TIGHTNESS OF FLUCTUATIONS OF FIRST PASSAGE PERCOLATION ON SOME LARGE GRAPHS

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ABSTRACT. The theorem of Dekking and Host [6] regarding tightness around the mean of first passage percolation on the binary tree, from the root to a boundary of a ball, is generalized to a class of graphs which includes all lattices in hyperbolic spaces and the lamplighter graph over $\mathbb{N}$. This class of graphs is closed under product with any bounded degree graph. Few open problems and conjectures are gathered at the end.

1. INTRODUCTION

In *First Passage Percolation* (FPP) random i.i.d lengths are assigned to the edges of a fixed graph. Among other questions one studies the distribution of the distance from a fixed vertex to another vertex or to a set, such as the boundary of a ball in the graph, see e.g. [9] for background. Formally, given a rooted, undirected graph $G = (V, E)$ with root o, let D_n denote the collection of vertices at (graph) distance n from the root. For $v \in D_n$, let $\mathcal{P}_v$ denote the collection of paths $(v_0 = o, v_1, v_2, v_3, \ldots, v_k = v)$ (with $(v_{i-1}, v_i) \in E$) from o to v. Given a collection of positive i.i.d. $\{X_e\}_{e\in E}$, define, for $v \in E$,

$$Z_v = \min_{p\in\mathcal{P}_v} \sum_{e\in p} X_e \,. \tag{1.1}$$

Because of the positivity assumption on the weights, we may and will assume that any path in $\mathcal{P}_v$ visits each vertex of G at most once.

For n integer, let $Z_n^* = \min_{v\in D_n} Z_v$. Under a mild moment condition on the law of the random lengths X_e, Dekking and Host [6] proved that for any regular tree, $Z_n^* - EZ_n^*$, the random distance from the root to D_n minus its mean, is tight. (Recall that a sequence of real valued random variables $\{X_n\}_{n\geq 0}$ is tight iff for any $\epsilon > 0$, there is some $r_\epsilon \in \mathbb{R}$, so that for all n, $P(|X_n| > r_\epsilon) < \epsilon$.)

We formulate here a simple and general property of the underling graph G and prove that for graphs satisfying this property and a mild condition on the law of X_e, the collection $\{Z_n^* - EZ_n^*\}_{n\geq 0}$ is tight. Lattices in real hyperbolic spaces $\mathbb{H}^d$, the graph

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of the lamplighter over $\mathbb{N}$, as well as graphs of the form $G \times H$ where G satisfies the conditions we list below and H is any bounded degree graph, are shown to possess this property. (In passing, we mention that the Euclidean case is wide open; it is known that in two dimensions the fluctuations of the distance are not tight, see [12, 13], however only very poor upper bounds are known [2]. For a special solved variant see [8].)

In the next section we formulate the geometric condition on the graph and the assumption on the distribution of the edge weights $\{X_e\}_{e \in V}$; we then state the tightness result, Theorem 2.1, which is proved in Section 3. We conclude with a few open problems.

2. A recursive structure in graphs and tightness

Throughout, let $dist_G$ denote the graph distance in G. The following are the properties of G and the law of X_e alluded to above.

(1) G contains two vertex-disjoint subgraphs G_1, G_2, which are isomorphic to G.
(2) There exists $K < \infty$ so that $EX_e < K$, and

$$dist_G(Root_G, Root_{G_1}) = dist_G(Root_G, Root_{G_2}).$$

One can replace Property (2) by the following.

(3) $X_e < K$ a.s., and every vertex at distance n from the root is connected to at least one vertex of distance $n+1$.

Properties (1) and (2) imply that the binary tree embeds quasi-isometrically into G, thus G has exponential growth. Property (3) is called having "no dead ends" in geometric group theory terminology.

Theorem 2.1. *Assume Property (1) and either Property (2) or Property (3). Then the sequence $\{(Z_n - EZ_n)\}_{n \geq 1}$ is tight.*

Note that a hyperbolic lattice in $\mathbb{H}^d, d \geq 2$, intersected with a half space, admits the graph part of Properties (1) and (2) above (and probably (3) as well but we don't see a general proof). This is due to topological transitivity of the action on the space of geodesics, i.e. pairs of point of the boundary. There exist elements g in the authomorphism group of the hyperbolic space that map the half space into arbitrarily small open sets of the boundary and elements of this group map the lattice orbit to itself. Note also that by the Morse lemma of hyperbolic geometry (see, e.g., [4] p. 175), if one assumes in addition that $X_e \geq \delta > 0$ a.s. then a path with minimal FPP length will be within a bounded distance from a hyperbolic geodesic and will not wind around, thus tightness for half space for weights that are bounded below by a uniform positive constant implies tightness for the whole space. (Recall also that the regular tree is a lattice in $\mathbb{H}^2$; see [10] for some nice pictures of other planar hyperbolic lattices.)

An example satisfying Properties (1) and (2) is given by the semi group of the lamplighter over $\mathbb{N}$. Recall the graph of the lamplighter over $\mathbb{N}$: a vertex corresponds to a scenery of 0's and 1's over $\mathbb{N}$, with finitely many 1's with a position of a lamplighter in $\mathbb{N}$; edges either change the bit at the position of the lamplighter or move the lamplighter one step to the left or the right, see, e.g., [10]. If we fix the left most bit and restrict the lamplighter to integers strictly bigger than 1, we get the required G_0 and G_1.

It easy to see that if G satisfies the properties in the theorem then $G \times H$ will too. In particular the theorem applies to $T \times T'$ for two regular trees. Note also that if G satisfies the property (1) in the theorem, then the lamplighter over G will admit it as well.

3. Proof of Theorem 2.1

The proof is based on a modification of an argument in [6]; a related modification was used in [1]. Note first that, by construction,

$$\text{(a) } EZ_{n+1} \geq EZ_n,$$

because to get to distance $n+1$ a path has to pass through distance n and the weights $\{X_e\}$ are positive.

Under Property (3), one has in addition

(a') Z_n and Z_{n+i} can be constructed on the same space so that

$$Z_{n+i} \geq Z_n \text{ while } Z_{n+i} \leq Z_n + Ki.$$

(The first inequality does not need Property (3), but the second does — one just goes forward from the minimum at distance n, i steps.)

On the other hand, from Property (1),

$$EZ_{n+1} \leq E(\min(Z_{n-R_1+1}, Z'_{n-R_2+1}) + KC,$$

where $R_i = dist_G(Root_G, Root_{G_i})$, $C = \max(R_1, R_2)$, and Z'_m denotes a identically distributed independent copy of Z_m .

Since $\min(a,b) = \frac{a+b}{2} - \frac{|a-b|}{2}$,

$$EZ_{n+1} \le (1/2)[EZ_{n-R_1+1} + EZ'_{n-R_2+1} - E|Z_{n-R_1+1} - Z'_{n-R_2+1}|] + KC.$$

Therefore, with $n_i = n+1-R_i$,

$$E|Z_{n_1} - Z'_{n_2}| \le [-2EZ_{n+1} + EZ_{n_1} + EZ_{n_2}] + 2KC.$$

If $R_1 = R_2$ (i.e. Property (2) holds), then, using (a),

$$E|Z_{n_1} - Z'_{n_1}| \le 2KC\,,$$

and the tightness follows by standard arguments (see, e.g., [6]). Otherwise, assume Property (3) with $n_2 > n_1$. By (a'), we can construct a version of Z'_{n_1}, independent of Z_{n_1}, so that $|Z'_{n_2} - Z'_{n_1}| \le K(n_2 - n_1)$. Therefore,

$$E|Z_{n_1} - Z'_{n_1}| \le [-2EZ_{n+1} + EZ_{n_1} + EZ_{n_2}] + 2KC + K(R_1 - R_2).$$

Applying again (a) we get, for some constant C',

$$E|Z_{n_1} - Z'_{n_1}| \le 2KC + K(R_1 - R_2) \le C'K,$$

and as before it is standard that this implies tightness. □

4. Questions

Question 1: Extend the theorem to the lamplighter group over Γ, for any finitely generated group Γ; start with $\mathbb{Z}$.

Question 2: Show that tightness of fluctuations is a quasi-isometric invariant. In particular, show this in the class of Cayley graphs.

Question 3: The lamplighter over $\mathbb{Z}$ is a rather small group among the finitely generated groups with exponential growth. It is solvable, amenable and Liouville. This indicates that all Cayley graphs of exponential growth are tight. We ask then which Cayley graphs admit tightness; is there an infinite Cayley graph, which is not quasi-isometric to $\mathbb{Z}$ or $\mathbb{Z}^2$, for which tightness does not hold? Start with a sub exponential example with tightness or even only variance smaller than on $\mathbb{Z}^2$.

Question 4: (Gabor Pete) Note that requiring (1) only quasi-isometrically (plus the root condition of (2)) does not imply exponential growth, because when one iterates, one may collect a factor (from quasiness) each time, killing the exponential growth. E.g., branch groups like Grigorchuk's group [7], where $G \times G$ is a subgroup of G, may have intermediate growth, see e.g. [11]. This condition is somewhat in the spirit of property (1). Bound the variance for FPP on the Grigorchuk's group.

Maybe ideas related to the one above will be useful in proving at least a sublinear variance?

The last two questions are regarding point to point FPP.

Question 5: We conjecture that in any hyperbolic lattice the point to point FPP fluctuations admit a central limit theorem with variance proportional to the distance. This is motivated by the fact that, due to the Morse lemma, the minimal path will be in a bounded neighborhood of the hyperbolic geodesic, and for cylinders a CLT is known to hold [5].

A related question is the following. Assume that for any pair of vertices in a Cayley graph the variance of point to point FPP is proportional to the distance, is the Cayley graph hyperbolic? Alternatively, what point to point variances can be achieved for Cayley graphs? As pointed out above, the only behavior known is linear in the distance (for $\mathbb{Z}$), the conjectured (and proved in some cases) behavior for $\mathbb{Z}^2$, which is the distance to the power 2/3. Can the bound or proof of theorem 2.1 be adapted to give point to point order 1 variance for $T \times \mathbb{Z}^d$ or $T \times T$ or some other graphs? Are other behaviors possible?

Question 6: In [3], among other things, tightness was proved for point to point FPP between random vertices in the configuration model of random d-regular graph with exponential weights. Does tightness hold for more general weights, or for point to point FPP between random vertices on expanders?

All the questions above are regarding the second order issue of bounding fluctuations. The fundamental fact regarding FPP on $\mathbb{Z}^d$ is the shape theorem, see e.g. [9]. That is, rescale the random FPP metric then the limiting metric space a.s. exists and is $\mathbb{R}^d$ with some deterministic norm. The subadditive ergodic theorem is a key in the proof. We conjecture that FPP on Cayley graph of groups of polynomial growth also admits a shape theorem. What can replace the subadditive ergodic theorem in the proof? Start with

Question 7: Prove a shape theorem for FPP on the Cayley graph of the discrete Heisenberg group.

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