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A LAW OF LARGE NUMBERS FOR RANDOM WALKS IN RANDOM MIXING ENVIRONMENTS

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We prove a law of large numbers for a class of ballistic, multidimensional random walks in random environments where the environment satisfies appropriate mixing conditions, which hold when the environment is a weak mixing field in the sense of Dobrushin and Shlosman. Our result holds if the mixing rate balances moments of some random times depending on the path. It applies in the nonnestling case, but we also provide examples of nestling walks that satisfy our assumptions. The derivation is based on an adaptation, using coupling, of the regeneration argument of Sznitman and Zerner.

1. Introduction and statement of results. Let S denote the $2d$-dimensional simplex, and set $\Omega = S^{\mathbb{Z}^d}$. We consider Ω as an "environment" for the random walk that we define in (1.1). We denote by $\omega(z,\cdot) = \{\omega(z, z+e)\}_{e\in\mathbb{Z}^d,|e|=1}$ the coordinate of $\omega \in \Omega$ corresponding to $z \in \mathbb{Z}^d$.

Conditional on a realization $\omega \in \Omega$, we define the Markov chain $\{X_n\} = \{X_n; n \geq 0\}$ with state space $\mathbb{Z}^d$ started at $z \in \mathbb{Z}^d$ as the process satisfying $X_0 \equiv z$ and

$$P_\omega^z(X_{n+1} = x + e | X_n = x) = \omega(x, x+e), \qquad e \in \mathbb{Z}^d, |e| = 1. \tag{1.1}$$

The law of the random walk in random environment (RWRE) $\{X_n\}$ under this transition kernel, denoted $P_\omega^z(\cdot)$, depends on the environment $\omega \in \Omega$ and is called the *quenched* law of $\{X_n\}$.

Let P be a probability measure on Ω, stationary and ergodic with respect to the shifts in $\mathbb{Z}^d$. With a slight abuse of notations, we write $\mathbb{P}^z = P \otimes P_\omega^z$ for both the joint law on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ of $\{X_n\}_n$ and ω, and for its marginal on $(\mathbb{Z}^d)^{\mathbb{N}}$; in the latter case, we refer to it as the *annealed* law of the process $\{X_n\}$. We will denote by $\mathbb{E}^z = E_{\mathbb{P}^z}$, $E_\omega^z = E_{P_\omega^z}$ the expectations corresponding to $\mathbb{P}^z$, P_ω^z, respectively. Considering the annealed law rather than the quenched law, one takes advantage of the smoothing from the ω-average, but the Markov property is lost.

The RWRE with $d = 1$ is by now well studied, see [21] for a recent review. The multidimensional case is much less understood. A crucial simplification in the case $d = 1$ is that a nearest neighbor random walk tending to $+\infty$ has to visit all

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positive sites; then one can use ergodic theorem to smooth the environment out. In constrast, it is not clear how to take advantage of ergodicity of the medium in dimension $d > 1$. When P is a product measure, laws of large numbers and central limit theorems for $\{X_n\}$ were derived in an impressive sequence of papers [18], [15], [16], focusing on the ballistic regime. Our goal in this paper is to present a technique, based on an appropriate coupling, for extending some of the results of [18], [15] to the case where P is not a product measure.

A motivation for this question relates to an example in [22], Proposition 2, of a RWRE in dimension 2 with the nonstandard asymptotics: $\mathbb{P}^o(\lim_n X_n/n = w) = 1/2$, $\mathbb{P}^o(\lim_n X_n/n = -w) = 1/2$ for some nonzero vector w. There, the environment is ergodic (but not mixing), and the RWRE is strictly elliptic (but not uniformly). In view of this example, it seems important to clarify the specific role of the various assumptions used to get a standard law of large numbers, for example, independent, identically distributed (i.i.d.) environment, uniform ellipticity and drift-condition in [18].

We work in the context of *ballistic* walks, that is, walks X_n which tend to infinity in some direction $\ell \in \mathbb{R}^d \setminus \{0\}$, with a nonvanishing speed. Conditions for the walk to escape to infinity in the direction ℓ have been explored by Kalikow [7] two decades ago, see assumption ($\mathcal{A}4$). With an i.i.d. environment, Sznitman and Zerner [18] introduced a sequence of *regeneration* times and showed, roughly, that the environments traversed by the walk between regeneration times, together with the path of the walk, form a sequence of i.i.d. random vectors under the *annealed* law $\mathbb{P}^o$. This allowed them to derive a law of large numbers under Kalikow's condition by studying the tail properties of these regeneration times.

In [21] a *coupling technique* was introduced that immediately allows one to adapt the construction of regeneration times to the case of measures P which are L-dependent, that is, such that coordinates of the environment at distance larger than L, some fixed deterministic L, are independent. This coupling covers, in particular, the setup in [14], that deals with a particular 1-dependent environment (note, however, that we do not attempt to recover here all the results of [14]). Intuitively, the coupling idea is that, due to the uniform ellipticity property, the walk has positive probability for travelling the L next steps whithout looking at the environment. One then is reduced to the study of tails of the regeneration times.

Our purpose in this work is to further modify the construction of regeneration times and allow for more general types of mixing conditions on P. A complication arises from the destruction of the renewal structure due to the dependence in the environment: in fact, the environments between regeneration times need not even define a stationary sequence any more. Our approach is based on suitably approximating this sequence by an i.i.d. sequence, namely, the so-called splitting representation [19].

Another approach to nonproduct measures P, but with a rather mild dependence structure, has been proposed in [8] and [9]. A comparison between their results and

ours is presented at the end of the article. Our results here cover natural examples of environment distributions as Gibbs measures in a mixing regime.

We now turn to a description of our results. We deal with environments subject to various mixing conditions, and it is appropriate to consider closed positive *cones*. For $\ell \in \mathbb{R}^d \setminus \{0\}$, $x \in \mathbb{R}^d$ and $\zeta \in (0,1)$, define the cone of vertex x, direction ℓ and angle $\cos^{-1}(\zeta)$,

$$C(x,\ell,\zeta) = \{y \in \mathbb{R}^d;\ (y-x)\cdot\ell \geq \zeta|y-x||\ell|\}. \tag{1.2}$$

Note that for $\zeta = 0$, this is just a usual half-space. (Throughout the paper $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, and $|\ell|_1 = \sum_i |\ell_i|$ the ℓ_1-norm.)

In the sequel we fix an $\ell \in \mathbb{R}^d \setminus \{0\}$ such that ℓ has *integer coordinates*. With $\operatorname{sgn}(0) = 0$ and $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{Z}^d$, let

$$\mathcal{E}_{\bar{\varepsilon}} = \{\operatorname{sgn}(\ell_i)e_i\}_{i=1}^d \setminus \{0\}. \tag{1.3}$$

Throughout, we make the following two assumptions on the environment:

ASSUMPTION 1.1.

($\mathcal{A}1$) P is stationary and ergodic, and satisfies the following mixing condition on ℓ-cones: for all positive ζ small enough there exists a function $\phi(r) \to_{r\to\infty} 0$ such that any two events A, B with $P(A) > 0$, $A \in \sigma\{\omega_z;\ z\cdot\ell \leq 0\}$ and $B \in \sigma\{\omega_z;\ z \in C(r\ell,\ell,\zeta)\}$, it holds that

$$\left|\frac{P(A\cap B)}{P(A)} - P(B)\right| \leq \phi(r|\ell|).$$

($\mathcal{A}2$) P is elliptic and uniformly elliptic with respect to ℓ: $P(\omega(0,e) > 0;\ |e| = 1) = 1$, and there exists a $\kappa > 0$ such that

$$P\left(\min_{e\in\mathcal{E}_{\bar{\varepsilon}}} \omega(0,e) \geq 2\kappa\right) = 1.$$

Note that ($\mathcal{A}1$) is equivalent to

$$|E(fg) - EfEg| \leq \phi(r|\ell|)\|f\|_1\|g\|_\infty \tag{1.4}$$

for all bounded functions, f being $\sigma\{\omega_z;\ z\cdot\ell \leq 0\}$-measurable and g being $\sigma\{\omega_z;\ z \in C(r\ell,\ell,\zeta)\}$-measurable. Properties of the type ($\mathcal{A}1$) are generically called ϕ-mixing or uniform mixing ([5], Section 1.1). When ($\mathcal{A}1$) holds for $\zeta = 0$, it is shown in [3] that the random field is finite-dependent, that is, there exists a finite L such that $\{\omega_x, x \in A\}$ and $\{\omega_y, y \in B\}$ are independent whenever $d(A,B) > L$. But this condition is too restrictive, and we give, in Section 4, examples of environments satisfying the mixing property ($\mathcal{A}1$), but not ϕ-mixing (with $\zeta = 0$). In most applications one can find such a ϕ not depending on $\ell \in \mathbb{R}^d \setminus \{0\}$, for which the condition holds for all ℓ.

Next, we turn to conditions on the environment ensuring the ballistic nature of the walk. In order to do so, we introduce (a natural extension of) Kalikow's Markov chain [7] as follows. Let U be a finite, connected subset of $\mathbb{Z}^d$, with $0 \in U$, let

$$\mathcal{F}_{U^c} = \sigma\{\omega_z : z \notin U\},$$

and define on $U \cup \partial U$ an auxiliary Markov chain with transition probabilities

$$\hat{P}_U(x, x+e) = \begin{cases} \dfrac{\mathbb{E}^o[\sum_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n = x\}} \omega(x, x+e) | \mathcal{F}_{U^c}]}{\mathbb{E}^o[\sum_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n = x\}} | \mathcal{F}_{U^c}]}, & x \in U, |e| = 1, \\ 1, & x \in \partial U, e = 0, \end{cases} \tag{1.5}$$

where $T_{U^c} = \min\{n \geq 0 : X_n \in \partial U\}$ [note that the expectations in (1.5) are finite due the Markov property and ℓ-ellipticity]. The transition kernel $\hat{P}_U$ weights the transitions $x \mapsto x + e$ according to the occupation time of the vertex x before exiting U. Define the Kalikow drift as $\hat{d}_U(x) = \sum_{|e|=1} e \hat{P}_U(x, x+e)$, with the RWRE's drift at x defined by $d(x, \omega) = \sum_{|e|=1} e\omega(x, x+e)$. Note that, unlike in the i.i.d. case, the Kalikow drift, as well as Kalikow's chain itself, here is random because it depends on the environment outside of U. This new Markov chain is useful because of the following property [7], which remains valid in our non-i.i.d. setup. Since U is finite and the walk is uniformly elliptic in the direction ℓ, under both $\hat{P}_U$ and $\mathbb{P}^o(\cdot | \mathcal{F}_{U^c})$, the exit time $T_{U^c} = \inf\{n \geq 0;\ X_n \in U^c\}$ is finite, and

$$X_{T_{U^c}} \text{ has the same law under } \hat{P}_U \text{ and } \mathbb{P}^o(\cdot | \mathcal{F}_{U^c}). \tag{1.6}$$

We consider one of following *drift* conditions, which ensure a ballistic behavior for the walk:

ASSUMPTION 1.2.

($\mathcal{A}3$) Kalikow's condition: There exists a $\delta(\ell) > 0$ deterministic such that

$$\inf_{U, x \in U} \hat{d}_U(x) \cdot \ell \geq \delta(\ell), \qquad P\text{-a.s.}$$

(The infimum is taken over all any connected finite subsets of $\mathbb{Z}^d$ containing 0.)

($\mathcal{A}4$) Nonnestling: There exists a $\delta(\ell) > 0$ such that

$$d(x, \omega) \cdot \ell \geq \delta(\ell), \qquad P\text{-a.s.}$$

We will always assume ($\mathcal{A}1$) and ($\mathcal{A}2$), and (except in the beginning of Section 3) also one of ($\mathcal{A}3$) or the stronger ($\mathcal{A}4$). Clearly, ($\mathcal{A}4$) implies that ($\mathcal{A}3$) holds with the same ℓ and $\delta(\ell)$. An inspection of the proof in [7] reveals that, under ($\mathcal{A}3$), the conclusion $\mathbb{P}^o(\liminf_{n\to\infty} X_n \cdot \ell = \infty) = 1$ remains valid in our non-i.i.d. setup (see, e.g., the exposition in [21]). Note that the requirement from ℓ to possess integer coordinates is not a restrictive one, for given an $\ell \in \mathbb{R}^d \setminus \{0\}$

satisfying either ($\mathcal{A}3$) or ($\mathcal{A}4$), one finds by continuity an ℓ with integer coordinates satisfying the same. We make this restriction for the convenience of defining the path $\bar{\varepsilon}$ in (2.2).

The statement of our fundamental result, Theorem 3.4, involves certain *modified regeneration times*, which are introduced in Section 2. The basic condition in this theorem is a trade-off between moments for these "regeneration times" and rate of mixing for the environment. A corollary can be readily stated here: it deals with the nonnestling case and does not require any assumption on the mixing rate other than $\phi \to 0$.

COROLLARY 1.3. *Assume* ($\mathcal{A}1$), ($\mathcal{A}2$) *and* ($\mathcal{A}4$) *hold for some* $\ell \in \mathbb{Z}^d \setminus \{0\}$. *Then there exists a deterministic* $v \neq 0$ *such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \qquad \mathbb{P}^o\text{-a.s.}$$

Our fundamental Theorem 3.4 does not restrict attention to nonnestling cases, but applies also when only Kalikow's condition holds. A class of nestling examples based on Ising-like environments, satisfying the conditions of Theorem 3.4 and, hence, the law of large numbers, is provided in Theorem 5.4. Dealing with nestling walks is much more delicate than with nonnestling ones due to the existence of so-called *traps*, that is, finite but large regions where the environment is atypical, confining the walk or creating an abnormal drift. In this case a renormalization procedure ("coarse graining") is needed to control the size of the traps and their effect.

The structure of the article is as follows: in Section 2 we introduce the coupling representation (2.1), which allows us to deal with nonindependent medium, and some *approximate regeneration times* $\tau_i^{(L)}$ parametrized by a parameter L: they are defined by (2.3) and (2.7), and they lead to an "approximate renewal" result, our Lemma 2.3. In Section 3 we prove the law of large numbers under ($\mathcal{A}1$)–($\mathcal{A}3$) and a suitable integrability condition ($\mathcal{A}5$). We then give the proof of Corollary 1.3, by showing that Condition ($\mathcal{A}5$) is trivially satisfied under the nonnestling assumption ($\mathcal{A}4$). In Section 4 we make a short digression to show that the mixing assumption ($\mathcal{A}1$) is satisfied in many cases of interest. In Section 5 we show how condition ($\mathcal{A}5$) can be checked in the nestling case, and we construct a class of nestling, mixing environments satisfying our conditions for the law of large numbers, see Theorems 5.1 and 5.4 for precise statements. Finally, Section 6 is devoted to concluding remarks and extensions.

2. Some random times. To implement our coupling technique, we begin, following [21], by constructing an extension of the probability space, depending on the vector ℓ with integer coordinates: recall that the RWRE was defined by means of the law $\mathbb{P}^o = P \otimes P_\omega^o$ on the canonical space $(\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$.

Set $W = \{0\} \cup \mathcal{E}_{\bar{\varepsilon}}$ [recall the notation (1.3)], and let $\mathcal{W}$ be the cylinder σ-algebra on $W^{\mathbb{N}}$. We now define the measure

$$\overline{\mathbb{P}}^o = P \otimes Q \otimes \overline{P}^o_{\omega,\varepsilon}$$

on

$$(\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}},\ \mathcal{F} \times \mathcal{W} \times \mathcal{G})$$

in the following way: Q is a product measure, such that with $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ denoting an element of $W^{\mathbb{N}}$, $Q(\varepsilon_1 = e) = \kappa$, for $e \in \mathcal{E}_{\bar{\varepsilon}}$, while $Q(\varepsilon_1 = 0) = 1 - \kappa|\mathcal{E}_{\bar{\varepsilon}}|$. For each fixed ω, ε, $\overline{P}^o_{\omega,\varepsilon}$ is the law of the Markov chain $\{X_n\}$ with state space $\mathbb{Z}^d$, such that $X_0 = 0$ and, for every $z, e \in \mathbb{Z}^d$, $|e| = 1$,

$$\begin{aligned}\overline{P}^o_{\omega,\varepsilon}&(X_{n+1} = z + e | X_n = z)\\ &= \mathbf{1}_{\{\varepsilon_{n+1} = e\}} + \frac{\mathbf{1}_{\{\varepsilon_{n+1}=0\}}}{1 - \kappa|\mathcal{E}_{\bar{\varepsilon}}|}\big[\omega(z, z+e) - \kappa \mathbf{1}_{\{e \in \mathcal{E}_{\bar{\varepsilon}}\}}\big].\end{aligned} \tag{2.1}$$

Clearly, the law of $\{X_n\}$ under $Q \otimes \overline{P}^o_{\omega,\varepsilon}$ coincides with its law under P^o_ω, while its law under $\overline{\mathbb{P}}^o$ coincides with its law under $\mathbb{P}^o$.

We fix now a particular sequence of ε in $\mathcal{E}_{\bar{\varepsilon}}$ of length $|\ell|_1$ with sum equal to ℓ: for definiteness, we take $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_{|\ell|_1})$ with

$$\begin{aligned}\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \cdots = \bar{\varepsilon}_{|\ell_1|} &= \operatorname{sgn}(\ell_1) e_1,\\ \bar{\varepsilon}_{\ell_1+1} = \bar{\varepsilon}_{\ell_1+2} = \cdots = \bar{\varepsilon}_{|\ell_1|+|\ell_2|} &= \operatorname{sgn}(\ell_2) e_2, \dots,\\ \bar{\varepsilon}_{|\ell|_1 - |\ell_d|+1} = \cdots = \bar{\varepsilon}_{|\ell|_1} &= \operatorname{sgn}(\ell_d) e_d.\end{aligned}$$

We fix, from now on through the whole paper, $\zeta > 0$ small enough such that

$$\bar{\varepsilon}_1, \bar{\varepsilon}_1 + \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_1 + \cdots + \bar{\varepsilon}_{|\ell|_1} = \ell \in C(0, \ell, \zeta), \tag{2.2}$$

and such that $(\mathcal{A}1)$ above is satisfied. Without mentioning it explicitly in the sequel, we always rotate the axes such that $\ell_1 \neq 0$.

For $L \in |\ell|_1 \mathbb{N}^*$ we will denote by $\bar{\varepsilon}^{(L)}$ the vector

$$\bar{\varepsilon}^{(L)} = (\bar{\varepsilon}, \bar{\varepsilon}, \dots, \bar{\varepsilon})$$

of dimension L. In particular, $\bar{\varepsilon} = \bar{\varepsilon}^{(|\ell|_1)}$, and for ε with $(\varepsilon_{n+1}, \dots, \varepsilon_{n+L}) = \bar{\varepsilon}^{(L)}$,

$$\overline{P}^o_{\omega,\varepsilon}\left(X_{n+L} = x + \frac{L}{|\ell|_1}\ell \middle| X_n = x\right) = 1,$$

and the path $X_n, X_{n+1}, \dots, X_{n+L}$ remains in the cone $C(x, \ell, \zeta)$.

Define

$$D' = \inf\{n \geq 0 : X_n \notin C(X_0, \ell, \zeta)\}. \tag{2.3}$$

(For $\zeta = 0$, this is D from [18].) Let us state a few direct consequences of $(\mathcal{A}3)$ and $(\mathcal{A}4)$.

LEMMA 2.1. *Assume* ($\mathcal{A}3$). *Let* $f(y) = y \cdot \ell - \zeta|y||\ell|$ *with* $\zeta \le \delta(\ell)/(3|\ell|)$.

1. *There exist a* $\lambda_0 = \lambda_0(\delta(\ell)) > 0$ *such that for all* $\lambda \in (0, \lambda_0]$ *and any connected, finite subset* U *of* $\mathbb{Z}^d$ *containing* 0,
$$M_n^\lambda = \exp\{-3\lambda f(X_n) + \lambda\delta(\ell)(n \wedge T_{U^c})\}$$
is a supermartingale for Kalikow's Markov chain $\hat{P}_U$ *from* (1.5) (T_{U^c} *is the exit time of* U *for* X_n).
2. *For* $m > |\ell|$, *consider the truncated cone* $V_m = C(0, \ell, \zeta) \cap \{y \in \mathbb{R}^d;\ y \cdot \ell \le m\}$. *We have* $\hat{P}_{V_m}(X_{T_{V_m^c}} \cdot \ell > m) \ge 2\eta$ *with some constant* $\eta > 0$ *depending on* $\delta(\ell), \zeta$ *but not on* m, ω.
3. *If also* ($\mathcal{A}4$) *holds* (*and* ℓ *is general*), *choosing* κ *small enough such that* $\delta(\ell) > 2\kappa$, *it holds that* M_n^λ *is a supermartingale under the quenched measure* $\overline{P}^o_{\omega,\varepsilon}$ *for all* $\omega, \varepsilon, \lambda \in (0, \lambda_0]$, *and* $\overline{P}^o_{\omega,\varepsilon}(X_{T_{V_m^c}} \cdot \ell > m) \ge 2\eta$. *Moreover, with* $W_r = \{x \in \mathbb{Z}^d;\ x \cdot \ell < r\}$, $r > 0$,
$$\overline{E}^o_{\omega,\varepsilon}(\exp\{\lambda\delta(\ell)T_{W_r^c}\}) \le \exp\{3\lambda r\}. \tag{2.4}$$
4. *Assume* ($\mathcal{A}3$) *holds with* $\ell = e_1$, *and* $\delta = \delta(e_1)$. *Then, there exists* $\lambda_1 = \lambda_1(\delta) > 0$ *with* $\lambda_1 \to +\infty$ *as* $\delta \to 1^-$ *such that* $\exp\{-3\lambda f(X_n)\}$ *is a supermartingale for Kalikow's Markov chain* ($\lambda \in [0, \lambda_1)$). *In particular,* $\inf_{m,\omega} \hat{P}_{V_m}(X_{T_{V_m^c}} \cdot \ell > m) \to 1$ *when* $\delta \to 1^-$.

We stress that the above constants do not depend on ω outside U, in contrast to Kalikow's Markov chain $\hat{P}_U$ itself. Recall that, due to (1.6), estimates on the exit distribution for Kalikow's Markov chain yield a similar estimate for the RWRE, but on the other hand, exit time distributions for Kalikow's Markov chain and RWRE may be quite different.

PROOF OF LEMMA 2.1. 1. Since the chain has unit jumps, ($\mathcal{A}3$) implies for $y \in U$,
$$\hat{E}_U\big[(f(X_{n+1}) - f(X_n))\big|X_n = y\big] \ge (2/3)\delta(\ell)$$
and $f(X_{n+1}) - f(X_n)$ is uniformly bounded. Choosing λ_0 such that $\max\{u^{-2}(e^u - 1 - u);\ 0 < |u| \le \lambda_0(3|\ell| + 2)\} \le \delta(\ell)/(\lambda_0[3|\ell| + 2]^2)$, we have for $\lambda \in [0, \lambda_0]$, that $\hat{E}_U(\exp\{-3\lambda(f(X_{n+1}) - f(X_n)) + \lambda\delta(\ell)\}|X_n = y) \le 1$ uniformly in $y \in U$. Note that we can choose λ_0 increasing in $\delta(\ell)$.

2. Applying the stopping theorem for the exit time $T_{V_m^c}$ to the above supermartingale, we get for $y \in V_m$,
$$\begin{aligned}\exp\{-3\lambda f(y)\} &\ge \hat{E}_{V_m}(\exp\{-3\lambda f(X_{T_{V_m^c}}) + \lambda\delta(\ell)T_{V_m^c}\}\big|X_0 = y)\\ &\ge \hat{P}_{V_m}(X_{T_{V_m^c}} \cdot \ell \le m\big|X_0 = y),\end{aligned}$$

where the second inequality is due to $f < 0$ on the boundary of the cone $C(0, \ell, \zeta)$ and $\lambda\delta(\ell) > 0$. With $y = \bar{\varepsilon}_1 \in V_m$, we have, for the chain starting from 0,

$$\begin{aligned}\hat{P}_{V_m}(X_{T_{V_m^c}} \cdot \ell > m) &\geq \hat{P}_{V_m}(X_1 = \bar{\varepsilon}_1)\hat{P}_{V_m}(X_{T_{V_m^c}} \cdot \ell \geq m \big| X_1 = \bar{\varepsilon}_1) \\ &\geq \kappa\big[1 - e^{-3\lambda f(\bar{\varepsilon}_1)}\big],\end{aligned} \tag{2.5}$$

which is positive since f is positive in the interior of the cone.

3. It is straightforward to check that the above computations apply to $\overline{P}^o_{\omega,\varepsilon}$ under the assumption ($\mathcal{A}4$) [the assumption $\delta(\ell) > 2\kappa$ is used to ensure that the modified environment appearing in the definition (2.1) of $\overline{P}^o_{\omega,\varepsilon}$ also is uniformly elliptic in the direction ℓ]. In addition, to prove (2.4), we apply the stopping theorem to the $\overline{P}^o_{\omega,\varepsilon}$-supermartingale M_n^λ at the exit time of the domain W_r intersected with large, finite boxes, and we get

$$\exp\{-3\lambda r\}\overline{E}^o_{\omega,\varepsilon}(\exp\{\lambda\delta(\ell)T_{W_r^c}\}) \leq 1.$$

4. With $\delta = \delta(e_1)$, set $\mathcal{A} = \{(\alpha, \beta) \subset [0,1]^2 : \alpha + \beta \leq 1, \alpha - \beta \geq \delta\}$. Then, for $y \in V_m$, and $\lambda > 0$, it follows from ($\mathcal{A}3$) that

$$\begin{aligned}&\hat{E}_U[\exp\{-3\lambda(X_{n+1} - X_n) \cdot e_1 + 3\lambda\zeta\} | X_n = y] \\ &\quad\leq \exp\{3\lambda\zeta\} \sup_{(\alpha,\beta)\in\mathcal{A}} [\alpha e^{-3\lambda} + \beta e^{3\lambda} + (1 - \alpha - \beta)] \\ &\quad= \exp\{3\lambda\zeta\}[\cosh(3\lambda) - \delta\sinh(3\lambda)] =: A(\delta, \lambda, \zeta).\end{aligned}$$

We see that $\lambda_1(\delta) := \sup\{\lambda : A(\delta, \lambda, \zeta) < 1, \forall\, \zeta \leq \delta/3\} \to \infty$ as $\delta \to 1$. In particular, the right-hand side of (2.5) can be arbitrarily close to 1 as $\delta \to 1$. □

Assumption ($\mathcal{A}3$) implies $\mathbb{P}^o(D' = \infty) > 0$. Indeed, consider the truncated cone $V_m = C(0, \ell, \zeta) \cap \{y \in \mathbb{R}^d; y \cdot \ell \leq m\}$ and Kalikow's Markov chain $\hat{P}_{V_m}$. From (1.6), the exit distribution for X out of V_m is the same under both $\hat{P}_{V_m}$ and $\mathbb{P}^o(\cdot | \mathcal{F}_{V_m^c})$. From part (2) of Lemma 2.1, it follows

$$\begin{aligned}\mathbb{P}^o(D' = \infty | \omega_x, x \cdot \ell \leq 0) &= \lim_{m\to\infty} \mathbb{P}^o(X_{T_{V_m^c}} \cdot \ell > m \big| \omega_x, x \cdot \ell \leq 0) \\ &\geq 2\eta, \qquad P\text{-a.s.}\end{aligned}$$

By integration, we get

$$\mathbb{P}^o(D' = \infty | \omega_x, x \cdot \ell \leq -r) \geq 2\eta, \qquad P\text{-a.s.} \tag{2.6}$$

Recall that ($\mathcal{A}3$) implies that $\mathbb{P}^o(X_n \cdot \ell \to_{n\to\infty} +\infty) = 1$. Set

$$\mathcal{G}_n = \sigma\big((\varepsilon_i, X_i),\ i \leq n\big),$$

fix $L \in |\ell|_1\mathbb{N}$ and, setting $\overline{S}_0 = 0$, define, using θ_n to denote time shift and $\bar{\theta}_x$ to denote space shift,

$$\begin{aligned}\overline{S}_1 = \inf\{&n \geq L : X_{n-L} \cdot \ell > \max\{X_m \cdot \ell : m < n - L\}, \\ &(\varepsilon_{n-L}, \ldots, \varepsilon_{n-1}) = \bar{\varepsilon}^{(L)}\} \leq \infty, \\ \overline{R}_1 = D' \circ \theta_{\bar{S}_1} + \overline{S}_1 \leq \infty.&\end{aligned} \tag{2.7}$$

Note that the random times $\overline{S}_1, \overline{R}_1$, depend on both $\{X_n\}_n$ and $\{\varepsilon_n\}_n$. Define further, by induction for $k \geq 1$,

$$\overline{S}_{k+1} = \inf\{n \geq \overline{R}_k : X_{n-L} \cdot \ell > \max\{X_m \cdot \ell : m < n - L\}, \\ (\varepsilon_{n-L}, \ldots, \varepsilon_{n-1}) = \bar{\varepsilon}^{(L)}\} \leq \infty,$$

$$\overline{R}_{k+1} = D' \circ \theta_{\overline{S}_{k+1}} + \overline{S}_{k+1} \leq \infty.$$

Clearly, these are $\mathcal{G}_n$ stopping times (depending on L), and

$$0 = \overline{S}_0 \leq \overline{S}_1 \leq \overline{R}_1 \leq \overline{S}_2 \leq \cdots \leq \infty$$

and the inequalities are strict if the left member is finite. On the set $A_\ell := \{X_n \cdot \ell \to_{n\to\infty} \infty\}$, it is straightforward to check (using the product structure of Q) that the time $\overline{S}_1$ is $\overline{\mathbb{P}}^o$-a.s. finite, as is $\overline{S}_{k+1}$ on the set $A_\ell \cap \{\overline{R}_k < \infty\}$. Define

$$K = \inf\{k \geq 1 : \overline{S}_k < \infty, \overline{R}_k = \infty\} \leq \infty,$$

$$\tau_1^{(L)} = \overline{S}_K \leq \infty.$$

This random time $\tau_1^{(L)}$ is the first time n when the walk performs as follows: at time $n - L$ it has reached a record value in the direction $+\ell$, then it travels using the ε-sequence only up to time n, and from time n on, it does not exit the positive cone $C(X_n, \ell, \zeta)$ with vertex X_n. In particular, $\tau_1^{(L)}$ is not a stopping time, and we emphasize its dependence on L. As in [21], the advantage in working with $\tau_i^{(L)}$ (as opposed to the more standard $\tau_i^{(0)}$) is that the $\{\varepsilon\}$ sequence creates a spacing where no information on the environment is gathered by the RWRE.

LEMMA 2.2. *Assume* ($\mathcal{A}1$)–($\mathcal{A}3$), *and* $\zeta \leq \delta(\ell)/(3|\ell|)$. *Then, there exists an* L_0 *such that for* $L \geq L_0$, $\tau_1^{(L)}$ *is finite* $\overline{\mathbb{P}}^o$*-a.s.*

PROOF. This amounts to proving $K < \infty$. Toward this end, write

$$\begin{aligned}
&\overline{\mathbb{P}}^o(\overline{R}_{k+1} < \infty) \\
&\quad = \overline{\mathbb{P}}^o\big(\overline{R}_k < \infty, \overline{S}_{k+1} < \infty, D' \circ \theta_{\overline{S}_{k+1}} < \infty\big) \\
&\quad = \sum_{z\in\mathbb{Z}^d} \sum_{n\in\mathbb{N}} \overline{\mathbb{P}}^o\big(\overline{R}_k < \infty, X_{\overline{S}_{k+1}} = z, \overline{S}_{k+1} = n, D' \circ \theta_n < \infty\big) \\
&\quad = \sum_{z\in\mathbb{Z}^d} \sum_{n\in\mathbb{N}} E_{P\otimes Q}\big(\overline{P}^o_{\omega,\varepsilon}\big(\overline{R}_k < \infty, X_{\overline{S}_{k+1}} = z, \\
&\qquad\qquad \overline{S}_{k+1} = n, D' \circ \theta_n < \infty\big)\big) \qquad (2.8) \\
&\quad = \sum_{z\in\mathbb{Z}^d} \sum_{n\in\mathbb{N}} E_{P\otimes Q}\big(\overline{P}^o_{\omega,\varepsilon}\big(\overline{R}_k < \infty, X_{\overline{S}_{k+1}} = z, \overline{S}_{k+1} = n\big) \\
&\qquad\qquad \times \overline{P}^o_{\bar{\theta}_z\omega,\theta_n\varepsilon}(D' < \infty)\big)
\end{aligned}$$

using the strong Markov property under $\overline{P}^o_{\omega,\varepsilon}$. The point here is that $\overline{P}^o_{\theta_z\omega,\theta_n\varepsilon}(D' < \infty)$ is measurable on $\sigma(\omega_x : x \in C(z,\ell,\zeta)) \otimes \sigma(\varepsilon_i, i \geq n)$, whereas $\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k < \infty, X_{\overline{S}_{k+1}} = z, \overline{S}_{k+1} = n)$ is measurable on $\sigma(\omega_x : x \cdot \ell \leq z \cdot \ell - L|\ell|^2/|\ell|_1) \otimes \sigma(\varepsilon_i, i < n)$. Hence, by the ϕ-mixing property on cones of P, by the product structure of Q and by stationarity,

$$
\begin{aligned}
&\overline{\mathbb{P}}^o(\overline{R}_{k+1} < \infty) \\
&\quad \leq \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} [E_{P\otimes Q}(\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k < \infty, X_{\overline{S}_{k+1}} = z, \overline{S}_{k+1} = n)) \\
&\qquad\qquad \times E_{P\otimes Q}(\overline{P}^o_{\omega,\varepsilon}(D' < \infty))] \\
&\qquad + \phi(L) \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q}(\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k < \infty, X_{\overline{S}_{k+1}} = z, \overline{S}_{k+1} = n)) \\
&\quad = \overline{\mathbb{P}}^o(\overline{R}_k < \infty)(\overline{\mathbb{P}}^o(D' < \infty) + \phi(L)) \\
&\quad \leq (\overline{\mathbb{P}}^o(D' < \infty) + \phi(L))^{k+1}
\end{aligned} \tag{2.9}
$$

by induction. Choosing L with $\phi(L) \leq \eta$, and using (2.6), we see that $\overline{\mathbb{P}}^o(K \geq k) \leq (1-\eta)^k$. $\square$

Consider now $\tau_1^{(L)}$ as a function of the path $\{X_n\}_n$ and of the sequence $\{\varepsilon_n\}_n$ and set

$$
\tau_{k+1}^{(L)} = \tau_k^{(L)}(X_\cdot, \varepsilon_\cdot) + \tau_1^{(L)}(X_{\tau_k^{(L)}+\cdot} - X_{\tau_k^{(L)}}, \varepsilon_{\tau_k^{(L)}+\cdot}), \tag{2.10}
$$

with $\tau_{k+1}^{(L)} = \infty$ on $\{\tau_k^{(L)} = \infty\}$.

Under ($\mathcal{A}3$), $\tau_k^{(L)}$ is $\overline{\mathbb{P}}^o$-a.s. finite for all k. Indeed, in view of the definition (2.10),

$$
\begin{aligned}
&\overline{\mathbb{P}}^o(\tau_1^{(L)} < \infty, \tau_2^{(L)} = \infty) \\
&\quad = \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q}\overline{P}^o_{\omega,\varepsilon}(\tau_1^{(L)} = n, X_n = z, \tau_2^{(L)} = \infty) \\
&\quad \leq \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q}\overline{P}^z_{\omega,\theta^n\varepsilon}(\tau_1^{(L)} = \infty) \\
&\quad = \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} \overline{\mathbb{P}}^z(\tau_1^{(L)} = \infty) = 0,
\end{aligned}
$$

since all summands are equal to $\overline{\mathbb{P}}^o(\tau_1^{(L)} = \infty) = 0$.

Define new σ-fields on $\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}$,

$$
\begin{aligned}
\mathcal{H}_1 = \sigma(&\tau_1^{(L)}, X_0, \varepsilon_0, X_1, \ldots, \varepsilon_{\tau_1^{(L)}-1}, X_{\tau_1^{(L)}}, \\
&\{\omega(y,\cdot); \ell \cdot y < \ell \cdot X_{\tau_1^{(L)}} - L|\ell|^2/|\ell|_1\}),
\end{aligned}
$$

$$\mathcal{H}_k = \sigma\big(\tau_1^{(L)}, \dots, \tau_k^{(L)},\ X_0, \varepsilon_0, X_1, \dots, \varepsilon_{\tau_k^{(L)}-1}, X_{\tau_k^{(L)}},$$
$$\{\omega(y,\cdot);\ \ell\cdot y < \ell\cdot X_{\tau_k^{(L)}} - L|\ell|^2/|\ell|_1\}\big).$$

Note that since $\{D' = \infty\} = \{(X_1, \dots, X_{\tau_1^{(L)}}) \in C(0,\ell,\zeta)\} \cap \{\tau_1^{(L)} < \infty\}$, see (2.2), we have that $\{D' = \infty\} \in \mathcal{H}_1$. Then, we have the following crucial lemma. Recall the variational distance $\|\mu - \nu\|_{\mathrm{var}} = \sup\{\mu(A) - \nu(A);\ A \text{ measurable}\}$ between two probability measures on the same space.

LEMMA 2.3. *Assume* (A1)–(A3), *and* $\zeta \le \delta(\ell)/(3|\ell|)$. *Set* $\phi'(L) = 2[\overline{\mathbb{P}}^o(D' = \infty) - \phi(L)]^{-1}\phi(L)$. [*Here, and in the following, we consider L large enough so that* $\phi(L) < \overline{\mathbb{P}}^o(D' = \infty)$.] *Then, it holds a.s.,*

$$\|\overline{\mathbb{P}}^o(\{X_{\tau_k^{(L)}+n} - X_{\tau_k^{(L)}}\}_{n\ge 0} \in \cdot\,|\,\mathcal{H}_k) - \overline{\mathbb{P}}^o(\{X_n\}_{n\ge 0} \in \cdot\,|D' = \infty)\|_{\mathrm{var}} \le \phi'(L).$$

PROOF. We start with the case $k = 1$. Let A be a measurable subset of the path space and write for short $\mathbf{1}_A = \mathbf{1}_{\{X_n - X_0\}_{n\ge 0}\in A}$. Let $h \ge 0$ be a $\mathcal{H}_1$-measurable nonnegative random variable. Then for all $l, n \ge 1$, $x \in \mathbb{Z}^d$, there exists a random variable $h_{x,l,n} \ge 0$, measurable with respect to $\sigma(\{\omega(y,\cdot);\ y\cdot\ell < x\cdot\ell - L|\ell|^2/|\ell|_1\}, \{X_i\})_{i\le n}$ such that, on the event $\{\tau_1^{(L)} = \overline{S}_l = n, X_{\overline{S}_l} = x\}$, it holds $h = h_{x,l,n}$. Recall that $\{K = l\} = \{\overline{S}_l < \infty, D'\circ\theta_{\overline{S}_l} = \infty\}$, and use the Markov property and shift invariance to write

$$\begin{aligned}
&\overline{\mathbb{E}}^o(h\mathbf{1}_A\circ\theta_{\tau_1^{(L)}})\\
&\quad= \sum_{l\ge 1}\overline{\mathbb{E}}^o(h\mathbf{1}_A\circ\theta_{\tau_1^{(L)}}\mathbf{1}_{K=l})\\
&\quad= \sum_{l\ge 1, x\in\mathbb{Z}^d, n\ge 1} E_{P\otimes Q}\overline{E}^o_{\omega,\varepsilon}(h_{x,l,n}\mathbf{1}_A\circ\theta_n\mathbf{1}_{\overline{S}_l=n, X_n=x, D'\circ\theta_n=\infty})\\
&\quad= \sum_{l\ge 1, x\in\mathbb{Z}^d, n\ge 1} E_{P\otimes Q}\big[\,\overline{E}^o_{\omega,\varepsilon}(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n, X_n=x})\\
&\qquad\qquad\qquad\qquad \times \overline{P}^x_{\omega,\theta_n\varepsilon}(A\cap\{D'=\infty\})\big] \qquad (2.11)\\
&\quad= \sum_{l\ge 1, x\in\mathbb{Z}^d, n\ge 1} \overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n, X_n=x})\overline{\mathbb{P}}^x(A\cap\{D'=\infty\}) + \rho_A\\
&\quad= \overline{\mathbb{P}}^o(A\cap\{D'=\infty\}) \sum_{l\ge 1, x\in\mathbb{Z}^d, n\ge 1} \overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n, X_n=x}) + \rho_A.
\end{aligned}$$

The quantity ρ_A is defined by the above equalities, that is, $\rho_A = \sum_{l,x,n}\mathrm{Cov}_{P\otimes Q}(f_{x,l,n}, g_{x,n})$ with $f_{x,l,n} = \overline{E}^o_{\omega,\varepsilon}(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n, X_n=x})$, $g_{x,n} =$

$\overline{P}^x_{\omega,\theta_{n\mathcal{E}}}(A\cap\{D'=\infty\})$. The point is that, from (1.4) it holds for a nonnegative h

$$|\rho_A|\le\phi(L)\sum_{l\ge1,x\in\mathbb{Z}^d,n\ge1}\overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x}) \tag{2.12}$$

uniformly in A. In particular, for A equal to the whole path space $(\mathbb{Z}^d)^{\mathbb{N}}$, one gets

$$\sum_{l\ge1,x\in\mathbb{Z}^d,n\ge1}\overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x})\le[\overline{\mathbb{P}}^o(D'=\infty)-\phi(L)]^{-1}\overline{\mathbb{E}}^o(h), \tag{2.13}$$

as well as a formula for the sum in the left member above. Plugging this formula in (2.11), one obtains

$$\begin{aligned}&\left|\overline{\mathbb{E}}^o(h\mathbf{1}_A\circ\theta_{\tau_1^{(L)}})-\overline{\mathbb{E}}^o(h)\overline{\mathbb{P}}^o(A|D'=\infty)\right|\\&\quad=\left|\rho_A-\rho_{(\mathbb{Z}^d)^{\mathbb{N}}}\overline{\mathbb{P}}^o(A|D'=\infty)\right|\\&\quad\le2[\overline{\mathbb{P}}^o(D'=\infty)-\phi(L)]^{-1}\phi(L)\overline{\mathbb{E}}^o(h),\end{aligned}$$

where the second inequality follows from (2.12) and (2.13). Since h is arbitrary, we have

$$\left|\overline{\mathbb{P}}^o(\{X_{\tau_1^{(L)}+n}-X_{\tau_1^{(L)}}\}_{n\ge0}\in A\big|\mathcal{H}_1)-\overline{\mathbb{P}}^o(\{X_n\}_{n\ge0}\in A|D'=\infty)\right|\le\phi'(L)$$

a.s. for all A's. But there are only countably many cylinders in the path space, so we can find a subset of Ω of P-measure one, where the previous inequality holds simultaneously for all measurable A. We have shown the lemma for $k=1$.

The case of a general $k\ge1$ follows similarly from the above computations and the definition (2.10). □

3. Law of large numbers. Throughout this section we assume ($\mathcal{A}1$) and ($\mathcal{A}2$) for some ℓ with integer coordinates, and we assume also that the conclusions of Lemma 2.3 hold. For $L\in|\ell|_1\mathbb{N}^*$ we define $\tau_0^{(L)}=0$, and for $k\ge1$,

$$\overline{\tau}_k^{(L)}=\kappa^L(\tau_k^{(L)}-\tau_{k-1}^{(L)}),\qquad\overline{X}_k^{(L)}=\kappa^L(X_{\tau_k^{(L)}}-X_{\tau_{k-1}^{(L)}}). \tag{3.1}$$

[A rescaling by the factor κ^L is needed in order to keep the variables $\overline{\tau}_k^{(L)},\overline{X}_k^{(L)}$ of order 1 as $L\to\infty$.]

The following uniform integrability condition is instrumental in our derivation:

ASSUMPTION 3.1.

($\mathcal{A}5$) There exist an $\alpha>1$ and $M=M(L)$ such that $\phi'(L)^{1/\alpha'}M(L)^{1/\alpha}\to_{L\to\infty}0$ (with $1/\alpha'=1-1/\alpha$), and

$$P(\overline{\mathbb{E}}^o((\overline{\tau}_1^{(L)})^\alpha|D'=\infty,\mathcal{F}_0^L)>M)=0, \tag{3.2}$$

where $\mathcal{F}_0^L=\sigma(\omega(y,\cdot):\ell\cdot y<-L)$.

We define

$$\beta_L := \overline{\mathbb{E}}^o(\overline{\tau}_1^{(L)} | D' = \infty) < \infty \tag{3.3}$$

and

$$\gamma_L := \overline{\mathbb{E}}^o(\overline{X}_1^{(L)} | D' = \infty) \in \mathbb{R}^d, \tag{3.4}$$

where the moments β_L and γ_L are finite due to (3.2). Further, we note that there exists a constant $t_{\text{av}} > 0$, independent of L, such that

$$\liminf_{L\to\infty} \beta_L \geq t_{\text{av}}. \tag{3.5}$$

Indeed, recall that $\overline{\mathbb{P}}^o(D' = \infty) > 0$. Define $z_L = \min\{n : (\varepsilon_n, \varepsilon_{n+1}, \ldots, \varepsilon_{n+L-1}) = \bar{\varepsilon}^{(L)}\}$. Clearly, $z_L \leq \tau_1^{(L)}$. Moreover, z_L can be represented as follows. Define a Markov chain $\{U_n\}$ with state space $S = \{0, 1, \ldots, L\}$, with $U_n = s$ if

$$s = \max\{k \geq 1 : (\varepsilon_{n-k+1}, \ldots, \varepsilon_n) = (\bar{\varepsilon}_1^{(L)}, \ldots, \bar{\varepsilon}_k^{(L)})\} \vee 0,$$

and with $U_0 = 0$. Then, $z_L = \min\{n : U_n = L\}$. Considering the successive times when $U_n = \ell_1$, it follows that z_L can be bounded below by a sum of a Geometric($\kappa^{L-|\ell_1|}$) number of independent random variables bounded below by 1. Therefore,

$$\liminf_{L\to\infty} \mathbb{P}^o(z_L \geq c\kappa^{-L}) \geq g(c), \tag{3.6}$$

where $g(c) \to_{c\to 0} 1$. Thus,

$$\begin{aligned}
\overline{\mathbb{E}}^o(\bar{\tau}_1^{(L)} | D' = \infty) &\geq \kappa^L \overline{\mathbb{E}}^o(\tau_1^{(L)} \mathbf{1}_{\{D'=\infty\}}) \\
&\geq \kappa^L \overline{\mathbb{E}}^o(\tau_1^{(L)} \mathbf{1}_{\{D'=\infty\}} \mathbf{1}_{\{z_L \geq c\kappa^{-L}\}}) \\
&\geq c\overline{\mathbb{E}}^o(\mathbf{1}_{\{D'=\infty\}} \mathbf{1}_{\{z_L \geq c\kappa^{-L}\}}) \\
&\geq c[\overline{\mathbb{P}}^o(D' = \infty) - \bar{\mathbb{P}}^o(z_L < c\kappa^{-L})].
\end{aligned}$$

Choosing c such that

$$2\limsup_{L\to\infty} \overline{\mathbb{P}}^o(z_L < c\kappa^{-L}) < \overline{\mathbb{P}}^o(D' = \infty),$$

which is possible by (3.6), the bound (3.5) follows.

From Lemma 2.3, we have a.s., that for $k \geq 2$,

$$\|\overline{\mathbb{P}}^o((\bar{\tau}_k^{(L)}, \overline{X}_k^{(L)}) \in \cdot | \mathcal{H}_{k-1}) - \mu^{(L)}(\cdot)\|_{\text{var}} \leq \phi'(L), \tag{3.7}$$

where $\mu^{(L)}$ is defined by

$$\mu^{(L)}(A \times B) = \overline{\mathbb{P}}^o(\bar{\tau}_1^{(L)} \in A, \overline{X}_1^{(L)} \in B | D' = \infty),$$

for any sets $A \subset \kappa^L\mathbb{N}^*$, $B \subset \kappa^L\mathbb{Z}^d$. This will allow us to implement a coupling procedure. We recall the following splitting representation: If $\overline{X}, \tilde{X}$ are random

variables of laws $\overline{P}$, $\tilde{P}$ such that $\|\overline{P}-\tilde{P}\|_{\text{var}} \le a$, then one may find, on an enlarged probability space, independent random variables $Y, \Delta, Z, \tilde{Z}$, where Δ is Bernoulli distributed on $\{0, 1\}$ with parameter a, and

$$\overline{X} = (1-\Delta)Y + \Delta Z, \qquad \tilde{X} = (1-\Delta)Y + \Delta\tilde{Z},$$

(see, e.g., [1], Appendix A.1, for the proof). In particular,

$$\overline{X} = (1-\Delta)\tilde{X} + \Delta Z, \qquad |\Delta Z| \le |\overline{X}|, \qquad |\Delta\tilde{Z}| \le |\tilde{X}|. \tag{3.8}$$

Note that, by recursive conditioning, this extends immediately to sequences (see [2], Lemma 2.1, or [19], Chapter 3, for similar arguments):

LEMMA 3.2. *Suppose $\{\overline{X}_i\}_{i\ge1}$ is a sequence of random vectors such that for some probability measure Q,*

$$\|P(\overline{X}_i \in A|\sigma\{\overline{X}_j\}_{j<i}) - Q\|_{\text{var}} \le \alpha. \tag{3.9}$$

Then, enlarging the probability space if necessary, there exists a sequence of i.i.d. random variables $\{(\tilde{X}_i, \Delta_i)\}_{i\ge1}$ such $\tilde{X}_1$ is distributed according to Q while $P(\Delta_1 = 1) = \alpha$, and a sequence of random vectors $\{Z_i\}_{i\ge1}$, such that Δ_i is independent of $\sigma(\tilde{X}_j, \Delta_j, j < i) \vee \sigma(Z_i)$, and

$$\overline{X}_i = (1-\Delta_i)\tilde{X}_i + \Delta_i Z_i.$$

PROOF. The proof proceeds by induction: for $i = 1$, this is (3.8). Suppose the lemma has been proved for $i-1$. Noting (3.9) and applying (3.8) with the laws $P(\overline{X}_i \in \cdot|\sigma\{\overline{X}_j\}_{j<i})$ and Q, one obtains the existence of the claimed $\Delta_i, Z_i, \tilde{X}_i$ with the required properties. □

In view of Lemma 3.2, we can enlarge our probability space, where is defined the sequence $\{(\overline{\tau}_i^{(L)}, \overline{X}_i^{(L)})\}_{i\ge1}$ in order to also support the following:

- a sequence $\{(\tilde{\tau}_i^{(L}, \tilde{X}_i^{(L)}, \Delta_i^{(L)})\}_{i\ge1}$ of i.i.d. random vectors (with values in $\kappa^L\mathbb{N}^* \times \kappa^L\mathbb{Z}^d \times \{0, 1\}$) such that $\{(\tilde{\tau}_1^{(L)}, \tilde{X}_1^{(L)})\}$ is distributed according to $\mu^{(L)}$, while $\Delta_1^{(L)} \in \{0, 1\}$ is such that $P(\Delta_1^{(L)} = 1) = \phi'(L)$,
- and another sequence $\{(Z_i^{(L)}, Y_i^{(L)})\}_{i\ge1}$ such that

$$(\overline{\tau}_i^{(L)}, \overline{X}_i^{(L)}) = (1-\Delta_i^{(L)})(\tilde{\tau}_i^{(L)}, \tilde{X}_i^{(L)}) + \Delta_i^{(L)}(Z_i^{(L)}, Y_i^{(L)}),$$

and such that, with

$$\tilde{\mathcal{G}}_i := \sigma(\{\tilde{\tau}_j^{(L}\}_{j\le i-1}, \{\tilde{X}_j^{(L)}\}_{j\le i-1}, \{\Delta_j^{(L)}\}_{j\le i-1}),$$

it holds that $\Delta_i^{(L)}$ is independent of $\tilde{\mathcal{G}}_i$ and of $(Z_i^{(L)}, Y_i^{(L)})$. (Here, and as above, the sigma-field generated by no variables is trivial, hence $\tilde{\mathcal{G}}_1$ is the trivial sigma-field.)

The joint law of the variables $\{(Z_i^{(L)}, Y_i^{(L)})\}_{i\geq 1}$ is complicated, but, because the walk is nearest neighbor, it holds that $|Y_i^{(L)}| \leq Z_i^{(L)}$, while, since $|\Delta_i^{(L)} Z_i^{(L)}| \leq \overline{\tau}_i^{(L)}$ and from (3.2), it holds almost surely that

$$M(L) \geq \overline{\mathbb{E}}^o[(\Delta_i^{(L)} Z_i^{(L)})^\alpha | \tilde{\mathcal{G}}_i] = \phi'(L)\overline{\mathbb{E}}^o[(Z_i^{(L)})^\alpha | \tilde{\mathcal{G}}_i]. \tag{3.10}$$

We next have the following lemma:

LEMMA 3.3. *Assume the integrability condition* (3.2). *Then, there exists a deterministic sequence* $\eta_L \to_{L\to\infty} 0$ *such that*

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n \overline{\tau}_i^{(L)} - \beta_L \right| < \eta_L, \qquad \overline{\mathbb{P}}^o\text{-}a.s., \tag{3.11}$$

and

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n \overline{X}_i^{(L)} - \gamma_L \right| < \eta_L, \qquad \overline{\mathbb{P}}^o\text{-}a.s. \tag{3.12}$$

PROOF. We prove (3.11), the proof of (3.12) being similar. Simply write

$$\frac{1}{n} \sum_{i=1}^n \overline{\tau}_i^{(L)} = \frac{1}{n} \sum_{i=1}^n \tilde{\tau}_i^{(L)} - \frac{1}{n} \sum_{i=1}^n \Delta_i^{(L)} \tilde{\tau}_i^{(L)} + \frac{1}{n} \sum_{i=1}^n \Delta_i^{(L)} Z_i^{(L)}.$$

Note first that by independence,

$$\frac{1}{n} \sum_{i=1}^n \tilde{\tau}_i^{(L)} \underset{n\to\infty}{\to} \beta_L, \qquad \overline{\mathbb{P}}^o\text{-a.s.},$$

while

$$\left| \frac{1}{n} \sum_{i=1}^n \Delta_i^{(L)} \tilde{\tau}_i^{(L)} \right| \leq \left(\frac{1}{n} \sum_{i=1}^n (\Delta_i^{(L)})^{\alpha'} \right)^{1/\alpha'} \left(\frac{1}{n} \sum_{i=1}^n (\tilde{\tau}_i^{(L)})^\alpha \right)^{1/\alpha} \tag{3.13}$$

and, hence,

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n \Delta_i^{(L)} \tilde{\tau}_i^{(L)} \right| \leq \phi'(L)^{1/\alpha'} M(L)^{1/\alpha}, \qquad \overline{\mathbb{P}}^o\text{-a.s.}$$

We next consider the term involving $Z_i^{(L)}$. Set $\overline{Z}_i^{(L)} := \overline{\mathbb{E}}^o(Z_i^{(L)} | \tilde{\mathcal{G}}_i)$, and note that $M_n := \sum_{i=1}^n \Delta_i^{(L)} (Z_i^{(L)} - \overline{Z}_i^{(L)})/i$ is a zero mean martingale with respect to

the filtration $\tilde{\mathcal{G}}_i$. By the Burkholder–Gundy maximal inequality ([20], 14.18), for $\gamma = \alpha \wedge 2$,

$$E\left|\sup_n M_n\right|^\gamma \le C_\gamma E\left(\sum_{i=1}^\infty \frac{(\Delta_i^{(L)}(Z_i^{(L)} - \overline{Z}_i^{(L)}))^2}{i^2}\right)^{\gamma/2}$$
$$\le C_\gamma \sum_{i=1}^\infty E\left(\frac{(\Delta_i^{(L)}(Z_i^{(L)} - \overline{Z}_i^{(L)}))^\gamma}{i^\gamma}\right)$$
$$\le C'_\gamma,$$

for some constants C_γ, C'_γ. Hence, M_n converges P-a.s. to an integrable random variable, and by the Kronecker lemma ([20], 12.7), it holds that $n^{-1}\sum_i \Delta_i^{(L)}(Z_i^{(L)} - \overline{Z}_i^{(L)}) \to 0$, almost surely. On the other hand, there is nothing to prove if $\phi'(L) = 0$ while, if $\phi'(L) > 0$, then

$$|\overline{Z}_i^{(L)}| \le (\overline{\mathbb{E}}^o(|Z_i^{(L)}|^\alpha | \tilde{\mathcal{G}}_i))^{1/\alpha} \le \left(\frac{M(L)}{\phi'(L)}\right)^{1/\alpha}, \qquad \overline{\mathbb{P}}^o\text{-a.s.}$$

by Jensen's inequality and by (3.10), and hence,

$$\left|\frac{1}{n}\sum_{i=1}^n \overline{Z}_i^{(L)}\Delta_i^{(L)}\right| \le \left(\frac{M(L)}{\phi'(L)}\right)^{1/\alpha} \frac{1}{n}\sum_{i=1}^n \Delta_i^{(L)}$$
$$\underset{n\to\infty}{\to} M(L)^{1/\alpha}\phi'(L)^{1/\alpha'}, \qquad \overline{\mathbb{P}}^o\text{-a.s.},$$

yielding (3.11) by choosing $\eta_L = 2M(L)^{1/\alpha}\phi'(L)^{1/\alpha'}$. □

Using that $\tilde{\tau}_i^{(L)} \ge \kappa^L$ and that $\beta_L \ge t_{\text{av}}$, compare with (3.5), we conclude from Lemma 3.3 that for all L large enough,

$$\limsup_{n\to\infty}\left|\frac{\frac{1}{n}\sum_{i=1}^n \overline{X}_i^{(L)}}{\frac{1}{n}\sum_{i=1}^n \overline{\tau}_i^{(L)}} - \frac{\gamma_L}{\beta_L}\right| \le C_1\eta_L, \qquad \overline{\mathbb{P}}^o\text{-a.s.},$$

for some constant $C_1 = C_1(t_{\text{av}})$, from which one deduces by standard arguments (e.g., pages 1864 and 1865 in [18]) that

$$\limsup_{n\to\infty}\left|\frac{X_n}{n} - \frac{\gamma_L}{\beta_L}\right| \le C_2\eta_L, \qquad \overline{\mathbb{P}}^o\text{-a.s.},$$

for some constant $C_2 = C_2(t_{\text{av}})$. Thus, we conclude both the existence of the limit $v := \lim_{L\to\infty}\gamma_L/\beta_L$ and the $\overline{\mathbb{P}}^o$ convergence of X_n/n to it. Summarizing, we have proved the following

THEOREM 3.4. *Assume the conclusion of Lemma* 2.3 *and the integrability condition* (3.2). *Then, there exists a deterministic vector* v *with* $v \cdot \ell > 0$ *such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \qquad \mathbb{P}^o\text{-a.s.}$$

PROOF. The only statement left to be shown is $v \cdot \ell > 0$. Let us fix L. We already know that $X_n/n \to v$, $\mathbb{P}^o$- a.s., and since $\tau_n^{(L)} \to \infty$, we have

$$\frac{X_{\tau_n^{(L)}}}{\tau_n^{(L)}} - v = \left(\frac{X_{\tau_n^{(L)}}}{n} - v\frac{\tau_n^{(L)}}{n}\right)\frac{n}{\tau_n^{(L)}} \underset{n\to\infty}{\to} 0, \qquad \mathbb{P}^o\text{-a.s.}$$

By (3.2) $\overline{\mathbb{E}}^o \tau_n^{(L)}/n \le M(L)\kappa^{-L}$, so that $\tau_n^{(L)}/n$ is bounded in probability, yielding

$$\lim_{n\to\infty}\left(\frac{X_{\tau_n^{(L)}}}{n} - v\frac{\tau_n^{(L)}}{n}\right) = 0, \qquad \mathbb{P}^o\text{-a.s.} \tag{3.14}$$

Again from (3.2) and Jensen's inequality, we have

$$\overline{\mathbb{E}}^o\left(\frac{1}{n}[\tau_{n+1}^{(L)} - \tau_1^{(L)}]\right)^\alpha \le \overline{\mathbb{E}}^o \frac{1}{n}\sum_{i=1}^n (\kappa^{-L}\bar{\tau}_{i+1}^{(L)})^\alpha \le M(L)\kappa^{-\alpha L}. \tag{3.15}$$

So the sequence of random variables $\{[\tau_n^{(L)} - \tau_1^{(L)}]/n\}_{n\ge 1}$ is uniformly integrable, and the similar conclusion holds for $\{[X_{\tau_n^{(L)}} - X_{\tau_1^{(L)}}]/n\}_{n\ge 1}$. Therefore, taking the expectation in (3.14), we obtain

$$\lim_{n\to\infty}\left|\frac{\overline{\mathbb{E}}^o(X_{\tau_n^{(L)}})}{n} - v\frac{\overline{\mathbb{E}}^o(\tau_n^{(L)})}{n}\right| \to 0.$$

Since $X_{\tau_n^{(L)}} \cdot \ell \ge n|\ell|_1$, one concludes, using (3.15) again, that $|v \cdot \ell| \ge |\ell|_1 \kappa^L / M(L)^{1/\alpha} > 0$. □

PROOF OF COROLLARY 1.3. Throughout, we write $\overline{P}_\omega^z$ for $E_Q \overline{P}_{\omega,\varepsilon}^z$, and assume without loss of generality, that κ is taken small enough such that $\delta(\ell) > 2\kappa$. (Though the statement of the corollary only involves the walk, it is necessary to consider the ε random sequence, and so to introduce $\overline{P}_\omega^z$ rather than P_ω^z.) Recall that from Lemma 2.1,

$$q := \inf_\omega \overline{P}_\omega^o(D' = \infty) = \inf_\omega P_\omega^o(D' = \infty) \in (0, 1].$$

For $a > 0$, set

$$\psi_0^L(a) = \sup_\omega \overline{E}_\omega^o(\exp a\kappa^L \overline{S}_1), \qquad \psi_1^L(a) = \sup_\omega \overline{E}_\omega^o(\exp a\kappa^L \overline{S}_1 | D' < \infty).$$

We derive below estimates on $\psi_0^L(a)$, $\psi_1^L(a)$ and, in particular, show that under a nonnestling assumption, these functions are bounded (uniformly in L) for a small

enough, and, hence, can be made arbitrarily close (again, uniformly in L) to 1 by reducing $a > 0$. Assuming that, we get from the Markov property and independence of the ε sequence,

$$
\begin{aligned}
&\overline{E}^o_\omega(\exp\{a\overline{\tau}^{(L)}_1\}) \\
&\quad= \sum_{k=1}^{\infty} \overline{E}^o_\omega(\exp\{a\overline{\tau}^{(L)}_1\}\mathbf{1}_{K=k}) \\
&\quad= \sum_{z\in\mathbb{Z}^d} \overline{E}^o_\omega(\exp\{a\kappa^L\overline{S}_1\}\mathbf{1}_{X_{\overline{S}_1}=z})\cdot \overline{P}^z_\omega(D'=\infty) \\
&\qquad+ \sum_{z,z_1\in\mathbb{Z}^d} \overline{E}^o_\omega(\exp\{a\kappa^L\overline{S}_1\}\mathbf{1}_{X_{\overline{S}_1}=z})\cdot \overline{P}^z_\omega(D'<\infty) \\
&\qquad\qquad \times \overline{E}^z_\omega(\exp\{a\kappa^L\overline{S}_1\}\mathbf{1}_{X_{\overline{S}_1}=z_1}|D'<\infty)\cdot \overline{P}^{z_1}_\omega(D'=\infty)+\cdots \qquad (3.16)\\
&\quad\le \sup_\omega \overline{P}^o_\omega(D'=\infty) \\
&\qquad\times\left[\psi^L_0(a)+\psi^L_0(a)\psi^L_1(a)\left(1-\inf_\omega \overline{P}^o_\omega(D'=\infty)\right)+\cdots\right] \\
&\quad\le \psi^L_0(a)\left[\sum_{k=0}^{\infty}\psi^L_1(a)^k(1-q)^k\right] \\
&\quad= \frac{\psi^L_0(a)}{(1-(1-q)\psi^L_1(a))} =: g(a,L),
\end{aligned}
$$

where $\sup_L g(a,L)<\infty$ for small enough, positive a. Thus,

$$
\begin{aligned}
\mathbb{E}^o((\overline{\tau}^{(L)}_1)^\alpha|D'=\infty,\mathcal{F}^L_0) &\le \frac{\sup_\omega \overline{E}^o_\omega[(\overline{\tau}^{(L)}_1)^\alpha]}{\inf_\omega \overline{P}^o_\omega(D'=\infty)} \\
&\le \frac{\text{Const.}}{qa^\alpha}\sup_{\omega,L}\overline{E}^o_\omega e^{a\overline{\tau}^{(L)}_1},
\end{aligned}
$$

which, together with (3.16) and choosing $a>0$ small enough, yields (3.2) with any $\alpha>1$ and some M not depending on L. Hence, for any rate $\phi\to 0$, assumption ($\mathcal{A}5$) holds, thus, Theorem 3.4 applies.

We thus turn to the proof of the claimed (uniform in L) finiteness of $\psi^L_0(a)$ and $\psi^L_1(a)$ for a small enough. Since $\psi^L_1(a)\le\frac{\psi^L_0(a)}{1-q}$, it clearly suffices to consider $\psi^L_0(a)$. Let us denote here by T_m ($m=1,2\ldots$) the hitting time of the half-space $\{x\cdot\ell\ge mL|\ell|^2/|\ell|_1\}$, limited by the hyperplanes through points $m(L/|\ell|_1)\ell$ and orthogonal to ℓ. Time T_m is called *L-successful* if

$$
(\varepsilon_{T_m+1},\varepsilon_{T_m+2},\ldots,\varepsilon_{T_m+L})=\bar{\varepsilon}^{(L)}.
$$

We denote by $I = \inf\{m \geq 1;\ T_m \text{ is } L\text{-successful}\}$, and we note that, by definition, $\overline{S}_1 \leq T_I + L$, and that I is geometrically distributed on $\mathbb{N}^*$ with failure probability κ^L. Let

$$\psi_0^L(a) = \sup_\omega \overline{E}_\omega^o(\exp a\kappa^L(T_1 + L)),$$

$$\psi_1^L(a) = \sup_\omega \overline{E}_\omega^o(\{\exp a\kappa^L T_1\}\mathbf{1}_{\{(\varepsilon_1,\ldots,\varepsilon_L)\neq\bar{\varepsilon}^{(L)}\}}).$$

Similar to (3.16),

$$\begin{aligned}
&\overline{E}_\omega^o(\exp\{a\kappa^L\overline{S}_1\}\\
&\quad\leq \sum_{m=1}^{\infty} \overline{E}_\omega^o(\exp\{a\kappa^L(T_m + L)\}\mathbf{1}_{I=m})\\
&\quad= \overline{E}_\omega^o(\exp\{a\kappa^L(T_1 + L)\})\kappa^L\\
&\qquad+ \sum_{z\in\mathbb{Z}^d} \overline{E}_\omega^o(\exp\{a\kappa^L(T_1 + L)\}\mathbf{1}_{X_{T_1}=z})\\
&\qquad\qquad\times \overline{E}_\omega^o(\exp\{a\kappa^L(T_2 - T_1)\}\mathbf{1}_{I>1})\kappa^L + \cdots\\
&\quad\leq \kappa^L\psi_0^L(a)\left[\sum_{k=0}^{\infty}\psi_1^L(a)^k\right] = \frac{\kappa^L\psi_0^L(a)}{(1-\psi_1^L(a))_+},
\end{aligned}$$

with $(r)_+ = \max\{r, 0\}$. But, if $a \leq \lambda_0\delta(\ell)\kappa^{-|\ell|_1}$,

$$\begin{aligned}
\psi_1^L(a) &\leq (1-\kappa^L)\sup_{\omega,\varepsilon}\overline{E}_{\omega,\varepsilon}^o(\{\exp a\kappa^L T_1\})\\
&\leq (1-\kappa^L)\exp\{3a\kappa^L/\delta(\ell)\},
\end{aligned}$$

from (2.4). Hence, we can choose $a > 0$ small enough so that $\psi_1^L(a) < \infty$ for all L and such that

$$\sup_L \frac{\kappa^L}{(1-\psi_1^L(a))_+} < \infty, \qquad \sup_L \psi_0^L(a) < \infty.$$

This implies that $\sup_L \psi_0^L(a) < \infty$ for $a > 0$ small. □

4. Mixing. In dimension $d > 1$, a stationary random field P for which there exists a function $\phi(r) \to_{r\to\infty} 0$, such for that any two r-separated events A, B with $P(A) > 0$,

$$\left|\frac{P(A\cap B)}{P(A)} - P(B)\right| \leq \phi(r),$$

is necessarily of finite dependence ([3]; see also Theorem 2 in Section I.3 in [5]). This fact is the reason for considering here cones with $\zeta > 0$ in our definition (2.3)

of the no backtrack event D', rather than hyperplanes as in [18]. Our ϕ-mixing condition on cones does not directly relate with any of the many classical ones, see, for example, [5]. We present next an important example of distributions P for the environment field, that satisfy this condition.

DEFINITION 4.1. Let $k \geq 1$, and let $\partial\Lambda^k = \{z \in \Lambda^c; \operatorname{dist}(z, \Lambda) \leq k\}$ be the k-boundary of $\Lambda \subset \mathbb{Z}^d$. (dist and $|\cdot|$ both denote the Euclidean distance). A random field P is *k-Markov* if there exists a family π of transition kernels—called *specification*—$\pi_\Lambda = \pi_\Lambda(\prod_{y\in\Lambda} d\omega_y | \mathcal{F}_{\partial\Lambda})$ for finite $\Lambda \subset \mathbb{Z}^d$ such that

$$P\big((\omega_x)_{x\in\Lambda} \in \cdot | \mathcal{F}_{\Lambda^c}\big) = \pi_\Lambda(\cdot | \mathcal{F}_{\partial\Lambda}), \quad P\text{-a.s.} \tag{4.1}$$

In addition, a k-Markov field P is called *weak-mixing* if there exist constants $\gamma > 0$, $C < \infty$ such that for all finite subsets $V \subset \Lambda \subset \mathbb{Z}^d$,

$$\begin{aligned} &\sup\{\|\pi_\Lambda(\cdot|\omega) - \pi_\Lambda(\cdot|\omega')\|_V;\ \omega, \omega' \in S^{\Lambda^c}\} \\ &\qquad \leq C \sum_{y\in V, z\in\partial\Lambda^k} \exp(-\gamma|z-y|), \end{aligned} \tag{4.2}$$

with $\|\cdot\|_V = \|\cdot\|_{\mathrm{var},V}$ the variational norm on V, $\|\mu - \nu\|_V = \sup\{\mu(A) - \nu(A);\ A \subset \sigma((\omega_x)_{x\in V})\}$.

The weak-mixing property holds for environments produced by a Gibbsian particle system at equilibrium in the uniqueness regime [4, 11].

PROPOSITION 4.2. *Assume P is stationary and ergodic. If P is L-dependent, then P is ϕ-mixing on cones, that is, assumption $(\mathcal{A}1)$ is satisfied, with some function ϕ, $\phi(r) = 0$ for $r \geq L$. When P is weak-mixing k-Markov of constant γ, then assumption $(\mathcal{A}1)$ is satisfied with the function $\phi(r) = \mathrm{Const.}(\zeta)e^{-\gamma' r}$, and $\gamma' = \gamma/\sqrt{2}$.*

PROOF. 1. The statements in the L-dependent case are obvious. We can even take $\zeta = 0$, that is, we can replace cones by hyperplanes as in [18], throughout the paper.

2. We assume now that P is a weak-mixing k-Markov field. Fix some $\zeta > 0$ and $\ell \in \mathbb{R}^d \setminus \{0\}$. For $m, M, N > 0$, define the truncated cone V,

$$V_m = V = C(r\ell, \ell, \zeta) \cap \{y \in \mathbb{R}^d; (y - r\ell)\cdot\ell \leq m|\ell|^2\},$$

and the cylinder Λ,

$$\begin{aligned} \Lambda = \{&y \in \mathbb{R}^d; 0 \leq y\cdot\ell \leq (r+m+M)|\ell|^2, \\ &\big|y - (y\cdot\ell)\ell/|\ell|^2\big| \leq N|\ell| + m|\ell|\tan(\cos^{-1}(\zeta))\}, \end{aligned}$$

depicted in Figure 1.

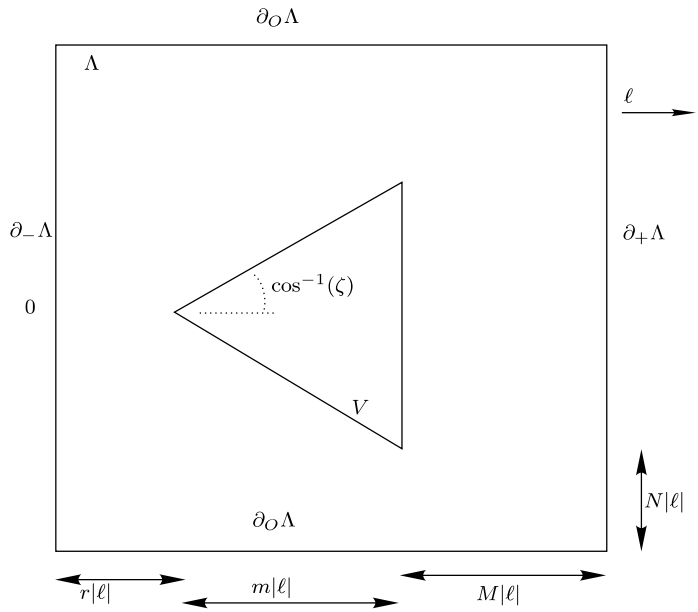

FIG. 1. *Cone definition.*

Then, $V \subset \Lambda$, and we split the sum over z in (4.2) into three terms, according to $\partial\Lambda^k = (\partial_-\Lambda) \cup (\partial_+\Lambda) \cup (\partial_O\Lambda)$ with

$$\partial_-\Lambda = \partial\Lambda^k \cap \{z \cdot \ell \le 0\},$$
$$\partial_+\Lambda = \partial\Lambda^k \cap \{z \cdot \ell \ge (r+m+M)|\ell|^2\},$$
$$\partial_O\Lambda = \partial\Lambda^k \cap \{|z - (z\cdot\ell)\ell/|\ell|^2| \ge N|\ell| + m|\ell|\tan(\cos^{-1}(\zeta))\}.$$

Since $\zeta > 0$, the number of points in the cone $C(r\ell, \ell, \zeta)$ at distance l from the hyperplane $z \cdot \ell = 0$ grows polynomially in l. Using $\sqrt{a^2+b^2} \ge (a+b)/\sqrt{2}$ and the previous remark, we find that

$$\sum_{y\in V, z\in\partial_-\Lambda} \exp(-\gamma|z-y|) \le \sum_{y\in V} \text{Const.}\exp(-\gamma' y\cdot\ell)$$
$$\le \text{Const.}\exp(-\gamma' r|\ell|)$$

with Const. some constant (depending on k, ζ) which may change from line to line. Similarly,

$$\sum_{y\in V, z\in\partial_O\Lambda} \exp(-\gamma|z-y|) \le \text{Const.}(N+m)^{d-2}\exp(-\gamma' N|\ell|)$$

and

$$\sum_{y\in V, z\in\partial_+\Lambda} \exp(-\gamma|z-y|) \le \text{Const.}\, m^{d-1}\exp(-\gamma' M|\ell|).$$

Letting now $M \to \infty$ and then $N \to \infty$ in (4.3), we get for all $A \in \sigma((\omega)_x, x \cdot \ell \leq 0)$, and all $B \in \sigma((\omega)_x, x \in V_m)$, from (4.2)

$$|P(B|A) - P(B)| \leq \text{Const.} \exp(-\gamma' r|\ell|),$$

with arbitrary m. But for any $B \in \sigma((\omega)_x, x \in C(r\ell, \ell, \zeta))$, there exists a sequence $B_m \in \sigma((\omega)_x, x \in V_m)$ with $\lim_m P(B \Delta B_m) = 0$. Therefore, such a B also satisfies the previous estimate, and P satisfies $(\mathcal{A}1)$ with exponentially vanishing ϕ. □

5. A nestling example. For simplicity, we work here with $\ell = e_1$. The example below can be modified to work for any ℓ at the expense of more cumbersome notations. Note that in this case, $(\mathcal{A}2)$ can be rephrased as the directional ellipticity condition:

$(\mathcal{A}2')$ There exists a $\kappa > 0$ such that $P(\omega(0, e_1) > \kappa) = 1$.

Our goal is to provide a family of *examples* of environments which are nestling and to which the results of this paper apply. The examples can be considered as a perturbation of the environment with $\omega(x, x + e_1) = 1$. Alternatively, they may also be considered as a perturbation of i.i.d. environments satisfying Kalikow's condition, perturbed by a slight dependence. (Indeed, it will appear from the proof that $\delta_0 \searrow 0$ as $\gamma' \to +\infty$ in the statement below.)

We claim the following theorem:

THEOREM 5.1. *Assume P satisfies $(\mathcal{A}1)$ with $\phi(L) \leq \text{Const.}(\zeta) e^{-\gamma' L}$ and $(\mathcal{A}2')$. Then there exists a $\delta_0 = \delta_0(\kappa, \gamma', d) < 1$ such that*

if P satisfies $(\mathcal{A}3)$ with $\delta(e_1) > \delta_0$, then P satisfies $(\mathcal{A}5)$.

A class of explicit examples satisfying the conditions of Theorem 5.1 is provided at the end of this section, after the following.

PROOF OF THEOREM 5.1. It is useful to consider the marked point process of fresh times and fresh points. Formally, fresh times are those times when the random walk achieves a new record value in the ℓ direction.

DEFINITION 5.2. A point $x \in \mathbb{Z}^d$ is called a *fresh point* for the RWRE (X_n), and a time s is called a *fresh time* for (X_n), if

$$\{X_n \cdot \ell < x \cdot \ell,\ n < s\} \cap \{X_s = x\}.$$

We label these random couples (s, x) according to increasing times, $0 = s_0 < s_1 < \cdots, x_n = X_{s_n} (n \geq 0)$. For transient walks in the direction ℓ, there are infinitely many fresh times $\{s_i\}_{i \geq 0}$ and fresh points $\{x_i\}_{i \geq 0}$, and, in the present case $\ell = e_1$, it holds $x_{i+1} \cdot \ell = x_i \cdot \ell + 1$.

Like in the proof of Corollary 1.3, call a fresh time s *L-successful* if $\varepsilon_{s+1} = \varepsilon_{s+2} = \cdots = \varepsilon_{s+L} = e_1$. Note that all $\overline{S}_k$ are L-successful fresh times, and that an L-successful fresh time s leads to an L-regeneration time (more accurately, an "approximate L-regeneration time") $s+L$ if $\theta_{s+L}D' = \infty$.

Define $F \geq 0$ by $L + s_F = \tau_1^{(L)}$, so that it holds that

$$X_{\tau_1^{(L)}} \cdot e_1 = F + L.$$

As a general feature, fresh points have much nicer tail properties than fresh times. The following summarizes some properties of fresh points and regeneration positions and does not require any additional assumptions. It is slightly stronger than what we need in the sequel.

LEMMA 5.3. *Assume* $(\mathcal{A}1)$, $(\mathcal{A}2)$, $(\mathcal{A}3)$. *Then there exist deterministic constants* $\zeta_0 > 0$ *and* $\lambda_2 = \lambda_2(\delta(\ell), \kappa, d)$ *and a function* $Q(\lambda)$ [*depending on* $\delta(\ell), \kappa$] *such that for any* $\zeta < \zeta_0$, $\lambda < \lambda_2$ *and all* $L > L_0(\lambda)$,

$$\begin{aligned}&\overline{\mathbb{E}}^o\bigl(\exp(\lambda\kappa^L X_{\tau_1^{(L)}} \cdot e_1)\bigr|\mathcal{F}_0^0\bigr)\\ &\quad = e^{\lambda L\kappa^L}\overline{\mathbb{E}}^o\bigl(\exp(\lambda\kappa^L F)\bigr|\mathcal{F}_0^0\bigr) < Q(\lambda) < \infty, \qquad P\text{-a.s.},\end{aligned}$$

with $\mathcal{F}_0^0$ *defined in* (3.2). *Further*, $\lambda_2 \to_{\delta(\ell)\to 1} 1$.

PROOF. Set $W := \min\{i : s_i \text{ is } L\text{-successful}\}$. By definition, $s_W + L = \overline{S}_1$ and $X_{\overline{S}_1} \cdot \ell = W + L$. We first evaluate the exponential moments of W. To every i, attach a random variable $\chi_i = \mathbf{1}_{\{\overline{\varepsilon}_{s_i+1} = e_1\}}$. Note that the χ_i are i.i.d., Bernoulli distributed with parameter κ and

$$W = \min\{j : \chi_j = \chi_{j+1} = \cdots = \chi_{j+L-1} = 1\}.$$

Consider the inter-failure times $\{\mu_i\}_{i\geq 1}$, that is, the sequence in $\mathbb{N}^*$ defined by

$$\{j > 0;\, \chi_j = 0\} = \{\mu_1, \mu_1 + \mu_2, \ldots\},$$

which is i.i.d., geometrically distributed with failure probability $(1-\kappa)$, and note that $W = \mu_1 + \cdots + \mu_i$ when $\mu_1 \leq L, \ldots, \mu_i \leq L, \mu_{i+1} > L$.

With the notation $\overline{E}_\omega^o = E_Q \overline{E}_{\omega,\varepsilon}^o$ as in the proof of Corollary 1.3, we have for all ω such that the walk is $\overline{P}_\omega^o$-a.s. transient in the direction ℓ,

$$\begin{aligned}\overline{E}_\omega^o \exp\{\lambda\kappa^L W\} &= \overline{E}_\omega^o \sum_{i\geq 0} \exp\{\lambda\kappa^L(\mu_1 + \cdots + \mu_i)\}\mathbf{1}_{\mu_1<L,\ldots,\mu_i<L,\mu_{i+1}\geq L}\\ &= \sum_{i\geq 0} \kappa^L \bigl(\overline{E}_\omega^o[\mathbf{1}_{\mu_1<L}\exp\{\lambda\kappa^L\mu_1\}]\bigr)^i\\ &= \frac{\kappa^L}{(1 - \overline{E}_\omega^o[\mathbf{1}_{\mu_1<L}\exp\{\lambda\kappa^L\mu_1\}])_+},\end{aligned}$$

while

$$\overline{E}^o_\omega[\mathbf{1}_{\mu_1<L}\exp\{\lambda\kappa^L\mu_1\}] = (1-\kappa)e^{\lambda\kappa^L}\frac{1-(\kappa e^{\lambda\kappa^L})^L}{1-(\kappa e^{\lambda\kappa^L})} = 1+(\lambda-1)\kappa^L+o(\kappa^L).$$

Hence, for all $\lambda<1$, there exists a finite $L_1(\lambda)$ with

$$\sup_{L\geq L_1(\lambda)}\operatorname*{ess\,sup}_{\omega\in\mathrm{supp}(P)}\overline{E}^o_\omega\exp\{\lambda\kappa^L W\}<\infty. \tag{5.1}$$

Note that the estimate in (5.1) is *quenched*, that is, for P-almost all environments.

Set

$$M_0=\max\{X_n\cdot\ell-X_0\cdot\ell,0\leq n<D'\}\in(0,\infty],$$

which is a.s. finite on the set $\{D'<\infty\}$ and infinite otherwise. Next, let $\overline{s}$ denote an L-successful fresh time (i.e., $\overline{s}=s_k$ for some k) and consider

$$\begin{aligned}M=M(\overline{s}) &:= M_0\circ\theta_{\overline{s}+L}\\ &=\max\{X_n\cdot\ell-X_{\overline{s}+L}\cdot\ell,\overline{s}+L\leq n<\overline{s}+L+D'\circ\theta_{\overline{s}+L}\}.\end{aligned}$$

Define

$$\overline{\mathcal{F}}_{\overline{s}}=\sigma(\omega_z:z\cdot\ell\leq X_{\overline{s}}\cdot\ell)\vee\sigma(X_t,t\leq\overline{s}),$$

and recall the definition from (3.2) $\mathcal{F}_0^L=\sigma(\omega(y,\cdot):\ell\cdot y<-L)$ and the notation $\bar\theta_x$ for space shift. It is useful to note that the annealed law of paths after fresh points has the following property,

$$\overline{\mathbb{P}}^o(M>r|\overline{\mathcal{F}}_{\overline{s}})=\overline{\mathbb{P}}^o(M_0>r|\mathcal{F}_0^L)\circ\bar\theta_{X_{\overline{s}+L}},\qquad r>0, \tag{5.2}$$

which implies also that $\overline{\mathbb{P}}^o(M>r,D'\circ\theta_{\overline{s}+L}<\infty|\overline{\mathcal{F}}_{\overline{s}})=\overline{\mathbb{P}}^o(M_0>r,D'<\infty|\mathcal{F}_0^L)\circ\bar\theta_{X_{\overline{s}+L}}$, since $\{M_0>r,D'<\infty\}=\{M_0\in(r,\infty)\}$ almost surely. Indeed, fix $n\geq L$, $\{x_i^*\}_{0\leq i<n}$ a path for walk with $x^*_{i+1}=x^*_i+e_1$ for $n-L\leq i\leq n$, fix some $K>0$ large and fix some $\omega^*\in S^{n+1}$ that belongs to the restriction of the support of the marginal of P on the path x^*, and fix an $A\in\sigma\{\omega(y):y\cdot\ell\leq x^*_{n-L}\cdot\ell\}$ such that

$$1-\frac{1}{\kappa Kn^2}\leq|\omega(x_i^*)/\omega_i^*-1|\leq1+\frac{1}{\kappa Kn^2},\qquad 0\leq i\leq n,\omega\in A. \tag{5.3}$$

[Note that $P(A)>0$ because ω^* belongs to the restriction of the support of the marginal of P on the path x^*. If $P(\omega^*)>0$, one may take $K=\infty$, and all inequalities below become equalities.] Hence, by the Markov property, and for arbitrary $\omega^{**}\in A$,

$$\begin{aligned}&\overline{\mathbb{P}}^o(M>r\,|\,\overline{s}=n-L,\{X_i\}_{0\leq i\leq n}=\{x_i^*\}_{0\leq i\leq n},\omega\in A)\\ &\quad=\frac{E_{P\otimes Q}[\,\overline{P}^o_{\omega,\varepsilon}(M>r,\overline{s}=n-L,\{X_i\}_{0\leq i\leq n}=\{x_i^*\}_{0\leq i\leq n})\mathbf{1}_{\omega\in A}]}{\overline{\mathbb{P}}^o(\overline{s}=n-L,\{X_i\}_{0\leq i\leq n}=\{x_i^*\}_{0\leq i\leq n},\omega\in A)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{E_{P\otimes Q}[\,\overline{P}^{x_n^*}_{\omega,\theta_n\varepsilon}(M_0>r)\overline{P}^{o}_{\omega,\varepsilon}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})\mathbf{1}_{\omega\in A}]}{\overline{\mathbb{P}}^{o}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n},\omega\in A)}\\
&\le \left(1+\frac{1}{\kappa K n^2}\right)^n E_{P\otimes Q}[\,\overline{P}^{x_n^*}_{\omega,\theta_n\varepsilon}(M_0>r)\\
&\qquad\qquad \times \overline{P}^{o}_{\omega^{**},\varepsilon}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})\mathbf{1}_{\omega\in A}]\\
&\quad\times(\overline{\mathbb{P}}^{o}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n},\omega\in A))^{-1}\\
&= \left(1+\frac{1}{\kappa K n^2}\right)^n E_Q[E_P[\,\overline{P}^{x_n^*}_{\omega,\theta_n\varepsilon}(M_0>r)\mathbf{1}_{\omega\in A}]\\
&\qquad\qquad \times \overline{P}^{o}_{\omega^{**},\varepsilon}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})]\\
&\quad\times(\overline{\mathbb{P}}^{o}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n},\omega\in A))^{-1}\\
&\le \frac{(1+1/(\kappa K n^2))^n}{(1-1/(\kappa K n^2))^n}\frac{E_{P\otimes Q}[\overline{P}^{x_n^*}_{\omega,\theta_n\varepsilon}(M_0>r)\mathbf{1}_{\omega\in A}]}{E_{P\otimes Q}[\mathbf{1}_{\omega\in A}]}\\
&\quad\times\frac{E_Q[\overline{P}^{o}_{\omega^{**},\varepsilon}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})]}{E_Q[\overline{P}^{o}_{\omega^{**},\varepsilon}(\overline{s}=n-L,\{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})]}\\
&\le \frac{(1+1/(\kappa K n^2))^n}{(1-1/(\kappa K n^2))^n}\frac{E_{P\otimes Q}\overline{P}^{x_n^*}_{\omega,\theta_n\varepsilon}(M_0>r,\omega\in A)}{P(\omega\in A)}\\
&\le C(K)\overline{\mathbb{P}}^{o}(M_0>r|\omega\in\overline{\theta}_{x_n^*}A)
\end{aligned}
$$

performing the same computations on the denominator, where $C(K)\to_{K\to\infty}1$ uniformly in n. The reverse inequality [with $1/C(K)$ replacing $C(K)$] is obtained similarly. One thus has (5.2), after taking $K\to\infty$.

We next derive tail estimates on M, considering only the case of M_0 in view of (5.2). For $x\in\mathbb{N}^*$, define

$$U_x^{(L)}=\{z\in\mathbb{Z}^d:-L\le z\cdot\ell\le x\}$$

and set $\tau_{x,L}=\min\{n>0:X_n\notin U_x^{(L)}\}$. Finally, let Π denote the orthogonal projection with respect to $\ell=e_1$, that is, $\Pi(z)=z-z_1e_1$, and K_0 a positive constant. Then, for any $\zeta>0$, using the Markov property and a union bound,

$$
\begin{aligned}
&\overline{\mathbb{P}}^{o}(M_0\in(x,\infty)|\mathcal{F}_0^L)\\
&\quad\le\overline{\mathbb{P}}^{o}(X_{\tau_{x,L}}\cdot\ell\ge x,|\Pi(X_{\tau_{x,L}})|>K_0x|\mathcal{F}_0^L)\\
&\qquad+(2K_0x)^{d-1}\max_{z:z\cdot\ell=x,|\Pi(z)|\le K_0x}\overline{\mathbb{P}}^{z}(X_\bullet\text{ hits }C(0,\ell,\zeta)^c|\mathcal{F}_0^L)\\
&\quad=:\mathrm{I}+\mathrm{II},
\end{aligned}
$$

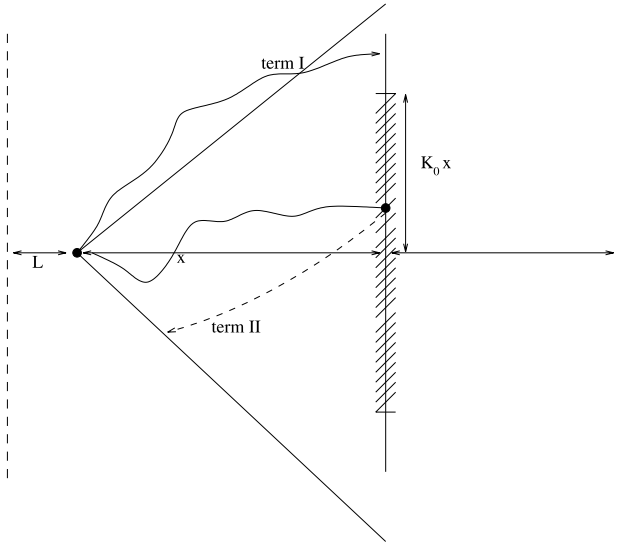

FIG. 2. *Escape events.*

as indicated in Figure 2.

Note that the term I does not depend on ζ. We argue below that, by Kalikow's condition,

$$\mathrm{I} \le e^{-g(K_0,\delta(\ell))x}, \qquad \bar{\mathbb{P}}^o\text{-a.s.}, \tag{5.4}$$

where $0 < g(K_0, \delta(\ell)) \to_{K_0 \to \infty} \infty$ is monotone nondecreasing in $\delta(\ell)$ and does not depend on ζ.

Fix now K_0 such that $g(K_0, \delta(\ell)) > 0$ and set ζ_0 such that

$$\{z : z \cdot \ell = x, |\Pi(z)| \le 2K_0 x\} \subset C(0, \ell, \zeta_0). \tag{5.5}$$

We will also argue that

$$\mathrm{II} \le e^{-\beta'_{K_0}(\delta(\ell))x}, \tag{5.6}$$

where $\beta'_{K_0} > 0$ comes from Kalikow's condition and $\beta'_{K_0}(\delta) \to_{\delta \to 1} \infty$. Therefore, we obtain for all x large, and all $\zeta < \zeta_0$,

$$\sup_{\omega} \overline{\mathbb{P}}^o\big(M \in (x, \infty) | \overline{\mathcal{F}}_{\overline{s}}\big) \le e^{-\beta(\delta(\ell))x} \tag{5.7}$$

for some function $\beta = \beta(\delta) > 0$ such that $\beta \to \infty$ as $\delta \to 1$.

Indeed, to see (5.4) and, thus, control I, we use (1.6) for the truncated strip $U_x^{(L)} \cap \{|\Pi(z)| \le \tilde{L}\}$ for large $\tilde{L} < \infty$ and apply the stopping theorem to the supermartingale $\exp\{-3\lambda_0 f(X_n)\}$, as in Lemma 2.1(1) [with the function $f(y) = y \cdot e_1 - \zeta' |y|$, $\zeta' = \delta(\ell)/2$] and to Kalikow's chain at the exit time from this strip.

Letting then $\tilde{L} \to \infty$, one readily gets (5.4) with $g(K_0) = 1.5\lambda_0(\delta(\ell)\sqrt{1+K_0^2} - 2)_+$. To prove (5.6), we proceed similarly with truncated cones, using (1.6) and, this time, the supermartingale $\exp\{-3\lambda_1 f(X_n)\}$ as in Lemma 2.1(4) [with the function $f(y) = y \cdot e_1 - \zeta_0|y|$ and assuming $\zeta_0 \le \delta(\ell)/3$ w.l.o.g.], to get

$$\overline{\mathbb{P}}^z(X_\bullet \text{ hits } C(0,\ell,\zeta)^c | \mathcal{F}_0^L) \times \exp\{0\} \le \exp(-3\lambda_1 x(1-\zeta_0\sqrt{K_0^2+1})),$$

which yields (5.6) with $\beta'_{K_0}(\delta) = 3\lambda_1(\delta)(1-\zeta_0\sqrt{K_0^2+1})$, where the factor $1-\zeta_0\sqrt{K_0^2+1} \ge 1 - (\sqrt{K_0^2+1}/\sqrt{4K_0^2+1}) > 0$ from (5.5).

Next, from Lemma 2.1(4), it follows also that, for any L-successful fresh time $\overline{s}$,

$$\overline{\mathbb{P}}^o(D' \circ \theta_{\overline{s}+L} = \infty | \overline{\mathcal{F}}_{\overline{s}}) \overset{(5.2)}{\ge} q_1 := \inf_\omega \overline{\mathbb{P}}^o(D' = \infty | \overline{\mathcal{F}}_0^L) > 0, \tag{5.8}$$

where $q_1 \to_{\delta\to 1} 1$ uniformly in L and $\zeta < \zeta_0$.

By definition, $X_{\overline{S}_1} \cdot e_1 = W + L$, and on $\{\overline{S}_k < \infty\}$, $(X_{\overline{S}_{k+1}} - X_{\overline{S}_k}) \cdot e_1 = M_k + W_{k+1} + L$, with the notations $M_k = M(\overline{S}_k)$, $W_{k+1} = W \circ \theta_{\overline{S}_k + M_k}$ if $M_k < \infty$, and $W_{k+1} = \infty$ on $\{M_k = \infty\} = \{D' \circ \theta_{\overline{S}_k} = \infty\}$. Put also $W_1 = W$, write

$$F + L = \sum_{k\ge 0} (M_k \mathbf{1}_{k\ne 0} + W_{k+1} + L)\mathbf{1}_{k<K},$$

(see Figure 3) and, with $\lambda > 0$,

$$\begin{aligned}
&\overline{\mathbb{E}}^o(e^{\lambda\kappa^L(F+L)} | \mathcal{F}_0^0) \\
&\quad \le \overline{\mathbb{E}}^o\left(\sum_{k\ge 0} e^{\lambda\kappa^L \sum_{l=0}^k (M_l \mathbf{1}_{l\ne 0} + W_{l+1} + L)} \mathbf{1}_{M_1,\dots,M_k<\infty} \middle| \mathcal{F}_0^0 \right) \\
&\quad = \sum_{k\ge 0} \overline{\mathbb{E}}^o\Big(e^{\lambda\kappa^L \sum_{l=0}^{k-1} (M_l \mathbf{1}_{l\ne 0} + W_{l+1} + L)} \mathbf{1}_{M_1,\dots,M_{k-1}<\infty} \\
&\qquad\qquad \times \overline{\mathbb{E}}^o\big[e^{\lambda\kappa^L(M_k \mathbf{1}_{k\ne 0} + W_{k+1} + L)} \mathbf{1}_{M_k<\infty} \big| \overline{\mathcal{F}}_{\overline{S}_k} \vee \mathcal{F}_0^0 \big] \big| \mathcal{F}_0^0 \Big) \\
&\quad \le \sum_{k\ge 0} \overline{\mathbb{E}}^o\Big(e^{\lambda\kappa^L \sum_{l=0}^{k-1} (M_l \mathbf{1}_{l\ne 0} + W_{l+1} + L)} \mathbf{1}_{M_1,\dots M_{k-1}<\infty} \big| \mathcal{F}_0^0 \Big) \\
&\qquad \times \overline{\mathbb{E}}^o\big[e^{\lambda\kappa^L(W+L)} \big] \sup_\omega \overline{\mathbb{E}}^o\big[e^{\lambda\kappa^L M_0} \mathbf{1}_{D'<\infty} \big| \mathcal{F}_0^0 \big] \\
&\quad \le \sum_{k\ge 0} \big(\overline{\mathbb{E}}^o\big[e^{\lambda\kappa^L(W+L)} \big] \big)^{k+1} \Big(\sup_\omega \overline{\mathbb{E}}^o\big[e^{\lambda\kappa^L M_0} \mathbf{1}_{D'<\infty} \big| \mathcal{F}_0^0 \big] \Big)^k \\
&\quad = \frac{e^{\lambda L\kappa^L} \overline{\mathbb{E}}^o[e^{\lambda\kappa^L W}]}{(1 - e^{\lambda L\kappa^L} \overline{\mathbb{E}}^o[e^{\lambda\kappa^L W}] \sup_\omega \overline{\mathbb{E}}^o[e^{\lambda\kappa^L M_0} \mathbf{1}_{D'<\infty} | \mathcal{F}_0^0])_+} =: Q(\lambda),
\end{aligned}$$

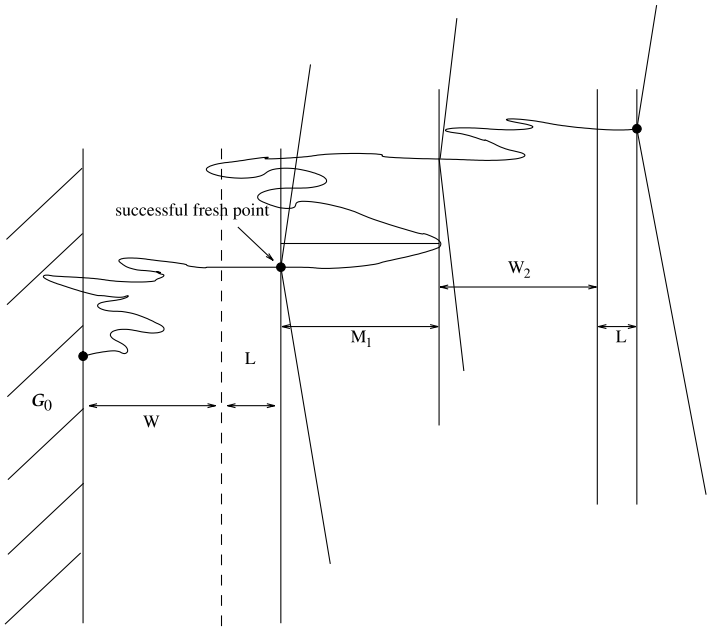

FIG. 3. *Successful fresh times leading to regeneration.*

where the second inequality comes from independence and from (5.2), and the third inequality from a recursion. From (5.1), $\sup_{L>L_1(\lambda)}\{e^{\lambda L\kappa^L}\overline{\mathbb{E}}^o[e^{\lambda\kappa^L W}]\} < \infty$ for $\lambda < 1$, while by Schwarz's inequality,

$$\sup_\omega \overline{\mathbb{E}}^o\big[e^{\lambda\kappa^L M_0}\mathbf{1}_{D'<\infty}\big|\mathcal{F}_0^0\big] \le \Big[(1-q_1)\sup_\omega \overline{\mathbb{E}}^o\big[e^{2\lambda\kappa^L M_0}\mathbf{1}_{M_0<\infty}\big|\mathcal{F}_0^0\big]\Big]^{1/2}.$$

From (5.7), we can choose L large enough so that the supremum in the right-hand side is finite, and, therefore, there exists some $\lambda_2 > 0$ such that $Q(\lambda) < \infty$ for $\lambda < \lambda_2$. This completes the proof of the first claim in the lemma. To obtain the second claim, we make $1 - q_1$ arbitrarily small by taking $\delta(\ell)$ close to 1, thus keeping Q finite for λ arbitrarily close to 1.

This completes the proof of Lemma 5.3. □

We now complete the proof of Theorem 5.1, by deriving probability estimates for tails of $\tau_1^{(L)}$. Fix $n > 0$, $T = 3\kappa^{-L}\ln n$, and a large constant $K_1 > 0$. Set

$$B = \{z \in \mathbb{Z}^d : -L \le z\cdot\ell \le T\},$$
$$\tilde{B}_{K_1} = \{z \in B : |\Pi(z)| \le K_1 T\},$$
$$\partial_+\tilde{B}_{K_1} = \{z \in \partial\tilde{B}_{K_1} : z\cdot\ell \ge T\}.$$

(Recall that Π is the projection on the hyperplane orthogonal to ℓ.) With T_{B^c} (resp. $T_{\tilde{B}^c_{K_1}}$), the exit time from B (resp. $\tilde{B}_{K_1}$), we decompose the set $\{\tau_1^{(L)} > \kappa^{-L}n\} \cap \{D' = \infty\}$ according to $X.$ exiting $\tilde{B}_{K_1}$ after time $\kappa^{-L}n$, and then decompose the latter case according to $X.$ exiting $\tilde{B}_{K_1}$ from $\partial_+\tilde{B}_{K_1}$ or not:

$$
\begin{aligned}
&\overline{\mathbb{P}}^o\big(\tau_1^{(L)} > \kappa^{-L}n, D' = \infty \big| \mathcal{F}_0^L\big) \\
&\quad \le \overline{\mathbb{P}}^o\big(T_{\tilde{B}^c_{K_1}} > \kappa^{-L}n, D' = \infty \big| \mathcal{F}_0^L\big) + \overline{\mathbb{P}}^o\big(X_{\tau_1^{(L)}} \cdot \ell > T \big| \mathcal{F}_0^L\big) \\
&\qquad + \overline{\mathbb{P}}^o\Big(X_{T_{\tilde{B}^c_{K_1}}} \notin \partial_+\tilde{B}_{K_1}, D' = \infty \big| \mathcal{F}_0^L\Big) \\
&\quad \le \overline{\mathbb{P}}^o\big(T_{\tilde{B}^c_{K_1}} > \kappa^{-L}n, D' = \infty \big| \mathcal{F}_0^L\big) + e^{-3\lambda \ln n} + 0,
\end{aligned}
\tag{5.9}
$$

where $\lambda < \lambda_2$ from Lemma 5.3, and where the last term is made equal to zero by fixing $K_1 > \zeta^{-1}(1-\zeta^2)^{1/2}$, in which case all paths contained in $C(0,\ell,\zeta)$ have to exit $\tilde{B}_{K_1}$ from $\partial_+\tilde{B}_{K_1}$.

In $\tilde{B}_{K_1}$, consider strips of width $\Delta = \delta_1 \ln n$, with $-\delta_1 \ln \kappa < 1$ and T/Δ integer,

$$
B_i = \big\{z \in \tilde{B}_{K_1} : (i-1)\Delta < z \cdot \ell < (i+1)\Delta\big\}, \qquad i = 0, 1, \ldots, T/\Delta,
$$

consider the truncated hyperplanes (slices) $A_i = \{z \in \tilde{B}_{K_1} : z \cdot \ell = [i\Delta]\}$ and define the random variables ($T_{B_i^c}$ = exit time of B_i),

$$
Y_i := \sup_{z \in A_i} P_\omega^z\big(X_{T_{B_i^c}} \cdot \ell < (i-1)\Delta\big),
$$

that is, the largest quenched probability starting from the middle slice A_i to exit the strip B_i at the left.

By Kalikow's condition, for $z \in A_i$,

$$
E\big(P_\omega^z(X_{T_{B_i^c}} \cdot \ell < (i-1)\Delta\big) \big| \omega(x,\cdot), x \notin B_i\big) \le e^{-c\Delta}
$$

with $c = c(\delta) \to_{\delta \to 1} \infty$ [using the supermartingale $\exp\{-2\lambda_1 f(X_n)\}$ from Lemma 2.1, see proof of (5.7)]. Hence,

$$
P\big(Y_i > e^{-c\Delta/2} \big| \omega(x,\cdot), x \notin B_i\big) \le (2K_1T)^{d-1} e^{-c\Delta/2}.
$$

In particular, the set

$$
\mathcal{A} := \big\{\exists i \le T/\Delta : Y_i > e^{-c\Delta/2}\big\}
$$

is such that $P(\mathcal{A}|\mathcal{F}_0^L) \le (T/\Delta)(2K_1T)^{d-1}e^{-c\Delta/2}$. Let now $\mathcal{B}$ denote the event that the walk, sampled at hitting times of neighboring slices, successively visits $A_1, A_2, \ldots, A_{T/\Delta}$ without backtracking to the neighboring slice on the left. Then, for $\omega \in \mathcal{A}^c$,

$$
P_\omega^0(\mathcal{B}^c) \le 1 - \big(1 - e^{-c\Delta/2}\big)^{(T/\Delta)} \le \frac{T}{\Delta} e^{-c\Delta/2},
$$

by convexity ($T/\Delta > 1$). On the other hand, the times spent inside each block are stochastically dominated by $2(\Delta+1)G_i$, where G_i are independent random variables, geometrically distributed with parameter $\kappa^{2(\Delta+1)}$; indeed, by ellipticity, the walk starting from any point in B_i has probability larger than $\kappa^{2(\Delta+1)}$ to exit B_i by traveling only with steps to the right, with at most $2(\Delta+1)$ steps. Now, since

$$\begin{aligned}&\overline{\mathbb{P}}^o\big(T_{\tilde{B}^c_{K_1}} > \kappa^{-L}n, D' = \infty \big| \mathcal{F}_0^L\big)\\ &\quad \le \overline{\mathbb{P}}^o(\mathcal{A}|\mathcal{F}_0^L) + \overline{\mathbb{P}}^o(\mathcal{A}^c \cap \mathcal{B}^c|\mathcal{F}_0^L) + E\big(P^o_\omega\big(T_{\tilde{B}^c_{K_1}} > \kappa^{-L}n, \mathcal{B}\big)\mathbf{1}_{\mathcal{A}^c}\big|\mathcal{F}_0^L\big),\end{aligned}$$

we have

$$\begin{aligned}&\overline{\mathbb{P}}^o\big(\{\tau_1^{(L)} > \kappa^{-L}n\} \cap \{D' = \infty\}\big|\mathcal{F}_0^L\big)\\ &\quad \le \left[n^{-3\lambda} + \frac{T}{\Delta}(2K_1T)^{d-1}e^{-c\Delta/2} + \frac{T}{\Delta}e^{-c\Delta/2}\right.\\ &\qquad\qquad \left. + P\left(\sum_{i\le T/\Delta} G_i > \frac{\kappa^{-L}n}{2(\Delta+1)}\right)\right] \wedge 1.\end{aligned}$$

But, on the last event, at least one of the G_i's is larger than $\Delta\kappa^{-L}n/[2(\Delta+1)T]$, so

$$\begin{aligned}P\left(\sum_{i\le T/\Delta} G_i > \frac{\kappa^{-L}n}{2(\Delta+1)}\right) &\le \frac{T}{\Delta}P\left(G_1 > \frac{n}{12\ln n}\right)\\ &= \frac{T}{\Delta}\big(1-\kappa^{2(\Delta+1)}\big)^{n/(12\ln n)}\\ &\le \frac{3\kappa^{-L}}{\delta_1}\exp\left\{\frac{-n}{12n^{-2\delta_1\ln\kappa}\ln n}\right\},\end{aligned}$$

and, finally,

$$\begin{aligned}&\overline{\mathbb{P}}^o\big(\{\tau_1^{(L)} > \kappa^{-L}n\} \cap \{D' = \infty\}\big|\mathcal{F}_0^L\big)\\ &\quad \le \left[n^{-3\lambda} + \frac{3\kappa^{-L}}{\delta_1}\left(n^{-\delta_1 c/2}[(6K_1\kappa^{-L}\ln n)^{d-1}+1]\right.\right.\\ &\qquad\qquad \left.\left. + \exp\left\{-\frac{n}{12n^{-2\delta_1\ln\kappa}\ln n}\right\}\right)\right] \wedge 1\\ &\quad \le \begin{cases}1, & \text{for } K_2\kappa^{-dL}(\ln n)^{d-1} > n^{+\delta_1 c/4},\\ 2n^{-[(\delta_1 c/4)\wedge(3\lambda)]}, & \text{else,}\end{cases}\end{aligned}$$

for some constant K_2, all n large enough and all $\lambda < \lambda_2$. Choose δ large enough such that $3\lambda_2 > 2$, which is always possible by Lemma 5.3 (we may need to take δ

even larger below). Then, with $\alpha = 2$,

$$M(L) = \mathbb{E}^o\big((\tau_1^{(L)}\kappa^L)^2 \mathbf{1}_{\{D=\infty\}} | \mathcal{F}_0^L\big) \le K_3 \kappa^{-8dL/c\delta_1},$$

for some constant K_3 independent of L, as soon as $c = c(\delta)$ is large enough, which happens if κ is kept fixed and $\delta \to 1$. Thus, as soon as $\phi'(\cdot)$ decreases exponentially one may find a $\delta(\ell)$ close enough to 1 such that $M(L)\phi'(L) \to_{L\to\infty} 0$. For such $\delta(\ell)$, we thus conclude that $(\mathcal{A}5)$ is satisfied with $\alpha = 2$. $\square$

We are now ready to describe the class of examples satisfying the assumptions of Theorem 5.1. Let $(\sigma(x), x \in \mathbb{Z}^d)$ be a d-dimensional nearest neighbor Ising model with $\beta \ge 0$, $h > 0$ and π the corresponding probability measure, that is,

$$\pi(\sigma(x) = \pm 1 | \sigma(y), y \ne x)$$
$$= \exp(\pm\{\beta \mathrm{S} + h\})/(\exp\{\beta \mathrm{S} + h\} + \exp(-\{\beta \mathrm{S} + h\})),$$

with $\mathrm{S} = \sum_{e:|e|=1} \sigma(x+e)$. Fix now two probability vectors $\omega^{\pm} = (\omega^{\pm}(e); e \in \mathbb{Z}^d, |e| = 1)$, and assume that

$$\omega^{\pm}(e) > 0, \qquad |e| = 1, \tag{5.10}$$

$$d^+ := \left(\sum_{|e|=1} \omega^+(e) e\right) \cdot e_1 > 0,$$
$$-d^- := \left(\sum_{|e|=1} \omega^-(e) e\right) \cdot e_1 < 0. \tag{5.11}$$

Consider the random environment given by

$$\omega(x, x+e) = \omega^{\pm}(e) \qquad \text{according to } \sigma(x) = \pm 1. \tag{5.12}$$

Note that the RWRE is nestling in the case where the local drifts points in opposite directions, for example, $\sum_{|e|=1} \omega^+(e) e \in (0,1) \cdot e_1$ and $\sum_{|e|=1} \omega^-(e) e \in (-1,0) \cdot e_1$.

THEOREM 5.4. *For all choices of $\omega^{\pm}$ satisfying* (5.10) *and* (5.11), *there exist a finite number h_0 and a positive function $\beta_0(h)$ with $\lim_{h\to+\infty} \beta_0(h) = \infty$ such that for $h > h_0$ and $\beta < \beta_0(h)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_n = v, \qquad \overline{\mathbb{P}}^o\text{-a.s.}$$

for some deterministic vector v $(v \cdot e_1 > 0)$.

Note that, since the above function β_0 is unbounded, the result applies to arbitrary low temperature–but with large external field.

PROOF OF THEOREM 5.4. We start by giving a sufficient condition for Kalikow's condition ($\mathcal{A}3$).

LEMMA 5.5. *If for a deterministic* $\delta > 0$, *we have P-a.s.*

$$\inf_{f:\{\pm e_i\}_1^d \to (0,1]} \left[E\left(\frac{1}{\sum_{|e|=1} f(e)\omega(x, x+e)} \middle| \mathcal{F}_{\{x\}^c} \right)^{-1} \times E\left(\frac{d(x,\omega)\cdot \ell}{\sum_{|e|=1} f(e)\omega(x, x+e)} \middle| \mathcal{F}_{\{x\}^c} \right) \right] \geq \delta, \tag{5.13}$$

then ($\mathcal{A}3$) *holds with* ℓ *and* $\delta(\ell) = \delta$.

The proof is similar to [7], pages 759 and 760, replacing μ therein with our Gibbs measure $\pi(\cdot|\sigma(y), y \neq x)$. We will omit it here, and we will use Lemma 5.5 with $\ell = e_1$.

LEMMA 5.6. *For condition* (5.13) *to hold with* $\delta > 0$ *and* $\ell = e_1$ *in the example* (5.12), *it is sufficient that*

$$\delta < d^+, \qquad 2h - 4\beta d \geq \ln \frac{(d^- + \delta)}{d^+ - \delta} + \ln \max_{|e|=1} \frac{\omega^+(e)}{\omega^-(e)}.$$

PROOF. To simplify notations, we set $\tilde{\pi}(\pm|\mathbb{S}) = \pi(\sigma(x) = \pm 1 | \sigma(y), y \neq x)$ on the set $\{\sum_{|e|=1} \sigma(x+e) = s\}$, and $\mathcal{S} = \{-2d, -2d+2, \ldots, 2d\}$, the set of possible values for s. For $\delta > 0$,

$$(5.13) \iff \forall f, s, \quad \frac{\tilde{\pi}(+|s)\frac{d^+}{\sum_{|e|=1} f(e)\omega^+(e)} + \tilde{\pi}(-|s)\frac{-d^-}{\sum_{|e|=1} f(e)\omega^-(e)}}{\tilde{\pi}(+|s)\frac{1}{\sum_{|e|=1} f(e)\omega^+(e)} + \tilde{\pi}(-|s)\frac{1}{\sum_{|e|=1} f(e)\omega^-(e)}} \geq \delta$$

$$\iff \forall f, s, \quad \frac{\tilde{\pi}(+|s)}{\tilde{\pi}(-|s)} \geq \frac{\sum_{|e|=1} f(e)\omega^+(e)}{\sum_{|e|=1} f(e)\omega^-(e)} \frac{d^- + \delta}{d^+ - \delta}, \tag{5.14}$$

since $\delta < d^+$. Note that for the Ising measure π,

$$\inf_{s \in \mathcal{S}} \frac{\tilde{\pi}(+|s)}{\tilde{\pi}(-|s)} = \inf_{s \in \mathcal{S}} \exp(2h + 2\beta s) = \exp(2h - 4\beta d),$$

while on the other hand,

$$\sup_{f:\{\pm e_i\}_1^d \to (0,1]} \frac{\sum_{|e|=1} f(e)\omega^+(e)}{\sum_{|e|=1} f(e)\omega^-(e)} = \max_{|e|=1} \frac{\omega^+(e)}{\omega^-(e)}.$$

Lemma 5.6 is proved. □

Let $c = c(\beta, h)$ be Dobrushin's contraction coefficient (e.g., definition (2.7) in [6]). If $c = c(h, \beta) < 1$, then π is weak-mixing, with a constant γ depending

only on c—as can be checked from (2.8) in [6]. (See also [5], Theorem 3, Section 2.2.1.3.) According to Proposition 4.2, P satisfies $(\mathcal{A}1)$ with $\phi(L) \le e^{-\gamma' L}$, where $\gamma' = \gamma/\sqrt{2}$ depends only on c. From Theorem 5.1 and Lemma 5.6, condition $(\mathcal{A}5)$ is implied by

$$\begin{aligned} c(h,\beta) &< 1, \\ \delta &< d^+, \\ 2h - 4\beta d &\ge \ln \frac{d^- + \delta}{d^+ - \delta} + \ln \max_{|e|=1} \frac{\omega^+(e)}{\omega^-(e)}, \\ \delta &> \delta_0(\omega^-(e_1), \gamma'(c(h,\beta)), d). \end{aligned}$$

Note that $c(h,\beta) \le c_0 < 1$ contains a region $0 \le \beta < \beta_0'(h)$ with $\beta_0'(h) \to \infty$ as $h \to \infty$. The proof of Theorem 5.4 is complete. □

6. Concluding remarks.

1. While working on the first draft of this work, we learned of [9], where the authors consider law of large numbers for L-dependent nonnestling environments. The approach of [9] is quite different from ours, as it relies on constructing an invariant measure, absolutely continuous with respect to the law P, which makes the regeneration sequence $\{\overline{\tau}_i^{(0)}, \overline{X}_i^{(0)}\}$ (with $L = 0$ and $\zeta = 0$ as introduced in [18]) a stationary sequence. These results are covered by our approach.

   In another paper [8] the same authors obtain the law of large numbers for L-dependent environments with Kalikow's condition, under the crucial assumption that with positive probability the walker jumps at a distance larger than L. An appropriate modification of the regeneration times introduced in [18] leads in that case to a renewal structure as in the independent environment.

   We also mention that [13] derived, for RWREs satisfying Kalikow's condition, results related to ours, by a clever adaptation of the method of the environment viewed from the point of view of the particle (see [10, 15]). The advantage of that approach is that it does not require explicitely extra integrability condition as in Assumption $(\mathcal{A}5)$.
2. It is reasonable to expect that under $(\mathcal{A}1), (\mathcal{A}2), (\mathcal{A}3)$, the integrability condition $(\mathcal{A}5)$, with $\alpha > 2$, implies that the annealed CLT for $(X_n - nv)/\sqrt{n}$ holds true. However, the proof of such a statement in this generality does present some challenges (there are particular cases where we can prove such a CLT). We hope to return to this question in future work.
3. It is worthwhile to note that in the case of i.i.d. environments, under Kalikow's condition it holds that $(\mathcal{A}5)$ with any $\alpha > 1$ is satisfied as soon as $d \ge 2$, by the results of [15]. It is not clear whether, under reasonable mixing conditions

which are not of the L-dependent type, ($\mathcal{A}5$) continues to hold true (the law of large numbers does hold true for the whole range of Kalikow's condition, by the work of [13] mentioned above). More generally, it is not clear what should the analogue of Sznitman's T'-condition (see [16]), be in the mixing setup.

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