Frequent points and covering times:

ON CONJECTURES OF ERDÖS-TAYLOR AND OF KESTEN-RÉVÉSZ

(A. Dembo, Y. Peres, J. Rosen, O. Zeitouni)

X_n- simple random walk on $\mathbb{Z}^2$

$$T_n(x) = \sum_{j=1}^{n} \mathbf{1}_{\{X_j = x\}} \Big(\# \text{ of visits to } x \text{ by time } n \Big)$$

$$T_n^* = \max_{x \in \mathbb{Z}^2} T_n(x)$$

Erdös-Taylor conjecture:

$$\boxed{\liminf_{n\to\infty} \frac{T_n^*}{(\log n)^2} = \limsup_{n\to\infty} \frac{T_n^*}{(\log n)^2} = \frac{1}{\pi}, \quad \text{a.s.}}$$

(They proved the upper bound and a lower bound $1/4\pi$).

Main result in this talk:

$$\boxed{\text{E-T conjecture is true.}}$$

$\tilde{X}_j$- simple random walk on the two-dimensional torus $\mathbb{Z}_n^2 = \mathbb{Z}^2/n\mathbb{Z}^2$

$$\mathcal{T}(x) = \min\{j \geq 0 : \tilde{X}_j = x\}$$

$$\mathcal{T}_n = \max_{x \in \mathbb{Z}_n^2} \mathcal{T}(x) \text{ (covering time)}$$

Main result:

$$\boxed{\lim_{n \to \infty} \frac{\mathcal{T}_n}{(n \log n)^2} = \frac{4}{\pi}, \quad \text{in probability}}$$

(Known: upper bound $\frac{4}{\pi}$, lower bound $\frac{2}{\pi}$:
Kesten, Révész, Lawler, Aldous)

Upper bound (sketch)

Truncated Green function:

$$G_n(0,0) = \sum_{j=0}^{n} P(X_j = 0) \sim \frac{\log n}{\pi}$$

(by Brownian scaling and local CLT).

Renewal argument yields

$$P(X_j \neq 0, j \in [1,n)) \geq \frac{(1-\delta)\pi}{\log n}, \quad n \text{ large enough}$$

$$\Rightarrow P(T_n(0) \geq \alpha(\log n)^2) \leq n^{-(1-\delta)\alpha\pi}$$

$$\Rightarrow P_\alpha := P(T_n \geq \alpha(\log n)^2)$$

$$\leq P(\max_{0\leq i\leq n} |X_i| > n^{(1+\delta)/2})$$

$$+ \pi n^{(1+\delta)} n^{-(1-\delta)\alpha\pi}$$

Thus, for $\alpha > \frac{1}{\pi}$, $P_\alpha \to 0$.
(a.s. by Borel-Cantelli).

Main issue – lower bound

Standard technique: second moment

$$Z_n = \sum_{x:|x|\le n^{1/2}} \mathbf{1}_{T_n(x)\ge\beta(\log n)^2}$$

Adapting the proof of the upper bound,

$$EZ_n \approx n^{(1-\beta\pi)}$$

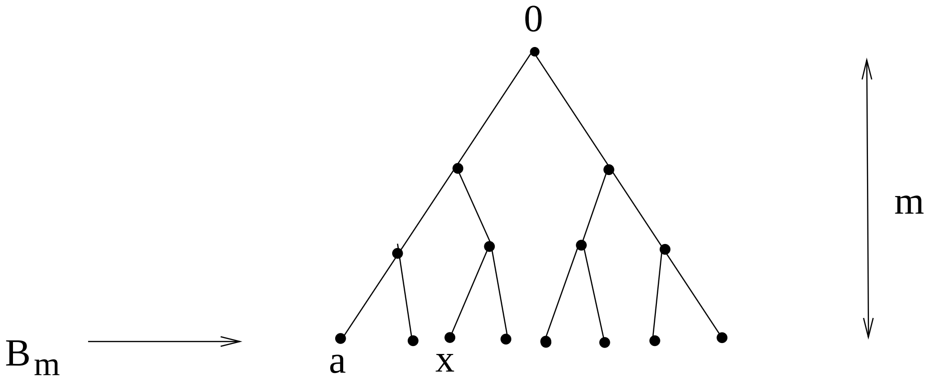
If $$\limsup_{n\to\infty} \frac{EZ_n^2}{(EZ_n)^2} < \infty,$$

then $$\liminf_{n\to\infty} P(T_n^* \ge \beta(\log n)^2) > 0$$

Unfortunately, for this to work need $\beta = \frac{1}{4\pi}$

(Erdös Taylor).

Move to a toy problem (on binary tree).



Y_n- nearest neighbour walk on tree, started at a.

$T_m(x) =$ time spent at x before hitting 0.

$T_m^* = \max_{x \in B_m} T_m(x)$

$T_m(a)$ is geometric, mean m

$$P(T_m(a) > \alpha m^2) \simeq e^{-\alpha m}$$

$\Rightarrow$ Upper bound as above:

$$P(T_m^* > \alpha m^2) \leq e^{-\alpha m} 2^m$$

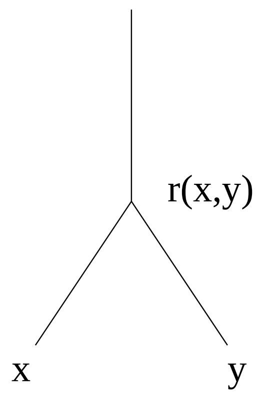
Modification of second moment method:
Fix some K large. A ray x is *successful* if

$$N_i(x) := \{\#\text{ excursions from level } i \text{ to } i+1 \text{ on ray } x\} \simeq \alpha i^2,$$
$$\text{for } i = 0, K-1, \ldots, m-1$$

$$P(N_{i+K}(x) \simeq \alpha(i+K)^2 | N_i(x) \simeq \alpha i^2) \simeq e^{-\alpha K}$$
$$\Rightarrow P(x\text{-ray is successful}) \simeq e^{-\alpha m}$$

Advantage is reduction of variance:

$$P((x,y) \text{ are successful}) \simeq e^{-2\alpha m} e^{\alpha r(x,y)}$$



If

$$Z_m = \sum_{x \in B_m} \mathbf{1}_{\{x\text{-ray successful}\}}$$

Then

$$\frac{EZ_m^2}{(EZ_m)^2} \simeq \sum_{r=1}^{m/K} e^{(\alpha - \log 2)Kr}$$

That is

$$\Rightarrow \alpha < \log 2 \text{ is good!}$$

How to adapt this to SRW?

I don't know!

But: Can adapt it to planar Brownian motion, w_t; let

$$\overline{\theta} = \min\{t : |w_t| = 1\}$$

$$\mu^w_{\overline{\theta}}(A) = \int_0^{\overline{\theta}} \mathbf{1}_A(w_t)dt$$

We will claim:

$$\lim_{\varepsilon \to 0} \sup_{|x|<1} \frac{\mu^w_{\overline{\theta}}(D(x,\varepsilon))}{\varepsilon^2(\log \frac{1}{\varepsilon})^2} = 2$$

(Perkins-Taylor conjecture)

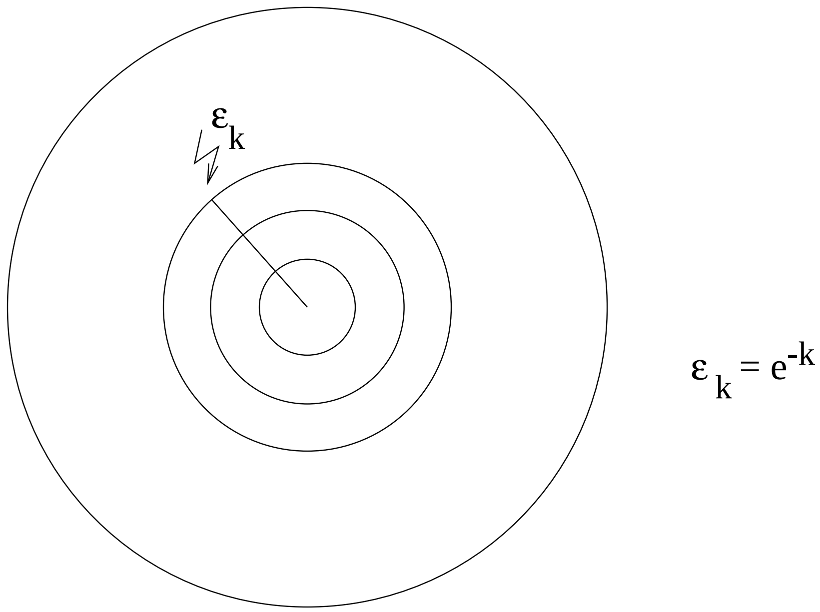
and then use strong (Hungarian!)

approximation of BM by SRW.

Idea of proof of the P-T conjecture:

– Upper bound: as for SRW (P-T)

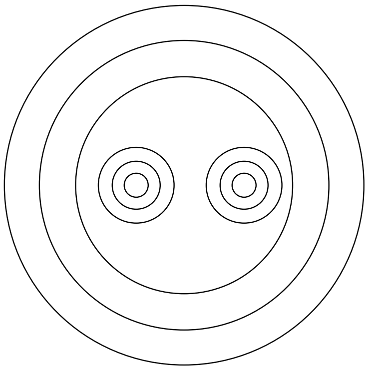
– Lower bound: again, difficulty is in correlation structure. Consider



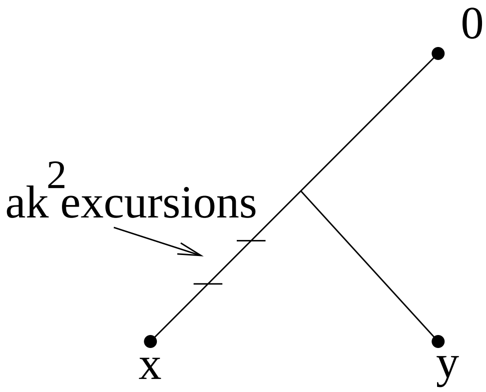
A point x is k *successful* if # of excursions of BM between ε_k and ε_{k+1} disc is ak^2.

Remark: if $\varepsilon = \varepsilon_{k+1}$, for x succesful the time spent at inner ball is $ak^2 \cdot \varepsilon^2 \sim a\varepsilon^2(\log \frac{1}{\varepsilon})^2$ with overwhelming probability.

Now, for points $x \neq y$, we have



$\Rightarrow$ tree structure

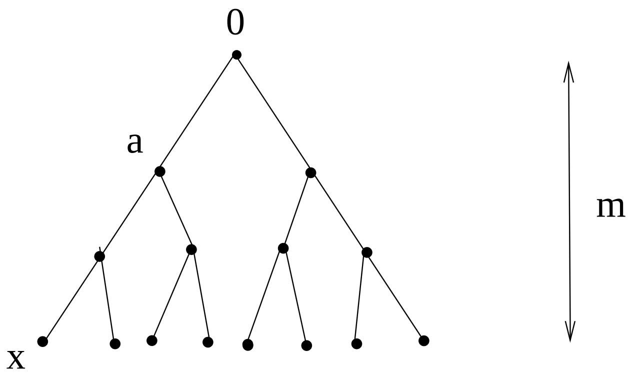


and same technique as for tree works!

Technical issues: Many!

From late points to covering times

Before handling question for SRW, deal with "toy" problem: covering time of binary tree



Upper bound

$$P^1(\text{hit } x \text{ before } 0) = \frac{1}{m}$$

$\Rightarrow P(\text{not cover } x \text{ during first } N \text{ visits to } 0)$

$$\simeq \left(1 - \frac{1}{m}\right)^{N/2}$$

$\Rightarrow P := P^0(\text{not cover } B_m \text{ during } N \text{ visits to } 0)$

$$\leq 2^m \left(1 - \frac{1}{m}\right)^{N/2}$$

$\Rightarrow$ For $N = 2(1+\delta)m^2 \log 2,\ P \to 0\,.$

But, time needed for N visits at root is $2^{m+1}N$.

$\Rightarrow P$ (not cover B_m before time

$$2(1+\delta)\log 2 \cdot m^2 2^{m+1}) \to 0.$$

Lower bound A ray x is called successful if # of excursions from level i to $i+1$ in ray is $a(m-i)^2$.

Apply now some second moment analysis to deduce that w.p. $\to 1$, before $2(1-\delta)m^2 \log 2$ visits to root, there are points which are not covered. Then, time needed to visit root that many times is about $2(1-\delta)m^2 \log 2 \cdot 2^{(m+1)}$.

$\square$

To solve SRW problem on $\mathbb{Z}_n^2$, first solve problem for BM X_t on T^2:

$$\mathcal{T}(x, \varepsilon) = \inf\{t > 0 : \ X_t \in D(x, \varepsilon)\}$$

$$C_\varepsilon = \sup_{x \in T^2} \mathcal{T}(x, \varepsilon)$$

Then,

$$\boxed{\lim_{\varepsilon \to 0} \frac{C_\varepsilon}{(\log \varepsilon)^2} = \frac{2}{\pi}, \quad \text{a.s.}}$$

Then use KMT.