DIFFERING AVERAGED AND QUENCHED LARGE DEVIATIONS FOR RANDOM WALKS IN RANDOM ENVIRONMENTS IN DIMENSIONS TWO AND THREE

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Abstract. We consider the quenched and the averaged (or annealed) large deviation rate functions I_q and I_a for space-time and (the usual) space-only RWRE on $\mathbb{Z}^d$. By Jensen's inequality, $I_a \leq I_q$.

In the space-time case, when $d \geq 3+1$, I_q and I_a are known to be equal on an open set containing the typical velocity ξ_o. When $d = 1+1$, we prove that I_q and I_a are equal only at ξ_o. Similarly, when $d = 2+1$, we show that $I_a < I_q$ on a punctured neighborhood of ξ_o.

In the space-only case, we provide a class of non-nestling walks on $\mathbb{Z}^d$ with $d = 2$ or 3, and prove that I_q and I_a are not identically equal on any open set containing ξ_o whenever the walk is in that class. This is very different from the known results for non-nestling walks on $\mathbb{Z}^d$ with $d \geq 4$.

1. Introduction

1.1. **The models.** Consider a discrete time Markov chain on the d-dimensional integer lattice $\mathbb{Z}^d$ with $d \geq 1$. For any $x, z \in \mathbb{Z}^d$, denote the transition probability from x to $x+z$ by $\pi(x, x+z)$. Refer to the transition vector $\omega_x := (\pi(x, x+z))_{z\in\mathbb{Z}^d}$ as the *environment* at x. If the environment $\omega := (\omega_x)_{x\in\mathbb{Z}^d}$ is sampled from a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, then this process is called *random walk in a random environment* (RWRE). Here, $\mathcal{B}$ is the Borel σ-algebra corresponding to the product topology.

For every $y \in \mathbb{Z}^d$, define the shift T_y on Ω by $(T_y\omega)_x := \omega_{x+y}$. In order to have some statistical homogeneity in the environment, $\mathbb{P}$ is generally assumed to be stationary and ergodic with respect to $(T_y)_{y\in\mathbb{Z}^d}$. In this paper, we will make the stronger assumption that

$$\mathbb{P} \text{ is a product measure with equal marginals.} \tag{1.1}$$

In other words, $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ is a collection of independent and identically distributed (i.i.d.) random vectors.

The set $\mathcal{R} := \{z \in \mathbb{Z}^d : \mathbb{P}(\pi(0, z) > 0) > 0\}$ is the *range* of allowed steps of the walk (here and throughout, we often use 0 to denote the origin in $\mathbb{Z}^d$ when no confusion occurs). Let $(e_i)_{i=1}^d$ denote the canonical basis for $\mathbb{Z}^d$. The walk is said to be *space-time* if

$$\mathcal{R} = \mathcal{R}_{st} := \{(z_1, \ldots, z_d) \in \mathbb{Z}^d : |z_1| + \cdots + |z_{d-1}| = 1, z_d = 1\}, \tag{1.2}$$

and it is said to be *space-only* if

$$\mathcal{R} = \mathcal{R}_{so} := \{\pm e_i\}_{i=1}^d. \tag{1.3}$$

In either case, we will assume throughout the paper that there exists a $\kappa > 0$ such that $\mathbb{P}(\pi(0, z) \geq \kappa) = 1$ for every $z \in \mathcal{R}$. This condition is known as *uniform ellipticity.*

Space-time is a natural term for the case (1.2) since then, the walk decomposes into two parts. Its projection on the e_d-axis is deterministic and can be identified with time. The motion in the span of $(e_i)_{i=1}^{d-1}$ can be thought of as a variation of space-only RWRE where the environment is freshly sampled at each time step. To emphasize this decomposition, we will write the dimension as $d = (d-1)+1$. For example, when $d = 3$, we will say that the dimension is $2+1$.

For every $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, the Markov chain with environment ω induces a probability measure P_x^ω on the space of paths starting at x. Statements about P_x^ω that hold for $\mathbb{P}$-a.e. ω are referred to as *quenched.* Statements about the semi-direct product $P_x := \mathbb{P} \times P_x^\omega$ are referred to as *averaged* (or *annealed*). Expectations under $\mathbb{P}, P_x^\omega$ and P_x are denoted by $\mathbb{E}, E_x^\omega$ and E_x, respectively.

See [25] for a survey of results and open problems on RWRE.

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It is clear that no model satisfies both (1.2) and (1.3). Nevertheless, it turns out that many of the results that hold for space-only RWRE are valid under also the space-time assumption, and it is fair to say that space-time RWRE is easier to analyze than space-only RWRE because (1.2) ensures that the walk never visits the same point more than once.

1.2. **Regeneration times.** In the next subsection, we will give a brief survey of the previous results on large deviations for RWRE in order to put the present work in context. Some of these results involve certain random times which are introduced below for convenience.

Let $(X_n)_{n\geq 0}$ denote the path of a space-only RWRE. Consider a unit vector $\hat{u}\in\mathcal{S}^{d-1}$. Define a sequence $(\tau_m)_{m\geq 0}$ of random times, which are referred to as *regeneration times* (relative to $\hat{u}$), by $\tau_o:=0$ and

$$\tau_m:=\inf\{j>\tau_{m-1}:\langle X_i,\hat{u}\rangle<\langle X_j,\hat{u}\rangle\leq\langle X_k,\hat{u}\rangle\text{ for all }i,k\text{ with }i<j<k\}$$

for every $m\geq 1$. (Regeneration times first appeared in the work of Kesten [9] on one-dimensional RWRE. They were adapted to the multidimensional setting by Sznitman and Zerner, c.f. [18].) Because we assumed the environment $\omega=(\omega_x)_{x\in\mathbb{Z}^d}$ to be an i.i.d. collection, if the walk is directionally transient relative to $\hat{u}$, i.e., if $P_o\left(\lim_{n\to\infty}\langle X_n,\hat{u}\rangle=\infty\right)=1$, then $P_o\left(\tau_m<\infty\right)=1$ for every $m\geq 1$. In this setup, as noted in [18], the significance of $(\tau_m)_{m\geq 1}$ is due to the fact that

$$\left(X_{\tau_m+1}-X_{\tau_m},X_{\tau_m+2}-X_{\tau_m},\ldots,X_{\tau_{m+1}}-X_{\tau_m},\tau_{m+1}-\tau_m\right)_{m\geq 1}$$

is an i.i.d. sequence under P_o.

The walk is said to satisfy Sznitman's transience condition (**T**) if

$$E_o\left[\sup_{1\leq i\leq\tau_1}\exp\{c_1|X_i|\}\right]<\infty\text{ for some }c_1>0.$$

(Here and throughout, the norm $|\cdot|$ denotes the ℓ_2 norm). When $d\geq 2$, Sznitman [17] proves that (1.1), (1.3) and (**T**) imply a *ballistic* law of large numbers (LLN), an averaged central limit theorem and certain large deviation estimates.

Condition (**T**) holds as soon as the walk is *non-nestling* relative to $\hat{u}$, i.e., when the random drift vector

$$v(\omega):=\sum_{z\in\mathcal{R}}\pi(0,z)z\qquad\text{satisfies}\qquad\operatorname*{ess\,inf}_{\mathbb{P}}\langle v(\cdot),\hat{u}\rangle>0.\tag{1.4}$$

The walk is said to be non-nestling if it is non-nestling relative to some unit vector. Otherwise, it is referred to as *nestling*. In the latter case, the convex hull of the support of the law of $v(\cdot)$ contains the origin.

In the case of space-time RWRE, regeneration times are defined naturally by taking $\hat{u}=e_d$ and $\tau_m=m$ for every $m\geq 1$. Clearly, the space-time walk is always non-nestling relative to $\hat{u}=e_d$.

1.3. **Previous results on large deviations for RWRE.** Recall that a sequence $(Q_n)_{n\geq 1}$ of probability measures on a topological space $\mathbb{X}$ is said to satisfy the *large deviation principle* (LDP) with a rate function $I:\mathbb{X}\to\mathbb{R}_+\cup\{\infty\}$ if I is lower semicontinuous and for any measurable set G,

$$-\inf_{x\in G^o}I(x)\leq\liminf_{n\to\infty}\frac{1}{n}\log Q_n(G)\leq\limsup_{n\to\infty}\frac{1}{n}\log Q_n(G)\leq-\inf_{x\in\bar{G}}I(x).$$

Here, G^o is the interior of G, and $\bar{G}$ its closure. See [4] for general background regarding large deviations.

We will focus on the following large deviation principles for walks in uniformly elliptic environments.

Theorem 1.1 (Quenched LDP). *For $\mathbb{P}$-a.e. ω, $\left(P_o^\omega\left(\frac{X_n}{n}\in\cdot\right)\right)_{n\geq 1}$ satisfies the LDP with a deterministic and convex rate function I_q.*

Theorem 1.2 (Averaged LDP). *$\left(P_o\left(\frac{X_n}{n}\in\cdot\right)\right)_{n\geq 1}$ satisfies the LDP with a convex rate function I_a.*

There are many works on large deviations for space-only RWRE. We briefly mention them in chronological order. Greven and den Hollander [7] prove Theorem 1.1 for walks on $\mathbb{Z}$ under the i.i.d. environment assumption. They provide a formula for I_q and show that its graph typically has flat pieces. Zerner [26] establishes Theorem 1.1 for nestling walks on $\mathbb{Z}^d$ in i.i.d. environments. Comets, Gantert and Zeitouni [3] generalize the result of [7] to walks on $\mathbb{Z}$ in stationary and ergodic environments. Also, they prove Theorem 1.2 for walks on $\mathbb{Z}$ in i.i.d. environments and give a formula that links I_a to I_q. Varadhan [20] generalizes Zerner's result to stationary and ergodic environments without any nestling assumption. He also proves Theorem 1.2 for walks on $\mathbb{Z}^d$ in i.i.d. environments and gives a variational formula for I_a. Rassoul-Agha

[12] generalizes the latter result of [20] to certain mixing environments. Rosenbluth [15] gives an alternative proof of Theorem 1.1 for walks on $\mathbb{Z}^d$ in stationary and ergodic environments, and provides a variational formula for I_q. Yilmaz [23] generalizes the result of [15] to a so-called level-2 LDP. Berger [1], Peterson and Zeitouni [11], and Yilmaz [21] obtain certain qualitative properties of I_a. Rassoul-Agha and Seppäläinen [14] generalize the result of [15] to a so-called level-3 LDP.

In the case of space-time RWRE, Rassoul-Agha and Seppäläinen [13] prove Theorem 1.1 by adapting the quenched argument in [20]. Theorem 1.2 does not require any work. Indeed, Assumption (1.2) implies that the walk under P_o is a sum of i.i.d. increments. The common distribution of these increments is $(q(z))_{z\in\mathcal{R}}$ where $q(z) := \mathbb{E}[\pi(0,z)]$ for every $z \in \mathcal{R}$. Therefore, Theorem 1.2 in the space-time setup is simply Cramér's theorem, c.f. [4].

In addition to the works mentioned in the last two paragraphs, there are two more results on large deviations for RWRE that are relevant to this paper. We state them in detail.

Theorem 1.3 (Yilmaz [22]). *Assume (1.1) and (1.2). If $d \geq 3+1$, then $I_q = I_a$ on a set $\mathcal{A}_{st} \times \{e_d\}$ containing the LLN velocity ξ_o, where $\mathcal{A}_{st}$ is an open subset of $\mathbb{R}^{d-1}$.*

Theorem 1.4 (Yilmaz [24]). *Assume (1.1), (1.3), $d \geq 4$, and that Sznitman's (**T**) condition holds for some $\hat{u} \in \mathcal{S}^{d-1}$.*

(a) *If the walk is non-nestling, then $I_q = I_a$ on an open set $\mathcal{A}_{so}$ containing the LLN velocity ξ_o.*
(b) *If the walk is nestling, then*
  (i) *$I_q = I_a$ on an open set $\mathcal{A}_{so}^+$,*
  (ii) *there exists a $(d-1)$-dimensional smooth surface patch $\mathcal{A}_{so}^b$ such that $\xi_o \in \mathcal{A}_{so}^b \subset \partial\mathcal{A}_{so}^+$,*
  (iii) *the unit vector η_o normal to $\mathcal{A}_{so}^b$ (and pointing inside $\mathcal{A}_{so}^+$) at ξ_o satisfies $\langle \eta_o, \xi_o \rangle > 0$, and*
  (iv) *$I_q(t\xi) = tI_q(\xi) = tI_a(\xi) = I_a(t\xi)$ for every $\xi \in \mathcal{A}_{so}^b$ and $t \in [0,1]$.*

It is worthwhile to emphasize that the equality $I_q = I_a$ does not extend, in the setup of Theorems 1.3 and 1.4, to the whole space. Indeed, for any $d \geq 1$,

$$I_a < I_q \text{ at the extremal points of the domain of } I_a. \tag{1.5}$$

By continuity, this inequality holds also at some interior points. See Proposition 4 of [24] for details.

1.4. **Our results.** For space-time RWRE, it is natural to ask whether Theorem 1.3 can be generalized to $d \geq 1+1$ or $2+1$. The answer turns out to be no.

Theorem 1.5. *Assume (1.1) and (1.2). If $d = 1+1$, then $I_q(\xi) = I_a(\xi) < \infty$ if and only if $\xi = \xi_o$, the LLN velocity.*

Theorem 1.6. *Assume (1.1) and (1.2). If $d = 2+1$, then $I_a < I_q$ on a set $(\mathcal{G}_{st} \times \{e_3\}) \setminus \{\xi_o\}$, where $\mathcal{G}_{st} \subset \mathbb{R}^2$ is open and $\mathcal{G}_{st} \times \{e_3\}$ contains ξ_o.*

In the case of space-only RWRE on $\mathbb{Z}$, a consequence of Comets et al. [3], Proposition 5, is that $I_q(\xi) = I_a(\xi) < \infty$ if and only if $\xi = 0$ or $I_a(\xi) = 0$. In particular, Theorem 1.4 cannot be generalized to $d \geq 1$. Our next result shows that the conclusion of Theorem 1.4 is false for a class of space-only RWRE's in dimensions $d = 2, 3$.

Definition 1.7. *Assume $d \geq 2$, and fix a triple $p = (p^+, p^o, p^-)$ of positive real numbers such that $p^- < p^+$ and $p^+ + p^o + p^- = 1$. For any $\epsilon > 0$, a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B})$ is said to be in class $\mathcal{M}_\epsilon(d,p)$ if*

(a) *(1.1) and (1.3) hold,*
(b) *$\mathbb{P}(\pi(0,e_d) = p^+,\ \pi(0,-e_d) = p^-) = 1$,*
(c) *$\mathbb{P}(\epsilon/2 < |\pi(0,e_1) - \frac{p^o}{2(d-1)}| < \epsilon) = 1$, and*
(d) *$\mathbb{P}$ is invariant under the rotations of $\mathbb{Z}^d$ that preserve e_d. (We will refer to this as isotropy.)*

Theorem 1.8. *Assume $d = 2$ or 3. Fix a triple $p = (p^+, p^o, p^-)$ as in Definition 1.7. Then there exists an $\epsilon_o = \epsilon_o(p)$ such that if $\epsilon < \epsilon_o$ and $\mathbb{P}$ is in class $\mathcal{M}_\epsilon(d,p)$, then the quenched and the averaged rate functions I_q and I_a are not identically equal on any open set containing the LLN velocity ξ_o.*

The proofs of our results are based on a technique that combines the so-called *fractional moment* method with a certain change of measure (which we will refer to as *tilting the environment*). This technique has been developed for analyzing the so-called polymer pinning model, c.f. [5, 19, 6], and it has been recently refined by Lacoin [10] for obtaining certain lower bounds for the free energy of directed polymers in random environments. Comparing with the polymer setup, an extra complication occurs in the RWRE model due to the dependence of the transition probabilities of the walk on the environment. (In the polymer model discussed above, the walk is a simple random walk, and the environment only appears in the evaluation of exponential moments with respect to the random walk.) The difficulty in the RWRE setup, and much of our work, lies in overcoming this dependency. For space-time RWRE, this task is greatly simplified because each site is visited at most once. For space-only RWRE, where this is not true, we employ a perturbative approach that unfortunately restricts the class of models considered, see Section 4 for further comments.

Here is how the rest of the paper is organized: In Section 2, we consider space-time RWRE and prove Theorems 1.5 and 1.6 by adapting the relevant arguments given in [10]. In Section 3, we focus on space-only walks that are non-nestling relative to e_d, and modify the previous proofs by making use of regeneration times. This way, we establish a result (see Theorem 3.4) analogous to Theorems 1.5 and 1.6. The only difference is that Theorem 3.4 is valid under a certain correlation condition, c.f. (3.17). Finally, we prove Theorem 1.8 by checking that (3.17) holds whenever $\mathbb{P}$ is in class $\mathcal{M}_\epsilon(d,p)$ with some triple p (as in Definition 1.7) and a sufficiently small $\epsilon > 0$.

2. Inequality of the rate functions for space-time RWRE

2.1. **Reducing to a fractional moment estimate.** Assume $d \geq 1+1$. Recall (1.2). Consider a space-time random walk on $\mathbb{Z}^d$ in a uniformly elliptic and i.i.d. environment. For every $\theta \in \mathbb{R}^d$, define

$$\phi(\theta) := \sum_{z\in\mathcal{R}} \mathrm{e}^{\langle\theta,z\rangle} q(z)$$

where $q(z) := \mathbb{E}[\pi(0,z)]$. Since the walk visits every point at most once, $E_o\left[\exp\{\langle\theta, X_N\rangle\}\right] = \phi(\theta)^N$ for every $N \geq 1$.

Define the logarithmic moment generating functions

$$\Lambda_q(\theta) := \lim_{N\to\infty} \frac{1}{N} \log E_o^\omega\left[\exp\{\langle\theta, X_N\rangle\}\right] \quad \text{and} \quad \Lambda_a(\theta) := \lim_{N\to\infty} \frac{1}{N} \log E_o\left[\exp\{\langle\theta, X_N\rangle\}\right] = \log\phi(\theta).$$

By Varadhan's Lemma, c.f. [4], $\Lambda_q(\theta) = \sup_{\xi\in\mathbb{R}^d}\{\langle\theta,\xi\rangle - I_q(\xi)\} = I_q^*(\theta)$, the convex conjugate of I_q at θ. Similarly, $\Lambda_a(\theta) = \log\phi(\theta) = I_a^*(\theta)$.

For every $N \geq 1$, $\theta \in \mathbb{R}^d$ and $\omega \in \Omega$, define

$$W_N(\theta,\omega) := E_o^\omega[\exp\{\langle\theta, X_N\rangle - N\log\phi(\theta)\}].$$

Given any $\alpha \in (0,1)$, Jensen's inequality and the bounded convergence theorem imply that

$$\begin{aligned}
\Lambda_q(\theta) - \log\phi(\theta) &= \lim_{N\to\infty} \frac{1}{N} \log W_N(\theta,\cdot) = \mathbb{E}\left[\lim_{N\to\infty} \frac{1}{N}\log W_N(\theta,\cdot)\right] \\
&= \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log W_N(\theta,\cdot)\right] = \lim_{N\to\infty} \frac{1}{N\alpha}\mathbb{E}\left[\log W_N(\theta,\cdot)^\alpha\right] \\
&\leq \limsup_{N\to\infty} \frac{1}{N\alpha} \log\mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right] \\
&\leq \lim_{N\to\infty} \frac{1}{N\alpha}\log\left(\mathbb{E}\left[W_N(\theta,\cdot)\right]\right)^\alpha = 0.
\end{aligned} \tag{2.1}$$

Lemma 2.1. *Assume (1.1) and (1.2). Fix any $\alpha \in (0,1)$. If $d = 1+1$, then*

$$\limsup_{N\to\infty} \frac{1}{N} \log\mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right] < 0 \tag{2.2}$$

whenever $\theta \notin sp\{e_2\}$, the one-dimensional vector space spanned by e_2.

Lemma 2.2. *Assume (1.1) and (1.2). Fix any $\alpha \in (0,1)$. If $d = 2+1$, then there exists a $\beta > 0$ such that (2.2) holds whenever* $\mathrm{dist}(\theta, sp\{e_3\}) \in (0,\beta)$.

Remark 2.3. *For every $\theta \in sp\{e_d\}$, (1.2) implies that $W_N(\theta,\cdot) = 1$ and $\Lambda_q(\theta) = \log\phi(\theta)$.*

When $d = 1+1$, it follows from (2.1) and Lemma 2.1 that $\Lambda_q(\cdot) < \log\phi(\cdot)$ on $\{\theta \in \mathbb{R}^2 : \theta \notin sp\{e_2\}\}$. By convex duality, $I_a < I_q$ on $\{\nabla\log\phi(\theta) : \theta \notin sp\{e_2\}\}$. It is easy to see that the latter set is equal to $((-1,1)\times\{e_2\})\setminus\{\xi_o\}$. In combination with (1.5), this proves Theorem 1.5.

Similarly, when $d = 2+1$, Lemma 2.2 implies that $I_a < I_q$ on $\{\nabla \log \phi(\theta) : \mathrm{dist}(\theta, sp\{e_3\}) \in (0,\beta)\}$. One can check that this set is of the form $(\mathcal{G}_{st} \times \{e_3\}) \setminus \{\xi_o\}$ where $\mathcal{G}_{st} \subset \mathbb{R}^2$ is open and $\mathcal{G}_{st} \times \{e_3\}$ contains ξ_o. This proves Theorem 1.6.

The rest of this section is devoted to proving Lemmas 2.1 and 2.2.

2.2. **Decomposing into paths.** Assume $d = 1+1$ or $2+1$. Let $\mathbb{V}_d := \mathbb{Z}^{d-1} \times \{0\} \subset \mathbb{Z}^d$. Fix an n of the form k^2, with k an integer to be determined later (e.g., for $d = 1+1$, this n is chosen so that the conclusion of Lemma 2.4 below holds). When $d = 1+1$, let

$$J_y := \left[(y' - \frac{1}{2})\sqrt{n}, (y' + \frac{1}{2})\sqrt{n}\right) \times \{0\} \subset \mathbb{R}^2 \tag{2.3}$$

for every $y = (y', 0) \in \mathbb{V}_2$. Similarly, when $d = 2+1$, let

$$J_y := \left[(y' - \frac{1}{2})\sqrt{n}, (y' + \frac{1}{2})\sqrt{n}\right) \times \left[(y'' - \frac{1}{2})\sqrt{n}, (y'' + \frac{1}{2})\sqrt{n}\right) \times \{0\} \subset \mathbb{R}^3$$

for every $y = (y', y'', 0) \in \mathbb{V}_3$.

Take $N = nm$ for some $m \geq 1$. For every $\theta \in \mathbb{R}^d$, $\omega \in \Omega$ and $Y = (y_1, \ldots, y_m) \in (\mathbb{V}_d)^m$, define

$$\bar{W}_N(\theta, \omega, Y) := E_o^\omega[\exp\{\langle \theta, X_N\rangle - N \log \phi(\theta)\}, X_{jn} - \lfloor jn\xi(\theta)\rfloor \in J_{y_j} \text{ for every } j \leq m] \tag{2.4}$$

where $\xi(\theta) = \nabla \log \phi(\theta)$. (For $u \in \mathbb{R}^d$, $\lfloor u \rfloor$ denotes the closest element of $\mathbb{Z}^d$ to u. If there is more than one closest element, then take the one whose index is the smallest with respect to the lexicographic order.) Note that $\langle \xi(\theta), e_d\rangle = 1$ because $\langle z, e_d\rangle = 1$ for every $z \in \mathcal{R}_{st}$.

Since $\mathbb{V}_d$ is contained in the disjoint union $\cup_{y \in \mathbb{V}_d} J_y$, we see that $W_N(\theta, \omega) = \sum_Y \bar{W}_N(\theta, \omega, Y)$. Hence, $W_N(\theta, \omega)^\alpha \leq \sum_Y \bar{W}_N(\theta, \omega, Y)^\alpha$ by subadditivity, and

$$\mathbb{E}[W_N(\theta, \cdot)^\alpha] \leq \sum_Y \mathbb{E}\left[\bar{W}_N(\theta, \cdot, Y)^\alpha\right]. \tag{2.5}$$

In the rest of this section, we will treat the cases $d = 1+1$ and $d = 2+1$ separately.

2.3. **Tilting along a path ($d = 1+1$).** Our aim is to prove Lemma 2.1 which states that $\mathbb{E}[W_N(\theta, \cdot)^\alpha]$ decays exponentially in N. Let us say a few words about our strategy. For any function $g(\theta, \cdot)$ on Ω,

$$\begin{aligned} \mathbb{E}[W_N(\theta, \cdot)^\alpha] &= \mathbb{E}\left[(W_N(\theta, \cdot) g(\theta, \cdot))^\alpha g(\theta, \cdot)^{-\alpha}\right] \\ &\leq \mathbb{E}\left[W_N(\theta, \cdot) g(\theta, \cdot)\right]^\alpha \mathbb{E}\left[g(\theta, \cdot)^{-\frac{\alpha}{1-\alpha}}\right]^{1-\alpha} \end{aligned} \tag{2.6}$$

by Hölder's inequality. For every $i \geq 1$, $E_{X_i}^\omega[\exp\{\langle \theta, X_{i+1} - X_i\rangle - \log \phi(\theta)\}]$ and $\langle \theta, v(T_{X_i}\omega) - \xi_o\rangle$ are correlated, c.f. (2.23), where $v(\cdot)$ denotes the random drift vector. We could try to exploit this fact by tilting the environment at the points on the path in a clever way, e.g., by choosing a $g(\theta, \cdot)$ that penalizes the environments for which $\frac{1}{N}\sum_{i=1}^N \langle \theta, v(T_{X_i}\omega) - \xi_o\rangle$ deviates from zero. This way, we could make the first expectation in (2.6) small. However, there is a problem: we do not know where the path is, and if we naively tilt the environment everywhere, then the second expectation in (2.6) might become too large. Fortunately, it is possible to resolve this issue by first decomposing $\mathbb{E}[W_N(\theta, \cdot)^\alpha]$ as in (2.5) (so that we know roughly where the path is), and then tilting the environment on a tube which contains most of the path with a high probability.

Given $m \geq 1$, $\theta \notin sp\{e_2\}$, $C_1 \geq 1$ and $Y = (y_1, \ldots, y_m) \in (\mathbb{V}_2)^m$, let

$$B_j := \{(s,i) \in \mathbb{Z}^2 : (j-1)n \leq i < jn, \left|(s,i) - \lfloor i\xi(\theta)\rfloor - \sqrt{n} y_{j-1}\right| \leq C_1\sqrt{n}\} \tag{2.7}$$

for every $j \in \{1, \ldots, m\}$. Here, $y_o = (0,0)$. Recall that $n = k^2$ for some integer k.

Fix a large K and a small δ_n, both to be determined later (depending on the choice of α, see (2.12), (2.13) and Lemma 2.4). Define $f_K(u) := -K\mathbb{1}_{u \geq \mathrm{e}^{K^2}}$ and

$$g(\theta, \omega, Y) := \exp\sum_{j=1}^m f_K\left(\delta_n D(B_j)\right) > 0, \tag{2.8}$$

where

$$D(B_j) := \sum_{(s,i) \in B_j} a(\theta, (s,i)) \qquad \text{for every } j \in \{1, \ldots, m\}, \tag{2.9}$$

and $a(\theta,x) := \langle\theta, v(T_x\omega) - \xi_o\rangle$ for every $x\in\mathbb{Z}^2$, c.f. (1.4). Note that $\mathbb{E}[a(\theta,x)] = 0$.

As before, take $N = nm$. By Hölder's inequality,

$$\begin{aligned}\mathbb{E}[\bar{W}_N(\theta,\cdot,Y)^\alpha] &= \mathbb{E}\left[(\bar{W}_N(\theta,\cdot,Y)g(\theta,\cdot,Y))^\alpha g(\theta,\cdot,Y)^{-\alpha}\right]\\ &\le \mathbb{E}\left[\bar{W}_N(\theta,\cdot,Y)g(\theta,\cdot,Y)\right]^\alpha \mathbb{E}\left[g(\theta,\cdot,Y)^{-\frac{\alpha}{1-\alpha}}\right]^{1-\alpha}. \end{aligned}\tag{2.10}$$

Let us control the second term in (2.10). B_j's are pairwise disjoint and they each have $n(2C_1\sqrt{n}+1)$ elements. Since the environment is i.i.d.,

$$\begin{aligned}\mathbb{E}\left[g(\theta,\cdot,Y)^{-\frac{\alpha}{1-\alpha}}\right] &= \mathbb{E}\left[\exp\left(-\frac{\alpha}{1-\alpha}\sum_{j=1}^m f_K\left(\delta_n D(B_j)\right)\right)\right] = \prod_{j=1}^m \mathbb{E}\left[\exp\left(-\frac{\alpha}{1-\alpha}f_K\left(\delta_n D(B_j)\right)\right)\right]\\ &= \mathbb{E}\left[\exp\left(-\frac{\alpha}{1-\alpha}f_K\left(\delta_n D(B_1)\right)\right)\right]^m \le \left(1+\mathrm{e}^{\frac{\alpha}{1-\alpha}K}\mathbb{P}\left(\delta_n D(B_1)\ge \mathrm{e}^{K^2}\right)\right)^m. \end{aligned}\tag{2.11}$$

Note that, by Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}\left(\delta_n D(B_1)\ge \mathrm{e}^{K^2}\right) &\le \mathrm{e}^{-2K^2}\delta_n^2\mathbb{E}\left[D(B_1)^2\right] = \mathrm{e}^{-2K^2}\delta_n^2\mathbb{E}\left[\sum_{(s,i)\in B_1} a(\theta,(s,i))^2\right]\\ &= \mathrm{e}^{-2K^2}\delta_n^2 n(2C_1\sqrt{n}+1)\mathbb{E}\left[a(\theta,(0,0))^2\right]\\ &\le \mathrm{e}^{-2K^2}\delta_n^2 3C_1 n^{3/2}\mathbb{E}\left[a(\theta,(0,0))^2\right]\end{aligned}$$

since, by the i.i.d. assumption on the environment, only the diagonal terms survive. Take

$$\delta_n = C_1^{-1/2}n^{-3/4}, \tag{2.12}$$

where C_1 is still to be defined (and will be chosen as in Lemma 2.4). Then, the RHS of (2.11) is bounded from above by

$$\left(1+3\mathbb{E}\left[a(\theta,(0,0))^2\right]\mathrm{e}^{\frac{\alpha}{1-\alpha}K-2K^2}\right)^m \le \left(1+12\mathrm{e}^{\frac{\alpha}{1-\alpha}K-2K^2}\right)^m \le 2^m$$

as soon as

$$12\mathrm{e}^{\frac{\alpha}{1-\alpha}K-2K^2}\le 1. \tag{2.13}$$

Recalling (2.5) and (2.10), we see that

$$\mathbb{E}[W_N(\theta,\cdot)^\alpha] \le 2^m\sum_Y \mathbb{E}\left[\bar{W}_N(\theta,\cdot,Y)g(\theta,\cdot,Y)\right]^\alpha. \tag{2.14}$$

2.4. **Estimating the expectation under the tilt ($d = 1+1$).** For every $m\ge 1$, $\theta\notin sp\{e_2\}$, $\omega\in\Omega$ and $Y\in(\mathbb{V}_2)^m$, let $N = nm$ as before. By the Markov property,

$$\begin{aligned}\bar{W}_N(\theta,\omega,Y) &= \sum_{x_1,\ldots,x_m\in\mathbb{Z}^2} E_o^\omega[\exp\{\langle\theta,X_N\rangle - N\log\phi(\theta)\}, X_{jn}-\lfloor jn\xi(\theta)\rfloor = x_j\in J_{y_j}\ \forall j\le m]\\ &= \sum_{x_1,\ldots,x_m\in\mathbb{Z}^2} E_o^\omega[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, X_n-\lfloor n\xi(\theta)\rfloor = x_1\in J_{y_1}]\\ &\qquad\times E^\omega_{x_1+\lfloor n\xi(\theta)\rfloor}[\exp\{\langle\theta, X_n-(x_1+\lfloor n\xi(\theta)\rfloor)\rangle - n\log\phi(\theta)\},\\ &\qquad\qquad X_n - \lfloor 2n\xi(\theta)\rfloor = x_2\in J_{y_2}]\\ &\qquad\times\cdots\\ &= \sum_{x_1,\ldots,x_m\in\mathbb{Z}^2} E_o^\omega[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, X_n-\lfloor n\xi(\theta)\rfloor = x_1\in J_{y_1}]\\ &\qquad\times E^{T_{\lfloor n\xi(\theta)\rfloor+\sqrt{n}y_1}\omega}_{x_1-\sqrt{n}y_1}[\exp\{\langle\theta, X_n-(x_1-\sqrt{n}y_1)\rangle - n\log\phi(\theta)\},\\ &\qquad\qquad X_n - \lfloor n\xi(\theta)\rfloor = x_2-\sqrt{n}y_1\in J_{y_2}-\sqrt{n}y_1]\\ &\qquad\times\cdots.\end{aligned}$$

Recall (2.8) and (2.9). It follows from the i.i.d. environment assumption that

$$\begin{aligned}
&\mathbb{E}\left[\bar{W}_N(\theta,\cdot,Y)g(\theta,\cdot,Y)\right]\\
&\quad= \sum_{x_1,\ldots,x_m} \mathbb{E}\left[E_o^\omega[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor = x_1 \in J_{y_1}]\right.\\
&\qquad\qquad \times E_{x_1-\sqrt{n}y_1}^{T_{\lfloor n\xi(\theta)\rfloor+\sqrt{n}y_1}\omega}\left[\exp\{\langle\theta, X_n - (x_1-\sqrt{n}y_1)\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\},\right.\\
&\qquad\qquad\qquad\qquad\left. X_n - \lfloor n\xi(\theta)\rfloor = x_2 - \sqrt{n}y_1 \in J_{y_2} - \sqrt{n}y_1\right]\\
&\qquad\qquad \left.\times\cdots\right]\\
&\quad= \sum_{x_1,\ldots,x_m} E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor = x_1 \in J_{y_1}]\\
&\qquad\quad \times E_{x_1-\sqrt{n}y_1}\left[\exp\{\langle\theta, X_n - (x_1-\sqrt{n}y_1)\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\},\right.\\
&\qquad\qquad\qquad\qquad\left. X_n - \lfloor n\xi(\theta)\rfloor = x_2 - \sqrt{n}y_1 \in J_{y_2} - \sqrt{n}y_1\right]\\
&\qquad\quad \times\cdots\\
&\quad\le E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_1}]\\
&\qquad \times \max_{x_1\in J_{y_1}} E_{x_1-\sqrt{n}y_1}\left[\exp\{\langle\theta, X_n - (x_1-\sqrt{n}y_1)\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\},\right.\\
&\qquad\qquad\qquad\qquad\left. X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_2} - \sqrt{n}y_1\right]\\
&\qquad \times\cdots\\
&\quad= E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_1}]\\
&\qquad \times \max_{x_1\in J_o} E_{x_1}\left[\exp\{\langle\theta, X_n - x_1\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_{y_2-y_1}\right]\\
&\qquad \times\cdots.
\end{aligned}$$

Plugging this in (2.14), we conclude that

$$\mathbb{E}[W_N(\theta,\cdot)^\alpha] \le \left(2\sum_{y\in\mathbb{V}_2}\max_{x\in J_o} E_x\left[\exp\{\langle\theta,X_n-x\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_y\right]^\alpha\right)^m.$$

The RHS of this inequality decays exponentially in m if the term in the parentheses is strictly less than 1. Since $N = nm$ and n was fixed, this proves Lemma 2.1 (and hence Theorem 1.5), provided that we have

Lemma 2.4. *Assume (1.1) and (1.2). If $d = 1+1$, $\alpha\in(0,1)$, $\theta\notin sp\{e_2\}$ and $\delta_n = C_1^{-1/2}n^{-3/4}$, then*

$$\sum_{y\in\mathbb{V}_2}\max_{x\in J_o} E_x\left[\exp\{\langle\theta,X_n-x\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}, X_n - \lfloor n\xi(\theta)\rfloor \in J_y\right]^\alpha < 1/2 \tag{2.15}$$

whenever n, K and C_1 are sufficiently large.

(The proof is valid with the constant 1/2 replaced by any arbitrarily small positive number.)

2.5. **Finishing the proof of Theorem 1.5.** It remains to give the

Proof of Lemma 2.4. We write the sum in (2.15) as

$$\sum_{y\in\mathbb{V}_2}\max_{x\in J_o} E_x\left[\cdots\right]^\alpha = \sum_{\substack{y\in\mathbb{V}_2:\\|y|>R}}\max_{x\in J_o} E_x\left[\cdots\right]^\alpha + \sum_{\substack{y\in\mathbb{V}_2:\\|y|\le R}}\max_{x\in J_o} E_x\left[\cdots\right]^\alpha \tag{2.16}$$

with some large constant R, to be determined. Since $f_K(u) = -K\mathbb{1}_{u\ge e^{K^2}} \le 0$, the first sum on the RHS of (2.16) is bounded from above by

$$\begin{aligned}
&\sum_{\substack{y\in\mathbb{V}_2:\\|y|>R}}\max_{x\in J_o} E_x\left[\exp\{\langle\theta,X_n-x\rangle - n\log\phi(\theta)\}, |X_n - \lfloor n\xi(\theta)\rfloor - \sqrt{n}y| \le \frac{\sqrt{n}}{2}\right]^\alpha\\
&\le \sum_{\substack{y\in\mathbb{V}_2:\\|y|>R}} E_o\left[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \left|\frac{X_n - \lfloor n\xi(\theta)\rfloor}{\sqrt{n}} - y\right| \le 1\right]^\alpha. \qquad (2.17)
\end{aligned}$$

Consider a tilted space-time walk on $\mathbb{Z}^2$ (in a deterministic environment) with transition probabilities $q^\theta(z) := q(z)\exp\{\langle\theta, z\rangle - \log\phi(\theta)\}$ for $z \in \mathcal{R}_{st}$. Let $\hat{P}_o^\theta$ denote the probability measure it induces on paths. Note that the LLN velocity under $\hat{P}_o^\theta$ is

$$\sum_{z\in\mathcal{R}_{st}} zq(z)\exp\{\langle\theta, z\rangle - \log\phi(\theta)\} = \nabla\log\phi(\theta) = \xi(\theta).$$

With this notation, (2.17) is equal to

$$\sum_{\substack{y\in\mathbb{V}_2:\\ |y|>R}} \hat{P}_o^\theta\left(\left|\frac{X_n - \lfloor n\xi(\theta)\rfloor}{\sqrt{n}} - y\right| \le 1\right)^\alpha \le \sum_{\substack{y\in\mathbb{V}_2:\\ |y|>R}} \hat{P}_o^\theta\left(\left|\frac{X_n - \lfloor n\xi(\theta)\rfloor}{\sqrt{n}}\right| \ge |y| - 1\right)^\alpha$$

which, by Chebyshev's inequality, can be made arbitrarily small (uniformly in large n) by choosing R sufficiently large.

The second sum on the RHS of (2.16) is bounded from above by

$$(2R+1)\max_{x\in J_o} E_x\left[\exp\{\langle\theta, X_n - x\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1))\}\right]^\alpha.$$

Therefore, to conclude the proof of Lemma 2.4, it suffices to show that

$$E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1 - x))\}\right] \le \left(\frac{1}{8R}\right)^{\alpha^{-1}} \tag{2.18}$$

for every $x \in J_o$.

Similar to B_1 defined in (2.7), introduce a new set

$$\bar{B}_1 := \{(s,i)\in\mathbb{Z}^2 : 0 \le i < n, |(s,i) - \lfloor i\xi(\theta)\rfloor| \le (C_1 - 1/2)\sqrt{n}\}.$$

Note that $\bar{B}_1 \subset B_1 - x$ for every $x \in J_o$ since $|x| \le \sqrt{n}/2$.

$$\begin{aligned}
&E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta) + f_K(\delta_n D(B_1 - x))\}\right]\\
&\quad= \mathrm{e}^{-K} E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n D(B_1 - x) \ge \mathrm{e}^{K^2}\right]\\
&\qquad + E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0\le i<n\} \not\subset \bar{B}_1, \delta_n D(B_1 - x) < \mathrm{e}^{K^2}\right]\\
&\qquad + E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0\le i<n\} \subset \bar{B}_1, \delta_n D(B_1 - x) < \mathrm{e}^{K^2}\right]\\
&\quad\le \mathrm{e}^{-K} + \hat{P}_o^\theta\left(\{X_i : 0\le i<n\} \not\subset \bar{B}_1\right)\\
&\qquad + E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0\le i<n\} \subset \bar{B}_1, \delta_n D(B_1 - x) < \mathrm{e}^{K^2}\right].
\end{aligned} \tag{2.19}$$

The first term in (2.19) is small when K is large. Donsker's invariance principle ensures that the second term can be made arbitrarily small (uniformly in n) by choosing C_1 sufficiently large.

Let us focus on the third term in (2.19). For any sequence $(A_n)_{n\geq 1}$ of natural numbers,

$$\begin{aligned}
&E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0\leq i<n\}\subset \bar{B}_1, \delta_n D(B_1-x) < \mathrm{e}^{K^2}] \\
&\quad\leq E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0\leq i<n\}\subset \bar{B}_1, \delta_n \sum_{\substack{(s,i)\in B_1-x\\(s,i)\neq X_i}} a(\theta,(s,i)) < -A_n] \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \{X_i : 0\leq i<n\}\subset \bar{B}_1, \delta_n \sum_{i=0}^{n-1} a(\theta,X_i) < \mathrm{e}^{K^2}+A_n] \\
&\quad\leq \sum_{x_1,\ldots,x_{n-1}} \mathbb{E}[E_o^\omega\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, X_i = x_i\ \forall i<n\right], \delta_n \sum_{\substack{(s,i)\in B_1-x\\(s,i)\neq x_i}} a(\theta,(s,i)) < -A_n] \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta,X_i) < \mathrm{e}^{K^2}+A_n] \\
&\quad= \sum_{x_1,\ldots,x_{n-1}} E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, X_i = x_i\ \forall i<n\right] \times \mathbb{P}(\delta_n \sum_{\substack{(s,i)\in B_1-x\\(s,i)\neq x_i}} a(\theta,(s,i)) < -A_n) \qquad (2.20)\\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta,X_i) < \mathrm{e}^{K^2}+A_n] \\
&\quad\leq \max_{x_1,\ldots,x_{n-1}} \mathbb{P}(\delta_n \sum_{\substack{(s,i)\in B_1-x\\(s,i)\neq x_i}} a(\theta,(s,i)) < -A_n) \\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta,X_i) < \mathrm{e}^{K^2}+A_n] \\
&\quad\leq A_n^{-2}\delta_n^2 2C_1 n^{3/2}\mathbb{E}\left[a(\theta,(0,0))^2\right] \qquad (2.21)\\
&\qquad + E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1} a(\theta,X_i) < \mathrm{e}^{K^2}+A_n].
\end{aligned}$$

Here, (2.20) follows from the independence assumption on the environment, and (2.21) is an application of Chebyshev's inequality. Since $\delta_n = C_1^{-1/2}n^{-3/4}$, the first term in (2.21) goes to zero as $n\to\infty$ if $A_n\to\infty$.

Choose A_n such that $A_n\to\infty$ and $A_n = o(n^{1/4})$ as $n\to\infty$. For any $\mu\in\mathbb{R}^+$, the second term in (2.21) is equal to

$$\begin{aligned}
&E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n \sum_{i=0}^{n-1}(a(\theta,X_i)-\mu) < \mathrm{e}^{K^2}+A_n-\mu n\delta_n] \\
&\quad\leq M_n E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}\sum_{i=0}^{n-1}(a(\theta,X_i)-\mu)^2] \qquad (2.22)\\
&\qquad + M_n E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}\sum_{i\neq j}(a(\theta,X_i)-\mu)(a(\theta,X_j)-\mu)]
\end{aligned}$$

by Chebyshev's inequality, where $M_n = \left(\frac{\delta_n}{\mu n\delta_n - A_n - \mathrm{e}^{K^2}}\right)^2 = O(n^{-2})$.

By the FKG inequality (c.f. [8]),

$$\begin{aligned}
E_o\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}a(\theta,(0,0))\right] &= \mathbb{E}\left[E_o^\omega\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}\right]a(\theta,(0,0))\right] \qquad (2.23)\\
&> \mathbb{E}\left[E_o^\omega\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}\right]\right]\mathbb{E}\left[a(\theta,(0,0))\right] = 0
\end{aligned}$$

since $E_o^\omega\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}\right]$ and $a(\theta,(0,0))$ are easily checked to be either both strictly increasing functions (when $\langle\theta, e_1\rangle > 0$) or both strictly decreasing functions (when $\langle\theta, e_1\rangle < 0$) of the random variable $\pi((0,0),(1,1))$. If we choose

$$\mu = E_o\left[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}a(\theta,(0,0))\right], \qquad (2.24)$$

then the second term in (2.22) vanishes by the independence assumption on the environment. Finally, observe that the first term in (2.22) is equal to

$$nM_nE_o\left[\exp\{\langle\theta,X_1\rangle-\log\phi(\theta)\}(a(\theta,(0,0))-\mu)^2\right]=O(n^{-1}).\quad\square$$

2.6. **Proof of Theorem 1.6.** Let us recall a few points regarding the arguments in Subsections 2.3 – 2.5. There, since $d=1+1$, the volume of B_1 (defined in (2.7)) is $O(n^{3/2})$. The variance of $D(B_1)$ (c.f. (2.9)) scales like that volume. We take $\delta_n=O(n^{-3/4})$ so that the variance of $\delta_nD(B_1)$ is $O(1)$. With this choice, $n\delta_n\to\infty$ as $n\to\infty$. As we saw, this fact is crucial in the proof of Theorem 1.5.

In this subsection, we will assume that $d=2+1$. For every $m\geq 1$, $1\leq j\leq m$, $\theta\notin sp\{e_3\}$, $C_1\geq 1$ and $Y=(y_1,\ldots,y_m)\in(\mathbb{V}_3)^m$, we define

$$B_j:=\{(r,k):r\in\mathbb{Z}^2,(j-1)n\leq k<jn,\left|(r,k)-\lfloor k\xi(\theta)\rfloor-\sqrt{n}y_{j-1}\right|\leq C_1\sqrt{n}\},\tag{2.25}$$

similar to (2.7). Note that the volume of this new set is $O(n^2)$. If we were to define $D(B_1)$ analogously to (2.9), then we would have to take $\delta_n\leq O(n^{-1})$ in order to make the variance of $\delta_nD(B_1)$ not grow with n, in which case $n\delta_n$ remains bounded. Hence, the proof for $d=1+1$ does not directly carry over to the case $d=2+1$.

To resolve this issue, following [10], we will modify the proof by redefining $D(B_1)$ and δ_n. (We will continue using these names so that we can refer to the parts of Subsections 2.3 – 2.5 that carry over word by word.) The modification amounts essentially to using a tilting that is quadratic, instead of linear, in the local drift, as follows.

For every (r,k) and (s,l) with $r,s\in\mathbb{Z}^2$ and $k,l\geq 1$, let

$$V((r,k),(s,l)):=\frac{1}{|k-l|}1\!\!1_{\{|(s,l)-(r,k)-\lfloor(l-k)\xi(\theta)\rfloor|<C_2\sqrt{|k-l|}\}}\tag{2.26}$$

if $k\neq l$, and set it to be equal to zero if $k=l$. Here, the constant $C_2\geq 1$ will be determined later. Given any n integer and $x_1,\ldots,x_n\in\mathbb{Z}^3$ with $\langle x_k,e_3\rangle=k$, it follows easily from (2.26) that

$$\text{for any } s\in\mathbb{Z}^2,\ l\in\{1,\ldots,n\},\qquad \sum_{k=1}^n V(x_k,(s,l))\leq 2\log n,$$

$$\sum_{k=1}^n\sum_{(s,l)\in B_1}V(x_k,(s,l))\leq\sum_{\substack{1\leq k,l\leq n\\k\neq l}}\frac{1}{|k-l|}\left(2C_2\sqrt{|k-l|}\right)^2\leq 4C_2^2n^2,$$

$$\begin{aligned}\sum_{(s,l)\in B_1}\left(\sum_{k=1}^n V(x_k,(s,l))\right)^2&=\left(\max_{(s',l')}\sum_{k=1}^n V(x_k,(s',l'))\right)\sum_{(s,l)\in B_1}\sum_{k=1}^n V(x_k,(s,l))\\&\leq(2\log n)(4C_2^2n^2)=8C_2^2n^2\log n,\quad\text{and}\end{aligned}\tag{2.27}$$

$$\begin{aligned}\sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}}V((r,k),(s,l))^2&\leq\sum_{k=1}^n(2C_1\sqrt{n})^2\sum_{l=1}^n\frac{1\!\!1_{\{k\neq l\}}}{|k-l|^2}(2C_2\sqrt{|k-l|})^2\\&=16C_1^2C_2^2n\sum_{\substack{1\leq k,l\leq n\\k\neq l}}\frac{1}{|k-l|}\leq 32C_1^2C_2^2n^2\log n.\end{aligned}\tag{2.28}$$

Recall the tilted law $\hat{P}_o^\theta$ introduced in the proof of Lemma 2.4.

Lemma 2.5. *For any $\delta>0$, there exists a $C_2\geq 1$ such that $\nu(n,X):=\sum_{1\leq i,j\leq n}V(X_i,X_j)$ satisfies*

$$\hat{P}_o^\theta(\nu(n,X)<n\log(n-1)/2)\leq\delta$$

for every $n\geq 2$.

Proof. For any realization of $X=(X_i)_{i\geq 1}$,

$$\nu(n,X)\leq\sum_{\substack{1\leq i,j\leq n\\i\neq j}}\frac{1}{|i-j|}=:H(n).$$

Observe that

$$\hat{E}_o^\theta[\nu(n,X)] = \sum_{1\leq i,j\leq n} \hat{E}_o^\theta[V(X_i,X_j)] = \sum_{\substack{1\leq i,j\leq n\\ i\neq j}} \frac{1}{|i-j|}\hat{P}_o^\theta(|X_i - X_j - \lfloor (i-j)\xi(\theta)\rfloor| < C_2\sqrt{|i-j|}).$$

When C_2 is sufficiently large, the CLT implies that

$$\hat{P}_o^\theta(|X_i - X_j - \lfloor (i-j)\xi(\theta)\rfloor| < C_2\sqrt{|i-j|}) \geq (1-\delta/2)$$

for any $i\neq j$. Therefore, $\hat{E}_o^\theta[\nu(n,X)] \geq (1-\delta/2)H(n)$. Applying Markov's inequality, we see that

$$\hat{P}_o^\theta(\nu(n,X) < H(n)/2) = \hat{P}_o^\theta(H(n)-\nu(n,X) > H(n)/2) \leq \delta.$$

This implies the desired result since $H(n)\geq n\log(n-1)$. □

For any $\theta\in\mathbb{R}^3$ and $x\in\mathbb{Z}^3$, define $a(\theta,x) := \langle\theta, v(T_x\omega)-\xi_o\rangle$ as before, where $v(\omega)=\sum_{z\in\mathcal{R}}\pi(0,z)z$.

Lemma 2.6. *There exists a $\beta>0$ such that*

$$\mu := E_o\left[\exp\{\langle\theta,X_1\rangle - \log\phi(\theta)\}a(\theta,(0,0,0))\right] > 0$$

whenever $\operatorname{dist}(\theta, sp\{e_3\})\in(0,\beta)$.

Proof. For every $\theta\notin sp\{e_3\}$, let

$$F(\theta) := \mathbb{E}\{E_o^\omega[\mathrm{e}^{\langle\theta,X_1\rangle}]E_o^\omega[\langle\theta,X_1\rangle]\}\quad\text{and}\quad G(\theta) := E_o[\mathrm{e}^{\langle\theta,X_1\rangle}]E_o[\langle\theta,X_1\rangle] = \phi(\theta)\langle\theta,\xi_o\rangle.$$

Our aim is to show that $F(\theta)>G(\theta)$.

Write $\theta = ce_3+\theta'$ for some $c\in\mathbb{R}$ and $\theta'\in\mathbb{R}^3$ such that $\langle\theta',e_3\rangle=0$. Then, $F(\theta)=\mathrm{e}^cF(\theta')+c\mathrm{e}^c\phi(\theta')$ and $G(\theta)=\mathrm{e}^cG(\theta')+c\mathrm{e}^c\phi(\theta')$. Therefore, it suffices to show that $F(\theta')>G(\theta')$.

Clearly, we have

$$\nabla F(\theta)|_{\theta=0} = \nabla G(\theta)|_{\theta=0} = E_o[X_1]=\xi_o.$$

Also, for any $u,u'\in\mathbb{R}^3$, with D^2F denoting the Hessian of F,

$$\langle u, D^2F(\theta)u'\rangle\big|_{\theta=0} = 2\mathbb{E}\{E_o^\omega[\langle X_1,u\rangle]E_o^\omega[\langle X_1,u'\rangle]\}$$

and

$$\langle u, D^2G(\theta)u'\rangle\big|_{\theta=0} = 2E_o[\langle X_1,u\rangle]E_o[\langle X_1,u'\rangle] = 2\langle\xi_o,u\rangle\langle\xi_o,u'\rangle.$$

By Schwarz' inequality (which is strict since the walk is uniformly elliptic in the directions other than e_3),

$$\inf_{\substack{|u|=1\\ \langle u,e_3\rangle=0}}\left(\langle u,D^2F(\theta)u\rangle\big|_{\theta=0} - \langle u,D^2G(\theta)u\rangle\big|_{\theta=0}\right) > 0.$$

Finally, Taylor's theorem implies the existence of a $\beta>0$ such that $F(\theta')>G(\theta')$ whenever $|\theta'|\in(0,\beta)$. □

Now, we are ready to give the new definition of $D(B_1)$ which is suitable for $d=2+1$. For any $\theta\in\mathbb{R}^3$ such that $\operatorname{dist}(\theta,sp\{e_3\})\in(0,\beta)$ (with β as in Lemma 2.6), let

$$D(B_1) := \sum_{\substack{(r,k)\in B_1,\\ (s,l)\in B_1}} V((r,k),(s,l))a(\theta,(r,k))a(\theta,(s,l)). \tag{2.29}$$

Note that $V((\cdot,k),(\cdot,k))=0$ for every $1\leq k\leq n$. Since $\mathbb{E}[a(\theta,0)]=0$, it follows from the independence of the environment that $\mathbb{E}[D(B_1)]=0$. Also, $\mathbb{E}[D(B_1)^2]\leq 1024|\theta|^4C_1^2C_2^2n^2\log n$ by (2.28) and the fact that $|a(\theta,0)|\leq 2|\theta|$.

If we choose

$$\delta_n := n^{-1}(\log n)^{-1/2},$$

then the variance of $\delta_nD(B_1)$ is $O(1)$. Once we have this fact, the arguments in Subsections 2.3 – 2.5 carry over until (2.18). So, it suffices to show that $E_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}, \delta_nD(B_1-x)<\mathrm{e}^{K^2}]$ is small for all $x\in J_o$ when n and K are large. In the estimate below, we will (WLOG) take $x=0$.

Let $\gamma = 1/2$, and observe that

$$\begin{aligned}
&E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \delta_n D(B_1) < \mathrm{e}^{K^2}] \\
&\quad\le E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \nu(n,X) \ge \gamma n \log(n-1), \delta_n D(B_1) < \mathrm{e}^{K^2}] \\
&\qquad + E_o\left[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \nu(n,X) < \gamma n \log(n-1)\right] \\
&\quad= E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}, \nu(n,X) \ge \gamma n \log(n-1), \\
&\qquad\qquad \delta_n(D(B_1) - \mu^2\nu(n,X)) < \mathrm{e}^{K^2} - \mu^2\delta_n\nu(n,X)] + \hat{P}_o^\theta(\nu(n,X) < \gamma n\log(n-1)) \\
&\quad\le M_n E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}(D(B_1) - \mu^2\nu(n,X))^2, \nu(n,X) \ge \gamma n\log(n-1)] \qquad (2.30)\\
&\qquad + \hat{P}_o^\theta(\nu(n,X) < \gamma n\log(n-1)) \\
&\quad\le M_n E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}(D(B_1) - \mu^2\nu(n,X))^2] + \hat{P}_o^\theta(\nu(n,X) < \gamma n\log(n-1)). \qquad (2.31)
\end{aligned}$$

Here, (2.30) follows from the elementary inequality $\mathbb{1}_{a<b} \le a^2/b^2$ with $a = \delta_n(D(B_1) - \mu^2\nu(n,X))$ and $b = \mathrm{e}^{K^2} - \mu^2\delta_n\nu(n,X) < 0$, and

$$M_n = \left(\frac{\delta_n}{\mu^2\delta_n\gamma n\log(n-1) - \mathrm{e}^{K^2}}\right)^2. \tag{2.32}$$

Choose C_2 sufficiently large so that the second term in (2.31) is small for all $n \ge 2$ by Lemma 2.5.

It remains to control the first term in (2.31). Note that

$$\begin{aligned}
&D(B_1) - \mu^2\nu(n,X) \\
&\quad= 2\mu\sum_{k=1}^{n}\sum_{(s,l)\in B_1} V(X_k,(s,l))(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}}) \\
&\qquad+ \sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}} V((r,k),(s,l))(a(\theta,(r,k)) - \mu\mathbb{1}_{\{X_k=(r,k)\}})(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}}),
\end{aligned}$$

and

$$\begin{aligned}
&E_o[\exp\{\langle\theta, X_n\rangle - n\log\phi(\theta)\}(D(B_1) - \mu^2\nu(n,X))^2] \\
&\quad\le 8\mu^2 E_o\left[\exp\{\cdots\}\left(\sum_{k=1}^{n}\sum_{(s,l)\in B_1} V(X_k,(s,l))(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}})\right)^2\right] \\
&\qquad+ 2E_o\left[\exp\{\cdots\}\left(\sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}} V((r,k),(s,l))(a(\theta,(r,k)) - \mu\mathbb{1}_{\{X_k=(r,k)\}})(a(\theta,(s,l)) - \mu\mathbb{1}_{\{X_l=(s,l)\}})\right)^2\right] \\
&\quad=: 8\mu^2\mathfrak{E}_1 + 2\mathfrak{E}_2
\end{aligned} \tag{2.33}$$

by the inequality $(a+b)^2 \le 2(a^2+b^2)$.

One should note at this stage that in fact, even though μ was chosen to equal the mean under the tilted measure of $a(\theta, 0)$, it is not necessarily the case that the mean of $D(B_1) - \mu^2\nu(n,X)$ under that tilted measure vanishes. This makes the control of $\mathfrak{E}_i$ somewhat messy, involving a local CLT (Lemma 2.7).

We turn to the details of the computation. $\mathfrak{E}_1$ can be written as a double sum over pairs $(s,l), (s',l') \in B_1$. If $(s,l) \ne (s',l')$, then it is clear from independence that this pair does not contribute to $\mathfrak{E}_1$ on the event

$\{X_l \neq (s,l)\} \cup \{X_{l'} \neq (s',l')\}$. Therefore,

$$
\begin{aligned}
\mathfrak{E}_1 &= E_o\left[\exp\{\cdots\}\sum_{(s,l)\in B_1}\left(\sum_{k=1}^n V(X_k,(s,l))(a(\theta,(s,l)) - \mu 1\!\!1_{\{X_l=(s,l)\}})\right)^2\right] \\
&\quad + \sum_{\substack{k,k',l,l':\\ l\neq l'}} E_o\left[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}V(X_k,X_l)V(X_{k'},X_{l'})(a(\theta,X_l)-\mu)(a(\theta,X_{l'})-\mu)\right] \\
&=: E_o\left[\exp\{\cdots\}\sum_{(s,l)\in B_1}\left(\sum_{k=1}^n V(X_k,(s,l))(a(\theta,(s,l)) - \mu 1\!\!1_{\{X_l=(s,l)\}})\right)^2\right] + \sum_{\substack{k,k',l,l':\\ l\neq l'}} \mathfrak{E}_1(k,k',l,l') \\
&\leq (2|\theta|+\mu)^2 E_o\left[\exp\{\cdots\}\sum_{(s,l)\in B_1}\left(\sum_{k=1}^n V(X_k,(s,l))\right)^2\right] + \sum_{\substack{k,k',l,l':\\ l\neq l'}} \mathfrak{E}_1(k,k',l,l') \\
&\leq (2|\theta|+\mu)^2 8C_2^2 n^2\log n + \sum_{\substack{k,k',l,l':\\ l\neq l'}} \mathfrak{E}_1(k,k',l,l') \qquad (2.34)
\end{aligned}
$$

by (2.27) and the fact that $|a(\theta,\cdot)| \leq 2|\theta|$.

If $l' > \max(k,k',l)$, then $\mathfrak{E}_1(k,k',l,l')$ is equal to zero since we can condition on the path up to l' and use the fact that, for any $(x_i)_1^{l'}$,

$$
E_o\left[\exp\{\langle\theta,X_n - X_{l'}\rangle - (n-l')\log\phi(\theta)\}(a(\theta,X_{l'})-\mu)\;\Big|\;(X_i)_1^{l'} = (x_i)_1^{l'}\right] = 0
$$

by the definition of μ, c.f. Lemma 2.6.

If $l < l' < k' < k$, then $V(X_k,X_l)$ and $V(X_{k'},X_{l'})$ create a slight complication since X_k and $X_{k'}$ are not independent of $X_{l'+1} - X_{l'}$. Indeed,

$$
\begin{aligned}
\mathfrak{E}_1(k,k',l,l') = \sum_{\substack{x_1,\ldots,x_{l'}\\ z\in\mathcal{R}}} & E_o\left[\exp\{\cdots\}(a(\theta,X_l)-\mu)(a(\theta,X_{l'})-\mu), (X_i)_1^{l'} = (x_i)_1^{l'}, X_{l'+1}-X_{l'} = z\right] \\
& \times \hat{E}_o^\theta[V(X_k,x_l)V(X_{k'},x_{l'}) \,|\, X_{l'+1} = x_{l'}+z],
\end{aligned}
$$

and the latter expectation depends on z. (If it were independent of z, we could simply take the sum over $z\in\mathcal{R}$ and conclude that $\mathfrak{E}_1(k,k',l,l') = 0$.) However, for any $z,z'\in\mathcal{R}$,

$$
\begin{aligned}
&\left|\hat{E}_o^\theta[V(X_k,x_l)V(X_{k'},x_{l'}) \,|\, X_{l'+1} = x_{l'}+z] - \hat{E}_o^\theta[V(X_k,x_l)V(X_{k'},x_{l'}) \,|\, X_{l'+1} = x_{l'}+z']\right| \\
&\leq \sum_{\substack{x_{k'}:\\ V(x_{k'},x_{l'})>0}} (k'-l')^{-1}\left|\hat{P}_o^\theta(X_{k'} = x_{k'}|X_{l'+1} = x_{l'}+z) - \hat{P}_o^\theta(X_{k'} = x_{k'}|X_{l'+1} = x_{l'}+z')\right| \\
&\qquad\qquad \times \hat{E}_o^\theta[V(X_k,x_l) \,|\, X_{k'} = x_{k'}] \\
&\leq 4C_2^2(k'-l')(k'-l')^{-1}O((k'-l')^{-3/2})(k-l)^{-1} = O((k-l)^{-1}(k'-l')^{-3/2})
\end{aligned}
$$

uniformly in $(x_i)_1^{l'}$, c.f. Lemma 2.7 (given below). Hence,

$$
\sum_{l<l'<k'<k} \mathfrak{E}_1(k,k',l,l') \leq O(n^2\log n).
$$

It is easy to see that this technique works for $\mathfrak{E}_1(k,k',l,l')$ in all other cases, and we get $\mathfrak{E}_1 \leq O(n^2\log n)$ by (2.34).

$\mathfrak{E}_2$ is a quadruple sum over $(r,k),(r',k'),(s,l),(s',l')\in B_1$ that is symmetric in (r,k) and (s,l) (as well as in (r',k') and (s',l')). Recall that $V((r,k),(s,l)) = 0$ when $(r,k) = (s,l)$. If $(r,k)\notin\{(r',k'),(s',l')\}$, then it is clear from independence that there is no contribution to $\mathfrak{E}_2$ on the event $\{X_k \neq (r,k)\}$. The contribution from the complementary event can be estimated using Lemma 2.7, just like in the case of $\mathfrak{E}_1$.

Putting everything together and recalling (2.28), we see that

$$\begin{aligned}
\mathfrak{E}_2 &\leq 2E_o\left[\exp\{\cdots\}\sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}}\big(V((r,k),(s,l))(a(\theta,(r,k))-\mu 1\!\!1_{\{X_k=(r,k)\}})(a(\theta,(s,l))-\mu 1\!\!1_{\{X_l=(s,l)\}})\big)^2\right]\\
&\quad + O(n^2\log n)\\
&\leq 2(2|\theta|+\mu)^4 E_o\left[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}\sum_{\substack{(r,k)\in B_1,\\(s,l)\in B_1}} V((r,k),(s,l))^2\right] + O(n^2\log n)\\
&\leq 2(2|\theta|+\mu)^4 32 C_1^2C_2^2 n^2\log n + O(n^2\log n)\\
&\leq O(n^2\log n).
\end{aligned}$$

Finally,

$$M_nE_o[\exp\{\langle\theta,X_n\rangle - n\log\phi(\theta)\}(D(B_1)-\mu^2\nu(n,X))^2] \leq M_n(8\mu^2\mathfrak{E}_1 + 2\mathfrak{E}_2) \leq O((\log n)^{-1})$$

by (2.32) and (2.33). This concludes the proof of Theorem 1.6, apart from

Lemma 2.7. *For any $z,z'\in\mathcal{R}$,*

$$\sup_{x\in\mathbb{Z}^3}|\hat{P}^\theta_z(X_m=x)-\hat{P}^\theta_{z'}(X_m=x)| \leq O(m^{-3/2}) \quad \textit{as } m\to\infty.$$

Proof. Let G^θ be the centered Gaussian density on $\mathbb{R}^2$ that has the same covariance with $(\langle X_1,e_1\rangle,\langle X_1,e_2\rangle)$ under $\hat{P}^\theta_o$. For any $z\in\mathcal{R}$, it is shown in Theorem 22.1 of [2] that

$$\sup_x\left|\hat{P}^\theta_z(X_m=x) - \frac{2}{m}G^\theta\left(\frac{\langle x-z-m\xi(\theta),e_1\rangle}{\sqrt{m}},\frac{\langle x-z-m\xi(\theta),e_2\rangle}{\sqrt{m}}\right)\right| \leq O(m^{-3/2}) \quad \text{as } m\to\infty.$$

Here, the supremum is taken over all $x=(x_1,x_2,x_3)\in\mathbb{Z}^3$ such that x_1+x_2+m+1 is even and $x_3=m+1$. (Otherwise, $\hat{P}^\theta_z(X_m=x)$ is equal to zero.) Since $\sup_{y\in\mathbb{R}^2}|\nabla_yG^\theta(y)|<\infty$, the desired result follows from the triangle inequality. □

3. Inequality of the rate functions for space-only RWRE

3.1. **Reducing to a fractional moment estimate.** Consider space-only RWRE on $\mathbb{Z}^d$ with $d\geq 1$. Assume that the walk is non-nestling relative to the canonical basis vector e_d. By Jensen's inequality, the quenched and the averaged logarithmic moment generating functions

$$\Lambda_q(\theta) := \lim_{N\to\infty}\frac{1}{N}\log E^\omega_o\left[\exp\{\langle\theta,X_N\rangle\}\right] \quad\text{and}\quad \Lambda_a(\theta) := \lim_{N\to\infty}\frac{1}{N}\log E_o\left[\exp\{\langle\theta,X_N\rangle\}\right]$$

satisfy $\Lambda_q(\theta)\leq\Lambda_a(\theta)\leq|\theta|$ for every $\theta\in\mathbb{R}^d$.

Recall the definition of regeneration times $(\tau_n)_{n\geq 0}$ (relative to e_d) given in Subsection 1.2. Let

$$\beta := \inf\{i\geq 0 : \langle X_i,e_d\rangle < \langle X_o,e_d\rangle\}\in[1,\infty].$$

By the non-nestling assumption, there exist constants $c_2,c_3>0$ such that

$$\operatorname*{ess\,inf}_{\mathbb{P}} P^\omega_o(\beta=\infty)\geq c_2 \quad\text{and}\quad \operatorname*{ess\,sup}_{\mathbb{P}} P^\omega_o(\tau_1>n)\leq \mathrm{e}^{-c_3n} \tag{3.1}$$

for every $n\geq 1$, c.f. [16]. These bounds clearly imply that

$$\operatorname*{ess\,sup}_{\mathbb{P}} E^\omega_o[\exp\{c\tau_1\}\,|\,\beta=\infty] \leq c_2^{-1}\operatorname*{ess\,sup}_{\mathbb{P}} E^\omega_o[\exp\{c\tau_1\}] =: H(c) < \infty \tag{3.2}$$

whenever $c<c_3$.

For every $c\in(0,c_3]$, introduce the set

$$\mathcal{C}(c) := \{\theta\in\mathbb{R}^d : 2|\theta|<c\}. \tag{3.3}$$

Lemma 3.1. *For every $\theta \in \mathcal{C}(c_3)$,*

$$E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} \,|\, \beta = \infty] = 1. \tag{3.4}$$

Λ_a is analytic on $\mathcal{C}(c_3)$. $\nabla\Lambda_a(0) = \xi_o$. The Hessian $\mathcal{H}_a$ of Λ_a is positive definite on $\mathcal{C}(c_3)$. For every $c < c_3$ and $\theta \in \mathcal{C}(c)$, the smallest eigenvalue of $\mathcal{H}_a(\theta)$ is bounded from below by a positive constant that depends only on c and the ellipticity constant κ of the walk.

Proof. See the proofs of Lemmas 6 and 12 of [21]. In particular, the desired lower bound for the smallest eigenvalue of $\mathcal{H}_a$ is evident from equation (2.10) of that paper. □

Given any $N \geq 1$, $\theta \in \mathcal{C}(c_3)$ and $\omega \in \Omega$, define

$$\hat{W}_N(\theta,\omega) := E_o^\omega[\exp\{\langle\theta, X_{\tau_N}\rangle - \Lambda_a(\theta)\tau_N\}] \qquad \text{and}$$
$$W_N(\theta,\omega) := E_o^\omega[\exp\{\langle\theta, X_{\tau_N}\rangle - \Lambda_a(\theta)\tau_N\} \,|\, \beta = \infty].$$

Lemma 3.2. *For every $\theta \in \mathcal{C}(c_3)$, if*

$$\limsup_{N\to\infty} \frac{1}{N} \log \hat{W}_N(\theta,\cdot) < 0$$

holds $\mathbb{P}$-a.s., then $\Lambda_q(\theta) < \Lambda_a(\theta)$.

Proof. Let $\theta \in \mathcal{C}(c_3)$. Then, $\theta \in \mathcal{C}(c)$ for some $c < c_3$. By hypothesis, for $\mathbb{P}$-a.e. ω, there exist $C_3 \geq 1$ and $c_4 > 0$ (both depending on ω) such that $\hat{W}_N(\theta,\omega) \leq C_3 \mathrm{e}^{-c_4 N}$ for every $N \geq 1$.

Given any $n \geq 1$ and $K \geq 1$, it follows from Chebyshev's inequality and (3.2) that

$$\begin{aligned}
&E_o^\omega[\exp\{\langle\theta, X_n\rangle - \Lambda_a(\theta)n\}]\\
&\quad= E_o^\omega[\exp\{\langle\theta, X_n\rangle - \Lambda_a(\theta)n\}, n < \tau_{\lfloor\frac{n}{K}\rfloor}] + \sum_{j=\lfloor\frac{n}{K}\rfloor}^{n} E_o^\omega[\exp\{\langle\theta, X_n\rangle - \Lambda_a(\theta)n\}, \tau_j \leq n < \tau_{j+1}]\\
&\quad\leq \mathrm{e}^{2|\theta|n} P_o^\omega(n < \tau_{\lfloor\frac{n}{K}\rfloor}) + \sum_{j=\lfloor\frac{n}{K}\rfloor}^{n} E_o^\omega[\exp\{\langle\theta, X_{\tau_j}\rangle - \Lambda_a(\theta)\tau_j\}] \operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{2|\theta|\tau_1\} \,|\, \beta = \infty]\\
&\quad\leq \mathrm{e}^{(2|\theta|-c)n} E_o^\omega[\exp\{c\tau_{\lfloor\frac{n}{K}\rfloor}\}] + \sum_{j=\lfloor\frac{n}{K}\rfloor}^{n} \hat{W}_j(\theta,\omega) \operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{c\tau_1\} \,|\, \beta = \infty]\\
&\quad\leq \mathrm{e}^{(2|\theta|-c)n} E_o^\omega[\exp\{c\tau_1\}] \left(\operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{c\tau_1\} \,|\, \beta = \infty]\right)^{\lfloor\frac{n}{K}\rfloor - 1}\\
&\qquad+ \sum_{j=\lfloor\frac{n}{K}\rfloor}^{n} \hat{W}_j(\theta,\omega) \operatorname*{ess\,sup}_{\mathbb{P}} E_o^{\omega'}[\exp\{c\tau_1\} \,|\, \beta = \infty]\\
&\quad\leq \mathrm{e}^{(2|\theta|-c)n} H(c)^{\lfloor\frac{n}{K}\rfloor} + H(c) \sum_{j=\lfloor\frac{n}{K}\rfloor}^{n} C_3 \mathrm{e}^{-c_4 j}.
\end{aligned}$$

Take K sufficiently large, and conclude that

$$\Lambda_q(\theta) - \Lambda_a(\theta) = \lim_{n\to\infty} \frac{1}{n} \log E_o^\omega[\exp\{\langle\theta, X_n\rangle - \Lambda_a(\theta)n\}] < 0. \quad \square$$

Lemma 3.3. *For every $\theta \in \mathcal{C}(c_3)$, if*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right] < 0 \tag{3.5}$$

for some $\alpha \in (0,1)$, then $\Lambda_q(\theta) < \Lambda_a(\theta)$. Hence, by convex duality, $I_a < I_q$ at $\xi = \nabla\Lambda_a(\theta)$.

Proof. For any $N \geq 1$ and $\theta \in \mathcal{C}(c_3)$, it follows from the renewal structure and (3.4) that

$$\begin{aligned}
\mathbb{E}[P_o^\omega(\beta = \infty) W_N(\theta,\cdot)] &= P_o(\beta = \infty) E_o[\exp\{\langle\theta, X_{\tau_N}\rangle - \Lambda_a(\theta)\tau_N\} \,|\, \beta = \infty]\\
&= P_o(\beta = \infty) \left(E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} \,|\, \beta = \infty]\right)^N\\
&= P_o(\beta = \infty).
\end{aligned}$$

Given any $\alpha \in (0,1)$, by the same reasoning as in (2.1),

$$\limsup_{N\to\infty} \frac{1}{N} \log \hat{W}_N(\theta,\cdot) \le \limsup_{N\to\infty} \frac{1}{N\alpha} \log \mathbb{E}\left[\hat{W}_N(\theta,\cdot)^\alpha\right], \quad \mathbb{P}\text{-a.s.} \tag{3.6}$$

On the other hand, if $2|\theta| < c < c_3$, then we see by subadditivity, Chebyshev's inequality, and (3.2) that

$$\begin{aligned}
\mathbb{E}\left[\hat{W}_{N+1}(\theta,\cdot)^\alpha\right] &= \mathbb{E}\left[\left(\sum_{x\in\mathbb{Z}^d} E_o^\omega[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}, X_{\tau_1} = x] W_N(\theta, T_x\cdot)\right)^\alpha\right]\\
&\le \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} \left(E_o^\omega[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}, X_{\tau_1} = x]\right)^\alpha W_N(\theta, T_x\cdot)^\alpha\right]\\
&\le \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} \left(E_o^\omega[\exp\{2|\theta|\tau_1\}, \tau_1 \ge |x|_1]\right)^\alpha W_N(\theta, T_x\cdot)^\alpha\right]\\
&\le \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} \left(\mathrm{e}^{(2|\theta|-c)|x|_1} E_o^\omega[\exp\{c\tau_1\}]\right)^\alpha W_N(\theta, T_x\cdot)^\alpha\right]\\
&\le H(c)^\alpha \mathbb{E}\left[W_N(\theta,\cdot)^\alpha\right] \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{(2|\theta|-c)\alpha|x|_1}. \qquad (3.7)
\end{aligned}$$

The desired result follows immediately from (3.6), (3.7) and Lemma 3.2. $\square$

3.2. **The correlation condition.** In this subsection, we will consider space-only RWRE on $\mathbb{Z}^d$ with $d = 2,3$, assume that the walk is non-nestling relative to e_d, and outline how one can modify the arguments given in Section 2 in order to reduce (3.5) to a simpler inequality.

We start with $d = 2$. For every $n \ge 1$ of the form k^2, and for every $y = (y', y'') \in \mathbb{Z}^2$, let

$$J_y := [(y' - \tfrac{1}{2})\sqrt{n}, (y' + \tfrac{1}{2})\sqrt{n}) \times [(y'' - \tfrac{1}{2})\sqrt{n}, (y'' + \tfrac{1}{2})\sqrt{n}) \subset \mathbb{R}^2,$$

c.f. (2.3). Take $N = nm$ for some $m \ge 1$. For every $\theta \in \mathcal{C}(c_3)$, $\omega \in \Omega$ and $Y = (y_1, \ldots, y_m) \in (\mathbb{Z}^2)^m$, define

$$\bar{W}_N(\theta,\omega,Y) := E_o^\omega[\exp\{\langle\theta, X_{\tau_N}\rangle - \Lambda_a(\theta)\tau_N\}, X_{\tau_{jn}} - \lfloor jn\zeta(\theta)\rfloor \in J_{y_j} \text{ for every } j \le m \big|\, \beta = \infty],$$

c.f. (2.4), where

$$\zeta(\theta) := E_o[X_{\tau_1} \exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} |\, \beta = \infty]. \tag{3.8}$$

By subadditivity,

$$\mathbb{E}[W_N(\theta,\cdot)^\alpha] \le \sum_Y \mathbb{E}\left[\bar{W}_N(\theta,\cdot,Y)^\alpha\right],$$

c.f. (2.5). Given any $C_1 \ge 1$, $Y = (y_1, \ldots, y_m) \in (\mathbb{Z}^2)^m$ and $j \in \{1, \ldots, m\}$, let

$$\begin{aligned}
B_j = B_j(y_{j-1}, y_j) := \{(s,i) \in \mathbb{Z}^2 : (j-1)n\langle\zeta(\theta), e_2\rangle + \sqrt{n}(y''_{j-1} + 1/2) \le i < jn\langle\zeta(\theta), e_2\rangle + \sqrt{n}(y''_j - 1/2),\\
|(s - \sqrt{n}y'_{j-1}) - \frac{\langle\zeta(\theta), e_1\rangle}{\langle\zeta(\theta), e_2\rangle}(i - \sqrt{n}y''_{j-1})| \le C_1\sqrt{n}\},
\end{aligned}$$

c.f. (2.7). Also, redefine $a(\theta,\cdot)$ by setting

$$a(\theta,x) := \langle\theta, v(T_x\omega)\rangle - \mathbb{E}[\langle\theta, v(\cdot)\rangle]$$

for every $x \in \mathbb{Z}^2$, where $v(\omega) = \sum_{z\in\mathcal{R}} \pi(0,z)z$ as before. Note that, under the assumptions stated in Definition 1.7, we have $\mathbb{E}[\langle\theta, v(\cdot)\rangle] = \langle\theta, \xi_o\rangle$. However, this equality does not necessarily hold in general.

With these modified definitions, the arguments in Subsections 2.3 and 2.4 easily carry over, once one replaces the i.i.d. random variables

$$E_o^{T_{X_i}\omega}[\exp\{\langle\theta, X_1\rangle - \log\phi(\theta)\}]$$

by the variables

$$E_o^{T_{X_{\tau_i}}\omega}[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} | \beta = \infty].$$

Therefore, in order to prove (3.5), it suffices to show that

$$\sum_{y\in\mathbb{Z}^2}\max_{x\in J_o} E_x\left[\exp\{\langle\theta, X_{\tau_n}-x\rangle-\Lambda_a(\theta)\tau_n+f_K(\delta_n D(B_1))\}, X_{\tau_n}-\lfloor n\zeta(\theta)\rfloor\in J_y\,|\,\beta=\infty\right]^\alpha<1/2 \tag{3.9}$$

when $\zeta(\theta)$ is as in (3.8) and n, K, C_1 are sufficiently large, c.f. Lemma 2.4. Here, $\alpha\in(0,1)$ is fixed, $f_K(u):=-K\mathbb{1}_{u\geq e^{K^2}}$ and $\delta_n=C_1^{-1/2}n^{-3/4}$, as before.

We imitate (2.16), and write the sum in (3.9) as

$$\sum_{y\in\mathbb{Z}^2}\max_{x\in J_o}E_x\left[\cdots\right]^\alpha=\sum_{\substack{y\in\mathbb{Z}^2:\\|y|>R}}\max_{x\in J_o}E_x\left[\cdots\right]^\alpha+\sum_{\substack{y\in\mathbb{Z}^2:\\|y|\leq R}}\max_{x\in J_o}E_x\left[\cdots\right]^\alpha \tag{3.10}$$

with some large constant R, to be determined. Just like in the space-time case, the first sum on the RHS of (3.10) is bounded from above by

$$\sum_{\substack{y\in\mathbb{Z}^2:\\|y|>R}}\hat{P}_o^\theta\left(\left|\frac{X_n-\lfloor n\zeta(\theta)\rfloor}{\sqrt{n}}\right|\geq|y|-1\right)^\alpha. \tag{3.11}$$

Here, $\hat{P}_o^\theta$ is redefined to be the probability measure on paths induced by the random walk (in a deterministic environment) whose transition probabilities are given by

$$q^\theta(x):=E_o[\exp\{\langle\theta,X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\},X_{\tau_1}=x|\,\beta=\infty],\quad x\in\mathbb{Z}^2.$$

(Note that $\sum_{x\in\mathbb{Z}^2}q^\theta(x)=1$ by (3.4).) If $\hat{E}_o^\theta$ denotes the corresponding expectation, it is clear that

$$\hat{E}_o^\theta[\exp\{c|X_1|\}]<\infty\quad\text{ for every } c\in(0,c_3-2|\theta|). \tag{3.12}$$

Therefore, by Chebyshev's inequality, (3.11) can be made arbitrarily small (uniformly in large n) by choosing R sufficiently large.

The second sum on the RHS of (3.10) can be controlled by showing that

$$\max_{\substack{y\in\mathbb{Z}^2:\\|y|\leq R}}\max_{x\in J_o}E_x\left[\exp\{\langle\theta,X_{\tau_n}-x\rangle-\Lambda_a(\theta)\tau_n+f_K(\delta_nD(B_1))\},X_{\tau_n}-\lfloor n\zeta(\theta)\rfloor\in J_y\,|\,\beta=\infty\right] \tag{3.13}$$

is small when n, K and C_1 are sufficiently large. In the space-time case, the verification of the analogous statement, i.e., (2.18), relied on the fact that

$$E_o[\exp\{\langle\theta,X_n\rangle-n\log\phi(\theta)\}\sum_{i=0}^{n-1}a(\theta,X_i)] \tag{3.14}$$

grows linearly in n, c.f. (2.21) and (2.22). In the space-only case, the drift vectors at the points off the path do not contribute to the mean of $D(B_1)$ under the tilted measure, and the drift vector at any point on the path contributes only once even if it is visited multiple times. Therefore, the statement concerning (3.14) needs to be replaced by the statement that

$$E_o[\exp\{\langle\theta,X_{\tau_n}\rangle-\Lambda_a(\theta)\tau_n\}\sum_{x\in S(X,\tau_n)}a(\theta,x)|\beta=\infty] \tag{3.15}$$

grows linearly in n. Here, for any $j\geq1$,

$$S(X,j):=\{X_i:0\leq i<j\}. \tag{3.16}$$

In the space-time case, the variance of $D(B_1)$ under the tilted measure was shown to be $O(n^{3/2})$ since the only non-vanishing terms were those corresponding to points $x,y\in\mathbb{Z}^2$ such that $x=y$. In the space-only case, steps of the walk between consecutive regeneration times are not independent, and we therefore need to also consider terms corresponding to x and y that are both on the path in the same regeneration block. However, since regeneration times have exponentially decaying tails, the total contribution of such terms is $O(n)$, and the variance of $D(B_1)$ under the tilted measure is still $O(n^{3/2})$.

With these modifications, the argument in Subsection 2.5 enables us to deduce (3.5) provided that (3.15) grows linearly in n. By the renewal structure, the latter is equivalent to the following *correlation condition*:

$$\mu:=E_o[\exp\{\langle\theta,X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\}\sum_{x\in S(X,\tau_1)}a(\theta,x)|\beta=\infty]>0. \tag{3.17}$$

(This replaces the choice of μ for the space-time case, see (2.24).) For $d = 3$, after modifying (3.13) by (i) taking the first maximum over $\{y \in \mathbb{Z}^3 : |y| \leq R\}$, (ii) replacing the sets J_y and B_1 by their three dimensional analogs, and (iii) redefining $D(B_1)$ as in (2.29), one can employ the reasoning above in order to reduce (3.5) to showing that (3.13) is small when n, K and C_1 are sufficiently large. After that, one can set $\delta_n := n^{-1}(\log n)^{-1/2}$, apply the same kind of modifications to the argument given in Subsection 2.6, and further reduce (3.5) to (3.17). In particular, note that Lemma 2.7 continues to hold under the new definition of $\hat{P}_o^\theta$, thanks to (3.12). We omit the (routine) details.

We have arrived at the following theorem.

Theorem 3.4. *Consider space-only RWRE on $\mathbb{Z}^d$ with $d = 2, 3$. Assume (1.1), (1.3) and that the walk is non-nestling relative to e_d. Then, there exists an open set $\mathcal{A}_{so} \subset \mathbb{R}^d$ with the following properties:*

- (i) *I_a is strictly convex and analytic on $\mathcal{A}_{so}$,*
- (ii) *$\xi_o \in \mathcal{A}_{so}$, and*
- (iii) *for every $\xi \in \mathcal{A}_{so}$, the strict inequality $I_a(\xi) < I_q(\xi)$ holds if (3.17) is satisfied at $\theta := \nabla I_a(\xi)$.*

Proof. Recall (3.3), and define

$$\mathcal{A}_{so} := \{\nabla\Lambda_a(\theta) : \theta \in \mathcal{C}(c_3)\}.$$

It follows from Lemma 3.1 and the inverse function theorem that I_a is strictly convex and analytic on $\mathcal{A}_{so}$ which is an open set containing ξ_o.

Take any $\xi \in \mathcal{A}_{so}$. Note that $\theta := \nabla I_a(\xi)$ satisfies $\xi = \nabla\Lambda_a(\theta)$ by convex duality. As outlined above, (3.17) implies (3.5). Hence, the desired result follows from Lemma 3.3. □

3.3. **Proof of Theorem 1.8.** Consider space-only RWRE on $\mathbb{Z}^d$ with $d = 2, 3$. Fix a triple $p = (p^+, p^o, p^-)$ of positive real numbers such that $p^- < p^+$ and $p^+ + p^o + p^- = 1$. Assume that $\mathbb{P}$ is in class $\mathcal{M}_\epsilon(d, p)$ for some small $\epsilon > 0$, c.f. Definition 1.7. Assume that $\epsilon \leq \frac{p^o}{4(d-1)}$ so that the ellipticity constant κ of the walk satisfies

$$\kappa \geq \min\left(p^+, p^-, \frac{p^o}{4(d-1)}\right). \tag{3.18}$$

Lemma 3.5. *There exist $C_4 \geq 1$ and $c_5 > 0$ (depending only on p) such that $|\Lambda_a(\theta) - \langle\theta, \xi_o\rangle| \leq C_4|\theta|^2$ holds for every $\theta \in \mathcal{C}(c_5)$.*

Proof. Recall (3.1). Note that c_3 depends only on the law of the regeneration times which, in turn, is determined by the fixed triple p. Moreover, the ellipticity constant κ of the walk satisfies (3.18). Fix any $c_5 < c_3$. The desired result follows immediately from Lemma 3.1. □

Consider the set

$$\mathcal{C}_t(c_5) := \{\theta \in \mathcal{C}(c_5) : \langle\theta, e_d\rangle = 0\}.$$

(Here, the subscript stands for *transversal*.) Take any $\theta \in \mathcal{C}_t(c_5)$. Recall the notation in (3.16). Since $\mathbb{P}$ is in class $\mathcal{M}_\epsilon(d, p)$, it is easy to see that

$$\xi_o = (p^+ - p^-)e_d, \qquad \langle\theta, \xi_o\rangle = 0, \qquad \text{and} \qquad |a(\theta, x)| = |\langle\theta, v(T_x\omega)\rangle| \leq 2\epsilon(d-1)|\theta| \tag{3.19}$$

for every $x \in \mathbb{Z}^d$. Similarly, the isotropy assumption ensures that

$$Z(\theta) = Z(\theta, X, \tau_1, \omega) := \sum_{x \in S(X, \tau_1)} a(\theta, x)$$

satisfies

$$E_o[Z(\theta)|\beta = \infty] = E_o[\tau_1 Z(\theta)|\beta = \infty] = 0. \tag{3.20}$$

Our aim is to show that

$$E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}Z(\theta)|\beta = \infty] > 0 \tag{3.21}$$

for certain choices of θ, to be determined later. Expanding the exponential on the LHS of (3.21), we see that

$$E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}Z(\theta)|\beta = \infty] \geq E_o[(1 + \langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1)Z(\theta)|\beta = \infty] - C_5|\theta|^3 \tag{3.22}$$

$$= E_o[\langle\theta, X_{\tau_1}\rangle Z(\theta)|\beta = \infty] - C_5|\theta|^3. \tag{3.23}$$

Indeed, (3.22) follows from $|Z(\theta)| \le 2\epsilon(d-1)|\theta|\tau_1$ and

$$1 + \langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1 \le \exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} \le 1 + \langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1 + 2|\theta|^2\tau_1^2 \exp\{2|\theta|\tau_1\},$$

C_5 is some constant that depends only on p and c_5 and finally, (3.20) implies (3.23).

In order to estimate (3.23), we first provide a more convenient representation of the RWRE. Let $(b_i)_{i\ge 0}$ be an i.i.d. sequence of random variables taking values in $\{e_d, 0, -e_d\}$, with

$$P(b_1 = e_d) = p^+, \quad P(b_1 = 0) = p^o, \quad \text{and} \quad P(b_1 = -e_d) = p^-.$$

Let $(f_i)_{i\ge 0}$ be another i.i.d. sequence of random variables (independent of $(b_i)_{i\ge 0}$) taking values in the set $\{\pm e_j : 1 \le j < d\} \cup \{0\}$, with

$$P(f_1 = 0) = \frac{2\epsilon(d-1)}{p^o} \quad \text{and} \quad P(f_1 = \pm e_j) = \frac{1}{2(d-1)} - \frac{\epsilon}{p^o} \quad \text{if } 1 \le j < d.$$

For any $\omega \in \Omega$, the walk $(X_i)_{i\ge 0}$ under P_o^ω can be constructed by setting

$$X_{i+1} - X_i := b_i + (1 - |b_i|)f_i + (1 - |b_i|)(1 - |f_i|)U_i\,,$$

where $(U_i)_{i\ge 0}$ is a sequence of independent random variables taking values in $\{\pm e_j : 1 \le j < d\}$, with

$$P^\omega(U_i = \pm e_j | \mathcal{F}_i) = \frac{\pi(X_i, X_i \pm e_j) - (\frac{p^o}{2(d-1)} - \epsilon)}{2\epsilon(d-1)}.$$

Here, $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$. Note that the laws of the sequences $(b_i)_{i\ge 0}$ and $(f_i)_{i\ge 0}$ do not depend on the environment, and that τ_1 is a function of $(b_i)_{i\ge 0}$ only.

Let

$$N_i := \sum_{j=0}^{i-1} \mathbb{1}_{1=(1-|b_i|)(1-|f_i|)}\,.$$

Introduce the events $L_0 := \{N_{\tau_1} = 0\}$, $L_1 := \{N_{\tau_1} = 1\}$, and $L_2 := \{N_{\tau_1} \ge 2\}$. Let $\mathcal{G} := \sigma((b_i, f_i)_{i\ge 0})$. Note that the events L_0, L_1 and L_2 are $\mathcal{G}$-measurable, and so is the event $\{\beta = \infty\}$. On the event L_0, the walker never sees the environment until τ_1, and thus X_{τ_1} is $\mathcal{G}$-measurable. Also, for any $i \ge 0$, on the event $\{X_i \notin S(X, i)\}$ (i.e., when X_i is a fresh point), $a(\theta, X_i)$ is independent of $\mathcal{F}_i$ and $\mathcal{G}$ under P_o. Therefore, by isotropy,

$$E_o[Z(\theta)|\mathcal{G}] = E_o\left[\left.\sum_{i=0}^{\tau_1-1} a(\theta, X_i)\mathbb{1}_{X_i \notin S(X,i)}\right| \mathcal{G}\right] = 0.$$

Putting these observations together, we see that

$$E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_0, \beta = \infty] = E_o[\langle \theta, X_{\tau_1}\rangle E_o[Z(\theta)|\mathcal{G}], L_0, \beta = \infty] = 0. \tag{3.24}$$

On the other hand, it is easy to check that $P_o(L_2) \le c_6\epsilon^2$ for some $c_6 = c_6(p)$. By Hölder's inequality,

$$|E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_2, \beta = \infty]| \le P_o(L_2)^{2/3} E_o[|\langle \theta, X_{\tau_1}\rangle Z(\theta)|^3, \beta = \infty]^{1/3} \le c_7\epsilon^{7/3}|\theta|^2 \tag{3.25}$$

for some $c_7 = c_7(p) > 0$. (Recall that $a(\theta, \cdot) \le 2\epsilon(d-1)|\theta|$, c.f. (3.19).)

Finally, let $L_1^\ell = L_1 \cap \{(1 - |b_\ell|)(1 - |f_\ell|) = 1, \ell < \tau_1\}$. Then,

$$E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_1, \beta = \infty] = \sum_{\ell=0}^{\infty} E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_1^\ell, \beta = \infty]. \tag{3.26}$$

For every $\ell \ge 0$,

$$\begin{aligned} E_o[\langle \theta, X_{\tau_1}\rangle Z(\theta), L_1^\ell, \beta = \infty] &= E_o[\langle \theta, X_\ell\rangle Z(\theta), L_1^\ell, \beta = \infty] \\ &\quad + E_o[\langle \theta, X_{\ell+1} - X_\ell\rangle Z(\theta), L_1^\ell, \beta = \infty] \\ &\quad + E_o[\langle \theta, X_{\tau_1} - X_{\ell+1}\rangle Z(\theta), L_1^\ell, \beta = \infty]. \end{aligned} \tag{3.27}$$

By computations similar to the one involving L_0, the first and the third terms on the RHS of (3.27) are zero. The second term is equal to

$$\begin{aligned}
E_o[\langle\theta, X_{\ell+1}-X_\ell\rangle\langle\theta, v(T_{X_\ell}\omega)\rangle, L_1^\ell, \beta=\infty] &= P_o(L_1^\ell,\beta=\infty)\mathbb{E}[E^\omega[\langle\theta,U_o\rangle]\langle\theta,v(\omega)\rangle] \\
&= P_o(L_1^\ell,\beta=\infty)\mathbb{E}\left[\left(\sum_{z\neq\pm e_d}\frac{\pi(0,z)-(\frac{p^o}{2(d-1)}-\epsilon)}{2\epsilon(d-1)}\langle\theta,z\rangle\right)\langle\theta,v(\omega)\rangle\right] \\
&= \frac{P_o(L_1^\ell,\beta=\infty)}{2\epsilon(d-1)}\mathbb{E}\left[\langle\theta,v(\omega)\rangle^2\right].
\end{aligned}$$

Therefore, by (3.26),

$$E_o[\langle\theta, X_{\tau_1}\rangle Z(\theta), L_1, \beta=\infty] = \frac{P_o(L_1,\beta=\infty)}{2\epsilon(d-1)}\mathbb{E}\left[\langle\theta,v(\omega)\rangle^2\right].$$

It is easy to see that $P_o(L_1,\beta=\infty)\geq c_8\epsilon$ for some $c_8=c_8(p)>0$ if ϵ is small enough. Also, part (c) of Definition 1.7 ensures that $\mathbb{E}\left[\langle\theta,v(\omega)\rangle^2\right]\geq c_9\epsilon^2|\theta|^2$ for some $c_9=c_9(p)>0$. Hence,

$$E_o[\langle\theta, X_{\tau_1}\rangle Z(\theta), L_1, \beta=\infty]\geq c_{10}\epsilon^2|\theta|^2 \tag{3.28}$$

for some $c_{10}=c_{10}(p)>0$. Combining (3.24), (3.25) and (3.28) gives

$$\begin{aligned}
E_o[\langle\theta, X_{\tau_1}\rangle Z(\theta)|\beta=\infty]-C_5|\theta|^3 &\geq c_{10}\epsilon^2|\theta|^2-c_7\epsilon^{7/3}|\theta|^2-C_5|\theta|^3 \\
&= \left((c_{10}-c_7\epsilon^{1/3})\epsilon^2-C_5|\theta|\right)|\theta|^2.
\end{aligned}\tag{3.29}$$

If $\epsilon<(c_{10}/c_7)^3$, then, for every $\theta\in\mathcal{C}_t(c_5)$ such that $0<|\theta|<(c_{10}-c_7\epsilon^{1/3})\epsilon^2/C_5$,

$$E_o[\exp\{\langle\theta,X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\}Z(\theta)|\beta=\infty]>0$$

by (3.23) and (3.29).

Finally, Theorem 3.4 implies that $I_a<I_q$ on the set

$$\{\nabla\Lambda_a(\theta):\theta\in\mathcal{C}_t(c_5),\ 0<|\theta|<(c_{10}-c_7\epsilon^{1/3})\epsilon^2/C_5\}$$

whose closure contains the LLN velocity $\xi_o=\nabla\Lambda_a(0)$. We have proved Theorem 1.8.

4. Open problems

Our technique of proof puts several restrictions on the class of models treated. The following are natural questions we have not addressed.

(1) Does Theorem 1.8 extend to all space-only RWRE in dimension $d=2,3$, or at least to those satisfying Sznitman's condition (**T**)? Note that, for non-nestling walks, it suffices to show that the correlation condition (3.17) is satisfied on a sequence $(\theta_n)_{n\geq1}$ that converges to zero, c.f. Theorem 3.4.

(2) In case $\sum\pi(0,z)\langle z,e\rangle$ is random for any $e\in\mathcal{R}_{so}$, is it true that $I_q(\xi)=I_a(\xi)$ only when $\xi=0$ or $I_a(\xi)=0$, as is the case in dimension $d=1$?

In our proof of Theorem 1.8 (specifically, in the proof of the correlation condition (3.17)), we used the isotropy assumption in order to get rid of a centering term under the (untilted) measure; this does not seem essential and probably, the lack of isotropy could be handled in the perturbative regime. However, getting rid of the perturbative restriction, or of the non-randomness in the e_d direction, requires additional arguments.

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