Addendum to: large deviations asymptotics for spherical integrals

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Abstract

We correct an omission in [3] and improve the main result there to a full large deviations trajectorial result.

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1 Addendum and Corrections

The paper [3] contains an annoying typo and a gap in the proof of Theorem 4.1. First, regarding the typo, the law of the Gaussian we considered in [3] is not (see [3, Section 5])

$$\mathbb{P}^{\beta}_{D_N}(dX_N) = \frac{1}{Z^{\beta}_N} e^{-\frac{N}{2}\mathrm{tr}(X_N - D_N)^2} dX_N \tag{1.1}$$

but rather should be

$$\mathbb{P}^{\beta}_{D_N}(dX_N) = \frac{1}{Z^{\beta}_N} e^{-\frac{N\beta}{4}\mathrm{tr}(X_N - D_N)^2} dX_N. \tag{1.2}$$

This error (which only concerns the case $\beta = 1$) spread over the whole paper and we shall indicate here how to correct it. Indeed, one of the main goals in [3] is to prove a large deviation principle for the law of the spectral measure of X_N under $\mathbb{P}^{\beta}_{D_N}$ with a good rate function $J_\beta(\mu_D, .)$ when the spectral measure of D_N converges towards μ_D. The rate function $J_\beta(\mu_D, .)$ defined in [3] corresponds to the definition of $\mathbb{P}^{\beta}_{D_N}$ given in (1.2) rather than (1.1) as stated in [3].

To prove the large deviations result, we consider matrix-valued paths $X_N(t) = D_N + H^{N,\beta}_t$ where $H^{N,\beta}$ is a Hermitian (if $\beta = 2$) or symmetric Brownian motion. In [3], we obtained a partial large deviation principle for the law of the spectral measure process $(\hat\mu^N_t, t \in [0,1])$ of $X_N(.)$ in the sense that the lower bound was not proved to be sharp. To deduce a full large deviation for the law of the spectral measure at time one, we used some approximation, and then showed in [3, Theorem 4.1] that this family of approximations is rich enough. However, in the proof of the latter theorem, c.f. (4.3) there, we use the results of Lemma 2.12

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without checking that the measure $\nu^{\delta,\epsilon,\Delta}$ satisfies Assumption (A). Instead of verifying this last point, we provide below an alternative proof of a stronger result, that bypasses the use of the free Brownian motion and has some nice consequences. Namely, we obtain directly the full large deviation principle for the law of $(\hat{\mu}_t^N, t \in [0,1])$, from which the contraction at time one becomes trivial. We emphasize that with the details provided below, Theorem 1.1 of [3] remains correct as stated (for $\beta = 2$), see Theorem 2.1 in the next section.

2 Missing $\frac{\beta}{2}$ factor

Let us first note that the definition of the spherical integrals in [3] should be

$$I_N^{(\beta)}(D_N, E_N) := \int \exp\{\frac{N\beta}{2} \mathrm{tr}(U D_N U^* E_N)\} dm_N^\beta(U),$$

where m_N^β denote the Haar measure on the orthogonal group $\mathcal{O}_N$ when $\beta = 1$ and on the unitary group $\mathcal{U}_N$ when $\beta = 2$, and D_N, E_N are diagonal real matrices whose spectral measures converge to μ_D, μ_E. In [3], we missed the factor $\frac{\beta}{2}$ coming in the law of the Gaussian variables we were considering. This point should be corrected in the whole article where all covariances terms should be multiplied by $\frac{\beta}{2}$. For instance, Theorem 1.1 in [3] becomes

Theorem 2.1 *Assume the existence of a compact subset $\mathcal{K}$ of $I\!R$ such that $\mathrm{supp}\,\hat{\mu}_{D_N}^N \subset \mathcal{K}$ for all $N \in I\!N$. Moreover, suppose that $\hat{\mu}_{E_N}^N(x^2)$ is uniformly bounded (in N). Suppose that $\hat{\mu}_{E_N}^N$ and $\hat{\mu}_{D_N}^N$ converge weakly towards μ_E and μ_D respectively. Then,*

$$I^{(\beta)}(\mu_D, \mu_E) := \lim_{N\to\infty} \frac{1}{N^2} \log I_N^{(\beta)}(D_N, E_N) = -J_\beta(\mu_D, \mu_E) + I_\beta(\mu_E) - \inf_{\mu \in M_1(I\!R)} I_\beta(\mu) + \frac{\beta}{4} \int x^2 d\mu_D(x)$$

where, for any $\mu \in M_1(I\!R)$,

$$I_\beta(\mu) = \frac{\beta}{4} \int x^2 d\mu(x) - \frac{\beta}{2} \int \log|x-y| d\mu(x) d\mu(y).$$

Other statements need to be corrected similarly.

3 Statements of the corrected results

We make throughout the following assumption on the sequence D_N:

Assumption 3.1

$$\lim_{N\to\infty} \hat{\mu}_{D_N}^N = \mu_D\,, \quad \sup_N \hat{\mu}_{D_N}^N(x^2) < \infty\,.$$

For both $\beta = 1$ and $\beta = 2$, the main large deviations result of [3] is contained in Theorem 1.4 there which states as follows:

Theorem 3.2 *1) For any $\mu \in M_1(I\!R)$, S_μ is a good rate function on $\mathcal{C}([0,1], M_1(I\!R))$.*

2) Assume Assumption 3.1. Then,

a) For any closed set $F \subset \mathcal{C}([0,1], M_1(\mathbb{R}))$

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}^N_. \in F\right) \le -\frac{\beta}{2} \inf_{\nu\in F} S_{\mu_D}(\nu).$$

b) Denote

$$\mathcal{A} = \{\nu \in \mathcal{C}([0,1], M_1(\mathbb{R}))\,;\ \ \exists \eta > 0,\ \sup_{t\in[0,1]} \nu_t(|x|^{5+\eta}) < \infty.\}$$

Then, for any open subset $O \in \mathcal{C}([0,1], M_1(\mathbb{R}))$*,*

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}^N_. \in O\right) \ge -\frac{\beta}{2} \inf_{\nu\in O\cap\mathcal{A}} S_{\mu_D}(\nu).$$

We improve here this theorem by showing the following :

Theorem 3.3 *Assume that* $\mu_D \in M_1(\mathbb{R})$ *satisfies, for some* $\epsilon > 0$*,*

$$\int |x|^{5+\epsilon} \mu_D(dx) < \infty\,. \tag{3.1}$$

Then, for any $\nu_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ *there exists a sequence* ν_n *in* $\mathcal{A}$ *such that*

$$\lim_{n\to\infty} \nu_n = \nu \quad \text{ and } \quad \lim_{n\to\infty} S_{\mu_D}(\nu_n) = S_{\mu_D}(\nu).$$

Consequently, for any open subset $O \in \mathcal{C}([0,1], M_1(\mathbb{R}))$*,*

$$\inf_{\nu\in O\cap\mathcal{A}} S_{\mu_D}(\nu) = \inf_{\nu\in O} S_{\mu_D}(\nu).$$

Hence, Theorems 3.2, 3.3 gives the full large deviation principle for the law of $(\hat{\mu}^N_t, t \in [0,1])$ with good rate function S_{μ_D}. As a consequence, the contraction principle shows that for $\mu_D, \hat{\mu}^N_{D_N}$ satisfying the assumptions of these theorems, the law of $\hat{\mu}^N_1$ satisfies a full large deviation principle with good rate function

$$J_\beta(\mu_D, \mu) = \frac{\beta}{2} \inf\{S_{\mu_D}(\nu); \nu_1 = \mu\}.$$

This yields Theorem 1.5, Theorem 4.1 and Corollary 1.6 in [3] under the additional assumption (3.1). All other theorems in [3] remain valid, except for the $(\beta/2)$ factor mentioned above.

4 Proof of Theorem 3.3

Let us recall the definition of S_{μ_D}; We set, for any $f, g \in \mathcal{C}^{2,1}_b(\mathbb{R}\times[0,1])$, any $s \le t \in [0,1]$, and any $\nu_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$,

$$\begin{aligned} S^{s,t}(\nu, f) \;&=\; \int f(x,t)d\nu_t(x) - \int f(x,s)d\nu_s(x) \\ &\quad - \int_s^t \int \partial_u f(x,u)d\nu_u(x)du - \frac{1}{2}\int_s^t \int\int \frac{\partial_x f(x,u) - \partial_x f(y,u)}{x-y} d\nu_u(x)d\nu_u(y)du, \end{aligned} \tag{4.1}$$

$$< f, g >^{\nu}_{s,t} \;=\; \int_s^t \int \partial_x f(x,u)\partial_x g(x,u)d\nu_u(x)du\,, \tag{4.2}$$

and

$$\bar{S}^{s,t}(\nu, f) = S^{s,t}(\nu, f) - \frac{1}{2} < f, f >^{\nu}_{s,t} \ . \tag{4.3}$$

Set, for any probability measure $\mu_D \in M_1(I\!\!R)$ and $0 \le u \le v \le 1$,

$$S^{u,v}(\nu) := \sup_{f \in \mathcal{C}_b^{2,1}(I\!\!R \times [0,1])} \sup_{u \le s \le t \le v} \bar{S}^{s,t}(\nu, f) \,,$$

and

$$S_{\mu_D}(\nu) := \begin{cases} +\infty \,, & \text{if } \nu_0 \neq \mu_D, \\ S^{0,1}(\nu)\,, & \text{otherwise.} \end{cases}$$

Since S_{μ_D} is a good rate function (see [3], Theorem 1.4(1)),for any sequence $\nu_n \in \mathcal{A}$ converging towards ν,

$$\limsup_{n \to \infty} S_{\mu_D}(\nu_n) \ge S_{\mu_D}(\nu),$$

so that we only need to prove the opposite inequality for a specific sequence $\nu_n \in \mathcal{A}$.

We begin by proving that there exists a sequence of paths ν^δ in $\mathcal{A}$ converging towards ν as δ converges to zero such that

$$\lim_{\delta \to 0} S^{0,1}(\nu^\delta) = S^{0,1}(\nu)\,. \tag{4.4}$$

To construct the sequence ν^δ, we proceed in two steps: first, fix $\eta > 0$ and define $\nu^\eta = \nu \boxed{+} \sigma_\eta$ to be the free convolution of ν with a semicircular variable of parameter η. Then [2], ν^η converges towards ν and $S^{0,1}(\nu^\eta)$ converges toward $S^{0,1}(\nu)$ as $\eta \to 0$. Further, by [1, Lemma 3 and Corollary 2], $\nu_t^\eta \ll dx$ with density $\rho_t^\eta(x)$ satisfying

$$\mathbf{1}_{\rho_t^\eta(x)>0} \left[\rho_t^\eta(x)^2 + (H\nu_t^\eta)^2(x)\right] \le \frac{1}{\eta}\,, \tag{4.5}$$

where, for $\mu \in M_1(I\!\!R)$, $H\mu(x) = PV \int (x-y)^{-1} \mu(dy)$ denotes the Stieltjes transform of μ.

Because $S^{0,1}(\nu^\eta) < \infty$ due to the assumption $S^{0,1}(\nu) < \infty$ (see [2, Theorem 4.1]), one may find, by Riesz's theorem, a field $k^\eta(x,t)$ such that $\partial_x k^\eta \in L^2(I\!\!R \times [0,1]; d\nu_t^\eta(x) \times dt)$ and

$$S^{0,1}(\nu^\eta, f) = \int_0^1 \int \partial_x f(x,s) \partial_x k^\eta(x,s) d\nu_s^\eta(x) ds\,,$$

for all $f \in \mathcal{C}_b^{2,1}(I\!\!R \times [0,1])$. Moreover,

$$S^{0,1}(\nu^\eta) = \frac{1}{2} \int_0^1 \int (\partial_x k^\eta(x,s))^2 d\nu_s^\eta(x) ds. \tag{4.6}$$

By [4, Pg. 170], since $\rho_t^\eta \in L^1 \cap L^\infty(I\!\!R) \subset L^3(I\!\!R)$ by (4.5), for any $f \in \mathcal{C}_b^{2,1}(I\!\!R \times [0,1])$,

$$\iint \frac{\partial_x f(x,s) - \partial_x f(y,s)}{x-y} d\nu_s^\eta(x) d\nu_s^\eta(y) = 2 \int \partial_x f(x,s) (H\nu_s^\eta)(x) d\nu_s^\eta(x) ds\,,$$

and, introducing $u_t^\eta(x) = u^\eta(x,t) := \partial_x k^\eta(x,t) + (H\nu_t^\eta)(x) \in L^2(I\!\!R \times [0,1]; d\nu_t^\eta(x) \times dt)$, it follows that,

$$\int f(x,1) d\nu_1^\eta(x) = \int f(x,0) d\nu_0^\eta(x) + \int_0^1 \int \partial_s f(x,s) d\nu_s^\eta(x) ds + \int_0^1 \int \partial_x f(x,s) u_s^\eta(x) d\nu_s^\eta(x) ds\,. \tag{4.7}$$

We next claim that in fact, for any $t \in (0,1]$, and any $f \in \mathcal{C}_b^{2,1}(I\!\!R \times [0,1])$,

$$\int f(x,t) d\nu_t^\eta(x) = \int f(x,0) d\nu_0^\eta(x) + \int_0^t \int \partial_s f(x,s) d\nu_s^\eta(x) ds + \int_0^t \int \partial_x f(x,s) u_s^\eta(x) d\nu_s^\eta(x) ds\,. \tag{4.8}$$

Indeed, given such f and t, fix $1-t>\delta>0$ and define

$$f^{\delta}(x,s)=\begin{cases} f(x,s), & s\le t\\ f(x,t)-(s-t)\frac{f(x,t)}{\delta}, & s\in(t,t+\delta]\\ 0, & s>t+\delta\,. \end{cases}$$

Substituting in (4.7), we find that

$$\begin{aligned} 0 &= \int f(x,0)d\nu_0^{\eta}(x)+\int_0^t\int\partial_s f(x,s)d\nu_s^{\eta}(x)ds-\frac{1}{\delta}\int_t^{t+\delta}\int f(x,t)d\nu_s^{\eta}(x)ds\\ &\quad+\int_0^t\int\partial_x f(x,s)u_s^{\eta}(x)d\nu_s^{\eta}(x)ds-\frac{1}{\delta}\int_t^{t+\delta}(s-t)\int\partial_x f(x,t)u_s^{\eta}(x)d\nu_s^{\eta}(x)ds\,. \end{aligned}\tag{4.9}$$

Because $\nu_{\cdot}\in\mathcal{C}([0,1],M_1(I\!R))$, it holds that $\int f(x,t)d\nu_s^{\eta}(x)\to_{s\to t}\int f(x,t)d\nu_t^{\eta}(x)$, uniformly. Because $u_t^{\eta}(x)\in L^2(I\!R\times[0,1];d\nu_t^{\eta}(x)\times dt)$ and $\partial_x f$ is bounded, it follows that

$$\frac{1}{\delta}\int_t^{t+\delta}(s-t)\int\partial_x f(x,t)u_s^{\eta}(x)d\nu_s^{\eta}(x)ds\to_{\delta\to0}0\,.$$

Combining these facts with (4.9) yields (4.8).

Fix next $a\in(0,1/2)$ (for the rest of the proof, $a=0.4$ will do) and an $\epsilon>0$ small and define $\nu_t^{\eta,\epsilon}=\nu_t^{\eta}\circ h_\epsilon^{-1}(x)$, where

$$h_\epsilon(x)=\frac{x}{(1+\epsilon x^2)^a}\,,$$

that is

$$\int f(x)\nu_t^{\eta,\epsilon}(dx)=\int f\left(\frac{x}{(1+\epsilon x^2)^a}\right)d\nu_t^{\eta}(x)\,,\ \text{for all } f\in\mathcal{C}_b^0(I\!R).$$

Define also

$$u_s^{\eta,\epsilon}(x)=\partial_x h_\epsilon\circ h_\epsilon^{-1}(x)\cdot u_s^{\eta}\circ h_\epsilon^{-1}(x)\,.$$

Note that

$$\int_0^1\int u_s^{\eta,\epsilon}(x)^2d\nu_s^{\eta,\epsilon}(x)ds=\int_0^1\int(\partial_x h_\epsilon(x))^2u_s^{\eta}(x)^2d\nu_s^{\eta}(x)ds\,,$$

and since

$$\partial_x h_\epsilon(x)=\frac{1+\epsilon x^2(1-2a)}{(1+\epsilon x^2)^{a+1}}$$

increases towards one as ϵ goes to zero, and is uniformly bounded, the dominated convergence theorem (note that $u^{\eta}\in L^2(d\nu_s^{\eta}(x)ds)$ by (4.5) and (4.6)) yields that

$$\lim_{\epsilon\to0}\int_0^1\int u_s^{\eta,\epsilon}(x)^2d\nu_s^{\eta,\epsilon}(x)ds=\int_0^1\int u_s^{\eta}(x)^2d\nu_s^{\eta}(x)ds<\infty.\tag{4.10}$$

Moreover, $\nu_s^{\eta}(dx)ds\ll dxds$ implies $\nu_s^{\eta,\epsilon}(dx)ds\ll dxds$ and the densities are given by

$$\rho_s^{\eta,\epsilon}(x)=(h'_\epsilon\circ h_\epsilon^{-1})^{-1}\rho_s^{\eta}\circ h_\epsilon^{-1}(x)\,.$$

Therefore,

$$\int_0^1\int\rho_t^{\eta,\epsilon}(x)^3dxdt=\int_0^1\int\frac{1}{h'_\epsilon(x)^2}\rho_t^{\eta}(x)^3dxdt\,.$$

Adapting the argument in [2, pages 1219–1220], the conditions $S^{0,1}(\nu) < \infty$ and $\nu_0(x^2) < \infty$ imply that $\sup_{t\in[0,1]} \nu_t(x^2) < \infty$, which implies in turn that $\sup_{\eta<1} \sup_{t\in[0,1]} \nu_t^\eta(x^2) < \infty$. Since $h'_\epsilon(x)^{-1} \le C(1+x^2)^a \le C(1+x^2)^{\frac{1}{2}}$, C independent of ϵ, it follows from the last two displays and the estimate in (4.5) that

$$\int_0^1 \int \rho_t^{\eta,\epsilon}(x)^3 dx dt \le \frac{C}{\eta} \int_0^1 \int (1+x^2)\nu_t^\eta(dx)dt < \infty \,. \tag{4.11}$$

Applying again [4, page 170] in the first and last displays, we have that

$$\begin{aligned}
\int_0^1 \int (H\nu_t^{\eta,\epsilon}(x))^2 d\nu_t^{\eta,\epsilon}(x)dt &= \frac{\pi^2}{3} \int_0^1 \int \rho_t^{\eta,\epsilon}(x)^3 dx dt \\
&= \frac{\pi^2}{3} \int_0^1 \int (h'_\epsilon(x))^{-2} \rho_t^\eta(x)^3 dx dt \\
&\to_{\epsilon\to 0} \frac{\pi^2}{3} \int_0^1 \int \rho_t^\eta(x)^3 dx dt \\
&= \int_0^1 \int (H\nu_t^\eta(x))^2 d\nu_t^\eta(x) dt < \infty \,,
\end{aligned} \tag{4.12}$$

where the limit is due to the pointwise convergence of h'_ϵ to 1 and the fact that $h'_\epsilon(x)^{-2} \le C(1+x^2) \in L^1(\rho_t^\eta(x)^3 dx dt)$ where C does not depend on ϵ. Next, write

$$\begin{aligned}
\int_0^1 \int u_s^{\eta,\epsilon}(x)(H\nu_s^{\eta,\epsilon}(x))d\nu_s^{\eta,\epsilon}(x)ds &= \int_0^1 \int u_s^\eta(x)\partial_x h_\epsilon(x) PV \int \frac{\nu^\eta(dy)}{h_\epsilon(x)-h_\epsilon(y)} d\nu_s^\eta(x)ds \\
&= \int_0^1 \int u_s^\eta(x)\partial_x h_\epsilon(x) PV \int \frac{\nu^\eta(dy)}{x-y} d\nu_s^\eta(x)ds \\
&\quad + \int_0^1 \int u_s^\eta(x)\partial_x h_\epsilon(x) PV \int \nu^\eta(dy) \left[\frac{1}{h_\epsilon(x)-h_\epsilon(y)} - \frac{1}{x-y}\right] d\nu_s^\eta(x)ds \\
&=: I_1 + I_2
\end{aligned}$$

Since $\partial_x h_\epsilon$ converges to 1 pointwise as ϵ converges to 0 and is uniformly (in ϵ, x) bounded, one deduces from (4.10, 4.12) and dominated convergence that

$$I_1 \to_{\epsilon\to 0} \int_0^1 \int u_s^\eta(x)(H\nu_s^\eta(x))d\nu_s^\eta(x) \,.$$

On the other hand, expanding

$$h_\epsilon(x) - h_\epsilon(y) = \partial_x h_\epsilon(x)(x-y) + \partial_{xx} h_\epsilon(\xi)(x-y)^2 \,, \text{ for some } \xi \in [x,y] \,,$$

one sees that

$$\left[\frac{1}{h_\epsilon(x)-h_\epsilon(y)} - \frac{1}{x-y}\right] = \frac{1}{x-y}\frac{g_\epsilon(x)}{\partial_x h_\epsilon(x)} + k_\epsilon(x,y) \,,$$

where the functions g_ϵ, k_ϵ are uniformly bounded and converge to 0 pointwise as $\epsilon \to 0$. In conclusion, one gets that

$$\int_0^1 \int u_s^{\eta,\epsilon}(x)(H\nu_s^{\eta,\epsilon}(x))d\nu_s^{\eta,\epsilon}(x)ds \to_{\epsilon\to 0} \int_0^1 \int u_s^\eta(x)(H\nu_s^\eta(x))d\nu_s^\eta(x) \,. \tag{4.13}$$

Also, because of (4.11), again by [4, Page 170], for any $g \in \mathcal{C}_b^{2,1}(I\!R \times [0,1])$, writing $g_t(x) = g(x,t)$,

$$\iint \frac{\partial_x g_s(x) - \partial_x g_s(y)}{x-y} d\nu_s^{\eta,\epsilon}(x) d\nu_s^{\eta,e}(y) = 2\int \partial_x g_s(x)(H\nu_s^{\eta,\epsilon})(x)d\nu_s^{\eta,\epsilon}(x)ds \,.$$

Taking $f(x,t) = (g_t \circ h)(x)$ and using (4.8), we find that

$$\int g_t(x) d\nu_t^{\eta,\epsilon}(x) = \int g_0(x) d\nu_0^{\eta,\epsilon}(x) + \int_0^t \int \partial_s g_s(x) d\nu_s^{\eta,\epsilon}(x) ds + \int_0^t \int u_s^{\eta,\epsilon}(x) \partial_x g_s(x) d\nu_s^{\eta,\epsilon}(x) ds \tag{4.14}$$

Therefore, from the last two displays,

$$\begin{aligned}
S^{0,1}(\nu^{\eta,\epsilon}) &= \sup_{g\in C_b^{2,1}(\mathbb{R}\times[0,1])} \sup_{0\le v\le t\le 1} \left(S^{v,t}(\nu^{\eta,\epsilon}, g) - \frac{1}{2}\langle g, g\rangle_{v,t}^{\nu^{\eta,\epsilon}} \right) \\
&= \sup_{\substack{g\in C_b^{2,1}(\mathbb{R}\times[0,1])\\ 0\le v\le t\le 1}} \left(\int_v^t [\int u^{\eta,\epsilon}(x)\partial_x g_s(x) d\nu_s^{\eta,\epsilon}(x) - \frac{1}{2}\iint \frac{\partial_x g_s(x) - \partial_x g_s(y)}{x-y} d\nu_s^{\eta,\epsilon}(x) d\nu_s^{\eta,\epsilon}(y)] ds \right. \\
&\qquad \left. - \frac{1}{2}\langle g, g\rangle_{v,t}^{\nu^{\eta,\epsilon}} \right) \\
&= \sup_{\substack{g\in C_b^{2,1}(\mathbb{R}\times[0,1])\\ 0\le v\le t\le 1}} \left(\int_v^t [\int u_s^{\eta,\epsilon}(x)\partial_x g_s(x) d\nu_s^{\eta,\epsilon}(x) - \int \partial_x g_s(x) (H\nu_s^{\eta,\epsilon})(x) d\nu_s^{\eta,\epsilon}(x)] ds \right. \\
&\qquad \left. - \frac{1}{2}\langle g, g\rangle_{v,t}^{\nu^{\eta,\epsilon}} \right) \qquad (4.15) \\
&\le \frac{1}{2}\int_0^1 \int (H\nu_s^{\eta,\epsilon}(x) - u_s^{\eta,\epsilon}(x))^2 d\nu_s^{\eta,\epsilon}(x) ds < \infty,
\end{aligned}$$

where first inequality is due to the fact that $\sup_{x\in\mathbb{R}}[xa - x^2/2] = a^2/2$ and the last inequality is due to (4.10) and (4.12). Expanding the square in the last expression and using (4.10), (4.12) and (4.13), one concludes that

$$\limsup_{\epsilon\to 0} S^{0,1}(\nu^{\eta,\epsilon}) \le S^{0,1}(\nu^{\eta}) \to_{\eta\to 0} S^{0,1}(\nu),$$

which implies (4.4) for an appropriate choice of $\nu^\delta = \nu^{\delta,\epsilon(\delta)}$. We shall see below that ν^δ belongs to $\mathcal{A}$ when $a > 1/3$ for any $\delta > 0$.

(4.4) almost completes the proof of the theorem, except that the measures ν^δ do not possess the right initial condition. To overcome this fact, we use a regularization near 0. Indeed, modify the definition of $\nu^{\eta,\epsilon}$ above to

$$\begin{aligned}
\nu_t^{\eta,\epsilon}(dx) &= \mu_D \boxed{+} \sigma_t, \quad \text{if } t \le \eta \\
&= [\mu_D \boxed{+} \sigma_\eta]^{(t-\eta)}, \quad \text{if } \eta \le t \le \eta+\epsilon \\
&= [\sigma_\eta \boxed{+} \nu_{t-\eta-\epsilon}]^\epsilon, \quad \text{if } t \ge \eta+\epsilon
\end{aligned}$$

with $[\mu]^\epsilon(f) = \mu \circ h_\epsilon^{-1}$, $\nu_0^{\eta,\epsilon} = \mu_D$. By continuity of $t \to \nu_t$, it is clear that $\nu^{\eta,\epsilon}$ converges towards ν as η, ϵ go to zero. Moreover, we have

$$\begin{aligned}
S_{\mu_D}(\nu^{\eta,\epsilon}) &= S^{0,1}(\nu^{\eta,\epsilon}) \\
&= \sup_{g\in C_b^{2,1}(\mathbb{R}\times[0,1])} \sup_{0\le s\le t\le 1} \{\bar{S}^{s,\eta\wedge t}(\nu^{\eta,\epsilon}, g) + \bar{S}^{\eta\vee s,(\eta+\epsilon)\wedge t}(\nu^{\eta,\epsilon}, g) + \bar{S}^{(\eta+\epsilon)\vee s,t}(\nu^{\eta,\epsilon}, g)\} \\
&\le S^{0,\eta}(\nu^{\eta,\epsilon}) + S^{\eta,\eta+\epsilon}(\nu^{\eta,\epsilon}) + S^{\eta+\epsilon,1}(\nu^{\eta,\epsilon}) = I(\eta) + II(\eta,\epsilon) + III(\eta,\epsilon)
\end{aligned}$$

where we took the convention that $\bar{S}^{u,v} = 0$ if $u \geq v$. By the previous argument, when $a < 1/2$, and since $S^{0,\eta+\epsilon}(\nu)$ goes to zero as η, ϵ do for $\nu \in \{S_{\mu_D} < \infty\}$,

$$\limsup_{\eta\to 0} \limsup_{\epsilon\to 0} III(\eta,\epsilon) = S^{0,1}(\nu).$$

Moreover,

$$S^{0,\eta}_{\mu_D}(\nu^{\eta,\epsilon}) = 0$$

since it is costless to move by a semicircular process. Finally, for $t \in [\eta, \eta+\epsilon]$, mimicking the argument in the proof above, with $h^\eta_t(x) = h_{t-\eta}(x)$, $g_{\eta,a}(x) = -a[(1+(t-\eta)x^2)^{-1-a}x^3]$, and

$$u^\eta_t(x) = g_{\eta,a} \circ (h^\eta_t)^{-1}(x)\,,$$

it holds that for any $f \in \mathcal{C}^1_b(I\!R)$,

$$\partial_t \nu^{\eta,\epsilon}_t(f) = \partial_t \mu_D \boxplus \sigma_\eta(f(h_{t-\eta}(x))) = \mu_D \boxplus \sigma_\eta(\partial_t h_{t-\eta}(x)(\partial_x f)(h_{t-\eta}(x))) = \nu^{\eta,\epsilon}_t(u^\eta_t\, \partial_x f)$$

resulting with

$$S^{\eta,\eta+\epsilon}_{\mu_D}(\nu^{\eta,\epsilon}) \quad \leq \quad \frac{1}{2}\int_\eta^{\eta+\epsilon}\int (H[\mu_D \boxplus \sigma_\eta]^{(t-\eta)}(x) - \partial_x u^\eta_t(x))^2 d[\mu_D \boxplus \sigma_\eta]^{(t-\eta)}(x)dt.$$

Now, recall that for any t, $[\mu_D \boxplus \sigma_\eta]^{(t-\eta)} \ll dx$ with density ρ^η_t given, if $\mu_D \boxplus \sigma_\eta(dx) = \rho^\eta(x)dx$, by

$$\rho^\eta_t(x) = \frac{\rho^\eta \circ (h^\eta_t)^{-1}(x)}{\partial_x(h^\eta_t) \circ (h^\eta_t)^{-1}}(x).$$

Hence, since ρ^η is uniformly bounded, we see as above that for any t,

$$\int (H[\mu_D \boxplus \sigma_\eta]^{(t-\eta)}(x))^2 d[\mu_D \boxplus \sigma_\eta]^{(t-\eta)}(x) \leq C(\eta)$$

with a constant $C(\eta)$ which only depends on η. Further, using the inequality

$$\frac{x^6}{(1+zx^2)^{1+a}} \leq \frac{x^{4+2a}}{z^{1-a}}\,, z \geq 0\,,$$

that can be verified by considering separately $x^2 \geq 1/z$ and $x^2 < 1/z$, one has

$$\begin{aligned}\int (u^\eta_t(x))^2 d[\mu_D \boxplus \sigma_\eta]^{(t-\eta)}(x) \quad &= \quad a^2 \int [(1+(t-\eta)x^2)^{-1-a}x^3]^2 d\mu_D \boxplus \sigma_\eta(x)\\ &\leq \quad a^2(t-\eta)^{-1+a}\int x^{4+2a} d\mu_D \boxplus \sigma_\eta(x)\end{aligned}$$

is integrable for $a > 0$ since $\mu_D \boxplus \sigma_\eta$ possesses 5 moments at least. Hence, for any $\eta > 0$,

$$\int (H[\mu_D \boxplus \sigma_\eta]^{(t-\eta)}(x) - u^\eta_t(x))^2 d[\mu_D \boxplus \sigma_\eta]^{(t-\eta)}(x) \leq (t-\eta)^{-1+a}C'(\eta)$$

and so

$$\lim_{\epsilon\to 0} S^{\eta,\eta+\epsilon}_{\mu_D}(\nu^{\eta,\epsilon}) = 0.$$

Putting together the above information, we get, for $a < 1/2$

$$\lim_{\eta\to 0}\lim_{\delta\to 0} S_{\mu_D}(\nu^{\eta,\delta}) \leq S_{\mu_D}(\nu) \tag{4.16}$$

We finally claim that $\nu^{\eta,\epsilon} \in \mathcal{A}$ for $a > 1/3$. Indeed, let $1 > \epsilon > 0$ be small enough such that

$$\mu_D(|x|^{5+\epsilon}) < \infty.$$

Since the semicircular law is compactly supported, it is clear that

$$\sup_{t\leq\eta} \nu_t^{\eta,\epsilon}(|x|^{5+\epsilon}) < \infty.$$

Now, by construction, for $t \in [\eta, \eta+\epsilon]$

$$\nu_t^{\eta,\epsilon}(|x|^{5+\epsilon}) \leq \nu_\eta^{\eta,\epsilon}(|x|^{5+\epsilon}) < \infty.$$

Finally, for $t > \eta+\epsilon$, since $\sup_{t\in[0,1]} \nu_t(x^2) < \infty$ (c.f. the argument before (4.11)), it holds that

$$\nu_t^{\eta,\epsilon}(|x|^{5+\epsilon}) = \int \left|\frac{x}{(1+\epsilon x^2)^a}\right|^{5+\epsilon} d\nu_{t-\eta-\epsilon}(x) \leq C(\epsilon)\int |x|^{(5+\epsilon)(1-2a)} d\nu_{t-\epsilon-\eta}(x)$$

is uniformly bounded as soon as $(5+\epsilon)(1-2a) \leq 2$, that is $a > 1/3$ for $\epsilon < 1$. $\square$

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