ON INCREASING SUBSEQUENCES OF I.I.D. SAMPLES

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Abstract. We study the fluctuations, in the large deviations regime, of the longest increasing subsequence of a random i.i.d. sample on the unit square. In particular, our results yield the precise upper and lower exponential tails for the length of the longest increasing subsequence of a random permutation.

§. 1 Introduction

Let $\{Z_i\}_{i=1}^n = \{(X_i, Y_i)\}_{i=1}^n$ denote a sequence of i.i.d. random variables with marginal law μ on the unit square $\mathbf{Q} = [0,1]^2$. Throughout, we make the assumption that μ possesses a strictly positive density $p \in C^1(\mathbf{Q})$ with respect to the Lebesgue measure λ on $\mathbf{Q}$.

A subsequence $\{Z_{i_1}, ..., Z_{i_\ell}\} \subseteq \{Z_i\}_{i=1}^n$ is called a *monotone increasing subsequence of length ℓ*, if

$$X_{i_j} < X_{i_{j+1}} \quad \text{and} \quad Y_{i_j} < Y_{i_{j+1}}, \quad \text{for} \quad j = 1, ..., \ell - 1.$$

Define next $\ell_{\max}(n)$ to be the length of the longest increasing subsequence in the sample $\{Z_i\}_{i=1}^n$. Note that we do not require that $i_j < i_{j+1}$.

In the case that $\mu = \lambda$, $\ell_{\max}(n)$ possesses the same law as the length of the longest increasing subsequence of a random permutation, denoted hereafter by $L_{\max}(n)$. Building on the fact that

$$\lim_{n\to\infty} \frac{L_{\max}(n)}{\sqrt{n}} = 2 \qquad \text{in probability},$$

c.f. [10],[11], we showed in [3] that

$$\lim_{n\to\infty} \frac{\ell_{\max}(n)}{\sqrt{n}} = 2\bar{J}_\mu \qquad \text{in probability}, \tag{1.1}$$

where $\bar{J}_\mu \in \mathbb{R}^+$ is the solution to the variational problem

$$\bar{J}_\mu = \sup_{\phi \in B^\uparrow} \int_0^1 \sqrt{p(x, \phi(x))\dot{\phi}(x)}\, dx \tag{1.2}$$

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with

$$B^{\uparrow} \equiv \{\phi : [0,1] \longrightarrow [0,1], \quad \text{non-decreasing, absolutely continuous}\}.$$

Furthermore, it follows from [3] that any longest increasing subsequence will concentrate on the solutions to the variational problem (1.2). See [4], Proposition 4.4, for an alternative expression for $\bar{J}_\mu$.

Note that $\bar{J}_\mu = 1$ for $p(x,y) = 1$, in which case the maximum is achieved on the diagonal $x \longrightarrow \phi(x) = x$.

The fluctuations of $\ell_{\max}(n)$ and $L_{\max}(n)$ are highly nontrivial and have been investigated in several papers, c.f. [2], [9], [1], [5]. In particular, Aldous and Diaconis have exhibited quite different behaviors in their upper and lower tails. Our goal in this paper is to provide information on *the large deviations of these fluctuations*. The results and techniques differ sharply in the study of lower and upper tail, and we divide the discussion in the rest of this introduction between these two cases.

Turning our attention to *the lower tail* we first show in Theorem 1 that for any $-2 < c < 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log P(L_{\max}(n) < (2+c)\sqrt{n}) = -2H_0(c), \tag{1.3}$$

with an explicit function H_0, first introduced by Logan and Shepp in [6],

$$H_0(c) = -\frac{1}{2} + \frac{(2+c)^2}{8} + \log\frac{c+2}{2} - \big(1 + \frac{(c+2)^2}{4}\big)\log\big(\frac{2(c+2)^2}{4+(c+2)^2}\big)\,.$$

See Fig. 1 for a plot of $H_0(\cdot)$. The proof based on the random Young tableau correspondence is purely combinatoric and sheds no light on the random mechanism responsible for the large deviations. In particular, we cannot prove that, conditioned on $L_{\max}(n) < (2+c)\sqrt{n}$, the longest increasing subsequence concentrates around a curve $\phi(\cdot)$.

While we could hope to use this result in order to prove an exponential lower tail for general μ, that is

$$\lim_{n\to\infty} \frac{1}{n} \log P(\ell_{\max}(n) < (2\bar{J}_\mu + c)\sqrt{n}) = -2H_\mu(c), \tag{1.4}$$

with $H_\mu(c) > 0$ for $-2\bar{J}_\mu < c < 0$, we were not able to compute H_μ explicitly, nor to prove the existence of the limit in (1.4). We thus present in Propositions 2.1 and 2.2 nontrivial upper and lower bounds on the left hand side of (1.4), avoiding the question of existence of the limit.

The situation is quite different for *the upper tail*, here an easy sub-additive argument shows that for $c > 0$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log P(L_{\max}(n) > (2+c)\sqrt{n}) = -U_0(c), \tag{1.5}$$

for some nontrivial convex rate function $U_0(\cdot)$. In the first version of this work, we presented only bounds on $U_0(\cdot)$, leaving open the explicit evaluation of this function. Subsequently, T. Seppäläinen has proved in [8], using Hammersley's particle system associated with the Poissonized version of $L_{\max}(n)$, that

$$U_0(c) = \beta(c) := 2(2+c)\cosh^{-1}(c/2+1) - 4\sqrt{c^2+4c}\,.$$

See Fig. 2 for a plot of $U_0(\cdot)$. In fact, Kim [5] had already observed, by combinatorial techniques, the upper bound in (1.5) with the function $U_0(c)$. For the sake of completeness, we will present in Section 3 a combinatorial proof of the lower bound in (1.5).

Our interest is in exploring the similar question for $\ell_{\max}(n)$, where the sub-additive argument is not applicable. Our main result in this direction (c.f. Theorem 3) is in fact that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log P(\ell_{\max}(n) > (2\bar{J}_\mu + c)\sqrt{n}) = -U_\mu(c) \tag{1.6}$$

where

$$U_\mu(c) = \bar{J}_\mu U_0\Big(\frac{c}{\bar{J}_\mu}\Big).$$

Moreover we show that, under the conditioning that

$$\{\ell_{\max}(n) > (2\bar{J}_\mu + c)\sqrt{n}\},$$

the longest increasing subsequences concentrate near the maximizing curves in (1.2).

The precise statement and proof of these theorems is followed in section 4 by a discussion and several conjectures and open questions.

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§. 2 The lower tail

In this section we first describe the large deviations for the uniform measure $p(x,y)=1$. Let us recall some notation from [6]: For $f \in \mathcal{F}$, the class of nonnegative, nondecreasing functions on $[0,\infty)$ of unit integral, define

$$H(f) \equiv \int_0^\infty \int_0^{f(x)} \log(f(x) - y + f^{-1}(y) - x)\, dy\, dx\,. \tag{2.1}$$

Let, for $c \in (-2, 0)$,

$$H_0(c) \equiv \inf\{H(f), \quad f \in \mathcal{F}, f(0) = 2 + c\} \;+\; \frac{1}{2},$$

c.f. [6]. Then $H(f) \geq 0$, and

$$H_0(c) = -\frac{1}{2} + \frac{(2+c)^2}{8} + \log\frac{c+2}{2} - \big(1 + \frac{(c+2)^2}{4}\big)\log\big(\frac{2(c+2)^2}{4+(c+2)^2}\big). \tag{2.2}$$

Note that H_0 is a strictly convex, monotone decreasing function with minimum 0 at $c=0$, c.f. Fig. 1 below. Our first result, which is an immediate consequence of [6], is based on Schensted's identity:

Theorem 1. *For any* $-2 < c \leq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log P(L_{\max}(n) < (2+c)\sqrt{n}) = -2H_0(c).$$

Proof. The basic idea is to use a combinatorial identity of Schensted, expressing the probability distribution of $L_{\max}(n)$ in terms of Young tableaux, c.f. [7] and [6], §1:

A Young shape τ of size $|\tau| = n$ is an array of n unit squares s, left and bottom justified, whose columns' lengths are nonincreasing from left to right. The *hook length* $\sigma(s)$ of a square s in the shape τ is just the number of squares in τ directly above and to the right of it, counting each square exactly once, c.f. Fig 3. Let $\pi(\tau) = \prod_{s\in\tau} \sigma(s)$ denotes the hook product, i.e. the product of all hook lengths in the tableau τ. Then, the Schensted identity states that

$$P(L_{\max}(n) = k) = \sum_{\tau:\tau(0)=k} \frac{n!}{(\pi(\tau))^2}, \qquad k = 1, ..., n\,, \tag{2.3}$$

where the sum is taken over all shapes τ containing n squares, possessing a first column of length k. In order to estimate $P(L_{\max}(n) \leq (2+c)\sqrt{n})$ for fixed $c \in (-2, 0)$ it suffices to find an optimal shape τ_n with $\tau_n(0) \leq (2+c)\sqrt{n}$ which maximizes the hook product $\pi(\tau_n)$. This is in essence the argument of [6] which yields the upper bound, c.f. (1.9), (1.10) and (3.2) there. We hence concentrate in the sequel in proving the lower bound

$$\liminf_{n\to\infty} \frac{1}{n} \log P(L_{\max}(n) < (2+c)\sqrt{n}) \geq -2H_0(c).$$

Our goal is to find for fixed $c \in (-2, 0)$ a sequence of shapes $\{\tau_n\}$ of maximal hook product such that $\lim_{n\to\infty} |\tau_n|/n = 1$ and $\lim_{n\to\infty} \tau_n(0)/n^{1/2} \leq (2+c)$. Let $f_0^c \in \mathcal{F}$ be such that

$$H(f_0^c) = \inf\{H(f) : \quad f \in \mathcal{F} \qquad \text{with} \qquad f(0) = 2 + c\}.$$

The curve f_0^c is constructed in [6], it has the support

$$b_0(c) = \frac{1}{(2+c)} - \frac{(2+c)}{4} + \sqrt{2 + \frac{(2+c)^2}{2}}.$$

Hence, the length of the curve $\{(x, f_0^c(x)), 0 \leq x \leq b_0(c)\}$ is bounded by some constant k_c.

We construct a particular Young tableau out of f_0^c. For $i = 1, ..., [b_0(c)\sqrt{n}] \equiv i_{\max}$ set $j(i) = [f_0^c(\frac{i}{\sqrt{n}})\sqrt{n}]$. Note that $j(i)$ is a decreasing sequence, and, because the length of $\{(x, f_0^c(x)), 0 \leq x \leq b_0(c)\}$ is bounded,

$$m_n = \sum_{i=1}^{i_{\max}} \sum_{j=1}^{j(i)} 1 \geq n - k_c\sqrt{n}.$$

The sequence $\{(i, j(i)), i = 1, ..., i_{\max}\}$ defines a Young tableau τ_n of size m_n. Moreover for any $y < f_0^c(x)$ such that $\bar{\imath} = [x\sqrt{n}] \le i_{\max}$ and $\bar{\jmath} = [y\sqrt{n}] \le j(i)$, denoting by $\pi_{\bar{\imath}\bar{\jmath}}$ the hook length of the square with indices $(\bar{\imath}, \bar{\jmath})$,

$$\log \pi_{\bar{\imath}\bar{\jmath}}(\tau_n) \le \log(f_0^c(x) - y + (f_0^c)^{-1}(y) - x) + \log n.$$

Hence, for some constant $C > 0$ independent of n, whose value may change from line to line,

$$\begin{aligned} nH(f_0^c) &= n \int_0^{b_0(c)} \int_0^{f_0^c(x)} \log\left(f_0^c(x) - y + (f_0^c)^{-1}(y) - x\right) dx\, dy \\ &\ge -\frac{n}{2}\log n + \log \pi(\tau_n) + n \sum_{i=1}^{i_{\max}} \left(f_0^c(\frac{i}{\sqrt{n}}) - f_0^c(\frac{i+1}{\sqrt{n}}) + \frac{1}{\sqrt{n}}\right) \int_0^{1/\sqrt{n}} \log x\, dx \\ &\ge -\frac{n}{2}\log n + \log \pi(\tau_n) - C\sqrt{n}\log n\,. \end{aligned} \tag{2.4}$$

It follows that for any $-2 < c < 0$,

$$\begin{aligned} P(L_{\max}(m_n) < (2+c)\sqrt{n}) &\ge \frac{m_n!}{(\pi(\tau_n))^2} \ge \frac{m_n!}{n!} n! e^{-2n\log\sqrt{n}} e^{-C\sqrt{n}\log n} e^{-2nH(f_0^c)} \\ &\ge e^{-C\sqrt{n}\log n} e^{-2n[H(f_0^c)+\frac{1}{2}]}. \end{aligned}$$

Finally, for any $\bar{c} < 2$, by rescaling,

$$P(L_{\max}(n) < \bar{c}\sqrt{n}) \le P(L_{\max}(m_n) < \bar{c}\sqrt{m_n}) = P(L_{\max}(m_n) < \bar{c}\sqrt{n}\frac{\sqrt{m_n}}{\sqrt{n}}),$$

and the conclusion follows from the continuity of $H(f_0^c)$ in c. $\square$

An immediate corollary, which will be useful below, is the following:

Corollary 1. *For any $-2 < c < 0$ there exists a function $\eta(c, \delta)$ satisfying*

$$\lim_{\delta \to 0} \eta(c, \delta) = 0$$

such that if $p(x, y)$ satisfies $(1 - \delta) \le p(x, y) \le (1 + \delta)$ then

$$\limsup_{n\to\infty} \left|\frac{1}{n}\log P(\ell_{\max}(n) < (2+c)\sqrt{n}) + 2H_0(c)\right| \le \eta(c, \delta)$$

Proof. The proof is based on the same idea as the proof of Lemma 7 in [3]. By a possible change of coordinates in the x axis, we may and will assume that $p(x) = \int_0^1 p(x, y)dy = 1$, and that $|p(y|x) - 1| \le \delta' = 2\delta/(1-\delta)$. Let P_i be the law on $[0, 1]$ with density $p(y|X_i)$. Note that P_i may be written as a mixture of a uniform law (with weight $(1 - \delta')$) and another law on $[0, 1]$, depending on X_i and denoted q_i, that is $P_i(dy) = (1-\delta')\lambda_1(dy) + \delta' q_i(dy)$. Thus, the sample $((X_1, Y_1), \ldots, (X_n, Y_n))$

possesses the same law as $\tilde{Z}_n = ((X_1, (m_1 U_1 + (1-m_1)W_1)), \ldots, (X_n, (m_n U_n + (1-m_n)W_n)))$, where $\{U_i\}_{i=1}^n$ is a sequence of i.i.d. uniform random variables, independent of the sequence $\{X_i\}_{i=1}^n$, $\{m_i\}_{i=1}^n$ is a sequence of i.i.d. Bernoulli$(1-\delta')$ random variables, independent of the sequences $\{U_i\}_{i=1}^n$ and $\{X_i\}_{i=1}^n$, and $\{W_i\}_{i=1}^n$ is a sequence of random variables whose law depends on the sequence $\{X_i\}_{i=1}^n$. Let I denote the set of indices with $m_i = 1$, and let $N_n = \sum_{i=1}^n 1_{m_i=1} = |I|$ denote the number of indices where a uniform random variable is chosen in the mixture. Note that one may find a $\delta'' = \delta''(\delta) \geq \delta'$, $\delta''(\delta) \to_{\delta\to 0} 0$ such that

(2.5)
$$\limsup_{n\to\infty} \frac{1}{n} \log P(N_n/n < 1-\delta'') = -\Big\{(1-\delta'')\log\frac{1-\delta''}{1-\delta'} + \delta\text{''} \log\frac{\delta''}{\delta'}\Big\} < -3H_0(c)$$

for all δ small enough. Let $\tilde{\ell}_{\max}(n)$ denote the length of the maximal increasing subsequence corresponding to $\tilde{Z}_n$, then $\tilde{\ell}_{\max}(n)$ possesses the same law as $\ell_{\max}(n)$ and, on the other hand, is not smaller than the length of the maximal increasing subsequence when one considers only those indices $i \in I$. The latter is distributed precisely as the length of the maximal increasing subsequence of a *uniform* sample of random length N_n which is independent of the uniform sequence. Therefore,

(2.6)
$$P(\ell_{\max}(n) < (2+c)\sqrt{n}) \leq P(L_{\max}(n(1-\delta'')) < (2+c)\sqrt{n}) + P(N_n/n < 1-\delta'')\,.$$

The continuity of $H_0(c)$ implies that for δ small enough,

$$2H_0(\frac{2+c}{\sqrt{1-\delta''}} - 2) < 3H_0(c)\,.$$

Hence, (2.5), (2.6) and Theorem 1 imply that for δ small enough,

$$\limsup_{n\to\infty} P(\ell_{\max}(n) < (2+c)\sqrt{n}) \leq -2H_0(\frac{2+c}{\sqrt{1-\delta''}} - 2) = -2H_0(c) + g(c,\delta)\,,$$

where the continuity of $H_0(\cdot)$ implies the required properties of $g(c,\delta)$. The complementary lower bound is proved by a similar coupling. $\square$

We now turn to general case and prove first a lower bound estimate: for fixed $d > 0$ set

$$I_\mu(d) \equiv \inf\{H(\nu|\mu) : \nu \in \mathcal{M}_1(\mathbf{Q}), 2\bar{J}_\nu = d\}\,, \tag{2.7}$$

where $\mathcal{M}_1(\mathbf{Q})$ is the set of probability measures on $\mathbf{Q}$ and $H(\nu|\mu)$ denotes the relative entropy of ν with respect to μ:

$$H(\nu|\mu) = \int_{\mathbf{Q}} \log\frac{q(x,y)}{p(x,y)}\nu(dx,dy)$$

if $\frac{d\nu}{d\lambda} = q$ and $H(\nu|\mu) = \infty$ otherwise.

Although an explicit computation for I_μ seems impossible, it is quite easy to verify that $I_\mu(d) = 0$ for $d \geq 2\bar{J}_\mu$, and $0 < I_\mu(d) < \infty$ if $0 < d < 2\bar{J}_\mu$, (e.g., by combining Lemma 1 and Proposition 2.2 below). Note that (1.1) implies that under $Q \equiv \prod \nu$, for each $\epsilon > 0$

$$\lim_{n\to\infty} Q(\ell_{\max}(n) \leq (2\bar{J}_\nu + \epsilon)\sqrt{n}) = 1.$$

Using a standard change of measure argument, we get from this:

Proposition 2.1. *For fixed* $0 < d < 2\bar{J}_\mu$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P(\ell_{\max}(n) \le d\sqrt{n}) \ge -I_\mu(d).$$

However, a simple comparison with $2H_0$ in case $\mu = \lambda$ shows that $I_\lambda(2-\cdot)$ is not the correct rate function:

Lemma 1. *Take* $\mu = \lambda$, *then*

$$\liminf_{\delta\searrow 0} \frac{I_\lambda(2-\delta)}{\delta^3} \ge \frac{4}{9} \tag{2.8}$$

Proof. Assume the existence of ν_δ such that $2\bar{J}_{\nu_\delta} < 2-\delta$ but $\lim_{\delta\to 0} H(\nu_\delta|\lambda)/\delta^3 < 4/9$. For a fixed $K > 0$ (independent of δ), let $q_\delta(x,y) = d\nu_\delta/d\lambda(x,y)$, and denote

$$A_K = \{(x,y) \in \mathbf{Q} : q_\delta(x,y) \le (1+\delta K)\}.$$

One easily checks that $\lambda(A_K^{\mathsf{c}}) < \delta/K^2$. Thus, we may assume that $q_\delta(x,y) = 1 + \delta m(x,y)$ for some m which, on A_K, is bounded above by K. Consider the set of curves $x : (0, 1-y) \longrightarrow \phi_y(x) = y + x$ where $0 < y < \delta/3$. Then

$$\begin{aligned} 1 - \frac{\delta}{2} > J_\nu(\phi_y) &= \int_0^{1-y} \sqrt{1 + \delta m(x, y+x)}\, dx \\ &\ge (1-y) + \frac{\delta}{2}\int_0^{1-y} (K \wedge m(x,y+x))\, dx + O(\delta^2) \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\delta}{3} - \frac{1}{6}\delta^2 &> \int_0^{\delta/3} J_\nu(\phi_y)\, dy \\ &\ge \frac{\delta}{3} - \frac{1}{18}\delta^2 + \frac{\delta}{2}\int_0^{\delta/3}\int_0^{1-y} (K \wedge m(x,y+x))\, dx\, dy + O(\delta^3)\,. \end{aligned}$$

Thus

$$\delta \int_0^{\delta/3}\int_0^{1-y} (K \wedge m(x,y+x))\, dx\, dy < -2\delta^2/9 + O(\delta^3)\,,$$

and, by symmetry, for $\Delta_\delta \equiv \{(x,y) \in \mathbf{Q} : -\delta/3 < x - y < \delta/3\} \cap A_K$,

$$\nu(\Delta_\delta) < \lambda(\Delta_\delta) - 4\delta^2/9 + O(\delta^3)\,,$$

with $\lim_{K\to\infty} \lim_{\delta\to 0} \lambda(\Delta_\delta)/\delta = 2/3$. Now the infimum of $H(\nu|\lambda)$ under the above condition is achieved at the constant density $1 - \delta'$, where

$$\delta' = \frac{4\delta^2/9 + O(\delta^3)}{\lambda(\Delta_\delta)} = \delta(2/3 + g_K)) + O(\delta^2)$$

on Δ_δ, and $g_K \to_{K\to\infty} 0$ is a constant independent of δ whose value may change from line to line. Substituting in $H(\nu|\lambda)$, one obtains

$$H(\nu_\delta|\lambda) \geq 4\delta^3/9 + g_K\delta^3 + O(\delta^4)\,.$$

Taking the limits as $\delta \to 0$ (first) and then $K \to \infty$ yields a contradiction. □

Note that our argument is quite rough and with additional work one could possibly identify the constant $b \in [\frac{4}{9}, \frac{3}{2}]$ such that $\lim_{\delta\searrow 0} \frac{I_\lambda(2-\delta)}{\delta^3} = b$, but this is quite irrelevant since a simple computation shows $2H_0(0) = 2H_0'(0) = 2H_0''(0) = 0$ and $2H_0'''(0) = \frac{1}{2}$ and therefore

$$\lim_{\delta\uparrow 0} \frac{2H_0(\delta)}{\delta^3} = \frac{1}{12} < \frac{4}{9} \leq \liminf_{\delta\searrow 0} \frac{I_\lambda(2-\delta)}{\delta^3}.$$

Our next result shows a volume upper bound:

Proposition 2.2. *Let $c < 0$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\ell_{\max}(n) \leq (2\bar{J}_\mu + c)\sqrt{n}) < 0.$$

Proof. Let ϕ denote an optimizer in (1.2) (whose existence is ensured by [3]). Fix $\Delta > 0$ with Δ^{-1} an integer, and for $i = 1, \ldots, \Delta^{-1}$ let

$$\mathbf{Q}_i = [(i-1)\Delta, i\Delta]\times[\phi((i-1)\Delta), \phi(i\Delta)], \quad \rho_i = \Delta(\phi(i\Delta)-\phi((i-1)\Delta))p(i\Delta, \phi(i\Delta)).$$

Set $n_i = n\rho_i$, and let m_i denote the (random) number of points in the sample $\{Z_i\}_{i=1}^n$ which belong to $\mathbf{Q}_i$. Then, for any $\epsilon > 0$, by Sanov's theorem, for $\Delta > 0$ small enough,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\cup_{i=1}^{\Delta^{-1}} A_i(\epsilon)) < 0 \quad \text{where} \quad A_i(\epsilon) = \{|n_i - m_i| > \epsilon n_i\},. \tag{2.10}$$

Let $\ell_i(m_i)$ denote the length of the longest increasing subsequence among the m_i points in $\mathbf{Q}_i$. Let P_{m_i} denote the law of the sample in $\mathbf{Q}_i$, conditioned on m_i. Then P_{m_i} possesses a product law with density p_i, satisfying $\lim_{\Delta\to 0} \sup_{x,y\in Q_i} |p_i(x,y) - 1| = 0$, c.f. Lemma 2 of [3]. Corollary 1 implies that for any $\epsilon' > 0$ and all i, and all Δ small enough,

$$\limsup_{m_i\to\infty} \frac{1}{m_i} \log P_{m_i}(\ell_i(m_i) < 2(1-\epsilon')\sqrt{m_i}) < 0.$$

Hence, for Δ small, on $\cap_i\{|n_i - m_i| \leq \epsilon n_i\}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\ell_i(m_i) < 2(1-\epsilon')\sqrt{m_i}\,|\{m_i\}_{i=1}^{\Delta^{-1}}) < 0\,. \tag{2.11}$$

Note that (c.f. [3]) $\sum_i \sqrt{\rho_i} \to_{\Delta\to 0} \bar{J}_\mu$. Choose Δ small enough such that $|\sum_i \sqrt{\rho_i} - \bar{J}_\mu| < |c|/2$. Then,

$$\begin{aligned}
P(\ell_{\max}(n) \le (2\bar{J}_\mu + c)\sqrt{n}) &\le P(\cup_{i=1}^{\Delta^{-1}} A_i(\epsilon)) \\
&\quad + E[\cup_{i=1}^{\Delta^{-1}} P(\ell_i(m_i) < 2\sqrt{n_i} + c\Delta\sqrt{n}/2 \,|\{m_j\}); \cap_{i=1}^{\Delta^{-1}} A_i(\epsilon)^{\mathsf{c}}] \\
&\le \Delta^{-1} \max_i P(|n_i - m_i| > \epsilon n_i) \\
&\quad + \Delta^{-1} \max_i \max_{m_i : |m_i - n_i| \le \epsilon n_i} P_{m_i}(\ell_i(m_i) < 2\sqrt{m_i}(1+\epsilon) + c'\sqrt{m_i}) ,
\end{aligned}$$

where $c' < 0$ is independent of ϵ and Δ. Choosing now ϵ small enough such that $2\epsilon + c' < 0$ and using (2.10) and (2.11), the proposition follows. $\square$

Remark: It is instructive to relate $\ell_{\max}(n)$ to $\bar{J}_\nu$ for an measure ν associated with $R_n \equiv \frac{1}{n}\sum_{i=1}^n \delta_{Z_i}$, the empirical measure of the sample. To this end, define for $\epsilon > 0$, the random measure $R_{n,\epsilon}$ with constant density ϵ^{-2} on the squares $\mathbf{Q}_\epsilon(Z_i) = [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]^2 + Z_i$, $i = 1, ..., n$, that is

$$\frac{d\, R_{n,\epsilon}}{d\,\lambda}(x,y) = \frac{1}{n}\sum_{i=1}^n \epsilon^{-2} \mathbf{1}_{\mathbf{Q}_\epsilon(Z_i)}(x,y) \,.$$

Note that, P almost surely $\epsilon_n \equiv \frac{1}{2}\min_{1\le i<j\le n}(|X_i - X_j| \wedge |Y_i - Y_j|) > 0$. A simple computation shows

$$\frac{\ell_{\max}(n)}{\sqrt{n}} = \bar{J}_{R_{n,\epsilon_n}} , \tag{2.12}$$

and therefore $\{\ell_{\max}(n) \le \sqrt{n} d\} = \{\bar{J}_{R_{n,\epsilon_n}} \le d\}$, for each $d > 0$. However a derivation of the large deviation principle using this equality fails, due to the discontinuity of the mapping $\nu \longrightarrow \bar{J}_\nu$. In particular, R_{n,ϵ_n} converges weakly to μ, on the other hand, we have $\lim_{n\to\infty} \frac{\ell_{\max}(n)}{\sqrt{n}} = 2\bar{J}_\mu$.

§. 3 The upper tail

Here the situation is quite different from the lower tail, and in some sense much simpler. Our first result is:

Theorem 2. *For all* $c \ge 0$

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} \log P(L_{\max}(n) \ge (2+c)\sqrt{n}) = -U_0(c), \tag{3.1}$$

where $U_0 : [0,\infty) \longrightarrow [0,\infty)$ *is a continuous, strictly increasing convex function with* $U_0(c) = 0$ *iff* $c = 0$, *and*

$$U_0(c) = \beta(c) := 2(2+c)\cosh^{-1}(c/2+1) - 2\sqrt{c^2+4c}\,. \tag{3.2}$$

Note that $U_0(c) = O(c^{3/2})$ as $c \to 0$, this is also predicted by the behavior $2H_0(c) = O(c^3)$ as $c \to 0$. The explicit computation of $U_0(c)$ was first done in [8] using Hammersley's particle system.

Proof. As pointed out by [1], the convergence in (3.1), and the convexity and monotonicity of U_0 follows from sub-additivity. We briefly recall the argument. Let $\mathcal{N}_n$ denote the number of points in a Poisson point process of rate $\lambda_n = n\lambda$ on the unit square, and let $\bar{L}_{\max}(\mathcal{N}_n)$ denote the longest increasing subsequence in that sample. Then, for any $\epsilon > 0$, a direct computation using the Poisson distribution yields

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log P(|\mathcal{N}_n/n - 1| > \epsilon) = -\infty \,. \tag{3.3}$$

On the other hand, conditioned on $\mathcal{N}_n$, the law of the sample is uniform and hence

$$P(\bar{L}_{\max}(\mathcal{N}_n) = x | \mathcal{N}_n = m) = P(L_{\max}(m) = x) \,.$$

Therefore,

$$P(\bar{L}_{\max}(\mathcal{N}_n) \geq (2+c)\sqrt{n})/P(\mathcal{N}_n > n(1-\epsilon)) \geq P(L_{\max}(n(1-\epsilon)) > (2+c)\sqrt{n})$$

while

$$P(\bar{L}_{\max}(\mathcal{N}_n) \geq (2+c)\sqrt{n}) - P(\mathcal{N}_n > n(1+\epsilon)) \leq P(L_{\max}(n(1+\epsilon)) > (2+c)\sqrt{n}) \,,$$

which implies (using (3.3)) that (3.1) holds as soon as it holds with $\bar{L}_{\max}(\mathcal{N}_n)$ replacing $L_{\max}(n)$.

On the other hand, consider the squares $\mathbf{Q}^1 = [0, \sqrt{n}/(\sqrt{n}+\sqrt{m})]^2$ and $\mathbf{Q}^2 = (\sqrt{n}/(\sqrt{n}+\sqrt{m}), 1]^2 \subseteq \mathbf{Q}$, and denote by $\bar{L}^1_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2})$ and $\bar{L}^2_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2})$ the length of the longest increasing subsequence in the squares $\mathbf{Q}^1$ and $\mathbf{Q}^2$, corresponding to $\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}$. The scaling and independence properties of the Poisson process imply that $\bar{L}^i_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2})$, $i = 1, 2$, are independent, and that the laws of $\bar{L}^1_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2})$ and $L_{\max}(\mathcal{N}_n)$, respectively $\bar{L}^2_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2})$ and $L_{\max}(\mathcal{N}_m)$, are identical. Therefore, since

$$\bar{L}_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) \geq \bar{L}^1_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) + \bar{L}^2_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) \,,$$

we deduce that

$$\begin{aligned}
&P(\bar{L}_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) > (2+c)(\sqrt{n}+\sqrt{m})) \\
&\quad \geq P(\bar{L}^1_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) + \bar{L}^2_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) > (2+c)(\sqrt{n}+\sqrt{m})) \\
&\quad \geq P(\bar{L}^1_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) > (2+c)\sqrt{n}) P(\bar{L}^2_{\max}(\mathcal{N}_{(\sqrt{n}+\sqrt{m})^2}) > (2+c)\sqrt{m}) \,, \\
&\quad = P(\bar{L}_{\max}(\mathcal{N}_n) > (2+c)\sqrt{n}) P(\bar{L}_{\max}(\mathcal{N}_m) > (2+c)\sqrt{m}) \,,
\end{aligned}$$

which immediately implies the existence and convexity of the limit

$$\bar{U}_0(c) = \lim_{k\to\infty} \frac{1}{k} \log P(\bar{L}_{\max}(\mathcal{N}_{k^2}) > (2+c)k) \,.$$

Next, since

$$\begin{aligned}
P(\bar{L}_{\max}(\mathcal{N}_{([\sqrt{n}]+1)^2} > (2+c)\sqrt{n}) &\geq P(\bar{L}_{\max}(\mathcal{N}_n) > (2+c)\sqrt{n}) \\
&\geq P(\bar{L}_{\max}(\mathcal{N}_{[\sqrt{n}]^2} > (2+c)\sqrt{n}) \,,
\end{aligned}$$

(3.1) follows with $U_0 = \bar{U}_0$. It thus remains only to explicitely compute $U_0(c)$.

In fact, Kim has already observed that $U_0(c) \geq \beta(c)$, $c \geq 0$, see equation (1.6) in [5]. We thus concentrate in the sequel in the reverse inequality. The proof is constructive: we exhibit an appropriate collection of Young shapes.

Fix n large enough, $\epsilon > 0$ small and $c > 0$, and let $\mathcal{T}_{c,n}$ denote the set consisting of Young shapes of size $n_c = n - \lceil (c+\epsilon)\sqrt{n} \rceil$. Recall the function $f_0^0 \in \mathcal{F}$ defined in Section 2, and define

$$\mathcal{T}_{c,n}^{\epsilon} = \{\tau \in \mathcal{T}_{c,n} : |\tau(0) - 2\sqrt{n}| \leq \epsilon\sqrt{n}, \sup_x |n_c^{-1/2}\tau(x\sqrt{n_c}) - f_0^0(x)| \leq \epsilon\} .$$

It follows easily from [5], [6] and (2.3) that for n large enough,

$$\sum_{\tau \in \mathcal{T}_{c,n}^{\epsilon}} \frac{n_c!}{(\pi(\tau))^2} \geq \frac{1}{2} . \tag{3.4}$$

For each $\tau \in \mathcal{T}_{c,n}^{\epsilon}$, define a new Young shape τ' obtained by increasing the height $\tau(0)$ by $\lceil (c+\epsilon)\sqrt{n} \rceil$. Note that $\tau'(0) \geq (2+c)\sqrt{n}$ while $|\tau'| = n$. From (2.3), we have

$$\begin{aligned} P(L_{\max}(n) \geq (2+c)\sqrt{n}) &\geq \sum_{\tau' : \tau \in \mathcal{T}_{c,n}^{\epsilon}} \frac{n!}{(\pi(\tau'))^2} \\ &= \frac{n!}{n_c!} \sum_{\tau \in \mathcal{T}_{c,n}^{\epsilon}} \frac{n_c!}{(\pi(\tau))^2} \Big(\frac{\pi(\tau)}{\pi(\tau')} \Big)^2 . \end{aligned}$$

For $\tau \in \mathcal{T}_{c,n}^{\epsilon}$,

$$\frac{\pi(\tau')}{\pi(\tau)} = \Big(\prod_{i=1}^{\lceil (c+\epsilon)\sqrt{n} \rceil} i \Big) \prod_{j=1}^{\tau(0)} \frac{(\tau(0) + \lceil (c+\epsilon)\sqrt{n} \rceil - j + \tau(j))}{(\tau(0) - j + \tau(j))} .$$

Note that, due to the definition of $\mathcal{T}_{c,n}^{\epsilon}$,

$$\begin{aligned} &\log \prod_{j=1}^{\tau(0)} \frac{(\tau(0) + \lceil (c+\epsilon)\sqrt{n} \rceil - j + \tau(j))}{(\tau(0) - j + \tau(j))} \\ &\leq \sqrt{n} \Big[\int_0^2 \log(2 + c - x + f_0^0(x)) dx - \int_0^2 \log(2 - x + f_0^0(x)) dx + C_\epsilon \Big] + o(\sqrt{n}) , \end{aligned}$$

where $C_\epsilon \to_{\epsilon \to 0} 0$ does not depend on n. Using the change of variables $x - f_0^0(x) = \xi, x = h_0(\xi)$ and $g_0(\xi) = h_0(\xi) - \xi(1 + \mathsf{sign}(\xi))/2$ as in Pg. 212 of [6], one obtains after some manipulations that

$$\begin{aligned} &\int_0^2 \log(2 + c - x + f_0^0(x)) dx - \int_0^2 \log(2 - x + f_0^0(x)) dx \\ &= \int_{-\infty}^{\infty} g_0'(\xi)(\log(2 + c - \xi) - \log(2 - \xi)) d\xi + \int_0^2 (\log(2 + c - \xi) - \log(2 - \xi)) d\xi \\ &= \pi\tilde{g}_0(2 + c) - \pi\tilde{g}_0(2) + (2 + c)\log(2 + c) - c\log c - 2\log 2 , \end{aligned} \tag{3.5}$$

where $\tilde{g}_0$ denotes the Hilbert transform of g_0 and is given by (2.31) in [6]. Note however that by (2.22) in [6], $\pi\tilde{g}_0(2) = 2 - 2\log 2$, while (2.22) and (2.32) in [6] imply $\pi\tilde{g}_0(c+2) = (2+c) - (2+c)\log(2+c) + \beta(c)/2$. Substituting in (3.5), and then using (3.4), one concludes that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log P(L_{\max}(n) \geq (2+c)\sqrt{n}) \geq -(\beta(c) + 2C_\epsilon)\,.$$

Taking $\epsilon \to 0$ yields the desired conclusion that $U_0(c) \leq \beta(c)$ for $c > 0$. $\square$

The following corollary follows from Theorem 2 in the same way that Corollary 1 followed from Theorem 1:

Corollary 2. *For any $c > 0$ there exists a function $\bar{\eta}(c,\delta)$ satisfying*

$$\lim_{\delta\to 0} \bar{\eta}(c,\delta) = 0$$

such that if $p(x,y)$ satisfies $(1-\delta) \leq p(x,y) \leq (1+\delta)$ then

$$\limsup_{n\to\infty} \left|\frac{1}{\sqrt{n}} \log P(\ell_{\max}(n) > (2+c)\sqrt{n}) + U_0(c)\right| \leq \bar{\eta}(c,\delta)\,.$$

Let $K \subseteq B^{\uparrow}$ be the set of solution to the variational problem (1.2).

Theorem 3. *For all $c \geq 0$*

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} \log P(\ell_{\max}(n) \geq (2\bar{J}_\mu + c)\sqrt{n}) = -\bar{J}_\mu U_0\big(c/\bar{J}_\mu\big)\,. \tag{3.6}$$

Next assume that $K = \{\phi_1, ..., \phi_r\}$. Then for each $\delta > 0$ and longest increasing subsequence $Z^{\max} = \{(X_{i_j}, Y_{i_j}), j = 1, ..., \ell_{\max}(n)\}$,

$$\lim_{n\to\infty} P(\min_{\alpha=1}^{r} \max_{j=1}^{\ell_{\max}(n)} |Y_{i_j} - \phi_\alpha(X_{i_j})| < \delta \big| \ell_{\max}(n) \geq (2\bar{J}_\mu + c)\sqrt{n}) = 1.$$

Proof. We begin by providing a lower bound in (3.6). Let ϕ denote a maximizer in (1.2), and define $\Delta, \rho_i, n_i, m_i, \mathbf{Q}_i, \ell_i(m_i)$ be as in the beginning of the proof of Proposition 2.2. Fix $\delta > 0$, and reduce Δ if necessary. By Sanov's theorem,

$$\limsup_{n\to\infty} \frac{1}{n^{1/2}} \log P(|n_i - m_i| > \delta n_i) = -\infty\,.$$

Hence,

$$\liminf_{n\to\infty} \frac{1}{n^{1/2}} \log P(\ell_{\max}(n) \geq (2\bar{J}_\mu + c)\sqrt{n}) \geq$$

$$\liminf_{n\to\infty} \frac{1}{n^{1/2}} \log P(\sum_{i=1}^{\Delta^{-1}} \ell_i(m_i) \geq (2\bar{J}_\mu + c)\sqrt{n}; \cap_{i=1}^{\Delta^{-1}} \{|n_i - m_i| \leq \delta n_i\})\,.$$

Next, for each i and $t_i > 0$, using $\lim_{\Delta\to 0}\sup_{x,y\in Q_i}|p_i(x,y)-1| = 0$, one has by Corollary 2 that for some $\delta'(\Delta)\to_{\Delta\to 0} 0$,

$$|\lim_{m_i\to\infty}\frac{1}{m_i^{1/2}}\log P_{m_i}(\ell_i(m_i) > (2+t_i)\sqrt{m_i}) + U_0(t_i)| \le \bar\eta(t_i,\delta')\,. \tag{3.7}$$

(P_{m_i} is the law of the sample in $\mathbf{Q}_i$ conditioned on m_i).

Recall $\rho_i = n_i/n = \Delta(\phi(i\Delta)-\phi((i-1)\Delta))p(i\Delta,\phi(i\Delta))$, and fix a sequence $\{t_i \ge 0\}_{i=1}^{\Delta^{-1}}$ such that

$$\sum_{i=1}^{\Delta^{-1}}(2+t_i)\sqrt{(1-\delta)\rho_i} \ge (2\bar J_\mu + c)\,. \tag{3.8}$$

Then,

$$\begin{aligned}
&\frac{1}{n^{1/2}}\log P(\sum_{i=1}^{\Delta^{-1}}\ell_i(m_i) \ge (2\bar J_\mu + c)\sqrt{n};\cap_{i=1}^{\Delta^{-1}}\{|n_i-m_i|\le\delta n_i\})\\
\ge&\frac{1}{n^{1/2}}\log\inf_{\{m_i:|n_i-m_i|\le\delta n_i\}}P(\sum_{i=1}^{\Delta^{-1}}\ell_i(m_i)\ge(2\bar J_\mu+c)\sqrt{n}|\{m_j\})\\
\ge&\frac{1}{n^{1/2}}\log\prod_{i=1}^{\Delta^{-1}}P_{n_i(1-\delta)}(\ell_i(m_i)\ge(2+t_i)\sqrt{(1-\delta)n_i})
\end{aligned}$$

where the last inequality is a consequence of (3.8), the monotonicity of $\ell_i(m_i)$ in m_i, and of the (conditional in $\{m_j\}$) independence of the $\ell_i(m_i)$. Thus, combining the above with (3.7), one concludes that

$$\liminf_{n\to\infty}\frac{1}{n^{1/2}}\log P(\ell_{\max}(n)\ge(2\bar J_\mu+c)\sqrt{n}) \ge -\sum_{i=1}^{\Delta^{-1}}\sqrt{\rho_i(1-\delta)}(U_0(t_i)+\bar\eta(t_i,\delta')\}\,.$$

Since the last bound is valid for any choice of $\{t_i\}$ satisfying (3.8), we conclude that

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n^{1/2}}\log P(\ell_{\max}(n)\ge(2\bar J_\mu+c)\sqrt{n}) \ge\\
&-\inf\{\sum_{i=1}^{\Delta^{-1}}\sqrt{\rho_i}(U_0(t_i)+\bar\eta(t_i,\delta')) : t_\cdot\ge 0, \sum_{i=1}^{\Delta^{-1}}(2+t_i)\sqrt{(1-\delta)\rho_i}\ge(2\bar J_\mu+c)\}\,.
\end{aligned} \tag{3.9}$$

Recall (c.f. [3]) that $\sum_{i=1}^{\Delta^{-1}}\sqrt{\rho_i}\to_{\Delta\to 0}\bar J_\mu$. The smoothness of ϕ proved in [3] and (3.9) imply therefore, by taking the limit $\Delta\to 0$ in the right hand side of (3.9), that

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n^{1/2}}\log P(\ell_{\max}(n)\ge(2\bar J_\mu+c)\sqrt{n})\\
&\ge -\inf\{\int_0^1\sqrt{\dot\phi(x)p(x,\phi(x))}U_0(t(x))dx :\\
&\qquad t\in C([0,1],\mathbb{R}^+)\,, \int_0^1 t(x)\sqrt{\dot\phi(x)p(x,\phi(x))}dx\ge c\}\,.
\end{aligned} \tag{3.10}$$

Making the change of variables $dy = \sqrt{\dot\phi(x)p(x,\phi(x))}dx, t(x) \to \bar t(y)$, the right hand side of (3.10) becomes

$$-\inf\{\int_0^{\bar J_\mu} U_0(\bar t(y))dy \,:\, \bar t \in C([0,\bar J_\mu],\mathbb{R}^+), \int_0^{\bar J_\mu} \bar t(y)dy \geq c\}\,. \tag{3.11}$$

Take now $\bar t(y) = c/\bar J_\mu$ to conclude from (3.10) that

$$\liminf_{n\to\infty} \frac{1}{n^{1/2}} \log P(\ell_{\max}(n) \geq (2\bar J_\mu + c)\sqrt{n}) \geq -\bar J_\mu U_0(\frac{c}{\bar J_\mu})\,.$$

The proof of the complimentary upper bound is only slightly more complicated, and involves the same tools as in [3]. Let $\Delta_y << \Delta$, with Δ_y^{-1} an integer. Define a "block curve" as an integer valued sequence $\{j(i)\}_{i=1}^{\Delta^{-1}}$, satisfying $j(i+1) > j(i)$ and $j(\Delta^{-1})\Delta_y \leq 1$. Let $\mathcal{B}^\Delta$ denote the set of all possible block curves, and note that the cardinality of $\mathcal{B}^\Delta$ is finite. To any block curve $b \in \mathcal{B}^\Delta$ associate naturally a (piecewise linear) curve ϕ_b, and define $\bar{\mathbf{Q}}_i = [(i-1)\Delta, i\Delta]\times[j(i-1)\Delta_y, (j(i)+1)\Delta_y]$, $\bar m_i$ as the number of points within $\bar{\mathbf{Q}}_i$, $\bar\rho_i = \Delta(j(i)-j(i-1)+1)\Delta_y p(i\Delta, j(i)\Delta_y)$, $\bar n_i = n\bar\rho_i$ and $\bar\ell_i(b,n)$ as the length of the longest increasing subsequence within $\bar{\mathbf{Q}}_i$. Clearly, $\ell_{\max}(n) \leq \max_{b\in\mathcal{B}^\Delta} \sum_{i=1}^{\Delta^{-1}} \bar\ell_i(b,n)$. Hence,

$$\begin{aligned}&\frac{1}{n^{1/2}} \log P(\ell_{\max}(n) \geq (2\bar J_\mu + c)\sqrt{n}) \leq \\ &\frac{1}{n^{1/2}} \log|\mathcal{B}^\Delta| + \max_{b\in\mathcal{B}^\Delta} \frac{1}{n^{1/2}} \log P(\sum_{i=1}^{\Delta^{-1}} \bar\ell_i(b,n) \geq (2\bar J_\mu + c)\sqrt{n})\,.\end{aligned}$$

Fix $\delta > 0$ small. Repeating the argument used in the proof of the lower bound, one finds (reducing $\Delta, \Delta_y/\Delta$ if necessary, but independently of n) that

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n^{1/2}} \log P(\sum_{i=1}^{\Delta^{-1}} \bar\ell_i(b,n) \geq (2\bar J_\mu + c)\sqrt{n}) \\ &\leq -\inf\{\sum_{i=1}^{\Delta^{-1}} \sqrt{\bar\rho_i}(U_0(t_i) + \bar\eta(\delta', t_i)) \,:\, t_i \geq 0, \sum_{i=1}^{\Delta^{-1}} (2+t_i))\sqrt{\bar\rho_i(1+\delta)} \geq 2\bar J_\mu + c\}\,.\end{aligned}$$

Let $J_\phi = \int_0^1 \sqrt{\dot\phi(x)p(x,\phi(x))}dx$. With Δ small enough,

$$|\sum_{i=1}^{\Delta^{-1}} \sqrt{\bar\rho_i} - J_{\phi_b}| \leq \delta\,.$$

Hence, taking now first $n \to \infty$ and then $\Delta \to 0$, followed by $\delta \to 0$, one concludes that

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n^{1/2}} \log P(\ell_{\max}(n) \geq (2\bar J_\mu + c)\sqrt{n}) \\ &\leq -\inf_{\phi\in B^\uparrow} \inf\{\int_0^1 \sqrt{\dot\phi(x)p(x,\phi(x))}U_0(t(x))dx \,: \\ &\quad t(\cdot) \geq 0, \int_0^1 t(x)\sqrt{\dot\phi(x)p(x,\phi(x))}dx \geq 2(\bar J_\mu - J_\phi) + c)\,.\end{aligned} \tag{3.12}$$

Making the same change of variables as in the proof of the lower bound, the right hand side of (3.12) equals

$$I = \inf_{\phi\in B^\uparrow} \inf\{\int_0^{J_\phi} U_0(\bar{t}(y))dy : \int_0^{J_\phi} \bar{t}(y)dy \geq c + 2(\bar{J}_\mu - J_\phi)\}$$

$$\geq \inf_{\phi\in B^\uparrow}\{J_\phi U_0(-2 + \frac{c+2\bar{J}_\mu}{J_\phi})\}, \tag{3.13}$$

where the last inequality follows from the convexity of U_0 and Jensen's inequality. Let $x = -2 + \frac{c+2\bar{J}_\mu}{J_\phi}$. Then, since $J_\phi \leq \bar{J}_\mu$, $x \geq c/\bar{J}_\mu$. Hence, using again the convexity of U_0 and the fact that $U_0(0) = 0$, $cU_0(x) \geq \bar{J}_\mu x U_0(c/\bar{J}_\mu)$ and hence

$$J_\phi U_0(x) \geq \frac{xJ_\phi\bar{J}_\mu}{c} U_0(c/\bar{J}_\mu) \geq \bar{J}_\mu U_0(c/\bar{J}_\mu), \tag{3.14}$$

with the second inequality being strict unless $J_\phi = \bar{J}_\mu$. (3.12), (3.13) and (3.14) imply the required upper bound.

Finally, the last statement of Theorem 3 follows from the fact that the inequality in (3.14) is strict unless $J_\phi = \bar{J}_\mu$, and the fact that the assumption of finite K implies A4 of [3] (the proof is similar to the proof of Lemma 4 in [3] and is thus omitted). □

§4 Open problems and remarks

We conclude this paper with a list of comments and open problems.

1) We have left open the question of existence of limit in (1.4) and of the computation of $H_\mu(c)$. After a discretization as used in Section 2, maybe techniques borrowed from percolation may allow one to control the interaction between overlapping "block curves".

2) We have seen in Theorem 3 that, under the conditioning $\{\ell_{\max}(n) \geq (2\bar{J}_\mu + c)\sqrt{n}\}$, $(c > 0)$, any longest increasing subsequence concentrates along the solution to the variational problem (1.2). The corresponding question for $\{\ell_{\max}(n) \leq (2\bar{J}_\mu + c)\sqrt{n}\}$, $(-2\bar{J}_\mu < c < 0)$, remains unsolved, even in case $\mu = \lambda$.

3) Under the assumption that K is finite, the strict convexity of $U_0(c)$ implies uniqueness of the minimizing function $t(\cdot)$ in (3.10) and gives the profile of the longest increasing subsequence under the conditioning that an upper tail deviation occurred, for any μ. Indeed, for an optimal curve ϕ, the minimizing function $t(x) = t_\phi(x)$ in the variational problem (3.10) is readily seen to have the interpretation as the (local) fluctuation from the mean behavior, and strict convexity of U_0 would imply that $t(x) = c/\bar{J}_\mu$, a constant.

4) It is natural to ask what happens when $\mathbf{Q}$ is replaced by $[0,1]^d$, $d > 2$. The subadditivity argument for the upper tail is the same, as well as the analog of Theorem 3 (with exponential speed $n^{1/d}$, and functional J_μ as given in [3], page 864). What about the lower tail? The lack of a direct probabilistic proof of Theorem 1, and the unavailability of the Schensted correspondence in higher dimension makes finding the analog of Theorem 1 challenging. One can still show, however, that the order of decay is exponential in n.

5) As pointed out to us by P. Baxendale, it seems reasonable to expect that under the conditioning $\ell_{\max}(n) \geq \alpha_n$, $\alpha_n/\sqrt{n} \to \infty$, the maximizing subsequences concentrate around the solutions of the optimization problem (1.2). For $\alpha_n = n$, this was proved in [3], and the technique of the proof seems to carry to the general case.

6) Let $\mathcal{N}_n$ be the number of points of a Poisson point process on $\mathbf{Q}$ with intensity $n\mu$ and denote by $\bar{\ell}_{\max}(\mathcal{N}_n)$ the length of the longest increasing subsequence of $\mathcal{N}_n$. Note that, conditioned on $\mathcal{N}_n = m$, the law of $\bar{\ell}_{\max}(\mathcal{N}_n)$ is the same as the law of $\ell_{\max}(m)$. Applying the same type of argument as in the first step of the proof of Theorem 2, one shows

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} \log P(\bar{\ell}_{\max}(\mathcal{N}_n) \geq (2\bar{J}_\mu + c)\sqrt{n}) = -\bar{J}_\mu U_0\big(c/\bar{J}_\mu\big).$$

The corresponding result for the lower tail (in the case $\mu = \lambda$) can also be read off Theorem 1, c.f. [8]. Note that in this case the rate function does differ from the uniform case due to fluctuations in the number of points in the Poisson sample.

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