THICK POINTS FOR INTERSECTIONS OF PLANAR SAMPLE PATHS

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ABSTRACT. Let $L_n^X(x)$ denote the number of visits to $x \in \mathbf{Z}^2$ of the simple planar random walk X, up till step n. Let X' be another simple planar random walk independent of X. We show that for any $0 < b < 1/(2\pi)$, there are $n^{1-2\pi b+o(1)}$ points $x \in \mathbf{Z}^2$ for which $L_n^X(x)L_n^{X'}(x) \geq b^2(\log n)^4$. This is the discrete counterpart of our main result, that for any $a < 1$, the Hausdorff dimension of the set of *thick intersection points* x for which $\limsup_{r\to 0} \mathcal{I}(x,r)/(r^2|\log r|^4) = a^2$, is almost surely $2-2a$. Here $\mathcal{I}(x,r)$ is the projected intersection local time measure of the disc of radius r centered at x for two independent planar Brownian motions run till time 1. The proofs rely on a 'multi-scale refinement' of the second moment method. In addition, we also consider analogous problems where we replace one of the Brownian motions by a transient stable process, or replace the disc of radius r centered at x by $x + rK$ for general sets K.

1. Introduction

Let X_n denote the simple random walk in the plane $\mathbf{Z}^2$ with

$$L_n^X(x) = \#\{i : X_i = x,\ 0 \leq i \leq n\}\,,$$

the number of visits to $x \in \mathbf{Z}^2$ during the first n steps of the walk and

$$T_n^X = \max_{x\in\mathbf{Z}^2} L_n^X(x)\,,$$

its maximal value. In [5, Theorem 1.1] we showed that

$$\lim_{n\to\infty} \frac{T_n^X}{(\log n)^2} = \frac{1}{\pi}\,,\quad a.s. \tag{1.1}$$

resolving a conjecture of Erdös-Taylor. Further, with $|A|$ counting the number of points in the set A, we show there that

$$\lim_{n\to\infty} \frac{\log|\{x \in \mathbf{Z}^2 : L_n^X(x) \geq a(\log n)^2\}|}{\log n} = 1 - \pi a\,,\quad a.s. \tag{1.2}$$

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for any fixed $0 < a < 1/\pi$.

It is natural to consider the situation for intersections of two (or more) independent simple random walks (SRW) in $\mathbf{Z}^2$. Thus, for two independent simple random walks X_n, X'_n and $x \in \mathbf{Z}^2$, define

$$T_n^{X,X'} = \max_{x \in \mathbf{Z}^2} L_n^X(x) L_n^{X'}(x).$$

Here is our analogue of (1.1)-(1.2) for two independent random walks.

Theorem 1.1.

$$\lim_{n\to\infty} \frac{T_n^{X,X'}}{(\log n)^4} = \frac{1}{4\pi^2}\,, \quad a.s. \tag{1.3}$$

Further, for $0 < b < 1/(2\pi)$,

$$\lim_{n\to\infty} \frac{\log |\{x \in \mathbf{Z}^2 : L_n^X(x) L_n^{X'}(x) \geq b^2 (\log n)^4\}|}{\log n} = 1 - 2\pi b\,, \quad a.s. \tag{1.4}$$

We now outline a heuristic leading to Theorem 1.1. With high probability the path $\{X_k : k \leq n\}$ is contained in a disc of radius $n^{1/2+o(1)}$. The proof of (1.2) suggests that, with probability $n^{-\pi a + o(1)}$, a point x in that disc is visited by the random walk $\{X_k : k \leq n\}$ at least $a(\log n)^2$ times, i.e. $L_n^X(x) \geq a(\log n)^2$. Similarly, the probability that $L_n^{X'}(x) \geq a'(\log n)^2$ is $n^{-\pi a' + o(1)}$. So, $L_n^X(x) L_n^{X'}(x) \geq b^2 (\log n)^4$ with probability $n^{-\pi(a+b^2/a)+o(1)}$ for some $a > 0$. Taking the optimal value $a = b$ suggests the upper bound given in (1.4). As the events $\{L_n^X(x) L_n^{X'}(x) \geq b^2(\log n)^4\}$ for $x \in \mathbf{Z}^2$ are highly dependent, establishing the complementary lower bound requires a 'multi-scale refinement' of the second moment method. Such a technique was developed in [5] for the planar Brownian occupation measure

$$\mu_{\bar\theta}^W(A) := \int_0^{\bar\theta} 1_A(W_s) ds\,,$$

where A is a measurable subset of $\mathbb{R}^2$, $\{W_s, s \geq 0\}$ denotes a planar Brownian path and $\bar\theta := \min\{s : |W_s| = 1\}$. Using strong approximations of Brownian motion by random walks, (1.2) is the outcome of a precise study of those points x in whose neighborhood the Brownian motion spends an unusually large amount of time, in all sufficiently small scales. We show that

$$\lim_{\epsilon\to 0} \sup_{x \in D(0,1)} \frac{\mu_{\bar\theta}^W(D(x,\epsilon))}{\epsilon^2 (\log \epsilon)^2} = 2 \tag{1.5}$$

where $D(x, r)$ is the open disc in $\mathbb{R}^2$ of radius r centered at x. Defining then the set of *thick points*

$$\mathsf{Thick}_a^W = \{x \in D(0,1) : \lim_{\epsilon\to 0} \frac{\mu_{\bar\theta}^W(D(x,\epsilon))}{\epsilon^2 (\log \epsilon)^2} = a\}$$

for any $a \in (0,2]$, another outcome of that study, [5, Theorem 1.4], shows that $\dim(\mathsf{Thick}_a^W) = 2 - a$, with dim denoting the Hausdorff dimension of a set.

Our goal in this paper is to better understand the essential reason for such a result. We do so by extending it in three different directions:

- Considering occupation measure of sets without radial symmetry.
- Considering the projected intersection local time for two (or more) independent planar Brownian motions.
- Considering also the projected intersection local time between the recurrent Brownian path and the sample path of a *transient* (stable) process with jumps.

Turning to the first of these directions, let $K \subseteq D(0,1)$ have area (i.e. Lebesgue measure) $|K|$, and let $K(x,r) = x + rK$. We next show how the results of [5] for thick points are modified when we replace the discs $D(x,\epsilon)$ by $K(x,\epsilon)$. The analogous modification of our study concerning thick points for intersections will then be straightforward and is left to the reader.

Theorem 1.2. *If* $|\partial K| = 0$ *then,*

$$\lim_{\varepsilon\to 0} \sup_{x\in \mathbb{R}^2} \frac{\mu_{\bar\theta}^W(K(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = \sup_{|x|<1} \limsup_{\varepsilon\to 0} \frac{\mu_{\bar\theta}^W(K(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = 2|K|/\pi \qquad a.s. \tag{1.6}$$

and for any $a \le 2|K|/\pi$,

$$\dim\left\{x \in \mathbb{R}^2 : \lim_{\varepsilon\to 0} \frac{\mu_{\bar\theta}^W(K(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = a\right\} = 2 - a\pi/|K| \qquad a.s. \tag{1.7}$$

Here one may replace $\bar\theta$ by any deterministic $0 < T < \infty$. We also note that all remarks following Theorem 1.3 in [5] extend verbatim to the situation where the set K is used instead of $D(0,1)$.

Our main example concerns those points x in whose neighborhood two independent planar Brownian motions have an unusually large 'amount' of intersections, in all sufficiently small scales. This is quantified by the projected intersection local time. Let $\{W_s\,;\, 0 \le s \le S\}$ and $\{W'_t\,;\, 0 \le t \le T\}$ be two independent planar Brownian motions started at the origin. For any Borel set $A \subseteq \mathbb{R}^2$ we define the projected intersection local time in A by

$$\mathcal{I}_{S,T}(A) = \lim_{\epsilon\to 0} \pi \int_0^S \int_0^T 1_A(W_s) f_\epsilon(W_s - W'_t)\, ds\, dt \tag{1.8}$$

where the factor π is a convenient normalization, and f_ϵ is any approximate δ-function, i.e. we take f to be a non-negative continuous function supported on the unit disc with $\int f\,dx = 1$ and set $f_\epsilon(x) = f(x/\epsilon)/\epsilon^2$. It is known that the limit (1.8) exists a.s. and in all L^p spaces, and that $\mathcal{I}_{S,T}(\cdot)$ is a measure supported on $\{x \in \mathbb{R}^2 |\, x = W_s = W'_t$ for some $0 \le s \le S$, $0 \le t \le T\}$, see [13, Chapter VIII, Theorem 1]. Further, Le Gall [14] shows that, with

$\bar\theta' = \inf\{t : |W'_t| = 1\}$, there exists a constant $0 < c < \infty$ such that, almost surely, for typical x in the support of $\mathcal{I}_{\bar\theta,\bar\theta'}$,

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2 \left(\log\frac{1}{\varepsilon}\log\log\log\frac{1}{\varepsilon}\right)^2} = c\,.$$

In contrast, our next result describes just how large the projected intersection local time can be in the neighborhood of any exceptional point.

Theorem 1.3.

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^2} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2 \left(\log\frac{1}{\varepsilon}\right)^4} = 1\,, \qquad a.s. \tag{1.9}$$

Here also one may replace $\bar\theta$, $\bar\theta'$ by any deterministic $0 < S, T < \infty$.

The next theorem describes the multi-fractal structure of the planar projected intersection local time.

Theorem 1.4. *For any* $0 < a \le 1$,

$$\dim\left\{x \in D(0,1) : \lim_{\varepsilon\to 0} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4} = a^2\right\} = 2 - 2a \qquad a.s. \tag{1.10}$$

Also,

$$\sup_{|x|<1}\limsup_{\varepsilon\to 0} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4} = 1 \qquad a.s. \tag{1.11}$$

Since $\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\epsilon)) = 0$ for any $x \notin \{W_t \,\big|\, 0 \le t \le \bar\theta\}$ and ϵ small enough, by the *uniform dimension doubling* property of Brownian motion (see [10] or [16, Eqn. (0.1)]), (1.10) is equivalent to

$$\dim\left\{0 \le t \le \bar\theta : \lim_{\varepsilon\to 0} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(W_t,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4} = a^2\right\} = 1 - a \qquad a.s. \tag{1.12}$$

We also have the following analogue of the coarse multi-fractal spectrum:

Proposition 1.5. *For all* $a < 1$,

$$\lim_{\varepsilon\to 0} \frac{\log \mathcal{L}eb(x : \mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon)) \ge a^2\varepsilon^2(\log\varepsilon)^4)}{\log\varepsilon} = 2a\,, \quad a.s.$$

Our last example concerns the intersections of planar Brownian paths with other random fractals. Let $\{X_t\}$ denote the symmetric stable process of index $0 < \beta < 2$ in the plane. As usual, we let

$$u^0(x) = \frac{c_\beta}{|x|^{2-\beta}} \tag{1.13}$$

denote the 0-potential density for $\{X_t\}$, where $c_\beta = 2^{-\beta}\pi^{-1}\Gamma(\frac{2-\beta}{2})/\Gamma(\frac{\beta}{2})$. Let $\Lambda = \Lambda_\beta$ denote the norm of

$$K_\beta f(x) = \int_{D(0,1)} u^0(x-y) f(y)\, dy \tag{1.14}$$

considered as an operator from $L^2\left(D(0,1),\,dx\right)$ to itself.

For any Borel set $A \subseteq I\!R^2$ we define the projected intersection local time in A for W and X by

$$\mathcal{I}^{W,X}_{S,T}(A) = \lim_{\epsilon\to 0}\ \pi\Lambda^{-1}\int_0^S\int_0^T 1_A(W_s) f_\epsilon(W_s - X_t)\,ds\,dt \tag{1.15}$$

where f_ϵ is any approximate δ-function, and the factor $\pi\Lambda^{-1}$ is a convenient normalization. As was the case for $\mathcal{I}_{S,T}(A)$ it can be shown that the limit (1.15) exists a.s. and in all L^p spaces, and that $\mathcal{I}^{W,X}_{S,T}(\cdot)$ is a measure supported on $\{x \in I\!R^2 | \, x = W_s = X_t \text{ for some } 0 \le s \le S,\, 0 \le t \le T\}$. When $T = \infty$ we set $\mathcal{I}^{W,X}_{S}(A) := \mathcal{I}^{W,X}_{S,\infty}(A)$.

We provide next the multi-fractal analysis of thick points for $\mathcal{I}^{W,X}$.

Theorem 1.6. *Fix $T \in (0,\infty]$. Then,*

$$\lim_{\varepsilon\to 0}\ \sup_{x\in I\!R^2} \frac{\mathcal{I}^{W,X}_{\bar\theta,T}(D(x,\varepsilon))}{\varepsilon^\beta\left(\log\frac{1}{\varepsilon}\right)^3} = \beta^2/4\,, \qquad a.s. \tag{1.16}$$

and for any $0 < a \le \beta/2$,

$$\dim\left\{x \in D(0,1) :\ \limsup_{\varepsilon\to 0} \frac{\mathcal{I}^{W,X}_{\bar\theta,T}(D(x,\varepsilon))}{\varepsilon^\beta\left(\log\frac{1}{\varepsilon}\right)^3} = a^2\right\} = \beta - 2a \qquad a.s. \tag{1.17}$$

Equivalently

$$\dim\left\{0 \le t \le \bar\theta :\ \limsup_{\varepsilon\to 0} \frac{\mathcal{I}^{W,X}_{\bar\theta,T}(D(W_t,\varepsilon))}{\varepsilon^\beta\left(\log\frac{1}{\varepsilon}\right)^3} = a^2\right\} = \beta/2 - a \qquad a.s. \tag{1.18}$$

Also,

$$\sup_{|x|<1}\ \limsup_{\varepsilon\to 0} \frac{\mathcal{I}^{W,X}_{\bar\theta,T}(D(x,\varepsilon))}{\varepsilon^\beta\left(\log\frac{1}{\varepsilon}\right)^3} = \beta^2/4 \qquad a.s. \tag{1.19}$$

In the next section we estimate the moments of $\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))$. Applying these estimates we then provide the upper bounds in Theorems 1.3 and 1.4. Section 3 constructs the complementary lower bounds, as well as the lower bounds for Theorem 1.6, dealing with $\mathcal{I}^{W,X}(D(x,\varepsilon))$. Certain key lemmas for deriving the lower bounds are stated and proved in Section 4. Proposition 1.5 is proved in Section 5. The upper bounds for Theorem 1.6 are derived in Section 6. Section 7 contains the proof of Theorem 1.2 and Section 8 contains that of Theorem 1.1. Complements and open problems are provided in the last section.

2. Intersection local time estimates and upper bounds

Throughout this section, fix $0 < r_1 \le r$, let $\bar\theta_r = \inf\{s > 0 : |W_s| = r\}$, $\bar\theta'_r = \inf\{t > 0 : |W'_t| = r\}$ and define

$$\bar{\mathcal{I}} = \mathcal{I}_{\bar\theta_r,\bar\theta'_r}(D(0,r_1))\,.$$

Lemma 2.1. *We can find $c<\infty$ such that for all $k\geq 1$, $r_1\leq r/2\leq 1$, and x_0,x_0' with $|x_0|=|x_0'|=r_1$*

$$\mathbb{E}^{x_0,x_0'}(\bar{\mathcal{I}}/r_1^2)^k \leq (k!)^2\left(\log(r/r_1)+c\right)^{2k}. \tag{2.1}$$

Proof of Lemma 2.1: Let $g_r(x,y)$ denote the Green's function for $D(0,r)$, i.e. the 0-potential density for planar Brownian motion killed when it first hits $D(0,r)^c$. It follows from (1.8) that

(2.2)
$$\mathbb{E}^{x_0,x_0'}(\bar{\mathcal{I}}^k)=\pi^k\sum_{\sigma,\sigma'}\int_{D(0,r_1)^k}\prod_{j=1}^k g_r(y_{\sigma(j-1)},y_{\sigma(j)})g_r(y_{\sigma'(j-1)},y_{\sigma'(j)})\,dy_j$$

where the sum runs over all pairs of permutations σ,σ' of $\{1,\dots,k\}$ and we use the convention that $y_{\sigma(0)}=x_0$, $y_{\sigma'(0)}=x_0'$. From [8, page 242] (note that our g is twice theirs), we have

$$g_r(x,y)=\frac{1}{\pi}\log\left(\frac{r}{|x-y|}|1-\frac{x\bar{y}}{r^2}|\right)$$

where $\bar{y}$ denotes the complex conjugate of y. Thus, there exists $c_o<\infty$ such that

$$\frac{1}{\pi}\log\left(\frac{r}{|x-y|}\right)-c_o\leq g_r(x,y)\leq\frac{1}{\pi}\log\left(\frac{r}{|x-y|}\right)+c_o \tag{2.3}$$

for all $|x|$, $|y|\leq r/2$. Thus, after scaling in r_1, (2.2) is bounded by

$$\begin{aligned}
&\pi^k r_1^{2k}\sum_{\sigma,\sigma'}\int_{D(0,1)^k}\prod_{j=1}^k \left(\frac{1}{\pi}\log(r/r_1)-\frac{1}{\pi}\log|y_{\sigma(j)}-y_{\sigma(j-1)}|+c_o\right)\\
&\qquad\qquad\left(\frac{1}{\pi}\log(r/r_1)-\frac{1}{\pi}\log|y_{\sigma'(j)}-y_{\sigma'(j-1)}|+c_o\right)dy_j \qquad (2.4)\\
&\leq r_1^{2k}\sum_{\sigma,\sigma'}\sum_{j,j'=0}^k (\log(r/r_1)+c_o\pi)^{j+j'}\sum_{\substack{|\mathcal{A}|=k-j\\|\mathcal{B}|=k-j'}}\\
&\int_{D(0,1)^k}\prod_{i\in\mathcal{A}}|\log\left(|y_{\sigma(i)}-y_{\sigma(i-1)}|\right)|\,|\prod_{l\in\mathcal{B}}|\log\left(|y_{\sigma'(l)}-y_{\sigma'(l-1)}|\right)|\,\frac{dy_1}{\pi}\cdots\frac{dy_k}{\pi}
\end{aligned}$$

where the last sum goes over all subsets $\mathcal{A},\mathcal{B}$ of $\{1,\dots,k\}$ of cardinality $k-j,k-j'$ respectively. We then bound the integral in the last line of (2.4) by integrating successively with respect to $dy_1\cdots dy_k$, using Hölder's inequality and the bound

$$L:=1\vee\max_{1\leq m\leq 4}\left(\sup_{x\in D(0,1)}\int_{D(0,1)}\left|\log\left(|x-y|\right)\right|^m\frac{dy}{\pi}\right)^{1/m}<\infty \tag{2.5}$$

for each integration, noting that the variable y_j never appears in more than four factors of $\log(\cdot)$. We thus bound the integral in the last line of (2.4) by $L^{(k-j)+(k-j')}$, (we use the fact that $L \geq 1$), so that (2.4) is bounded by

$$r_1^{2k} \sum_{\sigma,\sigma'} \sum_{j,j'=0}^{k} \big(\log(r/r_1) + c_o\pi\big)^{j+j'} \sum_{\substack{|\mathcal{A}|=k-j \\ |\mathcal{B}|=k-j'}} L^{(k-j)+(k-j')}$$

$$= r_1^{2k}(k!)^2 \sum_{j,j'=0}^{k} \big(\log(r/r_1) + c_o\pi\big)^{j+j'} \binom{k}{j}\binom{k}{j'} L^{(k-j)+(k-j')}$$

$$= r_1^{2k}(k!)^2 (\log(r/r_1) + L + c_o\pi)^{2k}\,, \tag{2.6}$$

and (2.1) then follows. $\square$

Lemma 2.2. *For any $\delta > 0$ we can find $c, y_0 < \infty$ and $\varepsilon_0 > 0$ so that for all $\varepsilon \leq \varepsilon_0$ and $y \geq y_0$*

$$P^{x_0,x_0'}(\mathcal{I}_{\bar\theta_2,\bar\theta_2'}(D(0,\varepsilon)) \geq y^2\epsilon^2|\log\epsilon|^2) \leq c\exp(-(1-\delta)2y) \tag{2.7}$$

for all x_0, x_0' with $|x_0| = |x_0'| = \varepsilon$.

Proof of Lemma 2.2: Fix $\delta > 0$. By (2.1) with $r = 2$ for all ε sufficiently small and k sufficiently large

$$\begin{aligned} \mathbb{E}^{x_0,x_0'}\left(\mathcal{I}_{\bar\theta_2,\bar\theta_2'}(D(0,\varepsilon))\right)^k &\leq (k!)^2\,(\varepsilon\log(1/\varepsilon)(1+\delta))^{2k} \\ &\leq (2k)!\left(\frac{1}{2}\varepsilon\log(1/\varepsilon)(1+2\delta)\right)^{2k} \end{aligned}$$

so that with $F = \sqrt{\frac{\mathcal{I}_{\bar\theta_2,\bar\theta_2'}(D(0,\varepsilon))}{\frac{1}{4}\varepsilon^2\log^2(1/\varepsilon)(1+2\delta)^2}}$ we have for all k large enough

$$\mathbb{E}^{x_0,x_0'}(F^{2k}) \leq (2k)!$$

Hence, $\mathbb{E}^{x_0,x_0'}(\exp(\theta F)) \leq c_1$ for some finite $c_1 = c_1(\theta)$ and all $|x_0| = |x_0'| = \varepsilon$, $\theta \in [0,1)$. Using this in Chebyscheff's inequality yields

$$P^{x_0,x_0'}(F \geq y) \leq ce^{-(1-\delta)y},$$

out of which (2.7) follows. $\square$

We next provide the required upper bounds in Theorems 1.3 and 1.4. Namely, with the notation

$$\mathsf{ThickInt}_{\geq a^2} = \left\{x \in D(0,1) :\ \limsup_{\varepsilon\to 0} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4} \geq a^2\right\}, \tag{2.8}$$

we will show that for any $a \in (0,1]$,

$$\dim(\mathsf{ThickInt}_{\geq a^2}) \leq 2 - 2a\,, \quad a.s.\,, \tag{2.9}$$

and

$$\limsup_{\varepsilon\to 0}\sup_{|x|<1}\frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4}\leq 1\,,\quad a.s. \tag{2.10}$$

(note that (2.10) provides the upper bound also for (1.11)).

Set $h(\epsilon)=\epsilon^2|\log\epsilon|^4$ and

$$z(x,\epsilon):=\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))/h(\varepsilon).$$

Fix $\delta>0$ small enough ($\delta<1/22$ will do), and choose a sequence $\tilde\epsilon_n\downarrow 0$ as $n\to\infty$ in such a way that $\tilde\epsilon_1<e^{-2}$ and

$$h(\tilde\epsilon_{n+1})=(1-\delta)h(\tilde\epsilon_n), \tag{2.11}$$

implying that $\tilde\epsilon_n$ is monotone decreasing in n. Since, for $\tilde\epsilon_{n+1}\leq\epsilon\leq\tilde\epsilon_n$ we have

$$z(x,\tilde\epsilon_n)=\frac{h(\tilde\epsilon_{n+1})}{h(\tilde\epsilon_n)}\frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\tilde\epsilon_n))}{h(\tilde\epsilon_{n+1})}\geq(1-\delta)z(x,\epsilon)\,, \tag{2.12}$$

it is easy to see that for any $a>0$,

$$\mathsf{ThickInt}_{\geq a^2}\subseteq D_{a^2}:=\{x\in D(0,1)\,\Big|\,\limsup_{n\to\infty}z(x,\tilde\epsilon_n)\geq(1-\delta)a^2\}.$$

Let $\{x_j : j=1,\dots,\bar K_n\}$, denote a maximal collection of points in $D(0,1)$ such that $\inf_{\ell\neq j}|x_\ell-x_j|\geq\delta\tilde\epsilon_n$. Let $\mathcal{A}_n$ be the set of $1\leq j\leq\bar K_n$, such that

$$\mathcal{I}_{\bar\theta,\bar\theta'}(D(x_j,(1+\delta)\tilde\epsilon_n))\geq(1-2\delta)a^2h(\tilde\epsilon_n). \tag{2.13}$$

Set $\tau_{x,n}=\inf\{s : W_s\in D(x,(1+\delta)\tilde\epsilon_n)\}$, $\tau'_{x,n}=\inf\{t : W'_t\in D(x,(1+\delta)\tilde\epsilon_n)\}$. Applying the strong Markov property and then Lemma 2.2 we have

$$\begin{aligned}
&\mathbf{P}(\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,(1+\delta)\tilde\epsilon_n))\geq(1-2\delta)a^2h(\tilde\epsilon_n))\\
=\;&\mathbb{E}\Big(\tau_{x,n}<\bar\theta,\,\tau'_{x,n}<\bar\theta'\,;\\
&\qquad\mathbf{P}^{W_{\tau_{x,n}},W'_{\tau'_{x,n}}}(\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,(1+\delta)\tilde\epsilon_n))\geq(1-2\delta)a^2h(\tilde\epsilon_n))\Big)\\
\leq\;&c\,\tilde\epsilon_n^{(1-10\delta)2a}
\end{aligned}$$

for some $c=c(\delta)<\infty$, all sufficiently large n and any $x\in D(0,1)$.

Thus, for all sufficiently large n, any j and $a>0$,

$$\mathbf{P}(j\in\mathcal{A}_n)\leq c\,\tilde\epsilon_n^{(1-10\delta)2a}\,, \tag{2.14}$$

implying that

$$\mathbb{E}|\mathcal{A}_n|\leq c'\,\tilde\epsilon_n^{(1-10\delta)2a-2}. \tag{2.15}$$

Let $\mathcal{V}_{n,j}=D(x_j,\delta\tilde\epsilon_n)$. For any $x\in D(0,1)$ there exists $j\in\{1,\dots,\bar K_n\}$ such that $x\in\mathcal{V}_{n,j}$, hence $D(x,\tilde\epsilon_n)\subseteq D(x_j,(1+\delta)\tilde\epsilon_n)$. Consequently, $\cup_{n\geq m}\cup_{j\in\mathcal{A}_n}\mathcal{V}_{n,j}$ forms a cover of D_{a^2} by sets of maximal diameter $2\delta\tilde\epsilon_m$.

Fix $a \in (0,2]$. Since $\mathcal{V}_{n,j}$ have diameter $2\delta\tilde{\epsilon}_n$, it follows from (2.14) that for $\gamma = 2 - (1-11\delta)2a > 0$,

$$\mathbb{E}\sum_{n=m}^{\infty}\sum_{j\in\mathcal{A}_n}|\mathcal{V}_{n,j}|^{\gamma} \le c'\,(2\delta)^{\gamma}\sum_{n=m}^{\infty}\tilde{\epsilon}_n^{\,\delta 2a} < \infty\,.$$

Thus, $\sum_{n=m}^{\infty}\sum_{j\in\mathcal{A}_n}|\mathcal{V}_{n,j}|^{\gamma}$ is finite a.s. implying that $\dim(D_{a^2}) \le \gamma$ a.s. Taking $\delta \downarrow 0$ completes the proof of the upper bound (2.9).

Turning to prove (2.10), set $a = (1+\delta)/(1-10\delta)$ noting that by (2.15)

$$\sum_{n=1}^{\infty}\mathbf{P}(|\mathcal{A}_n| \ge 1) \le \sum_{n=1}^{\infty}\mathbb{E}|\mathcal{A}_n| \le c'\sum_{n=1}^{\infty}\tilde{\epsilon}_n^{\,2\delta} < \infty\,.$$

By Borel-Cantelli, it follows that a.s. $\mathcal{A}_n$ is empty a.s. for all $n > n_0(\omega)$ and some $n_0(\omega) < \infty$. By (2.12) we then have

$$\sup_{\epsilon\le\tilde{\epsilon}_{n_0(\omega)}}\ \sup_{|x|<1}\frac{\mathcal{I}_{\bar{\theta},\bar{\theta}'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4} \le a^2\,,$$

and (2.10) follows by taking $\delta \downarrow 0$. □

In proving the lower bound in the following sections we will need a variant of Lemma 2.1 which we now discuss. Let ρ be a fixed measure on $D(0,1)$ with $\int g_1(x,y)\,d\rho(y)$ uniformly bounded and continuous on $D(0,1)$ and let L^{ρ}_t denote the continuous additive functional for $\{W_s\}$ with Revuz measure ρ. We can define L^{ρ}_t as

$$L^{\rho}_t = \lim_{\epsilon\to 0}\int_0^{\bar{\theta}}\int f_{\epsilon}(W_s - y)\,d\rho(y) \tag{2.16}$$

where f_{ϵ} is any approximate identity. Convergence in (2.16) holds a.s and in all L^p spaces. Our interest in such functionals stems from the fact, see [1, Section 2], that for any r

$$\mathcal{I}_{\bar{\theta}_r,\bar{\theta}'_r}(A) = L_{\bar{\theta}_r}^{\pi\mu^{W'}_{\bar{\theta}'_r}|A}$$

where $\mu^{W'}_{\bar{\theta}'_r}(B) = \int_0^{\bar{\theta}'_r} 1_B(W'_t)\,dt$ is the occupation measure with respect to W', and for any measure ν we define the restriction $\nu|A(B) = \nu(A\cap B)$. In the sequel, we write $a = (b \pm c)d$ to mean $|a/d - b| \le c$.

Lemma 2.3. *Assume that for some $\gamma_1 > 0$, $\gamma_2 > 0$ and $c_1 < \infty$*

$$\sup_{|x|\le 2}\frac{\rho(D(x,\varepsilon))}{\varepsilon^{\gamma_1}\left(\log\frac{1}{\varepsilon}\right)^{\gamma_2}} \le c_1 < \infty\,, \qquad \forall\varepsilon > 0. \tag{2.17}$$

We can then find $c < \infty$ such that for all $k \ge 1$, $r_1 \le r_2 \le r/2 \le 1/2$, and x_0 with $|x_0| = r_2$, both

$$\mathbb{E}^{x_0}(L_{\bar{\theta}_r}^{\pi\rho|\,D(0,r_1)})^k \le k!\,(\rho(D(0,r_1))\log(r/r_1) + c r_1^{\gamma_1}(\log(1/r_1))^{\gamma_2})^k\ , \tag{2.18}$$

and

$$\mathbb{E}^{x_0}(L^{\pi\rho|\,D(0,r_1)}_{\bar{\theta}_r}) = \rho(D(0,r_1))\log(r/r_2) \pm c r_1^{\gamma_1}(\log(1/r_1))^{\gamma_2}\,. \tag{2.19}$$

Proof of Lemma 2.3: We recall Kac's moment formula [7], which can be derived easily from (2.16):

$$\mathbb{E}^{x_0}(L^{\pi\rho|\,D(0,r_1)}_{\bar{\theta}_r})^k = k!\pi^k \int_{D(0,r_1)^k} \prod_{j=1}^{k} g_r(y_{j-1},y_j)\,d\rho(y_j). \tag{2.20}$$

Thus, to prove (2.19) we must show that

$$\pi\int_{D(0,r_1)} g_r(x_0,y)\,d\rho(y) = \rho(D(0,r_1))\log(r/r_2) \pm c r_1^{\gamma_1}(\log(1/r_1))^{\gamma_2}\,, \tag{2.21}$$

for all x_0 with $|x_0| = r_2$, and (2.18) will follow from (2.20) using

$$\sup_{x\in D(0,r_1)} \pi\int_{D(0,r_1)} g_r(x,y)\,d\rho(y) \le \rho(D(0,r_1))\log(r/r_1) + c r_1^{\gamma_1}(\log(1/r_1))^{\gamma_2}\,. \tag{2.22}$$

inductively for the $dy_k, dy_{k-1}, \ldots, dy_2$ integrals, and (2.21) for the dy_1 integral.

By (2.3) we can bound the left hand side in the last display by

$$\sup_{x\in D(0,r_1)}\int_{D(0,r_1)} \left(\log(r/r_1) + \log\left(\frac{r_1}{|y-x|}\right) + \pi c_o\right)\,d\rho(y).$$

Now $D(0,r_1)\subseteq D(x,e\,r_1)$ for any $x\in D(0,r_1)$, and breaking $D(x,e\,r_1)$ up into annuli $D(x,e\,r_1) = \bigcup_{j=-1}^{\infty}\left\{D(x,e^{-j}r_1)\setminus D(x,e^{-j-1}r_1)\right\}$ allows us to bound

$$\begin{aligned}\int_{D(0,r_1)} &|\log\left(\frac{r_1}{|x-y|}\right)|\,d\rho(y) \\ &\le \sum_{j=-1}^{\infty}\int_{D(x,e^{-j}r_1)\setminus D(x,e^{-j-1}r_1)} |\log\left(\frac{r_1}{|x-y|}\right)|\,d\rho(y) \\ &\le \sum_{j=-1}^{\infty}|j+2|\rho(D(x,e^{-j}r_1)) \le c_1\sum_{j=-1}^{\infty}|j+2|e^{-\gamma_1 j}r_1^{\gamma_1}(\log 1/r_1+|j|)^{\gamma_2}\end{aligned} \tag{2.23}$$

where we have used the upper bound (2.17) for the last inequality. This immediately gives (2.22).

Turning to (2.21), by (2.3) we can write the left hand side as

$$\begin{aligned}&\int_{D(0,r_1)}\left(\log(r/r_2) + \log\left(\frac{r_2}{|x_0-y|}\right) \pm \pi c_o\right)\,d\rho(y) \\ &= (\log(r/r_2)\pm\pi c_o)\,\rho(D(0,r_1)) + \int_{D(0,r_1)}\log\left(\frac{r_2}{|x_0-y|}\right)\,d\rho(y).\end{aligned}$$

When $r_1/r_2 \leq 1/2$, we have that $|\log \frac{r_2}{|x_0-y|}|$ is bounded on $D(0,r_1)$ so that (2.21) follows in that case. When $1/2 \leq r_1/r_2 \leq 1$ we can use

$$\int_{D(0,r_1)} |\log\left(\frac{r_2}{|x_0-y|}\right)|\,d\rho(y) \leq \int_{D(0,r_2)} |\log\left(\frac{r_2}{|x_0-y|}\right)|\,d\rho(y)$$

and (2.21) follows by the argument we applied in (2.23). □

3. Lower bounds

Fixing $a<1$, $c>0$ and $\delta>0$, let

$$\Gamma_c=\Gamma_c(\omega,\omega') := \{x\in D(0,1):\ \lim_{\varepsilon\to 0}\frac{\mathcal{I}_{\bar\theta_c,\bar\theta'_c}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4}=a^2\}\,,$$

and $\mathcal{E}_c := \{\omega,\omega' : \dim(\Gamma_c(\omega,\omega'))\geq 2-2a-\delta\}$.

In view of the results of Section 2, we will obtain Theorem 1.4 once we show that $\mathbf{P}\times\mathbf{P}'(\mathcal{E}_1)=1$ for any $a<1$ and $\delta>0$. Moreover, then the inequality

$$\liminf_{\epsilon\to 0}\sup_{|x|<1}\frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4}\geq\sup_{|x|<1}\liminf_{\epsilon\to 0}\frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4}$$

implies that for any $\eta>0$,

$$\liminf_{\varepsilon\to 0}\sup_{|x|<1}\frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4}\geq 1-\eta\,,\quad a.s.$$

In view of (2.10), these lower bounds establish Theorem 1.3.

The bulk of this section and the next will be dedicated to showing that $\mathbf{P}\times\mathbf{P}'(\mathcal{E}_1)>0$. Assuming this for the moment, let us show that this implies $\mathbf{P}\times\mathbf{P}'(\mathcal{E}_1)=1$. With $W^c_t := c^{-1}W_{c^2t}$ we have that $c^2\bar\theta(\omega^c)=\inf\{c^2t : |c^{-1}W_{c^2t}|=1\}=\bar\theta_c(\omega)$, and similarly for W' and hence

$$\begin{aligned}
(3.1)\quad & \mathcal{I}^{W^c,W'^c}_{\bar\theta,\bar\theta'}(D(x,\varepsilon)) \\
&=\lim_{\gamma\to 0}\ \pi\int_0^{\bar\theta(\omega^c)}\int_0^{\bar\theta(\omega'^c)} 1_{\{|W^c_s-x|\leq\epsilon\}}f_\gamma(W^c_s-W'^c_t)\,ds\,dt\\
&=\lim_{\gamma\to 0}\ \pi\int_0^{\bar\theta(\omega^c)}\int_0^{\bar\theta(\omega'^c)} 1_{\{|W_{c^2s}-cx|\leq c\epsilon\}}f_\gamma(c^{-1}W_{c^2s}-c^{-1}W'_{c^2t})\,ds\,dt\\
&=\lim_{\gamma\to 0}\ \frac{\pi}{c^2}\int_0^{c^2\bar\theta(\omega^c)}\int_0^{c^2\bar\theta(\omega'^c)} 1_{\{|W_s-cx|\leq c\epsilon\}}f_{c\gamma}(W_s-W'_t)\,ds\,dt\\
&=\frac{1}{c^2}\mathcal{I}_{\bar\theta_c,\bar\theta'_c}(D(cx,c\varepsilon)).
\end{aligned}$$

Consequently, $\Gamma_c(\omega,\omega')=c\Gamma_1(\omega^c,\omega'^c)$, so Brownian scaling implies that $p=\mathbf{P}\times\mathbf{P}'(\mathcal{E}_c)$ is independent of $c>0$. Let

$$\mathcal{E} := \limsup_{n\to\infty}\mathcal{E}_{n^{-1}}\ ,$$

so that $\mathbf{P} \times \mathbf{P}'(\mathcal{E}) \geq p$. Since $\mathcal{E}_c \in \mathcal{F}_{\bar{\theta}_c} \times \mathcal{F}'_{\bar{\theta}'_c}$ and $\bar{\theta}_{n^{-1}} \downarrow 0$, the Blumenthal $0-1$ law tells us that $\mathbf{P} \times \mathbf{P}'(\mathcal{E}) \in \{0,1\}$. Thus, $p > 0$ yields $\mathbf{P} \times \mathbf{P}'(\mathcal{E}) = 1$. Since $x \mapsto \mathcal{I}_{\bar{\theta}_b, \bar{\theta}'_b}(D(x,\varepsilon))$ are Borel measurable, it follows that Γ_b are Borel sets (hence analytic), and exactly as in [5, (3.1)] we find that the events $\mathcal{E}_c$ are essentially increasing in c, *i.e.*,

$$\forall 0 < b < c \quad \mathbf{P} \times \mathbf{P}'(\mathcal{E}_b \setminus \mathcal{E}_c) = 0 \,. \tag{3.2}$$

Thus, $\mathbf{P}\times\mathbf{P}'(\mathcal{E}\setminus\mathcal{E}_1) \leq \mathbf{P}\times\mathbf{P}'(\bigcup_n \{\mathcal{E}_{n^{-1}}\setminus\mathcal{E}_1\}) = 0$, so that also $\mathbf{P}\times\mathbf{P}'(\mathcal{E}_1) = 1$.

It thus remains to show that $\mathbf{P} \times \mathbf{P}'(\mathcal{E}_1) > 0$. We start by constructing a subset of Γ_1, the Hausdorff dimension of which is easier to bound below. To this end we recall some notation from [5]. Fix $a < 2$, $\epsilon_1 = 1/8$ and the squares $S = S_1 = [\epsilon_1, 2\epsilon_1]^2 \subset D(0,1)$. Note that for all $x \in S$ and $y \in S \cup \{0\}$ both $0 \notin D(x,\epsilon_1)$ and $0 \in D(x,1/2) \subset D(y,1) \subset D(x,2)$. Let $\epsilon_k = \epsilon_1 (k!)^{-3} = \epsilon_1 \prod_{l=2}^k l^{-3}$. For $x \in S$, $k \geq 2$ and $\rho > \epsilon_1$ let $N_k^x(\rho)$ denote the number of excursions of W. from $\partial D(x,\epsilon_{k-1})$ to $\partial D(x,\epsilon_k)$ prior to hitting $\partial D(x,\rho)$. Set $n_k = 3ak^2 \log k$. We will say that a point $x \in S$ is **n-perfect** if

$$n_k - k \leq N_k^x(1/2) \leq N_k^x(2) \leq n_k + k \,, \qquad \forall k = 2, \dots, n.$$

For $n \geq 2$ we partition S into $M_n = \epsilon_1^2/(2\epsilon_n)^2 = (1/4)\prod_{l=1}^n l^6$ non-overlapping squares of edge length $2\epsilon_n = 2\epsilon_1/(n!)^3$, which we denote by $S(n,i)\,;\, i = 1,\dots,M_n$ with $x_{n,i}$ denoting the center of each $S(n,i)$. Let $Y(n,i)\,;\, i = 1,\dots,M_n$ be the sequence of random variables defined by

$$Y(n,i) = 1 \quad \text{if } x_{n,i} \text{ is n-perfect}$$

and $Y(n,i) = 0$ otherwise. Set $q_{n,i} = \mathbf{P}(Y(n,i) = 1) = \mathbb{E}(Y(n,i))$. Define

$$A_n = \bigcup_{i:Y(n,i)=1} S(n,i),$$

and

$$F = F(\omega) = \bigcap_m \overline{\bigcup_{n\geq m} A_n} := \bigcap_m F_m \,. \tag{3.3}$$

Note that each $x \in F$ is the limit of a sequence $\{x_n\}$ such that x_n is n-perfect. Finally, we set

$$C_a := \Big\{ x \in S : \lim_{\epsilon\to 0} \frac{\mu_{\bar{\theta}'}^{w'}(D(x,\varepsilon))}{\varepsilon^2 (\log \epsilon)^2} = a \Big\}, \tag{3.4}$$

The next lemma (to be proved in Section 4), shows that $F \cap C_a \subset \Gamma_1$.

Lemma 3.1. *A.s. for all $x \in F \cap C_a$*

$$\lim_{\varepsilon\to 0} \frac{\mathcal{I}_{\bar{\theta},\bar{\theta}'}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^4} = a^2.$$

To complete the proof that $\mathbf{P} \times \mathbf{P}'(\mathcal{E}_1) > 0$ it thus suffices to show that

$$\mathbf{P} \times \mathbf{P}'(\dim(F \cap C_a) \geq 2 - 2a - 2\delta) > 0, \tag{3.5}$$

for any $a < 1$ and $\delta > 0$.

It was proved in [5, Section 3] that

$$\mathbf{P}'(\dim(C_a) \geq 2 - a - \delta) > 0,$$

for any $a < 2$ and $\delta > 0$. The following general lemma will thus imply (3.5).

Lemma 3.2. *Let $B \subseteq S$ be a closed $\mathcal{F}'$-measurable set with*

$$\mathbf{P}'(\dim(B) \geq b) > 0. \tag{3.6}$$

Then, for any $a < b$ and $\delta > 0$

$$\mathbf{P} \times \mathbf{P}'(\dim(F \cap B) \geq b - a - \delta) > 0. \tag{3.7}$$

Proof. Fixing $a < b$ and $\delta > 0$ such that $h := b - a - \delta > 0$, we establish (3.7) by finding a set $\mathcal{C} \times \mathcal{C}'$ of positive $\mathbf{P} \times \mathbf{P}'$ probability, such that for any $(\omega, \omega') \in \mathcal{C} \times \mathcal{C}'$ we can find a non-zero random measure $\rho_{\omega,\omega'}$ supported on $F(\omega) \cap B(\omega')$ with finite h-energy, where the h-energy of a measure ν is defined as

$$\mathcal{G}_h(\nu) = \int\int |x - y|^{-h} \, d\nu(x) \, d\nu(y)$$

(see e.g. [15, Theorem 8.7]).

By (3.6) we can find a set $\mathcal{C}'$ with $\mathbf{P}'(\mathcal{C}') > 0$ and a finite constant c such that for any $\omega' \in \mathcal{C}'$, we have a probability measure $\mu_{\omega'}$ supported on B with

$$\mathcal{G}_{b-\delta/2}(\mu_{\omega'}) < c\,.$$

In the sequel, we restrict attention to $\omega' \in \mathcal{C}'$ without mentioning it explicitly. The measure $\rho = \rho_{\omega,\omega'}$ will be constructed as a weak limit of measures ν_n, where $\nu_n = \nu_{n,\omega,\omega'}$ for $n \geq 2$ is the random measure supported on $A_n \subseteq F_n$ whose density with respect to $\mu_{\omega'}$ is

$$f_n(x) = \sum_{i=1}^{M_n} q_{n,i}^{-1} 1_{\{Y(n,i)=1\}} 1_{\{x \in S(n,i)\}}.$$

Note that

$$\mathbb{E}(\nu_n(S)) = \sum_{i=1}^{M_n} q_{n,i}^{-1} \mathbf{P}(Y(n,i) = 1) \mu_{\omega'}(S(n,i)) = \mu_{\omega'}([\epsilon_1, 2\epsilon_1]^2) = 1. \tag{3.8}$$

We next recall Lemma 3.2 of [5], combining it with the comments just below it. In the sequel, we let C_m denote generic finite constants that are independent of n.

Lemma 3.3. *Let $l(i,j) = \min\{m : D(x_{n,i}, \epsilon_m) \cap D(x_{n,j}, \epsilon_m) = \emptyset\} \leq n$, with $l(i,i) := n$. Then, there exists $\delta_n \to 0$ such that for all $n \geq 2$, i,*

$$q_{n,i} \geq Q_n := \inf_{x \in S} \mathbf{P}(x \text{ is } n\text{-perfect}) \geq \epsilon_n^{a+\delta_n}\ ,$$

whereas for all n,

$$\mathbb{E}(Y(n,i)Y(n,j)) \leq C_0 Q_n^2 \epsilon_{l(i,j)}^{-a-\delta_{l(i,j)}} .$$

Furthermore, $Q_n \geq c q_{n,i}$ for some $c>0$ and all $n \geq 2$ and i.

It follows from Lemma 3.3 that for all $\omega' \in \mathcal{C}'$,

$$\begin{aligned}
\mathbb{E}\left((\nu_n(S))^2\right) &= \sum_{i,j=1}^{M_n} q_{n,i}^{-1} q_{n,j}^{-1} \mathbb{E}(Y(n,i)Y(n,j)) \mu_{\omega'}(S(n,i)) \mu_{\omega'}(S(n,j)) \\
&\leq C_1 \sum_{i,j=1}^{M_n} \epsilon_{l(i,j)}^{-a-\delta_{l(i,j)}} \mu_{\omega'}(S(n,i)) \mu_{\omega'}(S(n,j)) \\
&\leq C_2 \sum_{i,j=1}^{M_n} \int_{x\in S(n,i)} \int_{y \in S(n,j)} \epsilon_{l(i,j)}^{-b+\delta-\delta_{l(i,j)}} \mu_{\omega'}(dx) \mu_{\omega'}(dy) \\
&\leq C_3 \int\int |x-y|^{-b+\delta/2} \mu_{\omega'}(dx) \mu_{\omega'}(dy) = C_3 \mathcal{G}_{b-\delta/2}(\mu_{\omega'}) \\
&\leq C_3 c < \infty \qquad (3.9)
\end{aligned}$$

is a bounded sequence (recall that $\delta_l \to 0$). Applying the Paley-Zygmund inequality (see [9, page 8]), (3.8) and (3.9) together guarantee that for some $r>0$, $v>0$ and all $\omega' \in \mathcal{C}'$,

$$\mathbf{P}(r^{-1} \geq \nu_n(S) \geq r) \geq 2v > 0, \qquad \forall n. \tag{3.10}$$

Similarly, for $h = b-a-\delta \in (0,2)$, and $\omega' \in \mathcal{C}'$,

$$\begin{aligned}
&\mathbb{E}(\mathcal{G}_h(\nu_n)) \\
\leq\ & C_4 \sum_{i,j=1}^{M_n} \frac{\mathbb{E}(Y(n,i)Y(n,j))}{q_{n,i} q_{n,j}} \int_{S(n,i)} \int_{S(n,j)} |x-y|^{-h} \mu_{\omega'}(dx)\, \mu_{\omega'}(dy) \\
\leq\ & C_5 \sum_{i,j=1}^{M_n} \epsilon_{l(i,j)}^{-a-\delta_{l(i,j)}} \int_{S(n,i)} \int_{S(n,j)} |x-y|^{-h} \mu_{\omega'}(dx)\, \mu_{\omega'}(dy) \\
\leq\ & C_6 \mathcal{G}_{b-\delta/2}(\mu_{\omega'}) \leq C_6 c < \infty.
\end{aligned}$$

is a bounded sequence. Thus we can find $d < \infty$ such that for all $\omega' \in \mathcal{C}'$,

$$\mathbf{P}(\mathcal{G}_h(\nu_n) \leq d) \geq 1 - v > 0, \qquad \forall n.$$

Combined with (3.10) this shows that for all $\omega' \in \mathcal{C}'$,

$$\mathbf{P}(r^{-1} \geq \nu_n(S) \geq r,\, \mathcal{G}_h(\nu_n) \leq d) \geq v > 0, \qquad \forall n. \tag{3.11}$$

Let $\mathcal{C}_n = \{\omega : r^{-1} \geq \nu_n(S) \geq r,\, \mathcal{G}_h(\nu_n) \leq d \text{ for all } \omega' \in \mathcal{C}'\}$ and set $\mathcal{C} = \limsup_n \mathcal{C}_n$. Then, (3.11) implies that

$$\mathbf{P}(\mathcal{C}) \geq v > 0.$$

Fixing $\omega \in \mathcal{C}$ there exists a subsequence $n_k \to \infty$ such that $\omega \in \mathcal{C}_{n_k}$ for all k and all $\omega' \in \mathcal{C}'$. Due to the lower semi-continuity of $\mathcal{G}_h(\cdot)$, the set of non-negative measures ν on S such that $\nu(S) \in [r, r^{-1}]$ and $\mathcal{G}_h(\nu) \leq d$ is compact with respect to weak convergence. Thus, for $(\omega, \omega') \in \mathcal{C} \times \mathcal{C}'$, the sequence $\nu_{n_k} = \nu_{n_k,\omega,\omega'}$ has at least one weak limit $\rho_{\omega,\omega'}$ which is a finite measure supported on $F(\omega) \cap B(\omega')$, having positive mass and finite h-energy. This completes the proof of Lemma 3.2. $\square$

Let $\{X_t\}$ denote the symmetric stable process of index β, with law denoted $\mathbf{P}'$. Fix $T < \infty$ and for $a < \beta/2$, define

$$C_a^X = \{x \in S : \limsup_{\epsilon \to 0} \frac{\mu_T^X(D(x,\epsilon))}{\Lambda \epsilon^\beta |\log \epsilon|} = a\}\,.$$

As in the case of two Brownian motions, the lower bound in Theorem 1.6 follows from two results. The first result is the next lemma, to be proved in Section 4.

Lemma 3.4. *A.s. for all $x \in F \cap C_a^X$*

$$\limsup_{\varepsilon \to 0} \frac{\mathcal{I}_{\bar{\theta},T}^{W,X}(D(x,\varepsilon))}{\varepsilon^\beta \left(\log \frac{1}{\varepsilon}\right)^3} = a^2. \tag{3.12}$$

The second result says that for any $\delta > 0$

$$\mathbf{P} \times \mathbf{P}'(\dim(F \cap C_a^X) \geq \beta - 2a - \delta) > 0\,. \tag{3.13}$$

Indeed, recall from [4, (1.5)] that $\mathbf{P}'(\dim(C_a^X) = \beta - a) = 1$ (by shift invariance and stable scaling, [4, (1.5)] is valid even when restricting to $x \in S$). We can extract a closed subset of C_a^X which still has dimension $\beta - a$, so that (3.13) follows from Lemma 3.2.

As in the case of two Brownian motions, we conclude from (3.12) and (3.13) that $\mathbf{P} \times \mathbf{P}'(\mathcal{E}_1) > 0$, where $\mathcal{E}_c := \{\omega, \omega' : \dim(\Gamma_c(\omega, \omega')) \geq \beta - 2a - \delta\}$ and

$$\Gamma_c(\omega, \omega') := \{x \in D(0,c) : \limsup_{\varepsilon \to 0} \frac{\mathcal{I}_{\bar{\theta}_c, c^\beta T}^{W,X}(D(x,\varepsilon))}{\varepsilon^\beta \left(\log \frac{1}{\varepsilon}\right)^3} = a^2\}\,.$$

With $W_t^c := c^{-1} W_{c^2 t}$ and $X_t^c := c^{-1} X_{c^\beta t}$, it follows by Brownian and stable scaling that

$$\mathcal{I}_{\bar{\theta},T}^{W^c,X^c}(D(x,\varepsilon)) = c^{-\beta} \mathcal{I}_{\bar{\theta}_c, c^\beta T}^{W,X}(D(cx, c\varepsilon))$$

(see (3.1) for a similar derivation). Consequently, $\Gamma_c(\omega, \omega') = c\Gamma_1(\omega^c, \omega'^c)$, with Brownian and stable scaling together implying that $p = \mathbf{P} \times \mathbf{P}'(\mathcal{E}_c) > 0$ is independent of $c > 0$. Since $\mathcal{E}_c \in \mathcal{F}_{\bar{\theta}_c} \times \sigma(X_t : 0 \leq t \leq c^\beta T)$, and $\bar{\theta}_{n^{-1}} \downarrow 0$, the Blumenthal $0-1$ law tells us that $\mathbf{P} \times \mathbf{P}'(\limsup_{n \to \infty} \mathcal{E}_{n^{-1}}) = 1$, resulting with $\mathbf{P} \times \mathbf{P}'(\mathcal{E}_1) = 1$ (see (3.2) for more details). This concludes the proof of the lower bound for any $T < \infty$. Next note that the random set Γ_1 for

$T = \infty$ is the same as Γ_1 for $T < \infty$ whenever $\inf_{s \geq T} |X_s| > 1$. Consequently, in case $T = \infty$, we see that

$$\mathbf{P} \times \mathbf{P}'(\mathcal{E}_1) \geq \lim_{T \to \infty} \mathbf{P}'(\inf_{s \geq T} |X_s| > 1) = 1$$

(by stable scaling and the transience of X_t). □

Remark: The alert reader might ask whether it is possible to get a statement similar to the one in Lemma 3.2 but rather holding with probability one. Recall that

$$\mathsf{Thick}_a^W = \{x \in D(0,1) : \lim_{\epsilon \to 0} \frac{\mu_{\bar{\theta}}^W(D(x,\epsilon))}{\epsilon^2 (\log \epsilon)^2} = a\}.$$

A variation of our proof yields the following statement:
Let $B \subset D(0,1)$ be a (possibly random) closed set, independent of Thick_a^W. Further assume there exist random probability measures $\{\mu_{n,\omega'}(\cdot)\}$ such that $\mathbf{P}'(\mathcal{G}_b(\mu_{1,\omega'}) < \infty) = 1$, $\mu_{n,\omega'}(\cdot)$ possesses the same law as $\mu_{1,\omega'}(\cdot/2^{n-1})$, and for any $n \geq 1$,

$$\mathbf{P}'(\mu_{n,\omega'}(B \cap D(0, 2^{-(n-1)})) = 1) = 1\,. \tag{3.14}$$

Finally, assume that the sequence of measures $\mu_{n,\omega'}$, viewed as measure-valued random variables, possesses a trivial tail. Then,

$$\mathbf{P} \times \mathbf{P}'(\dim(\mathsf{Thick}_a^W \cap B) \geq b - a) = 1\,. \tag{3.15}$$

4. From excursions to intersection local times

4.1. **Intersection of two Brownian motions.** Recall the sets F, C_a, introduced in (3.3) and (3.4) and $h(\varepsilon) = \varepsilon^2 (\log \varepsilon)^4$. Lemma 3.1 will follow from the next two lemmas.

Lemma 4.1. *For every $\delta > 0$, if $x \in F \cap C_a$ then*

$$a^2(1-\delta)^5 \leq \liminf_{\epsilon \to 0} \mathcal{I}_{\bar{\theta},\bar{\theta}'}(D(x,\epsilon))/h(\epsilon). \tag{4.1}$$

Lemma 4.2. *For every $\delta > 0$, if $x \in F \cap C_a$ then*

$$\limsup_{\epsilon \to 0} \mathcal{I}_{\bar{\theta},\bar{\theta}'}(D(x,\epsilon))/h(\epsilon) \leq a^2(1+\delta)^5. \tag{4.2}$$

Proof of Lemma 4.1: Let $\delta_k = \epsilon_k/k^6$ and let $\mathcal{D}_k$ be a δ_k-net of points in S. Let

$$\epsilon_k' = \epsilon_k e^{1/k^6}, \qquad \epsilon_{k-1}'' = \epsilon_{k-1} e^{-1/k^6},$$

so that

$$\epsilon_k' \geq \epsilon_k + \delta_k, \qquad \epsilon_{k-1}'' \leq \epsilon_{k-1} - \delta_k. \tag{4.3}$$

We will say that a point $x' \in \mathcal{D}_k$ is **lower k-successful** if there are at least $n_k - k$ excursions of W from $\partial D(x', \epsilon_{k-1}'')$ to $\partial D(x', \epsilon_k')$ prior to $\bar{\theta}$. Let

$$\epsilon_{k,j} = \epsilon_k e^{-j/k}\,,\ j = 0, 1, \ldots, 3k \log(k+1),$$

and let $\epsilon'_{k,j} = \epsilon_{k,j} e^{-2/k^3} = \epsilon'_k e^{-j/k} e^{-2/k^3 - 1/k^6}$. We say that $x' \in \mathcal{D}_k$ is **lower k,δ-successful** if it is lower k-successful and in addition,

$$a(1-\delta)\epsilon'^2_{k,j} |\log \epsilon'_{k,j}|^2 \le \mu^{W'}_{\bar\theta'}(D(x',\epsilon'_{k,j})),\ \forall j = 0,\dots,3k\log(k+1). \tag{4.4}$$

We now derive Lemma 4.1 from the following lemma.

Lemma 4.3. *There exists a $k_0 = k_0(\delta,\omega,\omega')$ such that for all $k \ge k_0$ and $x' \in \mathcal{D}_k$, if x' is lower k,δ-successful then*

$$a^2(1-\delta)^4 h(\epsilon'_{k,j}) \le \mathcal{I}_{\bar\theta,\bar\theta'}(D(x',\epsilon'_{k,j})), \qquad \forall j = 0,1,\dots,3k\log(k+1).$$

To begin our derivation of Lemma 4.1, note that if $x \in F$ then there exists a sequence of n-perfect points $x_n \to x$. Since an n-perfect point is k-perfect for all $k \le n$, it follows that one may find a point $\tilde x_k$ which is k-perfect and satisfies $|x - \tilde x_k| < \epsilon_k/k^6$. Let $x_k \in \mathcal{D}_k$ with $|x_k - \tilde x_k| \le \epsilon_k/k^6$. Using (4.3) and the fact that $\tilde x_k$ is k-perfect we can see that $x_k \in \mathcal{D}_k$ is lower k-successful. (c.f. [5, Section 6, figure 1]).

Since $|x - x_k| \le 2\epsilon_k/k^6$ we have

$$\mu^{W'}_{\bar\theta'}(D(x_k,\epsilon'_{k,j})) \ge \mu^{W'}_{\bar\theta'}(D(x,\epsilon'_{k,j} - 2\epsilon_k/k^6))$$

for all j,k, so that if $x \in C_a$ it is easy to see that there exists a $k_2 = k_2(x,\omega,\delta)$ such that for $k > k_2$, x_k is in fact lower k,δ-successful.

Applying Lemma 4.3 with $x' = x_k$ and using $\epsilon'_{k,j} + 2\epsilon_k/k^6 \le \epsilon_{k,j}$ so that

$$\mathcal{I}_{\bar\theta,\bar\theta'}(D(x_k,\epsilon'_{k,j})) \le \mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\epsilon'_{k,j} + 2\epsilon_k/k^6)) \le \mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\epsilon_{k,j}))$$

then shows that for all k sufficiently large

$$a^2(1-\delta)^4 h(\epsilon_{k,j}) \le \mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\epsilon_{k,j})), \qquad \forall j = 0,1,\dots,3k\log(k+1).$$

Now for any $\epsilon_{k+1} \le \epsilon \le \epsilon_k$, let j be such that $\epsilon_{k,j+1} \le \epsilon \le \epsilon_{k,j}$. Then,

$$\frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\epsilon))}{h(\epsilon)} \ge \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\epsilon_{k,j+1}))}{h(\epsilon_{k,j})} \ge \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\epsilon_{k,j+1}))}{h(\epsilon_{k,j+1})}(1-4/k)\ ,$$

and this completes the proof of Lemma 4.1.

Proof of Lemma 4.3: Suppose that $x' \in \mathcal{D}_k$ is k,δ-successful. Then there are at least $n'_k = n_k - k$ excursions of W between $\partial D(x',\epsilon'_k)$ and $\partial D(x',\epsilon''_{k-1})$, where $n'_k \to \infty$ as $k \to \infty$. Let $\tau_{l,k,j}$ denote the projected intersection local time measure of $D(x',\epsilon'_{k,j}) \subset D(x',\epsilon'_k)$ accumulated while W executes its l-th excursion between $\partial D(x',\epsilon'_k)$ and $\partial D(x',\epsilon''_{k-1})$, and W' runs from 0 until $\bar\theta'$.

Let

$$A(x',k,j) = \{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x',\epsilon'_{k,j})) \le a^2(1-\delta)^4 h(\epsilon'_{k,j})\}.$$

Note that conditional on W' the $\tau_{l,k,j}$ are i.i.d. and using P^W to denote probability with respect to W, i.e. conditional on W' we have

$$\begin{aligned} P_{x',k,j} &:= \mathbf{P}^W\left(A(x',k,j), x' \text{ is lower k,}\delta\text{-successful}\right) \\ &\leq \mathbf{P}^W_{x',s}\left(\sum_{l=1}^{n'_k}\tau_{l,k,j} \leq a^2(1-\delta)^4 h(\epsilon'_{k,j})\right) \end{aligned}$$

where the subscript x', s on P^W indicates that x' satisfies (4.4). In [5, Theorem 1.2] we show that

$$\lim_{\varepsilon\to 0}\sup_{x\in I\!R^2}\frac{\mu^{W'}_{\bar\theta'_r}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = 2\,, \qquad a.s.$$

Consequently, the measure $\rho(\cdot) = \mu^{W'}_{\bar\theta'}(x'+\cdot)$ satisfies (2.17) for $\gamma_1 = \gamma_2 = 2$. We now apply Lemma 2.3 with $r_1 = \epsilon'_{k,j}$, $r_2 = \epsilon'_k$, $r = \epsilon''_{k-1}$ and $\rho(\cdot)$ as above. Our condition (4.4) implies that

$$\rho(D(0,r_1)) = \mu^{W'}_{\bar\theta'}(D(x',\epsilon'_{k,j})) \geq a(1-\delta)r_1^{\gamma_1}(\log(1/r_1))^{\gamma_2},$$

and so, by (2.19), for all $k \geq k_1(a,\delta)$,

$$E^W_{x',s}(\tau_{l,k,j}) = \mathbb{E}^{x_0}(L^{\pi\rho|\,D(0,r_1)}_{\bar\theta_r}) \geq a(1-\delta)^2\log(r/r_2)r_1^{\gamma_1}(\log(1/r_1))^{\gamma_2}\,. \tag{4.5}$$

Using Stirling's approximation for $\log\epsilon_k = \log\epsilon_1 - 3\log k!$, it follows that for all $k \geq k_2(a,\delta)$ and all $j = 0,\dots,3k\log(k+1)$,

$$n'_k\log(r/r_2) = n'_k(3\log k - 2k^{-6}) \geq a(1-\delta)|\log(\epsilon'_{k,j})|^2 \tag{4.6}$$

Combining (4.5) and (4.6), we see that

$$a^2(1-\delta)^3 h(\epsilon'_{k,j}) \leq n'_k E^W_{x',s}(\tau_{l,k,j})\,.$$

This immediately results with

$$P_{x',k,j} \leq \mathbf{P}^W_{x',s}\left(\frac{1}{n'_k}\sum_{l=1}^{n'_k}\widehat{\tau}_{l,k,j} \leq 1-\delta\right)\,,$$

where $\widehat{\tau}_{l,k,j} := \tau_{l,k,j}/E^W_{x',s}(\tau_{l,k,j})$. Applying Lemma 2.3 as above, we see that for all k large enough,

$$\mathbb{E}^W_{x',s}(\widehat{\tau}_{l,k,j}) = 1\,, \qquad \mathbb{E}^W_{x',s}(\widehat{\tau}^2_{l,k,j}) \leq 10\,,$$

so that, with $\widetilde{\tau}_{l,k,j} := \widehat{\tau}_{l,k,j} - \mathbb{E}^W_{x',s}(\widehat{\tau}_{l,k,j})$ we have

$$P_{x',k,j} \leq \mathbf{P}^W_{x',s}\left(\frac{1}{n'_k}\sum_{l=1}^{n'_k}\widetilde{\tau}_{l,k,j} \leq -\delta\right)\,.$$

Since $\widetilde{\tau}_{l,k,j} \geq -1$, it follows that for all $0 < \theta < 1$,

$$\mathbb{E}^W_{x',s}(e^{-\theta\widetilde{\tau}_{l,k,j}}) \leq 1 + 2\theta^2\mathbb{E}^W_{x',s}(\widetilde{\tau}^2_{l,k,j}) \leq 1 + 20\theta^2 \leq e^{20\theta^2}\,.$$

Taking $\theta = \delta/40$, a standard application of Chebyscheff's inequality then shows that for some $\lambda = \lambda(a, \delta) > 0$, $C_1 < \infty$ and all $x' \in \mathcal{D}_k$, k, j

$$P_{x',k,j} \leq C_1 e^{-\lambda k^2 \log k} .$$

Since $|\mathcal{D}_k| \leq e^{C_2 k \log k}$ for some $C_2 < \infty$ and all k, it follows that

$$\sum_{k=1}^{\infty} \sum_{j=0}^{3k \log(k+1)} \sum_{x' \in \mathcal{D}_k} P_{x',k,j} \leq 3C_1 \sum_{k=1}^{\infty} k^2 e^{C_2 k \log k} e^{-\lambda k^2 \log k} < \infty .$$

The Borel-Cantelli lemma completes the proof of Lemma 4.3. $\square$

Proof of Lemma 4.2: The situation here is quite similar to the lower bound. Let now

$$\bar{\epsilon}'_k = \epsilon_k e^{-2/k^6}, \qquad \bar{\epsilon}''_{k-1} = \epsilon_{k-1} e^{1/k^6},$$

so that

$$\bar{\epsilon}'_k \leq \epsilon_k - \delta_k, \qquad \bar{\epsilon}''_{k-1} \geq \epsilon_{k-1} + \delta_k.$$

We now say that $x' \in \mathcal{D}_k$ is **upper k-successful** if there are at most $n_k + k$ excursions of W from $\partial D(x', \bar{\epsilon}''_{k-1})$ to $\partial D(x', \bar{\epsilon}'_k)$ prior to $\bar{\theta}$. We say it is **upper k,δ-successful** if it is upper k-successful and in addition,

$$a(1+\delta)\epsilon'^2_{k,j} |\log \epsilon'_{k,j}|^2 \geq \mu^{W'}_{\bar{\theta}'}(D(x', \epsilon'_{k,j})), \quad \forall j = 0, \ldots, 3k \log(k+1). \tag{4.7}$$

As above, we can derive Lemma 4.2 from the following lemma.

Lemma 4.4. *There exists a $k_0 = k_0(\delta, \omega, \omega')$ such that for all $k \geq k_0$ and $x' \in \mathcal{D}_k$, if x' is upper k,δ-successful then*

$$a^2(1+\delta)^4 h(\epsilon'_{k,j}) \geq \mathcal{I}_{\bar{\theta}, \bar{\theta}'}(D(x', \epsilon'_{k,j})), \quad \forall j = 0, 1, \ldots, 3k \log(k+1).$$

Since $0 \notin D(x, \epsilon_1)$ for all $x \in S$, with $n''_k = n_k + k$, the proof of Lemma 4.4, in analogy to that of Lemma 4.3, comes down to bounding

$$Q_{x',k,j} := \mathbf{P}^W_{x',us}\left(\sum_{l=1}^{n''_k} \tau_{l,k,j} \geq a^2(1+\delta)^4 h(\epsilon'_{k,j}) \right) ,$$

where the subscript x', us indicates that x' satisfies (4.7). We apply next Lemma 2.3 for $r_1 = \epsilon'_{k,j}$, $r_2 = \bar{\epsilon}'_k$, $r = \bar{\epsilon}''_{k-1}$ and $\rho(\cdot) = \mu^{W'}_{\bar{\theta}'}(x' + \cdot)$. Combining (2.19) and the condition (4.7), it follows that for all $k \geq k_3(a, \delta)$ and all $j = 0, \ldots, 3k \log(k+1)$,

$$E^W_{x',us}(\tau_{l,k,j}) \leq a(1+\delta)^2 \log(\bar{\epsilon}''_{k-1}/\bar{\epsilon}'_k)(\epsilon'_{k,j})^2 |\log \epsilon'_{k,j}|^2 ,$$

and that

$$n''_k \log(\bar{\epsilon}''_{k-1}/\bar{\epsilon}'_k) \leq a(1+\delta)|\log(\epsilon'_{k,j})|^2 .$$

Consequently,

$$a^2(1+\delta)^3 h(\epsilon'_{k,j}) \geq n''_k E^W_{x',us}(\tau_{l,k,j}),$$

and with $\log(\bar{\epsilon}''_{k-1}/\bar{\epsilon}'_k) \geq (1/3)\log(\bar{\epsilon}''_{k-1}/\epsilon'_{k,j})$, it follows that for some $c_2 > 0$ (for example, $c_2 = a\delta/3$ will do), and all such k, j,

$$Q_{x',k,j} \leq \mathbf{P}^W_{x',us}\left(\frac{1}{n''_k}\sum_{l=1}^{n''_k}\widetilde{\tau}_{l,k,j} \geq c_2\right) \leq e^{-c_2\lambda n''_k}\left(E^W_{x',us}\left(e^{\lambda\widetilde{\tau}_{1,k,j}}\right)\right)^{n''_k},$$

where now

$$\widetilde{\tau}_{l,k,j} := \widehat{\tau}_{l,k,j} - E^W_{x',us}(\widehat{\tau}_{l,k,j}), \quad \widehat{\tau}_{l,k,j} := \frac{\tau_{l,k,j}}{\log(\bar{\epsilon}''_{k-1}/\epsilon'_{k,j})\epsilon'^2_{k,j}|\log\epsilon'_{k,j}|^2}.$$

Applying Lemma 2.3, it follows from (2.17) and (2.18) that for some $C < \infty$, all $\lambda > 0$ small and k large enough,

$$\begin{aligned} E^W_{x',us}\left(e^{\lambda\widetilde{\tau}_{1,k,j}}\right) &= 1 + \sum_{n=2}^{\infty}\frac{\lambda^n}{n!}E^W_{x',us}\left(\widetilde{\tau}^n_{1,k,j}\right) \\ &\leq 1 + \sum_{n=2}^{\infty}\frac{(2\lambda)^n}{n!}E^W_{x',us}\left(\widehat{\tau}^n_{1,k,j}\right) \leq 1 + C\lambda^2. \end{aligned}$$

The proof of Lemma 4.4 then follows as in the proof of Lemma 4.3. □

4.2. **Intersection with stables.** Recall the sets F, C^X_a introduced in Section 3. In this subsection, we take $T \in (0,\infty)$ and $h(\epsilon) = \epsilon^\beta|\log\varepsilon|^3$. Lemma 3.4 is implied by the next two lemmas.

Lemma 4.5. *For every $\delta > 0$, if $x \in F \cap C^X_a$ then*

$$a^2(1-\delta)^5 \leq \limsup_{\epsilon\to 0}\mathcal{I}^{W,X}_{\bar{\theta},T}(D(x,\epsilon))/h(\epsilon). \tag{4.8}$$

Lemma 4.6. *For every $\delta > 0$, if $x \in F \cap C^X_a$ then*

$$\limsup_{\epsilon\to 0}\mathcal{I}^{W,X}_{\bar{\theta},T}(D(x,\epsilon))/h(\epsilon) \leq a^2(1+\delta)^5. \tag{4.9}$$

Proof of Lemma 4.5: Recall the notations $\epsilon_k, \epsilon_{k,j}$, etc. of Lemma 4.1, and say now that $x' \in \mathcal{D}_k$ is **lower k,δ-successful** if it is lower k-successful and

$$\Lambda a(1-\delta)(\epsilon'_{k,j})^\beta|\log\epsilon'_{k,j}| \leq \mu^X_T(D(x',\epsilon'_{k,j})),$$

for **some** $j \in \{0,1,\ldots,3k\log(k+1)\}$. In [4, Theorem 1.1] we show that a.s.

$$\lim_{\epsilon\to 0}\sup_{|x|\leq 2}\frac{\mu^X_T(D(x,\epsilon))}{\Lambda\epsilon^\beta|\log\epsilon|} = \beta < \infty.$$

Thus, following the arguments of the proof of Lemma 4.3, but now applying Lemma 2.3 for $\rho(\cdot) = \Lambda^{-1}\mu^X_T(x'+\cdot)$ with $\gamma_1 = \beta$ and $\gamma_2 = 1$ (instead of the scale $\gamma_1 = \gamma_2 = 2$ used in proving Lemma 4.3), we see that there exists a $k_0 = k_0(\delta,\omega,\omega')$ such that for all $k \geq k_0$ and $x' \in \mathcal{D}_k$, if x' is lower k,δ-successful then

$$a^2(1-\delta)^4 h(\epsilon'_{k,j}) \leq \mathcal{I}^{W,X}_{\bar{\theta},T}(D(x',\epsilon'_{k,j})), \tag{4.10}$$

for some $j \in \{0,1,\ldots,3k\log(k+1)\}$. If $x \in F$, then there exists a sequence of points $x_k \in \mathcal{D}_k$ with $|x_k - x| \le 2\epsilon_k/k^6$ such that x_k is lower k-successful. If further $x \in C_a^X$, then there exist a subsequence $k_n \to \infty$ and $j_n \in \{0,1,\ldots,3k_n\log(k_n+1)\}$, such that

$$\Lambda a(1-\delta)(\epsilon'_{k_n,j_n})^\beta |\log \epsilon'_{k_n,j_n}| \le \mu_T^X(D(x_{k_n},\epsilon'_{k_n,j_n}))\,.$$

Applying (4.10) and using the continuity of $h(\cdot)$, one concludes the proof of Lemma 4.5. □

Proof of Lemma 4.6: The proof is analogous to that of Lemma 4.2, now with $x' \in \mathcal{D}_k$, **upper k,δ-successful** if it is upper k-successful, and such that

$$\mu_T^X(D(x',\epsilon'_{k,j})) \le \Lambda a(1+\delta)(\epsilon'_{k,j})^\beta |\log \epsilon'_{k,j}|, \quad \forall j = 0,\ldots,3k\log(k+1).$$

Here again, the application of Lemma 2.3 is with $\gamma_1 = \beta$ and $\gamma_2 = 1$, otherwise using (2.18) and (2.19) as in the proof of Lemma 4.4. The task of completing the details is left to the reader. □

5. The coarse multi-fractal spectrum

Proof of Proposition 1.5: Fix $a \in (0,1)$ and let

$$C(\varepsilon,a^2) = \{x : \mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon)) \ge a^2\varepsilon^2(\log\varepsilon)^4\}\,.$$

With $\tilde\varepsilon_n$ as in (2.11), the bound (2.15) yields for some $c_i = c_i(\delta) < \infty$ and any $\eta > 0$,

$$\begin{aligned}\mathbf{P}(\mathcal{L}eb(C(\tilde\varepsilon_n,a^2)) \ge \tilde\varepsilon_n^\eta) &\le \tilde\varepsilon_n^{-\eta}\mathbb{E}(\mathcal{L}eb(C(\tilde\varepsilon_n,a^2))\\ &\le c_1\mathbb{E}|\mathcal{A}_n|\tilde\varepsilon_n^{2-\eta} \le c_2\tilde\varepsilon_n^{(1-10\delta)2a-\eta}\,.\end{aligned}$$

The Borel-Cantelli lemma and (2.12) then imply that

$$\liminf_{\varepsilon\to 0}\frac{\log\mathcal{L}eb(C(\varepsilon,a^2/(1-\delta)))}{\log\varepsilon} \ge 2a(1-10\delta)\,, \quad a.s.$$

Taking $\delta \to 0$ then yields the conclusion

$$\liminf_{\varepsilon\to 0}\frac{\log\mathcal{L}eb(C(\varepsilon,a^2))}{\log\varepsilon} \ge 2a\,, \quad a.s.$$

Turning to a complementary upper bound, fix $\delta > 0$ such that $a^2(1+\delta)^3 < 1$. Let $\epsilon_\delta = \epsilon\delta/(1+\delta)$, $C_\delta = C(\varepsilon/(1+\delta), a^2(1+\delta)^3)$ and $N(\epsilon)$ a (finite) maximal set of $x_i \in C_\delta$ such that $|x_i - x_j| > 2\epsilon_\delta$ for all $i \ne j$. Note that $\{D(x_i,\epsilon_\delta) : x_i \in N(\epsilon)\}$ are disjoint and if $x \in C_\delta$ then $D(x,\epsilon_\delta) \subset C(\varepsilon,a^2)$. Therefore,

$$\pi\epsilon_\delta^2|N(\epsilon)| \le \mathcal{L}eb(\bigcup_{x\in C_\delta} D(x,\epsilon_\delta)) \le \mathcal{L}eb(C(\varepsilon,a^2))\,.$$

With $d(\epsilon) = \log|N(\varepsilon)|/\log(1/\epsilon)$, we thus see that

$$\liminf_{\epsilon\to 0} d(\epsilon) \le 2 - \limsup_{\varepsilon\to 0}\frac{\log\mathcal{L}eb(C(\varepsilon,a^2))}{\log\varepsilon}\,. \tag{5.1}$$

Let

$$\mathsf{CThickInt}_{\geq a^2} = \{x \in D(0,1) : \liminf_{\varepsilon \to 0} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^4} \geq a^2\}\,, \tag{5.2}$$

and

$$\mathsf{CThickInt}_{\gamma,\geq a^2} = \{x \in D(0,1) : \inf_{\epsilon \leq \gamma} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^4} \geq a^2\}.$$

The sets $\mathsf{CThickInt}_{\gamma,\geq a^2}$ are monotone non-increasing in γ and

$$\mathsf{CThickInt}_{\geq a^2(1+\delta)^4} \subseteq \bigcup_n \mathsf{CThickInt}_{\gamma_n,\geq a^2(1+\delta)^3} \tag{5.3}$$

for any $\gamma_n \to 0$. Recall that $\mathcal{S}_\epsilon := \{D(x_i, 3\epsilon_\delta) \,:\, x_i \in N(\epsilon)\}$, forms a cover of C_δ, so à fortiori it is also a cover of $\mathsf{CThickInt}_{\varepsilon/(1+\delta),\geq a^2(1+\delta)^3}$. Fixing $\epsilon_n \downarrow 0$ it follows from (5.3) that $\cup_{n\geq m}\mathcal{S}_{\epsilon_n}$ is a cover of $\mathsf{CThickInt}_{\geq a^2(1+\delta)^4}$ by sets of maximal diameter $6\epsilon_m$. Hence, the η-Hausdorff measure of $\mathsf{CThickInt}_{\geq a^2(1+\delta)^4}$ is finite for any η such that

$$\sum_{n=1}^\infty |N(\epsilon_n)|\epsilon_n^\eta = \sum_{n=1}^\infty \epsilon_n^{\eta - d(\epsilon_n)} < \infty\,,$$

that is, whenever $\eta > \liminf_{\epsilon\to 0} d(\epsilon)$. Consequently, by (5.1)

$$\dim(\mathsf{CThickInt}_{\geq a^2(1+\delta)^4}) \leq \liminf_{\epsilon \to 0} d(\epsilon) \leq 2 - \limsup_{\varepsilon\to 0} \frac{\log \mathcal{L}eb(C(\varepsilon,a^2))}{\log \varepsilon}\,.$$

Since the set considered in (1.10) is contained in $\mathsf{CThickInt}_{\geq a^2}$, taking $\delta \to 0$ and using (1.10) yields that

$$\limsup_{\varepsilon\to 0} \frac{\log \mathcal{L}eb(C(\varepsilon,a^2))}{\log \varepsilon} \leq 2a\,, \quad a.s.\,,$$

as needed to complete the proof. ☐

6. Proof of Theorem 1.6: the upper bound

Throughout this section, fix $0 < r_1 \leq r$, let $\bar\theta_r = \inf\{s > 0 : |W_s| = r\}$, and define

$$\bar{\mathcal{I}}^{W,X} = \mathcal{I}^{W,X}_{\bar\theta_r}(D(0,r_1))\,.$$

Lemma 6.1. *For each $\delta > 0$ we can find $c < \infty$ such that for all $k \geq 1$, $r_1 \leq r/2 \leq 1$, and x_0, x_0' with $|x_0| = |x_0'| = r_1$*

$$\mathbb{E}^{x_0,x_0'}(\bar{\mathcal{I}}^{W,X}/r_1^\beta)^k \leq c(k!)^2\left((1+\delta)\log(r/r_1) + c\right)^k. \tag{6.1}$$

Proof of Lemma 6.1: It follows from (1.15) that

$$\begin{aligned} &\mathbb{E}^{x_0,x_0'}((\bar{\mathcal{I}}^{W,X})^k) \qquad (6.2)\\ &= k!(\pi\Lambda^{-1})^k \sum_\sigma \int_{D(0,r_1)^k} \prod_{j=1}^k g_r(y_{\sigma(j-1)}, y_{\sigma(j)}) u^0(y_{j-1} - y_j)\,dy_j \end{aligned}$$

where the sum runs over all permutations σ of $\{1,\dots,k\}$ and we use the convention that $y_{\sigma(0)} = x_0$ and $y_0 = x'_0$ (see (2.2) for a similar formula in case of two Brownian motions).

Recalling (2.3), after scaling in r_1, (6.2) can be bounded above by

$$(k!)^2(\Lambda^{-1})^k r_1^{k\beta} \sum_{l=0}^{k} \big(\log(r/r_1) + c_o\pi\big)^l \sum_{|\mathcal{A}|=k-l} J_{\mathcal{A}} \tag{6.3}$$

where the sum goes over all subsets $\mathcal{A}$ of $\{1,\dots,k\}$ of cardinality $k-l$, and

$$J_{\mathcal{A}} := \sup_{\sigma, |y_0|=|y_{\sigma(0)}|=1} \int_{D(0,1)^k} \prod_{j\in\mathcal{A}} \Big| \log\big(|y_{\sigma(j)} - y_{\sigma(j-1)}|\big)\Big| \prod_{j=1}^{k} u^0(y_{j-1}-y_j)\, dy_j\,.$$

Fix $p>1$ and $q = p/(p-1)$, and write

$$\bar{L}_q = \max_{m=1,2} \sup_{x\in D(0,1)} \left(\int_{D(0,1)} \Big| \log\big(|y-x|\big)\Big|^{mq} dy \right)^{1/mq}.$$

Let Λ_p denotes the $L^2(D(0,1),dx)$ norm of the operator $K^{(p)}f(x) = \int_{D(0,1)} u^0(x-y)^p f(y)dy$ with kernel $(u^0)^p$. Note first that for some fixed r, we have that

$$C_p := \sup_{|y_0|=1, ||f||_2\le 1} |(K^{(p)})^r f(y)| < \infty\,.$$

Noting that each variable y_j appears in at most a pair of logarithmic factors, we have using Hölder's inequality that

$$J_{\mathcal{A}} \le \bar{L}_q^{k-l} \pi^{l/q} (C_p \Lambda_p^{(k-r)} \pi^{1/2})^{1/p}\,.$$

Hence, by (6.3), we have that for some finite C'_p,

$$\mathbb{E}^{x_0,x'_0}((\bar{\mathcal{I}}^{W,X})^k) \le C'_p (k!)^2 \left(\frac{\Lambda_p^{1/p}}{\Lambda}\right)^k r_1^{k\beta} \sum_{l=0}^{k} \Big(\pi^{1/q}\log(r/r_1) + \pi^{1+1/q} c_o\Big)^l \binom{k}{l} \bar{L}_q^{k-l}$$

Taking now p large enough such that $\Lambda_p^{1/p}\pi^{1/q} < (1+\delta)\Lambda$ (which is possible since for $p>1$ small enough, $\Lambda_p < \infty$, and using interpolation, $\Lambda_p \to_{p\to 1} \Lambda$), we get (6.1) with $c = (c_o \pi^{1+1/q} + \bar{L}_q) \vee C'_p$. ☐

The next lemma then follows by the same arguments as in the proof of Lemma 2.2, now with $F = \sqrt{4\mathcal{I}_{\bar{\theta}_2}^{W,X'}(D(0,\varepsilon))/(\varepsilon^\beta |\log\varepsilon|(1+4\delta))}$.

Lemma 6.2. *For any $\delta > 0$ we can find $c, y_0 < \infty$ and $\varepsilon_0 > 0$ so that for all $\varepsilon \le \varepsilon_0$ and $y \ge y_0$*

$$P^{x_0,x'_0}(\mathcal{I}_{\bar{\theta}_2}^{W,X}(D(0,\varepsilon)) \ge y^2 \epsilon^\beta |\log\epsilon|) \le c\exp(-(1-\delta)2y) \tag{6.4}$$

for all x_0, x'_0 with $|x_0| = |x'_0| = \varepsilon$.

The upper bound in Theorem 1.6 is now derived via the same line of reasoning used in [4] (following [2, Section 5]). The details are omitted. ☐

7. Proof of Theorem 1.2

Throughout this section, fix $0 < r_1 \le r$, let $\bar\theta_r = \inf\{s > 0 : |W_s| = r\}$, and define

$$\bar\mu = \frac{\mu^W_{\bar\theta_r}(K(0,r_1))}{|K|/\pi}\,.$$

Lemma 7.1. *We can find $c < \infty$ such that for all $k \ge 1$, $r_1 \le r_2 \le r/2 \le 1$, and x_0 with $|x_0| = r_2$*

$$\mathbb{E}^{x_0}(\bar\mu/r_1^2)^k \le k!\,(\log(r/r_1) + c)^k\,. \tag{7.1}$$

Furthermore,

$$\mathbb{E}^{x_0}(\bar\mu/r_1^2) = (\log(r/r_2) \pm c)\,. \tag{7.2}$$

Proof of Lemma 7.1: Let $g_r(x,y)$ denote the Green's function for $D(0,r)$, i.e. the 0-potential density for planar Brownian motion killed when it first hits $D(0,r)^c$. We have

$$\mathbb{E}^{x_0}(\bar\mu^k) = k!(\frac{\pi}{|K|})^k \int_{K(0,r_1)^k} \prod_{j=1}^k g_r(y_{j-1},y_j)\,dy_j \tag{7.3}$$

where we use the convention that $y_{\sigma(0)} = x_0$.

Thus, to prove (7.2) we must show that

$$\frac{\pi}{|K|}\int_{K(0,r_1)} g_r(x_0,y)\,dy = (\log(r/r_2) \pm c)\,r_1^2 \tag{7.4}$$

for all x_0 with $|x_0| = r_2$. We will also show that

$$\sup_{x\in K(0,r_1)} \frac{\pi}{|K|}\int_{K(0,r_1)} g_r(x,y)\,dy \le (\log(r/r_1) + c)\,r_1^2. \tag{7.5}$$

If we use this inductively for the $dy_k, dy_{k-1}, \dots, dy_2$ integrals in (7.3), and then use (7.4) for the dy_1 integral, we will establish (7.1). It follows from (2.3) that the left hand side in (7.5) is bounded by

$$\begin{aligned}
&\sup_{x\in K(0,r_1)} \frac{1}{|K|}\int_{K(0,r_1)} \left(\log(r/r_1) + \log\left(\frac{r_1}{|x-y|}\right) + \pi c_o\right) dy \\
&= (\log(r/r_1) + \pi c_o)\,r_1^2 + r_1^2 \sup_{x\in K} \frac{1}{|K|}\int_K \log\left(\frac{1}{|x-y|}\right) dy.
\end{aligned}$$

and (7.5) follows.

Turning to (7.4), as above we can write the left hand side as

$$\begin{aligned}
&\frac{1}{|K|}\int_{K(0,r_1)} \left(\log(r/r_2) + \log\left(\frac{r_2}{|x_0-y|}\right) \pm \pi c_o\right) dy \\
&= (\log(r/r_2) \pm \pi c_o)\,r_1^2 + \frac{1}{|K|}\int_{K(0,r_1)} \log\left(\frac{r_2}{|x_0-y|}\right) dy.
\end{aligned}$$

When $r_1/r_2 \le 1/2$, we have that $|\log \frac{r_2}{|x_0-y|}|$ is bounded on $K(0,r_1)$ so that (7.4) follows in that case. When $1/2 \le r_1/r_2 \le 1$ we can use

$$\int_{K(0,r_1)} |\log\left(\frac{r_2}{|x_0-y|}\right)|\,dy \le \int_{D(0,r_2)} |\log\left(\frac{r_2}{|x_0-y|}\right)|\,dy$$

and (7.4) follows as above.

Continuing as in proof of Lemma 2.2, now with $F = \bar{\mu}/\{r_1^2|\log r_1|(1+\delta)\}$, this easily implies:

Lemma 7.2. *For any $\delta > 0$ we can find $c, y_0 < \infty$ and $\varepsilon_0 > 0$ so that for all $\varepsilon \le \varepsilon_0$ and $y \ge y_0$*

$$P^{x_0}(\mu^W_{\bar{\theta}_2}(K(0,\varepsilon)) \ge (|K|/\pi)y\epsilon^2|\log\epsilon|) \le c\exp(-(1-\delta)y)$$

for all x_0 with $|x_0| = \varepsilon$.

We first turn to the proof of the upper bounds. The proof requires a slight adaptation of the technique of [5, Section 2], because it is not true in general that $\epsilon K \subset \epsilon' K$ if $\epsilon < \epsilon'$.

Let $r_n = (1-\delta)^n$, and let $\{Q_{i,n}\}$ be a tiling of $[-1,1]^2$ by squares of side r_n, and let $Q^j_{i,n} = j + Q_{i,n}$ where $j = (j_1, j_2)$ and $j_i \in \{0, r_n/M, \dots, (M-1)r_n/M\}$ with $M = 2/\delta$. We first have the following immediate corollary of Lemma 7.2.

Corollary 7.3. *There exists an $n_0 = n_0(\omega,\delta)$ such that for all $n > n_0$,*

$$\#\{(i,j) : \mu^W_{\bar{\theta}}(Q^j_{i,n}) \ge \frac{ar_n^2|\log r_n|^2}{\pi}\} \le r_n^{a(1-\delta)-2}\,.$$

Our use of Corollary 7.3 is as follows. By the assumption on the boundary of K, $|K| = |\mathrm{cl}K|$, and hence it is enough to prove the upper bounds for compact K. Thus, let $K \subset D(0,1)$ be compact. Fix $\delta > 0$. Cover K by a finite number (say k) of squares Q_i, with $\sum_{i=1}^k |Q_i| \le |K|(1+\delta/2)$. Note that $K(x,\epsilon) \subset \cup_{i=1}^k (x+\epsilon Q_i)$, and hence

$$\mu^W_{\bar{\theta}}(K(x,\epsilon)) \le \sum_{i=1}^k \mu^W_{\bar{\theta}}(x+\epsilon Q_i)\,.$$

Hence,

$$\{x : \limsup_{\epsilon\to 0} \frac{\pi\mu^W_{\bar{\theta}}(K(x,\epsilon))}{|K|\epsilon^2|\log\epsilon|^2} \ge a(1+\delta/2)\} \subset \bigcup_{i=1}^k \{x : \limsup_{\epsilon\to 0}\frac{\pi\mu^W_{\bar{\theta}}(Q_i(x,\epsilon))}{|Q_i|\epsilon^2|\log\epsilon|^2} \ge a\}\,.$$

Hence, it is enough to prove the upper bounds on the dimension for an arbitrary square Q of side b. Note however that then, with $r_{n+1} \le \epsilon b \le r_n$, and any x, there exist i,j such that $Q(x,\epsilon) \subset Q^j_{i,n-1}$, whereas

$$\{x : \frac{\pi\mu^W_{\bar{\theta}}(Q(x,\epsilon))}{|Q|\epsilon^2|\log\epsilon|^2} \ge a\} \subset \{\bigcup_{i,j} Q^j_{i,n-1} : \frac{\pi\mu^W_{\bar{\theta}}(Q^j_{i,n-1})}{|Q|\epsilon^2|\log\epsilon|^2} \ge a\}$$

Given Corollary 7.3, and using that $r_{n+1}/r_{n-1} = (1-\delta)^2$, one gets immediately that

$$\dim\{x : \limsup_{\epsilon\to 0} \frac{\pi\mu_{\bar\theta}^W(Q_i(x,\epsilon))}{|Q_i|\epsilon^2|\log\epsilon|^2} \geq a\} \leq 2 - a(1-\delta)^3 + 2\delta\,, \quad a.s.,$$

which yields the required upper bound on the dimension. The estimate (1.6) is similarly proved.

Surprisingly, the lower bounds in Theorem 1.2 require no further computations. First, we note the following slight adaptation of [5, Lemma 3.1].

Lemma 7.4. *Let $Q \subset D(0,1)$ be a fixed square, not necessarily centered. Then there exist $\delta(\epsilon) = \delta(\epsilon,\omega,Q) \to 0$ a.s. such that for all m and all $x \in D(0,1/2) \setminus D(0,1/8)$, if x is m-perfect then*

$$a - \delta(\epsilon) \leq \frac{\pi\mu_{\bar\theta}^W(Q(x,\epsilon))}{\epsilon^2(\log\epsilon)^2|Q|} \leq a + \delta(\epsilon)\,, \ \forall\epsilon \geq \epsilon_m\,.$$

Proof of Lemma 7.4: Simply follow the arguments of [5, Section 6] using Lemma 7.1 wherever [5, Lemma 2.1] is used there. $\square$

Let

$$E = \{x \in D(0,1/2) \setminus D(0,1/8) : \exists x_n \to x \text{ such that } x_n \text{ is } n\text{-perfect}\}\,.$$

Recall that

$$\dim(E) = 2 - a\,, \quad a.s. \tag{7.6}$$

It follows from Lemma 7.4 that for any fixed square Q, a.s.

$$\limsup_{\epsilon\to 0} \frac{\pi\mu_{\bar\theta}^W(Q(x,\epsilon))}{\epsilon^2(\log\epsilon)^2|Q|} = a\,, \quad \forall x \in E\,.$$

Consequently, for any compact $F \subset D(0,1)$, a.s.

$$\limsup_{\epsilon\to 0} \frac{\pi\mu_{\bar\theta}^W(F(x,\epsilon))}{\epsilon^2(\log\epsilon)^2|F|} \leq a\,, \quad \forall x \in E\,. \tag{7.7}$$

(Just cover F by a finite number of squares $Q_i \subset D(0,1)$ with $\sum_i |Q_i| \leq |F|(1+\delta)$, considering first the lim sup as $\epsilon \to 0$, then taking $\delta \to 0$). For K satisfying the assumptions of the theorem we have from the above that

$$\limsup_{\epsilon\to 0} \frac{\pi\mu_{\bar\theta}^W(K(x,\epsilon))}{\epsilon^2(\log\epsilon)^2|K|} \leq \limsup_{\epsilon\to 0} \frac{\pi\mu_{\bar\theta}^W(\mathrm{cl}K(x,\epsilon))}{\epsilon^2(\log\epsilon)^2|K|} \leq a\,, \quad \forall x \in E\,. \tag{7.8}$$

Recall that, for $F = \mathrm{cl}(D(0,1) \setminus K)$ and all x,

$$\begin{aligned} \mu_{\bar\theta}^W(K(x,\epsilon)) &= \mu_{\bar\theta}^W(D(x,\epsilon)) - \mu_{\bar\theta}^W(D(x,\epsilon) \setminus K(x,\epsilon)) \\ &\geq \mu_{\bar\theta}^W(D(x,\epsilon)) - \mu_{\bar\theta}^W(F(x,\epsilon))\,, \end{aligned}$$

whereas by [5, Lemma 3.1], a.s.

$$\lim_{\epsilon\to 0} \frac{\mu_{\bar\theta}^W(D(x,\epsilon))}{\epsilon^2(\log\epsilon)^2} = a\,, \quad \forall x \in E\,.$$

Noting that $|F| = |D(0,1) \setminus K|$ by our assumption on the boundary of K, and using (7.7), we conclude that a.s.

$$\liminf_{\epsilon\to 0} \frac{\mu_\theta^W(K(x,\epsilon))}{\epsilon^2(\log\epsilon)^2} \ge a - \frac{a|D(0,1)\setminus K|}{\pi} = \frac{a|K|}{\pi}, \quad \forall x \in E .$$

When combined with (7.8), this implies that a.s.

$$\lim_{\epsilon\to 0} \frac{\pi\mu_\theta^W(K(x,\epsilon))}{\epsilon^2(\log\epsilon)^2|K|} = a , \quad \forall x \in E ,$$

so (7.6) yields the required lower bound on the dimension of the sets in (1.7). □

8. Proof of Theorem 1.1

Recall the local times

$$L_n(x) := \sum_{k=0}^{n} \mathbf{1}_{\{X_k = x\}} , \qquad L'_n(x) := \sum_{k=0}^{n} \mathbf{1}_{\{X'_k = x\}} , \quad (x \in \mathbf{Z}^2)$$

of the simple random walks $\{X_k\}$ and $\{X'_k\}$ and let $L_n^{X,X'}(x) = L_n^X(x)L_n^{X'}(x)$. We next compute upper bounds on $M_n^{X,X'}(b) = |\{x \in \mathbf{Z}^2 : L_n^X(x)L_n^{X'}(x) \ge b\}|$ and $T_n^{X,X'}$. To this end, using the Markov property and translation invariance of SRW in $\mathbf{Z}^2$, as well as the bound of [5, (5.11)], we have for any $\delta > 0$, all $n \ge n_0(\delta)$, $\alpha > 0$ and $x \in \mathbf{Z}^2$,

$$\mathbf{P}[L_n(x) \ge \alpha(\log n)^2] \le \mathbf{P}[L_n(0) \ge \alpha(\log n)^2] \le n^{-(1-\delta)\pi\alpha}.$$

Thus, fixing $0 < \delta < b$ and $K > (2b-\delta)/\delta$, by the independence of $\{X_k\}$ and $\{X'_k\}$, we have for $n \ge n_0(\delta)$ and all $x \in \mathbf{Z}^2$,

$$\begin{aligned}(8.1)\quad & \mathbf{P}(L_n^{X,X'}(x) \ge b^2(\log n)^4) \\ &\le \sum_{i=1}^{K} \mathbf{P}\Big[\frac{L_n(x)}{(\log n)^2} \ge (i-1)\delta,\ \frac{L'_n(x)}{(\log n)^2} \ge \frac{b^2}{i\delta}\Big] + \mathbf{P}\Big[\frac{L_n(x)}{(\log n)^2} \ge K\delta\Big] \\ &\le \sum_{i=1}^{K} n^{-(1-\delta)\pi((i-1)\delta + b^2/(i\delta))} + n^{-(1-\delta)\pi K\delta} \le 2Kn^{-(1-\delta)\pi(2b-\delta)}\end{aligned}$$

(as $s + b^2/s \ge 2b$ for all $s > 0$).

Let $R_n = \max_{k\le n}(|X_k| \vee |X'_k|)$, noting that for some $c > 0$ and all n large,

$$\mathbf{P}(R_n \ge n^{1/2+\delta}) \le e^{-cn^\delta} . \tag{8.2}$$

Fixing $0 < \delta < b$ set $\gamma = 1 + 4\delta - (1-\delta)\pi(2b-\delta)$. Then, by (8.1) and (8.2) we have for all n large enough,

$$\begin{aligned} &\mathbf{P}(M_n^{X,X'}(b^2(\log n)^4) \geq n^\gamma) \\ &\leq \mathbf{P}(R_n \geq n^{1/2+\delta}) + n^{-\gamma}\mathbb{E}(M_n^{X,X'}(b^2(\log n)^4);\, R_n < n^{1/2+\delta}) \\ &\leq e^{-cn^\delta} + n^{-\gamma} \sum_{|x| \leq n^{1/2+\delta}} \mathbf{P}(L_n^{X,X'}(x) \geq b^2(\log n)^4) \leq n^{-\delta}\,. \end{aligned} \tag{8.3}$$

For $b < 1/(2\pi)$, taking $\delta \downarrow 0$ it follows by (8.3) and the Borel-Cantelli lemma that almost surely

$$\limsup_{m\to\infty} \frac{\log M_{n_m}^{X,X'}(b^2(\log n_m)^4)}{\log n_m} \leq 1 - 2\pi b, \tag{8.4}$$

on the subsequence $n_m = 2^m$. By the monotonicity of $n \mapsto \log L_n^{X,X'}(x)$ and $n \mapsto \log n$, one easily checks that (8.4) holds also when replacing n_m with m, yielding the upper bound in (1.4). In case $b > 1/(2\pi)$ we note that $\gamma = \gamma(\delta) < 0$ when $\delta > 0$ is small enough, so (8.3) implies that for all n large enough,

$$\mathbf{P}(T_n^{X,X'} \geq b^2(\log n)^4) = \mathbf{P}(M_n^{X,X'}(b^2(\log n)^4) \geq 1) \leq n^{-\delta}\,.$$

Therefore, taking $b \downarrow 1/(2\pi)$, it follows by the Borel-Cantelli lemma that almost surely,

$$\limsup_{m\to\infty} \frac{T_{n_m}^{X,X'}}{(\log n_m)^4} \leq \frac{1}{4\pi^2}\,,$$

on the subsequence $n_m = 2^m$. The monotonicity of $n \mapsto T_n^{X,X'}$ and $n \mapsto (\log n)^4$ allows us to replace n_m with m, leading to the upper bound of (1.3).

It suffices to prove the complementary lower bounds for (1.4), because the lower bound in (1.3) then directly follows. As in [5], the proof uses the strong approximation results of [6].

For any $A, A' \subseteq \mathbf{Z}$ let

$$L_{A\times A'}^{X,X'}(z) = \#\{(i,j) \in A \times A' : X_i = X'_j = z\} \tag{8.5}$$

$$M\left\{A \times A',\, b\right\} = \#\{z \in \mathbf{Z}^2 : L_{A\times A'}^{X,X'}(z) \geq b\} \tag{8.6}$$

so that $M_n^{X,X'}(b) = M\left\{[0,n]^2,\, b\right\}$. Let $n_{j,i} = j^{8i}e^{j^2}$ and $\Delta n_{j,i} = n_{j,i} - n_{j,i-1}$. Fix $\epsilon > 0$. We claim that there exists some $j_0 = j_0(\omega) < \infty$ a.s. such that for all $j \geq j_0$

$$\max_{1\leq i\leq j} M\left\{[n_{j,i-1}, n_{j,i}]^2,\, b^2(\log \Delta n_{j,i})^4\right\} \geq (\Delta n_{j,i})^{1-2\pi b-\epsilon}. \tag{8.7}$$

Assuming this for the moment, we see from (8.7) that for any $n_{j,j} \leq n \leq n_{j+1,j+1}$ with j sufficiently large

$$M\left\{[0,n]^2,\, (1-\epsilon)^2 b^2(\log n)^4\right\} \geq n^{(1-\epsilon)^2(1-2\pi b-\epsilon)} \tag{8.8}$$

so that, replacing b by $b/(1-\epsilon)$ we have

$$\liminf_{n\to\infty} \frac{\log M_n^{X,X'}(b^2(\log n)^4)}{\log n} \geq (1-\epsilon)^2(1-\frac{2\pi b}{1-\epsilon}-\epsilon) \qquad a.s. \tag{8.9}$$

and taking $\epsilon \to 0$ completes the proof of the lower bound of Theorem 1.1 subject only to (8.7) which we now establish.

Note that with $\Psi_{j,i} = \{|X_{n_{j,i-1}}| \vee |X'_{n_{j,i-1}}| \leq \sqrt{\Delta n_{j,i}}/\log \Delta n_{j,i}\}$ we have

$$\begin{aligned} &\mathbf{P}\left(\max_{1\leq i\leq j} M\left\{[n_{j,i-1}, n_{j,i}]^2,\, b^2(\log \Delta n_{j,i})^4\right\} \leq (\Delta n_{j,i})^{1-2\pi b-\epsilon}\right) \\ &\leq j \max_{1\leq i\leq j} \mathbf{P}(\Psi^c_{j,i}) \\ &+\mathbf{P}\left(\bigcap_{1\leq i\leq j}\left\{M\left\{[n_{j,i-1}, n_{j,i}]^2,\, b^2(\log \Delta n_{j,i})^4\right\} \leq (\Delta n_{j,i})^{1-2\pi b-\epsilon}\,;\, \Psi_{j,i}\right\}\right) \end{aligned} \tag{8.10}$$

We will show that this is summable in j so that (8.7) will follow by the Borel-Cantelli Lemma. By [17, Theorem 17.5], (which is essentially the Central Limit Theorem), for some $c > 0$ and all j sufficiently large

$$\mathbf{P}(|X_{n_{j,i-1}}| > \sqrt{\Delta n_{j,i}}/\log \Delta n_{j,i}) \leq \mathbf{P}(\frac{|X_{n_{j,i-1}}|}{\sqrt{n_{j,i-1}}} > j) \leq e^{-cj} \tag{8.11}$$

so that the first term on the right hand side of (8.10) is summable in j. On the other hand, since

$$\begin{aligned} &M\left\{[n_{j,i-1}, n_{j,i}]^2,\, b^2(\log \Delta n_{j,i})^4\right\} \\ &= M\left\{[0, \Delta n_{j,i}]^2,\, b^2(\log \Delta n_{j,i})^4\right\} \circ (\theta_{n_{j,i-1}},\, \theta'_{n_{j,i-1}}), \end{aligned}$$

the Markov property together with the following lemma will bound the second term on the right hand side of (8.10) by $(1-p_0)^j$ which is also summable in j. Thus, with $D_0(r) = D(0,r) \cap \mathbf{Z}^2$, (8.7) and consequently the proof of the theorem is reduced to the following lemma.

Lemma 8.1. *For any $\epsilon > 0$ we can find $p_0 > 0$ such that for all n sufficiently large and all $x, x' \in D_0(\sqrt{n}/\log n)$*

$$\mathbf{P}^x \times \mathbf{P'}^{x'}\left(M_n^{X,X'}(b^2(\log n)^4) \geq n^{1-2\pi b-\epsilon}\right) \geq p_0 > 0. \tag{8.12}$$

□

Proof of Lemma 8.1. For any $B \subseteq \mathbf{Z}^2$, let

$$M\left\{A \times A',\, B,\, b\right\} = \#\{z \in B : L^{X,X'}_{A\times A'}(z) \geq b\} \tag{8.13}$$

and note that if $\tau_B = \inf\{i \geq 0 : X_i \in B\}$, $\tau'_B = \inf\{i \geq 0 : X'_i \in B\}$ we have

$$M\left\{A \times A',\, B,\, b\right\} = M\left\{(A-\tau_B) \times (A'-\tau'_B),\, B,\, b\right\} \circ (\theta_{\tau_B},\, \theta'_{\tau'_B}). \tag{8.14}$$

We will use the abbreviation $\widetilde{b} = 1 - 2\pi b$. Let $\tau_n = \tau_{D_0^c(2\sqrt{n}/\log n)}$ and $\tau_n' = \tau'_{D_0^c(2\sqrt{n}/\log n)}$. By (8.14) and the Markov property

$$\mathbf{P}\times\mathbf{P}'\left(M\left\{[0,n]^2,\, D_0^c(2\sqrt{n}/\log n),\, b^2(\log n)^4\right\} \geq n^{\widetilde{b}-\epsilon}\right) \tag{8.15}$$

$$\leq \mathbb{E}\times\mathbb{E}'\left\{\mathbf{P}^{X_{\tau_n}}\times\mathbf{P}'^{X'_{\tau_n}}\left(M\left\{[0,n]^2,\, D_0^c(2\sqrt{n}/\log n),\, b^2(\log n)^4\right\} \geq n^{\widetilde{b}-\epsilon}\right)\right\}$$

$$= \sum_{y,y'} H_n(0,y)H_n(0,y')$$

$$\mathbf{P}^{y}\times\mathbf{P}'^{y'}\left(M\left\{[0,n]^2,\, D_0^c(2\sqrt{n}/\log n),\, b^2(\log n)^4\right\} \geq n^{\widetilde{b}-\epsilon}\right)$$

where $H_n(x,y) = \mathbf{P}^x(X_{\tau_n} = y)$ is a harmonic measure. By Harnack's inequality, [12, Theorem 1.7.2], for some $C < \infty$ and all n, y

$$H_n(0,y) \leq C \inf_{x\in D_0(\sqrt{n}/\log n)} H_n(x,y) \tag{8.16}$$

Using (8.14) and the Markov property again shows that uniformly in $x, x' \in D_0(\sqrt{n}/\log n)$

$$\begin{aligned} &\mathbf{P}\times\mathbf{P}'\left(M\left\{[0,n]^2,\, D_0^c(2\sqrt{n}/\log n),\, b^2(\log n)^4\right\} \geq n^{\widetilde{b}-\epsilon}\right) \\ &\leq C\mathbf{P}^x\times\mathbf{P}'^{x'}\left(M\left\{[0,n+\tau_n]\times[0,n+\tau_n'],\, b^2(\log n)^4\right\} \geq n^{\widetilde{b}-\epsilon}\right) \\ &\leq C\mathbf{P}^x\times\mathbf{P}'^{x'}\left(M\left\{[0,(1+\epsilon)n]^2,\, b^2(\log n)^4\right\} \geq n^{\widetilde{b}-\epsilon}\right) \\ &\qquad + C\mathbf{P}^x(\tau_n > \epsilon n) + C\mathbf{P}^{x'}(\tau_n' > \epsilon n). \end{aligned} \tag{8.17}$$

Since $\mathbf{P}^x(\tau_n > \epsilon n) = \mathbf{P}^x(\max_{j\leq \epsilon n}|X_j| < 2\sqrt{n}/\log n)$, we see from the last line of [17, Theorem 17.5] that for n sufficiently large the last line of (8.17) is negligible, so that given the next lemma, we obtain Lemma 8.1 after some adjustment of b and ϵ. □

Lemma 8.2. *For any $\epsilon > 0$ we can find $p_0 > 0$ such that for all n sufficiently large*

$$\mathbf{P}\times\mathbf{P}'\left(M\left\{[0,n]^2,\, D_0^c(2\sqrt{n}/\log n),\, b^2(\log n)^4\right\} \geq n^{1-2\pi b-3\epsilon}\right) \geq p_0. \tag{8.18}$$

Proof of Lemma 8.2 This lemma will be derived from results about Brownian motion by using strong approximation. However, results about $M\left\{[0,n]^2,\, D_0^c(2\sqrt{n}/\log n),\, b^2(\log n)^4\right\}$ are not easy to obtain directly by strong approximation since $L^{X,X'}_{[0,n]^2}(z)$ does not correspond to a functional of Brownian motion. Instead, we will derive results about $L^{X,X'}_{[0,n]^2}(z)$ from results about excursions of random walks between concentric discs, and it is such results which can be obtained from our work on Brownian thick points. To this end we introduce notation which is meant to simplify the connection with Brownian motion.

Fix $a<1$ and $\delta>0$ small. Set $m_k=ak^2$ and $k(n)=[(1/2-\delta)\log n]$ so that $e^{-k(n)}\geq n^{\delta-1/2}$. Let $D_z(r)=D(z,r)\cap\mathbf{Z}^2$ denote the disc of radius r in $\mathbf{Z}^2$ centered at z and define its boundary $\partial D_z(r)=\{z'\notin D_z(r):|z'-y|=1$ for some $y\in D_z(r)\}$. For any fixed $K<\infty$ let

$$r_n=(1+3\delta)e^{-k(n)}\sqrt{n/(2K)},\qquad R_n=(1-3\delta)e^{-k(n)+1}\sqrt{n/(2K)}$$

We say that $z\in\mathbf{Z}^2$ is n,δ**-admissible** if at least $(1-2\delta)m_{k(n)}$ excursions between $\partial D_z(r_n)$ and $\partial D_z(R_n)$ are completed by both $\{X_i\}$ and $\{X'_i\}$ for $i\leq n$. Let

$$I_{n,\delta}:=\{z\in D_0(\sqrt{n/(2K)})\cap D_0^c(2\sqrt{n}/\log n):z\text{ is }n,\delta\text{-admissible}\}\,.$$

Lemma 8.2 now follows from the next lemma if we take $a=2\pi b/(1-\epsilon)$ and ϵ, δ sufficiently small. ☐

Lemma 8.3. *For any $\epsilon>0$ we can find $K_0<\infty$ and $\delta_0>0$ such that for all $K>K_0$ and $\delta<\delta_0$*

$$\liminf_{n\to\infty}\mathbf{P}\times\mathbf{P}'\left(|I_{n,\delta}|\geq n^{1-a-\epsilon}\right)\geq 1/2 \tag{8.19}$$

and

$$\lim_{n\to\infty}\mathbf{P}\times\mathbf{P}'(\inf_{x\in I_{n,\delta}}L_n^{X,X'}(x)\leq\frac{(1-\epsilon)^2a^2}{4\pi^2}(\log n)^4)=0\,. \tag{8.20}$$

Proof of Lemma 8.3: We first prove (8.20). Fixing n, let σ_z denote the time it takes $\{X_i\}$ to complete $(1-2\delta)m_{k(n)}$ excursions between $\partial D_z(r_n)$ and $\partial D_z(R_n)$, after first hitting $\partial D_z(R_n)$, and denote by $Y_j(z)$ the occupation measure of z by $\{X_i\}$ during its j-th such excursion. Thus, fixing the starting point of the j-th excursion, $Y_j(z)$ is distributed like $L_{T_{R_n}}(0)$ where T_{R_n} is the first hitting time of $\partial D_0(R_n)$ (when starting at the corresponding $z'\in\partial D_0(r_n)$). We note that

$$\mathbb{E}^0\left(\left\{L_{T_{R_n}}(0)\right\}^k\right)=k!\,\{G_{R_n}(0,0)\}^k\qquad k=1,2,\dots \tag{8.21}$$

where $G_{R_n}(x,y)$ is Green's function for $D_0(R_n)$. This is a simple case of Kac's moment formula, see [7]. We will only need $k=1$ and 2. We will use the abbreviation $G_n=G_{R_n}(0,0)$.

Set $\lambda=\delta^{1/4}$, $\ell_n=(1-2\lambda)^2/(\delta\log n)$. As

$$L_{\sigma_z}(z)\geq\sum_{j=1}^{(1-2\delta)m_{k(n)}}Y_j(z)\,,$$

it follows by the strong Markov property of $\{X_i\}$ at the start of each of these excursions that

$$
\begin{aligned}
(8.22)\qquad & \sup_{y\in\mathbf{Z}^2}\mathbf{P}^y\left(\frac{L_{\sigma_z}(z)}{G_n}\le(1-2\delta)m_{k(n)}\ell_n\right)\\
&\le e^{\lambda(1-2\delta)m_{k(n)}\ell_n}\sup_{y\in\mathbf{Z}^2}\mathbb{E}^y(e^{-\lambda\frac{L_{\sigma_z}(z)}{G_n}})\\
&\le\left[e^{\lambda\ell_n}\sup_{z'\in\partial D_z(r_n)}\mathbb{E}^{z'}(e^{-\lambda L_{T_{R_n}}(z)/G_n})\right]^{(1-2\delta)m_{k(n)}}\\
&=\left[e^{\lambda\ell_n}\sup_{z\in\partial D_0(r_n)}\mathbb{E}^{z}(e^{-\lambda L_{T_{R_n}}(0)/G_n})\right]^{(1-2\delta)m_{k(n)}},
\end{aligned}
$$

with the last identity following from the translation invariance of SRW. We first study the quantity appearing in the the first line of (8.22). By [12, Theorem 1.6.6], for all n large enough,

$$(8.23)\qquad G_{R_n}(0,0)=\frac{2}{\pi}\log R_n+O(1)=\delta\frac{2}{\pi}\log n+O(1).$$

Hence for n large enough,

$$
\begin{aligned}
(8.24)\qquad (1-2\delta)m_{k(n)}\ell_nG_n &\ge (1-2\delta)(1-2\delta^{1/4})^3\frac{2}{\pi}ak^2(n)\\
&\ge (1-2\delta^{1/4})^5\frac{a}{2\pi}(\log n)^2.
\end{aligned}
$$

If $z\in D_0(\sqrt{n})$ is n,δ-admissible, then necessarily

$$(8.25)\qquad L_n^{X,X'}(z)\ge L_{\sigma_z}(z)L'_{\sigma'_z}(z)\ ,$$

where σ'_z denotes the time it takes $\{X'_i\}$ to complete $(1-2\delta)m_k(n)$ excursions between $\partial D_z(r_n)$ and $\partial D_z(R_n)$, after first hitting $\partial D_z(R_n)$.

There are at most πn lattice points $z\in D_0(\sqrt{n})$, hence by (8.25) and (8.24) we conclude that for $\delta\le\delta_0$ and all n large enough

$$
\begin{aligned}
(8.26)\qquad & \mathbf{P}\times\mathbf{P}'(\inf_{z\in I_{n,\delta}}L_n^{X,X'}(z)\le\frac{(1-2\delta^{1/4})^{10}a^2}{4\pi^2}(\log n)^4)\\
&\le 2\pi n\sup_{y\in\mathbf{Z}^2}\mathbf{P}^y(L_{\sigma_z}(z)\le(1-2\delta)m_{k(n)}\ell_nG_n)
\end{aligned}
$$

and (8.20) will follow once we show that the right hand side of (8.22) is bounded by n^{-2}, which we now do.

Let T_0 be the first hitting time of 0 and set $q_{n,z}:=\mathbf{P}^z(T_0<T_{R_n})$. The Markov property followed by the inequality $e^{-t}\le 1-t+t^2/2\,;\,t\ge 0$ shows that

$$
\begin{aligned}
(8.27)\qquad & \mathbb{E}^z(e^{-\lambda L_{T_{R_n}}(0)/G_n})\\
&=\mathbf{P}^z(T_0>T_{R_n})+\mathbf{P}^z(T_0<T_{R_n})\mathbb{E}^0(e^{-\lambda L_{T_{R_n}}(0)/G_n})\\
&\le 1-q_{n,z}+q_{n,z}\left(1-\lambda\mathbb{E}^0\left(\frac{L_{T_{R_n}}(0)}{G_n}\right)+\frac{\lambda^2}{2}\mathbb{E}^0\left(\left\{\frac{L_{T_{R_n}}(0)}{G_n}\right\}^2\right)\right)
\end{aligned}
$$

and by (8.21) this gives

$$\mathbb{E}^z(e^{-\lambda L_{T_{R_n}}(0)}) \leq 1 - q_{n,z}(\lambda - \lambda^2) \leq e^{-q_{n,z}(\lambda-\lambda^2)}. \tag{8.28}$$

By [12, Proposition 1.6.7], for all n large enough, and all $z \in \partial D_0(r_n)$

$$q_{n,z} = \frac{\log(R_n/r_n)}{\log R_n} + O(\frac{1}{\log^2 R_n}) \geq (1-2\lambda)/(\delta \log n) \tag{8.29}$$

Then with $q_n := (1-2\lambda)/(\delta \log n)$ we see from (8.28) and (8.29) that

$$\sup_{z \in \partial D_0(r_n)} \mathbb{E}^z(e^{-\lambda L_{T_{R_n}}(0)/G_n}) \leq e^{-q_n(\lambda-\lambda^2)}. \tag{8.30}$$

Since $\ell_n = (1-2\lambda)q_n$ we deduce that

$$e^{\lambda \ell_n} \sup_{z \in \partial D_0(r_n)} \mathbb{E}^z(e^{-\lambda L_{T_{R_n}}(0)/G_n}) \leq e^{-\lambda^2 q_n} \tag{8.31}$$

and for all $\delta \leq \delta_0(a)$ and n large enough

$$e^{-\lambda^2 q_n (1-2\delta) m_{k(n)}} \leq e^{-ak^2(n)/(\lambda \log n)} \leq n^{-2}.$$

The last two displays show that the right hand side of (8.22) is bounded by n^{-2}. As mentioned after (8.26) this completes the proof of (8.20).

We next turn to the proof of (8.19). It is here that we use strong approximation, so we need to introduce further notation concerning Brownian paths. Let M_k^x denote the number of excursions of $\{W_t\}$ from $\partial D(x, e^{-k+1})$ to $\partial D(x, e^{-k})$ prior to $\bar{\theta}$ and $M_k'^{x}$ denote the corresponding number of excursions for the process $\{W_t'\}$. Fix δ, $\beta > 0$ and let $\mathcal{U}_k(\beta)$ be a fixed maximal collection of points $x_j \in D(0,1) \cap D^c(0,\beta)$ with $|x_i - x_j| \geq \delta^2 e^{-k}$ for all $x_i, x_j \in \mathcal{U}_k(\beta)$. We say that a point $x \in \mathcal{U}_k(\beta)$ is **k-admissible** if $M_k^x \wedge M_k'^{x} \geq (1-2\delta)m_k$ and denote by $\widehat{\mathcal{U}}_k(\beta)$ the set of k-admissible points.

We now show how to derive (8.19) from the following lemma whose proof is momentarily deferred.

Lemma 8.4. *For $\beta > 0$ sufficiently small*

$$\liminf_{k \to \infty} \mathbf{P} \times \mathbf{P}' \left(|\widehat{\mathcal{U}}_k(\beta)| \geq e^{2(1-a-2\delta)k} \right) \geq 7/8 \tag{8.32}$$

Using (8.32) together with the fact that $\bar{\theta} \vee \bar{\theta}' < \infty$ a.s. we can find $K < \infty$ so that

$$\liminf_{k \to \infty} \mathbf{P} \times \mathbf{P}' \left(|\widehat{\mathcal{U}}_k(\beta)| \geq e^{2(1-a-2\delta)k};\ \bar{\theta} \vee \bar{\theta}' \leq K \right) \geq 3/4. \tag{8.33}$$

By Brownian scaling and the multidimensional strong approximation of [6, Theorem 1] we may construct for each n independent SRW's $\{X_i\}$, $\{X_i'\}$ and independent Brownian motions $\{W_t : t \in [0,K]\}$, $\{W_t' : t \in [0,K]\}$ on the same probability space such that $\mathbf{P} \times \mathbf{P}'(\mathcal{B}_n) \to 1$ as $n \to \infty$, where $\mathcal{B}_n$ is the set

$$\{ \sup_{0 \leq t \leq K} |W_t - \frac{\sqrt{2K}}{\sqrt{n}} X_{[tn/K]}| \leq \delta e^{-k(n)},\ \sup_{0 \leq t \leq K} |W_t' - \frac{\sqrt{2K}}{\sqrt{n}} X'_{[tn/K]}| \leq \delta e^{-k(n)} \}.$$

Recall that if $\bar{\theta} \vee \bar{\theta}' \leq K$ then the event that $x \in D(0,1)$ is k-admissible is measurable on $\sigma(W_t, W_t' : t \leq K)$. Hence, for all n large enough, if in addition $\mathcal{B}_n$ holds, then to each $k(n)$-admissible $x \in D(0,1)$ corresponds $\widetilde{x} \in \mathbf{Z}^2$ nearest to $\sqrt{n/(2K)}x$ that is n, δ-admissible. Since $\mathbf{P} \times \mathbf{P}'(\mathcal{B}_n) \to 1$ as $n \to \infty$, (8.19) follows. □

Proof of Lemma 8.4: We begin by using the techniques of [5] to find many points in whose neighborhood both Brownian paths have large occupation measure. Let

$$\mathcal{C}_a := \Big\{ x \in D(0,1) : \lim_{\epsilon \to 0} \frac{\mu_{\bar{\theta}}^{w}(D(x,\varepsilon))}{\varepsilon^2 (\log \epsilon)^2} = a \Big\}, \tag{8.34}$$

and let $\mathcal{C}_a'$ denote the corresponding set for the process $\{W_t'\}$.

Lemma 8.5.

$$\mathbf{P} \times \mathbf{P}' \left\{ \dim(\mathcal{C}_a \cap \mathcal{C}_a') = 2 - 2a \right\} = 1. \tag{8.35}$$

Proof of Lemma 8.5. We follow the proof of [5, Theorem 1.3], but we now say that the indicator function $Y(n,i)$ introduced in [5, Section 3] is equal to 1 iff $x_{n,i}$ is n-perfect both for W and W'. By independence, the bounds for the first moment and covariance of $Y(n,i)$ which appear in [5, Lemma 3.2] now have a replaced by $2a$. The rest of the proof now proceeds exactly as in [5]. □

The following lemma allows us to obtain large excursion counts from the large occupation times provided by the previous lemma. We abbreviate $\rho_k = e^{-k}$.

Lemma 8.6. *For any $\delta, \beta, \gamma > 0$ we can find $k_0 = k_0(\omega) < \infty$ a.s. such that for all $k > k_0$ and $x \in \mathcal{U}_k(\beta)$, if*

$$\mu_{\bar{\theta}}^{W}(D(x,\rho_k)) \geq a(1-\gamma^2)|\rho_k \log \rho_k|^2 \tag{8.36}$$

then $M_k^x \geq (1-\gamma) m_k$.

Proof of Lemma 8.6. We will say that $x \in \mathcal{U}_k(\beta)$ is *-thick if it satisfies (8.36). Assuming k is large enough so that $0 \notin D(x,\rho_k)$ for all $x \in \mathcal{U}_k(\beta)$, we let $\tau_{l,k}$ denote the occupation measure of $D(x,\rho_k)$ during the l-th excursion of W between $\partial D(x,\rho_k)$ and $\partial D(x,\rho_{k-1})$. Then, with $\mathbb{E}(\tau_{l,k}) = \rho_k^2$ and $m_k' = (1-\gamma)m_k = a(1-\gamma)(\log \rho_k)^2$, we have that for some universal constant $C < \infty$ and all $x \in \mathcal{U}_k(\beta)$,

$$\begin{aligned} \mathbf{P}(M_k^x \leq m_k' \,,\, x \text{ is *-thick}) &\leq \mathbf{P}(\sum_{l=1}^{m_k'} \tau_{l,k} \geq a(1-\gamma^2)|\rho_k \log \rho_k|^2) \\ &= \mathbf{P}(\frac{1}{m_k'} \sum_{l=1}^{m_k'} \tilde{\tau}_{l,k} \geq \gamma) \leq e^{-\gamma^2 m_k / C} \,, \end{aligned}$$

where $\tilde{\tau}_{l,k} := \tau_{l,k}/\mathbb{E}(\tau_{l,k}) - 1$, and the last inequality follows by the methods used in the proof of [5, Lemma 6.4]. Consequently,

$$(8.37)\quad \sum_{k=1}^{\infty} \sum_{x \in \mathcal{U}_k(\beta)} \mathbf{P}(M_k^x \le m_k' \,,\, x \text{ is *-thick}) \le \sum_{k=1}^{\infty} |\mathcal{U}_k(\beta)| e^{-\gamma^2 m_k/C} < \infty.$$

□

We can now complete the proof of Lemma 8.4. It follows from (8.35) that for some $\beta > 0$ sufficiently small and all k sufficiently large, with probability $\ge 7/8$ the set of $x \in D(0,1) \cap D^c(0,2\beta)$ with

$$(8.38)\quad \min\left(\frac{\mu_{\bar\theta}^{w}(D(x,(1-\delta^2)\rho_k))}{\rho_k^2 (\log \rho_k)^2}, \frac{\mu_{\bar\theta'}^{w'}(D(x,(1-\delta^2)\rho_k))}{\rho_k^2 (\log \rho_k)^2} \right) \ge (1-2\delta^2)a$$

has Hausdorff dimension $\ge 2-2a-\delta$. Let $\widetilde{\mathcal{U}}_k$ be the set of points in $\mathcal{U}_k(\beta)$ which are within $\delta^2\rho_k$ of the set in (8.38). Using Lemma 8.6 it is easy to check that each point in $\widetilde{\mathcal{U}}_k$ is k-admissible. Since $\{D(x,\delta\rho_k) : x \in \widetilde{\mathcal{U}}_k\}$ is a cover of the set in (8.38) by sets of maximal diameter $\delta^2\rho_k$, it follows that

$$(8.39)\quad \liminf_{k\to\infty} |\widetilde{\mathcal{U}}_k| (\delta^2\rho_k)^{2-2a-2\delta} = \infty.$$

Our lemma now follows. □

9. Complements and unsolved problems

• By Brownian scaling, for any deterministic $0 < r < \infty$, the set $D(0,1)$ and $\bar\theta, \bar\theta'$ can be replaced by $D(0,r)$ and $\bar\theta_r = \inf\{s : |W_s| = r\}, \bar\theta_r' = \inf\{t : |W_t'| = r\}$, without changing the conclusion of Theorem 1.4. Similarly, one may replace $\mathcal{I}_{\bar\theta,\bar\theta'}$ by $\mathcal{I}_{S,T}$ in this theorem, for any deterministic $0 < S, T < \infty$. Moreover, from its proof we have that (1.10) remains valid when the limit in ε is replaced by lim inf or lim sup and when considering the set of points x for which this limit (lim inf, lim sup, respectively), is at least a^2.

• Next, we discuss briefly the packing dimension analogue of Theorem 1.4; consult Mattila (1995) for background on packing dimension, Minkowski dimension and their relation. The set of consistently thick points $\mathsf{CThickInt}_{\ge a^2}$, defined in (5.2), has different packing dimension from the set $\mathsf{ThickInt}_{\ge a^2}$, defined in (2.8). Namely, for every $a \in (0,1]$,

$$(9.1)\quad \dim_P(\mathsf{CThickInt}_{\ge a^2}) = 2 - 2a\,, \quad a.s.$$

$$(9.2)\quad \dim_P(\mathsf{ThickInt}_{\ge a^2}) = 2 \quad a.s.$$

To justify (9.1), we use the notation of Section 2. The sets $\mathcal{A}_n$, defined in (2.13), satisfy

$$(9.3)\quad |\mathcal{A}_n| \le (\tilde\epsilon_n)^{(1-11\delta)2a-2}$$

for all large n, by (2.15) and Borel-Cantelli.

Recall the discs $\mathcal{V}_{n,j} = D(x_j, \delta\tilde\epsilon_n)$ defined after (2.15), and denote $\mathcal{V}_n = \cup_{j\in\mathcal{A}_n} \mathcal{V}_{n,j}$. By (9.3), the upper Minkowski dimension of $\mathcal{V}_\ell^* = \cap_{n\ge\ell}\mathcal{V}_n$ is at most $2-(1-11\delta)2a$. It is easy to see that $\mathsf{CThickInt}_{\geq a^2} \subset \cup_{\ell\geq 1}\mathcal{V}_\ell^*$, whence $\dim_P(\mathsf{CThickInt}_{\geq a^2}) \leq 2-(1-11\delta)2a$. Since δ can be taken arbitrarily small, while $\dim_P(\mathsf{CThickInt}_{\geq a^2}) \geq \dim(\mathsf{CThickInt}_{\geq a^2})$, this proves (9.1).

To prove (9.2), it clearly suffices to consider $a=1$. Recall that $\bar\theta = \inf\{t : |W_t| = 1\}$. For each $n \geq 1$, let

$$V_n := \bigcup_{0<\varepsilon<1/n} \Big\{0 < t < \bar\theta : \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(W_t,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4} > 1-1/n\Big\}.$$

The sets V_n are open by the path continuity of Brownian motion. Moreover, it is easy to check, e.g. by applying Theorem 1.3 with an arbitrary T replacing $\bar\theta$ there, and using the shift invariance of Brownian motion, that for any $n \geq 1$, almost surely V_n is a dense subset of $(0,\bar\theta)$; by [2, Corollary 2.4, part (i)], $\dim_P(\cap_n V_n) = 1$ a.s. The set

$$\Big\{0 \leq t \leq \bar\theta : \limsup_{\varepsilon\to 0} \frac{\mathcal{I}_{\bar\theta,\bar\theta'}(D(W_t,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^4} \geq 1\Big\}, \tag{9.4}$$

contains $\cap_n V_n$, so it has packing dimension 1. Finally, $\mathsf{ThickInt}_{\geq 1}$ is the image under planar Brownian motion of the set in (9.4); hence the uniform doubling of packing dimension by planar Brownian motion, see [16, Corollary 5.8], yields (9.2).

• The situation for intersections of m independent planar Brownian motions is completely analogous to that of Theorems 1.3 and 1.4. Specifically, let $W^{(1)}_{s_1}, \dots, W^{(m)}_{s_m}$ denote m independent planar Brownian motions, and define the m-fold projected intersection local time by

$$\mathcal{I}_{m,S_1,\dots,S_m}(A) = \lim_{\epsilon\to 0} \pi^{m-1}\int_0^{S_1}\cdots\int_0^{S_m} 1_A(W^{(1)}_{s_1})\prod_{j=2}^m f_\epsilon(W^{(1)}_{s_1}-W^{(j)}_{s_j})\,ds_1\dots ds_m$$

where f_ϵ is any approximate δ-function. It is known that the limit (1.8) exists a.s. and in all L^p spaces, and that $\mathcal{I}_{m,S_1,\dots,S_m}(\cdot)$ is a measure supported on

$$\{x \in I\!\!R^2 \,|\, x = W^{(1)}_{s_1} = \cdots = W^{(m)}_{s_m} \text{ for some } 0 \leq s_1 \leq S_1, \dots, 0 \leq s_m \leq S_m\},$$

see [13, Theorem 1, Chapter VIII]. Note that $\mathcal{I}_{1,\bar\theta}$ is the occupation measure studied in [5]. Let $\bar\theta^{(j)} = \inf\{s : |W^{(j)}_s| = 1\}$. Then, for any $m \geq 1$

$$\lim_{\varepsilon\to 0}\sup_{x\in I\!\!R^2} \frac{\mathcal{I}_{m,\bar\theta^{(1)},\dots,\bar\theta^{(m)}}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^{2m}} = (2/m)^m\,, \qquad a.s. \tag{9.5}$$

and for any $0 < a \leq 2/m$,

(9.6)

$$\dim\Big\{x \in D(0,1) : \lim_{\varepsilon\to 0}\frac{\mathcal{I}_{m,\bar\theta^{(1)},\dots,\bar\theta^{(m)}}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^{2m}} = a^m\Big\} = 2-ma \qquad a.s.$$

• The Hausdorff dimension of the set of thick intersection points for two independent Brownian motions in $I\!R^3$ was recently determined by König and Mörters. Specifically, let $B(x,r)$ denote the ball of radius r centered at $x \in I\!R^3$ and $\mathcal{J}(B(x,r))$ the total intersection local time in $B(x,r)$ for two independent Brownian motions in $I\!R^3$ (see [11, Section 2.1] for various equivalent definitions of $\mathcal{J}(\cdot)$). In [11, Theorem 1.4], König and Mörters show that for any $0 < a \le \rho^*/2$, almost surely,

$$\dim\left\{x \in R^3 : \limsup_{\varepsilon\to 0} \frac{\mathcal{J}(B(x,\varepsilon))}{\varepsilon\left(\log\frac{1}{\varepsilon}\right)^2} = a^2\right\} = 1 - 2a/\rho^* , \tag{9.7}$$

where the non-random $\rho^* > 0$ is the solution of the explicit variational formula [11, (1.9)]. The analog of (9.7) for consistently thick points, that is with lim inf instead of lim sup, involves a different gauge function and remains an open problem.

• In [3] we analyzed 'thin points' for the Brownian occupation measure, establishing that

$$\lim_{\varepsilon\to 0} \inf_{t\in[0,1]} \frac{\mu_{\bar\theta}^W(D(W_t,\varepsilon))}{\varepsilon^2/\log\frac{1}{\varepsilon}} = 1 , \qquad a.s. \tag{9.8}$$

with the multi-fractal spectrum

$$\dim\{x \in D(0,1) : \liminf_{\varepsilon\to 0} \frac{\mu_{\bar\theta}^W(D(x,\varepsilon))}{\varepsilon^2/\log\frac{1}{\varepsilon}} = a\} = 2 - 2/a \qquad a.s. \tag{9.9}$$

for any fixed $a > 1$. In the present paper, we analyzed Brownian projected intersection local time where it is exceptionally 'thick'. The analysis of the corresponding 'thin intersection points' is needed to describe completely the multi-fractal structure of this measure, and remains an open problem.

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