A LAW OF LARGE NUMBERS FOR FINITE-RANGE DEPENDENT RANDOM MATRICES

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Abstract. We consider random hermitian matrices in which distant above-diagonal entries are independent but nearby entries may be correlated. We find the limit of the empirical distribution of eigenvalues by combinatorial methods. We also prove that the limit has algebraic Stieltjes transform by an argument based on dimension theory of noetherian local rings.

1. Introduction

Study of the empirical distribution of eigenvalues of random hermitian (or real symmetric) matrices has a long history, starting with the seminal work of Wigner [Wig55] and Wishart [Wis28]. Except in cases where the joint distribution of eigenvalues is explicitly known, most results available are asymptotic in nature and based on one of the following approaches: (i) the *moment method*, i. e., evaluation of expectations of traces of powers of the matrix; (ii) appropriate recursions for the resolvant, as introduced in [PM67]; or (iii) the free probability method (especially the notion of asymptotic freeness) originating with Voiculescu [Voi91]. A good review of the first two approaches can be found in [Ba99]. For the third, see [Voi00], and for a somewhat more combinatorial perspective, [Sp98]. These approaches have been extended to situations in which the matrix analyzed neither possesses i.i.d. entries above the diagonal (as in the Wigner case) nor is it the product of matrices with i.i.d. entries (as in the Wishart case). We mention in particular the papers [MPK92], [KKP96], [Sh96] and [Gu02] for results pertaining to the model of "random band matrices", all with independent above-diagonal entries.

In our recent work [AZ06] we studied convergence of the empirical distribution of eigenvalues of random band matrices, and developed a combinatorial approach, based on the moment method, to identify the limit (and also to provide central limit theorems for linear statistics). Here we develop the method further to handle a class of matrices with local dependence among entries (we postpone the precise definition of the class to Section 2). To each random matrix of the class we associate a random band matrix with the same limit of empirical distribution of eigenvalues by a process of "Fourier transformation", thus making it possible to describe the limit in terms of our previous work (see Theorem 2.5). We also prove that the Stieltjes transform of the limit is algebraic (see Theorem 2.6), doing so by a general "soft" (i. e., nonconstructive) method based on the theory of noetherian local rings (see Theorem 6.2) which ought to be applicable to many more random matrix problems.

Date: September 10, 2006.
O.Z. was partially supported by NSF grant number DMS-0503775.

To get the flavor of our results, the reader should imagine a Wigner matrix (i.e., an N-by-N real symmetric random matrix with i.i.d. above-diagonal entries, each of mean 0 and variance $1/N$), on which a local "filtering" operation is performed: each entry not near the diagonal or an edge is replaced by half the sum of its four neighbors to northeast, southeast, southwest and northwest. At the end of Section 3 (Theorem 2.5 taken for granted) we analyze the "(NE+SE+SW+NW)-filtered Wigner matrix" described above. We find that the limit measure is the free multiplicative convolution of the semicircle law (density $\propto \mathbf{1}_{|x|<2}\sqrt{4-x^2}$) and the arcsine law (density $\propto \mathbf{1}_{0<x<2}/\sqrt{x(2-x)}$). The appearance in this example of a free multiplicative convolution has a simple explanation (see Proposition 3.6). We also write down the quartic equation satisfied by the Stieltjes transform of the limit measure.

Recently other authors have considered the empirical distribution of eigenvalues for matrices with dependent entries, see [GoT05],[Ch05],[SSB05]. Their class of models does not overlap significantly with ours. In particular, in all these works and unlike in our model, the limit of the empirical measure is always the same as that of a semicircle law multiplied by a random or deterministic constant.

Closest to our work is the recent paper by [HLN05], that builds upon earlier work by [BDM96] and [Gi01]. They consider Gram matrices of the form $X_N X_N^*$ where X_N is a sequence of (non-symmetric) Gaussian matrices obtained by applying a filtering operation to a matrix with (complex) Gaussian independent entries. They also consider the case $(X_N + A_N)(X_N + A_N)^*$ with A_N deterministic and Toeplitz. The Gaussian assumption allows them to directly approximate the matrix X_N by a unitary transformation of a Gaussian matrix with independent (but not identically distributed) entries, to which the results of [Gi01] and [AZ06] apply. In particular, they do not need an assumption on the support of the filter. Unlike the present work, the approach in [HLN05] and [BDM96] is based on studying resolvants, rather than moments.

We mention now motivation from electrical engineering. The analysis of the limiting empirical distribution of eigenvalues of random matrices has recently played an important role in the analysis of communication systems, see [TV04] for an extensive review. In particular, when studying multi-antenna systems, one often makes the (unrealistic) assumption that gains between different pairs of antennas are uncorrelated. The models studied in this work would allow correlation between neighboring antenna pairs. We do not develop this application further here.

The structure of the paper is as follows. In the next section we describe the class of matrices we treat, and state our main results, namely Theorem 2.5 (asserting a law of large numbers) and Theorem 2.6 (asserting algebraicity of a Stieltjes transform). In Section 3 we discuss the limit measure in detail, and in particular write down algebro-integral equations for its Stieltjes transform, which we call *color equations*. Section 4 provides a computation of limits of traces of powers of the matrices under consideration. In Section 5 we complete the proof of Theorem 2.5 by a variance computation. In Section 6 we set up the algebraic machinery needed to prove Theorem 2.6. We finish proving the theorem in Section 7 by analyzing the color equations.

2. Formulation of the main result

After defining a class of kernels in §2.1, we define in §2.2 the class of matrices dealt with in this paper and in §2.3 describe the subclass of *filtered Wigner matrices*. To each kernel we associate a measure in §2.4. Finally, we state our main results, Theorems 2.5 and 2.6.

2.1. Kernels.

2.1.1. *Color space (motivation).* As in [AZ06], the spatial change in the variance structure of the entries of a random matrix is captured by an auxiliary variable, which we call "color". The difference between the derivation in [AZ06] and the present paper is that we append to color space an additional variable (related to the local averaging mentioned in the introduction), that one should think of as a Fourier variable. The precise definition follows.

2.1.2. *Color space (formal definition).* Let $C = [0,1] \times S^1$, where S^1 is the unit circle in the complex plane. We call C *color space*. We declare C to be a probability space under the product of uniform probability measures on $[0,1]$ and S^1, denoted P. We say that a C-valued random variable is *uniformly distributed* if its law is P. In the sequel, all $L^p(C)$ (resp., $L^p(C \times C)$) spaces, $p \geq 1$, are taken with respect to the measure P (resp., $P \times P$).

2.1.3. *The kernel s.* We fix a kernel

$$s : C \times C \to \mathbb{R}$$

which will govern the local covariance structure of our random matrix model. We impose on s the following conditions.

Assumption 2.1.4.

(I) *s is a nonnegative symmetric function, i.e.*

$$s(c, c') = s(c', c) \geq 0.$$

(II) *s has a Fourier expansion*

$$s(c, c') = \sum_{i,j \in \mathbb{Z}} s_{ij}(x, y) \xi^i \eta^j \quad (c = (x, \xi),\ c' = (y, \eta))$$

where all but finitely many of the coefficients

$$s_{ij} : [0,1] \times [0,1] \to \mathbb{C}$$

vanish identically.

(III) *There is a finite partition $\mathcal{I}$ of $[0,1]$ into subintervals of positive length such that every coefficient function s_{ij} is constant on every set of the form $I \times J$ with $I, J \in \mathcal{I}$.*

(IV) *s is nondegenerate: $\|s\|_{L^1(C \times C)} > 0$.*

2.1.5. *The sets* $Q_K^{(N)}$. For each $N \in \mathbb{N}$ (here and below $\mathbb{N}$ denotes the set of positive integers) and $K > 0$, we define

$$Q_K^{(N)} = \left\{ i \in \{1, \ldots, N\} \,\middle|\, \min_{x \in \partial\mathcal{I}} |i - Nx| > K \right\}$$

where $\partial\mathcal{I} \subset [0,1]$ is the finite set consisting of all endpoints of all intervals belonging to the family $\mathcal{I}$.

2.1.6. *Remarks.* We have

$$\bar{s}_{ij} = s_{-i,-j}, \quad s_{ji}(y,x) = s_{ij}(x,y) \tag{1}$$

because s is real-valued and symmetric. Assumptions (I,II,III) imply that

$$0 \le s \le \|s\|_{L^\infty(C\times C)} < \infty \tag{2}$$

holds everywhere (not just $P \times P$-a.e.).

2.2. **The model.** For each $N \in \mathbb{N}$, let

$$X^{(N)} = [X_{ij}^{(N)}]_{i,j=1}^N$$

be an N-by-N random hermitian matrix. We impose the following conditions, where s satisfies Assumption 2.1.4.

Assumption 2.2.1.

(I) (a) $\forall N \in \mathbb{N} \quad \forall i,j \in \{1,\ldots,N\} \quad EX_{ij}^{(N)} = 0$.

(b) $\forall k \in \mathbb{N} \quad \sup_{N=1}^{\infty} \max_{i,j=1}^{N} E|X_{ij}^{(N)}|^k < \infty$.

(II) *There exists* $K > 0$ *such that for all* $N \in \mathbb{N}$, *the following hold:*

(a) $\forall i,j \in \mathbb{Z} \quad \max(|i|,|j|) > K \Rightarrow s_{ij} \equiv 0$.

(b) *For all nonempty subsets*

$$A, B \subset \{(i,j) \in \{1,\ldots,N\}^2 \mid 1 \le i \le j \le N\}$$

such that

$$\min_{(i,j)\in A} \min_{(k,\ell)\in B} \max(|i-k|, |j-\ell|) > K,$$

the σ*-fields*

$$\sigma(\{X_{ij}^{(N)} \mid (i,j) \in A\}), \quad \sigma(\{X_{k\ell}^{(N)} \mid (k,\ell) \in B\})$$

are independent.

(c) $\forall i,j,k,\ell \in Q_K^{(N)}$ s.t. $\min(j-i, \ell-k) > K$
$EX_{ij}^{(N)}\overline{X}_{k\ell}^{(N)} = s_{i-k,\ell-j}(i/N, j/N)$.

2.2.2. *The empirical distribution of eigenvalues.* Let

$$\lambda_1^{(N)} \leq \lambda_2^{(N)} \leq \ldots \leq \lambda_N^{(N)}$$

denote the eigenvalues of the hermitian matrix $X^{(N)}/\sqrt{N}$, and let

$$L^{(N)} = N^{-1} \sum_{i=1}^{N} \delta_{\lambda_i^{(N)}}$$

denote the corresponding empirical distribution of the eigenvalues. We are concerned with the limiting behavior of $L^{(N)}$ as $N \to \infty$.

2.2.3. *Remarks.* (i) The existence of K satisfying Assumption 2.2.1(IIa) is assured by Assumption 2.1.4(IIb). (ii) Assumption 2.2.1(IIb) says that the on-or-above diagonal entries of $X^{(N)}$ form a finite-range dependent random field, with K a bound for the range of dependence. This explains the title of the paper. (iii) Assumption 2.2.1(IIc) fixes variances at each site and also short-range local correlations for sites in sufficiently general position. (iv) None of our assumptions rule out the possibility that all matrices $X^{(N)}$ are real. In other words, we can handle hermitian and real symmetric cases uniformly under Assumption 2.2.1. (v) The relations (15), which are equivalent to the reality and symmetry of the kernel s, play a key role in our analysis of the limiting behavior of $L^{(N)}$. New methods would be needed were the reality and symmetry conditions to be relaxed.

2.2.4. *Kernels of pure spatial type.* Let $s : [0,1] \times [0,1] \to \mathbb{R}$ be a kernel. If the kernel $\tilde{s} : C \times C \to \mathbb{R}$ defined by the formula $\tilde{s}((x,\xi),(y,\eta)) = s(x,y)$ satisfies Assumption 2.1.4, then by abuse of terminology we say that s is a kernel of *pure spatial type* satisfying Assumption 2.1.4, and with the evident modification of Assumption 2.2.1(IIc) we can use s to govern the covariance structure of our model. In the special case of kernels of pure spatial type our model essentially contains the model of [AZ06] in the special case in which color space is a finite set.

2.2.5. *Kernels of pure Fourier type.* Let $s : S^1 \times S^1 \to \mathbb{R}$ be a kernel. If the kernel $\tilde{s} : C \times C \to \mathbb{R}$ defined by the formula $\tilde{s}((x,\xi),(y,\eta)) = s(\xi,\eta)$ satisfies Assumption 2.1.4 with $\mathcal{I} = \{[0,1]\}$, then by abuse of terminology we say that s is a kernel of *pure Fourier type* satisfying Assumption 2.1.4, and with the evident modification of Assumption 2.2.1(IIc) we can use s to govern the covariance structure of our model. We suggest that the reader focus on the pure Fourier case when first approaching this paper because little would be lost in terms of grasping the main ideas. The main reason for us to work at a higher level of generality is to make sure that our theory handles not only "filtered Wigner matrices" (for which $|\mathcal{I}| = 1$) but also "filtered Wishart matrices" (for which $|\mathcal{I}| = 2$). (Here and below $|S|$ denotes the cardinality of a set S.) We then might as well allow $|\mathcal{I}| > 2$ as a possibility because there is no gain in simplicity by excluding it.

2.3. **Filtered Wigner matrices.** We describe now a natural class of random matrices fitting into the framework of Assumptions 2.1.4 and 2.2.1. This class should be considered the main motivation for the paper. Members of the class arise by "filtering" Wigner matrices. The corresponding kernels are of pure Fourier type and depend in a simple way on the "filter".

2.3.1. *Wigner matrices.* Let

$$\{Y_{ij}\}_{-\infty<i<j<\infty}$$

be an i.i.d. family of real random variables. Assume that Y_{01} has absolute moments of all orders. Assume that $EY_{01}=0$ and $EY_{01}^2=1$. Put

$$Y_{ii}=0 \text{ for } -\infty<i<\infty,\ \ Y_{ij}=Y_{ji} \text{ for } -\infty<j<i<\infty.$$

Then

$$Y_{ij}=Y_{ji},\quad EY_{ij}=0,\quad EY_{ij}Y_{k\ell}=(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})(1-\delta_{ij})(1-\delta_{k\ell}) \tag{3}$$

for all $i,j,k,\ell\in\mathbb{Z}$. Let $Y^{(N)}=[Y_{ij}^{(N)}]_{i,j=1}^N$ be the N-by-N matrix with entries Y_{ij}. Then $Y^{(N)}/\sqrt{N}$ (in the terminology of [AZ06]) is a *Wigner matrix*, and hence the empirical distribution of its eigenvalues for $N\to\infty$ tends to the semicircle law. In particular, if the Y_{ij} are standard normal random variables and one makes a suitable adjustment to the diagonal of $Y^{(N)}/\sqrt{N}$, the result is a Wigner matrix in the standard sense, i.e., a member of the Gaussian orthogonal ensemble.

2.3.2. *The filter, its Fourier transform, and associated kernel.* Let a *filter*

$$h:\mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$$

be given, with H denoting its Fourier transform, that is

$$H(\xi,\eta)=\sum_{i,j\in\mathbb{Z}}h(i,j)\xi^i\eta^j\quad(\xi,\eta\in S^1).$$

We assume that h does not vanish identically. We assume that there exists $K>0$ such that

$$\max(|i|,|j|)>K/2\Rightarrow h(i,j)=0, \tag{4}$$

and hence H is well-defined. We assume that h satisfies the symmetry condition

$$h(-i,-j)=h(j,i), \tag{5}$$

which implies the symmetry condition

$$\overline{H(\xi,\eta)}=H(\eta,\xi).$$

Put

$$s(\xi,\eta)=|H(\xi,\eta)|^2=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}s_{ij}\xi^i\eta^j\quad(s_{ij}\in\mathbb{R}).$$

Then $s:S^1\times S^1\to\mathbb{R}$ is a kernel of pure Fourier type satisfying Assumption 2.1.4. In particular, $\|s\|_{L^1(C\times C)}=\|h\|^2_{L^2(\mathbb{Z}\times\mathbb{Z})}>0$.

2.3.3. *Filtered Wigner matrices (definition).* For $i,j\in\{1,\dots,N\}$ set

$$X_{ij}^{(N)}=\sum_{k=1}^N\sum_{\ell=1}^N Y_{k\ell}h(i-k,\ell-j),$$

thus defining an N-by-N random matrix $X^{(N)}$ which in view of the symmetry (5) is real symmetric. We call $X^{(N)}/\sqrt{N}$ a *filtered Wigner matrix*, with *filter* h. We think of $X^{(N)}/\sqrt{N}$ as the result of filtering the Wigner matrix $Y^{(N)}/\sqrt{N}$ by h.

2.3.4. *Local covariance structure of $X^{(N)}$.* From (3) and (4) we deduce that

$$i,j,k,\ell \in Q_K^{(N)} \;\&\; \min(j-i,\ell-k) > K \Rightarrow EX_{ij}^{(N)}X_{k\ell}^{(N)} = s_{i-k,\ell-j}, \tag{6}$$

after a straightforward (extremely tedious) calculation. This is the main point in verifying that the real symmetric random matrices $X^{(N)}$ satisfy Assumption 2.2.1 with respect to the kernel s. The remaining details needed to check Assumption 2.2.1 are easy to supply. Thus we can put filtered Wigner matrices into the framework of our model.

2.3.5. *The (NE+SE+SW+NW)-filtered Wigner matrix.* Taking

$$h = (1/2)\mathbf{1}_{\{(1,-1),(1,1),(-1,1),(-1,-1)\}}, \text{ and hence } s(e^{i\theta_1}, e^{i\theta_2}) = 4\cos^2\theta_1\cos^2\theta_2,$$

we get a precisely defined version of the model mentioned in the introduction, which presently we will analyze in detail. Note that with θ uniformly distributed in $[0, 2\pi)$, the law of $2\cos^2\theta$ has density $\propto \mathbf{1}_{0<x<2}/\sqrt{x(2-x)}$.

2.3.6. *Remark.* In [AZ06] we handled Wishart matrices Z^*Z (and more general matrices formed from matrices Z with independent but perhaps not i.i.d. entries) in terms of band-type matrices $\left[\begin{array}{cc} 0 & Z \\ Z^* & 0 \end{array}\right]$. A similar trick in the present setting puts "filtered Wishart matrices" into the framework of our model. For the kernels arising in that connection, the associated partition $\mathcal{I}$ consists of two intervals. We do not discuss the Wishart case further here.

2.4. **The measure μ_s.** We make the last preparation to state our main results. Let s be a kernel satisfying Assumption 2.1.4. For each positive integer N, let

$$C^{(N)} = \{c_1^{(N)}, \ldots, c_{N^2}^{(N)}\} \subset C$$

be the set of pairs $(x,\xi) \in C$ where $x \in [0,1) \cap \frac{1}{N}\mathbb{Z}$ and $\xi^N = 1$. Then the empirical distribution $\frac{1}{N^2}\sum_{c\in C^{(N)}}\delta_c$ tends weakly as $N \to \infty$ to the uniform probability measure P. Let $C^{(\infty)}$ be the union of the sets $C^{(N)}$. Let

$$\{\tilde{Y}_e\}_{e\subset C^{(\infty)} \text{ s.t. } |e|=1,2}$$

be an i.i.d. family of standard normal (mean 0 and variance 1) random variables. (Recall from §2.2.5 that $|S|$ denotes the cardinality of S.) Let $\tilde{X}^{(N)}$ be the N^2-by-N^2 real symmetric random matrix with entries

$$\tilde{X}_{ij}^{(N)} = 2^{\delta_{ij}/2}\sqrt{s(c_i^{(N)}, c_j^{(N)})}\,\tilde{Y}_{\{c_i^{(N)}, c_j^{(N)}\}}.$$

Let $\tilde{\lambda}_1^{(N)} \le \cdots \le \tilde{\lambda}_{N^2}^{(N)}$ be the eigenvalues of $\tilde{X}^{(N)}/N$ and let $\tilde{L}^{(N)} = \frac{1}{N^2}\sum_{i=1}^{N^2}\delta_{\tilde{\lambda}_i^{(N)}}$ be the empirical distribution of the eigenvalues. By [AZ06, Thm. 3.2] the empirical distribution $\tilde{L}^{(N)}$ tends weakly in probability as $N \to \infty$ to a limit μ_s with bounded support. (The strange-looking factor $2^{\delta_{ij}/2}$ in the definition of $\tilde{X}^{(N)}$ could be dropped without changing the limit of $L^{(N)}$. More generally, within the theory of [AZ06], one has many ways to construct models with limiting measure μ_s. We made our concrete choice to simplify the proof of Proposition 3.6 below.) In Section 3 we will provide a combinatorial description of the moments of μ_s and write down algebro-integral equations (which we call *color equations*) satisfied by the Stieltjes transform of μ_s and certain auxiliary functions.

The following are our main results. In these results we fix a kernel s satisfying Assumption 2.1.4 and a family $[X^{(N)}]_{N=1}^{\infty}$ of random hermitian matrices satisfying Assumption 2.2.1 with respect to s. As defined in §2.2.2, let $L^{(N)}$ be the empirical distribution of the eigenvalues of $X^{(N)}/\sqrt{N}$. Let $\mu = \mu_s$ be the measure associated to s by the procedure of §2.4.

Theorem 2.5. *$L^{(N)}$ converges weakly in probability to μ.*

Theorem 2.6. *The Stieltjes transform $S(\lambda) = \int \frac{\mu(dx)}{\lambda - x}$ is an algebraic function of λ, i. e., there exists some not-identically-vanishing polynomial $F(X, Y)$ in two variables with complex coefficients such that $F(\lambda, S(\lambda))$ vanishes for all complex numbers λ not in the support of μ.*

To prove Theorem 2.5 we first prove "convergence in moments" in Section 4, and then finish the proof in Section 5 by considering variances. To prove Theorem 2.6 we first set up a general method for proving algebraicity in Section 6, and then finish the proof in Section 7 by analyzing the color equations. The method of proof does not yield an explicit polynomial $F(X, Y)$.

We remark that the general method of Section 6 ought to be applicable to many more random matrix problems. For example, it applies to the Stieltjes transforms of the limiting measures arising from the model of [AZ06] in the case of a finite color space, yielding algebraicity in all those cases.

3. The moments and Stieltjes transform of μ

We fix a kernel s satisfying Assumption 2.1.4 and put $\mu = \mu_s$. We provide a detailed description of the moments and Stieltjes transform of μ. We also introduce combinatorial tools needed throughout the paper.

3.1. **Quick review of key combinatorial notions.**

3.1.1. *Graphs.* For us a *graph* $G = (V, E)$ consists by definition of a finite set V of *vertices* and a set E of *edges*, where every element of E is a subset of V of cardinality 1 or 2. In other words, we are dealing here with graphs (i) which have finitely many vertices, (ii) which have unoriented edges, (iii) in which a vertex may be joined to itself by an edge, but (iv) in which no two vertices may be joined by more than one edge. A graph G is a *tree* if G is connected and $|V| = |E| + 1$. We emphasize that every edge of a tree joins two distinct vertices—it is not allowed for a vertex of a tree to be joined to itself by an edge.

3.1.2. *Set partitions.* We say that a set $\pi \subset 2^{\{1,\dots,k\}}$ is a *set-partition* of k if π is a disjoint family of nonempty sets with union equal to $\{1, \dots, k\}$. The elements of π are called the *parts* of π. For each $i \in \{1, \dots, k\}$, let $\pi(i)$ be the part of π to which i belongs. For convenience we extend $i \mapsto \pi(i)$ to a periodic function on $\mathbb{Z}$ by the rule $\pi(i) = \pi(i + k)$.

3.1.3. *The graph associated to a set partition.* To each set partition π of k there is canonically associated a graph $G_\pi = (V_\pi, E_\pi)$, where $V_\pi = \pi$ and

$$E_\pi = \{\{\pi(i), \pi(i + 1)\} \mid i = 1, \dots, k\}.$$

By construction G_π comes canonically equipped with a walk, namely

$$\pi(1), \dots, \pi(k), \pi(k + 1) = \pi(1),$$

whence in particular it follows that G_π is connected.

3.1.4. *Wigner set partitions.* We say that a set partition π of k is a *Wigner set partition* if the corresponding graph G_π has $k/2+1$ vertices and $k/2$ edges, in which case G_π, since connected, is a tree. We denote the set of such π by $\mathcal{W}_k$. For k odd the set $\mathcal{W}_k$ is empty. For $k=2\ell$ the set $\mathcal{W}_k$ is canonically in bijection with the set of rooted planar trees with $\ell+1$ nodes and hence, as is well-known [St99], the cardinality of $\mathcal{W}_k$ is the *Catalan number* $\frac{1}{\ell+1}\binom{2\ell}{\ell}$.

Lemma 3.1.5. *Fix $\pi \in \mathcal{W}_k$. (i) For each $i \in \{1,\dots,k\}$ we have $\pi(i) \neq \pi(i+1)$. (ii) For each $e \in E_\pi$ the equation $e = \{\pi(i),\pi(i+1)\}$ has exactly two solutions $i \in \{1,\dots,k\}$, say i_1 and i_2, and moreover $\pi(i_1) \neq \pi(i_2)$. (iii) For each $i \in \{1,\dots,k\}$ we have $\{\pi(i),\pi(i+1)\} = \{\pi(j-1),\pi(j)\}$, where j is the least of the integers $\ell > i$ such that $\pi(i) = \pi(\ell)$.*

Proof. The lemma formalizes facts about the tree G_π and the canonical walk on it which are clear from a graph-theoretic point of view. (i) No edge of G_π connects a vertex to itself. (ii) The canonical walk on G_π visits each edge of G_π exactly twice. More precisely, the canonical walk traverses each edge of G_π exactly once in each direction. (iii) The canonical walk on G_π extended by periodicity returns to a given vertex on the same edge by which it departed. $\square$

3.1.6. *Tree integrals.* Let $\{\kappa_A\}$ be an i.i.d. family of C-valued uniformly distributed random variables indexed by finite nonempty sets of positive integers. Expectations with respect to these variables are denoted by $\mathbb{E}$. For each $\pi \in \mathcal{W}_k$ we define a bounded random variable by the formula

$$M_\pi = \prod_{\{A,B\}\in E_\pi} s(\kappa_A,\kappa_B),$$

which is well-defined on account of the symmetry $s(c,c') = s(c',c)$. We call the expectation $\mathbb{E}M_\pi$ a *tree integral.*

Proposition 3.2 (Combinatorial description of the moments of μ)**.** *We have*

$$\langle \mu, x^k\rangle = \sum_{\pi\in\mathcal{W}_k} \mathbb{E}M_\pi \tag{7}$$

for every integer $k>0$.

The proposition is essentially just a special case of [AZ06, Thm. 3.2], but a fair amount of explanation is needed because the set up in this paper is (superficially) incompatible with that of [AZ06]—here we emphasize set partitions, whereas in [AZ06] we emphasized "words" and "spelling". We can immediately reduce the proposition to the following technical lemma. The lemma is slightly more detailed than needed for the proof of the proposition—part (ii) will be needed for the derivation of algebro-integral equations for the Stieltjes transform of μ.

Lemma 3.2.1. *Put $A = 2\|s\|^{1/2}_{L^\infty(C\times C)}$. (i) There exists a unique system $\{\Phi_n,\Psi_n\}_{n\in\mathbb{N}}$ of functions in $L^\infty(C)$ such that*

$$\Psi_n(c) = \int s(c,c')\Phi_n(c')P(dc') \tag{8}$$

holds P-a.e. for every n and there holds an identity

$$\sum_{n=1}^\infty \Phi_n t^n = t\left(1 - t\sum_{n=1}^\infty \Psi_n t^n\right)^{-1} \tag{9}$$

of formal power series in t with coefficients in $L^\infty(C)$. (ii) The bounds

$$0 \leq \Phi_n \leq A^{n-1}, \quad 0 \leq \Psi_n \leq A^{n+1}/4 \tag{10}$$

hold P-a.e. for every n. (iii) The formula

$$\langle P, \Phi_{k+1} \rangle = \sum_{\pi \in \mathcal{W}_k} \mathbb{E} M_\pi \tag{11}$$

holds for every integer $k > 0$.

Proof of the proposition granting the lemma. According [AZ06, Lemma 3.2] (taking there $\sigma = P$, $D = 0$, and $s^{(2)} = s$), there exists a unique probability measure on the real line with k^{th} moment $\langle P, \Phi_{k+1} \rangle$ for every $k \geq 0$. That measure according to [AZ06, Thm. 3.2] is none other than μ. □

Plan for the proof of the lemma. Part (i) of the lemma is nothing but an inductive definition of Φ_n and Ψ_n. So only parts (ii,iii) require proof. In principle, part (iii) follows from [AZ06, Lemmas 6.3 and 6.4], but because of the incompatibility of set-ups noted above, those lemmas cannot be directly applied here—some amplification is needed. Also part (ii) is most easily explained from from the point of view of [AZ06, loc. cit.] So, after recalling in §3.3 the needed background from [AZ06], we lightly sketch in §3.4 a proof of parts (ii,iii) of the lemma.

3.3. **The "verbal" approach.** We briefly recall the point of view of [AZ06] and compare it to the present one. The material reviewed here will be used in a substantial way only in Section 3, not in later sections of the paper.

3.3.1. *Words.* We fix a set of *letters* and define a *word* to be a finite nonempty sequence $w = \alpha_1 \cdots \alpha_k$ of letters. We say that words $w = \alpha_1 \cdots \alpha_k$ and $x = \beta_1 \cdots \beta_\ell$ are *equivalent* if $k = \ell$ (the words are the same length) and there exists a one-to-one-correspondence $\varphi : \{\alpha_i\} \to \{\beta_j\}$ such that $\varphi(\alpha_i) = \beta_i$ for $i = 1, \dots k$ (each word codes to the other under a simple substitution cipher). Each word $w = \alpha_1 \cdots \alpha_k$ of length k gives rise naturally to a set partition of k, namely the set of equivalence classes for the equivalence relation $i \sim j \Leftrightarrow \alpha_i = \alpha_j$. Two words are equivalent if and only if they have the same length and give rise to the same set partition. The upshot is that speaking of equivalence classes of words is equivalent to speaking of set partitions.

3.3.2. *Wigner words.* Let w be a word of at least two letters with same first and last letter, and let w' be the word obtained by dropping the last letter of w. The word w is a Wigner word in the sense of [AZ06] if and only if the set partition associated to w' is a Wigner set partition in the sense of this paper. In [AZ06] we also declared every one-letter word to be a Wigner word. The Wigner words have a simple inductive characterization [AZ06, Prop. 4.5 and §4.7] which is not so convenient to state in the set partition language. To wit, a word w is a Wigner word if and only if the following conditions hold:

- The first and last letters of w are the same.
- No letter appears twice in a row in w.
- Let α be the first letter of w. Write $w = \alpha w_1 \alpha \cdots \alpha w_r \alpha$, where α does not appear in any of the words w_i. Then each word w_i is a Wigner word, and moreover for $i \neq j$ the words w_i and w_j have no letters in common.

(If $r = 0$ then w consists of a single letter and is by definition a Wigner word.)

3.3.3. *"Verbal" description of tree integrals.* Let $\{\kappa_\alpha\}$ be a letter-indexed i.i.d. family of uniformly distributed C-valued random variables. Given a Wigner word w, we define a random variable M_w inductively by the following procedure:

- Writing $w = \alpha w_1 \alpha \cdots \alpha w_r \alpha$ as in the inductive characterization of Wigner words, and letting β_i be the initial letter of w_i for $i = 1, \ldots, r$, we set $M_w = \prod_{i=1}^r s(\kappa_\alpha, \kappa_{\beta_i}) M_{w_i}$. (When w is one letter long, we put $M_w = 1$.)

The formula (11) claimed in Lemma 3.2.1 can be rewritten

$$\langle P, \Phi_{k+1} \rangle = \sum_{w \in W_{k+1}} \mathbb{E} M_w \tag{12}$$

where the sum is extended over a set of representatives W_{k+1} for equivalence classes of Wigner words of length $k+1$. Further and crucially, notation as above in the inductive characterization of the random variables M_w, we have a relation

$$\mathbb{E}(M_w | \kappa_\alpha) = \prod_{i=1}^r \mathbb{E}(s(\kappa_\alpha, \kappa_{\beta_i}) \mathbb{E}(M_{w_i} | \kappa_{\beta_i}) | \kappa_\alpha) \tag{13}$$

among conditional expectations.

3.4. **Proof of Lemma 3.2.1(ii,iii).** It is enough to construct an example of a system $\{\Phi_n, \Psi_n\}$ in $L^\infty(C)$ satisfying (8,9,10,11), and to do so we follow the path of the proofs of [AZ06, Lemmas 6.3 and 6.4]. Fix a letter α. We may suppose that every word belonging to the set of representatives W_{k+1} figuring in (12) begins with α. There exist for each integer $k \geq 0$ well-defined $\Phi_{k+1}, \Psi_{k+1} \in L^\infty(C)$ such that

$$\Phi_{k+1}(\kappa_\alpha) = \sum_{w \in W_{k+1}} \mathbb{E}(M_w | \kappa_\alpha), \quad \Psi_{k+1}(\kappa_{\alpha'}) = \sum_{w \in W_{k+1}} \mathbb{E}(s(\kappa_{\alpha'}, \kappa_\alpha) M_w | \kappa_{\alpha'}),$$

where α' is a letter not appearing in any of the words belonging to the set W_{k+1}. The system $\{\Phi_n, \Psi_n\}$ has property (8) by construction, and has property (11) since the latter is equivalent to (12). Since

$$|\mathcal{W}_k| = |W_{k+1}| \leq 2^k, \quad 0 \leq M_w \leq \|s\|_{L^\infty(C \times C)}^{k/2} \text{ for } w \in W_{k+1},$$

for all integers $k \geq 0$, the system $\{\Phi_n, \Psi_n\}$ has property (10). Finally, from (13) and the inductive characterization of Wigner words, we obtain identities

$$\Phi_{k+1} = \sum_{r=0}^\infty \sum_{\substack{(\ell_1, \ldots, \ell_r) \in \mathbb{N}^r \\ \sum_{i=1}^r (\ell_i + 1) = k}} \prod_{i=1}^r \Psi_{\ell_i}$$

in $L^\infty(C)$ for all integers $k \geq 0$ which together imply that the system $\{\Phi_n, \Psi_n\}$ has property (9). The proofs of Lemma 3.2.1 and Proposition 3.2 are now complete. $\square$

3.5. **The color equations.** We continue in the setting of Proposition 3.2.

3.5.1. *Nice functions.* We say that a complex-valued function f on color space is *nice* (with respect to the kernel s and associated partition $\mathcal{I}$ of $[0,1]$) if f has a Fourier expansion

$$f(c) = \sum_{i\in\mathbb{Z}} f_i(x)\xi^i \quad (c = (x,\xi) \in C)$$

where all but finitely many of the coefficient functions $f_i : [0,1] \to \mathbb{C}$ vanish identically, and every coefficient function f_i is constant on every interval of the partition $\mathcal{I}$. It is not difficult to see that Φ_n and Ψ_n can be "corrected" on a set of P-measure zero in a unique way to become nice. So we may and we will assume hereafter without any loss of generality that the functions Φ_n and Ψ_n are nice and that all the relations asserted in Proposition 3.2 to hold P-a.e. in fact hold without exception.

3.5.2. *An auxiliary function.* For all complex numbers $|\lambda| > A$ and $c \in C$ put

$$\Psi(c,\lambda) = \sum_{n=1}^{\infty} \Psi_n(c)\lambda^{-n},$$

defining a function depending holomorphically on λ and satisfying estimates

$$|\Psi(c,\lambda)| \le \frac{1}{4}\frac{A^2}{|\lambda| - A}, \quad |\lambda| > 2A \Rightarrow |\Psi(c,\lambda)| < \frac{A^2}{2|\lambda|} \le \frac{A}{4} \tag{14}$$

uniform in c.

3.5.3. *The equations.* From the system of equations described in Proposition 3.2 we now deduce by the substitution $t = 1/\lambda$ and application of dominated convergence the relations

$$\int \frac{s(c,c')P(dc')}{\lambda - \Psi(c',\lambda)} = \Psi(c,\lambda), \quad \int \frac{P(dc)}{\lambda - \Psi(c,\lambda)} = S(\lambda) = \int \frac{\mu(dx)}{\lambda - x} \tag{15}$$

which hold for all $c \in C$ and $|\lambda| > 2A$. We call the relations (15) the *color equations.* Equations of this sort have appeared already in other contexts, see [Gi01], [HLN05, eq. 2.2], [KKP96].

Proposition 3.6. *In the setting of the color equations, assume further that for some nice nonnegative function f on color space*

$$s(c,c') = f(c)f(c'), \quad \|f\|_{L^\infty(C)} = A/2, \quad \|f\|_{L^1(C)} = 1.$$

Let μ_f be the law of f viewed as a random variable on C under P. Let $S_f(\lambda)$ be the Stieltjes transform of μ_f. Then: (i) There exists a function $w(\lambda)$ defined and holomorphic for $|\lambda| \gg 0$ such that

$$\lim_{|\lambda|\to\infty} w(\lambda)\lambda = 1, \tag{16}$$

$$\lambda S(\lambda) = 1 + w(\lambda)^2 = \frac{\lambda}{w(\lambda)} S_f\left(\frac{\lambda}{w(\lambda)}\right) \tag{17}$$

for $|\lambda| \gg 0$. (We emphasize that we do mean S on the LHS and S_f on the RHS.) (ii) The measure μ is the free multiplicative convolution of μ_f with the semicircle law of mean 0 and variance 1.

Proof. (i) Put

$$w(\lambda)=\int\frac{f(c)P(dc)}{\lambda-\Psi(c,\lambda)}=\int\Psi(c,\lambda)P(dc), \tag{18}$$

thus defining a holomorphic function in the domain $|\lambda|>2A$ such that

$$w(\lambda)=\frac{1}{\lambda}+O\left(\frac{1}{|\lambda|^2}\right),\quad \Psi(c,\lambda)=w(\lambda)f(c).$$

By definition

$$S_f(\lambda)=\int\frac{P(dc)}{\lambda-f(c)},$$

which is a function holomorphic in the domain $|\lambda|>A/2$. For $|\lambda|\gg 1$ we have by (15) and the first equality in (18) that

$$\begin{aligned}\lambda S(\lambda) &= \int\frac{\lambda P(dc)}{\lambda-w(\lambda)f(c)}=\frac{\lambda}{w(\lambda)}S_f\left(\frac{\lambda}{w(\lambda)}\right),\\ \lambda S(\lambda)-1 &= \int\frac{w(\lambda)f(c)P(dc)}{\lambda-w(\lambda)f(c)}=w(\lambda)^2,\end{aligned}$$

which proves the result.

(ii) We return to the setting of §2.4. Let $W^{(N)}$ be the N^2-by-N^2 matrix with entries

$$W^{(N)}_{ij}=2^{\delta_{ij}/2}\tilde{Y}_{\{c_i^{(N)},c_j^{(N)}\}}/N.$$

Note that $W^{(N)}$ belongs to the GOE (the factor $2^{\delta_{ij}/2}$ is needed for orthogonal invariance). Let $\Lambda^{(N)}$ be the N^2-by-N^2 deterministic diagonal matrix with diagonal entries

$$\Lambda^{(N)}_{ii}=\sqrt{f(c_i^{(N)})}.$$

Then we have

$$\tilde{X}^{(N)}/N=\Lambda^{(N)}W^{(N)}\Lambda^{(N)}.$$

Furthermore, as $N\to\infty$, the empirical distribution of eigenvalues of $W^{(N)}$ (resp., $(\Lambda^{(N)})^2$) tends to the semicircle law (resp., μ_f). Finally, since $W^{(N)}$ and $\Lambda^{(N)}$ are asymptotically freely independent, see [HP00, Corollary 4.3.6] and the discussion on page 157 there concerning the extension from the unitary to orthogonal case, μ has the claimed form of free multiplicative convolution. □

3.7. **Analysis of (NE+SE+SW+NW)-filtered Wigner matrix.** We consider the setup of §2.3 in the special case mentioned in §2.3.5. We are thus considering the model mentioned in the introduction. Proposition 3.6 applies, with μ_f equal to the law of $2\cos^2\theta=1+\cos 2\theta$ with θ distributed uniformly in $[0,2\pi)$. Integrating, we get

$$S_f(\lambda)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\lambda-1-\cos 2\theta}=\frac{1}{\sqrt{\lambda(\lambda-2)}}.$$

It follows from (17) that

$$(1+w^2)^2=\frac{\lambda}{\lambda-2w},$$

and hence (after taking out an irrelevant factor of w)

$$2w^4-\lambda w^3+4w^2-2\lambda w+2=0.$$

In turn, after forming the resultant of lefthand side above and $1 + w^2 - \lambda S$ with respect to w and taking out irrelevant factors, we get the equation

$$4\lambda^2 S^4 - \lambda^3 S^3 - \lambda^2 S^2 + \lambda S + 1 = 0. \tag{19}$$

Since the latter is quartic, it can principle be solved explicitly by root extractions. We omit the details, which are available from the authors. We can without great difficulty find the boundary of the support of μ, as follows. The discriminant $\Delta(\lambda)$ of the left side of (19) relative to S is

$$\Delta(\lambda) = -16\lambda^6(8\lambda^4 + 107\lambda^2 - 1024).$$

The zeroes of $\Delta(\lambda)$ are the only obstructions to analytic continuation of S in the λ-plane. The only nonzero real roots of $\Delta(\lambda)$ are

$$\pm\frac{1}{4}\sqrt{-107 + 51\sqrt{17}} \quad \text{(approximately } \pm 2.5406\text{)},$$

and so these have to be the endpoints of the interval in which μ is supported. We finally remark that it can be shown that $d\mu/dx$ has a spike $\sim 1/\sqrt{|x|}$ at the origin.

4. The main limit calculation

To the end of proving Theorem 2.5, we first prove the following result.

Proposition 4.1. *Let Assumption 2.2.1 hold. For each positive integer k,*

$$\lim_{N\to\infty} N^{-k/2-1} E\,\mathrm{trace}((X^{(N)})^k) = \sum_{\pi\in\mathcal{W}_k} \mathbb{E}M_\pi.$$

The proof requires some preparation and will not be completed until §4.6.

4.2. Notation, terminology and strategy.

4.2.1. *(N,k)-words.* Let N and k be positive integers. An *(N,k)-word* $\mathbf{i}$ is by definition a function

$$\mathbf{i} : \{1,\dots,k\} \to \{1,\dots,N\}.$$

To each (N,k)-word we attach a random variable

$$X_{\mathbf{i}}^{(N)} = \prod_{\alpha=1}^{k} X^{(N)}_{\mathbf{i}(\alpha),\mathbf{i}(\eta_k(\alpha))}$$

where

$$\eta_k = (12\cdots k) \in S_k.$$

(Here and below we employ cycle notation for permutations.) We have

$$\sum_{\mathbf{i}} EX_{\mathbf{i}}^{(N)} = E\,\mathrm{trace}((X^{(N)})^k),$$

where the sum on the left is extended over (N,k)-words $\mathbf{i}$.

4.2.2. *The set partition associated to an (N,k)-word.* Given an (N,k)-word $\mathbf{i}$, consider the graph $G^K_{\mathbf{i}} = (V^K_{\mathbf{i}}, E^K_{\mathbf{i}})$, where $V^K_{\mathbf{i}} = \{1,\dots,k\}$ and

$$E^K_{\mathbf{i}} = \{\{\alpha,\beta\} \subset \{1,\dots,k\} \mid |\mathbf{i}(\alpha) - \mathbf{i}(\beta)| \leq K\}.$$

Here and below K is the constant figuring in Assumption 2.2.1(II). We define $\pi_{\mathbf{i}}$ to be the set partition of k the parts of which are the equivalence classes for the relation "α and β belong to the same connected component of $G^K_{\mathbf{i}}$". (Although the set partition $\pi_{\mathbf{i}}$ depends on K, we suppress reference to K in the notation.) We have

$$|\mathbf{i}(\alpha) - \mathbf{i}(\beta)| \leq K \Rightarrow \pi_{\mathbf{i}}(\alpha) = \pi_{\mathbf{i}}(\beta), \quad |\mathbf{i}(\alpha) - \mathbf{i}(\beta)| > Kk \Rightarrow \pi_{\mathbf{i}}(\alpha) \neq \pi_{\mathbf{i}}(\beta),$$

for all $\alpha, \beta \in \{1,\dots,k\}$.

4.2.3. *Distinguished (N,k)-words.* Let $\mathbf{i}$ be an (N,k)-word. Consider the following conditions:

(I) $\pi_{\mathbf{i}} \in \mathcal{W}_k$.
(II) For all $A \in \pi_{\mathbf{i}}$ we have $\mathbf{i}(\min A) \in Q^{(N)}_{(k+1)K}$.
(III) For all distinct $A, B \in \pi_{\mathbf{i}}$ we have $|\mathbf{i}(\min A) - \mathbf{i}(\min B)| > 5kK$.
(IV) $\max\limits_{A \in \pi_{\mathbf{i}}} \max\limits_{\alpha,\beta \in A} |\mathbf{i}(\alpha) - \mathbf{i}(\beta)| \leq Kk$.
(V) $\min\limits_{\alpha=1}^{k} |\mathbf{i}(\alpha) - \mathbf{i}(\eta_k(\alpha))| > 3Kk$.
(VI) For $\alpha = 1,\dots,k$ we have $\mathbf{i}(\alpha) \in Q^{(N)}_K$.
(VII) For all $A \in \pi$ and $\alpha, \beta \in A$, the numbers $\frac{\mathbf{i}(\alpha)}{N}$ and $\frac{\mathbf{i}(\beta)}{N}$ belong to the same interval of the partition $\mathcal{I}$.

If conditions (I,II,III) hold we say that $\mathbf{i}$ is *distinguished*, in which case $\mathbf{i}$ automatically also satisfies conditions (IV,V,VI,VII).

4.2.4. *Strategy.* We will show that the only nonnegligible contributions to

$$\lim_{N\to\infty} N^{-k/2-1} \sum_{\mathbf{i}:\ (N,k)\text{-word}} EX^{(N)}_{\mathbf{i}} = \lim_{N\to\infty} N^{-k/2-1} E\,\mathrm{trace}((X^{(N)})^k) \tag{20}$$

come from distinguished (N,k)-words. Then we will evaluate $EX^{(N)}_{\mathbf{i}}$ for distinguished $\mathbf{i}$. Finally, we will calculate the limit on the left with the summation restricted to distinguished $\mathbf{i}$.

4.3. Negligibility of nondistinguished (N,k)-words.

Lemma 4.3.1. *Let π be a set partition of k. There exists $C_\pi > 0$ such that for every positive integer N the sum $\sum_{\mathbf{i}} E|X^{(N)}_{\mathbf{i}}|$ extended over (N,k)-words $\mathbf{i}$ such that $\pi = \pi_{\mathbf{i}}$ does not exceed $C_\pi N^{|\pi|}$.*

Proof. By Assumption 2.2.1(Ib) and the Hölder inequality, it suffices simply to estimate the number of (N,k)-words such that $\pi = \pi_{\mathbf{i}}$. A crude estimate of the latter is $(1+2Kk)^{k-|\pi|}N^{|\pi|}$. □

Lemma 4.3.2. *Let π be a set partition of k. Let $\mathbf{i}$ be an (N,k)-word such that $\pi = \pi_{\mathbf{i}}$. If $|\pi| \geq k/2+1$ and $EX^{(N)}_{\mathbf{i}} \neq 0$, then $\pi \in \mathcal{W}_k$ (and hence $|\pi| = k/2+1$).*

Proof. Let $G_\pi = (V_\pi, E_\pi)$ be the graph associated to π. For each $\alpha \in \{1, \dots, k\}$ put $e(\alpha) = \{\pi(\alpha), \pi(\alpha+1)\} \in E_\pi$. Now fix $e \in E_\pi$. It is enough to show that $e = e(\alpha)$ for at least two $\alpha \in \{1, \dots, k\}$, for then, since G_π is connected, we have

$$k/2 + 1 \leq |\pi| = |V_\pi| \leq |E_\pi| + 1 \leq k/2 + 1,$$

and hence π is a Wigner set partition. To derive a contradiction, suppose rather that $e = e(\alpha)$ for unique $\alpha \in \{1, \dots, k\}$. For every $\gamma \in \{1, \dots, k\}$ let $i(\gamma) \leq j(\gamma)$ be the integers $\mathbf{i}(\gamma)$ and $\mathbf{i}(\eta_k(\gamma))$ rearranged. Then for every $\beta \in \{1, \dots, k\} \setminus \{\alpha\}$ we have $\max(|i(\alpha) - i(\beta)|, |j(\alpha) - j(\beta)|) > K$, for otherwise $e(\alpha) = e(\beta)$. It follows by Assumption 2.2.1(IIb) that the random variable $X^{(N)}_{\mathbf{i}(\alpha), \mathbf{i}(\eta_k(\alpha))}$ is independent of the rest of the random variables appearing in the product $X^{(N)}_{\mathbf{i}}$, and hence $EX^{(N)}_{\mathbf{i}} = 0$ by Assumption 2.2.1(Ia). This contradiction proves the lemma. □

Lemma 4.3.3. *Let π be a Wigner set partition. There exists $C'_\pi > 0$ such that for every positive integer N the sum $\sum_{\mathbf{i}} E|X^{(N)}_{\mathbf{i}}|$ extended over (N, k)-words $\mathbf{i}$ such that $\pi = \pi_{\mathbf{i}}$ but $\mathbf{i}$ is not distinguished does not exceed $C'_\pi N^{k/2}$.*

Proof. The proof is similar to that of Lemma 4.3.1. We omit the details. □

4.4. Evaluation of $EX^{(N)}_{\mathbf{i}}$ for distinguished i.

4.4.1. *Definitions of τ_π and σ_π.* Let π be a set partition of k. Enumerate π and its parts thus:

$$\pi = \{I_1, \dots, I_{|\pi|}\}, \quad \min I_1 < \cdots < \min I_{|\pi|},$$
$$I_\alpha = \{i_{\alpha 1} < \cdots < i_{\alpha, |I_\alpha|}\} \text{ for } \alpha = 1, \dots, |\pi|.$$

Put

$$\tau_\pi = (i_{11} \cdots i_{1,|I_1|}) \cdots (i_{|\pi|,1} \cdots i_{|\pi|,|I_{|\pi|}|}) \in S_k, \quad \sigma_\pi = \eta_k^{-1} \tau_\pi \in S_k.$$

By construction

$$\pi(\tau_\pi(i)) = \pi(i)$$

for $i = 1, \dots, k$.

Lemma 4.4.2. *Let $\pi \in \mathcal{W}_k$ be a Wigner set partition. Then the permutation σ_π is fixed-point-free and squares to the identity. Furthermore, for all distinct $i, j \in \{1, \dots, k\}$, we have $\{\pi(i), \pi(i+1)\} = \{\pi(j), \pi(j+1)\} \Leftrightarrow j = \sigma_\pi(i)$.*

Proof. Let $G_\pi = (V_\pi, E_\pi)$ be the graph (tree) associated to π. For each $e \in E_\pi$, there are by Lemma 3.1.5(ii) exactly two indices i such that $e = \{\pi(i), \pi(i+1)\}$, and σ_π swaps them by Lemma 3.1.5(iii). □

Lemma 4.4.3. *Let $\pi \in \mathcal{W}_k$ be a Wigner set partition. Put $\sigma = \sigma_\pi$ and $\tau = \tau_\pi$. Then we have*

$$EX^{(N)}_{\mathbf{i}} = \prod_{\substack{\alpha \in \{1, \dots, k\} \\ \text{s.t. } \alpha \leq \sigma(\alpha)}} EX^{(N)}_{\mathbf{i}(\alpha), \mathbf{i}(\tau(\sigma(\alpha)))} X^{(N)}_{\mathbf{i}(\sigma(\alpha)), \mathbf{i}(\tau(\alpha))}$$

for every (N, k)-word $\mathbf{i}$ such that $\pi_{\mathbf{i}} = \pi$.

Proof. Put $\eta = \eta_k$. By definition we have $\eta\sigma = \tau$. By the previous lemma we have $\tau\sigma = \eta$. Therefore after rearranging the product

$$X^{(N)}_{\mathbf{i}} = \prod_{\alpha \in \{1, \dots, k\}} X_{\mathbf{i}(\alpha), \mathbf{i}(\eta(\alpha))},$$

we have
$$X_{\mathbf{i}}^{(N)} = \prod_{\substack{\alpha\in\{1,\dots,k\}\\ \text{s.t. } \alpha\leq\sigma(\alpha)}} X^{(N)}_{\mathbf{i}(\alpha),\mathbf{i}(\tau(\sigma(\alpha)))} X^{(N)}_{\mathbf{i}(\sigma(\alpha)),\mathbf{i}(\tau(\alpha))}.$$
It suffices to prove enough independence to justify pushing the expectation under the product. For every $\gamma \in \{1,\dots,k\}$ let $i(\gamma) \leq j(\gamma)$ be the integers $\mathbf{i}(\gamma)$ and $\mathbf{i}(\eta(\gamma))$ rearranged. By definition of $\pi_{\mathbf{i}}$ and the preceding lemma, for all distinct $\alpha,\beta \in \{1,\dots,k\}$, if $\max(|i(\alpha)-i(\beta)|,|j(\alpha)-j(\beta)|) \leq K$, then $\beta = \sigma(\alpha)$. By Assumption 2.2.1(IIb) we deduce the desired independence. □

4.4.4. *The difference operator associated to a set partition* π. Let π be a set partition of k. Let $\tau = \tau_\pi \in S_k$ be the canonically associated permutation. Let $f : \{1,\dots,k\} \to \mathbb{Z}$ be a function. We define $\partial_\pi f : \{1,\dots,k\} \to \mathbb{Z}$ by the formula $(\partial_\pi f)(i) = f(i) - f(\tau(i))$ for $i = 1,\dots,k$. Given a function $g : \{1,\dots,k\} \to \mathbb{Z}$, the equation $\partial_\pi f = g$ has a solution $f : \{1,\dots,k\} \to \mathbb{Z}$ if and only if $\sum_{\alpha\in A} g(\alpha) = 0$ for all parts $A \in \pi$, and f is unique up to the addition of a function $\{1,\dots,k\} \to \mathbb{Z}$ constant on every part of π.

Lemma 4.4.5. *Let $\pi \in \mathcal{W}_k$ be a Wigner set partition. Put $\sigma = \sigma_\pi$, $\tau = \tau_\pi$ and $\partial = \partial_\pi$. Fix $\alpha \in \{1,\dots,k\}$ such that $\alpha \leq \sigma(\alpha)$. We have*
$$EX^{(N)}_{\mathbf{i}(\alpha),\mathbf{i}(\tau(\sigma(\alpha)))} X^{(N)}_{\mathbf{i}(\sigma(\alpha)),\mathbf{i}(\tau(\alpha))} = s_{(\partial\mathbf{i})(\alpha),(\partial\mathbf{i})(\sigma(\alpha))}\left(\frac{\mathbf{i}(\min\pi(\alpha))}{N}, \frac{\mathbf{i}(\min(\pi(\sigma(\alpha))))}{N}\right) \tag{21}$$
for every distinguished (N,k)-word $\mathbf{i}$ such that $\pi_{\mathbf{i}} = \pi$.

In particular, the expectation in question vanishes unless $|\partial\mathbf{i}| \leq K$ by Assumption 2.2.1(IIa).

Proof. Let $E(\alpha)$ be the left side of (21). Assume at first that $\mathbf{i}(\alpha) \leq \mathbf{i}(\tau(\sigma(\alpha))$. Then $\mathbf{i}(\alpha) + 3Kk < \mathbf{i}(\tau(\sigma(\alpha))$ by property (V) of a distinguished (N,k)-word, and hence $\mathbf{i}(\tau(\alpha)) + Kk < \mathbf{i}(\sigma(\alpha))$ by property (IV) of a distinguished (N,k)-word. So we have
$$E(\alpha) = EX^{(N)}_{\mathbf{i}(\alpha),\mathbf{i}(\tau(\sigma(\alpha)))} \overline{X}^{(N)}_{\mathbf{i}(\tau(\alpha)),\mathbf{i}(\sigma(\alpha))} = s_{(\partial\mathbf{i})(\alpha),(\partial\mathbf{i})(\sigma(\alpha))}\left(\frac{\mathbf{i}(\alpha)}{N}, \frac{\mathbf{i}(\tau(\sigma(\alpha)))}{N}\right) \tag{22}$$
by Assumption 2.2.1(IIc) and property (VI) of a distinguished (N,k)-word. Then we deduce that the desired formula holds by Assumption 2.1.4(III) and property (VII) of a distinguished (N,k)-word. Assume next and finally that $\mathbf{i}(\alpha) \geq \mathbf{i}(\tau(\sigma(\alpha))$. Reasoning as above we have
$$E(\alpha) = EX^{(N)}_{\mathbf{i}(\sigma(\alpha)),\mathbf{i}(\tau(\alpha))} \overline{X}^{(N)}_{\mathbf{i}(\tau(\sigma(\alpha)),\mathbf{i}(\alpha)} = s_{(\partial\mathbf{i})(\sigma(\alpha)),(\partial\mathbf{i})(\alpha)}\left(\frac{\mathbf{i}(\sigma(\alpha))}{N}, \frac{\mathbf{i}(\tau(\alpha))}{N}\right).$$
Now we apply the symmetry (1), and then continue to reason as above. We deduce the desired formula just as before. □

4.5. Contribution of $EX_{\mathbf{i}}^{(N)}$ to the limit for distinguished i.

Lemma 4.5.1. *Let $\pi \in \mathcal{W}_k$ and an (N,k)-word $\mathbf{i}$ be given such that the following hold:*

- *For all $A \in \pi$ we have $\mathbf{i}(\min A) \in Q^{(N)}_{(k+1)K}$.*
- *For all distinct $A, B \in \pi$ we have $|\mathbf{i}(\min A) - \mathbf{i}(\min B)| > 5Kk$.*
- $|\partial\mathbf{i}| \leq K$.

Then $\pi_{\mathbf{i}} = \pi$ and (hence) π is distinguished.

No proof is needed, but this point bears emphasis as an important step in the proof of Proposition 4.1.

Lemma 4.5.2. *Fix a Wigner set partition $\pi \in \mathcal{W}_k$. Then we have*

$$\mathbb{E}M_\pi = \lim_{N\to\infty} N^{-k/2-1} \sum_{\mathbf{i}} EX_{\mathbf{i}}^{(N)}, \tag{23}$$

where the sum is extended over distinguished (N,k)-words $\mathbf{i}$ such that $\pi_{\mathbf{i}} = \pi$.

Proof. Let $\partial = \partial_\pi$, $\sigma = \sigma_\pi$, and $\tau = \tau_\pi$. Let $\{t_A\}$ (resp., $\{z_A\}$) be an i.i.d. family of random variables uniform in $[0,1]$ (resp., S^1), indexed by finite nonempty sets A of positive integers. We further suppose that the families $\{t_A\}$ and $\{z_A\}$ are defined on a common probability space and are independent. We denote expectations with respect to these variables by $\mathbf{E}$. We have by Lemma 4.4.2 and the definitions that

$$\begin{aligned}
\mathbb{E}M_\pi &= \mathbf{E} \prod_{\substack{i\in\{1,\dots,k\}\\ \text{s.t. } i\le\sigma(i)}} \left(\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} s_{mn}(t_{\pi(i)}, t_{\pi(\sigma(i))}) z_{\pi(i)}^m z_{\pi(\sigma(i))}^n \right) \\
&= \sum_{f:\{1,\dots,k\}\to\mathbb{Z}} \mathbf{E} \prod_{\substack{i\in\{1,\dots,k\}\\ \text{s.t. } i\le\sigma(i)}} s_{f(i),f(\sigma(i))}(t_{\pi(i)}, t_{\pi(\sigma(i))}) \cdot \prod_{A\in\pi} z_A^{\sum_{i\in A} f(i)} \\
&= \sum_{\substack{f:\{1,\dots,k\}\to\mathbb{Z}\cap[-K,K]\\ \forall A\in\pi,\ \sum_{i\in A} f(i)=0}} \mathbf{E} \prod_{\substack{i\in\{1,\dots,k\}\\ \text{s.t. } i\le\sigma(i)}} s_{f(i),f(\sigma(i))}(t_{\pi(i)}, t_{\pi(\sigma(i))}).
\end{aligned}$$

At the last step we integrate out the z's and take into account Assumption 2.2.1(IIa). We then have

$$\mathbb{E}M_\pi = \sum_{\substack{f:\{1,\dots,k\}\to\mathbb{Z}\\ |\partial f|\le K\\ \forall A\in\pi\ f(\min A)=f_0(A)}} \mathbf{E} \prod_{\substack{i\in\{1,\dots,k\}\\ \text{s.t. } i\le\sigma(i)}} s_{(\partial f)(i),(\partial f)(\sigma(i))}(t_{\pi(i)}, t_{\pi(\sigma(i))}),$$

where $f_0 : \pi \to \mathbb{Z}$ is any fixed function defined on the parts of π. After some straightforward bookkeeping which we omit, it follows by the preceding two lemmas that (23) holds with the summation on $\mathbf{i}$ restricted to distinguished (N,k)-words such that $\pi_{\mathbf{i}} = \pi$ and $|\partial\mathbf{i}| \le K$. But then by Assumption 2.2.1(IIa), the limit does not change if we drop the restriction $|\partial\mathbf{i}| \le K$ (the further terms all vanish), whence the result. □

4.6. **Completion of the proof of Proposition 4.1.** The suite of lemmas proved in §4.3 shows that the limit on the left side of equation (20) does not change if we restrict attention to distinguished (N,k)-words. The last lemma above evaluates the limit on the left side of (20) with $\mathbf{i}$ restricted to distinguished (N,k)-words, and gives the desired value. The proof of Proposition 4.1 is complete. □

5. Completion of the proof of Theorem 2.5

Fix a positive integer k. As in [AZ06, pf. of Thm. 3.2, Section 6, p. 305], Theorem 2.5 will follow as soon as we can prove that

$$\mathrm{Var}(\langle L^{(N)}, x^k\rangle) \to_{N\to\infty} 0\,. \tag{24}$$

We will prove this by lightly modifying the arguments of §4.3. Now

$$\mathrm{Var}(\langle L^{(N)}, x^k\rangle) = N^{-k-2}\sum_{(\mathbf{i},\mathbf{j})}\left(E[X_{\mathbf{i}}^{(N)}X_{\mathbf{j}}^{(N)}] - E[X_{\mathbf{i}}^{(N)}]E[X_{\mathbf{j}}^{(N)}]\right) \tag{25}$$

where the sum is extended over pairs $(\mathbf{i},\mathbf{j})$ of (N,k)-words. By Assumption 2.2.1(Ib) and the Hölder inequality, it is (more than) enough to show that the number of pairs $(\mathbf{i},\mathbf{j})$ making a nonzero contribution to the sum on the right side of (25) is $O(N^{k+1})$. Now fix a pair $(\mathbf{i},\mathbf{j})$ of (N,k)-words indexing a nonzero term in the sum on the right side of (25). Let $\mathbf{ij}$ be the $(N,2k)$-word obtained by concatenating $\mathbf{i}$ and $\mathbf{j}$, i. e.,

$$\mathbf{ij}(\alpha) = \left\{\begin{array}{cl} \mathbf{i}(\alpha) & \text{if } \alpha \in \{1,\dots,k\},\\ \mathbf{j}(\alpha-k) & \text{if } \alpha\in\{k+1,\dots,2k\}.\end{array}\right.$$

Consider the set partition $\pi = \pi_{\mathbf{ij}}$ defined in §4.2.2. We need also to consider a graph associated to π. The relevant graph is no longer G_π, but rather a slightly modified version $\tilde{G}_\pi = (\tilde{V}_\pi, \tilde{E}_\pi)$, where $\tilde{V}_\pi = \pi$ and

$$\tilde{E}_\pi = \{\{\pi(1),\pi(2)\},\dots,\{\pi(k),\pi(1)\}\}\cup\{\{\pi(k+1),\pi(k+2)\},\dots,\{\pi(2k),\pi(k+1)\}\}.$$

By construction $\tilde{G}_\pi$ comes equipped with two walks, namely

$$\pi(1),\dots,\pi(k),\pi(1) \text{ and } \pi(k+1),\dots,\pi(2k),\pi(k+1).$$

Arguing as in the proof of Lemma 4.3.2, we find that nonvanishing of the term on the right side of (25) indexed by $(\mathbf{i},\mathbf{j})$ implies that the walks jointly visit every edge of $\tilde{G}_\pi$ at least twice, and moreover there must exist some edge of $\tilde{G}_\pi$ visited by both walks. Thus $|\tilde{E}_\pi| \le k$ and $\tilde{G}_\pi$ is connected. It follows that $|\pi| = |\tilde{V}_\pi| \le k+1$. Finally, arguing as in the proof of Lemma 4.3.1, we find that the number of nonzero terms on the right side of (25) is indeed $O(N^{k+1})$. The proof of Theorem 2.5 is complete. □

6. An algebraicity criterion

In this section, which is completely independent of the preceding ones, we develop a "soft" method for proving that a holomorphic function is algebraic under hypotheses commonly encountered in random matrix theory.

6.1. **Formulation of an algebraicity criterion.**

6.1.1. *Notation.* Let $\{X_i\}_{i=1}^\infty$ be independent (algebraic) variables. Let $\mathbb{C}[X_1,\dots,X_n]$ denote the ring of polynomials in $X_1,\dots,X_n$ with coefficients in $\mathbb{C}$. We view $\mathbb{C}[X_1,\dots,X_n]$ as a subring of $\mathbb{C}[X_1,\dots,X_{n+1}]$. Given $F\in\mathbb{C}[X_1,\dots,X_n]$, let $F(0)\in\mathbb{C}$ be the result of setting $X_1=\cdots=X_n=0$. Let $\mathbb{C}(X_1,\dots,X_n)$ be the field of rational functions in the variables $X_1,\dots,X_n$, i. e., the ring consisting of fractions A/B where $A,B\in\mathbb{C}[X_1,\dots,X_n]$ and B does not vanish identically. We say that $F\in\mathbb{C}(X_1,\dots,X_n)$ is *defined at the origin* if $F=A/B$ with $A,B\in\mathbb{C}[X_1,\dots,X_n]$ such that $B(0)\neq 0$, in which case we put $F(0)=A(0)/B(0)$.

6.1.2. *DIRE families.* Let $\varphi_1,\dots,\varphi_N$ be a finite family of holomorphic functions each defined in a connected open neighborhood of the origin in $\mathbb{C}^n$; the domains need not be the same. We will say that $\varphi_1,\dots,\varphi_N$ are *defined implicitly by rational equations* (*DIRE* for short) if there exist $\Phi_1,\dots,\Phi_N\in\mathbb{C}(X_1,\dots,X_{n+N})$ such that

(I) Φ_i is defined at the origin and $\Phi_i(0)=0$ for $i=1,\dots,N$,

(II) $(\det_{i,j=1}^N \frac{\partial\Phi_i}{\partial X_{j+n}})(0)\neq 0$, and

(III) $\Phi_i(z_1,\dots,z_n,\varphi_1(z)-\varphi_1(0),\dots,\varphi_N(z)-\varphi_N(0))=0$ for $i=1,\dots,N$ and $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ sufficiently near the origin.

The relationship between this definition and the implicit function theorem for holomorphic functions [Ca63, Proposition 6.1] is close. Indeed, given $\Phi_1,\dots,\Phi_N\in\mathbb{C}(X_1,\dots,X_{n+N})$ with properties (I,II) above, the implicit function theorem for holomorphic functions says that there exist holomorphic functions $\varphi_1,\dots,\varphi_N$ each defined in a connected open neighborhood of the origin such that (III) holds, and the theorem further asserts uniqueness of these functions in the sense that if $\psi_1,\dots,\psi_N$ are holomorphic functions each defined in a connected open neighborhood of the origin in $\mathbb{C}^n$ and also satisfying (III), then for $i=1,\dots,N$ there exists a neighborhood of the origin in which ψ_i and φ_i differ by a constant.

In the sequel, for brevity, when we say "$\varphi_1,\dots,\varphi_N$ is an n-variable DIRE family", this is short for the assertion that "$\varphi_1,\dots,\varphi_N$ are holomorphic functions each defined in some connected open neighborhood of the origin in $\mathbb{C}^n$ which together have the property of being defined implicitly by rational equations".

6.1.3. *Algebraic functions.* A holomorphic function φ defined in a nonempty open subset $D\subset\mathbb{C}^n$ is called an *n-variable algebraic function* if there exists a not-identically-vanishing polynomial $F=F(X_1,\dots,X_{n+1})\in\mathbb{C}[X_1,\dots,X_{n+1}]$ such that $F(z_1,\dots,z_n,\varphi(z))=0$ for all $z=(z_1,\dots,z_n)\in D$. If D is connected and $U\subset D$ is any nonempty open subset, then algebraicity of F in U implies algebraicity of F in D by the principle of analytic continuation.

The main result of Section 6 is the following.

Theorem 6.2. *Let $\varphi_1,\dots,\varphi_N$ be an n-variable DIRE family. Then each φ_i is an n-variable algebraic function.*

The proof takes up the last several subsections of Section 6. Before turning to the proof we give key examples of DIRE families, and describe techniques for constructing new DIRE families from old.

Proposition 6.3. *Fix a positive integer ℓ and put $L=2\ell+1$. For $z=(z_1,\dots,z_L)\in\mathbb{C}^L$ such that $\sum_{i=1}^L|z_i|<1$ and integers $j=1,\dots,L$, consider the quantity*

$$\vartheta_j(z)=-\delta_{j,\ell+1}+\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-i(j-\ell-1)x)dx}{1-\sum_{k=1}^L z_k\exp(i(k-\ell-1)x)},\tag{26}$$

which depends holomorphically on z and vanishes for $z=0$. Then the family $\vartheta_1,\dots,\vartheta_L$ can be extended to an L-variable DIRE family $\vartheta_1,\dots,\vartheta_N$.

Proof. Fix $z=(z_1,\dots,z_L)\in\mathbb{C}^L$ such that $\sum_{i=1}^L|z_i|<1$. Let $p=[p_{ij}]_{i,j\in\mathbb{Z}}$ be the unique matrix of complex numbers with rows and columns indexed by $\mathbb{Z}$ with the following properties:

- $p_{ij}=0$ for all i and j such that $|i-j|>\ell$.
- $p_{ij}=z_{j-i+\ell+1}$ for all i and j such that $|i-j|\le\ell$.

Let $|p|$ be the $\mathbb{Z}$-by-$\mathbb{Z}$ matrix with entries $|p_{ij}|$. We have the crude estimate

$$(|p|^t)_{ij}\le\left(\sum_{i=1}^L|z_i|\right)^t\tag{27}$$

holding for all positive integers t.

Now supposing p were a Markov matrix (which of course it is not), we could view each entry p_{ij} as a transition probability

$$P(S_{t+1} = j \mid S_t = i) = p_{ij},$$

for a random walk $\{S_t\}_{t=0}^{\infty}$ on $\mathbb{Z}$ with step-length bounded by ℓ and we would have, for any subset $M \subset \mathbb{Z}^t$, an equality

$$P((S_1, \dots, S_t) \in M \mid S_0 = i_0) = \sum_{(i_1,\dots,i_t) \in M} p_{i_0 i_1} \cdots p_{i_{t-1} i_t}.$$

Because it is a valuable aid to intuition, we will make the line above a definition. Notice that by (27) the sum on the right is absolutely convergent, and that the sum of the absolute values of the terms is $\leq (\sum_{i=1}^{L} |z_i|)^t$. We will be able to calculate using the usual rules of probability, provided we never invoke positivity $p_{ij} \geq 0$ or the Markov property $\sum_j p_{ij} = 1$. We are interested in functions of z describable in the language of random walk because, as we explain presently, the functions ϑ_j belong to this class, and moreover within this class we can easily find the "extra" functions needed to extend $\vartheta_1, \dots, \vartheta_L$ to a DIRE family.

Let U, V, A, B, C be ℓ-by-ℓ matrices with complex entries defined as follows:

$$\begin{array}{rcl} U_{ij} & = & \sum_{t=1}^{\infty} P(S_t = j, \min_{u=0}^{t} S_u > 0 | S_0 = i), \\ V_{ij} & = & \sum_{t=1}^{\infty} P(S_t = j, \max_{u=0}^{t} S_u < \ell + 1 | S_0 = i), \\ A_{ij} & = & P(S_1 = j + \ell \mid S_0 = i), \\ B_{ij} & = & P(S_1 = j \mid S_0 = i), \\ C_{ij} & = & P(S_1 = j - \ell \mid S_0 = i). \end{array}$$

Let W, D be L-by-L matrices with complex entries defined as follows:

$$\begin{array}{rcl} W_{ij} & = & \sum_{t=1}^{\infty} P(S_t = j | S_0 = i), \\ D_{ij} & = & P(S_1 = j \mid S_0 = i). \end{array}$$

By (27), all the series in question here converge absolutely and define functions of z holomorphic in the domain $\sum_{i=1}^{L} |z_i| < 1$. Notice that A, B, C, D depend linearly on z, and that all matrices A, B, C, D, U, V, W vanish for $z = 0$.

Consider the expansion of the integrand of (26) by geometric series and then integrate term by term. One finds in this way a series expression for ϑ_j identical to the series expression defining some entry of the matrix W. In short, every ϑ_j appears in W.

By breaking paths down according to visits to the sets $\{-\ell, \dots, -1\}$, $\{0\}$ and $\{1, \dots, \ell\}$, we obtain in the usual way recursions

$$\begin{array}{rcl} 0 & = & -U + B + A(1+U)C(1+U), \\ 0 & = & -V + B + C(1+V)A(1+V), \\ 0 & = & -W + D + \left[\begin{array}{ccc} C(1+V)A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A(1+U)C \end{array}\right](1+W). \end{array} \tag{28}$$

Extend the given family $\vartheta_1, \dots, \vartheta_n$ to an enumeration $\vartheta_1, \dots, \vartheta_N$ of all entries of U, V and W. Rewrite the system of equations (28) as a system of N polynomial equations in $z_1, \dots, z_L, \vartheta_1, \dots, \vartheta_N$, in order to find polynomials

$$\Phi_i(X_1, \dots, X_{L+N}) \in \mathbb{C}[X_1, \dots, X_{L+N}] \text{ for } i = 1, \dots, N$$

such that

$$\Phi_i(0, \dots, 0, X_{L+1}, \dots, X_{L+N}) = -X_{i+L}, \quad \Phi_i(z_1, \dots, z_L, \vartheta_1, \dots, \vartheta_N) = 0.$$

Note that part (II) of the definition of DIRE family holds because $\frac{\partial \Phi_i}{\partial X_{j+L}}(0) = -\delta_{ij}$. Thus $\vartheta_1, \ldots, \vartheta_N$ is indeed a DIRE family. $\square$

6.4. **Natural operations on DIRE families.** We write down some lemmas which will be helpful in applying the notion of DIRE family. The first three are trivial but deserve being stated for the sake of emphasis. The last is the trick decisive for the application of Theorem 6.2 to the proof of Theorem 2.6.

Lemma 6.4.1. *Let $\varphi_1, \ldots, \varphi_N$ and $\psi_1, \ldots, \psi_M$ be n-variable DIRE families. Then the concatenation $\varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_M$ is an n-variable DIRE family.*

Lemma 6.4.2. *Let $\varphi_1, \ldots, \varphi_N$ be an n-variable DIRE family. Let φ_{N+1} be a $\mathbb{C}$-linear combination of $\varphi_1, \ldots, \varphi_N$. Then the extended family $\varphi_1, \ldots, \varphi_{N+1}$ is an n-variable DIRE family.*

Lemma 6.4.3. *Let $\varphi_1, \ldots, \varphi_N$ be an n-variable DIRE family. Let A be an n by m matrix with complex entries. Identify $\mathbb{C}^n$ and $\mathbb{C}^m$ with spaces of column vectors. Then there exists an m-variable DIRE family $\psi_1, \ldots, \psi_N$ such that for $i = 1, \ldots, N$ we have $\psi_i(z) = \varphi_i(Az)$ for all $z \in \mathbb{C}^m$ sufficiently near the origin.*

Lemma 6.4.4. *Let $\psi_1, \ldots, \psi_n$ be a family of holomorphic functions defined in an open disk centered at the origin in $\mathbb{C}$. Let $\varphi_1, \ldots, \varphi_N$ be an n_0-variable DIRE family, where $N \geq n \geq n_0$. Assume that*

$$\psi_i(z) = z\varphi_i(z\psi_1(z), \ldots, z\psi_n(z))$$

for $i = 1, \ldots, n$ and $z \in \mathbb{C}$ sufficiently near the origin. Then $\psi_1, \ldots, \psi_n$ can be extended to a 1-variable DIRE family.

Proof. By the preceding lemma we may assume without loss of generality that $n = n_0$. For $i = n+1, \ldots, n+N$ the formula

$$\psi_i(z) = \varphi_{i-n}(z\psi_1(z), \ldots, z\psi_n(z)) - \varphi_{i-n}(0)$$

defines a holomorphic function ψ_i in some open neighborhood of the origin in $\mathbb{C}$. We will prove that the extended family $\psi_1, \ldots, \psi_{N+n}$ is a 1-variable DIRE family. Note that all the functions ψ_i vanish at the origin. Let $\Phi_1, \ldots, \Phi_{n+N} \in \mathbb{C}(X_1, \ldots, X_{N+n})$ be with respect to $\varphi_1, \ldots, \varphi_N$ as called for by the definition of an n-variable DIRE family. For $i = 1, \ldots, N+n$ define $\Psi_i \in \mathbb{C}(X_1, \ldots, X_{N+n+1})$ by the formula

$$\begin{aligned} &\Psi_i(X_1, \ldots, X_{n+N+1}) \\ = &\left\{ \begin{array}{ll} X_{i+1} - X_1(X_{i+n+1} + \varphi_i(0)) & \text{if } 1 \leq i \leq n, \\ \Phi_{i-n}(X_1X_2, \ldots, X_1X_{n+1}, X_{n+2}, \ldots, X_{n+N+1}) & \text{if } n+1 \leq i \leq n+N. \end{array} \right. \end{aligned}$$

Then:

(I) Ψ_i is defined at the origin and $\Psi_i(0) = 0$ for $i = 1, \ldots, N+n$,
(II) $(\det_{i,j=1}^{N+n} \frac{\partial \Psi_i}{\partial X_{j+1}})(0) \neq 0$, and
(III) $\Psi_i(z, \psi_1(z), \ldots, \psi_{N+n}(z)) = 0$ for $i = 1, \ldots, N+n$ and $z \in \mathbb{C}$ sufficiently near the origin.

In other words, $\Psi_1, \ldots, \Psi_{n+N}$ are with respect to $\psi_1, \ldots, \psi_{n+N}$ as called for by the definition of 1-variable DIRE family. $\square$

We next formulate a purely algebraic result and explain how to deduce Theorem 6.2 from it.

Theorem 6.5. *Let n and N be positive integers. Let*

$$F_1, \ldots, F_{n+N} \in \mathbb{C}[X_1, \ldots, X_{N+n}]$$

be given with the following properties:

$$F_i(0) = 0 \text{ for } i = 1, \ldots, N. \tag{29}$$

$$\left(\det_{i,j=1}^{N} \frac{\partial F_i}{\partial X_{j+n}}\right)(0) \neq 0. \tag{30}$$

Let $I \subset \mathbb{C}[X_1, \ldots, X_{N+n}]$ be the ideal generated by $F_1, \ldots, F_N$. Then there exist polynomials $G \in \mathbb{C}[X_1, \ldots, X_{n+1}]$ and $H \in \mathbb{C}[X_1, \ldots, X_{n+N}]$ such that $G \neq 0$, $H(0) \neq 0$ and $GH \in I$.

Remark. The important point here is that G is a polynomial involving only the variables $X_1, \ldots, X_{n+1}$; the variables $X_{n+2}, \ldots, X_{n+N}$ are uninvolved. The proof of Theorem 6.5 is a routine application of the theory of noetherian local rings, and will be given in §6.8 after we review in §6.7 the needed material from commutative algebra.

6.6. **Deduction of Theorem 6.2 from Theorem 6.5.** By symmetry it is enough to show that φ_1 is algebraic. Let $\Phi_1, \ldots, \Phi_N \in \mathbb{C}(X_1, \ldots, X_{n+N})$ be as required to exist by the definition of n-variable DIRE family with respect to $\varphi_1, \ldots, \varphi_N$. Write

$$\Phi_i = F_i/D_i \quad (F_i, D_i \in \mathbb{C}(X_1, \ldots, X_{n+N}),\ D_i(0) \neq 0).$$

Without loss of generality we may simply assume that $D_i = 1$ and hence $\Phi_i = F_i$. Then conditions (I,II) are precisely the hypotheses (29,30) of Theorem 6.5. Let G and H be as provided by Theorem 6.5. Then $H(z_1, \ldots, z_n, \varphi_1(z), \ldots, \varphi_N(z))$ is nonvanishing for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ sufficiently near the origin, hence $G(z_1, \ldots, z_n, \varphi_1(z))$ vanishes for $z \in \mathbb{C}^n$ sufficiently near the origin, and hence φ_1 is indeed algebraic. □

6.7. **Review of dimension theory of noetherian local rings.** The material reviewed here is developed in detail in the texts [AM] and [Mat].

6.7.1. *The setting.* In our review *rings* are always commutative with unit. A ring R is *noetherian* if every ideal is finitely generated. All rings to be considered below are assumed to be noetherian. A *local ring* is a ring possessing a unique maximal ideal. For the rest of the discussion we fix a noetherian local ring R with maximal ideal M and denote the residue field R/M by k. We urge the reader to keep the following key example of triples (R, M, k) in mind:

- $R = \{F \in \mathbb{C}(X_1, \ldots, X_d) \mid F \text{ is defined at the origin}\}$,
- $M = \{F \in R \mid F(0) = 0\}$, and
- $k = \mathbb{C}$.

6.7.2. *Dimension of a noetherian local ring.* For each integer n the quotient M^n/M^{n+1} is a finite-dimensional vector space over k. (Here M^n stands for the ideal generated by all n-fold products of elements of M, and by convention $M^0 = R$.) Consider the nonnegative-integer-valued function

$$\chi(n) = \sum_{i=0}^{n-1} \dim_k M^i/M^{i+1}$$

of nonnegative integers n. There exists a unique polynomial $F(t)$ in a variable t with rational coefficients such that $\chi(n) = F(n)$ for all $n \gg 0$. (See [Mat, 12.C] or [AM, Cor. 11.5].) The polynomial $F(t)$ is called the *Hilbert-Samuel* polynomial of R. The degree in t of $F(t)$ is by definition the *dimension* of R, and denoted $d(R)$. In the example discussed in §6.7.1, $d(R) = d$.

6.7.3. *Regular local rings.* We say that R is *regular* if $d(R) = \dim_k M/M^2$, in which case necessarily $\chi(n) = \left(\begin{array}{c} n + d(R) \\ d(R) \end{array}\right)$ for all n, and R is an integral domain. (See [Mat, (17.E) Thm. 35 and (17.F) Thm. 36] or [AM, Thm. 11.22 and Lemma 11.23]. In the case of the example in §6.7.1, this can be verified directly, noting that $\dim M/M^2 = d$ in that case). Suppose for the rest of this paragraph that R is regular of dimension d. Elements $f_1, \ldots, f_d \in M$ forming a basis over k for the quotient M/M^2 are said to form a *regular system of parameters* for R. By a standard argument employing Nakayama's lemma (for the latter see [Mat, (1.M) Lemma] or [AM, Prop. 2.6]) any regular system of parameters for R necessarily generates the maximal ideal M. In the key example of §6.7.1, the variables $X_1, \ldots, X_d$ form a regular system of parameters.

6.7.4. *Cutting down regular local rings.* Again suppose that R is regular of dimension d. Given a regular system of parameters $f_1, \ldots, f_d$ in R, and also given $i = 0, \ldots, d$, the ideal $(f_1, \ldots, f_i) \subset R$ generated by $f_1, \ldots, f_i$ is prime and the quotient $R/(f_1, \ldots, f_i)$ is a regular local ring in which the images of $f_{i+1}, \ldots, f_d$ form a regular system of parameters. (See [Mat, (17.F) Thm. 36].) One should think of this fact as an algebraic version of the implicit function theorem.

6.7.5. *Algebraic independence of regular systems of parameters.* Again assume that R is regular of dimension d, and further assume that R contains a field k_0. Then every regular system of parameters $f_1, \ldots, f_d$ in R is algebraically independent over k_0, i. e., for every polynomial $F(X_1, \ldots, X_d)$ in independent variables $X_1, \ldots, X_d$ with coefficients in k_0, if $F(f_1, \ldots, f_d) = 0$, then $F = 0$. (See [AM, Cor. 11.21] or [Mat, (20.D) App. 1].) In the example of §6.7.1, we may take $k_0 = \mathbb{C}$.

6.7.6. *Relation of dimension to transcendence degree.* Assume now that (R, M, k) is of the form of the example from §6.7.1. Let P be any prime ideal of R. The quotient R/P is again a noetherian local ring (but maybe not regular). (The ring R/P admits interpretation as the local ring at a point, perhaps singular, of an algebraic variety in $\mathbb{C}^d$.) Let e be the *transcendence degree* of R/P over $\mathbb{C}$, i. e., the supremum of the set of integers $m \geq 0$ such that there exist m elements of R/P algebraically independent over $\mathbb{C}$. Then we have $e = d(R/P)$. (See [AM, Thm. 11.25].) One has this equality whether or not R/P is regular.

6.8. **Proof of Theorem 6.5.** We are ready to move rapidly through the proof. We will flag the relevant paragraphs above at each step. Consider the ring $R \subset \mathbb{C}(X_1, \ldots, X_{n+N})$ consisting of all fractions A/B where $A, B \in \mathbb{C}[X_1, \ldots, X_{n+N}]$ and $B(0) \neq 0$. Then R, see §6.7.3, is a regular local ring of dimension $n + N$. Hypotheses (29,30) imply that $X_1, \ldots, X_n, F_1, \ldots F_N$ form a regular system of parameters for R. Let P be the prime ideal of R generated by $F_1, \ldots, F_N$ and let $x_1, \ldots, x_{n+N}$ be the images in the quotient R/P of $X_1, \ldots, X_{n+N}$, respectively. By §6.7.4, the quotient R/P is a regular local ring of dimension n for which $x_1, \ldots, x_n$ forms a regular system of parameters. Necessarily, by §6.7.5, $x_1, \ldots, x_n$

are algebraically independent over $\mathbb{C}$ and furthermore, see §6.7.6, no set of elements of R/P algebraically independent over $\mathbb{C}$ can have cardinality exceeding n (we emphasize that equality to 0 is taken here in R/P, not R). So there exists $0 \neq G = G(X_1, \ldots, X_{n+1}) \in \mathbb{C}[X_1, \ldots, X_{n+1}]$ such that $G(x_1, \ldots, x_{n+1}) = 0$ and hence (equivalently) $G \in \mathbb{C}[X_1, \ldots, X_{n+1}] \cap P$. Now every element of P can be written A/B where $A \in I$ and $B \in \mathbb{C}[X_1, \ldots, X_{n+N}]$ is such that $B(0) \neq 0$. In particular we may write $G = A/B$ in such fashion. Taking $H = B$, we have $GH = A \in I$, as desired. $\square$

7. Proof of Theorem 2.6

Let ℓ be a large positive integer. Let $\{Z_m\}_{m=1}^M$ be an enumeration of all complex-valued functions f on color space of the form

$$f(c) = \mathbf{1}_I(x)\xi^j/\sqrt{\text{length of } I}; \quad (c = (x, \xi) \in C = [0,1] \times S^1, \quad I \in \mathcal{I}, \quad j = -\ell, \ldots, \ell).$$

Note that $\{Z_m\}_{m=1}^M$ is an orthonormal system in $L^2(C)$. We suppose ℓ is chosen large enough so that we have an expansion

$$s(c, c') = \sum_{i=1}^M \sum_{j=1}^M \rho_{ij} Z_i(c) \overline{Z}_j(c')$$

for some complex constants ρ_{ij}. Also write

$$1 = \sum_{m=1}^M \rho_m Z_m \quad \left(\rho_m = \int \overline{Z}_m(c) P(dc)\right).$$

With $\Psi(c, \lambda)$ as defined in the color equations (15), put

$$w_m(z) = \int \Psi(c, 1/z) \overline{Z}_m(c) P(dc) \quad (m = 1, \ldots, M), \qquad w_{M+1}(z) = S(1/z).$$

The functions $w_m(z)$ are defined and holomorphic for $|z|$ small and positive, and moreover are $O(|z|)$, and hence extend to holomorphic functions in a small disk about the origin which vanish at the origin. Consider the functions

$$\begin{aligned} F_m(z) &= \sum_{j=1}^M \rho_{mj} \int \frac{\overline{Z}_j(c) P(dc)}{1 - \sum_{m=1}^M z_m Z_m(c)} \quad (m = 1, \ldots, M), \\ F_{M+1}(z) &= \sum_{j=1}^M \rho_j \int \frac{\overline{Z}_j(c) P(dc)}{1 - \sum_{m=1}^M z_m Z_m(c)} \end{aligned}$$

defined and holomorphic for $z = (z_1, \ldots, z_M) \in \mathbb{C}^M$ sufficiently near the origin. The family $F_1, \ldots, F_{M+1}$ may be extended to a M-variable DIRE family by Proposition 6.3 along with Lemmas 6.4.1, 6.4.2, and 6.4.3. Now the general color equations can be rewritten in the form

$$w_m(z) = zF_m(zw_1(z), \ldots, zw_M(z)) \quad \text{for } m = 1, \ldots, K+1 \text{ and } |z| \text{ small.} \tag{31}$$

By Lemma 6.4.4 the family $w_1, \ldots, w_M$ may be extended to a 1-variable DIRE family, and hence each function w_m is algebraic by Theorem 6.2. Finally, $S(\lambda) = w_{M+1}(1/\lambda)$ is algebraic. $\square$

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