CUT POINTS AND DIFFUSIVE RANDOM WALKS IN RANDOM ENVIRONMENT

Preliminary Draft

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Abstract

We study in this work a special class of multi-dimensional random walks in random environment for which we are able to prove in a non-perturbative fashion both a law of large numbers and a functional central limit theorem. As an application we provide new examples of diffusive random walks in random environment. In particular we construct examples of diffusive walks which evolve in an environment for which the static expectation of the drift does not vanish.

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0 Introduction

Over the recent years there has been considerable interest in the study of random walks in random environment. The asymptotic behavior of this canonical model of random motion in a random medium remains quite mysterious, especially in the multi-dimensional situation. Recent advances have mainly been concerned with the ballistic situation where the walk has a non-degenerate asymptotic velocity, see [16], [13], [14], [17]. Diffusive behavior has remained largely unexplored, except for the the work of Lawler [8] when the walk has no local drift, and of Bricmont-Kupiainen [2], for small isotropic perturbations of the simple random walk in dimension $d \geq 3$. The present article provides new examples of walks with diffusive behavior. It studies a special class of walks for which we are able to derive in a non-perturbative fashion the law of large numbers as well as central limit theorems. Interestingly, proofs are simple when compared to [2].

We now describe the setting. We consider two integers $d_1 \geq 5$, $d_2 \geq 1$, and write $d = d_1 + d_2$. We view $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ as the respective subspaces of $\mathbb{Z}^d$ of vectors with vanishing last d_2 and vanishing first d_1 components. Throughout this work we study random walks in random environment for which the $\mathbb{Z}^{d_1}$-projection evolves according to a standard random walk, and the random environment only affects the $\mathbb{Z}^{d_2}$-component. Specifically we consider a number $\kappa \in (0, \frac{1}{2d_1})$ (the ellipticity constant for the $\mathbb{Z}^{d_1}$-component) and a $(2d_1+1)$-vector governing the jump-distribution of the $\mathbb{Z}^{d_1}$-components of the walk:

$$(0.1)\quad \begin{aligned} &(q(e))_{|e|\leq 1, e\in\mathbb{Z}^{d_1}}, \text{ with } \textstyle\sum q(e) = 1,\ q(e) = q(-e) > 0, \text{ for } |e| \leq 1, e \in \mathbb{Z}^{d_1}, \\ &\text{and } q(e) \geq \kappa, \text{ for } e \neq 0, \end{aligned}$$

and introduce

$$(0.2)\quad \begin{aligned} &\mathcal{P}_{q(\cdot)} \text{ the set of } (2d)\text{-vectors } (p(e))_{|e|=1}, \text{ with } p(e) \in [0,1], \text{ for all } e \in \mathbb{Z}^d,\ |e| = 1, \\ &\sum_{|e|=1} p(e) = 1, \text{ and } p(e) = q(e), \text{ for } e \in \mathbb{Z}^{d_1}, |e| = 1\,. \end{aligned}$$

The random environment is an element $\omega = (\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ of $\Omega = \mathcal{P}_{q(\cdot)}^{\mathbb{Z}^d}$, endowed with the product σ-algebra and the product measure $\mathbb{P} = \mu^{\otimes\mathbb{Z}^d}$, where μ is a probability on $\mathcal{P}_{q(\cdot)}$ governing the distribution of the environment at a single site. The random walk in the random environment ω is the canonical Markov chain $(X_n)_{n\geq 0}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ with state space $\mathbb{Z}^d$, and "quenched" law $P_{x,\omega}$ starting from $x \in \mathbb{Z}^d$, under which

$$(0.3)\quad \begin{aligned} &P_{x,\omega}[X_{n+1} = X_n + e \,|\, X_0, \dots, X_n] \overset{P_{x,\omega}\text{-a.s}}{=} \omega(X_n, e),\ n \geq 0, |e| = 1\,, \\ &P_{x,\omega}[X_0 = x] = 1\,. \end{aligned}$$

The annealed laws are then defined as the semi-direct products on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$:

$$(0.4)\quad P_x = \mathbb{P} \times P_{x,\omega},\ \text{ for } x \in \mathbb{Z}^d\,.$$

Our very choice of environments ω in Ω forces the $\mathbb{Z}^d$-projection of X_n to evolve under $P_{x,\omega}$ as a random walk with jump distribution $q(\cdot)$. We assume symmetry of $q(\cdot)$ for otherwise we would be in a non-nestling situation where the law of large numbers and the central limit theorem have been proven in [16], [13]. The assumption $d_1 \geq 5$, enables to exploit the presence of cut times of the random walk, where loosely speaking past and future of the random walk have no intersection, (for the precise definition, see (1.4)). These cut times play a somewhat similar role to the regeneration times employed in [16], [13], although they do not provide a renewal structure.

In the above setting we are able to derive a law of large numbers:

$$P_0\text{-a.s.}, \ \frac{X_n}{n} \to v, \ (\text{with a deterministic } v)\,. \tag{0.5}$$

Further assuming that either the law of the environment is invariant under the antipodal transformation (cf. (2.1), in this case $v = 0$), and $d_1 \geq 7$, or without symmetry assumption that $d_1 \geq 13$, we obtain a functional central limit theorem under the quenched measure:

$$\begin{array}{l}\mathbb{P}\text{-a.s., under } P_{0,\omega}, \text{ the Skorohod-space valued } B^n_{\cdot} = \dfrac{1}{\sqrt{n}}(X_{[\cdot\, n]} - [\cdot\, n]\, v) \\ \text{converges in law to a Brownian motion with deterministic covariance.}\end{array} \tag{0.6}$$

One can of course replace the quenched measure by the annealed measure P_0 in (0.6). The above result in particular provides examples of diffusive behavior beyond current knowledge. It can also be applied to certain small perturbations of the standard random walk. For $\epsilon \in (0,1)$, following [15], we define

$$\begin{aligned}\mathcal{S}_\epsilon = \ & \text{the set of } (2d)\text{-vectors } (p(e))_{|e|=1}, \text{ with } \Big|p(e) - \frac{1}{2d}\Big| \leq \frac{\epsilon}{4d}, \text{ for all } e, \\ & \text{and } \sum_e p(e) = 1\,,\end{aligned} \tag{0.7}$$

and write $d(x,\omega)$ for the local drift:

$$d(x,\omega) = \sum_e \omega(x,e)e\,. \tag{0.8}$$

It is shown in [15], that for $\eta > 0$, and small ϵ depending on d and η, when the single site distribution is concentrated on $\mathcal{S}_\epsilon$, and the static expectation of the local drift $\mathbb{E}[d(0,\omega)]$ has size bigger than $\epsilon^{\frac{5}{2}-\eta}$, when $d = 3$, $\epsilon^{3-\eta}$, when $d \geq 4$, the walk has a non-vanishing limiting velocity (much more is known, see [15]). One can wonder whether the same remains true for arbitrarily small non-vanishing $\mathbb{E}[d(0,\omega)]$. We show here that this is not the case and provide examples when $d \geq 7$, of single site distributions concentrated on $\mathcal{S}_\epsilon$, for arbitrarily small ϵ, with $\mathbb{E}[d(0,\omega)] \neq 0$, but vanishing limiting velocity v, and even with diffusive behavior, when $d \geq 15$. We also construct further examples of analogous behavior for walks which are not small perturbations of the simple random walk.

Let us now explain how this article is organized.

In Section 1, we provide an alternative representation of the law of the walk under the annealed measure which takes advantage of the cut times. We then derive the law of large numbers in Theorem 1.4.

In Section 2, we prove the functional central limit theorem under the annealed measure. The case with antipodal symmetry and $d_1 \geq 7$ is covered by Theorem 2.1, the general case with $d_1 \geq 13$, is treated in Theorem 2.2.

Section 3 explains how the functional central limit theorem under the annealed measure can be strengthened to a similar statement under the quenched measure.

Section 4 provides examples of walks which are small perturbations of the simple random walk, for which $\mathbb{E}[d(0,\omega)] \neq 0$, but the limiting velocity vanishes, $(d \geq 7)$, and which behave diffusively, $(d \geq 15)$.

Section 5 contains further examples of analogous behavior, which in a certain sense are small perturbations of a one-dimensional random walk in a random environment.

1 An alternative representation of P_0 and a law of large numbers

In this section we fist introduce some further notations and provide a special representation of the walk under the measure P_0, see Proposition 1.2. This representation will provide an easy comparison of the walk under P_0 with a process constructed as an additive functional over a probability space with an ergodic shift. This will lead to a law of large numbers, cf. Theorem 1.4.

We begin with some notations. We denote by $(e_i)_{1\leq i\leq d}$ the canonical basis of $\mathbb{R}^d$, and by $|\cdot|$ the Euclidean distance on $\mathbb{R}^d$. For U a subset of $\mathbb{Z}^d$, $|U|$ denotes the cardinality of U and ∂U the boundary of U: $\partial U = \{x \in \mathbb{Z}^d \backslash U,\, \exists y \in U,\, |x-y| = 1\}$. The drift will be the $\mathbb{R}^d$-valued function on $\mathcal{P}_{q(\cdot)}$:

$$d(p) = \sum_{|e|=1} p(e)e = \sum_{i>d_1} (p(e_i) - p(-e_i))e_i, \text{ for } p(\cdot) \in \mathcal{P}_{q(\cdot)}\,. \tag{1.1}$$

To represent the random walk governing the evolution of the $\mathbb{Z}^{d_1}$-projection of the RWRE, we consider the product space

$$W_* = \{e \in \mathbb{Z}^{d_1},\ |e| \leq 1\}^{\mathbb{Z}}\,,$$

endowed with the product σ-algebra $\mathcal{W}_*$ and the product measure $P = q^{\otimes \mathbb{Z}}$, (in the notation of (0.1)). We denote by $(\theta_n)_{n\in\mathbb{Z}}$ the canonical shift on W_* and by $(I_n)_{n\in\mathbb{Z}}$ the canonical coordinates. We then define

$$X_n^1 = \begin{cases} I_1 + \cdots + I_n, & n \geq 1\,, \\ 0, & n = 0\,, \\ -(I_{n+1} + \cdots + I_0), & n \geq -1\,. \end{cases} \tag{1.2}$$

Observe that $X_n^1, n \geq 0$, and $X_n^1, n \leq 0$, are two independent random walks on $\mathbb{Z}^{d_1}$ with jump-distribution q, and that

$$X_n^1 \circ \theta_k = X_{n+k}^1 - X_k^1,\ n, k \in \mathbb{Z}\,. \tag{1.3}$$

The set of cut times where "future" and "past" of $X^1_{\cdot}$ have no intersection will play an important role in this article. Specifically, for $w \in W_*$, we consider

$$\mathcal{D}(w) = \{n \in \mathbb{Z},\ X^1_{(-\infty,n-1]} \cap X^1_{[n,\infty)} = \emptyset\}\,, \tag{1.4}$$

as well as the stationary point process

$$N(w,dk) = \sum_{n\in\mathbb{Z}} \delta_n(dk)\ 1\{n \in \mathcal{D}(w)\}\,. \tag{1.5}$$

It will be convenient to restrict P to the shift-invariant set of full P-measure, (cf. Lemma 1.1 below)

$$W = \{w \in W_*,\ N(w,(-\infty,0]) = N(w,[0,\infty)) = \infty\}\,. \tag{1.6}$$

We will write $\mathcal{W}$ for the restriction of $\mathcal{W}_*$ to W. We collect some useful properties relative to the point process N in the following

Lemma 1.1.

$$P(0 \in \mathcal{D}) > 0\,. \tag{1.7}$$

$$P(W) = 1, \textit{ and on } W,\ N(w,dk) = \sum_{m\in\mathbb{Z}} \delta_{T^m}(dk)\,, \tag{1.8}$$

where $T^m(w), m \in \mathbb{Z}$ *are* $\mathbb{Z}$*-valued variables on* W*, increasing with* m *and such that* $T^0 \le 0 < T^1$.

$$\widehat{P} \overset{\text{def}}{=} P[\,\cdot\,|\,0 \in \mathcal{D}] \textit{ is invariant under } \widehat{\theta} \overset{\text{def}}{=} \theta_{T^1}\,. \tag{1.9}$$

$$\int T^1\, d\widehat{P} = P[0 \in \mathcal{D}]^{-1} \textit{ and} \tag{1.10}$$

$$\int f\, dP = \int \sum_0^{T^1-1} f \circ \theta_k\, d\widehat{P} \Big/ \int T^1\, d\widehat{P}, \textit{ for } f \textit{ bounded measurable on } W\,. \tag{1.11}$$

$$P[T^1 > n] \le c(\log n)^{1+\frac{d_1-4}{2}}\, n^{-\frac{(d_1-4)}{2}},\ n \ge 1, \textit{ for a positive constant } c \textit{ depending only on } d_1 \textit{ and } q(\cdot)\,. \tag{1.12}$$

Proof. The claim (1.7) follows from the fact that X^1_n, $n \ge 0$, and X^1_{-n}, $n \ge 0$, are independent random walks on $\mathbb{Z}^{d_1}$, $d_1 \ge 5$, with jump distribution $q(\cdot)$ using classical estimates on the decrease of the transition probability, cf. Spitzer [12], p. 75, and similar arguments as in Section 3.2 of Lawler [9] or Section 4 of Erdös-Taylor [5]. Using the ergodicity of θ and (1.7), $P(W) = 1$ follows and (1.8) is straightforward. Up to a different normalization $\widehat{P}$ is the Palm measure attached to the stationary point process N, cf. Neveu [10], chapter II, (see in particular (10), p. 317). The statements (1.9), (1.10), (1.11) are then standard. We now turn to the proof of (1.12).

We consider an integer $L \geq 1$, and write:

$$k_j = 1 + Lj, \text{ for } j \geq 0\,. \tag{1.13}$$

Then for $J \geq 1$:

$$\begin{aligned} P[T^1 > k_{2J}] &= P[N(w,[1,k_{2J}]) = 0] \\ &\leq \sum_{0 \leq j < 2J+1} P[X^1_{(-\infty,k_j-1]} \cap X^1_{[k_j+1,\infty)} \neq \emptyset] \; + \\ &\qquad P[\text{for all } 0 \leq j < 2J+1,\; X^1_{(-\infty,k_j-1]} \cap X^1_{[k_j+1,\infty)} = \emptyset, \\ &\qquad \text{and } N(w,[1,k_{2J}]) = 0] \\ &\stackrel{\text{def}}{=} a_1 + a_2\,. \end{aligned} \tag{1.14}$$

We first bound a_2. To this end note that when $N(w,[1,k_{2J}]) = 0$ and $X^1_{(-\infty,k_j-1]} \cap X^1_{[k_j+1,\infty)} = \emptyset$ for $0 \leq j < 2J+1$, then for any $1 \leq j \leq 2J$,

$$\emptyset \neq X^1_{(-\infty,k_j-1]} \cap X^1_{[k_j,\infty)} = X^1_{[k_{j-1},k_j-1]} \cap X_{[k_j,k_{j+1}-1]}\,.$$

Hence using independence, we see that

$$a_2 \leq P[X^1_{[-L,-1]} \cap X^1_{[0,L-1]} \neq \emptyset]^J \leq P[0 \notin \mathcal{D}]^J\,. \tag{1.15}$$

We now turn to the control of a_1. We observe that

$$\begin{aligned} a_1 &\leq (2J+1)\, P[X^1_{(-\infty,-1]} \cap X^1_{[L,\infty)} \neq \emptyset] \\ &\leq (2J+1) \sum_{i \geq 1, j \geq L} P[X^1_{i+j} = 0] \leq (2J+1) \sum_{k \geq L} k\, P[X^1_k = 0] \\ &\leq (2J+1)\,\text{const}\; L^{-\frac{(d_1-4)}{2}}\,, \end{aligned} \tag{1.16}$$

using [12], p. 75, in the last step. Choosing a large enough γ depending on d_1, $q(\cdot)$, and setting $J = [\gamma \log n]$, $L = [\frac{n}{3J}]$, (1.12) now follows from (1.15), (1.16). □

We will now provide an alternative representation of the law of the walk under the annealed measure P_0. We let $\widetilde{W} = (\mathbb{Z}^{d_2})^{\mathbb{N}}$ stand for the space of $\mathbb{Z}^{d_2}$-valued trajectories $(\widetilde{w}(k))_{k \geq 0}$ and

$$\mathcal{I}(w) = \{k \geq 0,\; X^1_k(w) = X^1_{k+1}(w)\}, \text{ for } w \in W\,, \tag{1.17}$$

denote the non-negative idle times of X^1. We specify a probability kernel $K(w, d\widetilde{w}\, d\omega)$ from W to $\widetilde{W} \times \Omega$ through:

$$\begin{cases} \omega \text{ is } \mathbb{P}\text{-distributed}, \\ \widetilde{w}(0) = 0\,, \\ \text{for any } k \geq 0, \text{ conditionally on } \omega,\, \widetilde{w}(0), \dots, \widetilde{w}(k)\,, \\ \widetilde{w}(k+1) - \widetilde{w}(k) \text{ equals } 0, \text{ when } k \geq T^1 \text{ or } k \notin \mathcal{I}(w)\,, \\ \qquad\qquad e, \text{ with probability } \dfrac{\omega(X^1_k + \widetilde{w}(k), e)}{q(0)}, \text{ for any} \\ \qquad\qquad\quad e = \pm e_i,\; i > d_1, \text{ if } k < T^1 \text{ and } k \in \mathcal{I}(w)\,. \end{cases} \tag{1.18}$$

We can then consider the spaces

$$\Gamma_0 = W \times (\widetilde{W} \times \Omega)^{\mathbb{N}} \text{ and } \Gamma_s = W \times (\widetilde{W} \times \Omega)^{\mathbb{Z}}, \tag{1.19}$$

endowed with their product σ-fields, (the subscript "0" refers to P_0 and the subscript "s" to stationary) and the probabilities

$$Q_0 = P \times M_0,\ Q_s = P \times M_s, \tag{1.20}$$

where M_0 and M_s stand for the respective kernels from W to $(\widetilde{W} \times \Omega)^{\mathbb{N}}$ and W to $(\widetilde{W} \times \Omega)^{\mathbb{Z}}$ defined by

$$M_0(w, d\gamma_0) = K(w, d\widetilde{w}_0\, d\omega_0) \otimes \bigotimes_{m \geq 1} K(\theta_{T^m}\, w, d\widetilde{w}_m\, d\omega_m), \tag{1.21}$$

(with $\boldsymbol{\gamma}_0 = (w, \gamma_0) = (w, (\widetilde{w}_m, \omega_m)_{m \geq 0})$), and similarly

$$M_s(w, d\gamma_s) = \bigotimes_{m \in \mathbb{Z}} K(\theta_{T^m}\, w,\, d\widetilde{w}_m\, d\omega_m), \tag{1.22}$$

(with $\boldsymbol{\gamma}_s = (w, \gamma_s) = (w, (\widetilde{w}_m, \omega_m)_{m \in \mathbb{Z}})$). We will shortly see that (Γ_0, Q_0) is helpful in providing a representation of $X_.$ under P_0, whereas (Γ_s, Q_s) processes useful stationarity properties.

We now define on Γ_0 the $\mathbb{Z}^{d_2}$-valued process X_k^2, $k \geq 0$, via

$$\begin{cases} X_0^2 = 0,\ X_k^2 = \widetilde{w}_0(k), \text{ for } 0 \leq k \leq T^1, \\ X^2_{(T^m + k) \wedge T^{m+1}} = X^2_{T^m} + \widetilde{w}_m(k \wedge (T^{m+1} - T^m)) \text{ for } m \geq 1,\, k \geq 0, \end{cases} \tag{1.23}$$

in the notations of (1.21). In the sequel we will especially be interested in the $\mathbb{Z}^d$-valued process defined on Γ_0:

$$Z_k = X_k^1 + X_k^2,\ k \geq 0, \tag{1.24}$$

and by the $\mathcal{P}_{q(\cdot)}$-valued process (see (0.2)):

$$\begin{aligned} \sigma_k = \omega_0(Z_k, \cdot), &\text{ when } 0 \leq k \leq T^1, \\ \omega_m(Z_k - Z_{T^m}, \cdot), &\text{ when } T^m \leq k < T^{m+1}, m \geq 1. \end{aligned} \tag{1.25}$$

The above processes will easily be compared with the processes defined on Γ_s:

$$Z_k^s = X_k^1 + X_k^{2,s},\ k \in \mathbb{Z}, \tag{1.26}$$

$$\sigma_k^s = \omega_m(Z_k^s - Z_{T^m}^s, \cdot),\ \text{ for } T^m \leq k < T^{m+1}, \tag{1.27}$$

in the notations of (1.22), with

$$X_0^{2,s} = 0 \text{ and } X^{2,s}_{(T^m + k) \wedge T^{m+1}} = X^{2,s}_{T^m} + \widetilde{w}_m(k \wedge (T^{m+1} - T^m)), \text{ for } m \in \mathbb{Z},\ k \geq 0. \tag{1.28}$$

The next two propositions clarify the interest of the above objects.

Proposition 1.2. *Under Q_0, $(Z_k, \sigma_k)_{k\ge 0}$ has the same law as $(X_k, \omega(X_k,\cdot))_{k\ge 0}$ under P_0.*

Proof. For $\omega \in \Omega$, the Z^{d_1}-projection of $X_.$ under $P_{0,\omega}$ has same law as $(X_k^1)_{k\ge 0}$ under P. Further for $\omega \in \Omega$ if $Y_k, k \ge 0$, is a $\mathbb{Z}^{d_2}$-valued process such that $Y_0 = 0$ and for $k \ge 0$, conditionally on $X_.^1$, $Y_0, \dots, Y_k$, the increment $Y_{k+1} - Y_k$ is

$$(1.29)\qquad \begin{cases} 0, \text{ when } k \in \mathcal{I}(w)\,, \\ \\ \text{takes the value } e \text{ with probability } \dfrac{\omega(X_k^1 + Y_k, e)}{q(0)}, \text{ for } e = \pm e_i,\ i > d_1, \\ \text{when } k \notin \mathcal{I}(w)\,, \end{cases}$$

then

$$(1.30)\qquad (X_k^1 + Y_k, \omega(X_k^1 + Y_k, \cdot))_{k\ge 0} \text{ is distributed as } (X_k, \omega(X_k,\cdot))_{k\ge 0} \text{ under } P_{0,\omega}\,.$$

Letting $(\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ be i.i.d. μ-distributed (see below (0.2)), and replacing $P_{0,\omega}$ with P_0 the above identity of laws holds true as well. But the subsets of $\mathbb{Z}^{d_1}$: $X^1_{[0,T^1-1]}$, $X^1_{[T^1,T^2-1]}, \dots, X^1_{[T^m,T^{m+1}-1]}, \dots$ are disjoint. Hence if $(\omega_m)_{m\ge 0}$ is an i.i.d. sequence with common distribution $\mathbb{P}$, and one replaces in (1.29), and in the first expression of (1.30) ω with ω_0, if $0 \le k < T^1$ and $\omega_m(\cdot - (X^1_{T^m} + Y_{T^m}), \cdot)$, if $T^m \le k < T^{m+1}$, the identity in law is still preserved. Our claim now follows straightforwardly. $\square$

To take advantage of the stationarity property on (Γ_s, Q_s), we introduce on Γ_s the flow $(\Theta_k)_{k\in\mathbb{Z}}$ via:

$$(1.31)\qquad \Theta_k(\boldsymbol{\gamma}) = (\theta_k w, (\widetilde{w}_{n+m}, \omega_{n+m})_{m\in\mathbb{Z}}), \ \text{ on } T_n(w) \le k < T_{n+1}(w)\,,$$

with $\boldsymbol{\gamma}$ as below (1.22). This is the natural flow extending $(\theta_k)_{k\in\mathbb{Z}}$, if one views (w_m, ω_m) as marks of the δ_{T^m}, for $m \in \mathbb{Z}$.

Proposition 1.3.

$$(1.32)\qquad Z_n^s = \sum_{k=0}^{n-1} Z_1^s \circ \Theta_k, \textit{ for } n \ge 1,$$

$$(1.33)\qquad \sigma_n^s = \sigma_0^s \circ \Theta_n, \textit{ for } n \in \mathbb{Z}\,,$$

$$(1.34)\qquad \Theta_1 \textit{ preserves } Q_s \textit{ and in fact } (\Gamma_s, \Theta_1, Q_s) \textit{ is ergodic.}$$

Proof. Both (1.32) and (1.33) follow by direct inspection using (1.26) - (1.28). The fact Θ_1 preserves Q_s is checked by a straightforward calculation. Let us show the ergodicity of $(\Gamma_s, \Theta_1, Q_s)$. The Palm measure

$$(1.35)\qquad \widehat{Q}_s \overset{\text{def}}{=} Q_s(\cdot \,|\, 0 \in \mathcal{D}) = \widehat{P} \times M_s$$

attached to the stationary point process N preserves

$$(1.36)\qquad \widehat{\Theta} = \Theta_{T^1}\,,$$

(see Neveu [10], p. 338), and the analogue of (1.11) with Θ, $Q_s, \widehat{Q}_s$ in place of $\theta, P, \widehat{P}$ and f bounded measurable holds as well. Our claim is equivalent to the ergodicity of $(\Gamma_s \cap \{0 \in \mathcal{D}\}, \widehat{\Theta}, \widehat{Q}_s)$. Let A be measurable subset of $\Gamma_s \cap \{0 \in \mathcal{D}\}$ invariant under $\widehat{\Theta}$ and $\epsilon > 0$. We can find an integer $m_\epsilon \geq 1$ and a measurable subset A_ϵ depending only on w, $(\widetilde{w}_m, \omega_m)_{|m| \leq m_\epsilon}$, such that:

$$E^{\widehat{Q}_s}\big[\,|1_A - 1_{A^\epsilon}|\,\big] \leq \epsilon\,. \tag{1.37}$$

Then for $L \geq 0$,

$$\widehat{Q}_s(A) = E^{\widehat{Q}_s}[1_A\, 1_A \circ \widehat{\Theta}_L] = E^{\widehat{Q}_s}[1_{A_\epsilon}\, 1_{A_\epsilon} \circ \widehat{\Theta}_L] + c_\epsilon\,, \tag{1.38}$$

with $|c_\epsilon| \leq 2\epsilon$. On the other hand if $L > 2m_\epsilon$, conditioning on the w component and using the fact that the $(\widetilde{w}_m, \omega_m)_{m \in \mathbf{Z}}$ are independent conditionally on w (see (1.22), the above equals

$$E^{\widehat{P}}\,[\widehat{Q}_s(A_\epsilon|w)\; \widehat{Q}_s(A_\epsilon|w) \circ \widehat{\theta}_L] + c_\epsilon\;.$$

As a result

$$2\epsilon \geq \varlimsup_{N\to\infty}\, |\,\widehat{Q}_s(A) - \frac{1}{N} \sum_{L=0}^{N-1} E^{\widehat{P}}[\widehat{Q}_s(A_\epsilon|w)\; \widehat{Q}_s(A_\epsilon|w) \circ \widehat{\theta}_L]|\,, \tag{1.39}$$

but $(W \cap \{0 \in \mathcal{D}\}, \widehat{\theta}, \widehat{P})$ is ergodic as a consequence of the ergodicity of (W, θ, P) and $\frac{1}{N}\sum_0^{N-1} Q_s(A_\epsilon|w) \circ \widehat{\theta}_L \overset{L^1(\widehat{P})}{\longrightarrow} \widehat{Q}_s(A_\epsilon)$. We thus find with (1.37) and the above that

$$|\widehat{Q}_s(A) - \widehat{Q}_s(A)^2| \leq |\widehat{Q}_s(A) - \widehat{Q}_s(A_\epsilon)^2| + 2\epsilon \leq 4\epsilon\,.$$

Letting ϵ tend to 0, we see that $\widehat{Q}_s(A) = 0$ or 1, and our claim follows. $\square$

We will now apply the above to the derivation of a law of large numbers. In particular this will prove the existence of a (possibly vanishing) asymptotic velocity for the walk under the annealed measure P_0, when the single site distribution μ is concentrated on $\mathcal{P}_{q(\cdot)}$, (see (0.1), (0.2), with $d_1 \geq 5$, $d_2 \geq 1$).

Theorem 1.4. *Let Ψ be a bounded measurable function on $\mathcal{P}_{q(\cdot)}$, then*

$$P_0\text{-}a.s.,\; \frac{1}{n}\sum_{k=0}^{n-1} \Psi(\omega(X_k,\cdot)) \underset{n\to\infty}{\longrightarrow} E^{Q_s}[\Psi(\sigma_0^s)]\,, \tag{1.40}$$

and moreover in the notation of (1.1),

$$P_0\text{-}a.s.,\; \frac{X_n}{n} \to v \overset{\text{def}}{=} E^{Q_s}[d(\sigma_0^s)] = E^{Q_s}[Z_1^s]\,. \tag{1.41}$$

Proof. In view of Proposition 1.2, it suffices to prove similar statements with $(Z_k)_{k\geq 0}$ and $(\sigma_k)_{k\geq 0}$ in place of $(X_k)_{k\geq 0}$ and $(\omega(X_k,\cdot))_{k\geq 0}$.

In the notations of (1.19), we consider the kernel M from W to $(\widetilde{W}\times\Omega)\times(\widetilde{W}\times\Omega)^{\mathbb{Z}}$:

$$M(w,d\gamma)=K(w,d\widetilde{w}_0'\,d\omega_0')\otimes\bigotimes_{m\in\mathbb{Z}}K(\theta_{T^m}w,d\widehat{w}_m\,d\omega_m)\,, \tag{1.42}$$

for $\boldsymbol{\gamma}=(w,\gamma)=(w,(\widetilde{w}_0',\omega_0'),\ (\widetilde{w}_m,\omega_m)_{m\in\mathbb{Z}})$, and the probability Q on the space $\Gamma=W\times(\widetilde{W}\times\Omega)\times(\widetilde{W}\times\Omega)^{\mathbb{Z}}$ defined as the semi-direct product $Q=P\times M$. Then the applications

$$\boldsymbol{\gamma}\in\Gamma\xrightarrow{\Pi_0}\boldsymbol{\gamma}_0=(w,(\widetilde{w}_0',\omega_0'),\ (\widetilde{w}_m,\omega_m)_{m\geq 1})\in\Gamma_0$$

$$\boldsymbol{\gamma}\in\Gamma\xrightarrow{\Pi_s}\boldsymbol{\gamma}_s=(w,(\widetilde{w}_m,\omega_m)_{m\in\mathbb{Z}})\in\Gamma_s\,,$$

respectively map Q onto Q_0 and Q_s. Moreover with a slight abuse of notations, we see that

$$Q\text{-a.s., } Z_{T^1+k}-Z_{T^1}=Z^s_{T^1+k}-Z^s_{T^1},\ \sigma_{k+T^1}=\sigma^s_{k+T^1},\ k\geq 0\,. \tag{1.43}$$

As a result we find that for Ψ as in (1.40)

$$Q\text{-a.s., } \frac{1}{n}\sum_{k=0}^{n-1}\Psi(\sigma_k)-\frac{1}{n}\sum_{k=0}^{n-1}\Psi(\sigma^s_k)\to 0\,. \tag{1.44}$$

In view of Proposition 1.3 we can apply the ergodic theorem to the second expression in (1.44), and (1.40) follows. By (1.43), we also see that

$$Q\text{-a.s., } |Z_n-Z^s_n|\leq 2(T^1\wedge n)\,, \tag{1.45}$$

and from Proposition 1.3 and the ergodic theorem we conclude that

$$P_0\text{-a.s., } \frac{X_n}{n}\to E^{Q_s}[Z^s_1]\,. \tag{1.46}$$

Moreover by a martingale argument (under $P_{0,\omega}$),

$$E_0[X_n]=E_0\Big[\sum_{k=0}^{n-1}d(\omega(X_k,\cdot))\Big]\,, \tag{1.47}$$

and by (1.40) we now conclude that

$$E^{Q_s}[Z^s_1]=E^{Q_s}[d(\sigma^s_0)]\,,$$

finishing the proof of Theorem 1.4. □

2 Central limit theorem under the annealed measure

In the setting of the previous sections, we now present two central limit theorems for the walk under the measure P_0. Theorem 2.1 requires a symmetry assumption on the law of the environment, cf. (2.1) below, and holds when $d_1\geq 7$, on the other hand Theorem 2.2 makes no symmetry assumption, but holds when $d_1\geq 13$. We will later use Theorem 2.2

when providing in Sections 4 and 5 examples of diffusive behavior of the walk in biased environments.

For the first theorem, we assume the following "antipodal symmetry" of the single site distribution (see below (0.2))

$$\mu \text{ is invariant under } (p(e))_{|e|=1} \to (p(-e))_{|e|=1} . \tag{2.1}$$

Note that when (2.1) holds, $E_0[X_n] = 0$, for $n \geq 0$, and the limiting velocity v in (1.41) necessarily vanishes. In what follows we denote by $D(\mathbb{R}_+, \mathbb{R}^d)$ the set of $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$, which are right continuous with left limits, which is tacitly endowed with the Skorohod topology and its Borel σ-algebra, (cf. Chapter 3 of Ethier-Kurtz [6]).

Theorem 2.1. *($d_1 \geq 7$, under (2.1))*

Under P_0, the $D(\mathbb{R}_+, \mathbb{R}^d)$-valued sequence $B^n_\cdot = \frac{1}{\sqrt{n}} X_{[\cdot n]}$ converges in law to a Brownian motion with covariance matrix A given in (2.14).

Proof. In view of Proposition 1.2 and (1.45), it suffices to show that

$$\begin{array}{l} \text{under } Q_s,\ \dfrac{1}{\sqrt{n}}\, Z^s_{[\cdot n]} \text{ converges in law to a Brownian motion with} \\ \text{covariance matrix } A\,. \end{array} \tag{2.2}$$

Define the non-decreasing sequence $k_n, n \geq 0$, Q_s-a.s. surely tending to infinity such that $T^{k_n} \leq n < T^{k_n+1}$, and

$$\Sigma_m = Z^s_{T^m} - Z^s_{T^0}, \ \text{ for } m \geq 0\,. \tag{2.3}$$

Note that Q_s-a.s., for any $T > 0$:

$$\sup_{t \leq T} \Big| \frac{1}{\sqrt{n}}\, Z^s_{[tn]} - \frac{1}{\sqrt{n}}\, \Sigma_{k_{[tn]}} \Big| \leq 2 \sup_{0 \leq k \leq k_{[Tn]}} \frac{(T^{k+1} - T^k)}{\sqrt{n}}\,. \tag{2.4}$$

From (1.12) and $d_1 \geq 7$, we see that for $\gamma < \frac{3}{2}$,

$$E^P[(T^1)^\gamma] < \infty \tag{2.5}$$

and using (1.11) we conclude that for $\gamma < \frac{5}{2}$,

$$E^{\widehat{P}}[(T^1)^\gamma] = E^{\widehat{Q}_s}[(T^1)^\gamma] < \infty\,. \tag{2.6}$$

Using stationarity, we see that for $u > 0$,

$$\begin{array}{l} \widehat{P}\Big(\displaystyle\sup_{0 \leq k \leq [Tn]} \frac{(T^{k+1} - T^k)}{\sqrt{n}} > u\Big) \leq (Tn+1)\, \widehat{P}(T^1 > \sqrt{n}\, u) \leq \\ \dfrac{(Tn+1)}{n}\, E^{\widehat{P}}[(T^1)^2,\ T^1 > \sqrt{n}\, u] \underset{n \to \infty}{\overset{(2.6)}{\longrightarrow}} 0\,. \end{array}$$

On the other hand $\sup_{0\le k\le [Tn]} \frac{(T^{k+1}-T^k)}{\sqrt{n}}$ is invariant under θ_{T^0}, and by (1.11) the image of P under θ_{T^0} is $T^1\,\widehat{P}/\int T^1 d\widehat{P}$, so that the above calculation also proves that

$$\text{(2.7)} \qquad \sup_{0\le k\le [Tn]} \frac{(T^{k+1}-T^k)}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0 \text{ in } P \text{ (or } Q_s\text{)-probability}\,.$$

Since Q_s-a.s., $k_n \le n$ for all n, we see from (2.4), (2.7) that our claim will follow if we show (2.2) with $\frac{1}{\sqrt{n}}\ \Sigma_{k[tn]}$ in place of $\frac{1}{\sqrt{n}}\, Z^s_{[tn]}$.

Observe then that conditionally on w, under $\widehat{Q}_s$, the variables $Z^s_{T^{k+1}} - Z^s_{T^k}$, $k\ge 0$, are independent, cf. (1.22), (1.26), (1.28), with zero mean thanks to (2.1). Further from the ergodic theorem:

$$\text{(2.8)} \qquad \widehat{Q}_s\text{-a.s.},\ \frac{1}{n}\sum_{0\le k<n} (Z^s_{T^{k+1}} - Z^s_{T^k})(Z^s_{T^{k+1}} - Z^s_{T^k})^t \longrightarrow E^{\widehat{Q}_s}[(Z^s_{T^1})(Z^s_{T^1})^t] \overset{\text{def}}{=} \widetilde{A}\,.$$

Using the martingale central limit theorem, see Durrett [4], p. 334, or Ethier-Kurtz [6], p. 340, it follows from (2.6), (2.8) that

(2.9) for $\widehat{P}$-a.e. w, conditionally on w under $\widehat{Q}_s$, $\frac{1}{\sqrt{n}}\,\Sigma_{\cdot n}$ converges in law to a Brownian motion with covariance matrix $\widetilde{A}$, provided $\Sigma_s, s\ge 0$, stands for the linear interpolation of $\Sigma_m, m\ge 0$.

Noting that $\frac{1}{\sqrt{n}}\,\Sigma_{\cdot n}$ is invariant under Θ_{T^0} and the image of Q_s under Θ_{T^0} is $T^1\,\widehat{Q}_s/\int T^1\,d\widehat{P}$, it follows that

(2.10) under Q_s, $\frac{1}{\sqrt{n}}\,\Sigma_{\cdot n}$ converges in law to a Brownian motion with covariance matrix $\widetilde{A}$.

From the ergodic theorem, we know that

$$\text{(2.11)} \qquad \frac{T^m}{m} \to E^{\widehat{P}}[T^1] \ \ \widehat{Q}_s\text{-a.s.},$$

and by similar arguments as above the same holds true Q_s-a.s.. It then follows that Q_s-a.s. $\frac{k_n}{n} \to 1/\int T^1\,d\widehat{P}$, and with the help of Dini's theorem:

$$\text{(2.12)} \qquad Q_s\text{-a.s., for all } T>0,\ \sup_{0\le t\le T} \left|\frac{k_{[tn]}}{n} - \frac{t}{E^{\widehat{P}}[T^1]}\right| = 0\,.$$

From (2.10) and (2.12), we then conclude that

$$\text{(2.13)} \qquad \text{under } Q_s,\ \frac{1}{\sqrt{n}}\,\Sigma_{k_{[\cdot n]}} \text{ converges in law to a Brownian motion}$$

with covariance matrix

$$\text{(2.14)} \qquad A = E^{\widehat{Q}_s}[(Z^s_{T^1})(Z^s_{T^1})^t]/E^{\widehat{P}}[T^1]\ (= \widetilde{A}/E^{\widehat{P}}[T^1])\,,$$

which finishes the proof of our claim. $\square$

We now turn to the second theorem which does not require the symmetry assumption (2.1), and covers situations with possibly non-vanishing limiting velocity v, see (1.41).

Theorem 2.2. *($d_1 \geq 13$)*

Under P_0, the $D(\mathbb{R}_+, \mathbb{R}^d)$-valued sequence $B^n_{\cdot} = \frac{1}{\sqrt{n}}\,(X_{[\cdot n]} - [\cdot n]v)$ converges in law to a Brownian motion with covariance matrix A given in (2.20).

Proof. By Proposition 1.2 and (1.45), it suffices to prove a similar result for the sequence

$$\frac{1}{\sqrt{n}}\,(Z^s_{[\cdot n]} - [\cdot n]v) \overset{(1.32)-(1.41)}{=} \frac{1}{\sqrt{n}} \sum_{k=0}^{[\cdot n]-1} Y \cdot \Theta_k\,, \tag{2.15}$$

with the notation

$$Y = Z^s_1 - E^{Q_s}[Z^s_1]\,. \tag{2.16}$$

We now introduce on Γ_s, see (1.19), the filtration

$$\begin{aligned} &\mathcal{G}_k = \sigma(Z^s_{n+1} - Z^s_n,\ n < k,\ \text{ for }\ k \geq 0\,, \\ &(= \sigma(Z^s_n,\ n \leq k,\ \text{ since }\ Z^s_0 = 0)\,. \end{aligned} \tag{2.17}$$

□

The main step in proving Theorem 2.2 is provided by

Lemma 2.3. *There is a $G \in L^2(\Gamma^s, \mathcal{G}_0, Q_s)$ such that*

$$M_n \overset{\text{def}}{=} G \circ \Theta_n - G + Z^s_n - nv = G \circ \Theta_n - G + \sum_{k=0}^{n-1} Y \circ \Theta_k \ \textit{ is a } (\mathcal{G}_n)\textit{-martingale.} \tag{2.18}$$

Let us for the time being admit Lemma 2.3 and explain how we conclude the proof of Theorem 2.2. Observe that for any $\epsilon > 0$:

$$\begin{aligned} Q_s(\sup_{1 \leq m \leq n} G \cdot \Theta_m > \epsilon\sqrt{n}) &\leq n\, Q_s(G > \epsilon\sqrt{n}) \\ &\leq \epsilon^{-2}\, E^{Q_s}[G^2, G > \epsilon\sqrt{n}] \underset{n\to\infty}{\longrightarrow} 0\,, \end{aligned} \tag{2.19}$$

so that it suffices to prove that $\frac{1}{\sqrt{n}}\, M_{[\cdot n]}$ converges in law to conclude that $\frac{1}{\sqrt{n}}(Z^s_{[\cdot n]} - [\cdot n]v)$ converges in law to the same limit. However

$$M_n = \sum_{k=0}^{n-1} (G \circ \Theta_1 - G + Y) \circ \Theta_k$$

is a martingale with stationary increments and from the theorem of Billingsley and Ibragimov, see Durrett [4], p. 375, it follows that

$$\begin{aligned} &\text{under } Q_s,\ \tfrac{1}{\sqrt{n}}\, M_{[\cdot n]} \text{ converges in law to a Brownian motion with} \\ &\text{covariance matrix } A = E^{Q_s}[(G \circ \Theta_1 - G + Y)(G \cdot \Theta_1 - G + Y)^t]\,, \end{aligned} \tag{2.20}$$

which proves Theorem 2.2.

Proof of Lemma 2.3: To simplify notations, we drop the superscript Q_s when writing expectations or conditional expectations. It follows from (1.12) that

$$T^1 \in L^4(Q_s) (\text{or } L^4(P)) . \tag{2.21}$$

As we now explain the claim will follow once we show that

$$\sum_{p \geq 0} \| E[(H\, 1\{0 \in \mathcal{D}\}) \circ \Theta_p \,|\mathcal{G}_0]\|_2 < \infty , \tag{2.22}$$

where we recall the notation (1.4) and

$$H = \sum_{k=0}^{T^1-1} Y \circ \Theta_k\,, \quad (\text{note that } |H| \leq 2T^1, \text{ since } |Z_1^s| \leq 1) . \tag{2.23}$$

Indeed, if we define for $m \geq 1$,

$$G^m = E[H \,|\, \mathcal{G}_0] + \sum_{1 \leq p < m} E[(H\, 1\{0 \in \mathcal{D}\}) \cdot \Theta_p \,|\, \mathcal{G}_0] , \tag{2.24}$$

then G^m converges in $L^2(Q_s)$ towards $G \in L^2(\Gamma_s, \mathcal{G}_0, Q_s)$. Moreover, for $m \geq 1$, we can define in the notation of (1.5), $N = N(w, [1, m-1]) + 1$, so that for $n \geq 0$,

$$G \circ \Theta_n = \lim_{m \to \infty} E\Big[\Big(\sum_{k=0}^{T^N-1} Y \circ \Theta_k\Big) \circ \Theta_n \,|\, \mathcal{G}_n\Big] , \tag{2.25}$$

where the limit holds in L^2 and we have used stationarity. Hence

$$\begin{aligned}
&E\Big[G \circ \Theta_{n+1} + \sum_{k=0}^{n} Y \circ \Theta_k - G \circ \Theta_n - \sum_{k=0}^{n-1} Y \circ \Theta_k \,|\, \mathcal{G}_n\Big] \\
&= \lim_{m \to \infty} E\Big[E\Big[\Big(\sum_{k=0}^{T^N-1} Y \circ \Theta_k\Big) \circ \Theta_{n+1} \,|\, \mathcal{G}_{n+1}\Big] + Y \circ \Theta_n \\
&\quad - \Big(\sum_{k=0}^{T^N-1} Y \circ \Theta_k\Big) \circ \Theta_n | \mathcal{G}_n\Big] \\
&= \lim_{m \to \infty} E\Big[\Big(\sum_{k=0}^{T^N-1} Y \circ \Theta_k\Big) \circ \Theta_{n+1} + Y \circ \Theta_n - \Big(\sum_{k=0}^{T^N-1} Y \circ \Theta_k\Big) \circ \Theta_n | \mathcal{G}_n\Big]
\end{aligned} \tag{2.26}$$

The quantity under the conditional expectation in the above expression equals

$$1\{n + m \in \mathcal{D}\}\, H \circ \Theta_{n+m} = (H\, 1\{0 \in \mathcal{D}\}) \circ \Theta_{n+m} \tag{2.27}$$

and using (2.22) and stationarity we see that the last line of (2.26) vanishes. This proves that M_n, with the notation of (2.18), is a $(\mathcal{G}_n)$-martingale.

We are thus reduced to proving (2.22). To this end, we consider $B \in L^2(\Gamma_s, \mathcal{G}_0, Q_s)$ with unit L^2-norm. Then for $p \ge 1$,

$$E[(H\,1\{0 \in \mathcal{D}\}) \circ \Theta_p\, B] = \sum_{m \ge 1} E\Big[\Big(\sum_{T^m \le k < T^{m+1}} Y \circ \Theta_k\Big)\, B,\ T^m = p\Big]. \tag{2.28}$$

Note that B is $\mathcal{G}_0$-measurable and hence a function of w and $(\widetilde{w}_m, \omega_m)_{m \le 0}$, and $\sum_{T^m \le k < T^{m+1}} Y \circ \Theta_k = Z^s_{T^{m+1}} - Z^s_{T^m} - (T^{m+1} - T^m)v \overset{(1.26),(1.28)}{=} X^1_{T^{m+1}} - X^1_{T^m} + \widetilde{w}_m(T^{m+1} - T^m) - (T^{m+1} - T^m)v$. Hence conditioning on w in the right member of (2.28), and using the notation of (1.22), we find that for $p \ge 1$:

$$\begin{aligned} &E[(H\,1\{0 \in \mathcal{D}\}) \circ \Theta_p\, B] = \sum_{m \ge 1} E^P[(M_s H) \circ \theta_p\, M_s B,\ T^m = p] = \\ &E^P[((M_s H)\,1\{0 \in \mathcal{D}\}) \circ \theta_p\, M_s B]\,. \end{aligned} \tag{2.29}$$

Then observe that we can find measurable functions ψ and φ such that

$$M_s B = \psi(T^0, (X^1_i)_{i \le 0}),\ (M_s H)\,1\{0 \in \mathcal{D}\} = \varphi(T^1, (X^1_i)_{i \ge 0})\,1\{0 \in \mathcal{D}\}\,. \tag{2.30}$$

To take advantage of decoupling effects, we define

$$L = \Big[\frac{p}{3}\Big]\,, \tag{2.31}$$

and introduce two copies (X^-_n) and (X^+_n) of (X^1_n), such that X^-_n coincides with X^1_n for $n \le L$ and then "evolves" independently, whereas X^+_n coincides with $X^1_{n+p} - X^1_p$, for $n \ge -L$, and for $n < -L$, "evolves" independently. We then define

$$\begin{cases} U &= M_s B,\ U^- = \psi(T^-, (X^-_i)_{i \le 0})\,, \\ V &= ((M_s H)\,1\{0 \in \mathcal{D}\}) \circ \theta_p = \varphi(T^1 \circ \theta_p,\ (X^1_{i+p} - X^1_p)_{i \ge 0})\,1\{p \in \mathcal{D}\}\,, \\ V^+ &= \varphi(T^+, (X^+_i)_{i \ge 0})\,1\{0 \in \mathcal{D}^+\}\,, \end{cases} \tag{2.32}$$

where T^- and T^+ are respectively defined like T^0 and T^1 relatively to $(X^-_.)$ and $(X^+_.)$ and $\mathcal{D}^+$ is defined analogously to $\mathcal{D}$ with $(X^+_.)$ in place of $X_.$. We of course tacitly abuse the notations since the above objects are defined on an extension of the space $(W, \mathcal{W}, P)$. Note that

$$U \overset{\text{law}}{=} U^-,\ \ V \overset{\text{law}}{=} V^+\,. \tag{2.33}$$

We now find that for $p \ge 1$:

$$\begin{aligned} &E[(H\,1\{0 \in \mathcal{D}\}) \circ \Theta_p\, B] \overset{(2.29)}{=} E^P[VU] = \\ &E^P[V^+U^-] + E^P[V^+(U - U^-)] + E^P[(V - V^+)U]\,. \end{aligned} \tag{2.34}$$

Note also that:

$$E^P[V] = E^P[V^+] = E[H\,1\{0 \in \mathcal{D}\}] = E[Y]\,E^P[T_1] = 0\,, \tag{2.35}$$

using the analogue of (1.11) for Q_s, $\widehat{Q}_s$ and (2.16) in the third equality. Note that V^+ and U^- are independent. Hence the first term in the last member of (2.34) vanishes. Keeping in mind that B has unit L^2-norm we find

$$|E[(H\,1\{0 \in \mathcal{D}\}) \circ \Theta_p\, B]| \le \|V^+(U-U^-)\|_1 + \|V - V^+\|_2\,. \tag{2.36}$$

In view of (2.32) and the inequality $|H| \le 2T^1$, we find

$$\begin{aligned} &|V| \le 2|T^1 \circ \theta_p|,\ |V^+| \le 2|T^+|\,,\\ &|V - V^+| \le 2(1\{T^+ \neq T^1 \circ \theta_p\} + |1\{p \in \mathcal{D}\} - 1\{0 \in \mathcal{D}^+\}|)(T^+ + T^1 \circ \theta_p)\,. \end{aligned} \tag{2.37}$$

Using Cauchy-Schwarz's inequality and stationarity, we find

$$\|V - V^+\|_2 \le 4\|T^1\|_2\, (P[T^+ \neq T^1 \circ \theta_p]^{\frac{1}{2}} + 2\, P[\{p \in \mathcal{D}\}\backslash\{0 \in \mathcal{D}^+\}]^{\frac{1}{2}})\,. \tag{2.38}$$

Since X_n^+ and $X^1_{n+p} - X^1_p$ coincide for $n \ge -L$, we see that:

$$\{T^1 \circ \theta_p \neq T^+\} \subseteq \{X^+_{(-\infty,-L]} \cap X^+_{[0,\infty)} \neq \emptyset\} \cup \{(X^1 \circ \theta_p)_{(-\infty,-L]} \cap (X^1 \circ \theta_p)_{[0,\infty)]} \neq \emptyset\}\,, \tag{2.39}$$

and by a similar argument $\{p \in \mathcal{D}\}\backslash\{0 \in \mathcal{D}^+\}$ is included in the right hand side of (2.39). As a result we obtain:

$$\|V - V^+\|_2 \le 24\, \|T^1\|_2\, P[X^1_{(-\infty,-L]} \cap X^1_{[0,\infty)]} \neq \emptyset]^{\frac{1}{2}}\,. \tag{2.40}$$

By analogous arguments we also have

$$\begin{aligned} |U - U^-| &\le (|U| + |U^-|)\, 1\{T^0 \neq T^-\}\\ &\le (|U| + |U^-|)(1\{X^1_{(-\infty,0]} \cap X^1_{[L,\infty)} \neq \emptyset\} + 1\{X^-_{(-\infty,0]} \cap X^-_{[L,\infty)} \neq \emptyset\})\,. \end{aligned} \tag{2.41}$$

Using Hölder's inequality and $\|U\|_2 = \|U^-\|_2 \le 1$, we find

$$\|V^+(U-U^-)\|_1 \le 4\, \|T^1\|_4\ P[X^1_{(-\infty,0]} \cap X^1_{[L,\infty)}]^{\frac{1}{4}}\,. \tag{2.42}$$

Collecting (2.36), (2.40), (2.42), and using the fact that (X^1_n) and (X^1_{-n}) have same law (see (1.2)), we find

$$\|E[(H\,1\{0 \in \mathcal{D}\}) \circ \Theta_p | \mathcal{G}_0]\|_2 \le 28 \|T^1\|_4\, P[X^1_{(-\infty,-L]} \cap X^1_{[0,\infty)} \neq \emptyset]^{\frac{1}{4}}\,. \tag{2.43}$$

By the calculation in (1.16) we know that the rightmost expression is bounded by const $p^{-\frac{(d_1-4)}{8}}$, (recall (2.31)), and hence summable in p since $d_1 \ge 13$. This concludes the proof of (2.22) and consequently of Lemma 2.3. $\square$

3 Central limit theorem under the quenched measure

In this section we will explain how the central limit theorems of the previous section can be strengthened into statements under the quenched measure $P_{0,\omega}$, for $\mathbb{P}$-a.e. ω.

Theorem 3.1. *Assume $d_1 \geq 7$ and (2.1) or $d_1 \geq 13$. Then for $\mathbb{P}$-a.e. ω, under $P_{0,\omega}$, the $D(\mathbb{R}_+, \mathbb{R}^d)$-valued $B^n_\cdot = \frac{1}{\sqrt{n}}\,(X_{[\cdot n]} - [\cdot n]v)$ converges in law to a Brownian motion with covariance A given in Theorem 2.1 and 2.2 respectively.*

Proof. The claim will follow from a variance calculation. It is convenient to introduce the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$, and the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued variable

$$\beta^n_\cdot = \text{the polygonal interpolation of } \tfrac{k}{n} \to B^n_{\frac{k}{n}},\ k \geq 0\,. \tag{3.1}$$

It will also be useful to consider the analogously defined space $C([0,T], \mathbb{R}^d)$, of continuous $\mathbb{R}^d$-valued functions on $[0,T]$, for $T > 0$, which we endow with the distance

$$d_T(v, v') = \sup_{s \leq T} |v(s) - v'(s)| \wedge 1\,. \tag{3.2}$$

From Lemma 4.1 of [1], the claim will follow once we show that for all $T > 0$, for all bounded Lipschitz functions F on $C([0,T], \mathbb{R}^d)$ and $b \in (1,2]$:

$$\sum_m \operatorname{var}_{\mathbb{P}}(E_{0,\omega}[F(\beta^{[b^m]})]) < \infty\,, \tag{3.3}$$

(with a slight abuse of notations).

Before proving (3.3) we still need to introduce some further notations. Given $\omega \in \Omega$, we consider two independent copies $(X_k)_{k\geq 0}$ and $(\widetilde{X}_k)_{k\geq 0}$ evolving according to $P_{0,\omega}$. The respective $\mathbb{Z}^{d_1}$-projections $(X^1_k)_{k\geq 0}$ and $(\widetilde{X}^1_k)_{k\geq 0}$ are then independent and with distribution given in (1.2). We then denote by $\mathcal{C}$ the set of one-sided cut-times of $X^1_\cdot$:

$$\mathcal{C} = \{k \geq 1,\ X^1_{[0,k-1]} \cap X^1_{[k,\infty)} = \emptyset\}\,, \tag{3.4}$$

with an analogously defined $\widetilde{\mathcal{C}}$ attached to $\widetilde{X}^1$. We then pick:

$$b \in (1,2],\ \ 0 < \mu < \nu < \frac{1}{2}\,, \tag{3.5}$$

and for $m \geq 1$, we define $n = [b^m]$,

$$\tau_m = \inf\{\mathcal{C} \cap [n^\mu, \infty)\} < \infty,\ P_{0,\omega}\text{-a.s. (cf. Lemma 1.1)}\,, \tag{3.6}$$

as well as the corresponding variable $\widetilde{\tau}_m$ attached to $\widetilde{X}^1$. We will also need the event:

$$\mathcal{A}_m = \{\tau_m \vee \widetilde{\tau}_m \leq n^\nu \text{ and } X^1_{[\tau_m,\infty)} \cap \widetilde{X}^1_{[\widetilde{\tau}_m,\infty)} = \emptyset\}\,. \tag{3.7}$$

We now prove (3.3). Without loss of generality, we assume that $|F| \le 1$ and the Lipschitz constant of F is smaller than 1. Then for $m \ge 1$:

$$
\begin{aligned}
\text{(3.8)} \qquad & \operatorname{var}_{\mathbb{P}}(E_{0,\omega}[F(\beta^n_\cdot)]) = \mathbb{E}\big[E_{0,\omega} \otimes E_{0,\omega}[F(\beta^n_\cdot)\, F(\widetilde{\beta}^n_\cdot)]\big] - E_0 \otimes E_0[F(\beta^n_\cdot)\, F(\widetilde{\beta}^n_\cdot)] \\
& = \mathbb{E}\big[E_{0,\omega} \otimes E_{0,\omega}[F(\beta^n_\cdot)\, F(\widetilde{\beta}^n_\cdot), \mathcal{A}_m]\big] - E_0 \otimes E_0[F(\beta^n_\cdot)\, F(\widetilde{\beta}^n_\cdot), \mathcal{A}_m] + d_m \,,
\end{aligned}
$$

and with a slight abuse of notations

$$
\text{(3.9)} \qquad |d_m| \le 2(P \times P)(\mathcal{A}_m^c)\,.
$$

Moreover observe that P_0-a.s.

$$
\text{(3.10)} \qquad \sup_{s \ge 0} \Big|\Big(\beta^n_{s+\frac{\tau_m}{n}} - \beta^n_{\frac{\tau_m}{n}}\Big) - \beta^n_s\Big| \le \frac{2}{\sqrt{n}}\,(\tau_m + 1), \ \text{ and}
$$

$$
\text{(3.11)} \qquad \beta^n_{\cdot+\frac{\tau_m}{n}} - \beta^n_{\frac{\tau_m}{n}} = \text{the polygonal interpolation of } \frac{k}{n} \to \frac{1}{\sqrt{n}}\,(X_{k+\tau_m} - X_{\tau_m} - kv)\,.
$$

From the Lipschitz property of F and (3.7) we see that the first two terms of the last member of (3.8) equal

$$
\begin{aligned}
\text{(3.12)} \qquad & \mathbb{E}\big[E_{0,\omega} \otimes E_{0,\omega}[F(\beta^n_{\cdot+\frac{\tau_m}{n}} - \beta^n_{\frac{\tau_m}{n}})\, F(\widetilde{\beta}^n_{\cdot+\frac{\tilde{\tau}_m}{n}} - \widetilde{\beta}^n_{\frac{\tilde{\tau}_m}{n}}),\ \mathcal{A}_m]\big] \\
& - E_0 \otimes E_0\big[F(\beta^n_{\cdot+\frac{\tau_m}{n}} - \beta^n_{\frac{\tau_m}{n}})\, F(\widetilde{\beta}^n_{\cdot+\frac{\tilde{\tau}_m}{n}} - \widetilde{\beta}^n_{\frac{\tilde{\tau}_m}{n}}),\ \mathcal{A}_m\big] + e_m, \ \text{ with}
\end{aligned}
$$

$$
\text{(3.13)} \qquad |e_m| \le \frac{8}{\sqrt{n}}\,(n^\nu + 1)\,.
$$

Keeping in mind the definition of $\mathcal{A}_m$ in (3.7), we see by conditioning on $X^1_\cdot$ and $\widetilde{X}^1_\cdot$ that the difference of the first two terms of (3.12) vanishes. Since clearly $\sum_m |e_m| < \infty$, (recall $n = [b^m]$), we only need to observe that

$$
\text{(3.14)} \qquad \sum_m (P \times P)(\mathcal{A}_m^c) < \infty\,.
$$

By a similar calculation as in (1.16), we see that

$$
\text{(3.15)} \qquad P \times P[X^1_{[n^\mu,\infty)} \cap \widetilde{X}^1_{[n^\mu,\infty)} \ne \emptyset] \le \text{const}\; n^{-\mu\frac{(d_1-4)}{2}}\,,
$$

moreover $\tau_m - n^\mu$ is stochastically dominated by T^1 (under the P-measure) so that from (1.12), for large m:

$$
\text{(3.16)} \qquad P[\tau_m > n^\nu] \le \text{const}(\log n^\nu)^{1+\frac{d_1-4}{2}}\; n^{\frac{-\nu(d_1-4)}{2}} \le e^{-\text{const}\, m}\,.
$$

Combining (3.15) and (3.16) we deduce (3.14). $\square$

4 Diffusive behavior in a slightly biased environment

As explained in the introduction, it was shown in [15], that when the single-site distribution is concentrated on ϵ-perturbations of the d-dimensional simple random walk and $\mathbb{E}[d(0,\omega)]$ has size bigger than $\epsilon^{\frac{5}{2}-\eta}$, when $d=3$, $\epsilon^{3-\eta}$, where $d \geq 4$, then for small ϵ, depending on d and $\eta \in (0,1)$, the walk has non-vanishing limiting velocity (in fact much more is known, see [15]). In this section we provide examples of ϵ-perturbations of the simple random walk for which $\mathbb{E}[d(0,\omega)] \neq 0$, but the ballistic behavior is lost, when $d \geq 7$, and the diffusive behavior is even demonstrated when $d \geq 15$. We keep the notations of the previous sections, and specialize here κ to $\frac{1}{8d}$, and $q(\cdot)$ in (0.1) to

$$\begin{aligned} q(e) &= \frac{d_2}{d}, \text{ if } e = 0\,, \\ &= \frac{1}{2d}, \text{ if } e = \pm e_i,\ 1 \leq i \leq d_1\,. \end{aligned} \tag{4.1}$$

Recall the definition of $\mathcal{S}_\epsilon$ in (0.7). Note that when $p(\cdot) \in \mathcal{S}_\epsilon$, $p(e) \geq \kappa$, for all e, and κ is a global ellipticity constant. The main object of this section is the following

Theorem 4.1. *Assume $d \geq 7$, then for all $\epsilon \in (0,1)$, we can find μ concentrated on $\mathcal{S}_\epsilon$ such that*

$$\mathbb{E}[d(0,\omega)] \neq 0, \text{ but} \tag{4.2}$$

$$P_0\text{-a.s.}, \frac{X_n}{n} \to 0, \text{ as } n \to \infty\,. \tag{4.3}$$

In addition when $d \geq 15$, we can make sure that for $\mathbb{P}$-a.e. ω,

$$\begin{aligned} &\text{under } P_{0,\omega}, \tfrac{1}{\sqrt{n}}\, X_{[\cdot n]} \text{ converges in law towards a Brownian motion} \\ &\text{with covariance matrix } A \text{ (independent of } \omega). \end{aligned} \tag{4.4}$$

Proof. With the help of Theorem 1.4 and Theorem 3.1, it suffices to show that for any $\epsilon \in (0,1)$, $d_1 \geq 5$, $d_2 \geq 2$, we can find μ concentrated on $\mathcal{P}_{q(\cdot)} \cap \mathcal{S}_\epsilon$, for which the limiting velocity v of (1.41) vanishes, but $\int d(p)\, d\mu(p) \neq 0$, (see (1.1), and recall $d = d_1 + d_2$).

Let us denote by $\mathcal{P}^s_{q(\cdot)}$ the set of symmetric vectors in $\mathcal{P}_{q(\cdot)}$:

$$\mathcal{P}^s_{q(\cdot)} = \{p(\cdot) \in \mathcal{P}_{q(\cdot)}, \text{ such that } p(e) = p(-e) \text{ for all } e\}\,, \tag{4.5}$$

and define $\Omega_0 = (\mathcal{P}^s_{q(\cdot)} \cap \mathcal{S}_{\frac{\epsilon}{2}})^{\mathbb{Z}^d}$. We will use the following

Lemma 4.2. *Suppose φ is a measurable function on $\mathcal{P}^s_{q(\cdot)} \cap \mathcal{S}_{\frac{\epsilon}{2}}$ with values in $[-1,1]$, and μ_0 a probability on $\mathcal{P}^s_{q(\cdot)} \cap \mathcal{S}_{\frac{\epsilon}{2}}$ such that:*

$$\int \varphi(p)\, d\mu_0(p) = 0, \text{ and} \tag{4.6}$$

$$E^{Q^0_s}[\varphi(\sigma^s_0)] \neq 0\,, \tag{4.7}$$

where Q^0_s denotes the probability constructed in (1.20) when the single site distribution is μ_0. Then one can find a μ concentrated on $\mathcal{P}_{q(\cdot)} \cap \mathcal{S}_\epsilon$ for which $\int d(p)\, d\mu(p) \neq 0$, but $v = 0$.

Proof. We will look for environments of the form

$$\omega_{\rho,\lambda}(x,e) = \omega_0(x,e) + \rho(\varphi(\omega_0(x,\cdot)) + \lambda)e_d \cdot e,\ x \in \mathbb{Z}^d,\ |e| = 1\,, \tag{4.8}$$

with $\rho \in [0, \frac{\epsilon}{16d}]$, $\lambda \in [-1,1]$, two parameters and ω_0 distributed according to $\mathbb{P}_0 = \mu_0^{\otimes \mathbb{Z}^d}$. The distribution μ will correspond to the single site distribution $\mu_{\rho,\lambda}$ of the above $\omega_{\rho,\lambda}$ for small ρ and an appropriate choice of λ. Note that $\mu_{\rho,\lambda}$ is automatically concentrated on $\mathcal{P}_{q(\cdot)} \cap \mathcal{S}_\epsilon$.

For ρ, λ as above, we consider the kernel $K^{\rho,\lambda}$ from W to $\widetilde{W} \times \Omega_0$, defined as in (1.18) with the difference that ω is replaced by $\omega_{\rho,\lambda}$, and denote by $v_{\rho,\lambda}$ the asymptotic velocity corresponding to the single site distribution $\mu_{\rho,\lambda}$, see (1.41). We now find that

$$\begin{aligned} v_{\rho,\lambda} &= E^{\widehat{P}\times K^{\rho,\lambda}}\Big[\sum_{k=0}^{T^1-1} d(\omega_{\rho,\lambda}(X_k^1 + \widetilde{w}(k),\cdot))\Big]/E^{\widehat{P}}[T^1] \\ &\overset{(4.8)}{=} 2\rho\Big(\frac{E^{\widehat{P}\times K^{\rho,\lambda}}}{E^{\widehat{P}}[T^1]}\Big[\sum_{k=0}^{T^1-1} \varphi_0(\omega_0(X_k^1 + \widetilde{w}(k),\cdot))\Big] + \lambda\Big)\, e_d\,. \end{aligned} \tag{4.9}$$

From the above formula one deduces that

$$(\rho,\lambda) \in \Big[0, \frac{\epsilon}{16d}\Big] \times [-1,1] \to v_{\rho,\lambda} \text{ is a continuous function}\,. \tag{4.10}$$

Indeed given (ρ_0,λ_0) and (ρ_1,λ_1), one can couple the two kernels K^{ρ_0,λ_0} and K^{ρ_1,λ_1} so that when both walks are at time $k < T^1$ in the same location x, they simultaneously jump to $x+e$ with probability $\omega_{\rho_0,\lambda_0}(x,e) \wedge \omega_{\rho_1,\lambda_1}(x,e)$. The asserted continuity follows then from dominated convergence. Note also by direct inspection of the last line of (4.9) that

$$v_{\rho,1} \cdot e_d \ge 0 \text{ and } v_{\rho,-1} \cdot e_d \le 0, \text{ for } 0 < \rho \le \frac{\epsilon}{16d}\,. \tag{4.11}$$

We can hence define for $0 < \rho \le \frac{\epsilon}{16d}$:

$$\lambda_\rho \overset{\text{def}}{=} \text{the largest zero of the continuous function } \lambda \to v_{\rho,\lambda}\,. \tag{4.12}$$

We see that for $0 < \rho \le \frac{\epsilon}{16d}$:

$$\begin{cases} v_{\rho,\lambda_\rho} = 0, \\ \lambda_\rho = -E^{\widehat{P}\times K^{\rho,\lambda}}\Big[\sum_{k=0}^{T_1-1} \varphi(\omega_0(X_k^1 + \widetilde{w}(k),\cdot))\Big]/E^{\widehat{P}}[T^1]\,, \\ \int d(p)\, d\mu_{\rho,\lambda_\rho} = 2\rho\Big(\int \varphi(p)\, d\mu_0(p) + \lambda_\rho\Big)e_d \overset{(4.6)}{=} 2\rho\,\lambda_\rho\, e_d\,. \end{cases} \tag{4.13}$$

On the other hand a similar coupling argument as above shows that

$$\begin{aligned} \lim_{\rho\to 0} \frac{E^{\widehat{P}\times K^{\rho,\lambda_\rho}}}{E^{\widehat{P}}[T^1]}\Big[\sum_{k=0}^{T^1-1} \varphi(\omega_0(X_k^1 + \widetilde{w}(k),\cdot))\Big] &= \frac{E^{\widehat{P}\times K^0}}{E^{\widehat{P}}[T^1]}\Big[\sum_{k=0}^{T^1-1} \varphi(\omega_0(X_k^1 + \widetilde{w}(k),\cdot))\Big] \\ &= E^{Q_s^0}[\varphi(\sigma_0^s)]\,. \end{aligned} \tag{4.14}$$

As a result, we obtain that

$$\lim_{\rho\to 0} \lambda_\rho = -E^{Q_s^0}[\varphi(\sigma_0^s)] \overset{(4.7)}{\neq} 0\,, \tag{4.15}$$

so that for small ρ, μ_{ρ,λ_ρ} satisfies the claims of Lemma 4.2. $\square$

Remark 4.3. With minor modifications one obtains a similar statement for an $\mathbb{R}^{d_2}$-valued φ, with $|\varphi| \le 1$, and analogous assumptions as in (4.6), (4.7). One now chooses for $0 \le \rho \le \frac{\epsilon}{16d}$ and λ in the closed unit ball of $\mathbb{R}^{d_2}$,

$$\omega_{\rho,\lambda}(x,e) = \omega_0(x,e) + \rho(\varphi(\omega_0(x,\cdot)) + \lambda)\cdot e\,,$$

in place of (4.8), and uses Brouwer's fixed point theorem, cf. Dugundji [3], p. 341, to find λ_ρ for $0 < \rho \le \frac{\epsilon}{16d}$, satisfying the second equality of (4.13). This remark may be helpful if one wishes that the distribution μ of Theorem 4.1 accommodates a genuinely vector-valued local drift. $\square$

We now proceed with the proof of Theorem 4.1. We are reduced to checking the assumptions of Lemma 4.2. To this end we will use the general

Lemma 4.4. *(under the assumptions of Section 1)*

For Ψ bounded measurable on $\mathcal{P}_{q(\cdot)}$

$$E^{Q_s}[\Psi(\sigma_0^s)] = \lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big[e_n(x,0,\omega)\, \frac{\Psi(\omega(0,\cdot))}{\sum_{|e|=1} \omega(0,e)P_{e,\omega}[H_0=\infty]}\Big]\,, \text{ with} \tag{4.16}$$

$$e_n(x,y,\omega) = E_{x,\omega}[u^{H_y}, H_y < \infty]\,, \text{ for } x,y \in \mathbb{Z}^d,\ \omega\in\Omega,\ n\ge 1, \text{ where } u = 1-\frac{1}{n} \text{ and } H_z = \inf\{k \ge 0, X_k = z\}, \text{ for } z \in \mathbb{Z}^d\,. \tag{4.17}$$

Proof. We write $S_m = \sum_{k=0}^m \Psi(\omega(X_k,\cdot))$, for $m \ge 0$, and $S_{-1} = 0$, so that

$$\sum_{m=0}^\infty u^m \Psi(\omega(X_m,\cdot)) = \sum_{m=0}^\infty u^m(S_m - S_{m-1}) = \frac{1}{n} \sum_{m=0}^\infty u^m\, S_m\,.$$

Noting that $\frac{1}{n^2}\sum_{m=0}^\infty m u^m = 1 - \frac{1}{n}$, it follows from (1.40) that:

$$\lim_{n\to\infty} \frac{1}{n}\, E_0\Big[\sum_{m=0}^\infty u^m\, \Psi(\omega(X_m,\cdot))\Big] = E^{Q_s}[\Psi(\sigma_0^s)]\,. \tag{4.18}$$

On the other hand for $\omega\in\Omega$, setting

$$g_n(x,y,\omega) = E_{x,\omega}\Big[\sum_{m\ge 0} u^m\, 1\{X_m = y\}\Big]\,, \tag{4.19}$$

we find

$$
\begin{aligned}
E_{0,\omega}\Big[\sum_{m=0}^{\infty} u^m\, \Psi(\omega(X_m,\cdot))\Big] &= \sum_{x\in\mathbb{Z}^d} g_n(0,x,\omega)\, \Psi(\omega(x,\cdot))\,. \\
&= \sum_{x\in\mathbb{Z}^d} e_n(0,x,\omega)\, \frac{\Psi(\omega(x,\cdot))}{1-E_{x,\omega}[u^{\widetilde{H}_x}]}\,,
\end{aligned}
\tag{4.20}
$$

by a classical Markov chain calculation, provided

$$
\widetilde{H}_z = \inf\{k\ge 1,\ X_k = z\} \text{ for } z\in\mathbb{Z}^d\,. \tag{4.21}
$$

Since the $\mathbb{Z}^{d_1}$-projection of $X_.$ under $P_{0,\omega}$ is distributed as $X^1_.$ under P, we have:

$$
1-E_{0,\omega}[u^{\widetilde{H}_x}] \ge 1 - P_{0,\omega}[\widetilde{H}_x<\infty] \ge P[X^1_k\neq 0, \text{ for all } k\ge 1] > 0\,. \tag{4.22}
$$

Moreover for any $|e|=1$,

$$
\lim_{n\to\infty} \sup_{\omega} \big|E_{e,\omega}[u^{H_0}] - P_{e,\omega}[H_0<\infty]\big| = 0\,, \tag{4.23}
$$

since for $M>0$,

$$
\begin{aligned}
0 &\le P_{e,\omega}[H_0<\infty] - E_{e,\omega}[u^{H_0}] = E_{e,\omega}[(1-u^{H_0}), H_0<\infty] \\
&\le 1-u^M + P_{e,\omega}[M<H_0<\infty] \le 1-u^M + P[X^1_n = 0, \text{ for some } n\ge M]\,,
\end{aligned}
$$

from which (4.23) follows by letting n and then M tend to infinity. From (4.20) we see by choosing $\Psi=1$, that for $\omega\in\Omega$,

$$
\sum_{x\in\mathbb{Z}^d} \frac{1}{n}\, e_n(0,x,\omega) \le 1\,. \tag{4.24}
$$

Integrating over the environment in (4.20) and using translation invariance, as well as (4.22), (4.23), we obtain:

(4.25)

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\, E_0\Big[\sum_{m=0}^{\infty} u^m\, \Psi(\omega(X_m,\cdot))\Big] = \lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big[e_n(x,0,\omega)\, \frac{\Psi(\omega(0,\cdot))}{1-E_{0,\omega}[u^{\widetilde{H}_0}]}\Big] = \\
&\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \frac{1}{n}\, \mathbb{E}\Big[e_n(x,0,\omega)\, \frac{\Psi(\omega(0,\cdot))}{P_{0,\omega}[\widetilde{H}_0=\infty]}\Big]\,,
\end{aligned}
$$

which together with (4.18), finishes the proof of (4.16). $\square$

The distribution μ_0 of Lemma 4.2, that we now construct, will be concentrated on small perturbations of

$$
\begin{aligned}
p_\nu(e) =& \frac{1}{2d}, \text{ for } e=\pm\, e_i,\ i\le d-2\,, \\
& \frac{\nu}{2d}, \text{ for } e=\pm\, e_{d-1}, \qquad\qquad \text{with } \nu = 1-\frac{\epsilon}{8}\,, \\
& \frac{2-\nu}{2d}, \text{ for } e=\pm\, e_d\,.
\end{aligned}
\tag{4.26}
$$

Note that $p_\nu(\cdot) \in \mathcal{P}^s_{q(\cdot)} \cap \mathcal{S}_{\frac{\epsilon}{4}}$. We denote by P^ν_x, for $x \in \mathbb{Z}^d$, the canonical law of the random walk with jump distribution $p_\nu(\cdot)$, starting from x. Let us admit for the time being the fact that for small ϵ,

$$\Delta(\epsilon) \stackrel{\text{def}}{=} P^\nu_{e_d}[H_0 < \infty] - P^\nu_{e_{d-1}}[H_0 < \infty] > 0\,, \tag{4.27}$$

and explain how we complete the construction of μ_0 and φ of Lemma 4.2. We choose μ_0 concentrated on $\mathcal{P}^s_{q(\cdot)} \cap \mathcal{S}_{\frac{\epsilon}{2}}$ such that

$$\begin{aligned} &\mu_0\text{-a.s., } p(e) = p_\nu(e), \text{ for } e = \pm e_i,\ 1 \le i \le d-2, \text{ and} \\ &\widetilde{\delta} \stackrel{\text{def}}{=} p(e_d) - p_\nu(e_d) = -(p(e_{d-1}) - p_\nu(e_{d-1})) \text{ is such that} \end{aligned} \tag{4.28}$$

$$0 < \|\widetilde{\delta}\|_\infty \le \frac{\epsilon}{64d},\ \int \widetilde{\delta}\, d\mu_0 = 0,\ \int \widetilde{\delta}^2\, d\mu_0 \ge \frac{\|\widetilde{\delta}\|^2_\infty}{2}\,. \tag{4.29}$$

Such a choice is of course possible. We then define

$$\varphi(p) = \widetilde{\delta}\,, \tag{4.30}$$

so that $|\varphi| \le 1$, and (4.6) is satisfied. Writing $\widetilde{\delta}(x)$ for $\omega(x, e_d) - p_\nu(e_d)$, $x \in \mathbb{Z}^d$, we deduce from (4.16) that

$$E^{Q^0_s}[\varphi(\sigma^s_0)] = \lim_{n\to\infty} \frac{1}{n} \sum_x \mathbb{E}\Big[e_n(x,0,\omega)\, \frac{\widetilde{\delta}(0)}{P_{0,\omega}[\widetilde{H}_0 = \infty]}\Big]\,, \tag{4.31}$$

where $\mathbb{E}$ stands for the $\mu_0^{\otimes \mathbb{Z}^d}$-expectation. Note that

$$P_{0,\omega}[\widetilde{H}_0 = \infty] = \overline{P}_{0,\omega}[\widetilde{H}_0 = \infty] \Big(1 + \sum_{|e|=1} (\omega(0,e) - p_\nu(e))\, \frac{P_{e,\omega}[H_0 = \infty]}{\overline{P}_{0,\omega}[\widetilde{H}_0 = \infty]}\Big)\,,$$

where $\overline{P}_{0,\omega}$ denotes the probability corresponding to the environment $\overline{\omega}$, which coincides with ω outside 0 and such that $\overline{\omega}(0,\cdot) = p_\nu(\cdot)$. Note that the sum inside the parenthesis in the above expression is a.s. bounded by $\frac{4\|\widetilde{\delta}\|_\infty}{\kappa} \stackrel{(4.29)}{\le} \frac{1}{2}$, (see also the remark below (4.1) about κ). Using the inequality $|\frac{1}{1+\gamma} - 1 + \gamma| \le 2\gamma^2$, for $|\gamma| \le \frac{1}{2}$, we see that for $x \in \mathbb{Z}^d$,

$$\begin{aligned} &\mathbb{E}\Big[e_n(x,0,\omega)\, \frac{\widetilde{\delta}(0)}{P_{0,\omega}[\widetilde{H}_0 = \infty]}\Big] = \mathbb{E}\Big[e_n(x,0,\omega)\, \frac{\widetilde{\delta}(0)}{\overline{P}_{0,\omega}[\widetilde{H}_0 = \infty]}\Big] \\ &-\mathbb{E}\Big[\frac{e_n(x,0,\omega)}{\overline{P}_{0,\omega}[\widetilde{H}_0 = \infty]^2}\, \widetilde{\delta}(0) \sum_{|e|=1} (\omega(0,e) - p_\nu(e))\, P_{e,\omega}[H_0 = \infty]\Big] \\ &+\mathbb{E}\Big[\frac{e_n(x,0,\omega)}{\overline{P}_{0,\omega}[\widetilde{H}_0 = \infty]}\, B(x,\omega)\Big], \text{ with } |B(x,\omega)| \le \frac{32\|\widetilde{\delta}\|^3_\infty}{\kappa^2}\,. \end{aligned} \tag{4.32}$$

Using independence we see the first term in the right hand side of (4.32) vanishes, and

$$\begin{aligned}
&\frac{1}{n}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\Big[e_n(x,0,\omega)\frac{\widetilde{\delta}(0)}{P_{0,\omega}[\widetilde{H}_0=\infty]}\Big]=\\
&-\frac{2}{n}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\Big[\frac{e_n(x,0,\omega)}{\overline{P}_{0,\omega}[\widetilde{H}_0=\infty]^2}(P_{e_d,\omega}[H_0=\infty]-P_{e_{d-1},\omega}[H_0=\infty])\Big]\ \mathbb{E}[\widetilde{\delta}^2]\\
&+C,\ \text{with } |C|\le\frac{32\|\widetilde{\delta}\|_\infty^3}{\kappa^2}\ \frac{1}{n}\sum_{x\in\mathbb{Z}^d}\Big[\frac{e_n(x,0,\omega)}{\overline{P}_{0,\omega}[\widetilde{H}_0=\infty]}\Big].
\end{aligned}\tag{4.33}$$

Note that by choosing $\|\widetilde{\delta}\|_\infty$ sufficiently small, we can make sure that $\mu_0^{\otimes\mathbb{Z}^d}$-a.s.

$$P_{e_d,\omega}[H_0<\infty]-P_{e_{d-1},\omega}[H_0<\infty]\ge\frac{1}{2}\,\Delta(\epsilon),\ \text{cf. } (4.27)\,,\tag{4.34}$$

so that using (4.29) as well, the first term in the left member of (4.33) is bigger than:

$$\frac{1}{2}\,\Delta(\epsilon)\|\delta\|_\infty^2\ \frac{1}{n}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\Big[\frac{e_n(x,0,\omega)}{\overline{P}_{0,\omega}[\widetilde{H}_0=\infty]^2}\Big].\tag{4.35}$$

Observe that $\kappa\,\overline{P}_{0,\omega}[\widetilde{H}_0=\infty]\le P_{0,\omega}[\widetilde{H}_0=\infty]\le\frac{1}{\kappa}\,\overline{P}_{0,\omega}[\widetilde{H}_0=\infty]$ and $\lim_n\frac{1}{n}\sum_x\mathbb{E}[\frac{e_n(x,0,\omega)}{P_{0,\omega}[\widetilde{H}_0=\infty]}]\overset{(4.16)}{=}1$. As a result we see that

$$E^{Q_s^0}[\varphi(\sigma_0^s)]\ge\frac{\kappa}{2}\,\Delta(\epsilon)\|\widetilde{\delta}\|_\infty^2-\frac{32\|\widetilde{\delta}\|_\infty^3}{\kappa^3}>0,\ \text{when } \|\widetilde{\delta}\|_\infty \text{ is small}.\tag{4.36}$$

Hence (4.7) holds as well and Theorem 4.1 follows.

There now remains to prove (4.27). Let us denote by $g_\nu(\cdot,\cdot)$ the Green function of the random walk with jump distribution $p_\nu(\cdot)$ and by $\varphi_\nu(\cdot)$ the characteristic function of $p_\nu(\cdot)$. Then for $|e|=1$ or 0,

$$P_e^\nu[H_0<\infty]=\frac{g_\nu(e,0)}{g_\nu(0,0)}\ ,\ \text{and}\tag{4.37}$$

$$g_\nu(e,0)=\int_T\frac{e^{-it\cdot e}}{1-\varphi_\nu(t)}\,\frac{dt}{(2\pi)^d}\,,\ \text{with } t=(t_1,\dots,t_d)\in T=(-\pi,\pi)^d\,.\tag{4.38}$$

Using the symmetry of φ_ν we find:

$$\begin{aligned}
\frac{\partial}{\partial\nu}\,g_\nu(e,0)\ &=\ \int_T\frac{\partial\varphi_\nu}{\partial\nu}\,\frac{e^{-it\cdot e}}{(1-\varphi_\nu)^2}\,\frac{dt}{(2\pi)^d}\overset{\text{symmetry}}{=}\int\frac{\partial\varphi_\nu}{\partial\nu}\,\frac{\cos(t\cdot e)}{(1-\varphi_\nu)^2}\,\frac{dt}{(2\pi)^d}\\
&\overset{(4.26)}{=}\frac{1}{2d}\int_T(\cos t_{d-1}-\cos t_d)\,\frac{\cos(t\cdot e)}{(1-\varphi_\nu)^2}\,\frac{dt}{(2\pi)^d}\,.
\end{aligned}\tag{4.39}$$

Note in particular that $\frac{\partial}{\partial \nu}\, g^\nu(0,0)|_{\nu=1} = 0$, so that by (4.37)

$$\frac{\partial}{\partial \nu}\,(P^\nu_{e_d}[H_0<\infty] - P^\nu_{e_{d-1}}[H_0<\infty])|_{\nu=1} = \\ -\frac{1}{g_{\nu=1}(0,0)} \int \frac{(\cos t_{d-1} - \cos t_d)^2}{(1-\varphi_{\nu=1})^2}\, \frac{dt}{(2\pi)^d} < 0\,. \tag{4.40}$$

On the other hand $P^{\nu=1}_{e_d}[H_0<\infty] - P^{\nu=1}_{e_{d-1}}[H_0<\infty] = 0$, by symmetry, and the claim (4.27) follows. □

Remark 4.5. We know from Lawler[8], that for $\mu_0^{\otimes \mathbb{Z}^d}$-a.e. ω, $P_{0,\omega}$-a.s. $\frac{1}{\sqrt{n}} X_{[n\cdot]}$ converges in law to a Brownian motion with diagonal covariance matrix $A = \mathrm{diag}(a_i)$, where

$$a_i = 2\int_{\Omega_0} \omega(0,e_i)\, d\mathbb{Q}(\omega), \text{ for } 1 \le i \le d\,, \tag{4.41}$$

and $\mathbb{Q}$ is the unique invariant measure for the Markov chain of the environment viewed from the particle, which is absolutely continuous with respect to $\mu_0^{\otimes \mathbb{Z}^d}$. The measure $\mathbb{Q}$ is known to be an ergodic invariant measure and from (1.40), we see that $\omega(0,\cdot)$ under $\mathbb{Q}$ has same law as σ_0^s under Q_s^0. As a by-product of the above example, cf. the choice (4.30), we see that one cannot in general replace the dynamic measure $\mathbb{Q}$ with the static measure $\mu_0^{\otimes \mathbb{Z}^d}$ when calculating the limiting diffusion coefficient in (4.41). □

5 Perturbations of one-dimensional RWRE and velocity reversal

We construct in this section another class of examples of multidimensional walks that satisfy the law of large numbers with a velocity which has an opposite direction to the expected local drift, or can vanish even if the latter does not vanish. The examples in this section can be considered as perturbations of one-dimensional random walks in random environment, as opposed to the examples in Section 4 which were obtained as perturbation of the simple random walk in dimension d.

It is useful to first recall some known facts about one-dimensional random walks in random environment. Let $\bar\mu$ denote a Borel probability measure on $(0,1)$, set $\bar\Omega := (0,1)^{\mathbb{Z}}$, and define the measure $\bar{\mathbb{P}} = \bar\mu^{\otimes \mathbb{Z}}$ on the environment $\bar\Omega$. For every $\bar\omega \in \bar\Omega$, the one-dimensional walk $\bar X_n$ under the law $\bar P_0 = \bar{\mathbb{P}} \times P_{0,\bar\omega}$ is defined as in (0.3). Set $\rho_z = (1-\bar\omega_z)/\bar\omega_z$, define $d_0 = 2E_{\bar{\mathbb{P}}}(\bar\omega_0) - 1$ and $t_0 = E_{\bar{\mathbb{P}}}(\log \rho_0)$. The following facts are well known:

Lemma 5.1. *1. If $t_0 > 0$ then $\bar P_0$-a.s., $\lim \bar X_n = -\infty$. Further, if there exists a constant $\kappa > 0$ such that $\bar\mu[\bar\omega_0 \in (\kappa, 1-\kappa)] = 1$, then $\bar E_0(\bar X_n) < 0$ for all n large enough.*
2. One may construct a law $\bar\mu$ with $d_0 > 0$, $\kappa > 0$, but $t_0 > 0$.

Proof. The first part is a consequence of [11] and [7]. Concerning the second part, take $\delta \in (0,1)$ small enough such that

$$\frac{1}{5}\log\frac{1-\delta}{\delta} - \frac{4}{5}\log 2 > 0\,,$$

and define $\bar{\mu}(\{\delta\}) = 1/5$ and $\bar{\mu}(\{2/3\}) = 4/5$. $\square$

Fix a $\bar{\mu}$ as in part 2 of Lemma 5.1, and an $\epsilon_0 > 0$ small enough such that, if G_ϵ denotes a modified geometric random variable of parameter ϵ independent of $\{\bar{X}_n\}$, then

$$A_0 := \bar{E}_0(\bar{X}_{G_{\epsilon_0}}) < 0 \tag{5.1}$$

(this is always possible due to part 1 of Lemma 5.1). For every $1 > \epsilon \geq \epsilon_0$ and $d_1 \geq 5$, set $d_2 = 1$, $q(e) = \epsilon/2d_1, e \in \mathbb{Z}^{d_1}$, and $\mu \in \mathcal{P}_{q(\cdot)}$ such that $\bar{\mu}$ governs the law of the single site jump distribution conditioned on non-vanishing of the $\mathbb{Z}^{d_2}$-component. Let X_n denote the random walk in random environment corresponding to the law $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$, and let $v = v(\bar{\mu}, d_1, \epsilon)$ be the limiting velocity appearing in Theorem 1.4. Note that $v \cdot e = 0$ for every $e \in \mathbb{Z}^{d_1}$. Let $v_2 = v \cdot e_d$ denote the projection of v into the direction corresponding to the $\mathbb{Z}^{d_2}$ subspace. We now claim the following:

Theorem 5.2. *There exists an integer $\bar{d} = \bar{d}(\bar{\mu}, \epsilon_0)$ such that for any $d_1 > \bar{d}$, it holds that $v_2(\bar{\mu}, d_1, \epsilon_0) < 0$ while $\lim_{\epsilon \to 1} v_2(\bar{\mu}, d_1, \epsilon)/(1-\epsilon) = d_0 > 0$.*

By the continuity of $v_2(\bar{\mu}, d_1, \epsilon)$ in ϵ, which follows from similar considerations as in (4.10), we see that for every $d_1 > \bar{d}$ one may find an $\epsilon > \epsilon_0$ such that $v(\bar{\mu}, d_1, \epsilon) = 0$. Moreover, when $d_1 > \bar{d} \vee 13$, Theorem 2.2 implies that the corresponding walk X_n exhibits a diffusive behavior.

It is interesting to comment on the nature of the phenomenon described in Theorem 5.2: for ϵ close to 1, between consecutive cut points of the $\mathbb{Z}^{d_1}$ walk, X_n does not spend much time moving in the d-th direction, and with high probability makes at most one step in that direction. This then averages out to give a positive displacement since $d_0 > 0$. On the other hand, when d_1 is large, most moves in the $\mathbb{Z}^{d_1}$-walk are cut points. If also ϵ is small enough, the walker effectively executes in the d-direction a one-dimensional random walk in random environment between cut points, for a geometric time of mean $1/\epsilon$. That one-dimensional random walk in random environment is constructed such that while it does not have a negative speed (this is impossible since $d_0 > 0$), it is transient to $-\infty$ and hence leads to a negative displacement.

Proof. Recall the cut times T^i. From (1.41) and similar considerations as in (4.9),

$$v_2(\bar{\mu}, d_1, \epsilon_0) = \frac{E^{\widehat{Q}_s}[Z^s_{T^1} \cdot e_d]}{E^{\widehat{P}}[T^1]}.$$

Hence, the first part of the theorem follows as soon as we show that for $\epsilon = \epsilon_0$ and d_1 large enough it holds that

$$E^{\widehat{Q}_s}[Z^s_{T^1} \cdot e_d] < 0. \tag{5.2}$$

Define $\mathcal{J} = \{n : X^1_n \neq X^1_{n-1}\}$, and let $\ldots < j_{-1} < j_0 \leq 0 < j_1 < \ldots$ denote the elements of $\mathcal{J}$. Set $V^1_n = X^1_{j_n}$, and note that under $P = q^{\otimes \mathbb{Z}}$, $\{V^1_n\}$ is a d_1-dimensional simple random walk, independent of the i.i.d., geometric(ϵ_0) random variables $\{j_{i+1} -$

$j_i\}_{i\in\mathbb{Z}\setminus 0}, j_1, -j_0+1$. Recall the cut times T^i, note that $T^i \in \mathcal{J}$, and write $J_i = j_{c_i}$ for the element of $\mathcal{J}$ corresponding to T^i. Note that the c_i are precisely the cut times for the walk $\{V_n^1\}$.

Call a cut time T^i *good* if $X_n^1 = X_{T^i}^1$ for $n \in [T^i, T^{i+1}-1]$, that is if $J_{i+1} = j_{c_i+1}$. To prove (5.2), note first that

$$E^{\widehat{Q}_s}[Z_{T^1}^s \cdot e_d] = E^{\widehat{Q}_s}[Z_{T^1}^s \cdot e_d \mathbf{1}_{\{T^0 \text{is good}\}}] + E^{\widehat{Q}_s}[Z_{T^1}^s \cdot e_d \mathbf{1}_{\{T^0 \text{is not good}\}}] := A + B\,.$$

We claim that under the measure $\widehat{P}[\cdot|T^0 \text{ is good}]$, T^1 is geometric(ϵ_0). Indeed, with $\mathcal{D}^V = \{c_i\}_{i\in\mathbb{Z}}$ denoting the cut times of $\{V_n^1\}$,

$$\begin{aligned} P[T^1 = k, 0 \in \mathcal{D}, T^0 \text{ is good}] &= P[0 \in \mathcal{D}^V, 1 \in \mathcal{D}^V, j_0 = 0, j_1 = k] \\ &= P[0 \in \mathcal{D}^V, 1 \in \mathcal{D}^V](1-\epsilon_0)^{k-1}\epsilon_0^2\,, \end{aligned}$$

implying that

$$\widehat{P}[T^1 = k | T^0 \text{ is good}] = (1-\epsilon_0)^{k-1}\epsilon_0\,.$$

On the other hand, under the law $\widehat{Q}^s$, on the event $\{T^0 \text{ is good}\}$, X_n^2 performs, for $n \in [0, T^1-1]$, a one dimensional random walk in random environment, with environment generated by $\bar{\mu}$ (c.f. (1.22)). Hence,

$$\begin{aligned} A &= \widehat{Q}_s[T^0 \text{ is good}] E^{\widehat{Q}_s}[Z_{T^1}^s \cdot e_d | T^0 \text{ is good}] \\ &= \widehat{P}[T^0 \text{ is good}] \sum_{k=1}^{\infty} \widehat{Q}_s[T_1 = k | T^0 \text{ is good}] E^{\widehat{Q}_s}[Z_{T^1}^s \cdot e_d | T^0 \text{ is good}, T_1 = k] \\ &= \widehat{P}[T^0 \text{ is good}] \sum_{k=1}^{\infty} \bar{E}_0[\bar{X}_{k-1}] \widehat{P}[T_1 = k | T^0 \text{ is good}] \\ &= A_0 \widehat{P}[T^0 \text{ is good}] \end{aligned}$$

where $A_0 < 0$ is as in (5.1). We next note that

$$\widehat{P}[T^0 \text{ is good}] = \frac{P[0 \in \mathcal{D}, 1 \in \mathcal{D}^V]}{P[0 \in \mathcal{D}]} \geq 1 - \frac{P[1 \notin \mathcal{D}^V]}{P[0 \in \mathcal{D}]} \to_{d_1\to\infty} 1\,, \tag{5.3}$$

because (see [5], Remark 3, p. 248) $P[1 \in \mathcal{D}^V] = P[0 \in \mathcal{D}^V] \to_{d_1\to\infty} 1$ while

$$P[0 \in \mathcal{D}] = P[0 \in \mathcal{D}^{\mathcal{V}}, j_0 = 0] = \epsilon_0 P[0 \in \mathcal{D}^V]$$

is uniformly bounded below for $d_1 \geq 5$. Thus, $A \to_{d_1\to\infty} A_0 < 0$. On the other hand, a repeat of the proof of (1.12), using the fact that $P[X_n^1 = 0]$ decreases with d_1 as can be checked via characteristic functions, shows that, as a function of $d_1 \geq 9$, $E^P[(T^1)^2]$ is uniformly bounded. Hence, $E^{\widehat{Q}_s}[(T^1)^2]$ is uniformly bounded for $d_1 \geq 9$. The estimate (5.3) and the Cauchy-Schwarz inequality imply then that

$$|B| \leq E^{\widehat{Q}_s}[T^1 \mathbf{1}_{\{T^0 \text{is not good}\}}] \to_{d_1\to\infty} 0\,.$$

Choosing d_1 large enough such that $A + B < 0$, the first part of the theorem follows.

The second part is actually easier: with the notations of (1.2),

$$E^{\widehat{Q}_s}[Z^s_{T^1}\cdot e_d] = \widehat{Q}_s[|\{n\in[1,T^1]:I_n=0\}|=1]d_0+E^{\widehat{Q}_s}[\mathbf{1}_{\{|\{n\in[1,T^1]:I_n=0\}|>1\}}Z^s_{T^1}\cdot e_d] := d_0C+D\,.$$

But, setting $\widetilde{j}_i=j_i$ for $i\geq 1$ and $\widetilde{j}_0=0$,

$$\widehat{P}[|\{n\in[1,T^1]:I_n=0\}|=0] = \frac{P[\sum_{i=0}^{c_1-1}(\widetilde{j}_{i+1}-\widetilde{j}_i-1)=0; j_0=0; 0\in\mathcal{D}^V]}{P(0\in\mathcal{D})} = E^{\widehat{P}}[\epsilon^{c_1}]\,,$$

while, similarly,

$$\begin{aligned}\widehat{P}[|\{n\in[1,T^1]:I_n=0\}|>1] &\leq \widehat{P}[\exists\, 0\leq i<k\leq c_1-1:\widetilde{j}_{i+1}-\widetilde{j}_i-1=1,\widetilde{j}_{k+1}-\widetilde{j}_k-1=1]\\ &\quad+\widehat{P}[\exists\, 0\leq i\leq c_1-1:\widetilde{j}_{i+1}-\widetilde{j}_i-1\geq 2]\\ &\leq (1-\epsilon)^2E^{\widehat{P}}[(c_1)^2+c_1]\,.\end{aligned}$$

Note that the law $\widehat{P}[c_1\in\cdot]$ does not depend on ϵ. Since for $d_1\geq 7$ it holds that $E^{\widehat{P}}(c_1^2)<\infty$, we conclude that $D/(1-\epsilon)\to_{\epsilon\to 1}0$. Further, we also get

$$\lim_{\epsilon\to 1}\frac{C}{1-\epsilon} = \lim_{\epsilon\to 1}\frac{1-E^{\widehat{P}}(\epsilon^{c_1})}{1-\epsilon} = E^{\widehat{P}}[c_1]\,.$$

Since also $\lim_{\epsilon\to 1}P(c_1\neq T^1)=0$, one has that $E^{\widehat{P}}(\widehat{T}^1)\to_{\epsilon\to 1}E^{\widehat{P}}(c_1)$, and the theorem follows. □

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