Large Deviations in the Geometry of Convex Lattice Polygons

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Abstract We provide a full large deviation principle (LDP) for the uniform measure on certain ensembles of convex lattice polygons. This LDP provides for the analysis of concentration of the measure on convex closed curves. In particular, convergence to a limiting shape results in some particular cases, including convergence to a circle when the ensemble is defined as those centered convex polygons, with vertices on a scaled two dimensional lattice, and with length bounded by a constant. The Gauss-Minkowskii transform of convex curves plays a crucial role in our approach.

1 Introduction

The problem of finding the limit shape of convex lattice polygons in the unit square with respect to the uniform distribution, posed by the first author, was recently solved (see [9],[1],[8]). A variational principle describing the solution in terms of a maximization of affine length for smooth strictly convex curves is further developed in [9], see also [2],[10].

Our goal in this paper is twofold: first, we extend, using a technical lemma borrowed from [1],

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the result of [9] to more general ensembles of convex curves and put it in the context of the theory of large deviations. This then allows us to generalize the results of [9] and [1] concerning limit shapes to other situations, see for a representative example Corollary 2 below. We thus recover and extend the results in [2], and provide a tool for handling the concentration question under a variety of constraints.

The space of convex curves does not possess a natural linear structure, making it harder to formulate Legendre duality and its associated LDP. Therefore, an important tool in our approach is the mapping, via the Gauss-Minkowskii transform (area measure), of the problem from a question involving convex curves to a problem involving a subset of the set of positive measures on S^1. The duality between continuous functions and measures on S^1 allows then one to identify the rate function of the large deviations principle with a Legendre transform of an appropriate function. This latter function is related to a "pressure", c.f. Corollary 3 below.

Our large deviations approach allows one to define the notion of limit shape in greater generality than in [1],[2],[8],[9], and to put on the problem various constraints, such as length, area, etc. In particular, we show (c.f. Corollary 1 below) that the uniform measure on the set of convex lattice polygons of total length bounded by L concentrates on a limit shape which is a circle. This is in contrast to the limit shape for the uniform measure on the set of convex lattice polygons contained in the unit square, obtained in [1],[8],[9], which is a concatenation of four parabolas. The discrepancy between the two solutions is due to the influence of the boundary of the square on the shape of polygons. We also identify certain situations where the minimizer in the large deviation principle is not unique, leaving open the question of existence of concentration in those case.

The structure of the article is as follows: in the next section, we define precisely the set-up, define the Gauss-Minkovskii transformation, state our main results, and make various comments. The last section is devoted to proofs.

2 Definitions and statements of results

For any convex closed curve in $\mathbb{R}^2$, we denote by C_γ the subset of $\mathbb{R}^2$ encircled by γ, by A_γ the area of C_γ and by L_γ the length of γ. Let $\Gamma_{L,A}$ denote the set of closed convex curves γ in $\mathbb{R}^2$ satisfying $0 \in C_\gamma$, $A_\gamma \leq A$ and $L_\gamma \leq L$. Let $\overline{\Gamma}_{L,A}$ denote the quotient of $\Gamma_{L,A}$ with respect to shifts in $\mathbb{R}^2$ obtained by imposing that the barycenter of $\gamma \in \overline{\Gamma}_{L,A}$ is 0. Let $\overline{\Gamma}_L = \cup_{i=1}^{\infty} \overline{\Gamma}_{L,i}$ (due to the

Euclidean isoperimetric inequality, $\overline{\Gamma}_{L,i} = \overline{\Gamma}_{L,i+1}$ for $i > L^2/4\pi$). We equip $\Gamma_{L,A}$, $\overline{\Gamma}_{L,A}$ and $\overline{\Gamma}_L$ with the topology induced by the Hausdorff distance on subsets of $\mathbb{R}^2$. This makes both $\Gamma_{L,A}$ and $\overline{\Gamma}_{L,A}$ into compact Polish spaces.

Our study concerns polygonal curves with vertices on the two dimensional lattice. Thus, let $\mathrm{CLP}_{n,L,A}$ denote the (finite) set of all convex polygons γ in $\mathbb{R}^2$ satisfying the following conditions

A1) The vertices of γ belong to $\frac{1}{n}\,\mathbb{Z} \times \frac{1}{n}\,\mathbb{Z}$.

A2) $0 \in C_\gamma$.

A3) $A_\gamma :=$ Area enclosed by $\gamma \leq A$.

A4) $L_\gamma :=$ Length of the perimeter of $\gamma \leq L$.

By the Euclidean isoperimetric inequality, one always has $L^2 \geq 4\pi A$. In what follows, we will always omit A from the notations if it is not an active constraint, i.e. if $4\pi A = L^2$. We denote by $\nu_n^{L,A}$ the uniform measure on $\mathrm{CLP}_{n,L,A}$.

It is convenient to consider $\overline{\mathrm{CLP}}_{n,L,A}$, the quotient of $\mathrm{CLP}_{n,L,A}$ with respect to the translation by a vector in $\mathbb{R}^2$ obtained by imposing the barycenter of $\gamma \in \overline{\mathrm{CLP}}_{n,L,A}$ to be 0. Denote by $\overline{\nu}_n^{L,A}$ the uniform measure on $\overline{\mathrm{CLP}}_{n,L,A}$.

In order to study concentration properties of $\{\nu_n^{L,A}\}$ and $\{\overline{\nu}_n^{L,A}\}$, we introduce the Gauss-Minkowskii map. Let $M(S^1)$ denote the linear space of signed measures of bounded variation on S^1, equipped with the topology of weak convergence corresponding to the duality of $C(S^1)$ and $M(S^1)$. We denote the Lebesgue decomposition of any $\mu \in M(S^1)$ by $\mu = \mu^a + \mu^s$, where μ^a is the absolutely continuous part of μ and μ^s is the singular part.

Let $M_+(S^1)$ ($M_L(S^1)$) denote the subset of positive measures (respectively, with total mass bounded by L). Let $M_+^o(S^1)$ ($M_L^o(S^1)$) denote the subset of $M_+(S^1)$ (respectively, $M_L(S^1)$) of measures possessing zero barycenter. There exists a natural map (the Gauss-Minkowskii transformation, c.f. [4] and Lemma 1 below) from the set of convex sets $\overline{\Gamma}_L$ to the set of measures $M_L^o(S^1)$, defined by

$$\mu_\gamma(A) = \mathrm{Leb}\{x \in \gamma : \ \theta_x \in A\}$$

for any open interval $A \subset S^1$, where θ_x denotes the supporting hyperplane to γ at x. μ_γ is a rotation of the push-forward of the Lebesgue measure on γ by the Gauss map, which attaches to each normal point of γ its outward normal. The following is well known. For a proof, see [4, Section 8].

Lemma 1 *The map $\gamma \mapsto \mu_\gamma$ defines a homeomorphism between the compact spaces $\overline{\Gamma}_L$ and $M_L^o(S^1)$.*

In order to state our main result, we need to introduce some notation. Let $\zeta(\cdot)$ denote the Riemann zeta function, and define $c_\zeta = \frac{3\zeta(3)^{1/3}}{2^{2/3}\zeta(2)^{1/3}}$. For $\psi \in C(S^1)$, define the functional

$$\Lambda(\psi) = \begin{cases} \infty\,, & \mathrm{Leb}\{\psi \geq 0\} > 0 \text{ or } \psi^{-2} \notin L^1 \\ \frac{4}{27}\int \psi(x)^{-2}\,dx\,, & \text{otherwise}\,. \end{cases} \tag{1}$$

Note that the constraint on ψ to be negative a.e. corresponds to it belonging to the polar set to $M_+(S^1)$, that is to the set

$$C_- = \{\psi : \langle \psi, \mu \rangle < 0 \ \ \forall \mu \in M_+(S^1), \mu \neq 0\}\,.$$

Let $\bar{I}(\mu)$ denote the Legendre transform of $\Lambda(\cdot)$ with respect to the usual duality of $C(S^1)$ and $M(S^1)$, that is for $\mu \in M_+(S^1)$,

$$\overline{I}(\mu) = \sup_{\psi \in C(S^1)} \Big(\langle \psi, \mu \rangle - \Lambda(\psi) \Big)\,. \tag{2}$$

We recall (c.f. [5]) that a sequence of Borel probability measures $\{\nu_n\}$ on a topological space $\mathcal{X}$ satisfies the Large Deviations Principle (LDP) with speed a_n and rate function $I : \mathcal{X} \to [0, \infty]$ if

a) I is lower-semicontinuous.

b) For any open set G,

$$-\inf_{x \in G} I(x) \leq \liminf_{n \to \infty} a_n^{-1} \log \nu_n(G)\,.$$

c) For any closed set F,

$$\limsup_{n \to \infty} a_n^{-1} \log \nu_n(F) \leq -\inf_{x \in F} I(x)\,.$$

See [5] for basic results concerning the existence, uniqueness, and various properties of the LDP.

Our main result is the following

Theorem 1 *The sequence of measures* $\{\nu_n^{L,A}\}$ *(*$\{\overline{\nu}_n^{L,A}\}$*) satisfy in* $\Gamma_{L,A}$ *(respectively,* $\overline{\Gamma}_{L,A}$*) the LDP with speed* $n^{2/3}$ *and rate function* $I(\gamma) = c_\zeta(J(\gamma) - K_{L,A})$*, where*

$$\begin{aligned} J(\gamma) &:= \overline{I}(\mu_\gamma) = -\int_{S^1} m_\gamma^{2/3}(\theta)d\theta\,, \\ K_{L,A} &:= \min_{\gamma\in\Gamma_{L,A}} J(\gamma) = -(8\pi^2\min(A, L^2/4\pi))^{1/3}\,, \end{aligned} \tag{3}$$

and $m_\gamma(\theta) := (d\mu_\gamma^a/d\theta)(\theta)$ *denotes the density of the absolutely continuous part of* μ_γ *with respect to Lebesgue's measure on* S^1*.*

It is useful to note that for γ strictly convex and smooth, the rate function $J(\gamma)$ possesses a natural geometric interpretation: Lemma 2 below implies (c.f. [7, pg. 419]) that

$$J(\gamma) = -\oint \kappa^{1/3}\Big(\gamma(s)\Big)\,ds \tag{4}$$

where ds denotes the (Euclidean) arclength, and κ is the curvature. Thus, $-J(\gamma)$ is the affine arclength of the curve γ, denoted also L_γ^a. This functional appeared in a variational problem and was actually suggested as the correct rate function in [9] (see also [2]).

Next, recall that we omit A from our notations if A3) is not an active constraint. Let $\tilde{\nu}_n^L$ denote the measure induced by $\overline{\nu}_n^L$ on $M_L^o(S^1)$ through the bijection $\gamma \in \overline{\Gamma}_L \mapsto \mu_\gamma \in M_L^o(S^1)$. Theorem 1 is then an easy consequence, by the contraction theorem of large deviations theory (c.f. [5, Theorem 4.2.1] and the proof in Section 3) of the following LDP:

Theorem 2 *The sequence of measures* $\{\tilde{\nu}_n^L\}$ *satisfies in* $M_L^o(S^1)$ *the LDP with speed* $n^{2/3}$ *and rate function* $c_\zeta(\overline{I}(\cdot) - K_L)$*. Here,* $K_L = -(2\pi L^2)^{1/3}$*.*

An immediate corollary is the following.

Corollary 1 *The sequence of measures* $\{\tilde{\nu}_n^L\}$ *converges weakly to the uniform measure on* S^1*, while, again in the sense of weak convergence,* $\bar{\nu}_n^L \to \delta_{S_{L/2\pi}}$*, the Dirac measure on the circle of radius* $R = L/2\pi$*.*

The following corollary is a direct consequence of the large deviations principle. It extends some results announced in [1], which correspond to the choice of $\Gamma = \Gamma_F$ consisting of all convex curves contained inside a convex compact subset F of $\mathbb{R}^2$ (see Remark 5 below).

Corollary 2 *Let $\Gamma \subset \Gamma_{L,A}$ satisfy*

$$J_\Gamma := \inf_{\gamma \in \Gamma^o} J(\gamma) = \inf_{\gamma \in \overline{\Gamma}} J(\gamma)\,,$$

and equip Γ with the topology induced from $\Gamma_{L,A}$. Let ν_n^Γ denote the uniform measure on $\mathrm{CLP}_{n,L,A} \cap \Gamma$. Then the sequence of measures $\{\nu_n^\Gamma\}$ satisfies the LDP in Γ with speed $n^{2/3}$ and rate function $I_\Gamma(\gamma) = c_\zeta(J(\gamma) - J_\Gamma)$. In particular, if the minimizer in the definition of J_Γ is unique, then a limit shape exists.

Set next

$$\tilde{\Lambda}_L(\psi) = \sup_{\mu \in M_L^o(S^1)} \left(\langle \psi, \mu \rangle - \bar{I}(\mu) \right), \tag{5}$$

which is the Legendre transform of the modification of $\bar{I}$ obtained by setting it to ∞ outside $M_L^o(S^1)$. Note that in general, $\tilde{\Lambda}_L(\psi) \leq \Lambda(\psi) = \sup_{\mu \in M(S^1)}(\langle \psi, \mu \rangle - \bar{I}(\mu))$, where the last equality is due to Fenchel–Legendre duality and trhe convexity and lower semi-continuity of $\Lambda(\cdot)$ in $C(S^1)$. It is not hard to check by an explicit computation in (5) similar to that given in Lemma 2 below that

$$\tilde{\Lambda}_L(\psi) = \int \psi(\theta) m_L^\psi(\theta) d\theta + \int m_L^\psi(\theta)^{2/3} d\theta\,,$$

where

$$m_L^\psi(\theta) = -\frac{8}{27\left(\psi(\theta) + \lambda_1 \cos(\theta) + \lambda_2 \sin(\theta) + \eta\right)^3} \mathbf{1}_{\{\psi(\theta) + \lambda_1 \cos(\theta) + \lambda_2 \sin(\theta) + \eta < 0\}}\,,$$

and $\lambda_1, \lambda_2, \eta$ are the unique constants which satisfy $\int m_L^\psi(\theta) d\theta = L$, $\int \cos(\theta) m_L^\psi(\theta) d\theta = \int \sin(\theta)\, m_L^\psi(\theta)\, d\theta = 0$. In view of Theorem 2, the function $\tilde{\Lambda}_L(\cdot)$ in (5) can be given an interpretation of a generalized pressure:

Corollary 3 *It holds that*

$$\lim_{n \to \infty} \frac{1}{n^{2/3}} \log \int \exp(n^{2/3} \langle \psi, \mu \rangle - c_\zeta K_L) \tilde{\nu}_n^L(d\mu) = c_\zeta \tilde{\Lambda}_L(\psi)\,.$$

Remarks

1. By the affine isoperimetric inequality [3, Page 56], the minimizing γ in (3) are the ellipses. This explains the value of $K_{L,A}$ in the statement of Theorem 1.

2. For $A < L^2/4\pi$, the minimizer in (3) is not unique, as all ellipses of area A possess the same affine length which achieves the equality, and thus the set of minimizers of (3) consists of all ellipses of area A and Euclidean length bounded above by L. Thus, any limit point of the sequences $\{\nu_n^{L,A}\}$ or $\{\overline{\nu}_n^{L,A}\}$ is a measure supported on such ellipses. We do not know whether in this case a limiting shape exits, or even whether these sequences of measures converge on subsequences to Dirac measures. Solving this problem seems to require refined estimates.

3. For $A = L^2/4\pi$, one can apply the standard isoperimetric inequality to conclude that the maximizer of the area given L, which also maximizes the affine length, is the circle of radius $R = L/2\pi$. This is essentially the content of Corollary 1.

4. Theorem 1 is strongly related to [9, Theorem 2.4]. Indeed, the statement of the local estimates around *smooth* curves already appear there. The main new ingredient needed in order to extend it to general curves and a full LDP is the relation to $M_L^o(S^1)$, which provides a natural framework for proving lower-semicontinuity and dealing with approximations. In fact, a direct consequence of the LDP is that for any fixed γ,

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n^{2/3}} \log \overline{\nu}_n(B(\gamma, \epsilon)) = -c_\zeta(J(\gamma) - K_{L,A}) ,$$

where $B(\gamma, \epsilon)$ denotes the Hausdorff tube of size ϵ around γ.

5. Our choice of dealing with $\mathrm{CLP}_{n,L,A}$ is highly arbitrary. In fact, as a look at the proof of Theorem 2 demonstrates, one could take general ensembles of convex polygons in such a way that their image under $\gamma \mapsto \mu_\gamma$ is compact. Indeed, a particularly interesting use of Corollary 2 is the case when Γ is the subset of $\Gamma_{L,A}$ consisting of all convex polygons contained within the area enclosed by a given convex curve γ_0. This provides in particular a proof of the statement, mentioned in [1, pg. 283] and proved in [2], that the uniform measure over all polygons satisfying A1)-A2) and in addition contained in some compact $S \subset \mathbb{R}^2$ concentrates on the curve contained in S maximizing the affine length, which is unique by [2].

6. The extension to $\mathbb{Z}^d$ seems challenging. Indeed, most of the analytical arguments go through to higher dimension. What is missing is a link of the geometry with appropriate integer partitions, leading to an analog of the local estimates of [1],[9].

3 Proofs

Proof of Theorem 2

The proof consists of a sequence of lemmas. We first check that $\overline{I}(\cdot)$ possesses the required properties. Recall that we denote by $\mu = \mu_s + m_\mu(\theta)d\theta$ the Lebesgue decomposition of $\mu \in M(S^1)$. Lemma 2 below provides a representation of $\bar{I}(\gamma)$ in (2), which will be useful in the sequel.

Lemma 2 *$\overline{I}(\cdot)$ is a lower semi-continuous function on $M(S^1)$. Further, for non-negative μ,*

$$\overline{I}(\mu) = -\oint m_\mu(\theta)^{2/3}\, d\theta\,, \tag{6}$$

while $\overline{I}(\mu) = \infty$ for μ possessing a negative component. Finally, for $A \leq L^2/4\pi$,

$$K_{L,A} = \inf_{\mu \in M_L^o(S^1):\, \gamma_\mu \in \Gamma_{L,A}} \overline{I}(\mu) = -(8\pi^2 A)^{1/3}\,.$$

Proof: The proof follows the arguments in [6, Lemma 5]. Note that $C(S^1)$ is the dual space to $M(S^1)$, and $M_L^o(S^1)$ is equipped with the topology induced as a closed subset of $M(S^1)$. Hence, the representation of $\bar{I}(\cdot)$ as a Legendre transform implies the required lower semicontinuity. To see the rest of the claims, assume first that $\mu(A) < 0$ for some Borel $A \subset S^1$. One may then find a sequence of continuous functions $0 \leq \psi_k \leq 1$ approximating $\mathbf{1}_A$ in $L^1(\mu)$. Define $\Psi_k = -k\psi_k - 1$, one obtains

$$\Big(\langle \Psi_k, \mu\rangle - \Lambda(\Psi_k)\Big) \to_{k\to\infty} \infty\,.$$

We may thus consider only the case of non-negative μ. In that case, $\bar{I}(\mu) \leq \bar{I}(m_\mu(\theta)d\theta)$. Let $-c_k \geq \psi_k \geq -c_k^{-1}$ be a sequence of $C(S^1)$ functions satisfying

$$\Big(\langle \psi_k, m_\mu(\theta)d\theta\rangle - \Lambda(\psi_k)\Big) \to_{k\to\infty} \bar{I}(m_\mu(\theta)d\theta)\,.$$

Let B be a Borel set such that $\mathrm{Leb}(B)=0$ and $\mu_s(B)=\mu_s(S^1)$. For each $\epsilon>0$, let ψ_k^ϵ denote the ϵ continuous modification of $\psi_k \mathbf{1}_{B^c}$, that is $-c_k^{-1}\le \psi_k^\epsilon \le 0$ is continuous and $\psi_k^\epsilon=\psi_k\mathbf{1}_{B^c}$ on a set C with $\mu(C^c)<\epsilon$ and $\mathrm{Leb}(C^c)<\epsilon$. Then, letting $\tilde{\psi}_k^\epsilon=\psi_k^\epsilon-\epsilon^{1/4}$,

$$\begin{aligned}
\Big(\langle \psi_k, m_\mu(\theta)d\theta\rangle - \Lambda(\psi_k)\Big) &= \Big(\langle \psi_k \mathbf{1}_{B^c}, m_\mu(\theta)d\theta\rangle - \Lambda(\psi_k\mathbf{1}_{B^c})\Big)\\
&= \langle \tilde{\psi}_k^\epsilon, m_\mu(\theta)d\theta\rangle - \frac{4}{27}\int (\tilde{\psi}_k^\epsilon)^{-2}dx + \langle(\psi_k-\tilde{\psi}_k^\epsilon), m_\mu(\theta)d\theta\rangle\\
&\qquad -\frac{4}{27}\int \Big((\psi_k)^{-2}-(\tilde{\psi}_k^\epsilon)^{-2}\Big)\,dx\\
&\le \bar{I}(\mu)+2\epsilon^{1/4}\mu(S^1)+\mu(C^c)c_k^{-1}+\frac{2\epsilon^{1/4}}{c_k^4}+2\epsilon^{-1/2}c_k^{-4}\mathrm{Leb}(C^c)\to_{\epsilon\to 0}\bar{I}(\mu)\,.
\end{aligned}$$

It follows that $\bar{I}(\mu)=\bar{I}(m_\mu(\theta)d\theta)$. Thus, it remains to compute $\bar{I}(\mu)$ for absolutely continuous $\mu=m_\mu(\theta)d\theta$. It is easy to check that

$$\begin{aligned}
\sup_{0\ge\psi\in C(S^1)}\Big(\langle\psi,\mu\rangle-\Lambda(\psi)\Big) &= \sup_{0\ge\psi\in\mathcal{M}(S^1)}\Big(\langle\psi,\mu\rangle-\Lambda(\psi)\Big)\\
&= \sup_{0\ge\psi\in\mathcal{M}(S^1)}\int(\psi(\theta)m_\mu(\theta)-\frac{4}{27}\psi^{-2}(\theta))d\theta\,,
\end{aligned}$$

where $\mathcal{M}(S^1)$ denotes the space of measurable functions on S^1. The claim follows by optimizing pointwise over the value of ψ.

Finally, the identification of the constant $K_{L,A}$ follows from (4) above and the affine isoperimetric inequality (for one version of the latter, see [3, pg. 56] or [7, pg. 419]). □

We need next a local result borrowed from [9]. For any $\gamma\in\Gamma_{L,A}$, let $B(\gamma,\delta)$ denote the Hausdorff tube of size δ around γ. Let $Z_n^{\gamma,\delta}=\{\#\eta\in\mathrm{CLP}_{n,L,A}:\ \eta\in B(\gamma,\delta)\}$.

Lemma 3 *Assume that $\gamma\in\overline{\Gamma}_{L,A}$ is strictly convex and C^2. Then*

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n^{2/3}}\log Z_n^{\gamma,\delta}=-\bar{I}(\mu_\gamma)\,.$$

Proof: See [9, Theorem 4] or [1], pg. 282, and use Lemma 2 and (4). □

While Lemma 3 already hints at the existence of an LDP, some additional care is needed around non-smooth curves. The next lemma is the key to the LDP upper bound:

Lemma 4 *Let $\gamma \in \overline{\Gamma}_{L,A}$ be given. Then*

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n^{2/3}}\log Z_n^{\gamma,\delta} \leq -\overline{I}(\mu_\gamma)\,.$$

Proof: Let δ be given, and fix $\tilde{\gamma} \in B(\gamma,\delta)$ a convex closed curve. Fix $\Delta, \epsilon > 0$ (small enough). Throughout this proof, C denotes a universal constant (depending on A, L only but not on $n, \delta, \Delta, \epsilon$) whose value may change from line to line. Let $z_i := (x_i, y_i)$, $i = 0, \ldots, 3$ denote points where the supporting hyperplane to $\tilde{\gamma}$ at (x_i, y_i) has angle $i\pi/2$. Let $t_0 = x_0$, and define $\{t_i\}_{i=1}^{I}$ recursively by requiring that t_{i+1} be the least $\theta > t_i$ such that $\tilde{\gamma}(\theta) - \tilde{\gamma}(t_i) \geq \Delta$, or, if such θ does not exist and $t_i < x_1$, $t_{i+1} = x_1$. Since $\tilde{\gamma}$, being convex, is differentiable almost everywhere, we may and will assume that $\tilde{\gamma}$ is differentiable at all the t_i, $i = 0, \ldots, I-1$. Let $N_\epsilon = \{i \in (1, \ldots, I) : t_{i+1} - t_i \geq \Delta^{1/3}/\epsilon\}$. Note that $|N_\epsilon| \leq 2L\epsilon/\Delta^{1/3}$. Fix $s_i := (t_i, \tilde{\gamma}(t_i)) \in \mathbb{R}^2$. Define the parallelogram T_i as the parallelogram whose vertices include s_i, s_{i+1} and whose edges have slopes $\tilde{\gamma}'(t_i), \tilde{\gamma}'(t_{i+1})$. Let $\tilde{Z}_{n,i}^{\tilde{\gamma}}$ denote the number of increasing, convex polygonal lines with vertices in $(\frac{1}{n}\mathbb{Z})^2 \cap T_i$. By [1, Theorem C], there exists a universal constant C such that, for $i \in N_\epsilon$,

$$\tilde{Z}_{n,i}^{\tilde{\gamma}} \leq \exp(C(L\Delta)^{1/3}n^{2/3})\,.$$

On the other hand, for $i \notin N_\epsilon$, one may apply [1, Theorem B] and a linear transformation to conclude that

$$\tilde{Z}_{n,i}^{\tilde{\gamma}} \leq n^2 \exp\left(4^{1/3}c_\zeta n^{2/3}\left(A^{1/3}(T_i) + \frac{C}{\epsilon^{1/2}n^{1/4}}\right)\right)\,,$$

where $A(T_i)$ denotes the area of the parallelogram T_i.

Let $\bar{Z}_n^{\tilde{\gamma},0}$ denote the number of monotone convex curves connecting z_0 and z_1, with vertices in the lattice $(\frac{1}{n}\mathbb{Z})^2$, possessing linear edges of slopes $\tilde{\gamma}'(t_i)$ passing through s_i, $i = 0, \ldots, I$. It follows that

$$\bar{Z}_n^{\tilde{\gamma},0} \leq \prod_{i=0}^{I-1}\tilde{Z}_{n,i}^{\tilde{\gamma}} \leq \exp\left(4^{1/3}c_\zeta n^{2/3}\sum_{i=0}^{I-1}A(T_i)^{1/3} + CL\log n/\Delta + Cn^{1/2}\epsilon^{-1/2}\Delta^{-1} + C\epsilon n^{2/3}\right)\,, \quad (7)$$

where the last term is due to the contribution of the terms in N_ϵ and to the bound on $|N_\epsilon|$.

Let C_p^2 denote the space of piecewise C^2 curves. We need the following:

Lemma 5 *There exists $\phi(t) \in C_p^2$, which consists of a concatenation of parabolas, satisfying:*

1. $\phi(t_{2i}) = \tilde{\gamma}(t_{2i})$,

2. $\phi'(t_{2i}) = \tilde{\gamma}'(t_{2i})$,

3. $4^{1/3} \sum_{i=0}^{I-1} A(T_i)^{1/3} \leq L^a_\phi$, *where* L^a_ϕ *denotes the affine length of the curve* ϕ.

Proof of Lemma 5 Use an affine transformation on [9, Lemma 3] to conclude that there exists a parabola $\phi(\cdot)$ connecting $\phi(t_{2i})$ and $\phi(t_{2(i+1)})$ satisfying $\phi(t_{2i}) = \tilde{\gamma}(t_{2i})$, $\phi(t_{2(i+1)}) = \tilde{\gamma}(t_{2(i+1)})$, $\phi'(t_{2i}) = \tilde{\gamma}'(t_{2i})$ and $\phi'(t_{2(i+1)}) = \tilde{\gamma}'(t_{2(i+1)})$, and furthermore that for any value of $\tilde{\gamma}(t_{2i+1})$ and $\tilde{\gamma}'(t_{2i+1})$, $A(T_{2i})^{1/3} + A(T_{2i+1})^{1/3} \leq 4^{-1/3} L^a_i$ where L^a_i denotes the affine length of this parabola. Lemma 5 follows. $\square$

Returning to the proof of Lemma 4, assume that Δ is small enough ($\Delta < \delta/C(L)$ will do, with some fixed large $C(L)$), and let p_1 denote the concatenation of the above parabolas. Note that p_1 is convex and, for all $\tilde{\gamma} \in B(\gamma, \delta)$, it holds that $p_1 \in B(\tilde{\gamma}, \delta)$ and hence $p_1 \in B(\gamma, 2\delta)$. Let $L^a_{p_1}$ denote the affine length of p_1. Then,

$$\bar{Z}_n^{\tilde{\gamma},0} \leq \prod_{i=0}^{I-1} \tilde{Z}_{n,i}^{\tilde{\gamma}} \leq \exp\left(c_\zeta n^{2/3} L^a_{p_1} + CL \log n/\Delta + Cn^{1/2}\epsilon^{-1/2}\Delta^{-1} + C\epsilon n^{2/3}\right). \qquad (8)$$

Define

$$I_0 = \sup\{L^a_\phi : \phi \in B(\gamma, 2\delta) \cap C^2_p, \phi(x_0) = y_0, \phi(x_1) = y_1, \phi'(x_0) = 0, \phi'(x_1) = \infty\}.$$

Then $L^a_{p_1} \leq I_0$, hence (8) implies that

$$\bar{Z}_n^{\tilde{\gamma},0} \leq \exp\left(c_\zeta n^{2/3} I_0 + CL \log n/\Delta + Cn^{1/2}\epsilon^{1/2}/\Delta^{1/3} + C\epsilon n^{2/3}\right).$$

Define analogously $\bar{Z}_n^{\tilde{\gamma},j}$, $j = 1, 2, 3$ with $z_j, z_{(j+1) \bmod 4}$ replacing z_0, z_1, making the obvious modifications for $\{t_i\}, \{s_i\}$. Let $\bar{Z}_n^{\tilde{\gamma}} = \prod_{j=0}^{3} \bar{Z}_n^{\tilde{\gamma},j}$, and define I_j analogously to I_0. Then,

$$\bar{Z}_n^{\tilde{\gamma}} \leq \exp\left(c_\zeta n^{2/3}(I_0 + I_1 + I_2 + I_3) + CL \log n/\Delta + Cn^{1/2}\epsilon^{-1/2}\Delta^{-1} + C\epsilon n^{2/3}\right).$$

Clearly,

$$I_0 + I_1 + I_2 + I_3 \leq \sup\{L^a_\phi : \phi \in B(\gamma, 4\delta) \cap C^2_p \cap \overline{\Gamma}_{L,A}\}.$$

Therefore,

$$
\begin{aligned}
\bar{Z}_n^{\tilde{\gamma}} &\leq \exp\left(c_\varsigma n^{2/3} \sup_{\phi \in \bar{\Gamma}_{L,A} \cap B(\gamma,4\delta) \cap C_p^2} L_\phi^a + CL \log n/\Delta + Cn^{1/2}\epsilon^{-1/2}\Delta^{-1} + C\epsilon n^{2/3}\right) \\
&\leq \exp\left(-c_\varsigma n^{2/3} \inf_{\phi \in \bar{\Gamma}_{L,A} \cap B(\gamma,4\delta) \cap C_p^2} J(\phi) + CL \log n/\Delta + Cn^{1/2}\epsilon^{-1/2}\Delta^{-1} + C\epsilon n^{2/3}\right) \\
&\leq \exp\left(-c_\varsigma n^{2/3} \inf_{\phi \in \bar{\Gamma}_{L,A} \cap B(\gamma,4\delta)} J(\phi) + CL \log n/\Delta + Cn^{1/2}\epsilon^{-1/2}\Delta^{-1} + C\epsilon n^{2/3}\right). \qquad (9)
\end{aligned}
$$

Note next that any polygon with vertices in $(\frac{1}{n}\mathbb{Z})^2$ lying in $B(\gamma,\delta)$ must be counted in $\bar{Z}_n^{\tilde{\gamma}}$ for some $\tilde{\gamma} \in B(\gamma,\delta)$. Further, the total number of possible values of t_i is bounded by $n^{C(L,\Delta)}$ while the total number of possible slopes $\bar{\gamma}'(t_i)$, which must be of the form ℓ/m with ℓ, m integers bounded by $C(L)n$, is bounded by a similar bound. (Here and in the sequel, $C(L), C(L,\Delta)$ denote constants which now may depend on L or L,Δ, may change from line to line, but are still independent of n). Therefore, one still has that

$$
\begin{aligned}
Z_n^{\gamma,\delta} &\leq n^{C(L,\Delta)} \exp\left(-c_\varsigma n^{2/3} \inf_{\phi \in \bar{\Gamma}_{L,A} \cap B(\gamma,4\delta)} J(\phi) + CL \log n/\Delta + Cn^{1/2}\epsilon^{-1/2}\Delta^{-1} + C\epsilon n^{2/3}\right) \\
&\leq \exp\left(-c_\varsigma n^{2/3} \inf_{\phi \in \bar{\Gamma}_{L,A} \cap B(\gamma,4\delta)} J(\phi) + C(L,\Delta,\epsilon) \log n + Cn^{1/2}\epsilon^{1/2}/\Delta^{1/3} + C\epsilon n^{2/3}\right).
\end{aligned}
$$

The rest of the proof of Lemma 4 is standard by taking $n \to \infty$, followed by $\Delta \to 0$, $\epsilon \to 0$, and then $\delta \to 0$, using Lemma 2. $\square$

Returning to the proof of Theorem 2, Lemma 4 and the compactness of $M_L^o(S^1)$ imply, by taking a finite covering, that for any closed set $F \subset M_L^o(S^1)$,

$$
\lim_{n\to\infty} \frac{1}{n^{2/3}} \log |\{\gamma \in \overline{\mathrm{CLP}}_{n,L,A} : \mu_\gamma \in F\}| \leq -\inf_{\mu \in F} \bar{I}(\mu). \qquad (10)
$$

To complete the proof of Theorem 2, we first extend the lower bound in the statement of Lemma 3 to non-smooth curves γ. Let $\gamma \in \bar{\Gamma}_{L,A}$ be given, and let μ_γ denote its Gauss-Minkowskii transform. Obviously, by localizing Lemma 3 to piecewise strictly convex, smooth curves, we have that for $\mu_\gamma = m_\mu(\theta)d\theta + \mu_s$ with (finitely supported) atomic μ_s,

$$
\lim_{\delta\to 0} \lim_{n\to\infty} \frac{1}{n^{2/3}} \log Z_n^{\gamma,\delta} \geq -\bar{I}(\mu_\gamma),
$$

On the other hand, whenever μ_s is singular with respect to Lebesgues measure but not necessarily finitely supported atomic, one may approximate it by a sequence of finitely supported atomic measures μ_s^k. Then, for any $\delta > 0$, one may find a k large enough such that the curve corresponding to $m_\mu(\theta)d\theta + \mu_s^k$ lies in $B(\gamma, \delta)$, while $\bar{I}(\mu_\gamma) = \bar{I}(m_\mu(\theta)d\theta) = \bar{I}(m_\mu(\theta)d\theta + \mu_s^k)$ by Lemma 2. We conclude that, for any μ_γ,

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n^{2/3}} \log Z_n^{\gamma,\delta} \geq -\overline{I}(\mu_\gamma) \,. \tag{11}$$

Let now $G \subset M_L^o(S^1)$ be open, with $\nu \in G$. Then, for some $\gamma \in \overline{\Gamma}_L$, $\nu = \mu_\gamma$, and for some $\delta > 0$,

$$\{\mu_{\gamma'} : \gamma' \in B(\gamma, \delta)\} \subset G.$$

Therefore,

$$\begin{aligned} \tilde{\nu}_n^L(G) &\geq \frac{|\#\eta \in \overline{\mathrm{CLP}}_{n,L} \cap B(\gamma,\delta)|}{|\#\eta \in \mathrm{CLP}_{n,L}|} \\ &\geq \frac{Z_n^{\gamma,\delta}/4n^2L^2}{|\#\eta \in \mathrm{CLP}_{n,L}|} \,, \end{aligned} \tag{12}$$

where the second inequality is due to the fact that there are at most $4n^2L^2$ curves in $\mathrm{CLP}_{n,L}$ corresponding to a single curve in $\overline{\mathrm{CLP}}_{n,L}$. Since we already know from (10) that

$$\limsup_{n \to \infty} \frac{1}{n^{2/3}} \log |\#\eta \in \mathrm{CLP}_{n,L,A}| \leq -K_{L,A} \,,$$

we obtain, combining (12) and (11), that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \tilde{\nu}_n^L(G) \geq -\bar{I}(\mu_\gamma) + K_L \,,$$

proving the lower bound since $\nu \in G$ was arbitrary. Finally, the LDP upper bound follows in a similar way by combining (10) and (11). $\square$

Proof of Theorem 1 The LDP concerning $\bar{\nu}_n^{L,A}$ follows directly from Theorem 2 by the contraction principle [5, Theorem 4.2.1] due to the continuity of the bijection $\gamma \mapsto \mu_\gamma$ and of the map $\gamma \mapsto A_\gamma$. On the other hand, for any Borel set $C \subset \Gamma_{L,A}$, let

$$\bar{C} = \{\gamma \in \bar{\Gamma}_{L,A} : \ \gamma \text{ differs from an element of } C \text{ by a shift in } \mathbb{R}^2\} \,.$$

Since the number of possible shifts of a curve satisfying A1)–A4) is at most polynomial (more precisely, bounded by $4n^2L^2$), one has that for any $\delta > 0$,

$$\nu_n^{L,A}(C) \leq 4n^2L^2\bar{\nu}_n^{L,A}(\bar{C}^\delta)\,,$$

where $\bar{C}^\delta = \overline{\{\gamma : d(\gamma, C) \leq \delta\}}$. This proves the upper bound, since the rate function $\bar{I}$ is invariant under shifts. The lower bound is proved similarly by noting that for any $\gamma \in \Gamma_{L,A}$, denoting by $\bar{\gamma}$ the corresponding element of $\bar{\Gamma}_{L,A}$, one has that for n large enough ($n^{-1} < \delta/4$ suffices),

$$|\gamma' \in \mathrm{CLP}_{n,L,A} : d(\gamma,\gamma') < \delta| \geq |\gamma' \in \overline{\mathrm{CLP}}_{n,L,A} : d(\bar{\gamma},\gamma') < \delta/2|\,.$$

Hence,

$$\nu_n^{L,A}(\gamma' \in \mathrm{CLP}_{n,L,A} : d(\gamma,\gamma') < \delta) \geq \frac{1}{4n^2L^2}\bar{\nu}_n^{L,A}(\gamma' \in \overline{\mathrm{CLP}}_{n,L,A} : d(\bar{\gamma},\gamma') < \delta/2)\,,$$

which, in conjunction with Theorem 2, is more than enough to complete the proof of the lower bound. □

Proof of Corollary 1 While a proof can be drawn using the Euclidean and then affine isoperimetric inequalities, a direct proof follows by observing that the minimizer of (6) under the constraint on the total mass of μ is obtained by a uniform density. □

Proof of Corollary 2 The proof is a direct consequence of the LDP by noting that, for any measurable set $F \in \Gamma$,

$$\nu_n^{\Gamma}(F) = \frac{\nu_n^{L,A}(F \cap \Gamma)}{\nu_n^{L,A}(\Gamma)}\,.$$

□

Proof of Corollary 3 By Varadhan's Lemma of large deviations theory, c.f. [5, Theorem 4.3.1], and Theorem 2,

$$\lim_{n\to\infty} \frac{1}{n^{2/3}} \log \int \exp(n^{2/3}\,(\langle\psi,\mu\rangle - K_L))\tilde{\nu}_n^L(d\mu) = \sup_{\mu\in M_L^o(S^1)} (\langle\mu,\psi\rangle - \bar{I}(\mu)) = \tilde{\Lambda}_L(\psi)\,.$$

□

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