On the Annealed Large Deviation Rate Function for a Multi-Dimensional Random Walk in Random Environment

Jonathon Peterson*　　Ofer Zeitouni†

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Abstract

We derive properties of the rate function in Varadhan's (annealed) large deviation principle for multidimensional, ballistic random walk in random environment, in a certain neighborhood of the zero set of the rate function. Our approach relates the LDP to that of regeneration times and distances. The analysis of the latter is possible due to the i.i.d. structure of regenerations.

1 Introduction and Statement of Main Results

This paper studies annealed large deviations for multidimensional random walks in random environments (RWRE), in the ballistic regime. We will be concerned with nearest neighbor RWRE with uniformly elliptic i.i.d. environments, modeled as follows. Let $\mathcal{E}_d := \{x \in \mathbb{Z}^d : \|x\| = 1\}$ and let $\Omega := (\mathcal{M}(\mathcal{E}_d))^{\mathbb{Z}^d}$, where $\mathcal{M}(\mathcal{E}_d)$ is the space of all probability measures on $\mathcal{E}_d$. Let $\mathcal{F}$ be the σ-field generated by the cylinder sets of Ω, and let P be a probability measure on $(\Omega, \mathcal{F})$. A *random environment* $\omega = \{\omega(x,\cdot)\}_{x\in\mathbb{Z}^d}$ is an Ω-valued random variable with distribution P. Given an environment ω, the *quenched* law P_ω of a RWRE X_n starting at the origin $\mathbf{0}$ is given by

$$P_\omega(X_0 = \mathbf{0}) = 1 \quad \text{and} \quad P_\omega\left(X_{n+1} = x + y | X_n = x\right) = \omega(x,y).$$

The *annealed* (also called the averaged) law $\mathbb{P}$ of a RWRE starting at the origin is defined by

$$\mathbb{P}(\cdot) = \int P_\omega(\cdot) P(d\omega).$$

Expectations with respect to the measures P_ω and $\mathbb{P}$ will be denoted by E_ω and $\mathbb{E}$, respectively.

For the remainder of this paper we will assume that the law on environments is *uniformly elliptic* and *i.i.d.* That is, we will make the following assumptions:

Assumption 1 (Uniformly Elliptic)**.** *There exists an $\varepsilon > 0$ such that $P\left(\omega(0,x) \geq \varepsilon,\ \forall x \in \mathcal{E}_d\right) = 1$.*

Assumption 2 (i.i.d. environments)**.** *The law on environments P is an i.i.d. product measure. That is, $\{\omega(x,\cdot)\}_{x\in\mathbb{Z}^d}$ are i.i.d. under P.*

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*School of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53705. The research of the author was partially supported by NSF grants DMS-0503775 and DMS-0802942 and by a Doctoral Dissertation Fellowship from the University of Minnesota.

†School of Mathematics, University of Minnesota, 206 Church St. SE, Minneapolis, MN 55455 and Faculty of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel. The research of the author was partially supported by NSF grants DMS-0503775 and DMS-0804133, and by a grant from the Israel Science Foundation.

Recall the following classification of laws on the environment that was first introduced in [Zer98].

Definition 1. Let $d(\omega) := E_\omega X_1$ be the drift at the origin of the environment, and let $\mathcal{K}$ be the closure of the convex hull of the support, under P, of all possible drifts. If $\mathbf{0} \in \mathcal{K}$, then P is **nestling**. If P is nestling but $\mathbf{0}$ does not belong to the interior of $\mathcal{K}$, then P is **marginally nestling**. If $\mathbf{0} \notin \mathcal{K}$, then P is **non-nestling**. If $\ell \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ and $\inf_{x\in\mathcal{K}} x \cdot \ell > 0$ then P is **non-nestling in direction** ℓ.

Varadhan has proved the following annealed large deviation principle (LDP) for RWRE.

Theorem 1.1 (Varadhan [Var03])**.** *Let Assumptions 1 and 2 hold. Then, there exists a convex function $H(v)$ such that $\frac{X_n}{n}$ satisfies an annealed large deviation principle with good rate function $H(v)$. That is, for any Borel subset $\Gamma \subset \mathbb{R}^d$, with Γ° denoting its interior and $\overline{\Gamma}$ its closure,*

$$-\inf_{v\in\Gamma^\circ} H(v) \le \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{X_n}{n} \in \Gamma\right) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{X_n}{n} \in \Gamma\right) \le -\inf_{v\in\overline{\Gamma}} H(v).$$

Moreover, the zero set of the rate function $Z := \{v : H(v) = 0\}$ is a single point if P is nestling and a line segment containing the origin if P is non-nestling.

Remark: As shown in [Var03], the conclusion of Theorem 1.1 holds more generally for RWRE with bounded jumps in i.i.d. environments with certain strong uniform ellipticity conditions.

Until recently, other than this description of the zero set of H and the fact that H is convex, no other qualitative properties of the annealed rate function were known. In contrast, much more is known about the qualitative behavior of the annealed rate function when $d = 1$. In [CGZ00], a rather detailed description of the one-dimensional annealed rate function was given. In particular, intervals were identified on which the annealed rate function is strictly convex, and sufficient conditions were given for the annealed rate function to have linear pieces in a neighborhood of the origin.

Recently, Yilmaz [Yil08a], [Yil08b] has made some progress on the understanding of the annealed rate function for multidimensional RWRE and on the distribution of paths leading to large deviations. He has shown that under certain conditions on the environment, there exist regions where the annealed rate function is strictly convex and analytic. In this paper, we provide a different proof of these results, and also provide further information about the annealed rate function when the environment is nestling. In particular, we show in the latter case that there is an open set which has the origin in its boundary and on which the annealed rate function is analytic and 1-homogeneous (that is, $H(cv) = cH(v)$ if v and cv are both in the open set).

Our approach to analyzing the annealed large deviations of multidimensional RWRE utilizes what are known as *regeneration times*. Recall that for $\ell \in S^{d-1} := \{\xi \in \mathbb{R}^d : \|\xi\| = 1\}$ such that $c\ell \in \mathbb{Z}^d$ for some $c > 0$, regeneration times in the direction ℓ may be defined by

$$\tau_1 := \inf\{n > 0 : X_k \cdot \ell < X_n \cdot \ell \le X_m \cdot \ell, \quad \forall k < n, \quad \forall m \ge n\},$$

and

$$\tau_i := \inf\{n > \tau_{i-1} : X_k \cdot \ell < X_n \cdot \ell \le X_m \cdot \ell, \quad \forall k < n, \quad \forall m \ge n\}, \quad \text{for } i > 1.$$

Our final assumption is what is known as Sznitman's condition $\mathbf{T}$. To introduce it, define the event *escape to $+\infty$ in direction ℓ*: $A_\ell := \{\lim_{n\to\infty} X_n \cdot \ell = +\infty\}$.

Assumption 3 (Condition $\mathbf{T}$)**.** *Let $\ell \in S^{d-1}$ be such that $c\ell \in \mathbb{Z}^d$ for some $c > 0$, and such that the following hold. Either P is non-nestling in direction ℓ, or P is nestling and* *(i)* $\mathbb{P}(A_\ell) = 1$.

(ii) *There exists a constant* $C_1 > 0$ *such that*

$$\mathbb{E} \exp \left\{ C_1 \sup_{0 \le n \le \tau_1} \|X_n\| \right\} < \infty.$$

Remarks: 1. When P is non-nestling in direction ℓ, then it is straightforward to check that (i) and (ii) above hold. See Section 2.1 for more information.
2. We require $c\ell \in \mathbb{Z}^d$ for some $c > 0$ in order to allow for a simpler definition of regeneration times that agrees with the one given by Sznitman and Zerner [SZ99] (set $a = \frac{1}{c}$ in the definition of regeneration times in [SZ99]). This restriction is not essential, as [Szn01, Theorem 2.2] implies that Assumption 3 is equivalent to the version of condition **T** given in [Szn01] which does not require that $c\ell \in \mathbb{Z}^d$.

When P is non-nestling or $d \ge 2$, Assumptions 1, 2 and 3 imply a law of large numbers with non-zero limiting velocity (see [Szn01]). That is, there exists a point $\mathrm{v}_P \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ such that

$$\lim_{n \to \infty} \frac{X_n}{n} =: \mathrm{v}_P, \quad \mathbb{P} - a.s. \tag{1}$$

Varadhan's description of the zero-set of the annealed rate function in Theorem [Var03] implies that under Assumptions 1, 2 and 3, if P is non-nestling, then v_P is the unique zero of the annealed rate function, and if P is nestling then $[0, \mathrm{v}_P]$ is the zero set of the annealed rate function. Our main results are the following:

Theorem 1.2. *Let Assumptions 1, 2, and let* P *be non-nestling. Then, the annealed rate function* $H(v)$ *is analytic and strictly convex in a neighborhood* $\mathcal{A}'$ *of* v_P.

Theorem 1.3. *Let Assumptions 1, 2, and 3 hold, let* P *be nestling, and let* $d \ge 2$. *Then, there exists an open set* $\mathcal{A}$ *with the following properties:*

1. *The half open interval* $(0, \mathrm{v}_P] \subset \mathcal{A}$.
2. $\mathcal{A}$ *can be written as the disjoint union* $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^0 \cup \mathcal{A}^-$, *where* $\mathcal{A}^+$ *is open,* $\mathrm{v}_P \in \mathcal{A}^0 \subset \partial \mathcal{A}^+$, *and* $\mathcal{A}^- = \{cv : c \in (0,1), v \in \mathcal{A}^0\}$.
3. *The annealed rate function* $H(v)$ *is strictly convex and analytic on* $\mathcal{A}^+$.
4. *The annealed rate function* $H(v)$ *is analytic and* 1*-homogeneous on* $\mathcal{A}^-$.
5. *The annealed rate function* $H(v)$ *is continuously differentiable on* $\mathcal{A}$.

Remarks: 1. Theorem 1.2 was proved in [Pet08]. Independently, Yilmaz proved in his thesis [Yil08a] Theorem 1.2 and part 3 of Theorem 1.3. (Although stated under Kalikow's condition, which is stronger than Assumption 3, his proof carries over verbatim to the case where only Assumption 3 holds.) While Yilmaz's proof also uses regeneration times, his approach is different in that he does not introduce the rate function associated with regeneration times and distances. (He also depends on Theorem 1.1, although this dependence can probably be eliminated.) In contrast, our approach develops a new proof of (a local version of) the annealed large deviations, independent of Theorem 1.1. This alternative approach also allows one to make explicit an alternative description of the rate function, yielding additional properties of the annealed rate function.
2. Theorem 1.3 still holds when $d = 1$ (with $\mathcal{A}^0 = \mathrm{v}_P$). This follows from the fact that $H(v) = 0$

for all $v \in [0, \mathrm{v}_P]$ and the recent results of Yilmaz [Yil08a], [Yil08b] which show that $H(v)$ is strictly convex and analytic on an open set bordering v_P. Our proof of Theorem 1.3 can be easily modified to also cover the $d = 1$ case, but for simplicity we will restrict ourselves in this paper to $d \geq 2$.

The structure of this paper is as follows: In Section 2 we use the large deviations of regeneration times and distances to define a new function $\bar{J}(v)$. Most of Section 2 is devoted to proving qualitative properties of the function $\bar{J}(v)$. Section 3 provides an easy large deviation lower bound with rate function $\bar{J}(v)$ for both nestling and non-nestling RWRE. Then, in Section 4 we derive matching large deviation upper bounds in a neighborhood of v_P when P is non-nestling and in a neighborhood of $(0, \mathrm{v}_P]$ when P is nestling. The proofs of Theorems 1.2 and 1.3 are then completed by noting that the large deviation upper and lower bounds proved in Sections 3 and 4 imply that $\bar{J}(v) = H(v)$ for v in appropriate subsets of $\mathbb{R}^d$, and thus on these subsets, the annealed rate function $H(v)$ has the same properties that were proved for $\bar{J}(v)$ in Section 2. The Appendix contains a technical lemma on the analyticity of Legendre transforms that is used in Section 2.

2 Regeneration Times and the Rate Function $\bar{J}$

For the remainder of the paper, Assumptions 1, 2 and 3 (with respect to a fixed direction ℓ) will hold. Additionally, if P is nestling, we will assume that $d \geq 2$. The regeneration times τ_i are obviously not stopping times since they depend on the future of the random walk. They do however introduce an i.i.d. structure, described next. Let $D := \{X_n \cdot \ell \geq 0, \ \forall n \geq 0\}$. When $\mathbb{P}(D) > 0$, let $\overline{\mathbb{P}}$ be the annealed law of the RWRE conditioned on the event D (i.e., $\overline{\mathbb{P}}(\,\cdot\,) := \mathbb{P}(\cdot\,|D)$). Expectations under the measure $\overline{\mathbb{P}}$ will be denoted by $\overline{\mathbb{E}}$.

Theorem 2.1 (Sznitman and Zerner [SZ99])**.** *Assume $\mathbb{P}(A_\ell) = 1$, and let τ_i be the regeneration times in direction ℓ. Then $\mathbb{P}(D) > 0$, and*

$$(X_{\tau_1}, \tau_1), (X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \ldots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k), \ldots$$

are independent random variables. Moreover, the above sequence is i.i.d. under $\overline{\mathbb{P}}$.

Remark: The assumption that $\mathbb{P}(A_\ell) = 1$ in Theorem 2.1 is only needed to ensure that $\tau_1 < \infty$. In fact, what is shown in [SZ99] is that $\mathbb{P}(A_\ell) > 0$ implies that $\mathbb{P}(D) > 0$ and that $(X_{\tau_1}, \tau_1), (X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \ldots$ are i.i.d. under $\overline{\mathbb{P}}$.

Since Assumption 3 requires that $\mathbb{P}(A_\ell) = 1$, the conclusion of Theorem 2.1 is valid for the regeneration times τ_i in direction ℓ. A consequence of Theorem 2.1 is the following useful formula for the limiting velocity v_P:

$$\mathrm{v}_P = \lim_{n \to \infty} \frac{X_n}{n} = \frac{\overline{\mathbb{E}} X_{\tau_1}}{\overline{\mathbb{E}} \tau_1}, \qquad \mathbb{P} - a.s. \tag{2}$$

Since either P is non-nestling or $d \geq 2$ and condition **T** was assumed, it is known that $\overline{\mathbb{E}} \tau_1 < \infty$ and thus $\mathrm{v}_P \cdot \ell > 0$. In fact, P non-nestling or $d \geq 2$ imply that $\overline{\mathbb{E}} \tau_1^\gamma < \infty$ for all $\gamma < \infty$ (see [Szn00, Theorem 2.1] and [Szn01, Theorem 3.4]).

Under the measure $\overline{\mathbb{P}}$, $(X_{\tau_k}, \tau_k) = (X_{\tau_1}, \tau_1) + \sum_{i=2}^k (X_{\tau_i} - X_{\tau_{i-1}}, \tau_i - \tau_{i-1})$ is the sum of i.i.d. random variables. Therefore, a generalization of Cramér's Theorem [DZ98, Theorem 6.1.3] implies that $\frac{1}{n}(X_{\tau_n}, \tau_n)$ satisfies a weak large deviation principle under $\overline{\mathbb{P}}$ with convex rate function

$$\bar{I}(x, t) := \inf_{\eta \in \mathbb{R}^d, \ \lambda \in \mathbb{R}} \left[(\eta, \lambda) \cdot (x, t) - \overline{\Lambda}(\eta, \lambda) \right],$$

where $\cdot$ denotes inner product and

$$\overline{\Lambda}(\eta,\lambda) := \log \overline{\mathbb{E}} e^{(\eta,\lambda)\cdot(X_{\tau_1},\tau_1)} \qquad \text{for } \eta \in \mathbb{R}^d,\ \lambda \in \mathbb{R}.$$

In particular, for any open, convex subset $G \subset \mathbb{R}^{d+1}$,

$$\lim_{k\to\infty} \frac{1}{k} \log \overline{\mathbb{P}} \left(\frac{1}{k}(X_{\tau_k}, \tau_k) \in G \right) = - \inf_{(x,t)\in G} \bar{I}(x,t). \tag{3}$$

Let $H_\ell := \{v \in \mathbb{R}^d : v \cdot \ell > 0\}$. Then, for $v \in H_\ell$, let

$$\bar{J}(v) := \inf_{0<s\le 1} s\bar{I}\left(\frac{v}{s}, \frac{1}{s}\right).$$

Our goal is to show that $\bar{J}(v) = H(v)$, at least for certain $v \in H_\ell$. The reasoning behind this is as follows. We assume that when $X_n \approx nv$ for some $v \in H_\ell$, the regeneration times occur in a somewhat regular manner (that is, there are no extremely large regeneration times). If this is the case, then it should be true that

$$\mathbb{P}(X_n \approx nv) \approx \overline{\mathbb{P}}(\tau_k \approx n,\ X_{\tau_k} \approx nv), \qquad \text{for } k = sn. \tag{4}$$

However, the large deviations of $(X_{\tau_k}, \tau_k)/k$ imply that the latter probability is approximately $\exp\{-ns\bar{I}\left(\frac{v}{s}, \frac{1}{s}\right)\}$. The optimal s for which (4) would hold must be the s which minimizes $s\bar{I}\left(\frac{v}{s}, \frac{1}{s}\right)$.

The main difficulty in making the above heuristic argument precise comes in proving that there are no extremely long regeneration times when $X_n \approx nv$. In Section 4 we resolve this difficulty for certain v by showing that the least costly way to obtain a large deviation of $X_n \approx nv$ is to have all the regeneration times or distances relatively small.

Having defined the function $\bar{J}$, we now mention a few of basic properties.

Lemma 2.2. *$\bar{J}$ is a convex function on H_ℓ, and $\bar{J}(\mathrm{v}_P) = 0$.*

Proof. For $s \in (0,1]$ and $v \in H_\ell$, let

$$f(v,s) := s\bar{I}\left(\frac{v}{s}, \frac{1}{s}\right) = \sup_{\eta\in\mathbb{R}^d,\ \lambda\in\mathbb{R}} (\eta,\lambda)\cdot(v,1) - s\overline{\Lambda}(\eta,\lambda).$$

Since $f(\cdot,\cdot)$ is the supremum of a family of linear functions, $f(\cdot,\cdot)$ is a convex function on $H_\ell \times (0,1]$. Therefore, $\bar{J}(\cdot) = \inf_{s\in(0,1]} f(\cdot, s)$ is a convex function on H_ℓ.

For the second part of the lemma, note that (2) implies that $\overline{\mathbb{E}} X_{\tau_1} = \mathrm{v}_P \overline{\mathbb{E}}\tau_1$. Then, the law of large numbers and (3) imply that $\bar{I}(\mathrm{v}_P\overline{\mathbb{E}}\tau_1, \overline{\mathbb{E}}\tau_1) = 0$. The definition of $\bar{J}$ and the fact that $\bar{I}$ is non-negative imply that $\bar{J}(\mathrm{v}_P) = 0$. $\square$

We next evaluate some derivatives of $\bar{J}$. For any function $g : \mathbb{R}^k \to (-\infty, \infty]$, let $\mathcal{D}_g = \{z \in \mathbb{R}^k : g(z) < \infty\}$ denote the *domain* of g, and let $\mathcal{D}_g^\circ$ denote its interior.

Lemma 2.3. *Assume that v_0 and s_0 are such that $\bar{J}(v_0) = s_0\bar{I}\left(\frac{v_0}{s_0}, \frac{1}{s_0}\right)$ and $(v_0/s_0, 1/s_0) \in \mathcal{D}_{\bar{I}}^\circ$. Then,*

$$\frac{\partial \bar{J}}{\partial v_i}(v_0) = \frac{\partial \bar{I}}{\partial x_i}\left(\frac{v_0}{s_0}, \frac{1}{s_0}\right).$$

Proof. Since (X_{τ_1}, τ_1) is, by Assumption 1, a non-degenerate $d+1$-dimensional random variable, $\overline{\Lambda}(\eta, \lambda)$ is a strictly convex function on $\mathcal{D}_{\overline{\Lambda}}$. Since $\bar{I}$ is the Legendre transform of $\overline{\Lambda}$, this implies that $\bar{I}(x,t)$ is continuously differentiable in $\mathcal{D}_{\bar{I}}^{\circ}$ (see [Roc70, Theorem 26.3]). Therefore, $f(v,s)$ is continuously differentiable in the interior of $\mathcal{D}_f = \{(v,s) : (v/s, 1/s) \in \mathcal{D}_I\}$.

Since $(v_0, s_0) \in \mathcal{D}_f^{\circ}$, we have $\frac{\partial f}{\partial s}(v_0, s_0) = 0$. Also, since $f(v,s)$ is convex as a function of (v,s),

$$f(v_0 + he_i, s) \geq f(v_0, s_0) + \nabla f(v_0, s_0) \cdot (he_i, s - s_0) = f(v_0, s_0) + \frac{\partial f}{\partial v_i}(v_0, s_0) h,$$

where in the second equality we used that $\frac{\partial f}{\partial s}(v_0, s_0) = 0$. Since the right side of the above equation does not depend on s, we have

$$\bar{J}(v_0 + he_i) \geq f(v_0, s_0) + \frac{\partial f}{\partial v_i}(v_0, s_0) h. \tag{5}$$

On the other hand, a Taylor expansion of f near (v_0, s_0) implies that

$$\bar{J}(v_0 + he_i) \leq f(v_0 + he_i, s_0) = f(v_0, s_0) + \frac{\partial f}{\partial v_i}(v_0, s_0) h + o(h). \tag{6}$$

Recalling that $\bar{J}(v_0) = f(v_0, s_0)$, (5) and (6) imply that $\frac{\partial \bar{J}}{\partial v_i}(v_0, s_0) = \frac{\partial f}{\partial v_i}(v_0, s_0)$. The proof is completed by noting that the definition of $f(v,s)$ implies that $\frac{\partial f}{\partial v_i}(v,s) = \frac{\partial \bar{I}}{\partial x_i}(v/s, 1/s)$. $\square$

We now prove some more detailed properties of the function $\bar{J}(v)$ in the non-nestling and nestling cases, respectively. In particular, for certain v we are able to identify the minimizing s in the definition of $\bar{J}(v)$, and we are able to determine certain differentiability properties of $\bar{J}$.

2.1 Properties of $\bar{J}$ - Non-nestling Case

When P is non-nestling in direction ℓ, the regeneration time τ_1 has exponential tails [Szn00, Theorem 2.1]. That is, Sznitman proved that there exists a constant $C_2 > 0$ such that

$$\overline{\mathbb{E}} e^{C_2 \tau_1} < \infty. \tag{7}$$

Let

$$\mathcal{C} := \{\eta \in \mathbb{R}^d : \|\eta\| < C_2/2\}.$$

If $\eta \in \mathcal{C}$, then $-C_2\tau_1/2 \leq \eta \cdot X_{\tau_1} < C_2 \tau_1/2$ since $\|X_{\tau_1}\| \leq \tau_1$. Thus,

$$1 = \overline{\mathbb{E}} e^{-C_2\tau_1/2 + C_2\tau_1/2} < \overline{\mathbb{E}} e^{\eta \cdot X_{\tau_1} + C_2\tau_1/2} < \overline{\mathbb{E}} e^{C_2\tau_1/2 + C_2\tau_1/2} < \infty,$$

and so $\overline{\Lambda}(\eta, C_2/2) \in (0, \infty)$ for all $\eta \in \mathcal{C}$. Since $\overline{\Lambda}(\eta, \lambda)$ is strictly increasing in λ and since $\lim_{\lambda \to -\infty} \overline{\Lambda}(\eta, \lambda) = -\infty$, we may define a function $\lambda(\eta)$ on $\mathcal{C}$ by

$$\lambda(\eta) \text{ is the unique solution to } \overline{\Lambda}(\eta, \lambda(\eta)) = 0, \quad \forall \eta \in \mathcal{C}.$$

Since $\overline{\Lambda}$ is analytic in a neighborhood of $(\eta, \lambda(\eta))$ for any $\eta \in \mathcal{C}$, a version of the implicit function theorem [FG02, Theorem 7.6] implies that $\lambda(\eta)$ is analytic as a function of $\eta \in \mathcal{C}$. Differentiating the equality $\overline{\Lambda}(\eta, \lambda(\eta)) = 0$, we obtain that

$$\nabla \lambda(\eta) = -\frac{\overline{\mathbb{E}} X_{\tau_1} e^{\eta \cdot X_{\tau_1} + \lambda(\eta)\tau_1}}{\overline{\mathbb{E}} \tau_1 e^{\eta \cdot X_{\tau_1} + \lambda(\eta)\tau_1}}.$$

This is useful in the proof of the following lemma.

Lemma 2.4. *Let P be a non-nestling law on environments, and let $\mathcal{A} := -\nabla\lambda(\mathcal{C}) = \{-\nabla\lambda(\eta) : \eta \in \mathcal{C}\}$. Then, $\mathrm{v}_P \in \mathcal{A}$ and $\bar{J}$ is analytic and strictly convex on the open set $\mathcal{A}$. Moreover, if*

$$v_0 = -\nabla\lambda(\eta_0) = \frac{\overline{\mathbb{E}} X_{\tau_1} e^{\eta_0 \cdot X_{\tau_1} + \lambda(\eta_0)\tau_1}}{\overline{\mathbb{E}} \tau_1 e^{\eta_0 \cdot X_{\tau_1} + \lambda(\eta_0)\tau_1}} \qquad \text{and} \qquad s_0 = \frac{1}{\overline{\mathbb{E}} \tau_1 e^{\eta_0 \cdot X_{\tau_1} + \lambda(\eta_0)\tau_1}} \tag{8}$$

for some $\eta_0 \in \mathcal{C}$, then

$$\bar{J}(v_0) = s_0 \bar{I}\left(\frac{v_0}{s_0}, \frac{1}{s_0}\right), \qquad \text{and} \qquad \nabla \bar{J}(v_0) = \eta_0,$$

and s_0 is the unique value of s which attains the minimum in the definition of $\bar{J}(v_0)$.

Proof. Due to uniform ellipticity (Assumption 1), $\overline{\Lambda}$ is strictly convex, and thus $\overline{\Lambda}(\eta, \lambda(\eta)) = 0$ implies that $\lambda(\eta)$ is strictly concave as a function of η. Therefore, $\nabla\lambda(\eta)$ is a one-to-one function on $\mathcal{C}$. Thus, $\mathcal{A}$ is an open set, and $\mathrm{v}_P = \overline{\mathbb{E}} X_{\tau_1} / \overline{\mathbb{E}} \tau_1 = -\nabla\lambda(\mathbf{0}) \in \mathcal{A}^\circ$.

Since $\overline{\Lambda}$ is analytic and strictly convex in $\mathcal{D}_{\overline{\Lambda}}^\circ$ and $\bar{I}$ is the Legendre transform of $\overline{\Lambda}$, we have that $\bar{I}$ is analytic and strictly convex in the interior of $\mathcal{D}'_{\overline{\Lambda}} = \nabla\overline{\Lambda}(\mathcal{D}_{\overline{\Lambda}})$ (see Lemma A.1 in Appendix A). Moreover, for any $(\eta, \lambda) \in \mathcal{D}_{\overline{\Lambda}}^\circ$,

$$\bar{I}\left(\nabla\overline{\Lambda}(\eta,\lambda)\right) = (\eta,\lambda) \cdot \nabla\overline{\Lambda}(\eta,\lambda) - \overline{\Lambda}(\eta,\lambda), \qquad \text{and} \qquad \nabla\bar{I}\left(\nabla\overline{\Lambda}(\eta,\lambda)\right) = (\eta,\lambda). \tag{9}$$

Letting v_0 and s_0 be defined as in (8), we have that $(v_0/s_0, 1/s_0) = \nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0))$. Recalling the definition of $f(v,s)$, we obtain that

$$\frac{\partial f}{\partial s}(v,s) = \bar{I}\left(\frac{v}{s}, \frac{1}{s}\right) - \nabla\bar{I}\left(\frac{v}{s}, \frac{1}{s}\right) \cdot \left(\frac{v}{s}, \frac{1}{s}\right).$$

Therefore,

$$\begin{aligned}
\frac{\partial f}{\partial s}(v_0, s_0) &= \bar{I}\left(\frac{v_0}{s_0}, \frac{1}{s_0}\right) - \nabla\bar{I}\left(\frac{v_0}{s_0}, \frac{1}{s_0}\right) \cdot \left(\frac{v_0}{s_0}, \frac{1}{s_0}\right) \\
&= \bar{I}\left(\nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0))\right) - \nabla\bar{I}\left(\nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0))\right) \cdot \nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0)) \\
&= \bar{I}\left(\nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0))\right) - (\eta_0, \lambda(\eta_0)) \cdot \nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0)) \\
&= -\overline{\Lambda}(\eta_0, \lambda(\eta_0)) = 0,
\end{aligned}$$

where the second and third equalities follow from (9). Since $f(v,s)$ is convex as a function of (v,s), it follows that $\bar{J}(v_0) = f(v_0, s_0) = s_0 \bar{I}(v_0/s_0, 1/s_0)$.

Now, with $D^2\bar{I}$ denoting the Hessian of $\bar{I}$,

$$\frac{\partial^2 f}{\partial s^2}(v,s) = \frac{1}{s^3}(v,1) \cdot D^2\bar{I}\left(\frac{v}{s}, \frac{1}{s}\right)(v,1)^t.$$

Since $\bar{I}(x,t)$ is strictly convex in a neighborhood of $\nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0)) = (v_0/s_0, 1/s_0)$, $D^2\bar{I}(x,t)$ is strictly positive definite in a neighborhood of $(v_0/s_0, 1/s_0)$. Thus $\frac{\partial^2 f}{\partial s^2}(v_0, s_0) > 0$, and because $f(v,s)$ is analytic in a neighborhood of (v_0, s_0), another use of the implicit function theorem [FG02, Theorem 7.6] implies that there exists an analytic function $s(v)$ in a neighborhood of v_0 such that $s(v_0) = s_0$ and $\frac{\partial f}{\partial s}(v, s(v)) = 0$. Thus, $\bar{J}(v) = f(v, s(v))$, and therefore $\bar{J}(v)$ is analytic in a neighborhood of v_0. Moreover, since $\frac{\partial^2 f}{\partial s^2}(v_0, s_0) > 0$, s_0 is the unique value of s obtaining the minimum in the definition of $\bar{J}(v_0)$.

Since $\bar{J}(v) = f(v, s(v))$ in a neighborhood of v_0, $\bar{J}$ is strictly convex in a neighborhood of v_0 if $f(v,s)$ is strictly convex in a neighborhood of (v_0, s_0). To see that $f(v,s)$ is strictly convex in a neighborhood of (v_0, s_0), note that the definition of $f(v,s)$ implies that for $z \in \mathbb{R}^d$ and $w \in \mathbb{R}$,

$$(z,w)^t \cdot D^2 f(v,s) \cdot (z,w) = \frac{1}{s}\left(z - \frac{w}{s}v, \frac{-w}{s}\right) \cdot D^2 \bar{I}\left(\frac{v}{s}, \frac{1}{s}\right) \cdot \left(z - \frac{w}{s}v, \frac{-w}{s}\right)^t.$$

Since $D^2\bar{I}(x,t)$ is strictly positive definite in a neighborhood of $(v_0/s_0, 1/s_0)$, this implies that $D^2 f(v,s)$ is strictly positive definite in a neighborhood of (v_0, s_0), and thus $f(v,s)$ is strictly convex in a neighborhood of (v_0, s_0).

Finally, since $\bar{J}(v_0) = f(v_0, s_0)$ and $(v_0/s_0, 1/s_0) = \nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0)) \in \mathcal{D}^\circ_{\bar{I}}$, Lemma 2.3 implies that

$$\nabla \bar{J}(v_0) = \left(\frac{\partial \bar{I}}{\partial x_i}\left(\frac{v_0}{s_0}, \frac{1}{s_0}\right)\right)_{i=1}^d.$$

However, since $\nabla\bar{I}(v_0/s_0, 1/s_0) = \nabla\bar{I}(\nabla\overline{\Lambda}(\eta_0, \lambda(\eta_0))) = (\eta_0, \lambda(\eta_0))$, we obtain that $\nabla\bar{J}(v_0) = \eta_0$. $\square$

2.2 Properties of $\bar{J}$ - Nestling Case

In this subsection, we will assume that P is nestling, $d \geq 2$, and Assumptions 1, 2, and 3 hold.

Lemma 2.5. *If P is nestling, then $\overline{\Lambda}(\eta, \lambda) = \infty$ for any $\lambda > 0$.*

Proof. Sznitman has shown [Szn00, Theorem 2.7] that when Assumptions 1, 2, and 3 hold and P is nestling and not marginally nestling, then

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}(\tau_1 > n)}{(\log n)^d} > -\infty. \tag{10}$$

Sznitman proves (10) by constructing a "trap" of radius $\log n$ around the origin and then forcing the random walk to stay in the trap for at least the first n steps of the walk. If instead we construct the trap centered around a point near $(\log n)\ell$, then we can adapt Sznitman's argument (using Assumption 1) to show that when P is nestling but not marginally nestling,

$$\liminf_{n\to\infty} \frac{\log \overline{\mathbb{P}}(\tau_1 > n)}{(\log n)^d} > \liminf_{n\to\infty} \frac{\log \overline{\mathbb{P}}(\tau_1 > n,\ \|X_{\tau_1}\| < 3\log n)}{(\log n)^d} > -\infty. \tag{11}$$

In the marginally nestling case, we get immediately by approximating a marginally nestling walk by a nestling walk for the first n step (at exponential cost $e^{-\varepsilon n}$), that for any $\varepsilon > 0$,

$$\liminf_{n\to\infty} \frac{\log \overline{\mathbb{P}}(\tau_1 > n,\ \|X_{\tau_1}\| < 3\log n)}{n} > -\varepsilon. \tag{12}$$

The statement of the lemma follows easily from (11) and (12). $\square$

For any $\eta \in \mathbb{R}^d$, let

$$\overline{\Lambda}_X(\eta) = \overline{\Lambda}(\eta, 0) = \log \overline{\mathbb{E}} e^{\eta \cdot X_{\tau_1}}. \tag{13}$$

If P is non-nestling, recall the constant C_1 in Assumption 3, and define the following subsets of $\mathbb{R}^d$:

$$\mathcal{C} = \{\eta \in \mathbb{R}^d : \|\eta\| < C_1\}, \qquad \mathcal{C}^+ = \mathcal{C} \cap \{\overline{\Lambda}_X(\eta) > 0\} \qquad \text{and} \qquad \mathcal{C}^0 = \mathcal{C} \cap \{\overline{\Lambda}_X(\eta) = 0\}.$$

As in the non-nestling case, for any $\eta \in \mathcal{C}^+ \cup \mathcal{C}^0$, let $\lambda(\eta)$ be the unique solution to $\overline{\Lambda}(\eta, \lambda(\eta)) = 0$. (Lemma 2.5 implies that $\overline{\Lambda}(\eta, \lambda(\eta)) = 0$ does not have a solution when $\eta \in \mathcal{C} \backslash (\mathcal{C}^+ \cup \mathcal{C}^0)$). Note that $\lambda(\eta)$ is analytic and strictly concave on $\mathcal{C}^+$, and that $\lambda(\eta) = 0$ for all $\eta \in \mathcal{C}^0$. Define

$$\gamma(\eta) := \frac{\overline{\mathbb{E}} X_{\tau_1} e^{\eta \cdot X_{\tau_1} + \lambda(\eta)\tau_1}}{\overline{\mathbb{E}} \tau_1 e^{\eta \cdot X_{\tau_1} + \lambda(\eta)\tau_1}},$$

so that $\gamma(\eta) = -\nabla\lambda(\eta)$ for $\eta \in \mathcal{C}^+$, and $\gamma(\eta)$ is continuous as a function of η. Also, since $\lambda(\eta)$ is strictly concave as a function of η in $\mathcal{C}^+$, then $\gamma(\eta)$ must be a one-to-one function. Let

$$\mathcal{A}^+ := \gamma(\mathcal{C}^+) = \{\gamma(\eta) : \eta \in \mathcal{C}^+\}, \qquad \text{and} \qquad \mathcal{A}^0 := \gamma(\mathcal{C}^0) = \{\gamma(\eta) : \eta \in \mathcal{C}^0\}.$$

Then $\mathcal{A}^+$ is an open subset, and since $\mathrm{v}_P = \overline{\mathbb{E}} X_{\tau_1} / \overline{\mathbb{E}} \tau_1 = \gamma(\mathbf{0})$, then $\mathrm{v}_P \in \mathcal{A}^0 \subset \partial\mathcal{A}^+$.

Lemma 2.6. *Let P be nestling. Then, $\bar{J}$ is analytic and strictly convex on the open set $\mathcal{A}^+$. Moreover, if*

$$v_0 = \gamma(\eta_0) = \frac{\overline{\mathbb{E}} X_{\tau_1} e^{\eta_0 \cdot X_{\tau_1} + \lambda(\eta_0)\tau_1}}{\overline{\mathbb{E}} \tau_1 e^{\eta_0 \cdot X_{\tau_1} + \lambda(\eta_0)\tau_1}}, \qquad \text{and} \qquad s_0 = \frac{1}{\overline{\mathbb{E}} \tau_1 e^{\eta_0 \cdot X_{\tau_1} + \lambda(\eta_0)\tau_1}} \tag{14}$$

for some $\eta_0 \in \mathcal{C}^+$, then

$$\bar{J}(v_0) = s_0 \bar{I}\left(\frac{v_0}{s_0}, \frac{1}{s_0}\right), \qquad \text{and} \qquad \nabla J(v_0) = \eta_0,$$

and s_0 is the unique value of s which attains the minimum in the definition of $\bar{J}(v_0)$.

Proof. The proof is exactly the same as the proof of Lemma 2.4, and follows from the fact that $\overline{\Lambda}(\eta, \lambda(\eta)) = 0$ for $\eta \in \mathcal{C}^+$ and the fact that $\overline{\Lambda}(\eta, \lambda)$ is analytic and strictly convex in a neighborhood of $(\eta_0, \lambda(\eta_0))$ for any $\eta_0 \in \mathcal{C}^+$. $\square$

Since the sequence $X_{\tau_1}, X_{\tau_2} - X_{\tau_1}, \ldots$ is i.i.d. under $\overline{\mathbb{P}}$, Cramér's Theorem [DZ98, Theorem 6.1.3] implies that X_{τ_k}/k satisfies a large deviation principle under the measure $\overline{\mathbb{P}}$ with rate function $\bar{I}_1(x)$ given by

$$\bar{I}_1(x) = \sup_{\eta \in \mathbb{R}^d} \left[\eta \cdot x - \overline{\Lambda}_X(\eta)\right].$$

Lemma 2.7. *$\bar{I}_1(x) \leq \inf_{t \in \mathbb{R}} \bar{I}(x, t)$.*

Proof. The large deviation lower bound (3) for $(X_{\tau_n}/n, \tau_n/n)$ implies that

$$\liminf_{n \to \infty} \frac{1}{n} \log \overline{\mathbb{P}}(\|X_{\tau_n} - \xi n\| < \delta n) \geq -\inf_{\|x - \xi\| < \delta, t \in \mathbb{R}} \bar{I}(x, t).$$

On the other hand, the large deviation upper bound for X_{τ_n}/n implies that

$$\limsup_{n \to \infty} \frac{1}{n} \log \overline{\mathbb{P}}(\|X_{\tau_n} - \xi n\| < \delta n) \leq -\inf_{\|x - \xi\| \leq \delta} \bar{I}_1(x).$$

The above two inequalities and the lower semicontinuity of $\bar{I}$ and $\bar{I}_1$ imply that $\bar{I}_1(x) \leq \inf_{t \in \mathbb{R}} \bar{I}(x, t)$. $\square$

As mentioned above, when $d \geq 2$, Assumptions 1, 2, and 3 imply that $\overline{\mathbb{E}}\tau_1^p < \infty$ for all $p < \infty$. Then, for any $\eta \in \mathcal{C}$, by choosing p large enough so that $\|\eta\| < \frac{p-1}{p} C_1$ we have that

$$\overline{\mathbb{E}}\tau_1 e^{\eta\cdot X_{\tau_1}} \leq \left(\overline{\mathbb{E}}\tau_1^p\right)^{1/p} \left(\overline{\mathbb{E}} e^{p/(p-1)\eta\cdot X_{\tau_1}}\right)^{(p-1)/p} < \infty.$$

Then, for $\eta \in \mathcal{C}$, let $h(\eta) := \frac{\overline{\mathbb{E}}\tau_1 e^{\eta\cdot X_{\tau_1}}}{\overline{\mathbb{E}} e^{\eta\cdot X_{\tau_1}}}$, so that $\nabla\overline{\Lambda}(\eta, 0) = \left(\nabla\overline{\Lambda}_X(\eta), h(\eta)\right)$ (where the derivatives with respect to λ are one sided derivatives as $\lambda \to 0^-$).

Lemma 2.8. *If $x = \nabla\overline{\Lambda}_X(\eta)$ for some $\eta \in \mathcal{C}$, then $\bar{I}(x,t) = \bar{I}_1(x)$ for all $t \geq h(\eta)$.*

Proof. Since $\nabla\overline{\Lambda}(\eta, 0) = (x, h(\eta))$, we have using (13) that

$$\bar{I}(x, h(\eta)) = (x, h(\eta)) \cdot (\eta, 0) - \overline{\Lambda}(\eta, 0) = x \cdot \eta - \overline{\Lambda}_X(\eta).$$

Similarly, $\nabla\overline{\Lambda}_X(\eta) = x$ implies that $\bar{I}_1(x) = x \cdot \eta - \overline{\Lambda}_X(\eta)$. Thus, $\bar{I}(x, h(\eta)) = \bar{I}_1(x)$.

If $t > h(\eta)$, then since Lemma 2.5 implies that $\overline{\Lambda}(\eta, \lambda) = \infty$ for any $\lambda > 0$,

$$\begin{aligned}
\bar{I}(x,t) &= \sup_{\eta\in\mathbb{R}^d,\ \lambda\leq 0} (x,t)\cdot(\eta,\lambda) - \overline{\Lambda}(\eta,\lambda) \\
&\leq \sup_{\eta\in\mathbb{R}^d,\ \lambda\leq 0} (x,h(\eta))\cdot(\eta,\lambda) - \overline{\Lambda}(\eta,\lambda) \\
&= \bar{I}(x, h(\eta)) = \bar{I}_1(x).
\end{aligned}$$

This, along with Lemma 2.7 implies that $\bar{I}(x,t) = \bar{I}_1(x)$ for all $t \geq h(\eta)$. □

Let $\mathcal{A}^- := \{\theta v : \theta \in (0,1),\ v \in \mathcal{A}^0\}$. In [Yil08b] (proof of Theorem 3, bottom of page 7), Yilmaz shows that the unit vector $\hat{n}$ normal to $\partial\mathcal{A}^+$ (pointing into $\mathcal{A}^+$) at v_P satisfies $\hat{n} \cdot \mathrm{v}_P > 0$. In fact, this argument gives that for any $v_0 \in \mathcal{A}^0$ the unit vector $\hat{n}_0$ normal to $\partial\mathcal{A}^+$ (pointing into $\mathcal{A}^+$) at v_0 satisfies $\hat{n}_0 \cdot v_0 > 0$. This implies that $\mathcal{A}^-$ is an open set and that $\mathcal{A}^-$ and $\mathcal{A}^0$ are disjoint.

Remark: The above referenced argument of Yilmaz on the shape of $\mathcal{A}^+$ appears in a different form in [Yil08a] than it does here. Yilmaz defines a function $\Lambda_a(\eta)$ to be the Legendre transform of the large deviation rate function $H(v)$. He then shows that the equality $\overline{\Lambda}(\eta, -\Lambda_a(\eta)) = 0$ holds for all $\eta \in \mathcal{C}^+$. Note that our definition of $\lambda(\eta)$ implies that $\Lambda_a(\eta) = -\lambda(\eta)$ for all $\eta \in \mathcal{C}^+$, and thus

$$\mathcal{A}^+ = \{-\nabla\lambda(\eta) : \eta \in \mathcal{C}^+\} = \{\nabla\Lambda_a(\eta) : \eta \in \mathcal{C}^+\}.$$

Since Yilmaz's proof of the properties of the normal vectors at points in $\mathcal{A}^0$ only uses the fact that $\overline{\Lambda}(\eta, -\Lambda_a(\eta)) = 0$, it may be repeated here with $-\lambda(\eta)$ in place of $\Lambda_a(\eta)$.

We wish to identify the shape of the function $\bar{J}$ on the set $\mathcal{A}^-$ as well. For this we first need the following lemma.

Lemma 2.9. *Let $\bar{J}_1(v) := \inf_{s>0} s\bar{I}_1(v/s)$. Then $\bar{J}_1(v) \leq \bar{J}(v)$ for all v, and $\bar{J}_1(cv) = c\bar{J}_1(v)$ for all $c > 0$. Moreover, if $v_0 = \gamma(\eta_0)$ for some $\eta_0 \in \mathcal{C}^0$ and $c > 0$, then $\bar{J}_1(v)$ is analytic in a neighborhood of cv_0.*

Proof. Since $\bar{I}_1(x) \leq \inf_t \bar{I}(x,t)$, it follows immediately from the definitions of $\bar{J}$ and $\bar{J}_1$ that $\bar{J}_1(v) \leq \bar{J}(v)$. Also, if $c > 0$, then

$$\bar{J}_1(cv) = \inf_{s>0} s\bar{I}_1\left(\frac{cv}{s}\right) = c\inf_{s>0}(s/c)\bar{I}_1\left(\frac{v}{s/c}\right) = c\inf_{s'>0} s'\bar{I}_1\left(\frac{v}{s'}\right) = c\bar{J}_1(v).$$

Let $f_1(v,s) := s\bar{I}_1(v/s)$, so that $\bar{J}_1(v) = \inf_{s>0} f_1(v,s)$. Since $\bar{I}_1$ is a convex function, $f_1(v,s)$ is a convex function of (v,s). Let $v_0 \in \mathcal{A}^0$ so that $v_0 = \gamma(\eta_0) = \frac{\nabla\overline{\Lambda}_X(\eta_0)}{h(\eta_0)}$ for some $\eta_0 \in \mathcal{C}^0$. As in the proof of Lemma 2.4, to show that $\bar{J}_1$ is analytic in a neighborhood of v_0, by the implicit function theorem it is enough to show that there exists an s_0 such that $f(v,s)$ is analytic in a neighborhood of (v_0,s_0), $\frac{\partial f_1}{\partial s}(v_0,s_0) = 0$, and $\frac{\partial^2 f_1}{\partial s^2}(v_0,s_0) \neq 0$. If $s_0 = \frac{1}{h(\eta_0)}$, then $v_0/s_0 = \nabla\overline{\Lambda}_X(\eta_0)$. Since $\overline{\Lambda}_X$ is analytic and strictly convex in a neighborhood of η_0, it follows that $\bar{I}_1(x)$ is analytic and strictly convex in a neighborhood of $v_0/s_0 = \nabla\overline{\Lambda}_X(\eta_0)$ (see Lemma A.1 in the Appendix). Thus, $f_1(v,s)$ is analytic in a neighborhood of (v_0,s_0). When $\bar{I}_1$ is twice differentiable at v/s, then

$$\frac{\partial f_1}{\partial s}(v,s) = \bar{I}_1\left(\frac{v}{s}\right) - \nabla\bar{I}_1\left(\frac{v}{s}\right)\cdot\left(\frac{v}{s}\right), \tag{15}$$

and

$$\frac{\partial^2 f_1}{\partial s^2}(v,s) = \frac{1}{s^3} v\cdot D^2\bar{I}_1\left(\frac{v}{s}\right)\cdot v^t. \tag{16}$$

Since $\nabla\overline{\Lambda}_X(\eta_0) = v_0/s_0$, we obtain that $\bar{I}_1(v_0/s_0) = \eta_0\cdot(v_0/s_0) - \overline{\Lambda}_X(\eta_0)$ and thus (15) implies

$$\frac{\partial f_1}{\partial s}(v_0,s_0) = \bar{I}_1\left(\frac{v_0}{s_0}\right) - \eta_0\cdot\left(\frac{v_0}{s_0}\right) = -\overline{\Lambda}_X(\eta_0) = 0, \tag{17}$$

where the last equality is because $\eta_0 \in \mathcal{C}^0$. Also, since $\bar{I}(x)$ is strictly convex in a neighborhood of v_0/s_0, $D^2\bar{I}_1(v_0/s_0)$ is strictly positive definite, and thus (16) implies that $\frac{\partial^2 f_1}{\partial s^2}(v_0,s_0) > 0$. Therefore, $\bar{J}_1$ is analytic in a neighborhood of v_0. Since $\bar{J}_1(cv) = c\bar{J}_1(v)$ for all $c>0$, this implies that $\bar{J}_1$ is also analytic in a neighborhood of cv_0 for any $c>0$. $\square$

Lemma 2.10. *Let P be nestling. Then, $\bar{J}(v) = \bar{J}_1(v)$ for all $v \in \mathcal{A}^0 \cup \mathcal{A}^-$, and so $\bar{J}(v)$ is analytic and homogeneous on the open set $\mathcal{A}^-$. Moreover, if for some $\eta_0 \in \mathcal{C}^0$,*

$$v_0 = \gamma(\eta_0) = \frac{\overline{\mathbb{E}}X_{\tau_1}e^{\eta_0\cdot X_{\tau_1}}}{\overline{\mathbb{E}}\tau_1 e^{\eta_0\cdot X_{\tau_1}}} \qquad \textit{and} \qquad s_0 = \frac{1}{h(\eta_0)} = \frac{1}{\overline{\mathbb{E}}\tau_1 e^{\eta_0\cdot X_{\tau_1}}}, \tag{18}$$

then for any $\theta \in (0,1]$,

$$\bar{J}(\theta v_0) = \theta s_0 \bar{I}_1\left(\frac{v_0}{s_0}\right) = \theta s_0 \bar{I}\left(\frac{v_0}{s_0}, \frac{1}{\theta s_0}\right), \qquad \nabla\bar{J}(\theta v_0) = \eta_0,$$

and θs_0 is the unique value of s which attains the minimum in the definition of $\bar{J}(\theta v_0)$.

Proof. Let v_0 and s_0 be defined as in (18) for some $\eta_0 \in \mathcal{C}^0$. Recalling that $f_1(v,s) = s\bar{I}_1(v/s)$, then

$$\frac{\partial f_1}{\partial s}(\theta v_0, \theta s_0) = \frac{\partial f_1}{\partial s}(v_0,s_0) = 0,$$

where the first equality holds because $\frac{\partial f_1}{\partial s}(v,s)$ depends only on v/s by (15), and the second equality follows from (17). Therefore, $\bar{J}_1(\theta v_0) = f_1(\theta v_0, \theta s_0) = \theta s_0 \bar{I}_1(v_0/s_0)$. However, since $v_0/s_0 = \nabla\overline{\Lambda}_X(\eta_0)$ and $h(\eta_0) = 1/s_0 \le 1/(\theta s_0)$ for any $\theta \in (0,1]$, we have by Lemma 2.8 that

$$\theta s_0 \bar{I}_1\left(\frac{v_0}{s_0}\right) = \theta s_0 \bar{I}\left(\frac{v_0}{s_0}, \frac{1}{\theta s_0}\right) = \theta s_0 \bar{I}\left(\frac{\theta v_0}{\theta s_0}, \frac{1}{\theta s_0}\right).$$

Thus, $\bar{J}_1(\theta v_0) \ge \bar{J}(\theta v_0)$. Since $\bar{J}_1(v) \le \bar{J}(v)$ for all v, this implies that $\bar{J}(v) = \bar{J}_1(v)$ for all $v \in \mathcal{A}^0 \cup \mathcal{A}^-$. As in the proof of Lemma 2.4, since $(v_0/s_0, 1/s_0) = \nabla\overline{\Lambda}(\eta_0, 0)$ is in the interior of

$\mathcal{D}_{\bar{I}}$, we can apply Lemma 2.3 to show that $\nabla \bar{J}(v_0) = \eta_0$. Since $\bar{J}(\theta v_0) = \theta \bar{J}(v_0)$ for all $\theta \in (0,1]$ this implies that $\nabla \bar{J}(\theta v_0) = \eta_0$ as well.

Since $v_0/s_0 = \nabla \overline{\Lambda}_X(\eta_0)$, $\bar{I}_1$ is strictly convex in a neighborhood of v_0/s_0, and thus (16) implies that $f_1(v,s)$ is strictly convex in s in a neighborhood of $(\theta v_0, \theta s_0)$. Therefore, θs_0 is the unique minimizing value of s in the definition of $\bar{J}_1(\theta v_0)$. Since $f_1(v,s) \le f(v,s)$, this implies that θs_0 is the unique minimizing value of s in the definition of $\bar{J}(\theta v_0)$ as well. □

Corollary 2.11. *If P is non-nestling, then $\bar{J}(\theta \mathrm{v}_P) = 0$ for all $\theta \in (0,1]$.*

Proof. Since $\mathrm{v}_P = \gamma(\mathbf{0}) \in \mathcal{A}^0$, Lemma 2.10 implies that $\bar{J}(\theta \mathrm{v}_P) = \theta \bar{J}(\mathrm{v}_P)$. However, $\bar{J}(\mathrm{v}_P) = 0$ by Lemma 2.2. □

Corollary 2.12. *$\bar{J}(v)$ is continuously differentiable on the open set $\mathcal{A} := \mathcal{A}^- \cup \mathcal{A}^0 \cup \mathcal{A}^+$, and $\|\nabla \bar{J}(v)\| < C_1$ for all $v \in \mathcal{A}$.*

Proof. This is a direct application of the formulas given for $\nabla \bar{J}(v)$ in Lemmas 2.6 and 2.10 and the fact that $\gamma(\eta)$ is continuous and one-to-one on $\mathcal{C}^+ \cup \mathcal{C}^0$. □

3 LDP Lower Bound

We now prove, in both the nestling and non-nestling cases, the large deviation lower bound.

Proposition 3.1 (Lower Bound)**.** *Let Assumptions 1, 2, and 3 hold. For any $v \in H_\ell$,*

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\|X_n - nv\| < n\delta) \ge -\bar{J}(v).$$

Proof. Let $\|\xi\|_1$ denote the L^1 norm of the vector ξ. Then, it is enough to prove the statement of the proposition with $\|\cdot\|_1$ in place of $\|\cdot\|$. Also, since $\mathbb{P}(\|X_n - nv\|_1 < n\delta) \ge \mathbb{P}(D)\overline{\mathbb{P}}(\|X_n - nv\|_1 < n\delta)$, it is enough to prove the statement of the proposition with $\overline{\mathbb{P}}$ in place of $\mathbb{P}$. That is, it is enough to show

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \overline{\mathbb{P}}(\|X_n - nv\|_1 < n\delta) \ge -\bar{J}(v).$$

Now, for any $\delta > 0$ and any integer k, since the walk is a nearest neighbor walk,

$$\overline{\mathbb{P}}(\|X_n - nv\|_1 < 4n\delta) \ge \overline{\mathbb{P}}\left(\|X_{\tau_k} - nv\|_1 < 2n\delta,\ |\tau_k - n| < 2n\delta\right).$$

For any $t \ge 1$, let $k_n = k_n(t) := \lfloor n/t \rfloor$, so that $n - t < k_n t \le n$ for all n. Thus, for any $\delta > 0$ and $t \ge 1$, and for all n large enough (so that $n\delta > t$),

$$\begin{aligned} \overline{\mathbb{P}}(\|X_n - nv\|_1 < 4n\delta) &\ge \overline{\mathbb{P}}\left(\|X_{\tau_{k_n}} - nv\|_1 < 2n\delta,\ |\tau_{k_n} - n| < 2n\delta\right) \\ &\ge \overline{\mathbb{P}}\left(\|X_{\tau_{k_n}} - k_n t v\|_1 < k_n t\delta,\ |\tau_{k_n} - k_n t| < k_n t\delta\right). \end{aligned}$$

Therefore, for any $\delta > 0$ and $t \geq 1$,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log \overline{\mathbb{P}}(\|X_n - nv\|_1 < 4n\delta) \\
&\quad \geq \liminf_{n\to\infty} \frac{1}{n} \log \overline{\mathbb{P}}\left(\|X_{\tau_{k_n}} - k_n tv\|_1 < k_n t\delta,\ |\tau_{k_n} - k_n t| < k_n t\delta\right) \\
&\quad \geq \frac{1}{t} \liminf_{n\to\infty} \frac{1}{k_n} \log \overline{\mathbb{P}}\left(\|X_{\tau_{k_n}} - k_n tv\|_1 < k_n t\delta,\ |\tau_{k_n} - k_n t| < k_n t\delta\right) \\
&\quad = \frac{1}{t} \liminf_{k\to\infty} \frac{1}{k} \log \overline{\mathbb{P}}\left(\|X_{\tau_k} - ktv\|_1 < kt\delta,\ |\tau_k - kt| < kt\delta\right) \\
&\quad = -\frac{1}{t} \inf_{\substack{\|x-tv\|_1 < t\delta \\ |y-t|<t\delta}} \bar{I}(x,y),
\end{aligned}$$

where the last equality is from (3). Taking $\delta \to 0$ we get that for any $t \geq 1$,

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \overline{\mathbb{P}}(\|X_n - nv\|_1 < 4n\delta) \geq -\frac{1}{t}\bar{I}(vt,t).$$

Since the last inequality holds for any t, the proof is completed by taking the supremum of the right side over all $t \geq 1$ and recalling the definition of $\bar{J}$. $\square$

4 LDP Upper Bound

We now wish to prove a matching large deviation upper bound to Proposition 3.1, still working under Assumptions 1, 2 and 3. Ideally, we would like for the upper bound to be valid for all $v \in H_\ell$. This is possible for $d = 1$ (see the remarks at the end of the paper), but for $d > 1$ we are only able to prove a matching upper bound to Proposition 3.1 in a neighborhood of the set where $\bar{J}(v)$ equals zero. However, this is enough to be able to prove Theorems 1.2 and 1.3.

A key step in proving the large deviation upper bound in both the non-nestling and nestling cases is the following upper bound involving regeneration times:

Lemma 4.1. *For any $t, k \in \mathbb{N}$ and any $x \in \mathbb{Z}^d$,*

$$\overline{\mathbb{P}}(X_{\tau_k} = x,\ \tau_k = t) \leq e^{-t\bar{J}\left(\frac{x}{t}\right)}.$$

Proof. Chebychev's inequality implies that, for any $\lambda \in \mathbb{R}^{d+1}$,

$$\overline{\mathbb{P}}\left(X_{\tau_k} = x, \tau_k = t\right) \leq e^{-\lambda\cdot(x,t)} \overline{\mathbb{E}} e^{\lambda\cdot(X_{\tau_k},\tau_k)} = e^{-k\left(\lambda\cdot(x/k,t/k) - \overline{\Lambda}(\lambda)\right)},$$

where in the last equality we used the i.i.d. structure of regeneration times from Theorem 2.1. Thus, taking the infimum over all $\lambda \in \mathbb{R}^{d+1}$ and using the definition of $\bar{J}$ (with $s = \frac{k}{t}$),

$$\overline{\mathbb{P}}\left(X_{\tau_k} = x, \tau_k = t\right) \leq e^{-k\bar{I}\left(\frac{x}{k},\frac{t}{k}\right)} = e^{-t\frac{k}{t}\bar{I}\left(\frac{x}{t}\frac{t}{k},\frac{t}{k}\right)} \leq e^{-t\bar{J}\left(\frac{x}{t}\right)}.$$

$\square$

4.1 LDP Upper Bound - Non-nestling Case

We are now ready to give a matching upper bound to Proposition 3.1 in a neighborhood of v_P. Let $\mathcal{A}' := \{v \in \mathbb{R}^d : \|\nabla\bar{J}(v)\| < \frac{C_2}{4}\}$. Note that Lemma 2.4 implies that $\mathcal{A}' \subset \mathcal{A}$.

Proposition 4.2 (Upper Bound)**.** *Let Assumptions 1, 2, and 3 hold, and let P be non-nestling in direction ℓ. Then, if $v \in \mathcal{A}'$ and $\delta > 0$ is sufficiently small,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\|X_n - nv\| < n\delta\right) \le - \inf_{\|x-v\|<\delta} \bar{J}(x).$$

Proof. Since $\bar{J}$ is convex, $0 = \bar{J}(\mathrm{v}_P) \ge \bar{J}(v) + \nabla \bar{J}(v) \cdot (\mathrm{v}_P - v)$. Then, since $\|\nabla \bar{J}(v)\| < C_2/4$ for any $v \in \mathcal{A}'$ we have that $\bar{J}(v) \le C_2/4\|\mathrm{v}_P - v\| < C_2/2$. Thus, for a fixed $v \in \mathcal{A}'$ we can choose a $\delta > 0$ and an $\varepsilon \in (0, 1/2)$ such that $\bar{J}(v') < \varepsilon C_2$ and $\nabla \bar{J}(v') < C_2/4$ for all $\|v' - v\| < \delta$.

Recalling (7), we obtain that there exist constants $C_3, C_2 > 0$ such that

$$\max\left\{\mathbb{P}(\tau_1 > t), \overline{\mathbb{P}}(\tau_1 > t)\right\} \le C_3 e^{-C_2 t}, \qquad \forall t > 0.$$

Let $v \in \mathcal{A}'$, and let $\varepsilon, \delta > 0$ be chosen as above. Now,

$$\begin{aligned} \mathbb{P}(\|X_n - nv\| < n\delta) &\le \mathbb{P}(\exists k \le n : \tau_k - \tau_{k-1} \ge \varepsilon n) \\ &\quad + \mathbb{P}\left(\exists k : \tau_1 < \varepsilon n,\ \tau_k \in (n - \varepsilon n, n],\ \|X_n - nv\| < n\delta,\ \tau_{k+1} > n\right). \end{aligned} \tag{19}$$

Then, since $\bar{J}(v) \le \alpha(\eta) < \varepsilon C_2$,

$$\mathbb{P}(\exists k \le n : \tau_k - \tau_{k-1} \ge \varepsilon n) \le C_3 n e^{-C_2 \varepsilon n} \le C_3 n e^{-n\bar{J}(v)}.$$

Thus, we need only to bound the second term in (19).

Since the random walk is a nearest neighbor walk, $\|X_{\tau_k} - nv\| \le \|X_n - nv\| + |n - \tau_k|$. Thus,

$$\begin{aligned} &\mathbb{P}\left(\exists k : \tau_1 < \varepsilon n,\ \tau_k \in (n - \varepsilon n, n],\ \|X_n - nv\| < n\delta,\ \tau_{k+1} > n\right) \\ &\quad \le \sum_{k\le n} \sum_{u\in(0,\varepsilon)} \sum_{s\in[0,\varepsilon)} \mathbb{P}\left(\tau_1 = un,\ \tau_k = (1-s)n,\ \|X_{\tau_k} - nv\| < n(\delta + s),\ \tau_{k+1} > n\right), \end{aligned}$$

where the above sums are only over the finite number of possible u and s such that the probabilities are non-zero. However,

$$\begin{aligned} &\mathbb{P}\left(\tau_1 = un,\ \tau_k = (1-s)n,\ \|X_{\tau_k} - nv\| < n(\delta+s),\ \tau_{k+1} > n\right) \\ &\quad \le \mathbb{P}\left(\tau_1 = un,\ \tau_k - \tau_1 = (1-s-u)n,\ \|X_{\tau_k} - X_{\tau_1} - nv\| \le n(\delta+s+u),\ \tau_{k+1} - \tau_k > ns\right) \\ &\quad = \mathbb{P}(\tau_1 = un)\overline{\mathbb{P}}\left(\tau_{k-1} = (1-s-u)n,\ \|X_{\tau_{k-1}} - nv\| \le n(\delta+s+u)\right)\overline{\mathbb{P}}(\tau_1 > ns), \end{aligned}$$

where the first inequality again uses the fact that the random walk is a nearest neighbor random walk, and the last equality uses the independence structure of regeneration times from Theorem 2.1. Thus, since $\mathbb{P}(\tau_1 = un) \le C_3 e^{-C_2 un}$ and $\overline{\mathbb{P}}(\tau_1 > ns) \le C_3 e^{-C_2 sn}$,

$$\begin{aligned} &\mathbb{P}\left(\exists k : \tau_1 < \varepsilon n,\ \tau_k \in (n - \varepsilon n, n],\ \|X_n - nv\| < n\delta,\ \tau_{k+1} > n\right) \\ &\quad \le \sum_{k\le n} \sum_{u\in(0,\varepsilon)} \sum_{s\in[0,\varepsilon)} C_3^2 e^{-C_2(u+s)n} \overline{\mathbb{P}}\left(\tau_{k-1} = (1-s-u)n,\ \|X_{\tau_{k-1}} - nv\| < n(\delta+s+u)\right). \end{aligned} \tag{20}$$

By Lemma 4.1, the last expression is bounded above by

$$\begin{aligned} &\sum_{k\le n} \sum_{u\in(0,\varepsilon)} \sum_{s\in[0,\varepsilon)} \sum_{\|x-v\|<\delta+u+s} e^{-n(1-s-u)\bar{J}\left(\frac{x}{1-s-u}\right)} C_3^2 e^{-C_2(s+u)n} \\ &\quad \le C_4 n^{d+3} \sup_{s\in[0,2\varepsilon)} \sup_{\|x-v\|<\delta+s} e^{-n\left((1-s)\bar{J}\left(\frac{x}{1-s}\right) + C_2 s\right)} \\ &\quad = C_4 n^{d+3} \exp\left\{-n\left(\inf_{s\in[0,2\varepsilon)} \inf_{\|x-v\|<\delta+s} (1-s)\bar{J}\left(\frac{x}{1-s}\right) + C_2 s\right)\right\}, \end{aligned} \tag{21}$$

for some constant C_4.

To finish the proof of the proposition, it is enough to show that the infimum in (21) is achieved when $s = 0$. That is, it is enough to show the infimum is larger than $\inf_{\|x-v\|<\delta} \bar{J}(x)$. To this end, note first that

$$\inf_{s\in[0,2\varepsilon)} \inf_{\|x-v\|<\delta+s} (1-s)\bar{J}\left(\frac{x}{1-s}\right) + C_2 s = \inf_{\|x-v\|<\delta} \inf_{s\in[0,2\varepsilon)} \inf_{\|y-x\|<s} (1-s)\bar{J}\left(\frac{y}{1-s}\right) + C_2 s. \quad (22)$$

Since $\bar{J}$ is convex,

$$\bar{J}\left(\frac{y}{1-s}\right) \geq \bar{J}(x) + \nabla \bar{J}(x) \cdot \left(\frac{y}{1-s} - x\right). \geq \bar{J}(x) - \|\nabla \bar{J}(x)\| \left\| \frac{y}{1-s} - x \right\|.$$

If $\|y - x\| < s$ and $\|\nabla \bar{J}(x)\| < C_2/4$ this implies that

$$(1-s)\bar{J}\left(\frac{y}{1-s}\right) + C_2 s \geq (1-s)\bar{J}(x) - \frac{C_2}{2}s + C_2 s = \bar{J}(x) + \left(\frac{C_2}{2} - \bar{J}(x)\right) s.$$

Recalling (22), we obtain

$$\begin{aligned}\inf_{s\in[0,2\varepsilon)} \inf_{\|x-v\|<\delta+s} (1-s)\bar{J}\left(\frac{x}{1-s}\right) + C_2 s &\geq \inf_{\|x-v\|<\delta} \inf_{s\in[0,2\varepsilon)} \bar{J}(x) + s\left(\frac{C_2}{2} - \bar{J}(x)\right) \\ &= \inf_{\|x-v\|<\delta} \bar{J}(x),\end{aligned}$$

where the last inequality is because our choice of δ and $\|x - v\| < \delta$ imply that $\bar{J}(x) < \varepsilon C_2 < \frac{C_2}{2}$. This completes the proof of the proposition. □

4.2 LDP Upper Bound - Nestling Case

Before proving a large deviation upper bound in the nestling case, we need the following lemma.

Lemma 4.3. *Assume P is non-nestling. If $xn \in \mathbb{Z}^d$ and $k \leq n$, then*

$$\overline{\mathbb{P}}(X_{\tau_k} = xn, \tau_k \leq n) \leq n e^{-n\bar{J}(x)}.$$

Proof. Lemma 4.1 implies that

$$\overline{\mathbb{P}}(X_{\tau_k} = xn, \tau_k \leq n) = \sum_{\theta\in(0,1],\ \theta n\in\mathbb{Z}} \overline{\mathbb{P}}(X_{\tau_k} = xn, \tau_k = \theta n) \leq \sum_{\theta\in(0,1],\ \theta n\in\mathbb{Z}} e^{-n\theta \bar{J}\left(\frac{x}{\theta}\right)}.$$

Then, we will be finished if we can show that $\theta \bar{J}\left(\frac{x}{\theta}\right) \geq \bar{J}(x)$. The convexity of $\bar{J}$ implies that $\theta \bar{J}\left(\frac{x}{\theta}\right) \geq \bar{J}(x + (1-\theta)z) - (1-\theta)\bar{J}(z)$ for any z. Letting $z = c\mathrm{v}_P$ for some $c \in (0, 1]$, Lemma 2.11 implies that $\theta \bar{J}\left(\frac{x}{\theta}\right) \geq \bar{J}(x + (1-\theta)c\mathrm{v}_P)$. Letting $c \to 0^+$ completes the proof. □

We are now ready to prove a matching large deviation upper bound to Proposition 3.1 in the nestling case. The proof is similar to the proof of the upper bound in the non-nestling case. However, instead of forcing regeneration times to be small, we instead force regeneration distances to be small.

Proposition 4.4. *Let Assumptions 1, 2, and 3 hold, let P be nestling, and let $d \geq 2$. Then, if $v \in \mathcal{A}$ and $\delta > 0$ is sufficiently small,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\|X_n - nv\| < \delta n) \leq -\inf_{\|x-v\|\leq\delta} \bar{J}(x).$$

Proof. Corollary 2.12 implies that $\|\nabla \bar{J}(v)\| < C_1$ for all $v \in \mathcal{A}$. Since $\bar{J}$ is convex, $\bar{J}(z) \geq \bar{J}(v) + \nabla \bar{J}(v) \cdot (z - v)$ for any z. Letting $z = \theta \mathrm{v}_P$ for any $\theta \in (0, 1]$, Lemma 2.11 implies that $\bar{J}(v) \leq -\nabla \bar{J}(v) \cdot (\theta \mathrm{v}_P - v) \leq \|\nabla \bar{J}(v)\| \|\theta \mathrm{v}_P - v\|$. Letting $\theta \to 0^+$, we obtain that $\bar{J}(v) \leq \|\nabla \bar{J}(v)\| \|v\| < C_1 \|v\|$.

Now, for a fixed $v \in \mathcal{A}$, choose a $\delta > 0$ and a $c < \|v\| - \delta$ such that $\bar{J}(v') < cC_1$ and $\|\nabla \bar{J}(v')\| < C_1$ for all $\|v' - v\| < \delta$. Letting $\tau_0 := 0$, we define $S_k := \sup_{\tau_k < n \leq \tau_{k+1}} \|X_n - X_{\tau_k}\|$. By Assumption 3, it is clear that there exists a constant $C_5 > 0$ such that

$$\max\left\{\overline{\mathbb{P}}\left(S_0 > t\right), \mathbb{P}\left(S_0 > t\right)\right\} = \max\left\{\overline{\mathbb{P}}\left(\sup_{n<\tau_1} \|X_n\| > t\right), \mathbb{P}\left(\sup_{n<\tau_1} \|X_n\| > t\right)\right\} \leq C_5 e^{-C_1 t} \quad (23)$$

Then,

$$\begin{aligned}
&\mathbb{P}(\|X_n - nv\| < \delta n) \\
&\quad \leq \mathbb{P}(S_0 \geq cn) + n\overline{\mathbb{P}}(S_0 \geq cn) + \mathbb{P}(\|X_n - nv\| < \delta n,\ S_i < cn \quad \forall i = 0, 1, \dots n) \\
&\quad \leq C_5(n+1)e^{-C_1 cn} + \mathbb{P}(\|X_n - nv\| < \delta n,\ S_i < cn \quad \forall i = 0, 1, \dots n) \\
&\quad \leq C_5(n+1)e^{-n\bar{J}(v)} + \mathbb{P}(\|X_n - nv\| < \delta n,\ S_i < cn \quad \forall i = 0, 1, \dots n), \qquad (24)
\end{aligned}$$

where the last inequality is because $\bar{J}(v) < cC_1$. Thus, it is enough to bound the second term on the right side of (24). Since $c < \|v\| - \delta$, the event $\{\|X_n - nv\| < \delta n,\ S_0 < cn\}$ implies that $\tau_1 < n$. Decomposing according to the last regeneration time before n, we obtain that

$$\begin{aligned}
&\mathbb{P}(\|X_n - nv\| < \delta n,\ S_i < cn \quad \forall i = 0, 1, \dots n) \\
&\quad = \sum_{k=1}^{n} \mathbb{P}(\tau_k \leq n < \tau_{k+1},\ \|X_n - nv\| < \delta n,\ S_i < cn \quad \forall i = 0, 1, \dots n) \\
&\quad \leq \sum_{k=1}^{n} \sum_{\|x\|<c} \sum_{\|y\|<c} \sum_{\|z\|<\delta} \mathbb{P}(X_{\tau_1} = xn,\ X_{\tau_k} = n(v+z-y),\ X_n = n(v+z),\ \tau_k \leq n < \tau_{k+1}),
\end{aligned} \quad (25)$$

where the above sums are only over the finite number of possible x, y, and z such that the probabilities are non-zero. The i.i.d. structure of regeneration times and distances from Theorem 2.1 implies that

$$\begin{aligned}
&\mathbb{P}(X_{\tau_1} = xn,\ X_{\tau_k} = n(v+z-y),\ X_n = n(v+z),\ \tau_k \leq n < \tau_{k+1}) \\
&\quad \leq \mathbb{P}(X_{\tau_1} = xn)\overline{\mathbb{P}}(X_{\tau_{k-1}} = n(v+z-y-x),\ \tau_{k-1} \leq n)\overline{\mathbb{P}}(S_0 \geq \|y\|n) \\
&\quad \leq C_5 e^{-C_1\|x\|n} e^{-n\bar{J}(v+z-y-x)} C_5 e^{-C_1\|y\|n},
\end{aligned}$$

where in the last inequality we used (23) and Lemma 4.3. Since there are at most $C_6 n^{3d+1}$ terms in the sum in (25) for some constant C_6 depending only on c, δ, and d, we obtain that

$$\begin{aligned}
&\mathbb{P}(\|X_n - nv\| < \delta n,\ S_i < cn \quad \forall i = 0, 1, \dots n) \\
&\quad \leq C_6 n^{3d+1} \exp\left\{-n\left(\inf_{\|z\|<\delta} \inf_{\|x\|<c} \inf_{\|y\|<c} \bar{J}(v+z-x-y) + C_1(\|x\| + \|y\|)\right)\right\}. \qquad (26)
\end{aligned}$$

However, the convexity of $\bar{J}$ and the fact that $\|\nabla \bar{J}(v+z)\| < C_1$ for all $\|z\| < \delta$ imply that

$$\bar{J}(v+z-x-y) \geq \bar{J}(v+z) + \nabla \bar{J}(v+z) \cdot (-x-y) \geq \bar{J}(v+z) - C_1(\|x\| + \|y\|).$$

Thus, the infimum in (26) is acheived when $\|x\| = \|y\| = 0$, and therefore,

$$\mathbb{P}(\|X_n - nv\| < \delta n,\ S_i < cn \quad \forall i = 0, 1, \dots n) \leq C_6 n^{3d+1} \exp\left\{-n \inf_{\|z\|<\delta} \bar{J}(v + z)\right\}.$$

This, combined with (24) completes the proof of the proposition. □

Finally, we give the proofs of the main results of this paper.

Proof of Theorems 1.2 and 1.3:
The annealed large deviation principle in Theorem 1.1 implies that

$$\lim_{\delta\to 0} \liminf_{n\to 0} \frac{1}{n} \log \mathbb{P}(\|X_n - nv\| < n\delta) = -H(v).$$

Then, if the law on environments is non-nestling, Propositions 3.1 and 4.2 imply that $\bar{J}(v) = H(v)$ for all $v \in \mathcal{A}'$ (where $\mathcal{A}'$ is defined as in the beginning of subsection 4.1). Similarly, if P is nestling, Propositions 3.1 and 4.4 imply that $\bar{J}(v) = H(v)$ for all $v \in \mathcal{A}$ (where $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^0 \cup \mathcal{A}^-$ was defined as in Subsection 2.2). The properties of $\bar{J}(v)$ given in Subsections 2.1 and 2.2 are then also true of $H(v)$. □

5 Concluding Remarks and Open Problems

1. The function $\bar{J}$ depends implicitly on the direction ℓ chosen for the definition of the regeneration times. Write $\bar{J}^\ell$ to make this dependence explicit. A consequence of our proofs of Theorems 1.2 and 1.3 is that for any $\ell, \ell' \in \{\xi \in S^{d-1} : \xi \cdot \mathrm{v}_P > 0,\ c\xi \in \mathbb{Z}^d \text{ for some } c > 0\}$, $\bar{J}^\ell(v) = \bar{J}^{\ell'}(v)$ for all v in some neighborhood of where the functions are zero.

   Question 5.1. *Recall that $\bar{J}^\ell$ is defined on $H_\ell = \{v \in \mathbb{R}^d : v \cdot \ell > 0\}$. Is it true that $\bar{J}^\ell(v) = \bar{J}^{\ell'}(v)$ for all $v \in H_\ell \cap H_{\ell'}$?*

2. The large deviations lower bound in Proposition 3.1 holds for all $v \in H_\ell$, but we were only able to prove a matching upper bound in a neighborhood of the set where $\bar{J}(v) = 0$. However, if $d = 1$ then we are able to prove a matching upper bound for all $v \in H_\ell$:

   Proposition 5.2 (Proposition 6.3.11 in [Pet08])**.** *Let X_n be a RWRE on $\mathbb{Z}$. Let Assumptions 1 and 2 hold, and assume that $\mathbb{P}(\lim_{n\to\infty} X_n = +\infty) = 1$. Define $\bar{J}$ as above in terms of regeneration times in the direction $\ell = 1$. Then, for any $v > 0$ and $\delta < v$,*

   $$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(|X_n - nv| < n\delta) = -\inf_{x:|x-v|<\delta} \bar{J}(x).$$

   The following remains an open question:

   Question 5.3. *Do the large deviation upper bounds in Propositions 4.2 and 4.4 hold for all $v \in H_\ell$?*

   Note: An affirmative answer to Question 5.3 would imply that $H(v) = \bar{J}(v)$ for all $v \in H_\ell$. This would therefore imply that the answer to Question 5.1 is also affirmative.

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A Analyticity of Legendre Transforms

Let $F : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Then, the Legendre transform F^* of F is defined by

$$F^*(x) = \sup_{\lambda \in \mathbb{R}^d} \lambda \cdot x - F(\lambda). \tag{27}$$

Lemma A.1. *Let F be strictly convex and analytic on an open subset $U \subset \mathbb{R}^d$. Then, F^* is strictly convex and analytic in $U' := \{y \in \mathbb{R}^d : y = \nabla F(\lambda)$ for some $\lambda \in U\}$.*

Proof. Since F is strictly convex on U, ∇F is one-to-one on U. Therefore, for any $x \in U'$, there exists a unique $\lambda(x) \in U$ such that $\nabla F(\lambda(x)) = x$. (That is, $x \mapsto \lambda(x)$ is the inverse function of ∇F restricted to U.) This implies, since $\lambda \mapsto \lambda \cdot x - F(\lambda)$ is a concave function in λ, that the supremum in (27) is achieved with $\lambda = \lambda(x)$ when $x \in U'$. That is,

$$F^*(x) = \lambda(x) \cdot x - F\left((\lambda(x)\right), \qquad \forall x \in U'. \tag{28}$$

Since F is analytic on U, then ∇F is also analytic on U. Then, a version of the inverse function theorem [FG02, Theorem 7.5] implies that $\lambda(\cdot)$ is analytic on U' if

$$\det\left(D^2 F(x)\right) \neq 0, \qquad \forall x \in U, \tag{29}$$

where $D^2 F$ is the matrix of second derivatives of F. However, since F is strictly convex on U, $D^2 F(x)$ is strictly positive definite for all $x \in U$. Thus, (29) holds and so $x \mapsto \lambda(x)$ is analytic on U'. Recalling (28), we then obtain that F^* is also analytic on U'.

An application of the chain rule to (28) implies that

$$\nabla F^*(x) = \lambda(x) \quad \text{and} \quad D^2 F^*(x) = D\lambda(x) = \left(D^2 F(\lambda(x))\right)^{-1}, \qquad \forall x \in U'.$$

Since $D^2 F$ is strictly positive definite on U, the above implies that $D^2 F^*(x)$ is strictly positive definite for all $x \in U'$. Thus F^* is strictly convex on U'. ☐

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