_2,b_2}^{(n)} \mathbb{P}^{(n,1)}(S^{(n,1)} \in (0,\varepsilon)),\end{aligned} \tag{113}$$

with $\gamma_{a_2,b_2}^{(n)}, \gamma_{a_2,b_2,d}^{(n)}$ satisfying $1 - Cn^{-1/50} \leqslant \gamma_{a_2,b_2,d}^{(n)}/\gamma_{a_2,b_2}^{(n)} \leqslant 1 + Cn^{-1/50}$, for appropriate $C > 0$ depending on c but not on a_2. (Note that $\ell_{1,2}^{(n)} \leqslant C' \log n$, $|\theta^{(n,i)}| \leqslant C' n^{-1/48}$, and $|\theta^{(n,2)} - \theta^{(n,1)}| \leqslant C'(\log n)n^{-1/12}$ for appropriate C', and so, for $|a_2| \leqslant n^{1/16}$,

$$\exp\{\theta^{(n,2)}|(a_2 + \ell_{1,2}^{(n)}) - \theta^{(n,1)}a_2|\} \leqslant C''(\log n)n^{-1/48}$$

for appropriate C'', with C' and C'' depending on c.) Therefore, to demonstrate (110), it suffices to show that

$$\mathbb{P}^{(n,2)}(S^{(n,2)} \in (0,\varepsilon))/\mathbb{P}^{(n,1)}(S^{(n,1)} \in (0,\varepsilon)) \to 1 \qquad \text{as } n \to \infty, \tag{114}$$

uniformly in $|a_2| \leqslant n^{1/16}$.

In order to show (114), we will apply the central limit asymptotic expansion given in Feller [[9], Theorem 16.4.1], which states that

$$F_n(x) - \mathfrak{N}(x) - \frac{\mu_3}{6\sigma^3\sqrt{n}}(1 - x^2)\mathfrak{n}(x) = o(\frac{1}{\sqrt{n}}), \tag{115}$$

with $F_n(x) := F^{*n}(x\sigma\sqrt{n})$, where $F^{*n}(\cdot)$ is the n-fold convolute of a nonlattice, mean 0 random variable with variance σ^2 and third moment μ_3, and $\mathfrak{N}(\cdot)$ and $\mathfrak{n}(\cdot)$ denote the distribution and density of a standard normal random variable. It is not difficult to show that the variance and third moments of the summands $X_k^{(n,i)}$ of $S^{(n,i)}$ (with respect to $\mathbb{P}^{(n,i)}$), $i = 1, 2$, converge uniformly over $|a_2| \leqslant n^{1/16}$ to the variance and third moment of X_k. One can use this to show that the error on the right hand side of (115) is uniform when the formula is applied to $S^{(n,i)}$ over this range of a_2; the limit (114) will then follow from (115). We summarize the arguments for these steps in the next two paragraphs.

Let $\mu_k^{(i)}$ denote the kth moment of $X_k^{(n,i)}$. Since $|\theta^{(n,i)}| \leqslant C' n^{-1/48}$, it is not difficult to show that

$$|\mu_k^{(i)} - \mu_k| \leqslant C_k n^{-1/48}, \qquad i = 1, 2, \tag{116}$$

for any k and appropriate C_k, for all $|a_2| \leqslant n^{1/16}$. The bound (116) will be used to adapt the proof of Theorem 16.4.1 to our setting.

The argument for Theorem 16.4.1 consists of comparing $F_n(\cdot)$ with the remaining terms on the left hand side of (115) (denoted by $G(x)$ in Feller [9]) by applying the inversion inequality (16.4.4) of [9], which integrates the difference of their respective characteristic functions over an appropriate interval. In the proof of Theorem 16.4.1, there are three contributions to the error term on the right hand side of (115): (a) a truncation error ε that arises by restricting the interval of integration to that in (16.4.4) and that depends on $G(\cdot)$; (b) a rapidly decreasing error term in n that depends on the lower bound of the difference between 1 and the maximum of the absolute value of the characteristic function of $F_n(\cdot)$, on an appropriately chosen subinterval of the interval of integration in (16.4.4); and (c) a three-term Taylor expansion for the characteristic function of

$F_n(\cdot)$ that is compared with the characteristic function of $G(\cdot)$ over a third subinterval. Employing (116), with $k = 2, 3$, it is not difficult to check that one obtains uniform bounds on the errors in (a)–(c) for $S^{(n,i)}$ over $|a_2| \leqslant n^{1/16}$. Hence, the analog of (115) will also hold, with the uniform error bound $o(1/\sqrt{n})$ over $|a_2| \leqslant n^{1/16}$. The limit (114) follows from this error bound and another application (116), which is applied to the left hand side of (115) . This completes the demonstration of (110) and hence the proof of Lemma A.3. □

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