A sharp estimate for cover times on binary trees

Jian Ding* U. C. Berkeley

Ofer Zeitouni† University of Minnesota & Weizmann institute

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Abstract

We compute the second order correction for the cover time of the binary tree of depth n by (continuous-time) random walk, and show that with probability approaching 1 as n increases, $\sqrt{\tau_{\rm cov}} = \sqrt{|E|}[\sqrt{2\log 2}\cdot n - \log n/\sqrt{2\log 2} + O((\log\log n)^8]$, thus showing that the second order correction differs from the corresponding one for the maximum of the Gaussian free field on the tree.

1 Introduction

The cover time of a random walk on a graph, which is the time it takes the walk to visit every vertex in the graph, is a basic parameter and has been researched intensively over the last several decades (see [3, 12, 13] for background). One often studied aspect concerns precise estimates for cover times on specific graphs including 2D lattices and regular trees. For the 2D discrete torus, the asymptotics of the cover time were established by Dembo, Peres, Rosen and Zeitouni [9]. For regular trees, the asymtotics of the cover time were evaluated by Aldous [4] and a tightness result for the cover time after suitable normalization was demonstrated by Bramson and Zeitouni [7]. It was conjectured in [7] that the cover time of 2D discrete torii exhibits a similar tightness behavior.

Meanwhile, the supremum of the Gaussian free field (GFF) was also heavily studied. For squares in the 2D lattice, the first order asymptotics were evaluated by Bolthausen, Deuschel and Giacomin [5]. Interestingly, both [5] and [9] are based on the study of similar tree structures for the 2D lattice; in fact, the square of the GFF has the same first order asymptotics as the cover time after proper normalization. Recently, Ding, Lee, and Peres [10] demonstrated a useful connection between cover times and GFFs, by showing that, for any graph, the cover time is equivalent, up to a universal multiplicative constant, to the product of the number of edges and the supremum of the GFF. An important ingredient in [10] is a version of the so-called Dynkin Isomorphism theorem, which completely characterizes the distribution of local times (closely related to the cover time) using GFFs. All these connections seem to suggest that a detailed study of fluctuations for one model should carry over to the other with moderate work. A particular motivating example in this direction is the case of squares in the 2D discrete lattice. Recently, Bramson and Zeitouni

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[6] established a tightness result for the supremum of GFF there (with proper centering, but no other normalization), and further computed the centering up to an additive constant. One could hope that transfering this result to the cover time problem is now "purely technical"; an essential part of such a program would be to verify that the supremum of the GFF correctly predicts the second order correction for the (rescaled) cover time. The present paper is a cautionary note in that direction.

We study the cover time on binary trees and obtain the sharp second order term. Interestingly, we demonstrate that the latter is larger than the corresponding one for the binary tree GFF. Our result improves the estimate in [4], and complements the result of [7]. We focus here on binary trees, but it should be clear from the proof that the method applies to more general Galton–Watson trees.

Let $T = (V, E)$ be a binary tree rooted at ρ of height n, and consider a continuous-time random walk (X_t) started at ρ. Let τ_{cov} be the first time when the random walk visited every single vertex in the tree. Our main result is the following.

Theorem 1.1. *Consider a random walk on a binary tree $T = (V, E)$ of height n, started at the root ρ. Then, with high probability,*

$$\sqrt{\tfrac{\tau_{\text{cov}}}{|E|}} = \sqrt{2\log 2}\cdot n - \tfrac{\log n}{\sqrt{2\log 2}} + O((\log\log n)^8)\,. \tag{1}$$

At the cost of a more refined analysis, we believe that the error term $O((\log\log n)^8)$ can be improved to $O(1)$.

To relate Theorem 1.1 to the GFF $\{\eta_v\}_{v\in V}$ on the tree, recall that the latter can be defined as follows. Let $\{X_e\}_{e\in E}$ be i.i.d. standard Gaussian variables and set

$$\eta_v = \sum_{e: e\in\rho\leftrightarrow v} X_e,$$

where the sum is over all the edges that belong to the path from ρ to v. By adapting Bramson's arguments on branching Brownian motion [8] to the discrete setup, as in Addario-Berry and Reed [2], one can show that

$$\mathbb{E}\sup_v \eta_v = \sqrt{2\log 2}\cdot n - \tfrac{3\log n}{2\sqrt{2\log 2}} + O(1)\,. \tag{2}$$

(The lower bound in (2) follows directly from [2, Theorem 3]. The upper bound, that involves also the internal nodes of the tree, requires the use of [2, Lemma 13] and a union bound over the levels.)

Comparing (1) and (2), we do observe agreement in the first order and a discrepancy in the second order terms.

Our proof uses ideas from [8] and is based on the study of the local times associated with the random walk. For any $v \in V$, we define the local time L_t^v to be the time that the random walk spends at v up to t, with a normalization by the degree of v. More precisely,

$$L_t^v = \frac{1}{d_v}\int_0^t \mathbf{1}_{\{X_s=v\}}ds\,.$$

Define the inverse local time $\tau(t)$ to be the first time when the local time at the root achieves t, by

$$\tau(t) = \inf\{s \geqslant 0 : L_s^\rho \geqslant t\}\,.$$

We will let $\tau(t)$ be defined as above throughout the paper. We also set

$$t^+ = \left(\sqrt{\log 2}n - \tfrac{\log n}{2\sqrt{\log 2}} + 100 \log\log n\right)^2 \text{ and } t^- = \left(\sqrt{\log 2}n - \tfrac{\log n}{2\sqrt{\log 2}} - 100(\log\log n)^8\right)^2. \quad (3)$$

The following is the key to the proof of Theorem 1.1.

Theorem 1.2. *Consider a random walk on a binary tree T of height n, started at the root ρ. Then,*

$$\mathbb{P}(\tau(t^-) \leqslant \tau_{\mathrm{cov}} \leqslant \tau(t^+)) = 1 + o(1)\,, \ \text{ as } n \to \infty\,.$$

In the next two sections, we prove the upper and lower bounds for the preceding theorem respectively; we conclude the paper by deriving Theorem 1.1 from Theorem 1.2.

Notation and convention: Throughout, C, c denote generic constants that may change from line to line, but are independent of n. Further, the phrase *with high probability* should be understood as the statement *with probability approaching* 1 *as* $n \to \infty$.

2 Upper bound

We establish an upper bound on the cover time in this section, as formulated in the next theorem.

Theorem 2.1. *With notation as in Theorem 1.2, we have*

$$\mathbb{P}(\tau_{\mathrm{cov}} \leqslant \tau(t^+)) = 1 + o(1)\,, \ \text{ as } n \to \infty\,.$$

Theorem 2.1 is equivalent to the statement that at time $\tau(t^+)$, all the leaf-nodes have positive local times, with high probability. To this end, we consider a leaf-node of local time 0 with typical and non-typical profiles, respectively. For the latter, we show its unlikeliness directly; for the former, we prove it is a rare event by comparing to the same type of event for Gaussian free field.

2.1 Unlikeliness for non-typical profile

As preparation, we prove a large deviation result which will be used to control the pairwise concentration of local times.

Definition 2.2. *For $r, \lambda > 0$, let N be a Poisson variable with mean r and Y_i be i.i.d. exponential variables with mean λ. Then, the random variable $Z = \sum_{i=1}^N Y_i$ is said to follow the distribution* $\mathrm{PoiGamma}(r, \lambda)$*, and we write $Z \sim \mathrm{PoiGamma}(r, \lambda)$.*

Lemma 2.3. *For $\alpha, r > 0$, let $Z \sim \mathrm{PoiGamma}(r, \lambda)$. Then for $\alpha < \lambda r$,*

$$\mathbb{P}(Z \leqslant \lambda r - \alpha) \leqslant \exp\left(2\sqrt{r(r - \alpha/\lambda)} + \alpha/\lambda - 2r\right). \quad (4)$$

Furthermore, for all $\alpha > 0$,

$$\mathbb{P}(Z \geqslant \lambda r + \alpha) \leqslant \exp\left(2\sqrt{r(r + \alpha/\lambda)} - 2r - \alpha/\lambda\right). \quad (5)$$

Proof. As in the definition of the PoiGamma(r,λ) distribution, let N be Poisson variable with mean r and let Y be an independent exponential variable with mean λ. For $\theta > 0$, we have

$$\mathbb{E}\mathrm{e}^{-\theta Z/\lambda} = \mathbb{E}(\mathbb{E}\mathrm{e}^{-\theta Y/\lambda})^N = \mathbb{E}(1/(1+\theta))^N = \mathrm{e}^{-\frac{\theta r}{1+\theta}}\,.$$

Combined with Markov's inequality, it follows that

$$\mathbb{P}(Z \leqslant \lambda r - \alpha) = \mathbb{P}(\mathrm{e}^{-\theta Z/\lambda} \geqslant \mathrm{e}^{-\theta(\lambda r-\alpha)/\lambda}) \leqslant \mathrm{e}^{-\frac{\theta r}{1+\theta}} \cdot \mathrm{e}^{\theta(r-\alpha/\lambda)} = \exp\left(\tfrac{\theta^2 r}{1+\theta} - \tfrac{\theta\alpha}{\lambda}\right).$$

For $\alpha < \lambda r$, optimizing the exponent at $\theta = \sqrt{\frac{r}{r-\alpha/\lambda}} - 1$ leads to inequality (4).

To prove (5), consider $0 < \theta < 1$. We have

$$\mathbb{E}\mathrm{e}^{\theta Z/\lambda} = \mathbb{E}(\mathbb{E}\mathrm{e}^{\theta Y/\lambda})^N = \mathbb{E}(1/(1-\theta))^N = \mathrm{e}^{\frac{\theta r}{1-\theta}}\,.$$

Another application of Markov's inequality gives that

$$\mathbb{P}(Z \geqslant \lambda r + \alpha) \leqslant \mathbb{P}(\mathrm{e}^{\theta Z/\lambda} \geqslant \mathrm{e}^{\theta(\lambda r+\alpha)/\lambda}) = \exp\left(\tfrac{\theta^2 r}{1-\theta} - \tfrac{\theta\alpha}{\lambda}\right).$$

Optimizing the exponent at $\theta = 1 - \sqrt{\frac{r}{r+\alpha/\lambda}}$, we deduce the inequality (5). □

Remark. The right side of (4) can be bounded by $\mathrm{e}^{-\alpha^2/4\lambda^2 r}$. In this form, it is closely related to the discrete time bound in [11, Lemma 5.2].

We have the following immediate and useful corollary.

Corollary 2.4. *With notation as in Lemma 2.3, we have for any $\beta > 0$,*

$$\mathbb{P}(\sqrt{Z} \leqslant (1-\beta)\sqrt{\lambda r}) \leqslant \mathrm{e}^{-r\beta^2}\,, \tag{6}$$

and

$$\mathbb{P}(\sqrt{Z} \geqslant (1+\beta)\sqrt{\lambda r}) \leqslant \mathrm{e}^{-r\beta^2}\,. \tag{7}$$

For $k \in \mathbb{N}$, we denote by $V_k \subseteq V$ the set of vertices in k-th level of the tree. We next show that it is unlikely to have a too small local time for a vertex in intermediate levels.

Lemma 2.5. *With notation as in Theorem 1.2, define*

$$A = \cup_{k=1}^{n-\log^2 n} \cup_{u\in V_k} \{L^u_{\tau(t^+)} \leqslant ((1-k/n)\sqrt{t^+} - 3\log n)^2\}\,. \tag{8}$$

Then, $\mathbb{P}(A) = o(1)$ as $n \to \infty$.

Proof. Throughout the proof, we write t for t^+. Consider $u \in V_k$ such that $k \leqslant n - \log^2 n$. It is clear that $L^u_{\tau(t)}$ has the distribution PoiGamma$(t/k, k)$. Applying (4), we obtain that

$$\mathbb{P}\left(Z \leqslant ((1-k/n)\sqrt{t} - 3\log n)^2\right) \leqslant \exp\left(-\tfrac{1}{k}(\sqrt{t}k/n + 3\log n)^2\right) \leqslant 2^{-k}n^{-2}\,.$$

Now a simple union bound gives that

$$\mathbb{P}(A) \leqslant \sum_{k=1}^{n-\log^2 n} 2^k 2^{-k} n^{-2} \leqslant 1/n = o(1)\,.$$ □

For $v \in V$ and $1 \leqslant k < n$, let $v_k \in V_k$ be the ancestor of v in the k-th level. Define

$$\gamma(k) = \min\{\sqrt{k}\log k,\ \sqrt{n-k}\log(n-k)\} + 2\,. \tag{9}$$

Lemma 2.6. *With notation as in Theorem 1.2, define*

$$B = \left\{\exists v \in V_n, \exists k < n - \log^2 n : L^v_{\tau(t^+)} = 0, \left|\sqrt{L^{v_{k+1}}_{\tau(t^+)}} - \sqrt{L^{v_k}_{\tau(t^+)}}\right| \geqslant \frac{\sqrt{L^{v_k}_{\tau(t^+)}}}{\gamma(k)}\right\} \cap A^c\,. \tag{10}$$

Then, $\mathbb{P}(B) = o(1)$ *as* $n \to \infty$.

Proof. We continue to write $t = t^+$. Consider $v \in V_n$ and $k < n - \log^2 n$. Note that conditioned on $L^{v_k}_{\tau(t)}$, the collection of random variables $\{L^{v_{k+j}}_{\tau(t)}\}_{j\geqslant 0}$ possess the same law as $\{L^{v_{k+j}}_{\tau^{v_k}(L^{v_k}_{\tau(t)})}\}_{j\geqslant 0}$ (this is an instance of the second Ray-Knight theorem in this context). Abusing notation, this implies in particular that conditioned on $\{L^{v_k}_{\tau(t)} = x\}$, $L^{v_{k+1}}_{\tau(t)}$ has distribution $\mathrm{PoiGamma}(x^2, 1)$. (We will employ such an abuse of notation repeatedly throughout the paper.) Fixing $x \geqslant (1 - k/n)\sqrt{t} - 3\log n$, an application of Corollary 2.4 gives for $j \geqslant 1$,

$$\mathbb{P}\left(j\tfrac{x}{\gamma(k)} \leqslant \left|\sqrt{L^{v_{k+1}}_{\tau(t)}} - x\right| \leqslant (j+1)\tfrac{x}{\gamma(k)}, L^v_{\tau(t)} = 0\middle|L^{v_k}_{\tau(t)} = x^2\right) \leqslant 2 \cdot \mathrm{e}^{-j^2x^2/(\gamma(k))^2} \cdot \mathrm{e}^{-\frac{x^2(1-(j+1)/\gamma(k))^2}{n-k}}\,.$$

Note that the right hand side in the above decays geometrically with j. Thus, summing over j, we obtain that

$$\mathbb{P}\left(\left|\sqrt{L^{v_{k+1}}_{\tau(t)}} - x\right| \geqslant \tfrac{x}{\gamma(k)}, L^v_{\tau(t)} = 0\middle|L^{v_k}_{\tau(t)} = x^2\right) \leqslant 4\mathrm{e}^{-\frac{x^2}{n-k}}\mathrm{e}^{\frac{4x^2}{(n-k)\gamma(k)}}\mathrm{e}^{-\frac{x^2}{(\gamma(k))^2}} \leqslant 4\mathrm{e}^{-\frac{x^2}{n-k}}\mathrm{e}^{-\frac{x^2}{2(\gamma(k))^2}}\,.$$

Noting that $\mathbb{P}(L^v_{\tau(t)} = 0 \mid L^{v_k}_{\tau(t)} = x^2) = \mathrm{e}^{-\frac{x^2}{n-k}}$, we obtain that

$$\mathbb{P}\left(\left|\sqrt{L^{v_{k+1}}_{\tau(t)}} - x\right| \geqslant \tfrac{x}{\gamma(k)}\middle|L^v_{\tau(t)} = 0, L^{v_k}_{\tau(t)} = x^2\right) \leqslant 2\mathrm{e}^{-\frac{x^2}{2(\gamma(k))^2}} \leqslant \mathrm{e}^{-\log^{3/2} n}\,,$$

where the last nequality follows from the fact that $x \geqslant (1 - k/n)\sqrt{t} - 3\log n)$ and $k \leqslant n - \log^2 n$. Therefore,

$$\mathbb{P}\left(A^c, L^v_{\tau(t)} = 0, \left|\sqrt{L^{v_{k+1}}_{\tau(t)}} - \sqrt{L^{v_k}_{\tau(t)}}\right| \geqslant \frac{\sqrt{L^{v_k}_{\tau(t)}}}{\gamma(k)}\right) \leqslant \mathbb{P}(L^v_{\tau(t)} = 0) \cdot \mathrm{e}^{-\log^{3/2} n} = \mathrm{e}^{-t/n}\mathrm{e}^{-\log^{3/2} n} \leqslant \frac{2^{-n}}{n^2}\,.$$

At this point, a simple union bound completes the proof. □

2.2 Unlikeliness for typical profile

We next compare the density of local times and Gaussian variables. This comparison of density is of significance for the proof of both upper and lower bounds.

Lemma 2.7. *For* $\ell > 0$, *let* $Z \sim \mathrm{PoiGamma}(\ell^2, 1)$ *and let* $f(\cdot)$ *denote the density function of* $\sqrt{Z}$ *on* R_+, *with* $f(0) = \mathbb{P}(Z = 0)$. *Denote by* W *a standard Gaussian variable, and denote by* $g(\cdot)$ *the density function of* $W/\sqrt{2}$. *Then, for any* w *such that* $|w| \leqslant \ell/2$, *we have*

$$f(\ell + w) = \left(1 - \tfrac{w}{2\ell} + O\left(\tfrac{w^2+1}{\ell^2}\right)\right) \cdot g(w)\,.$$

Proof. Write $y = \ell + w$, and let $h(\cdot)$ be the density function of Z. Then for $z > 0$, we have

$$h(z) = \sum_{k=1}^{\infty} \mathrm{e}^{-\ell^2} \frac{\ell^{2k}}{k!} \mathrm{e}^{-z} \frac{z^{k-1}}{(k-1)!}\,.$$

Applying a change of variables, we obtain that

$$f(y) = 2y \sum_{k=1}^{\infty} \mathrm{e}^{-\ell^2} \frac{\ell^{2k}}{k!} \mathrm{e}^{-y^2} \frac{y^{2(k-1)}}{(k-1)!} = 2\ell \mathrm{e}^{-(\ell^2+y^2)} \sum_{k=0}^{\infty} \frac{(y\ell)^{2k+1}}{k!(k+1)!} = 2\ell \mathrm{e}^{-(\ell^2+y^2)} I_1(2y\ell)\,,$$

where $I_1(x)$ is a modified Bessel function defined by

$$I_1(x) \triangleq \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!}\,.$$

For the modified Bessel function $I_1(x)$, the following expansion is known when $|x|$ is large (see [1]):

$$I_1(x) = \frac{\mathrm{e}^x}{\sqrt{2\pi x}} \big(1 - \tfrac{3}{8x} + O\big(\tfrac{1}{x^2}\big)\big)\,.$$

Plugging into the preceding expansion, we get that

$$f(y) = 2\ell \mathrm{e}^{-(\ell^2+y^2)} \frac{\mathrm{e}^{2y\ell}}{\sqrt{2\pi 2y\ell}} \Big(1 - \tfrac{3}{8 \cdot 2y\ell} + O\big(\tfrac{1}{y^2\ell^2}\big)\Big) = \frac{\mathrm{e}^{-w^2}}{\sqrt{\pi}} \big(1 - \tfrac{w}{2\ell} + O\big(\tfrac{w^2+1}{\ell^2}\big)\big)\,.$$

Combined with the fact that $g(w) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-w^2}$, the desired estimate follows immediately. ☐

We single out the next calculation, which will be used repeatedly.

Claim 2.8. *Consider $z_i, \ell_i \in \mathbb{R}$ with $\ell_{i+1} = \ell_i + z_i$ for $i = 0, \ldots, m-1$ such that $|z_i| \leqslant \ell_i/2$ for all i. Assume that $\sum_i \frac{z_i^2+1}{\ell_{i-1}^2} = O(1)$. Then,*

$$\textstyle\prod_{i=1}^m \big(1 - \frac{z_i}{2\ell_{i-1}} + O\big(\frac{z_i^2+1}{\ell_{i-1}^2}\big)\big) = \Theta(1) \cdot \dfrac{\sqrt{\ell_0}}{\sqrt{\ell_m}}\,.$$

Proof. On one hand, note that

$$\frac{\ell_m}{\ell_0} = \prod_{i=1}^m \frac{\ell_i}{\ell_{i-1}} = \textstyle\prod_{i=1}^m \big(1 + \frac{z_i}{\ell_{i-1}}\big) = \exp\big(\sum_{i=1}^m \frac{z_i}{\ell_{i-1}} + O\big(\sum_{i=1}^m \frac{z_i^2}{\ell_{i-1}^2}\big)\big) = \exp\big(\sum_{i=1}^m \frac{z_i}{\ell_{i-1}} + O(1)\big)\,.$$

On the other hand, we have

$$\begin{aligned}\textstyle\prod_{i=1}^m \big(1 - \frac{z_i}{2\ell_{i-1}} + O\big(\frac{z_i^2+1}{\ell_{i-1}^2}\big)\big) &= \textstyle\exp\big(-\sum_{i=1}^m \frac{z_i}{2\ell_{i-1}} + O\big(\sum_{i=1}^m \frac{z_i^2+1}{\ell_{i-1}^2}\big)\big)\\ &= \textstyle\exp\big(-\sum_{i=1}^m \frac{z_i}{2\ell_{i-1}} + O(1)\big) = \sqrt{\frac{\ell_0}{\ell_m}}\exp(O(1))\,.\end{aligned}$$

Combining these estimates completes the proof. ☐

We next demonstrate that it is unlikely to have a leaf-node of local time 0, even with a typical profile for local times along the path from ρ to the leaf.

Lemma 2.9. *With notation as in Theorem 1.2 and A, B as in (8) and (10), define*

$$D_v = \{L^v_{\tau(t^+)} = 0\} \setminus (A \cup B)\,, \text{ for } v \in V_n\,. \tag{11}$$

Then, $\mathbb{P}(D_v) = o(2^{-n})$.

Proof. Again, we write $t = t^+$. Write $n' = n - \log^2 n$. Let $\Omega \subseteq \mathbb{R}^{n'}$ be such that for $z_1, \ldots, z_{n'} \in \Omega$, we have

$$\cap_{k=1}^{n'} \left\{ \sqrt{L^{v_k}_{\tau(t)}} - \sqrt{L^{v_{k-1}}_{\tau(t)}} = z_k \right\} \subseteq D_v\,.$$

Let $\alpha(\cdot)$ and $\beta(\cdot)$ be density functions for $(\sqrt{L^{v_k}_{\tau(t)}} - \sqrt{L^{v_{k-1}}_{\tau(t)}})_{1\leqslant k\leqslant n'}$ and $(\eta_{v_k}/\sqrt{2} - \eta_{v_{k-1}}/\sqrt{2})_{1\leqslant k \leqslant n'}$, respectively. Denote by $\ell_k = \sqrt{t} + \sum_{i=1}^k z_i$. Note that for $(z_1, \ldots, z_{n'}) \in \Omega$, we have

$$\textstyle\sum_{i=1}^{n'} \frac{1+z_i^2}{\ell_{i-1}^2} = O(1) \sum_{i=1}^{n'} \left(\frac{1}{(n-i)^2} + \frac{1}{(\gamma(i))^2}\right) = O(1)\,.$$

Applying Lemma 2.7 and Claim 2.8, we obtain that

$$\textstyle\frac{\alpha(z_1,\ldots,z_{n'})}{\beta(z_1,\ldots,z_{n'})} = \prod_{i=1}^{n'} \left(1 - \frac{z_i}{2\ell_{i-1}} + O\left(\frac{z_i^2+1}{\ell_{i-1}^2}\right)\right) = \Theta(1) \frac{\sqrt{n}}{\log n}\,.$$

Therefore, we obtain that

$$\mathbb{P}(D_v) = \int_\Omega \alpha(z_1, \ldots, z_{n'}) \mathbb{P}(L^v_{\tau(t)} = 0 \mid \sqrt{L^{v_{n'}}_{\tau(t)}} = \ell_{n'}) dz \leqslant O(1) \frac{\sqrt{n}}{\log n} \int_\Omega \beta(z_1, \ldots, z_{n'}) \mathrm{e}^{-\frac{\ell_{n'}^2}{n-n'}} dz\,. \tag{12}$$

Write $s = -(n'/n)\sqrt{t} - 3\log n$. Let $\beta(x) = \int_{\{\ell_{n'} = x\}} \beta(z_1, \ldots, z_{n'}) dz$ for $x \geqslant s$. Note that

$$\beta(x) = \frac{1}{\sqrt{\pi n'}} \mathrm{e}^{-\frac{x^2}{n'}} \mathbb{P}(\eta_{v_k}/\sqrt{2} \geqslant -(k/n)\sqrt{t} - 3\log n \text{ for } 1 \leqslant k \leqslant n' \mid \eta_{v_{n'}}/\sqrt{2} = x)\,. \tag{13}$$

Conditioning on $\eta_{v_{n'}}/\sqrt{2} = x$, we have

$$\{(\eta_{v_k}/\sqrt{2})_{1\leqslant k \leqslant n'} \mid \eta_{v_{n'}}/\sqrt{2} = x\} \overset{law}{=} \{(W_k/\sqrt{2} + (k/n')x)_{1\leqslant k\leqslant n'}\}\,,$$

where $(W_r)_{0\leqslant r\leqslant n'}$, is a Brownian Bridge of length n', i.e., a Brownian motion conditioned on hitting 0 at both time 0 and n'. It is well-known that the maximum of a Brownian bridge (W_r) on $[0, q]$ follows the Rayleigh distribution (see, e.g., [14]), i.e.,

$$\mathbb{P}(\max_{0\leqslant r\leqslant q} W_r \geqslant \lambda) = \mathrm{e}^{-\frac{2\lambda^2}{q}}\,, \text{ for all } \lambda \geqslant 0\,. \tag{14}$$

Therefore, we obtain that

$$\begin{aligned}\mathbb{P}(\eta_{v_k}/\sqrt{2} \geqslant -(k/n)\sqrt{t} - 3\log n \text{ for } 1 \leqslant k \leqslant n' \mid \eta_{v_{n'}}/\sqrt{2} = x) &\leqslant \mathbb{P}(\min_{r\leqslant n'} W_r/\sqrt{2} \geqslant -3\log n - (x-s)) \\ = \mathbb{P}(\max_{r\leqslant n'} W_r \leqslant \sqrt{2}(3\log n + (x-s))) &\leqslant \tfrac{4(3\log n + (x-s))^2}{n'}\,.\end{aligned}$$

Plugging the above estimate into (13), we obtain that

$$\beta(x) \leqslant \frac{4(3\log n + (x-s))^2}{(n')^{3/2}} \mathrm{e}^{-\frac{x^2}{n'}}\,.$$

Together with (12), we obtain that

$$\mathbb{P}(D_v) \leqslant O(1) \int_s^\infty \frac{(\log n + (x-s))^2}{n' \log n} \mathrm{e}^{-\frac{x^2}{n'}} \mathrm{e}^{-\frac{(\sqrt{t}+x)^2}{n-n'}} dx \leqslant O(1) \int_{-\infty}^\infty \frac{(\log n + (x-s))^2}{n' \log n} \mathrm{e}^{-\frac{x^2}{n'}} \mathrm{e}^{-\frac{(\sqrt{t}+x)^2}{n-n'}} dx\,.$$

Using the change of variables $y = x + \frac{\sqrt{t}n'}{n}$, we obtain that

$$\mathbb{P}(D_v) \leqslant O(1) \frac{\mathrm{e}^{-\frac{t}{n}}}{n \log n} \cdot \int_{-\infty}^\infty (4 \log n + y)^2 \mathrm{e}^{-(\frac{1}{n'} + \frac{1}{n-n'})y^2} dy = 2^{-n} \cdot o(\log^{-6} n)\,,$$

where we used the fact that $n' = n - \log^2 n$, completing the proof. $\square$

Proof of Theorem 2.1. The proof now follows trivially. Since $\sum_{v \in V_n} \mathbb{P}(D_v) = 2^n \cdot 2^{-n} o(1) = o(1)$ as well as $\mathbb{P}(A) = o(1)$ and $\mathbb{P}(B) = o(1)$, we see that with high probability, every leaf-node has positive local time by $\tau(t)$, implying the desired upper bound on cover time.

3 Lower bound

This section is devoted to the proof of the following lower bound on the cover time for a binary tree T.

Theorem 3.1. *With notation as in Theorem 1.2,*

$$\mathbb{P}(\tau_{\mathrm{cov}} \geqslant \tau(t^-)) = 1 + o(1)\,, \text{ as } n \to \infty\,.$$

The proof consists of an analysis for exceptionally large values in the Gaussian free field and a comparison argument based on Lemma 2.7.

3.1 Exceptional points for Gaussian free field

We first study the Gaussian free field $\{\eta_v\}_{v \in V}$ on the tree T of height n, with $\eta_\rho = 0$. For $1 \leqslant k < n$, let $\psi(k) = \frac{\log(k \wedge (n-k))}{2\sqrt{\log 2}}$. Denote by

$$a_k = (k/n)\big(\sqrt{\log 2} n - \tfrac{\log n}{2\sqrt{\log 2}}\big) - \psi(k) + 2\,, \text{ for } 1 \leqslant k < n\,, \text{ and } a_n = \sqrt{\log 2} n - \tfrac{\log n}{2\sqrt{\log 2}}\,. \tag{15}$$

Consider $\Delta = a_n + \log^4 n$. Recall the definition of $\gamma(k)$ in (9). For $v \in V_n$, define

$$E_v = \{\eta_{v_k}/\sqrt{2} \leqslant a_k, \text{ for all } 1 \leqslant k < n, a_n \leqslant \eta_v/\sqrt{2} \leqslant a_n + 1\}\,, \tag{16}$$

$$F_v = \{E_v, \exists k \leqslant n : |\eta_{v_k} - \eta_{v_{k-1}}| \geqslant |\Delta - \eta_{v_{k-1}}/\sqrt{2}|/\gamma(k)\}\,. \tag{17}$$

We start with a lower bound on the probability for event E_v.

Lemma 3.2. *There exists a constant $c > 0$ such that for all $v \in V_n$, we have*

$$\mathbb{P}(E_v) \geqslant \frac{c}{\sqrt{n}} 2^{-n}\,.$$

Proof. It is clear that

$$\mathbb{P}(E_v) \geqslant \mathbb{P}(a_n \leqslant \eta_v/\sqrt{2} \leqslant a_n+1) \min_{a_n \leqslant x \leqslant a_n+1} \mathbb{P}(E_v \mid \eta_v = \sqrt{2}x) \geqslant \frac{\sqrt{n}}{5\cdot 2^n} \cdot \min_{a_n \leqslant x \leqslant a_n+1} \mathbb{P}(E_v \mid \eta_v = \sqrt{2}x)\,,$$

where the second inequality follows from a bound on the Gaussian density. Denote by $(W_t)_{0\leqslant t\leqslant n}$ a Brownian bridge. We note that

$$\Big(\{\eta_{v_\ell} : 0 \leqslant \ell \leqslant n\} \mid \eta_v = \sqrt{2}x\Big) \stackrel{law}{=} \Big\{W_\ell + \tfrac{\ell}{n}\sqrt{2}x : 0 \leqslant \ell \leqslant n\Big\}\,.$$

This implies that, for $x \geqslant a_n$,

$$\mathbb{P}(E_v \mid \eta_v = \sqrt{2}x) \geqslant \mathbb{P}(W_\ell \leqslant 1 - \sqrt{2}\psi(\ell) \text{ for } 0 \leqslant \ell \leqslant n)\,.$$

By [8, Proposition 2'], we have that $\mathbb{P}(W_\ell \leqslant 1 - \sqrt{2}\psi(\ell)$ for $0 \leqslant \ell \leqslant n) \geqslant c/n$ for a constant $c > 0$. Altogether, we obtain that

$$\mathbb{P}(E_v) \geqslant \frac{c}{5\sqrt{n}} 2^{-n}\,. \qquad \square$$

We now show that the event F_v is extremely rare.

Lemma 3.3. *For any $v \in V_n$, we have*

$$\mathbb{P}(F_v) = 2^{-n} o(1/n), \ \ \text{as } n \to \infty\,.$$

Proof. Take $v \in V_n$. It is clear that

$$\mathbb{P}(F_v) \leqslant \mathbb{P}(a_n \leqslant \eta_v/\sqrt{2} \leqslant a_n + 1) J \leqslant 2^{-n} n J\,,$$

where

$$J = \max_{\substack{a_n \leqslant x \leqslant a_n+1 \\ y \leqslant a_{k-1}}} \sum_{k=1}^{n} \mathbb{P}(|\eta_{v_k} - \eta_{v_{k-1}}| \geqslant |\Delta - \eta_{v_{k-1}}/\sqrt{2}|/\gamma(k) \mid \eta_v = \sqrt{2}x, \eta_{v_{k-1}} = \sqrt{2}y)\,.$$

Conditioning on $\eta_v = \sqrt{2}x, \eta_{v_{k-1}} = \sqrt{2}y$, we have $\eta_{v_k} - \eta_{v_{k-1}}$ distributed as a Gaussian variable with mean $\frac{\sqrt{2}}{n-k+1}(x-y)$ and variance $\frac{n-k}{n-k+1}$. For x, y that is under consideration, we have $\frac{\sqrt{2}}{n-k+1}(x-y) = o(\frac{\Delta - y}{\gamma(k)})$. Therefore, we obtain that

$$\mathbb{P}(|\eta_{v_k} - \eta_{v_{k-1}}| \geqslant |\Delta - \eta_{v_{k-1}}/\sqrt{2}|/\gamma(k) \mid \eta_v = \sqrt{2}x, \eta_{v_{k-1}} = \sqrt{2}y) \leqslant \mathrm{e}^{-\frac{(\Delta-y)^2}{4(\gamma(k))^2}} \leqslant \mathrm{e}^{-\log^2 n}\,,$$

for large enough n. This implies that $J \leqslant n\mathrm{e}^{-\log^2 n}$, and thus $\mathbb{P}(F_v) = 2^{-n} o(1/n)$. $\square$

We next study the correlation for events E_u and E_v. For $u, v \in V$, denote by $u \wedge v$ the least common ancestor of u and v.

Lemma 3.4. *Consider $u, v \in V_n$ and assume that $u \wedge v \in V_k$. Then,*

$$\mathbb{P}(E_u \cap E_v) \leqslant \mathbb{P}(E_u) \frac{20 \log^2 n}{\sqrt{n-k}\cdot((n-k)\wedge k)} 2^{-(n-k)}\,.$$

Proof. Denote by $w = u \wedge v$, and let $f(\cdot)$ be the density function of $\eta_w/\sqrt{2}$. For $i < j$, write $E_v^{i,j} = \{\eta_{v_\ell}/\sqrt{2} \leqslant a_\ell, \text{ for all } i \leqslant \ell < j\}$. Then,

$$\begin{aligned}\mathbb{P}(E_u \cap E_v) = \mathbb{P}(E_u)\mathbb{P}(E_v \mid E_u) &\leqslant \mathbb{P}(E_u) \max_{x \leqslant a_k} \mathbb{P}(E_v^{k,n}, a_n \leqslant \eta_v \leqslant a_n + 1 \mid \eta_w/\sqrt{2} = x) \\ &\leqslant \mathbb{P}(E_u) \max_{x \leqslant a_k} \int_{a_n}^{a_n+1} \frac{1}{\sqrt{n-k}} \mathrm{e}^{-\frac{(y-x)^2}{n-k}} \mathbb{P}(E_v^{k,n} \mid \eta_w/\sqrt{2} = x, \eta_v/\sqrt{2} = y) dy\,. \end{aligned} \tag{18}$$

For $x \leqslant a_k$ and $a_n \leqslant y a_n + 1$, we first analyze the probability $\mathbb{P}(E_v^{k,n} \mid \eta_w/\sqrt{2} = x, \eta_v/\sqrt{2} = y)$. Let $(W_s)_{0 \leqslant s \leqslant n-k}$ be a Brownian bridge. It is clear that

$$\left(\left\{\eta_{v_\ell}/\sqrt{2} : k \leqslant \ell \leqslant n\right\} \mid \eta_w/\sqrt{2} = x, \eta_v/\sqrt{2} = y\right) \overset{law}{=} \left\{W_{\ell-k}/\sqrt{2} + \tfrac{\ell-k}{n-k} y + \tfrac{n-\ell}{n-k} x : k \leqslant \ell \leqslant n\right\}.$$

Combined with (14), it follows that

$$\mathbb{P}(E_v^{k,n} \mid \eta_w/\sqrt{2} = x, \eta_v/\sqrt{2} = y) \leqslant \mathbb{P}(\max_s W_s \leqslant 2(\log n + (a_k - x))) \leqslant \frac{4(\log n + (a_k - x))^2}{n-k}. \tag{19}$$

By a straightforward calculation, we have that

$$\begin{aligned}\mathrm{e}^{-\frac{(y-x)^2}{n-k}} \leqslant \mathrm{e}^{-\frac{(a_n-a_k)^2}{n-k}} \mathrm{e}^{-\frac{2(a_n-a_k)(a_k-x)}{n-k}} &\leqslant 2^{-(n-k)} \mathrm{e}^{\frac{n-k}{n}\log n - \log((n-k)\wedge k)} \mathrm{e}^{2\sqrt{\log 2}(x-a_k)} \\ &\leqslant 2^{-(n-k)} \frac{n-k}{(n-k)\wedge k} \mathrm{e}^{2\sqrt{\log 2}(x-a_k)}\,.\end{aligned}$$

Combined with (19), it follows that

$$\begin{aligned}\int_{a_n}^{a_n+1} \frac{1}{\sqrt{n-k}} \mathrm{e}^{-\frac{(y-x)^2}{n-k}} \mathbb{P}(E_v^{k,n} \mid \eta_w = \sqrt{2}x, \eta_v = \sqrt{2}y) dy &\leqslant 2^{-(n-k)} \frac{4(\log n + (a_k - x))^2}{\sqrt{n-k} \cdot ((n-k)\wedge k)} \mathrm{e}^{2\sqrt{\log 2}(x-a_k)} \\ &\leqslant 2^{-(n-k)} \frac{20 \log^2 n}{\sqrt{n-k} \cdot ((n-k)\wedge k)}\,.\end{aligned}$$

Together with (18), we deduce that

$$\mathbb{P}(E_u \cap E_v) \leqslant \mathbb{P}(E_u) \frac{20 \log^2 n}{\sqrt{n-k} \cdot ((n-k)\wedge k)} 2^{-(n-k)}\,. \qquad \square$$

3.2 Lower bound for cover times

We now turn to study the cover time. The key estimate lies in the following proposition.

Proposition 3.5. *With notation as in Theorem 1.2, let* $s = (\sqrt{\log 2} n - \frac{\log n}{2\sqrt{\log 2}} + \log^4 n)^2$. *Then there exists a constant* $c > 0$ *such that*

$$\mathbb{P}(\min\nolimits_{v \in V_n} L^v_{\tau(s)} \leqslant \log^8 n) \geqslant \frac{c}{\log^5 n}\,.$$

Proof. Let $Z_v = \sqrt{L^v_{\tau(s)}}$ for $v \in V$. Let a_k be defined as in (15). For $v \in V_n$, define

$$\tilde{E}_v = \{\sqrt{s} - Z_{v_k} \leqslant a_k, \text{ for all } 1 \leqslant k < n, a_n \leqslant \sqrt{s} - Z_v \leqslant a_n + 1\}, \tag{20}$$
$$\tilde{F}_v = \{\tilde{E}_v, \exists k \leqslant n : |Z_{v_k} - Z_{v_{k-1}}| \geqslant Z_{v_{k-1}}/\gamma(k)\}. \tag{21}$$

Define $\Omega_v \subseteq \mathbb{R}^n$ such that for any $(z_1, \ldots, z_n) \in \Omega_v$

$$\{Z_{v_k} - Z_{v_{k-1}} = z_k \text{ for all } 1 \leqslant k \leqslant n\} \subseteq \tilde{E}_v \setminus \tilde{F}_v.$$

It is clear from (16) and (17) that

$$\{\eta_{v_{k-1}}/\sqrt{2} - \eta_{v_k}/\sqrt{2} = z_k \text{ for all } 1 \leqslant k \leqslant n\} \subseteq E_v \setminus F_v.$$

Let $\alpha_v(\cdot), \beta_v(\cdot)$ be density functions over Ω for $(Z_{v_k} - Z_{v_{k-1}})_{1\leqslant k\leqslant n}$ and $(\eta_{v_{k-1}}/\sqrt{2} - \eta_{v_k}/\sqrt{2})_{1\leqslant k\leqslant n}$, respectively. Consider $(z_1, \ldots, z_n) \in \Omega_v$. By Lemma 2.7, we have

$$\alpha_v(z_1, \ldots, z_n) = \beta_v(z_1, \ldots, z_n)\textstyle\prod_{k=1}^n \big(1 - \frac{z_k}{2\ell_{k-1}} + O(\frac{z_k^2+1}{\ell_{k-1}})\big), \tag{22}$$

where $\ell_k = \sqrt{s} - \sum_{i=1}^k z_k$. Since $(z_1, \ldots, z_n) \in \Omega_v$, we have that

$$\sum_{k=1}^n \tfrac{z_k^2+1}{\ell_{k-1}^2} \leqslant \sum_{k=1}^n \big(\tfrac{1}{(\gamma(k))^2} + \tfrac{4}{(n-k)^2+\log^4 n}\big) = O(1).$$

Applying Claim 2.8, we obtain that

$$\alpha_v(z_1, \ldots, z_n) = \Theta(1)\frac{\sqrt{n}}{\log^2 n}\beta_v(z_1, \ldots, z_n).$$

Integrating over both sides and recalling Lemmas 3.3 and 3.2, we obtain that

$$\mathbb{P}(\tilde{E}_v \setminus \tilde{F}_v) = \Theta(1)\frac{\sqrt{n}}{\log^2 n}\mathbb{P}(E_v \setminus F_v) = \Theta(1)\frac{c_1\sqrt{n}}{2\log^2 n}\mathbb{P}(E_v) \geqslant \Theta(1) \cdot \frac{1}{2\log^2 n}2^{-n}. \tag{23}$$

We next analyze the correlation of $\tilde{E}_v \setminus \tilde{F}_v$ and $\tilde{E}_u \setminus \tilde{F}_u$. Consider $u, v \in V_n$ and assume that $u \wedge v \in V_k$. We write

$$\underline{Z} = (Z_{v_1} - Z_{v_0}, \ldots, Z_{v_n} - Z_{v_{n-1}}, Z_{u_{k+1}} - Z_{u_k}, \ldots, Z_{u_n} - Z_{u_{n-1}}),$$
$$\underline{\eta} = \frac{1}{\sqrt{2}}(\eta_{v_0} - \eta_{v_1}, \ldots, \eta_{v_{n-1}} - \eta_{v_n}, \eta_{u_k} - \eta_{u_{k+1}}, \ldots, \eta_{u_{n-1}} - \eta_{u_n}).$$

Define $\Omega_{u,v} \subseteq \mathbb{R}^{2n-k}$ such that for all $\underline{z} = (z_{v,1}, \ldots, z_{v,n}, z_{u,k+1}, \ldots, z_{u,n}) \in \Omega_{u,v}$,

$$\{\underline{Z} = \underline{z}\} \subseteq (\tilde{E}_v \setminus \tilde{F}_v) \cap (\tilde{E}_u \setminus \tilde{F}_u).$$

It is then clear that $\{\underline{\eta} = \underline{z}\} \subseteq (E_v \setminus F_v) \cap (E_u \setminus F_u)$. Let $\alpha_{u,v}(\cdot)$ and $\beta_{u,v}(\cdot)$ be density functions for $\underline{Z}$ and $\underline{\eta}$, respectively. Let $z_{u,i} = z_{v,i}$ for all $1 \leqslant i \leqslant k$. For $w \in \{u, v\}$, write $\ell_{w,j} = \sqrt{t} - \sum_{i=1}^j z_{w,j}$. By Lemma 2.7, we get that that

$$\alpha_{u,v}(\underline{z}) = \beta_{u,v}(\underline{z}) \cdot \prod_{j=1}^n \big(1 - \tfrac{z_{v,j}}{2\ell_{v,j-1}} + O\big(\tfrac{z_{v,j}^2+1}{\ell_{v,j-1}^2}\big)\big) \cdot \prod_{j=k}^n \big(1 - \tfrac{z_{u,j}}{2\ell_{u,j-1}} + O\big(\tfrac{z_{u,j}^2+1}{\ell_{u,j-1}^2}\big)\big).$$

Applying Claim 2.8 again, we obtain that

$$\alpha(u,v)(\underline{z}) = O(1)\frac{\sqrt{n(n-k)}}{\log^2 n}\beta_{u,v}(\underline{z})\,.$$

Integrating over both sides and applying Lemma 3.4, we get that

$$\mathbb{P}(\tilde{E}_v \setminus \tilde{F}_v) \cap (\tilde{E}_u \setminus \tilde{F}_u) = O(1)\frac{\sqrt{n}}{(n-k)\wedge k}\mathbb{P}(E_v)2^{-(n-k)}\,.$$

This implies that for a constant $C > 0$

$$\mathbb{E}\big(\textstyle\sum_{w\in V_n}\mathbf{1}_{\tilde{E}_w\setminus\tilde{F}_w}\big)^2 \leqslant C\sqrt{n}2^n\mathbb{P}(E_v)\displaystyle\sum_{j=1}^{n}\sum_{w:w\wedge v\in V_j}\frac{2^{-(n-j)}}{(n-j)\wedge j} \leqslant C\sqrt{n}2^n\mathbb{P}(E_v)\cdot 4\log n\,.$$

At this point, an application of the second moment method together with (23) gives that

$$\mathbb{P}(\exists w\in V_n : \tilde{E}_w\setminus\tilde{F}_w) \geqslant \frac{\big(\mathbb{E}\sum_{w\in V_n}\mathbf{1}_{\tilde{E}_w\setminus\tilde{F}_w}\big)^2}{\mathbb{E}\big(\sum_{w\in V_n}\mathbf{1}_{\tilde{E}_w\setminus\tilde{F}_w}\big)^2} \geqslant \frac{(2^n\mathbb{P}(\tilde{E}_v\setminus\tilde{F}_v))^2}{C\sqrt{n}2^n\mathbb{P}(E_v)} \geqslant \frac{1}{C'\log^5 n}\,,$$

for a constant $C' > 0$. Recalling the definition of $\tilde{E}_v$, we complete the proof of the proposition. ☐

Next, we bootstrap the above estimate and prove the main result in this section.

Proof of Theorem 3.1. Throughout the proof, we write $t = t^-$. Let $n_1 = 30\log\log n$, $n_3 = \frac{\log^4 n}{\sqrt{\log 2}}$, $n_4 = 10(\log\log n)^8$, and $n_2 = n - n_1 - n_3 - n_4$. For $k \in \mathbb{N}$, write $b_k = \sqrt{\log 2}k - \frac{\log k}{2\sqrt{\log 2}}$. Note that $\sqrt{t} + 50n_1 \leqslant b_{n_2} + b_{n_3}$. Our proof is divided into 4 steps.

Step 1. Write $t_1 = (\sqrt{t} + 2n_1)^2$. Since for all $v \in V_{n_1}$ we have $L^v_{\tau(t)} \sim \mathrm{PoiGamma}(t/n_1, n_1)$, an application of (7) yields that

$$\mathbb{P}(\exists v\in V_{n_1} : L^v_{\tau(t)} \geqslant t_1) \leqslant 2^{n_1}\mathbb{P}(\mathrm{PoiGamma}(t/n_1,n_1) \geqslant t_1) \leqslant 2^{n_1}\mathrm{e}^{-4n_1} = o(1)\,. \tag{24}$$

Step 2. For $v \in V_{n_1}$, let T_v be the subtree rooted at v of height n_2. Write $t_2 = (\sqrt{t_1} - b_{n_2})^2$. By (24), we assume in what follows that $L^v_{\tau(t)} \leqslant t_1$. Applying Proposition 3.5 to the subtree T_v, we deduce that for a constant $c > 0$,

$$\mathbb{P}(\min_{u\in T_v\cap V_{n_1+n_2}} L^u_{\tau(t)} \leqslant t_2) \geqslant \frac{c}{\log^5 n}\,.$$

Let $S_1 = \{v \in V_{n_1} : \min_{u\in T_v\cap V_{n_2}} L^u_{\tau(t)} \leqslant t_2\}$. By independence of the random walk on different subtrees, we obtain that with high probability $|S_1| \geqslant 2^{n_1}/\log^6 n \geqslant 2^{2\log\log n}$. We assume this in what follows. Define $S_2 = \{u \in V_{n_1+n_2} : L^u_{\tau(t)} \leqslant t_2\}$. We see that $|S_2| \geqslant |S_1| \geqslant 2^{2\log\log n}$.

Step 3. For $u \in S_2$, consider the subtree T_u rooted at u of height n_3. Since $(\sqrt{t_2} - b_{n_3})^2 \leqslant (\log\log n)^8$. We can apply Proposition 3.5 again to the subtree T_u and obtain that for a constant $c > 0$

$$\mathbb{P}(\min_{w\in T_u\cap V_{n_1+n_2+n_3}} L^w_{\tau(t)} \leqslant (\log\log n)^8) \geqslant \frac{c}{(\log\log n)^5}\,.$$

Let $S_3 = \{w \in T_u \cap V_{n_1+n_2+n_3} : L^w_{\tau(t)} \leqslant (\log\log n)^8\}$. We can then obtain that with high probability $|S_3| \geqslant 2^{\log\log n}$, and we assume this in what follows.
Step 4. For $w \in S_3$, let T_w be the subtree rooted at w that contains all its descendants. We trivially have that

$$\mathbb{P}(\min_{w' \in T_w} L^{w'}_{\tau(t)} = 0) \geqslant \frac{1}{2}\,.$$

Since $|S_3| \geqslant 2^{\log\log n}$, we see that with high probability there exists a vertex $w' \in \cup_{w\in S_3} T_w \subseteq V$ with $L^{w'}_{\tau(t)} = 0$, completing the proof. □

3.3 Concentration of inverse local time

We have been measuring the cover time via the inverse local time so far. In this subsection, we prove that the inverse local is well-concentrated around the mean and thus it indeed yields a good estimate on the cover time.

Lemma 3.6. *Consider a random walk on a rooted binary tree $T = (V, E)$ of height n. Then,*

$$\mathrm{Var}(\tau(t)) = O(1) \cdot 2^{2n} t\,.$$

Proof. Note that $\tau(t) = \sum_{v\in V} d_v L^v_{\tau(t)}$. Consider $u, v \in V$ and write $w = u \wedge v$. Assume that $w \in V_k$. Then,

$$\mathbb{E}(L^v_{\tau(t)} \cdot L^u_{\tau(t)}) = \mathbb{E}(\mathbb{E}(L^v_{\tau(t)} \cdot L^u_{\tau(t)}) \mid L^w_{\tau(t)}) = \mathbb{E}((L^w_{\tau(t)})^2) = t^2 + \mathrm{Var}(L^w_{\tau(t)}) \leqslant t^2 + 16tk\,,$$

where the last transition follows from the fact that $L^w_{\tau(t)} \sim \mathrm{PoiGamma}(t/k, k)$ and a simple application of the total variance formula $\mathrm{Var}\, X = \mathbb{E}(\mathrm{Var}(X \mid Y)) + \mathrm{Var}(\mathbb{E}(X \mid Y))$. We then have $\mathrm{Cov}(L^v_{\tau(t)}, L^u_{\tau(t)}) \leqslant 16tk$. Therefore,

$$\mathrm{Var}(\tau(t)) = \sum_{u,v\in V} d_v d_u \mathrm{Cov}(L^v_{\tau(t)}, L^u_{\tau(t)}) \leqslant \sum_{k=1}^{n} 2^k 2^{2(n-k)} 3^2 \cdot 16tk = O(1) \cdot 2^{2n} t\,,$$

where we used the fact that $d_v \leqslant 3$. □

Now it is obvious that Theorems 2.1, 3.1 imply Theorem 1.2. Together with Lemma 3.6, we complete the proof of Theorem 1.1, by noting that $\mathbb{E}(\tau(t)) = t \cdot 2|E| = (2^{n+2} - 4)t$.

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