Central Limit Theorem and Large Deviations of the Fading Wyner Cellular Model via Product of Random Matrices Theory

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Abstract

We apply the theory of products of random matrices to the analysis of multi-users communication channels similar to the Wyner model, that are characterized by short-range intra-cell broadcasting. We study the fluctuations of the per-cell sum-rate capacity in the non-ergodic regime and provide results of the type of central limit theorem (CLT) and large deviations (LD). Our results show that the CLT fluctuations of the per-cell sum-rate C_m are of order $1/\sqrt{m}$, where m is the number of cells, whereas they are of order $1/m$ in classical random matrix theory. We also show a LD regime of the form $\mathbb{P}(|C_m - C| > \varepsilon) \leq e^{-m\alpha}$ with $\alpha = \alpha(\varepsilon) > 0$ and $C = \lim_{m\to\infty} C_m$, as opposed to the rate $e^{-m^2\alpha}$ in classical random matrix theory.

I. Introduction

The Wyner model was introduced in [1]; one its extensions, the fading Wyner model, was extensively studied and the existing literature (see [2]-[5] and references therein) focuses on the ergodic regime, which is a good approximation of the inter-symbol interference model but can fail to represent the cellular model. Indeed, within acceptable communication delay, often, the channel does not exhibit the adequate variability to be faithfully approximated by an ergodic assumption, namely the delay in the communications can not accommodate many independent realizations of the fading coefficients. See [6] and [7] for the relevant background on the ergodic and non-ergodic regimes.

In this contribution, we focus on the non-ergodic regime where the channel coefficients are assumed to be fixed during the transmission of a message. We consider the uplink of a generalized “Wyner-like” cellular setup. According to Wyner’s setup, the cells are arranged on a circle (or a line), and the mobile users “see” only a fixed number of Base Stations (BSts), which are located close to their cell’s boundaries. All the BSts are assumed to be connected through an ideal back-haul network to a central multi-cell processor (MCP), that can *jointly* process the up-link received signals of all cell-sites, as well as pre-process the signals to be transmitted by all cell-sites in the down-link channel. Under the assumption that the channel varies quickly enough in an ergodic fashion, the per-cell sum-rate capacity was addressed in [8].

Using the tools of the later article and results concerning the product of random matrices, we consider the non-ergodic channel, that is, the fading coefficients are chosen randomly at the beginning of all time and are known only at the receiver. As noted in [9], since the transmitter does not know the Channel State Information (CSI), the Shannon capacity of the channel is not the relevant quantity as it can be 0, in case of Rayleigh fading, for example, because whatever rate R is chosen for broadcasting, there is a non-zero probability that the realized channel is incapable of supporting it, even with arbitrarily long codeword length. The relevant quantity is the outage probability, which is related to the study of fluctuations of the sum-rate. We present results of type Central Limit Theorem (CLT) and Large Deviations (LD) for the per-cell sum-rate.

Note that standard random matrices techniques ([10]) are not applicable because they concern random matrices built from a number of independent random variables of the order of the number of entries. As noted in [8], the limiting per-cell sum-rate depends on the underlying fading distributions, unlike associated results in classical random matrix theory. Our results show that the CLT fluctuations of the per-cell sum-rate C_m are of order $1/\sqrt{m}$, where m is the number of cells, whereas they are of order $1/m$ in classical random matrix theory (see [11] and references therein). We also show a LD regime of the form $\mathbb{P}(|C_m - C| > \varepsilon) \le e^{-m\alpha}$ with $\alpha = \alpha(\varepsilon) > 0$ and $C = \lim_{m\to\infty} C_m$ as opposed to the rate $e^{-m^2\alpha}$ in classical random matrix theory. (See Section B of the Appendix for review of the relevant result in classical random matrix theory.)

The rest of the paper is organized as follows. In Section II, we state the problem and the main results. In Section III, we prove the main results. Part of the proof requires heavy computation, we therefore provide a partially computer-based proof. Concluding remarks are given in Section IV. Refer to Sections C and D of the Appendix for the relevant background on Lyapunov exponents theory and exterior products respectively.

II. Problem statement and main results

We first describe the communication setup and provide the necessary definitions and notation. The main results are stated in Sub-section II-C.

A. Communication setup

In this paper we consider the following setup. $m+d$ cells with K single antenna users per cell are arranged on a line, where the m single antenna BSts are located in the cells. Starting with the wideband (WB) transmission scheme where all bandwidth is devoted for coding and all K users are transmitting simultaneously each with average power ρ, and assuming synchronized communication, a vector baseband representation of the signals received at the system's BSts is given for an arbitrary time index i by

$$y(i) = H_m x(i) + z(i),$$

where $x(i)$ is the $(m+d)K$ complex Gaussian symbols vector, $z(i)$ is the unitary complex Gaussian additive noise vector. Note that the SNR is ρ. From now on, we omit the time index i. H_m is the following $m \times K(m+d)$ channel transfer matrix, which is a $d+1$ block diagonal matrix defined by

$$H_m = \begin{pmatrix} \zeta_{1,1} & \zeta_{1,2} & \cdots & \zeta_{1,d+1} & 0 & \cdots & 0 \\ 0 & \zeta_{2,2} & \cdots & \zeta_{2,d+1} & \zeta_{2,d+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_{m,m} & \zeta_{m,m+1} & \cdots & \zeta_{m,d+m} \end{pmatrix},$$

where $\zeta_{i,j}$ are $1 \times K$ row vectors. Recall that H_m is chosen randomly at the beginning of all time and kept fixed thereafter. For $s \geq d+1$, we will denote by ζ^s the vector $(\zeta_{s-d,s}, \dots, \zeta_{s,s})$ and we denote by π it distribution. We assume in the rest of the paper that for $n \geq d+1$ and $0 \leq i \leq d$ the vectors $(\zeta_{n-i,n})$ are distributed according to π_i. We define $\Omega = (\zeta^n)_{n \geq d+1}$ and $\mathbb{P}$, the probability distribution on Ω associated to the above problem. We denote by $\mathbb{E}$ the associated expectation. We also use the 2 norm for vectors and matrices. For matrices, it is the Froebenius norm (i.e for a matrix $A = (a_{i,j})_{1 \leq i,j \leq m}$ $\|A\| = \sqrt{\sum_{i,j=1}^m |a_{i,j}|^2}$), which is a sub-multiplicative norm (i.e, for two matrices A and B, $\|AB\| \leq \|A\| \, \|B\|$.)

Throughout this paper, we assume a subset of the following hypotheses.

(H1) The vectors $(\zeta^j)_{j \geq d+1}$ form an i.i.d sequence.

(H2) There exists $\varepsilon > 0$ such that for $0 \leq i \leq d$, $\mathbb{E}_{\pi_i} |x|^{\varepsilon} < \infty$ and $\mathbb{E}_{\pi_i} |x|^{-\varepsilon} < \infty$.

(H3) If $(x_0, \dots, x_d)$ is distributed according to π, then almost surely, $x_0 x_d^\dagger \neq 0$.

(H4) The support of π is $\mathbb{C}^{K(d+1)}$.

(H5) The support of π is $\mathbb{R}^{K(d+1)}$.

B. *Definitions and notations*

Under (H4), we define $\mathbb{F} = \mathbb{C}$ and under (H5), we define $\mathbb{F} = \mathbb{R}$.

For $m \geq 1$ and $\lambda > 0$, we set $G_m = H_m H_m^\dagger + \lambda \,\mathrm{Id}_m$, where Id_m is the $m \times m$ identity matrix. Although G_m depends on λ, we will not write that dependence unless there is an ambiguity. Under our assumptions, the system is a multiple access channel. Since we are using Gaussian code-words, the per-cell sum-rate capacity is given by

$$C_m(\rho) = \frac{1}{m} \log\det\left(\mathrm{Id}_m + \rho H_m H_m^\dagger\right) = \log\rho + \frac{1}{m}\left(\log\det G_m(\lambda)\right), \tag{II.1}$$

where $\lambda = 1/\rho$. See [8] and references therein for the relevant background.

We set for $i \geq 1$

$$C_i = \begin{pmatrix} \zeta_{d(i-1)+1,d(i-1)+1} & \zeta_{d(i-1)+1,d(i-1)+2} & \cdots & \zeta_{d(i-1)+1,di} \\ 0 & \zeta_{d(i-1)+2,d(i-1)+2} & \cdots & \zeta_{d(i-1)+2,di} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \zeta_{di,di} \end{pmatrix}$$

and

$$D_i = \begin{pmatrix} {\zeta_{d(i-2)+1,d(i-1)+1}}^{\dagger} & {\zeta_{d(i-2)+2,d(i-1)+1}}^{\dagger} & \cdots & {\zeta_{d(i-1),d(i-1)+1}}^{\dagger} \\ 0 & {\zeta_{d(i-2)+2,d(i-1)+2}}^{\dagger} & \cdots & {\zeta_{d(i-1),d(i-1)+2}}^{\dagger} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & {\zeta_{d(i-1),di}}^{\dagger} \end{pmatrix}.$$

For all $i \geq 1$, C_i are $d \times dK$ matrices and D_i are $dK \times d$ matrices. We fix $\zeta_{i,j}$ with $i \leq 0$ or $j \leq 0$ so that $C_1 D_1 = \mathrm{Id}_d$.

For $i \geq 1$, we denote by Δ_i the following matrix

$$\left(\begin{array}{c|c} -\left(C_i C_i^{\dagger} + \lambda \,\mathrm{Id}_d\right)(C_i D_i)^{-1\dagger} & -C_i D_i + \left(C_i C_i^{\dagger} + \lambda \,\mathrm{Id}_d\right)(C_i D_i)^{-1\dagger} D_i^{\dagger} D_i \\ \hline (C_i D_i)^{-1\dagger} & -(C_i D_i)^{-1\dagger} D_i^{\dagger} D_i \end{array}\right)$$

and define $\Xi_i = \bigwedge^d \Delta_i$. Note that Ξ_i has size $\binom{2d}{d}$. See Section C of the Appendix for the relevant background on exterior products.

For a given integer $k \geq 1$, we denote by $Gl(k, \mathbb{F})$ the group of invertible matrices on $\mathbb{F}$ of size k. A semi-group of $Gl(k, \mathbb{F})$ is a subset of $Gl(k, \mathbb{F})$ which is stable under multiplication. Let T be the smallest closed (in the topological sense) semi-group in $Gl(\binom{2d}{d}, \mathbb{F})$ which contains the support of the law of the matrices Ξ_i. We denote by $e_1, \ldots, e_{2d}$ the canonical basis of $\mathbb{F}^{2d}$ and we denote $f \triangleq e_1 \wedge \cdots \wedge e_d$ and $g \triangleq e_{d+1} \wedge \cdots \wedge e_{2d}$.

We denote by $\mathcal{H}_0$ the minimal subspace of $\mathbb{F}^{\binom{2d}{d}}$ that contains f and that is stable under the action of T. We denote by d_0 its dimension. Moreover, for $i \geq 1$, since $\mathcal{H}_0$ is stable under Ξ_i, we can define the restriction of Ξ_i to $\mathcal{H}_0$: $\widetilde{\Xi}_i$. Finally, let $\widetilde{T}$ be the smallest closed semi-group in $Gl(d_0, \mathbb{F})$ which contains the support of the law of the matrices $\widetilde{\Xi}_i$.

Definition II.2. *Given a subset S of $Gl(k, \mathbb{F})$, we say that S is strongly irreducible if there does not exist a finite family of proper linear subspaces of $\mathbb{F}^k$, $V_1, V_2, \ldots, V_k$ such that for any M in S,*

$$M\left(V_1 \cup \cdots \cup V_k\right) = V_1 \cup \cdots \cup V_k.$$

Definition II.3. *Given a subset S of $Gl(k, \mathbb{F})$, we say that S is contracting if there exists a sequence $(M_n)_{n \in \mathbb{N}}$ in S such that $\|M_n\|^{-1} M_n$ converges to a matrix of rank one.*

Remark II.4. Assume [(H4) or (H5)]. We consider $1 \leq K_1 \leq K_2$. Denote by $\mathcal{H}_0(K_1)$ and $\widetilde{T}(K_1)$ (resp. $\mathcal{H}_0(K_2)$ and $\widetilde{T}(K_2)$) the sets $\mathcal{H}_0$ and $\widetilde{T}$ for $K = K_1$ (resp. $K = K_2$). Then

$\mathcal{H}_0(K_1) \subset \mathcal{H}_0(K_2)$ and $\widetilde{T}(K_1) \subset \widetilde{T}(K_2)$. In particular, if $\mathcal{H}_0(K_1) = \mathcal{H}_0(K_2)$ and $\widetilde{T}(K_1)$ is strongly irreducible, then $\widetilde{T}(K_2)$ is also strongly irreducible.

C. *Main results*

We first state in Theorem II.5 the results for all d under the condition that $\widetilde{T}$ is strongly irreducible. This condition is verified for $d = 1, 2$ as stated in Corollary II.6. See Section C of the Appendix for the definition of the Lyapunov exponent $\gamma\left(\widetilde{\Xi}\right)$.

Theorem II.5. *Assume (H1), (H2), (H3) and [(H4) or (H5)], and set $\lambda = 1/\rho$. We assume moreover that $\widetilde{T}$ is strongly irreducible.*

1. *Almost surely*
$$C_m(\rho) \xrightarrow[m\to\infty]{} \log\rho + \mathbb{E}_\pi \log\left|\zeta_0\zeta_d^\dagger\right| + \frac{1}{d}\gamma\left(\widetilde{\Xi}\right) \triangleq C(\rho),$$
where the expectation is taken such that $(\zeta_0, \ldots, \zeta_d)$ is distributed according to π.
2. $\sqrt{m}\left(C_m(\rho) - C_\rho\right)$ *converges in law to a centered normal random variable of variance $\sigma^2(\rho) > 0$.*
3. *For all $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon) > 0$ such that*
$$\limsup_{m\to\infty} \frac{1}{m} \log\mathbb{P}\left(|C_m(\rho) - C(\rho)| > \varepsilon\right) < -\alpha.$$

With Propositions III.14 and III.18, we get the following corollary.

Corollary II.6. *Assume (H1), (H2), (H3) and [(H4) or (H5)]. For $d = 1, 2$, for all $\rho \in (0, \infty)$, for all $K \geq 1$, $\widetilde{T}$ is strongly irreducible, therefore the conclusions of Theorem II.5 hold.*

Point 3 of Theorem II.5 shows an upper bound in the LD regime of the form $\mathbb{P}(|C_m - C| > \varepsilon) \leq e^{-m\alpha}$ with $\alpha = \alpha(\varepsilon) > 0$. This rate is indeed the correct rate as shown by the following proposition.

Proposition II.7. *Assume (H1) and [(H4) or (H5)]. There exists $\alpha' > 0$ such that*
$$\liminf_{n\to\infty} \frac{1}{m} \log\mathbb{P}\left(|C_m(\rho) - C(\rho)| \geq \frac{C(\rho)}{2}\right) \geq -\alpha'. \tag{II.8}$$

The proof is postponed to Section A of the Appendix

III. Proof of the main results

A. Proof of Theorem II.5

In order to prove Theorem II.5, we combine Lemma III.2 from [8] with results from [12]. In order to apply those result, we need the contractivity of $\widetilde{T}$ (see Definition II.3), which is given by the following lemma.

Lemma III.1. *$\widetilde{T}$ is contracting.*

Proof: Taking $\zeta_{i,i} = (1, 0, \ldots, 0)$, $\zeta_{i,i+d} = (\varepsilon, 0, \ldots, 0)$, with $\varepsilon > 0$ and $\zeta_{i,i+s} = (0, \ldots, 0)$ for $1 \leq s \leq d-1$ (that is $C^\varepsilon = \mathrm{Id}_d$ and $D^\varepsilon = \varepsilon \, \mathrm{Id}_d$), we get that the following matrix belongs to T.

$$\Xi^\varepsilon \triangleq \bigwedge^d \left(\begin{array}{c|c} -(1+\lambda)\varepsilon^{-1} \, \mathrm{Id}_d & \lambda\varepsilon \, \mathrm{Id}_d \\ \hline \varepsilon^{-1} \, \mathrm{Id}_d & -\varepsilon \, \mathrm{Id}_d \end{array} \right).$$

When ε goes to 0, $\|\Xi^\varepsilon f\|$ grows like ε^{-d} whereas for f' another vector of the canonical basis of $\mathbb{F}^{\binom{2d}{d}}$, $\|\Xi^\varepsilon f'\|$ grows like $\varepsilon^{d'}$ with $d' > -d$. Therefore,

$$\frac{\|\Xi^\varepsilon f'\|}{\|\Xi^\varepsilon f\|} \xrightarrow[\varepsilon \to 0]{} 0.$$

Denote by $\widetilde{f}'$ the orthogonal projection of f' on $\mathcal{H}_0$. Since $\widetilde{\Xi}^\varepsilon f = \Xi^\varepsilon f$ and $\left\| \widetilde{\Xi}^\varepsilon \widetilde{f}' \right\| \leq \|\Xi^\varepsilon f'\|$,

$$\frac{\left\| \widetilde{\Xi}^\varepsilon \widetilde{f}' \right\|}{\left\| \widetilde{\Xi}^\varepsilon f \right\|} \xrightarrow[\varepsilon \to 0]{} 0.$$

Since the orthogonal projection of the canonical basis of $\mathbb{F}^{\binom{2d}{d}}$ on $\mathcal{H}_0$ is a generating system of $\mathcal{H}_0$, $\widetilde{\Xi}^\varepsilon / \left\| \widetilde{\Xi}^\varepsilon \right\|$ converges to a matrix of rank 1.

∎

The following lemma is proved in [8, (A.9) and proof of Proposition IV.2].

Lemma III.2. *Define for $i \geq 1$,*

$$P_1(i) = \begin{pmatrix} -C_i D_i & -C_i C_i^\dagger - \lambda \, \mathrm{Id}_d \\ 0_d & \mathrm{Id}_d \end{pmatrix}$$

and

$$P_2(i) = \begin{pmatrix} 0_d & \mathrm{Id}_d \\ (C_i D_i)^{-1\dagger} & -\left(C_i D_i\right)^{-1\dagger} D_i^\dagger D_i \end{pmatrix},$$

then,

$$\frac{1}{nd}\log\det(G_{nd}) = \frac{1}{nd}\sum_{i=1}^{nd}\log\left|\zeta_{i,i+d}\zeta^{\dagger}_{i+d,i+d}\right|$$
$$+\frac{1}{nd}\log\left|g^{\dagger}\bigwedge^{d}P_2(n+1)\Xi_n\cdots\Xi_2\bigwedge^{d}P_1(1)g\right|.$$

Lemma III.2 is based on the Thouless formula, which relates the determinant of a large random bande matrix to the product of fixed-size random matrices (See [8] and references therein).

We continue with the proof of Theorem II.5. Note that for $i \geq 1$, $\Delta_i = P_1(i)P_2(i)$, therefore,

$$g^{\dagger}\bigwedge^{d}P_2(n+1)\Xi_n\cdots\Xi_2\bigwedge^{d}P_1(1)g = g^{\dagger}\bigwedge^{d}P_1^{-1}(n+1)\Xi_{n+1}\cdots\Xi_1\bigwedge^{d}P_2^{-1}(1)g.$$

However,

$$P_1^{-1}(i) = \begin{pmatrix} -(C_iD_i)^{-1} & -(C_iD_i)^{-1}(C_iC_i^{\dagger}+\lambda\,\mathrm{Id}_d) \\ 0_d & \mathrm{Id}_d \end{pmatrix}$$

and

$$P_2^{-1}(i) = \begin{pmatrix} D_i^{\dagger}D_i & (C_iD_i)^{\dagger} \\ \mathrm{Id}_d & 0_d \end{pmatrix}.$$

therefore,

$$g^{\dagger}\bigwedge^{d}P_1^{-1}(n+1) = g^{\dagger} \text{ and } \bigwedge^{d}P_2^{-1}(1)g = \det\left(C_{n+1}D_{n+1}\right)^{\dagger}f.$$

Using that

$$\log\left|\det C_{n+1}D_{n+1}\right| = \sum_{i=nd+1}^{(n+1)d}\log\left|\zeta_{i,i+d}\zeta^{\dagger}_{i+d,i+d}\right|,$$

we get

$$\frac{1}{nd}\log\det(G_{nd}) = \frac{1}{nd}\sum_{i=1}^{(n+1)d}\log\left|\zeta_{i,i+d}\zeta^{\dagger}_{i+d,i+d}\right| + \frac{1}{nd}\log\left|g^{\dagger}\Xi_{n+1}\cdots\Xi_1 f\right|.$$

Denoting by $\widetilde{g}$ the orthogonal projection of g on $\mathcal{H}_0$, we get

$$\frac{1}{nd}\log\det(G_{nd}) = \frac{1}{nd}\sum_{i=1}^{(n+1)d}\log\left|\zeta_{i,i+d}\zeta^{\dagger}_{i+d,i+d}\right| + \frac{1}{nd}\log\left|\widetilde{g}^{\dagger}\widetilde{\Xi}_{n+1}\cdots\widetilde{\Xi}_1 f\right|. \tag{III.3}$$

By the Law of Large Number (LLN), almost surely,

$$\frac{1}{nd}\sum_{i=1}^{(n+1)d}\log\left|\zeta_{i,i+d}\zeta_{i+d,i+d}^{\dagger}\right| \xrightarrow[n\to\infty]{} \mathbb{E}_{\pi}\log\left|\zeta_0\zeta_d^{\dagger}\right|. \tag{III.4}$$

By [12, Corollary A.VI.2.3(i)], almost surely

$$\frac{1}{nd}\log\left|\widetilde{g}^{\dagger}\widetilde{\Xi}_{n+1}\cdots\widetilde{\Xi}_1 f\right| \xrightarrow[n\to\infty]{} \frac{1}{d}\gamma\left(\widetilde{\Xi}\right),$$

which together with (III.3) and (III.4) proves point 1.

Remark III.5. Note that [12] consider real matrices but as stated in Remark A.V.8.3, the results apply verbatim to the complex case.

Continuing with the proof of Theorem II.5, denote for $i \geq 1$,

$$\Xi_i' = \left(\prod_{s=(i-1)d+1}^{id}\zeta_{i,i+d}\zeta_{i+d,i+d}^{\dagger}\right)\widetilde{\Xi}_i,$$

and let T' be the smallest closed semi-group in $Gl(d_0,\mathbb{F})$ which contains the support of the law of the matrices Ξ_i'. Then

$$\frac{1}{nd}\log\det(G_{nd}) = \frac{1}{nd}\log\left|\widetilde{g}^{\dagger}\Xi_{n+1}'\cdots\Xi_1' f\right|.$$

Note that $\widetilde{T}$ is strongly irreducible and contracting. Therefore, by (H3), T' is also strongly irreducible and contracting, therefore, by [12, Corollary A.VI.2.3(i)], $1/(\sqrt{n}d)\log\det(G_{nd})$ converges in law to a centered Gaussian random variable with non-zero variance. That finishes the proof of point 2.

Fix $\varepsilon > 0$. By [12, Theorem A.VI.6.2], there exist $\alpha > 0$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\left|\frac{1}{nd}\log\left\|\Xi_{n+1}'\cdots\Xi_1' f\right\| - \frac{1}{d}\gamma\left(\Xi'\right)\right| > \varepsilon/2\right) < -\alpha.$$

Moreover, as show in [12] in the course of the proof of Proposition A.VI.2.2, there exists $\eta > 0$ and $\beta > 0$ such that for n large enough, almost surely,

$$\frac{\left\|\Xi_{n+1}'\cdots\Xi_1' f\right\|}{\left|\widetilde{g}^{\dagger}\Xi_{n+1}'\cdots\Xi_1' f\right|} \leq \eta n^{\beta},$$

that is

$$0 \leq \frac{1}{nd}\log\frac{\left\|\Xi_{n+1}'\cdots\Xi_1' f\right\|}{\left|\widetilde{g}^{\dagger}\Xi_{n+1}'\cdots\Xi_1' f\right|} \leq \frac{\log\eta + \beta\log n}{nd}.$$

Therefore, for n large enough,

$$\left|\frac{1}{nd}\log\left\|\Xi'_{n+1}\cdots\Xi'_1 f\right\| - \frac{1}{nd}\log\left|\widetilde{g}^\dagger\Xi'_{n+1}\cdots\Xi'_1 f\right|\right| \leq \varepsilon/2.$$

This finishes the proof of point 3.

■

B. Strong irreducibility and proof of Corollary II.6 for $d = 1$

We deal with the case $d = 1$ separately because effective computation is possible. We will deal with the general case in the next sub-section.

Proposition III.6. *Set $d = 1$. Assume (H1), (H2), (H3) and [(H4) or (H5)]. Then $\mathcal{H}_0 = \mathbb{F}^2$ and $\widetilde{T}$ is strongly irreducible.*

Proof: Using Remark II.4, it is enough to do the proof in the case $K = 1$.

For $i \geq 1$, we get:

$$\Xi_i = \frac{1}{\zeta_{i-1,i}\zeta_{i,i}^\dagger}\begin{pmatrix} -|\zeta_{i,i}|^2 - \lambda & -\left|\zeta_{i,i}\zeta_{i-1,i}^\dagger\right|^2 + \left(|\zeta_{i,i}|^2 + \lambda\right)|\zeta_{i-1,i}|^2 \\ 1 & -|\zeta_{i-1,i}|^2 \end{pmatrix}.$$

For $u, v \in \mathbb{F}$, not both equal to 0,

$$\Xi_i\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\zeta_{i-1,i}\zeta_{i,i}^\dagger}\begin{pmatrix} -u\left(|\zeta_{i,i}|^2 + \lambda\right) + v\lambda\,|\zeta_{i-1,i}|^2 \\ u - v\,|\zeta_{i-1,i}|^2 \end{pmatrix}.$$

For any given $\zeta_{i-1,i}$ the vector space generated by $\left(\Xi_i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)_{\zeta_{i,i}\in\mathbb{F}}$ is $\mathbb{F}^2$, therefore, $\mathcal{H}_0 = \mathbb{F}^2$.

Let us show that $\widetilde{T}$ is strongly irreducible by contradiction. Assume that there exist $V_1, \ldots, V_k$ linear subspaces of $\mathbb{F}^2$ of dimension 1 such that for all M in $\widetilde{T}$,

$$M(V_1 \cup \cdots \cup V_k) = V_1 \cup \cdots \cup V_k.$$

Take $\begin{pmatrix} u \\ v \end{pmatrix} \in V_1$ such that $(u, v) \neq (0, 0)$. If $u \neq 0$, then for any given $\zeta_{i-1,i}$ the vector space generated by $\left(\Xi_i\begin{pmatrix} u \\ v \end{pmatrix}\right)_{\zeta_{i,i}\in\mathbb{F}}$ is $\mathbb{F}^2$, which gives a contradiction. If $u = 0$, then $\Xi_i\begin{pmatrix} u \\ v \end{pmatrix}$ and

$\begin{pmatrix} \lambda \\ -1 \end{pmatrix}$ are collinear, therefore,

$$\begin{pmatrix} \lambda \\ -1 \end{pmatrix} \in V_1 \cup \cdots \cup V_k.$$

For any given $\zeta_{i-1,i}$, the vector space generated by $\left(\Xi_i \begin{pmatrix} \lambda \\ -1 \end{pmatrix}\right)_{\zeta_{i,i} \in \mathbb{F}}$ is $\mathbb{F}^2$, which gives a contradiction. ■

C. Proof of Corollary II.6 by a computer based proof for general d

We first give an algorithm that allows us to prove irreducibility if we already know $\mathcal{H}_0$. Then, we give an algorithm that allows us to find $\mathcal{H}_0$.

1) Proving strong irreducibility: For $d > 1$, it is difficult to prove strong irreducibility by direct study of Ξ_i as we have done for $d = 1$. Indeed, the size of Ξ_i, $\binom{2d}{d}$ grows very quickly. Therefore, we provide here a computerized proof of strong irreducibility.

Consider $\widehat{\mathcal{H}}$, a subspace of $\mathbb{F}^{\binom{2d}{d}}$ that contains f and is stable under the action of T. We denote by $\widehat{d}$ its dimension. Moreover, we can define the restriction of Ξ_i to $\widehat{\mathcal{H}}$: $\widehat{\Xi}_i$. Finally, let $\widehat{T}$ be the smallest closed semi-group in $Gl\left(\widehat{d}, \mathbb{F}\right)$ which contains the support of the law of the matrices $\widehat{\Xi}_i$. Denote by $\mathcal{X}$ the support of the law of the $\widehat{\Xi}_i$. If the following algorithm succeeds then, for all $\lambda \in (0, \infty)$, $\widehat{T}$ is strongly irreducible.

The heart of the algorithm is Lemma III.10 that states that $\widehat{T}$ is strongly irreducible if we can find elements of $\mathcal{X}$ that verify a certain condition. The algorithm generates elements of $\mathcal{X}$ at random and checks whether they verify that condition. Once such elements have been found, the algorithm stops.

Take $p \geq 1$ a parameter. We denote by ϕ the function that to a matrix of size $\widehat{d} \times \widehat{d}$ associates a vector of size $\widehat{d}^2$, which is the list of its entries.

Algorithm III.7.

1. Produce at random $2p\widehat{d}$ samples from $\mathcal{X}$, $\widehat{\Xi}_{i,j,k}$ for $1 \leq i \leq p$, $1 \leq j \leq \widehat{d}^2$ and $k = 1, 2$.
2. For $k = 1, 2$, compute formally (i.e. as explicit function of λ)
$$\psi_k(\lambda) := \det\left(\phi\left(\widehat{\Xi}_{p,1,k} \cdots \widehat{\Xi}_{1,1,k}\right), \ldots, \phi\left(\widehat{\Xi}_{p,\widehat{d}^2,k} \cdots \widehat{\Xi}_{1,\widehat{d}^2,k}\right)\right).$$
3. For $k = 1, 2$, $\psi_k(\lambda)$ is a polynomial in λ. Define
$$\widetilde{\psi}_k(\lambda) = \frac{\psi_k(\lambda)}{\lambda^l},$$
where l is the largest integer such that λ^l divides $\psi_k(\lambda)$.
4. Compute the discriminant of $\widetilde{\psi}_1(\lambda)$ and $\widetilde{\psi}_2(\lambda)$
   - If the discriminant is not 0, return "SUCCES"
   - If the discriminant is 0, return "FAILURE"

Remark III.8. In step 2, the computation is formal because λ is a parameter. It is done using the Mathematica software.

Proposition III.9. *If there exists $p \geq 1$ such that Algorithm III.7 is successful, then for all $\lambda \in (0, \infty)$, $\widehat{T}$ is strongly irreducible. Moreover, $\widehat{\mathcal{H}} = \mathcal{H}_0$.*

Proof: Let us use the following lemma whose proof is postponed to the end of the section.

Lemma III.10. *For a given λ, if there exist $\widehat{\Xi}_{i,j} \in \mathcal{X}$ for $1 \leq i \leq p$ and $1 \leq j \leq \widehat{d}^2$ such that*
$$\det\left(\phi\left(\widehat{\Xi}_{p,1} \cdots \widehat{\Xi}_{1,1}\right), \ldots, \phi\left(\widehat{\Xi}_{p,\widehat{d}^2} \cdots \widehat{\Xi}_{1,\widehat{d}^2}\right)\right) \neq 0,$$
then $\widehat{T}$ is strongly irreducible for this λ.

The discriminant of two polynomials vanishes if and only if those polynomials have a common root. Assume that Algorithm III.7 is successful. By Step 4, $\widetilde{\psi}_1(\lambda)$ and $\widetilde{\psi}_2(\lambda)$ have no common root. By Step 3, $\psi_1(\lambda)$ and $\psi_2(\lambda)$ have no common root except for 0. Therefore, for a given $\lambda \in (0, \infty)$, $\psi_1(\lambda) \neq 0$ or $\psi_2(\lambda) \neq 0$, hence, by Lemma III.10, $\widehat{T}$ is strongly irreducible for this λ. Let us show that $\widehat{\mathcal{H}} = \mathcal{H}_0$. $\widehat{\mathcal{H}}$ contains f and is stable under the action of T, therefore $\mathcal{H}_0 \subset \widehat{\mathcal{H}}$. $\mathcal{H}_0$ is stable under the action of $\widehat{T}$ therefore, if it is a strict subset of $\widehat{\mathcal{H}}$, then it contradicts the fact that $\widehat{T}$ is strongly irreducible. ■

Remark III.11. Note that a posteriori, the fact that $\widehat{\mathcal{H}} = \mathcal{H}_0$ tells us that $\widehat{T} = \widetilde{T}$ and $\widehat{d} = d_0$

Proof of Lemma III.10: In this proof, λ is fixed. Let us assume that $\widehat{T}$ is not strongly irreducible and let us show that for all $\widehat{\Xi}_{i,j}$ ($1 \leq i \leq p$ and $1 \leq j \leq \widehat{d}^2$),

$$\det\left(\phi\left(\widehat{\Xi}_{p,1}\cdots\widehat{\Xi}_{1,1}\right),\ldots,\phi\left(\widehat{\Xi}_{p,\widehat{d}^2}\cdots\widehat{\Xi}_{1,\widehat{d}^2}\right)\right) = 0.$$

Consider a finite family of proper linear subspaces of $\mathbb{C}^d$, $V_1, V_2, \ldots, V_k$ such that for any M in $\widehat{T}$,

$$M\left(V_1 \cup \cdots \cup V_k\right) = V_1 \cup \cdots \cup V_k. \tag{III.12}$$

Denote by x a non zero vector in V_1 and by y_i a non zero vector in $V_i^{\perp}$ for $1 \leq i \leq k$. Define moreover the following function on $\mathcal{X}^p$:

$$R_i\left(\widehat{\Xi}_1,\ldots,\widehat{\Xi}_p\right) = y_i^{\dagger}\widehat{\Xi}_p\cdots\widehat{\Xi}_1 x.$$

By (III.12), the function $R_1\cdots R_k$ is uniformly 0. The matrices of $\mathcal{X}$ are polynomial functions of the real and imaginary parts of the fading coefficients, therefore, R_i ($1 \leq i \leq k$) are polynomial functions of the real and imaginary parts of the fading coefficients, hence, since the product of the functions R_i ($1 \leq i \leq k$) is zero, there exists $1 \leq i_0 \leq k$ such that the function R_{i_0} is uniformly 0. Thus, for all $1 \leq j \leq \widehat{d}^2$, there is a linear dependency between the entries of $\widehat{\Xi}_{p,j}\cdots\widehat{\Xi}_{1,j}$, that is, between the entries of $\phi\left(\widehat{\Xi}_{p,j}\cdots\widehat{\Xi}_{1,j}\right)$. We have proved that if $\widehat{T}$ is not strongly irreducible, then

$$\det\left(\phi\left(\widehat{\Xi}_{p,1}\cdots\widehat{\Xi}_{1,1}\right),\ldots,\phi\left(\widehat{\Xi}_{p,\widehat{d}^2}\cdots\widehat{\Xi}_{1,\widehat{d}^2}\right)\right) = 0.$$

■

Remark III.13. We want the computation to be exact, therefore, we draw samples of $\mathcal{X}$ such that the $\zeta_{i,j}$ are complex integers (of the form $a + ib$ such that a and b are real integers).

We applied Algorithm III.7 with $p = 2$, $\widehat{\mathcal{H}} = \mathbb{C}^{\binom{4}{2}}$ and for $K = 1$ and $d = 2$ and it was successful. See Section E of the Appendix for the matrices $\widehat{\Xi}_{i,j,k}$ that were randomly generated by the algorithm. With Remark II.4, we therefore get the following proposition.

Proposition III.14. *Assume (H1), (H2), (H3), (H4), and $d = 2$, then, for all $\lambda \in (0, \infty)$, for all $K \geq 1$, $\mathcal{H}_0 = \mathbb{C}^{\binom{2d}{d}}$ and $\widetilde{T}$ is irreducible.*

2) Finding $\mathcal{H}$: In the cases when $\mathcal{H}_0$ is a proper subspace of $\mathbb{F}^{\binom{2d}{d}}$, it is not obvious how to get a hold of it. We therefore give an algorithm that allows us to do so, if the following algorithm stops, it gives $\mathcal{H}_0$.

The idea is to randomly generate vectors in $\mathcal{H}_0$ by applying Ξ to f until we get enough of them in the sense that they generate an invariant subspace that contains f.

For $K \geq 1$, denote by $\underline{\Xi}^K$ the matrix valued function such that for $i \in \mathbb{N}$, $\Xi_i = \underline{\Xi}^K\left(\zeta^i\right)$. Note that $\underline{\Xi}^K$ is also a function of λ although we do not write it explicitly. Denote by $\mathcal{X}(1)$ the support of the law of the $\Xi_i(\lambda = 1)$ for $K = 1$.

Algorithm III.15.

1. Define $E = \emptyset$.
2. Produce at random $\binom{2d}{d}$ samples of Ξf for $\Xi \in \mathcal{X}(1)$, and add them to the set E.
3. Compute the span of E and denote it by $\mathcal{H}$. Denote by d_1 its dimension.
4.
   - If $f \in \mathcal{H}$, continue.
   - Else, go back to step 2.
5. Compute a base of $\mathbb{F}^{\binom{2d}{d}}$ which is a union of a base of $\mathcal{H}$ and a base of its orthogonal. Denote by $\mathcal{B}$ the base-changing matrix
6. Formally compute the matrix valued function $\mathcal{B}^{-1}\underline{\Xi}^K\mathcal{B}$.
7.
   - If the last $\binom{2d}{d} - d_1$ elements of the first d_1 columns are 0, STOP.
   - Else, go back to step 2.

Note that the formal computations are done using the Mathematica software.

Proposition III.16. *If Algorithm III.15 stops, then for all λ and K, $\mathcal{H}$ is a subspace of $\mathbb{F}^{\binom{2d}{d}}$ that contains f and that is invariant under the action of $\mathcal{M}$. Moreover, for all λ except for maybe a finite number of values, for all K, $\mathcal{H}$ is minimal, that is $\mathcal{H} = \mathcal{H}_0(\lambda)$.*

Proof: Step 4 ensure that $f \in \mathcal{H}$. Step 7 ensure that for all λ, $\mathcal{H}$ is invariant under the action of T.

For $\lambda = 1$ and $K = 1$, any subspace of $\mathbb{F}^{\binom{2d}{d}}$ that contains f and that is invariant under the action of T must contain E and therefore must contain $\mathcal{H}$, hence, $\mathcal{H}$ is minimal for $\lambda = 1$ and $K = 1$. Let us show that it is true for all λ, except for a finite number of values and for all K. Using remark II.4, we only need to prove that $\mathcal{H}$ is minimal for all λ except for maybe a finite number of values and $K = 1$.

Consider the determinant of the orthogonal projections of the vectors $\Xi(\zeta^{d+1})f_1, \ldots, \Xi(\zeta^{d+d_1})f_1$ onto $\mathcal{H}$. It is a deterministic polynomial function of λ and of the real and imaginary parts of the $\zeta_{i,j}$ for $d+1 \leq j \leq d+d_1$ and $j-d \leq i \leq j$. We denote by $Q(\zeta^{d+1}, \ldots, \zeta^{d+d_1}, \lambda)$ that polynomial function. Note that

$$Q(\cdot, \lambda) \not\equiv 0 \Rightarrow \mathcal{H} \text{ is minimal for } \lambda.$$

The fact that $\mathcal{H}$ is minimal for $\lambda = 1$ means that Q is a non zero polynomial. We write

$$Q(\zeta^{d+1}, \ldots, \zeta^{d+d_1}, \lambda) = \sum_{i=1}^{p} \widetilde{Q}_i(\lambda) R_i(\zeta^{d+1}, \ldots, \zeta^{d+d_1}),$$

where the R_i are non-zero monomials in the real and imaginary parts of the $\zeta_{i,j}$ and the $\widetilde{Q}_i$ are polynomials in λ. $Q(\cdot, \lambda) \equiv 0$ if and only if for all i, $\widetilde{Q}_i(\lambda) = 0$. Since only a finite number of values can satisfy the second condition, for all λ, except for a finite number of values, $\mathcal{H}$ is minimal. ∎

In the real case, we applied Algorithm III.15 for $d = 2$ and it stops, giving the following basis of $\mathcal{H}_0$:

$$\{e_1 \wedge e_2, e_1 \wedge e_4, e_2 \wedge e_3, e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4\}, \tag{III.17}$$

where $\{e_1, e_2, e_3, e_4\}$ is the canonical basis of $\mathbb{R}^4$.

We then applied Algorithm III.7 with $p = 2$ and for $K = 1$ and $d = 2$ and it was successful. See Section E of the Appendix for the matrices $\widehat{\Xi}_{i,j,k}$ that were randomly generated by the algorithm. We therefore get the following proposition.

Proposition III.18. *Assume (H1), (H2), (H3), (H5), and $d = 2$, then, for all $\lambda \in (0, \infty)$, for all $K \geq 1$, $\mathcal{H}_0$ has dimension 5 and $\widetilde{T}$ is irreducible. Moreover, a basis of $\mathcal{H}_0$ is*

$$\{e_1 \wedge e_2, e_1 \wedge e_4, e_2 \wedge e_3, e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4\}.$$

Remark III.19. It may seem surprising that under (H4), $\mathcal{H}_0$ is $\mathbb{C}^{\binom{4}{2}}$ whereas under (H5), $\mathcal{H}_0$ is a proper subspace. As an explanation, let us give the entry $(6, 1)$ of $\mathcal{B}^{-1}\Xi^K\mathcal{B}$ where $\mathcal{B}$ is the base-changing matrix for the base

$$\{e_1 \wedge e_2, e_1 \wedge e_4, e_2 \wedge e_3, e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4\}.$$

$$\left(\mathcal{B}^{-1}\Xi^{K}\mathcal{B}\right)_{6,1} = \frac{\zeta^{\dagger}_{2(i-1)+2,2(i-1)+2}\zeta_{2(i-2)+2,2(i-1)+2}}{\zeta^{\dagger}_{2(i-1)+1,2(i-1)+1}\zeta_{2(i-2)+1,2(i-1)+1}} \mathrm{Im}\left(\zeta^{\dagger}_{2(i-1)+1,2(i-1)+2}\zeta_{2(i-1)+2,2(i-1)+2}\right).$$

We can see that under (H5), that entry is identically 0.

IV. Concluding remarks

Using the tools of [8] and strong results in product of random matrices theory, we have proved results of type Central Limit Theorem (CLT) and Large Deviations (LD) for the non-ergodic uplink channel in a Wyner-type setup.

We first proved general CLT and LD result under a condition of strong irreducibility (Theorem II.5) and then showed that this condition is verified for $d = 1, 2$ (Corollary II.6) by a direct proof for $d = 1$ and by a computer based proof for $d = 2$. We conjecture that this condition is verified for all d.

Conjecture IV.1. *For all $d \geq 1$, $\rho \in (0, \infty)$, $K \geq 1$, $\widetilde{T}$ is strongly irreducible, therefore the conclusions of Theorem II.5 hold.*

Our results show that the CLT fluctuations of the per-cell sum-rate are of order $1/\sqrt{m}$, where m is the number of cells, whereas they are of order $1/m$ in classical random matrix theory. We also show a LD regime of the form $\mathbb{P}(|C_m - C| > \varepsilon) \leq e^{-m\alpha}$ with $\alpha = \alpha(\varepsilon) > 0$ as opposed to the rate $e^{-m^2\alpha}$ in classical random matrix theory.

Added in Proof: The conjecture has now been proved by P. Bougerol (personal communication).

Acknowledgments

This work was supported by the REMON consortium and by the European Commission in the framework of the FP7 Network of Excellence in Wireless COMmunications NEWCOM++

The work of O. Z. was partially supported by a grant from the Israel Science foundation, and by NSF grant DMS-0804133.

Appendix

A. Proof of Proposition II.7

Denote by B the ball of center 0 and radius $\sqrt{\frac{C(\rho)}{3\rho}}$ in $\mathbb{F}^{K(d+1)}$. By [(H4) or (H5)], $\pi(B) > 0$. We will show that (II.8) holds with $\alpha' = -\log(\pi(B))$.

Let us first show that if for all $1 \leq i \leq m+d$, $\zeta^i \in B$, then, $|C_m(\rho) - C(\rho)| > C(\rho)/2$. Indeed, we denote $M_m = \mathrm{Id}_{K(m+d)} + \rho H_m^\dagger H_m$ so that $C_m(\rho) = 1/m \log\det M_m$. By Hadamard's inequality for semi-positive definite Hermitian matrices,

$$\begin{aligned}
C_m(\rho) &\leq \frac{1}{m}\sum_{i=1}^{(m+d)K} \log M_m(i,i) \\
&= \frac{1}{m}\sum_{j=1}^{m+d}\sum_{k=1}^{K} \log M_m((j-1)K+k,(j-1)K+k).
\end{aligned}$$

Note that

$$M_m((j-1)K+k,(j-1)K+k) = 1 + \rho\sum_{s=0}^{d}\left|\zeta_{j-s,j}(k)\right|^2,$$

therefore,

$$\log M_m((j-1)K+k,(j-1)K+k) \leq \rho\sum_{s=0}^{d}\left|\zeta_{j-s,j}(k)\right|^2.$$

Thus

$$C_m(\rho) \leq \frac{1}{m}\sum_{j=1}^{m+d}\sum_{k=1}^{K}\sum_{s=0}^{d}\rho\left|\zeta_{j-s,j}(k)\right|^2 \leq \frac{1}{m}\sum_{j=1}^{m+d}\rho\left\|\zeta^j\right\|_2^2.$$

We take m large enough such that $(m+d)/m \leq 3/2$. If for all $1 \leq i \leq m+d$, $\zeta^i \in B$, then, $C_m(\rho) \leq C(\rho)/2$ and therefore,

$$|C_m(\rho) - C(\rho)| \geq C(\rho)/2.$$

By (H1), the probability that for all $1 \leq i \leq m+d$, $\zeta^i \in B$ is exactly $e^{-\alpha'(m+d)}$, therefore, for m large enough

$$\frac{1}{m}\log\mathbb{P}\left(|C_m(\rho) - C(\rho)| \geq \frac{C(\rho)}{2}\right) \geq -\frac{m+d}{m}\alpha'.$$

Since $(m+d)/m$ converges to 1 as m goes to infinity, the claim follows.

■

B. Concentration for Wishart-type random matrices

In this section, we present a result of [13] that will allow us to compare the LD result of Theorem II.5.3 with result of classical random matrix theory. In order to facilitate the comparison, we reformulate it in a form similar to Theorem II.5.3.

We consider a random channel H of size $N \times M$. Denote by $(H_{i,j})$ for $1 \leq i \leq N$ and $1 \leq j \leq M$ the entries of H. We assume that for $1 \leq i \leq M$ and $1 \leq j \leq N$,

$$H_{i,j} = \frac{1}{\sqrt{N+M}} h_{i,j},$$

where $h_{i,j}$ is a complex random variable whose distribution is $P_{i,j}$. We suppose moreover that under $P_{i,j}$, real and imaginary parts are independent. We consider the per-cell sum-rate capacity

$$Cap_{N,M}(\rho) \triangleq \frac{1}{N} \operatorname{Tr} \left\{ \log \left(\operatorname{Id} + \rho H H^{\dagger} \right) \right\}.$$

We moreover assume that N and M go to infinity such that N/M converge to a non-zero constant. By [14], $\mathbb{E} Cap_{N,M}(\rho)$ converges to a constant that we denote by $Cap(\rho)$.

Refer to [13] for the definition of the logarithmic Sobolev inequality. Note that the Gaussian law satisfies this inequality.

Proposition A.1 ([13])**.** *Assume that the* $(P_{ij}, 1 \leq i \leq N, 1 \leq j \leq M)$ *satisfy the logarithmic Sobolev inequality with uniform constant* c*. Then for any* $\delta > 0$*,*

$$\limsup_{n \to \infty} \frac{1}{N^2} \log \left(\mathbb{P}^{N,M} (|Cap_{N,M}(\rho) - Cap(\rho)| \geq \delta) \right) \leq -\frac{1}{8c\rho} \delta^2.$$

Proof: We apply [13, Corollary 1.] with $f(x) = \log(1 + \rho x)$. The Lipschitz constant of $g(x) = f(x^2)$ is $\sqrt{\rho}$ therefore

$$\mathbb{P}^{N,M} (|Cap_{N,M}(\rho) - \mathbb{E} Cap_{N,M}(\rho)| \geq \delta/2) \leq 2 e^{-\frac{1}{8c\rho} N^2 \delta^2}.$$

Then, for N and M large enough, $|\mathbb{E} Cap_{N,M}(\rho) - Cap(\rho)| \leq \delta/2$ and therefore,

$$\begin{aligned} \mathbb{P}^{N,M} (|Cap_{N,M}(\rho) - Cap(\rho)| \geq \delta) &\leq \mathbb{P}^{N,M} (|Cap_{N,M}(\rho) - \mathbb{E} Cap_{N,M}(\rho)| \geq \delta/2) \\ &\leq 2 e^{-\frac{1}{8c\rho} N^2 \delta^2}. \end{aligned}$$

■

Note that by using the tools of the proof of Proposition II.7, we can show that the right decay rate is indeed $e^{-N^2 \alpha}$ with $\alpha > 0$.

C. Lyapunov exponents theory

We use the theory of product of random matrices. For a general introduction to the aspects of the theory we use here, the reader may consult [15], [12], [16], [17], [18] or [19]. See Section D of the Appendix for the relevant background on exterior products.

Theorem A.2 (Furstenberg H., Kesten H. (1960))**.** *Consider a stationary ergodic sequence of complex random matrices $(X_i)_{i\geq 1}$ of size p and any norm on the matrices. Assume moreover that*

$$\mathbb{E}\log^+\|X_1\| < \infty,$$

then a.s, $n^{-1}\log\|X_n\cdots X_1\|$ converges to a constant:

$$\lim_{n\to\infty}\frac{1}{n}\log\|X_n\cdots X_1\| \triangleq \gamma(X).$$

We define p constants $\gamma_1(X),\ldots,\gamma_p(X)$ such that for $1\leq i\leq p$,

$$\gamma\left(\bigwedge^i X\right) = \gamma_1(X)+\cdots+\gamma_i(X).$$

Proposition A.3.

$$\gamma_1(X)\geq\cdots\geq\gamma_p(X).$$

The constants $\gamma_1(X)\geq\cdots\geq\gamma_p(X)$ are called the Lyapunov exponents and $\gamma(X)=\gamma_1(X)$ is called the top Lyapunov exponent.

We will also use the three following properties:

1. For any sub-multiplicative norm, for $p\geq 1$

$$\gamma(X)\leq\frac{1}{p}\mathbb{E}\log\|X_p\cdots X_1\|, \tag{A.4}$$

   and the limit of the RHS as p goes to infinity is $\gamma(X)$.

2.
$$\frac{1}{p}\mathbb{E}\log|\det X_1|\leq\gamma(X). \tag{A.5}$$

3. Assume that the matrices $(X_i)_{i\geq 1}$ are i.i.d, then for all $1\leq i\leq p$, $\gamma_i(X)=\gamma_i(X^\dagger)$.

Finally, we quote the following proposition [20, Proposition 1].

Proposition A.6. *Consider a stationary ergodic sequence of complex random matrices $(X_i)_{i\geq 1}$ of size p and any norm on the matrices. Assume moreover that*

$$\mathbb{E}\log^+\|X_1\| < \infty.$$

Finally, assume that there exist three sequences of random matrices $(X_i^1)_{i\geq 1}$, $(X_i^2)_{i\geq 1}$, $(X_i^3)_{i\geq 1}$, of respective sizes $k\times k$, $(p-k)\times k$ and $(p-k)\times(p-k)$, for $1\leq k\leq p-1$, such that almost surely, for all $i\geq 1$

$$X_i = \left(\begin{array}{c|c} X_i^1 & 0_{k,p-k} \\ \hline X_i^2 & X_i^3 \end{array}\right).$$

Then, $\gamma_1(X),\dots,\gamma_p(X)$ is equal up to the order to the sequence

$$\gamma_1(X^1),\dots,\gamma_k(X^1),\gamma_1(X^3),\dots,\gamma_{p-k}(X^3).$$

D. Exterior product

In this section we give the material on exterior products. We provide only the properties relevant to the paper, see [21, Chapter XVI.6-7] and [12, Chapter A.III.5] for more details.

Proposition A.7. *For $0\leq k\leq p$, the exterior product of k vectors in $\mathbb{F}^p$, $v_1,\dots,v_k$ is denoted by $v_1\wedge\cdots\wedge v_k$. It is a vector of the exterior product of degree k of $\mathbb{F}^p$ that we denote by $\bigwedge^k\mathbb{F}^p$. $\bigwedge^k\mathbb{F}^p$ is a $\mathbb{F}$-vector space of dimension $\binom{k}{p}$.*

The exterior product $v_1,\dots,v_k$ is a multi-linear (i.e. linear in every v_i, $1\leq i\leq k$) and anti-symmetric (i.e. $v_{\sigma(1)}\wedge\cdots v_{\sigma(k)} = \varepsilon(\sigma)$ for σ permutation of $\{1,\dots,k\}$ and $\varepsilon(\sigma)$ its signature) function.

If $e_1,\dots,e_p$ is a basis of $\mathbb{F}^p$, then $(e_{i_1}\wedge\cdots\wedge e_{i_k}|1\leq i_1<\cdots<i_k\leq p)$ is a basis of $\bigwedge^k\mathbb{F}^p$. The later is called the canonical basis *of $\bigwedge^k\mathbb{F}^p$ if $e_1,\dots,e_p$ is the canonical basis of $\mathbb{F}^p$.*

If M is a matrix of size $p\times q$, the exterior product of M that we denote by $\bigwedge^k M$ is a map from $\bigwedge^k\mathbb{F}^q$ to $\bigwedge^k\mathbb{F}^p$ such that

$$\bigwedge^k M\,(v_1\wedge\cdots\wedge v_k) = Mv_1\wedge\cdots\wedge Mv_k.$$

Finally, for two matrices M and N, $\bigwedge^k(MN) = \bigwedge^k(M)\bigwedge^k(N)$.

Proposition A.8. *If X is a square matrix of size p, then*

$$\bigwedge^p X = \det X.$$

Moreover, for $q \leq p$,

$$\det \bigwedge^q X = (\det X)^{\binom{p-1}{q-1}}.$$

Proposition A.9. *For p vectors $e_1, \dots, e_p$, we denote by $\left(\begin{array}{c|c|c} e_1 & \cdots & e_p \end{array} \right)$ the matrix whose columns are $e_1, \dots, e_p$. Then*

$$e_1 \wedge \cdots \wedge e_p = \bigwedge^p \left(\begin{array}{c|c|c} e_1 & \cdots & e_p \end{array} \right).$$

The last proposition can be used to actually compute the entries of the wedge product of a matrix. Indeed, if X is a matrix of size p, we denote by $e_1, \dots, e_p$ the canonical basis of $\mathbb{F}^p$. For $q \leq p$, we want to compute the entries of $\bigwedge^q X$ along the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_q};\ 1 \leq i_1 < \cdots < i_q \leq p\}$.

$$\begin{aligned}(e_{j_1} \wedge \cdots \wedge e_{j_q})^\dagger \bigwedge^q X\, (e_{i_1} \wedge \cdots \wedge e_{i_q}) = \\ \bigwedge^q \left(\begin{array}{c|c|c} e_{j_1} & \cdots & e_{j_q} \end{array} \right)^\dagger \bigwedge^q X \bigwedge^q \left(\begin{array}{c|c|c} e_{i_1} & \cdots & e_{j_q} \end{array} \right) \\ \det\left(\left(\begin{array}{c|c|c} e_{j_1} & \cdots & e_{i_q} \end{array} \right)^\dagger X \left(\begin{array}{c|c|c} e_{i_1} & \cdots & e_{i_q} \end{array} \right) \right)\end{aligned}$$

E. Matrices generated in the proof of Propositions III.14 and III.18

In this section, we give the matrices that where generated by the computerized proof of Propositions III.14 and III.18 by Algorithm III.7. For $i = 1, 2$, $1 \leq j \leq \widehat{d}^2$ and $k = 1, 2$,

$$\Xi_{i,j,k} = \bigwedge^2 \Delta_{i,j,k},$$

$$\Delta_{i,j,k} = \left(\begin{array}{c|c} -C_{i,j,k} D^{-1\dagger}_{i,j,k} - \lambda (C_{i,j,k} D_{i,j,k})^{-1\dagger} & \lambda C^{-1\dagger}_{i,j,k} D_{i,j,k} \\ \hline (C_{i,j,k} D_{i,j,k})^{-1\dagger} & -C^{-1\dagger}_{i,j,k} D_{i,j,k} \end{array} \right),$$

$$C_{i,j,k} = \begin{pmatrix} \zeta_{3,3_{i,j,k}} & \zeta_{3,4_{i,j,k}} \\ 0 & \zeta_{4,4_{i,j,k}} \end{pmatrix} \qquad \text{and} \qquad D_{i,j,k} = \begin{pmatrix} \zeta_{1,3}{}^\dagger_{i,j,k} & \zeta_{2,3}{}^\dagger_{i,j,k} \\ 0 & \zeta_{2,4}{}^\dagger_{i,j,k} \end{pmatrix}.$$

Note that the expression of $\Delta_{i,j,k}$ as a function of $C_{i,j,k}$ and $D_{i,j,k}$ was simplified using the fact that if $K = 1$, then $(C_{i,j,k} D_{i,j,k})^{-1\dagger} = C^{-1\dagger}_{i,j,k} D^{-1\dagger}_{i,j,k}$. In the proof of Proposition III.14, $\widehat{\Xi}_{i,j,k} = \Xi_{i,j,k}$. In the proof of Proposition III.18, $\widehat{\Xi}_{i,j,k}$ is $\Xi_{i,j,k}$ restricted to the basis given by (III.17). The coefficients ζ are given in the following tables.

For the proof of Proposition III.14, $\widehat{d} = 6$ and the coefficients ζ are as follows:

	$k=1$											
	$i=1$						$i=2$					
	$\zeta_{1,3_{i,j,k}}$	$\zeta_{2,3_{i,j,k}}$	$\zeta_{3,3_{i,j,k}}$	$\zeta_{2,4_{i,j,k}}$	$\zeta_{3,4_{i,j,k}}$	$\zeta_{4,4_{i,j,k}}$	$\zeta_{1,3_{i,j,k}}$	$\zeta_{2,3_{i,j,k}}$	$\zeta_{3,3_{i,j,k}}$	$\zeta_{2,4_{i,j,k}}$	$\zeta_{3,4_{i,j,k}}$	$\zeta_{4,4_{i,j,k}}$
$j=1$	$2+2i$	$2+2i$	$2+i$	$1+2i$	$2+i$	$1+2i$	$2+2i$	$2+2i$	$2+i$	$2+i$	$2+i$	$1+i$
$j=2$	$1+i$	$1+2i$	$1+i$	$1+i$	$1+2i$	$2+i$	$1+2i$	$1+i$	$2+i$	$2+2i$	$1+i$	$2+2i$
$j=3$	$1+i$	$1+2i$	$2+2i$	$2+i$	$2+i$	$2+i$	$2+i$	$1+i$	$2+i$	$2+2i$	$1+i$	$2+i$
$j=4$	$1+2i$	$2+i$	$1+2i$	$1+i$	$2+i$	$2+i$	$2+i$	$2+2i$	$2+2i$	$2+i$	$2+2i$	$2+i$
$j=5$	$2+i$	$2+2i$	$2+i$	$2+2i$	$2+2i$	$1+i$	$1+i$	$2+i$	$2+i$	$2+i$	$1+2i$	$2+i$
$j=6$	$2+i$	$1+i$	$2+i$	$2+2i$	$1+2i$	$2+i$	$1+2i$	$1+i$	$2+2i$	$1+2i$	$2+2i$	$2+i$
$j=7$	$2+i$	$2+2i$	$1+i$	$1+i$	$2+2i$	$2+2i$	$2+2i$	$1+2i$	$1+2i$	$1+2i$	$1+i$	$2+i$
$j=8$	$2+i$	$1+i$	$1+2i$	$1+i$	$1+2i$	$2+2i$	$2+2i$	$1+2i$	$1+2i$	$1+2i$	$1+2i$	$2+i$
$j=9$	$1+2i$	$1+2i$	$1+i$	$2+2i$	$2+2i$	$2+2i$	$1+2i$	$2+i$	$1+i$	$2+i$	$1+2i$	$1+2i$
$j=10$	$1+i$	$2+2i$	$2+2i$	$1+i$	$2+2i$	$2+2i$	$2+i$	$1+i$	$1+2i$	$1+i$	$1+2i$	$2+2i$
$j=11$	$1+i$	$1+i$	$1+i$	$2+2i$	$1+i$	$2+2i$	$2+i$	$1+2i$	$2+2i$	$2+2i$	$2+2i$	$2+2i$
$j=12$	$1+2i$	$1+i$	$1+i$	$1+2i$	$1+2i$	$1+i$	$2+i$	$1+2i$	$1+2i$	$2+i$	$2+i$	$1+i$
$j=13$	$1+i$	$2+2i$	$2+2i$	$1+2i$	$2+i$	$2+i$	$1+i$	$2+2i$	$2+2i$	$1+2i$	$2+i$	$1+2i$
$j=14$	$2+2i$	$1+2i$	$2+2i$	$1+i$	$1+2i$	$2+2i$	$1+i$	$2+2i$	$1+i$	$2+i$	$1+2i$	$1+i$
$j=15$	$2+2i$	$1+2i$	$2+i$	$1+2i$	$2+i$	$2+2i$	$1+2i$	$2+2i$	$1+i$	$1+i$	$2+2i$	$1+2i$
$j=16$	$1+i$	$2+i$	$2+i$	$1+2i$	$1+2i$	$1+i$	$2+2i$	$1+i$	$2+i$	$1+i$	$1+2i$	$1+2i$
$j=17$	$1+2i$	$2+i$	$2+i$	$1+i$	$2+2i$	$2+2i$	$1+i$	$2+2i$	$2+i$	$2+i$	$2+i$	$2+2i$
$j=18$	$1+2i$	$2+2i$	$2+2i$	$1+2i$	$2+i$	$2+i$	$2+2i$	$2+i$	$2+i$	$1+i$	$2+i$	$1+2i$
$j=19$	$1+i$	$1+i$	$2+2i$	$2+2i$	$2+i$	$2+i$	$2+i$	$1+i$	$2+i$	$1+2i$	$1+i$	$2+2i$
$j=20$	$2+2i$	$2+2i$	$2+2i$	$2+i$	$2+i$	$1+i$	$1+2i$	$2+2i$	$2+2i$	$1+i$	$1+i$	$2+i$
$j=21$	$2+i$	$2+2i$	$1+2i$	$2+2i$	$2+2i$	$2+2i$	$1+i$	$2+i$	$2+2i$	$2+2i$	$1+2i$	$2+2i$
$j=22$	$2+i$	$2+2i$	$1+2i$	$2+i$	$1+2i$	$1+i$	$2+2i$	$2+2i$	$1+2i$	$1+i$	$1+2i$	$1+i$
$j=23$	$2+2i$	$2+2i$	$1+2i$	$2+2i$	$2+2i$	$2+i$	$2+i$	$2+i$	$1+2i$	$2+2i$	$1+2i$	$2+2i$
$j=24$	$1+i$	$1+i$	$1+i$	$2+2i$	$2+2i$	$1+i$	$2+2i$	$1+2i$	$1+2i$	$1+2i$	$1+i$	$2+i$
$j=25$	$2+2i$	$1+i$	$1+i$	$2+2i$	$1+i$	$1+2i$	$2+i$	$2+i$	$2+i$	$1+2i$	$2+2i$	$2+2i$
$j=26$	$1+i$	$1+i$	$2+i$	$1+i$	$1+2i$	$1+2i$	$1+i$	$1+i$	$2+i$	$1+2i$	$1+2i$	$1+i$
$j=27$	$2+i$	$1+2i$	$1+2i$	$1+2i$	$2+2i$	$1+i$	$2+i$	$2+2i$	$1+2i$	$2+2i$	$2+2i$	$2+2i$
$j=28$	$1+i$	$2+2i$	$2+i$	$1+2i$	$1+2i$	$1+2i$	$2+i$	$2+2i$	$2+i$	$1+i$	$2+i$	$2+2i$
$j=29$	$2+2i$	$1+2i$	$1+i$	$2+i$	$2+2i$	$2+2i$	$2+2i$	$2+2i$	$2+2i$	$1+2i$	$2+2i$	$1+2i$
$j=30$	$2+2i$	$1+2i$	$1+i$	$1+2i$	$1+i$	$2+i$	$2+i$	$1+2i$	$1+2i$	$1+2i$	$1+2i$	$2+i$
$j=31$	$1+2i$	$2+2i$	$2+2i$	$1+i$	$2+2i$	$2+i$	$2+2i$	$1+2i$	$1+i$	$1+i$	$1+2i$	$1+2i$
$j=32$	$1+2i$	$2+2i$	$2+2i$	$1+i$	$2+i$	$1+2i$	$2+i$	$1+i$	$2+2i$	$2+2i$	$1+i$	$1+2i$
$j=33$	$1+2i$	$1+i$	$1+2i$	$1+i$	$2+2i$	$1+2i$	$1+i$	$2+i$	$1+2i$	$1+2i$	$1+2i$	$1+2i$
$j=34$	$1+2i$	$2+2i$	$2+2i$	$2+2i$	$2+i$	$1+i$	$1+i$	$1+2i$	$1+i$	$2+i$	$1+i$	$1+2i$
$j=35$	$2+2i$	$2+i$	$2+i$	$1+2i$	$1+i$	$2+2i$	$2+2i$	$1+2i$	$1+2i$	$1+2i$	$1+2i$	$2+i$
$j=36$	$2+2i$	$2+2i$	$2+2i$	$2+i$	$2+2i$	$2+2i$	$1+2i$	$2+2i$	$1+i$	$2+2i$	$1+2i$	$1+2i$

	$k=2$											
	$i=1$						$i=2$					
	$\zeta_{1,3_{i,j,k}}$	$\zeta_{2,3_{i,j,k}}$	$\zeta_{3,3_{i,j,k}}$	$\zeta_{2,4_{i,j,k}}$	$\zeta_{3,4_{i,j,k}}$	$\zeta_{4,4_{i,j,k}}$	$\zeta_{1,3_{i,j,k}}$	$\zeta_{2,3_{i,j,k}}$	$\zeta_{3,3_{i,j,k}}$	$\zeta_{2,4_{i,j,k}}$	$\zeta_{3,4_{i,j,k}}$	$\zeta_{4,4_{i,j,k}}$
$j=1$	$1+2i$	$2+i$	$2+i$	$2+2i$	$2+2i$	$1+i$	$1+2i$	$1+i$	$1+2i$	$1+i$	$1+2i$	$2+2i$
$j=2$	$1+2i$	$1+i$	$2+2i$	$1+2i$	$2+i$	$1+2i$	$1+i$	$2+2i$	$1+2i$	$1+i$	$2+2i$	$2+2i$
$j=3$	$1+i$	$2+i$	$2+2i$	$2+2i$	$1+i$	$1+i$	$2+2i$	$1+i$	$1+2i$	$2+2i$	$1+2i$	$1+i$
$j=4$	$2+2i$	$2+2i$	$1+2i$	$1+i$	$1+2i$	$2+2i$	$2+i$	$1+2i$	$2+i$	$2+2i$	$2+2i$	$2+i$
$j=5$	$1+i$	$1+i$	$2+i$	$2+i$	$1+2i$	$1+2i$	$1+i$	$2+2i$	$1+i$	$2+2i$	$1+i$	$1+i$
$j=6$	$2+i$	$1+i$	$2+2i$	$2+i$	$2+i$	$1+i$	$1+2i$	$2+i$	$2+2i$	$1+2i$	$1+i$	$1+i$
$j=7$	$2+2i$	$1+i$	$2+2i$	$1+i$	$1+i$	$2+i$	$2+i$	$1+2i$	$2+2i$	$1+2i$	$1+2i$	$2+2i$
$j=8$	$2+i$	$2+2i$	$2+i$	$2+i$	$1+i$	$1+2i$	$1+2i$	$2+2i$	$2+2i$	$2+2i$	$2+i$	$1+i$
$j=9$	$2+2i$	$1+i$	$2+i$	$1+i$	$1+2i$	$2+2i$	$1+i$	$1+2i$	$2+2i$	$1+2i$	$2+i$	$2+2i$
$j=10$	$1+i$	$2+i$	$2+i$	$1+2i$	$2+i$	$1+i$	$1+2i$	$1+i$	$2+i$	$2+i$	$2+i$	$1+i$
$j=11$	$1+2i$	$2+2i$	$1+2i$	$1+2i$	$1+2i$	$2+i$	$1+i$	$2+i$	$2+2i$	$2+i$	$1+i$	$2+i$
$j=12$	$1+2i$	$2+i$	$2+2i$	$1+2i$	$1+i$	$1+i$	$2+i$	$2+i$	$2+i$	$1+i$	$2+i$	$1+i$
$j=13$	$1+2i$	$2+i$	$2+i$	$2+2i$	$1+i$	$1+2i$	$1+2i$	$1+2i$	$2+2i$	$2+2i$	$1+2i$	$2+i$
$j=14$	$1+i$	$1+2i$	$1+i$	$2+i$	$1+i$	$2+2i$	$1+2i$	$2+2i$	$2+i$	$2+i$	$1+i$	$1+2i$
$j=15$	$1+2i$	$2+i$	$1+2i$	$2+2i$	$1+2i$	$2+2i$	$2+i$	$2+i$	$2+i$	$2+2i$	$2+2i$	$2+i$
$j=16$	$1+2i$	$1+2i$	$1+i$	$2+2i$	$1+i$	$1+i$	$2+2i$	$2+i$	$1+2i$	$1+2i$	$2+2i$	$2+i$
$j=17$	$2+2i$	$2+i$	$2+i$	$2+i$	$1+i$	$1+2i$	$2+2i$	$1+i$	$2+2i$	$2+2i$	$1+i$	$2+i$
$j=18$	$2+i$	$2+i$	$1+2i$	$1+2i$	$1+i$	$1+2i$	$2+i$	$2+2i$	$1+2i$	$1+2i$	$1+i$	$1+i$
$j=19$	$1+2i$	$2+2i$	$1+2i$	$2+2i$	$1+i$	$1+i$	$2+i$	$1+2i$	$1+i$	$1+i$	$2+i$	$1+i$
$j=20$	$1+2i$	$2+2i$	$1+2i$	$1+i$	$1+i$	$1+2i$	$1+2i$	$2+2i$	$1+2i$	$1+2i$	$2+i$	$2+2i$
$j=21$	$1+2i$	$2+i$	$2+i$	$2+i$	$2+i$	$1+i$	$1+i$	$1+2i$	$2+i$	$1+i$	$1+i$	$2+2i$
$j=22$	$2+i$	$2+i$	$1+i$	$2+i$	$1+2i$	$1+i$	$1+i$	$2+i$	$1+i$	$1+2i$	$1+i$	$2+i$
$j=23$	$1+i$	$2+i$	$1+2i$	$2+2i$	$1+2i$	$1+2i$	$1+i$	$1+2i$	$2+i$	$1+2i$	$2+i$	$2+2i$
$j=24$	$1+2i$	$1+2i$	$1+i$	$2+i$	$1+2i$	$2+2i$	$2+i$	$1+2i$	$1+2i$	$2+i$	$1+2i$	$2+i$
$j=25$	$2+2i$	$2+i$	$1+2i$	$2+2i$	$1+2i$	$2+i$	$2+2i$	$1+i$	$2+i$	$2+2i$	$1+i$	$2+2i$
$j=26$	$1+i$	$1+i$	$1+i$	$1+i$	$2+2i$	$2+2i$	$1+2i$	$2+i$	$1+i$	$1+2i$	$1+i$	$1+2i$
$j=27$	$1+2i$	$1+i$	$1+2i$	$2+i$	$2+2i$	$1+2i$	$1+i$	$1+2i$	$1+2i$	$1+2i$	$2+2i$	$1+i$
$j=28$	$2+i$	$2+i$	$2+i$	$1+2i$	$2+2i$	$1+2i$	$1+i$	$1+i$	$1+i$	$2+2i$	$1+2i$	$2+2i$
$j=29$	$2+i$	$2+2i$	$2+2i$	$2+2i$	$2+i$	$2+2i$	$1+i$	$1+2i$	$1+2i$	$1+2i$	$2+i$	$2+2i$
$j=30$	$2+i$	$2+2i$	$1+i$	$1+2i$	$2+2i$	$2+2i$	$1+i$	$1+i$	$2+i$	$1+2i$	$1+2i$	$1+2i$
$j=31$	$2+2i$	$1+2i$	$1+i$	$1+i$	$1+i$	$2+i$	$1+2i$	$2+i$	$2+2i$	$1+2i$	$1+i$	$1+i$
$j=32$	$2+2i$	$1+i$	$2+i$	$2+2i$	$2+i$	$2+i$	$2+2i$	$1+i$	$2+i$	$1+i$	$2+2i$	$2+2i$
$j=33$	$2+2i$	$1+2i$	$1+2i$	$1+2i$	$1+i$	$1+i$	$1+i$	$2+2i$	$2+i$	$2+2i$	$1+i$	$2+i$
$j=34$	$2+2i$	$1+2i$	$1+i$	$2+2i$	$1+2i$	$2+i$	$2+2i$	$2+2i$	$1+i$	$1+2i$	$2+2i$	$1+2i$
$j=35$	$2+i$	$1+i$	$2+i$	$1+2i$	$1+2i$	$2+i$	$2+i$	$2+2i$	$2+i$	$2+i$	$1+2i$	$2+i$
$j=36$	$2+2i$	$2+i$	$2+2i$	$2+2i$	$2+i$	$1+i$	$1+i$	$1+i$	$1+i$	$2+i$	$2+2i$	$1+i$

For the proof of Proposition III.18, $\widehat{d} = 5$ and the coefficients ζ are as follows:

	$k=1$											
	$i=1$						$i=2$					
	$\zeta_{1,3_{i,j,k}}$	$\zeta_{2,3_{i,j,k}}$	$\zeta_{3,3_{i,j,k}}$	$\zeta_{2,4_{i,j,k}}$	$\zeta_{3,4_{i,j,k}}$	$\zeta_{4,4_{i,j,k}}$	$\zeta_{1,3_{i,j,k}}$	$\zeta_{2,3_{i,j,k}}$	$\zeta_{3,3_{i,j,k}}$	$\zeta_{2,4_{i,j,k}}$	$\zeta_{3,4_{i,j,k}}$	$\zeta_{4,4_{i,j,k}}$
$j=1$	1	2	2	1	2	1	1	1	2	2	2	2
$j=2$	1	2	1	2	2	1	1	2	1	1	2	1
$j=3$	1	2	2	1	2	2	2	2	1	1	2	2
$j=4$	2	1	2	1	1	1	2	2	2	1	1	1
$j=5$	1	2	1	1	1	2	2	2	2	2	1	2
$j=6$	1	2	2	1	2	2	1	1	1	2	2	1
$j=7$	2	2	1	1	2	2	1	2	1	1	1	2
$j=8$	2	1	2	1	2	2	1	1	1	1	1	1
$j=9$	1	2	1	1	2	1	2	2	2	2	2	1
$j=10$	1	2	2	2	2	1	2	1	2	2	2	1
$j=11$	1	2	1	2	1	1	1	1	1	1	1	1
$j=12$	2	1	2	1	2	2	1	2	2	1	2	1
$j=13$	2	1	1	2	1	1	2	1	2	1	1	2
$j=14$	1	2	1	2	1	2	2	1	1	2	2	1
$j=15$	1	1	2	1	2	1	2	2	1	2	2	2
$j=16$	2	1	1	2	1	2	1	2	2	1	1	1
$j=17$	1	2	2	1	2	1	1	1	2	2	2	1
$j=18$	2	1	2	1	2	1	1	1	1	1	1	2
$j=19$	1	2	1	1	1	1	1	1	2	2	2	1
$j=20$	1	1	2	2	1	2	1	1	2	1	1	2
$j=21$	1	2	2	2	2	1	2	2	1	2	2	1
$j=22$	2	2	1	2	1	1	2	2	1	1	1	1
$j=23$	1	2	2	1	2	1	1	2	1	1	2	1
$j=24$	2	1	2	1	1	2	1	1	2	2	2	2
$j=25$	1	2	2	1	2	2	1	1	2	1	1	1

	$k=2$											
	$i=1$						$i=2$					
	$\zeta_{{1,3}_{i,j,k}}$	$\zeta_{{2,3}_{i,j,k}}$	$\zeta_{{3,3}_{i,j,k}}$	$\zeta_{{2,4}_{i,j,k}}$	$\zeta_{{3,4}_{i,j,k}}$	$\zeta_{{4,4}_{i,j,k}}$	$\zeta_{{1,3}_{i,j,k}}$	$\zeta_{{2,3}_{i,j,k}}$	$\zeta_{{3,3}_{i,j,k}}$	$\zeta_{{2,4}_{i,j,k}}$	$\zeta_{{3,4}_{i,j,k}}$	$\zeta_{{4,4}_{i,j,k}}$
$j=1$	2	1	2	2	2	1	1	2	1	2	2	2
$j=2$	2	2	2	2	1	2	2	1	2	1	2	1
$j=3$	2	1	2	2	2	2	2	2	1	1	2	1
$j=4$	2	1	1	2	2	2	1	1	2	2	1	1
$j=5$	1	2	1	2	2	1	2	1	2	1	2	1
$j=6$	1	2	2	1	2	2	2	2	2	1	1	1
$j=7$	1	1	2	1	1	1	2	1	2	2	1	2
$j=8$	2	1	2	1	2	2	2	2	2	2	2	2
$j=9$	1	1	1	2	2	1	1	1	1	1	2	2
$j=10$	2	2	1	2	2	2	2	1	1	1	2	1
$j=11$	2	1	2	1	1	1	1	2	1	1	2	2
$j=12$	2	2	1	1	1	1	2	2	2	1	1	1
$j=13$	1	1	1	2	1	1	2	2	2	1	1	2
$j=14$	2	2	2	2	2	1	1	2	1	1	1	2
$j=15$	1	2	2	1	2	2	2	2	2	2	2	1
$j=16$	2	1	2	1	2	2	1	2	2	1	2	2
$j=17$	2	2	1	2	2	1	1	1	1	2	2	2
$j=18$	1	1	1	1	1	1	2	2	2	2	2	1
$j=19$	2	2	1	2	2	2	2	1	1	2	1	2
$j=20$	1	2	1	1	2	2	2	1	2	1	2	1
$j=21$	1	2	2	2	1	2	2	1	2	2	1	1
$j=22$	1	1	1	1	1	1	1	1	1	1	1	1
$j=23$	1	1	1	1	2	2	2	2	1	1	1	1
$j=24$	2	2	2	2	2	1	1	1	1	1	1	1
$j=25$	2	2	1	2	1	1	1	1	1	2	2	1

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