Thin Points for Brownian Motion

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Abstract

Let $\mathcal{T}(x,r)$ denote the occupation measure of the ball of radius r centered at x for Brownian motion $\{W_t\}_{0\le t\le 1}$ in $I\!\!R^d$, $d \ge 2$. We prove that for any analytic set E in $[0,1]$, we have $\inf_{t\in E} \liminf_{r\to 0} \mathcal{T}(W_t,r)/(r^2/|\log r|) = 1/\dim_{\rm P}(E)$, where $\dim_{\rm P}(E)$ is the packing dimension of E. We deduce that for any $a \ge 1$, the Hausdorff dimension of the set of "thin points" x for which $\liminf_{r\to 0} \mathcal{T}(x,r)/(r^2/|\log r|) = a$, is almost surely $2 - 2/a$; this is the correct scaling to obtain a nondegenerate "multifractal spectrum" for the "thin" part of Brownian occupation measure. The methods of this paper differ considerably from those of our work on Brownian thick points, due to the high degree of correlation in the present case. To prove our results, we establish general criteria for determining which deterministic sets are hit by random fractals of 'limsup type' in the presence of long-range correlations. The hitting criteria then yield lower bounds on Hausdorff dimension. This refines previous work of Khoshnevisan, Xiao and the second author, that required decay of correlations.

1 Introduction

Let $B(x,r)$ denote the ball in $I\!\!R^d$ of radius r centered at x. In this paper we study thin points for Brownian motion, *i.e.,* points x on the Brownian path such that for some sequence of radii

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$r_i \to 0$, the balls $B(x, r_i)$ have unusually small occupation measure $\mu_T(B(x, r_i))$. Here

$$\mu_T(A) = \int_0^T \mathbf{1}_A(W_t)\, dt$$

for any Borel sets $A \subseteq \mathbb{R}^d$, where $\{W_t\}_{t \geq 0}$ is Brownian motion in $\mathbb{R}^d$. Lévy's uniform modulus of continuity provides a lower bound on the size of $\mu_T^W(B(W_t, \epsilon))$: there exists an absolute constant $0 < c < \infty$, such that almost surely for all times $0 \leq t \leq T$ and all positive $\epsilon \leq \epsilon_0(\omega)$,

$$\mu_T^W(B(W_t, \epsilon)) \geq c\epsilon^2 / |\log \epsilon| \,. \tag{1.1}$$

Our first result shows that there actually exist times t for which this lower bound provides the right scale, and says precisely how small the occupation measure can get.

Theorem 1.1 *Let $\{W_t\}$ be a Brownian motion in $\mathbb{R}^d$, $d \geq 2$. Then, for any $0 < T < \infty$,*

$$\inf_{t \in (0,T)} \liminf_{\epsilon \to 0} \frac{\mu_T(B(W_t, \epsilon))}{\epsilon^2 / |\log \epsilon|} = 1 \quad \text{a.s.} \tag{1.2}$$

Furthermore, for any analytic set $E \subseteq (0, T)$

$$\inf_{t \in E} \liminf_{\epsilon \to 0} \frac{\mu_T(B(W_t, \epsilon))}{\epsilon^2 / |\log \epsilon|} = \frac{1}{\dim_{\mathrm{P}}(E)} \quad \text{a.s.} \tag{1.3}$$

Here $\dim_{\mathrm{P}}(E)$ denotes the **packing dimension** *of the set E.*

We refer to [10] for background on packing dimension. For any fixed $T \in (0, \infty)$ and $a > 0$, let

$$\mathsf{Thin}_a := \left\{0 < t < T \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mu_T(B(W_t, \epsilon))}{\epsilon^2 / |\log \epsilon|} = a\right\} , \tag{1.4}$$

Theorem 1.1 follows from our main result which also gives the dimension of Thin_a:

Theorem 1.2 *Let $\{W_t\}$ be a Brownian motion in $\mathbb{R}^d$, $d \geq 2$. Fix $T \in (0, \infty)$. Then, for all $a \geq 1$,*

$$\dim(\mathsf{Thin}_a) = 1 - 1/a \;, \quad \text{a.s.} \tag{1.5}$$

whereas the packing dimension of Thin_a is a.s. 1 for all $a \geq 1$. Moreover, for any analytic set $E \subseteq (0, T)$ and any $a > 1/\dim_{\mathrm{P}}(E)$, it holds that $\mathsf{Thin}_a \cap E \neq \varnothing$ a.s., whereas a.s. $\mathsf{Thin}_a \cap E = \varnothing$ for all $a < 1/\dim_{\mathrm{P}}(E)$.

A time $t > 0$ is called a $\mathsf{thin\ time}$ if it is in the set Thin_a of (1.4) for some $a > 0$ and $T > 0$. Similarly, a point $x \in I\!R^d$ on the Brownian path is called a *thin point* if $x = W_t$ for some thin time $t > 0$.

Remarks.

- In particular, Theorem 1.2 shows that the sets Thin_a are empty for all $a < 1$, but non-empty (and of zero Hausdorff dimension) at the critical value $a = 1$, thereby implying (1.2).

- In case $d \geq 3$, Theorem 1.2 applies also for $T = \infty$.

- Fix $T > 0$ and $a > 1$. For any $x \notin \{W_t \,\big|\, 0 \leq t \leq T\}$ and ϵ small enough, $\mu_T(B(x, \epsilon)) = 0$. Hence, (1.5) implies by the *uniform dimension doubling* property of Brownian motion, due to Kaufman [7] (see also, [13, Eqn. (0.1)]) that

$$\dim\{x \in I\!R^d \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mu_T(B(x,\epsilon))}{\epsilon^2/|\log \epsilon|} = a\} = 2 - 2/a \quad \text{a.s.}$$

  Since $\dim_{\mathrm{P}}(\mathsf{Thin}_a) = 1$, we similarly deduce from the uniform doubling of packing dimension by Brownian motion in $I\!R^d$, $d \geq 2$, established by Perkins-Taylor [13, Cor. 5.8], that

$$\dim_{\mathrm{P}}\{x \in I\!R^d \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mu_T(B(x,\epsilon))}{\epsilon^2/|\log \epsilon|} = a\} = 2 \quad \text{a.s.}$$

- We shall also consider the set

$$\mathsf{Thin}_{\leq a} := \bigcup_{a' \leq a} \mathsf{Thin}_{a'} = \{0 < t < T \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mu_T(B(W_t,\epsilon))}{\epsilon^2/|\log \epsilon|} \leq a\}\,. \tag{1.6}$$

  As in Theorem 1.2, $\dim(\mathsf{Thin}_{\leq a}) = 1 - 1/a$ and $\dim_{\mathrm{P}}(\mathsf{Thin}_{\leq a}) = 1\,,\quad$ a.s.

The upper bound in (1.5) is by now a relatively standard chore given the asymptotics of the lower tail of the two-sided exit time of a ball recently obtained in [4]. The real novelty in our paper lies in our method of obtaining the lower bound in (1.5). Typically, in obtaining lower bounds on the Hausdorff dimension of a set A, one constructs a very regular subset of A and shows that this subset has Hausdorff dimension equal to the upper bound obtained for A. In constructing this regular subset, the 'discrete limsup random fractal' described in Section 3, one builds the subset up from small pieces, which in the simplest cases are independent. In our work [2], [3] where we studied thick points for Brownian motion, *i.e.,* points on the Brownian path that have neighborhoods with unusually large occupation measure at infinitely many scales, we developed a general approach to handle dependence among the pieces of a discrete limsup random fractal. In the present case of thin points, the dependence is much greater and has necessitated a new approach: rather than construct a regular subset of Thin_a, we construct a discrete limsup random fractal which is 'close' to a subset of Thin_a but whose 'pieces' have some independence, the 'quasi-locality' of Section 3. Section 3 provides a general exposition of this approach which we expect will be of use in many other situations with long range dependence. The actual application to thin points in Section 4 illustrates the delicate balancing needed to construct a discrete limsup random fractal that enjoys sufficient independence to give almost sure results, yet is still sufficiently close to a subset of Thin_a.

Our upper bound on the dimension of $\mathsf{Thin}_{\leq a}$ is obtained by establishing an upper bound on a superset of times which we call the bilateral fast times, $\mathsf{BiFast}_{\leq a}$, and our lower bound on the dimension of $\mathsf{Thin}_{\leq a}$ is obtained by establishing a lower bound on a subset of times which we call the times of quick escape, $\mathsf{Qscape}_{\leq a}$. Turning first to the upper bound, note that for $\mu_T(B(W_t,\epsilon)) \leq a\epsilon^2/|\log\epsilon|$ it is clearly necessary for the two-sided path segment $\{W_{t+s},\ -t \leq s \leq T-t\}$ to have a small two-sided first exit time from the ball of radius ϵ. To be more precise, let

$$\tau_r(t) := \inf\{s \geq 0 \,\Big|\, |W_{t+s} - W_t| \geq r\}$$

denote the amount of time needed for the path to reach a distance r from its position at time t. Similarly, with $\{\bar{W}_t\}_{-\infty}^{\infty}$ denoting two-sided Brownian motion in $\mathbb{R}^d$, let

$$\bar{\tau}_r(t) := \inf\{s \geq 0 \,\Big|\, |\bar{W}_{t-s} - \bar{W}_t| \geq r\}$$

denote the amount of time, running backwards, needed for the two-sided path to reach a distance r from its position at time t, and define $\mathcal{T}_r(t) := \tau_r(t) + \bar{\tau}_r(t)$ to be the corresponding *bilateral first exit time.* Define the random set

$$\mathsf{BiFast}_a := \{0 < t < T \,\Big|\, \liminf_{\epsilon\to 0} \frac{\mathcal{T}_\epsilon(t)}{\epsilon^2/|\log\epsilon|} = a\}\,, \tag{1.7}$$

A time $t > 0$ is called a $\mathsf{bilateral\ fast\ time}$ if it is in the set BiFast_a of (1.7) for some $a > 0$ and $T > 0$, and a point $x \in I\!\!R^d$ on the Brownian path is called a *bilateral fast point* if $x = W_t$ for some $\mathsf{bilateral\ fast\ time}$ $t > 0$. Finally, define

$$\mathsf{BiFast}_{\leq a} := \bigcup_{a' \leq a} \mathsf{BiFast}_{a'} = \{0 < t < T \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mathcal{T}_\epsilon(t)}{\epsilon^2/|\log \epsilon|} \leq a\} \,, \tag{1.8}$$

and note that $\mathsf{Thin}_{\leq a} \subseteq \mathsf{BiFast}_{\leq a}$.

Turning next to the lower bound, we observe that $\mu_T(B(W_t, \epsilon)) \leq a\epsilon^2/|\log \epsilon|$ will surely hold if two-sided path segment $\{W_{t+s},\ -t \leq s \leq T - t\}$ has a quick two-sided escape from the ball of radius ϵ (never returning again). To be more precise, for any $\xi > 0$ and $t \in I\!\!R$, let

$$\sigma_r^\xi(t) := \sup\{s \in [0, \xi] \,\Big|\, |W_{t+s} - W_t| \leq r\}$$

denote the amount of time till the last visit of $B(W_t, r)$ by the path killed at time $t + \xi$, with $\bar{\sigma}_r^\xi(t) := \sup\{s \in [0, \xi] \,\Big|\, |\bar{W}_{t-s} - \bar{W}_t| \leq r\}$ the corresponding time-reversed object, and $\mathcal{S}_r^{\eta,\xi}(t) = \bar{\sigma}_r^\eta(t) + \sigma_r^\xi(t)$, denoting the length of the minimal time interval containing all visits to $B(W_t, r)$ within $[t - \eta, t + \xi]$. Define

$$\mathsf{Qscape}_a := \{0 < t < T \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mathcal{S}_\epsilon^{T,T}(t)}{\epsilon^2/|\log \epsilon|} = a\} \,. \tag{1.9}$$

A time $t > 0$ is called a $\mathsf{time\ of\ quick\ escape}$ if it is in the set Qscape_a of (1.9) for some $a > 0$ and $T > 0$, and a point $x \in I\!\!R^d$ on the Brownian path is a *point of quick escape* if $x = W_t$ for some $\mathsf{time\ of\ quick\ escape}$ $t > 0$. Finally, define

$$\mathsf{Qscape}_{\leq a} := \bigcup_{a' \leq a} \mathsf{Qscape}_{a'} = \{0 < t < T \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mathcal{S}_\epsilon^{T,T}(t)}{\epsilon^2/|\log \epsilon|} \leq a\} \subseteq \mathsf{Thin}_{\leq a} \quad . \tag{1.10}$$

We thus have the set inclusions

$$\mathsf{Qscape}_{\leq a} \subseteq \mathsf{Thin}_{\leq a} \subseteq \mathsf{BiFast}_{\leq a} \,, \tag{1.11}$$

Theorem 1.2 will be obtained by showing:

Theorem 1.3 *The conclusions of Theorem 1.2 and the remarks which immediately follow it remain true if the sets* Thin_a*,* $\mathsf{Thin}_{\leq a}$ *are replaced by the sets* Qscape_a*,* $\mathsf{Qscape}_{\leq a}$ *or* BiFast_a*,* $\mathsf{BiFast}_{\leq a}$*.*

Remarks.

- The set $\mathsf{BiFast}_{\le a}$ of (1.8) can also be written as

$$\mathsf{BiFast}_{\le a} = \{0 < t < T \,\Big|\, \limsup_{\substack{h+h' \to 0 \\ h,h'>0}} \frac{|W_t - W_{t+h}| \wedge |W_t - W_{t-h'}|}{\sqrt{2(h+h')|\log(h+h')|}} \ge \frac{1}{2\sqrt{a}}\}.$$

  Thus, Theorem 1.3 is to be contrasted with Orey and Taylor results about the dimension of the set of fast times (see [12]),

$$\dim\{0 < t < T \,\Big|\, \limsup_{h \to 0} \frac{|W_t - W_{t+h}|}{\sqrt{2h|\log h|}} \ge \frac{1}{2\sqrt{a}}\} = 1 - 1/(4a) \quad \text{a.s.}$$

  and that of the set of two-sided fast times (see also [8] for some finer calculations),

$$\dim\{0 < t < T \,\Big|\, \limsup_{\substack{h+h' \to 0 \\ h,h'>0}} \frac{|W_{t+h} - W_{t-h'}|}{\sqrt{2(h+h')|\log(h+h')|}} \ge \frac{1}{2\sqrt{a}}\} = 1 - 1/(4a) \quad \text{a.s.}$$

- A slight modification of the proof of Theorem 1.3 shows that actually, for all $a \ge 1$, $\mathsf{BiFast}_a \setminus \mathsf{Qscape}_a \ne \varnothing$.

Our next result is about the coarse multifractal spectrum. It is the analog of [2, Corollary 1.5]. Unfortunately, we have not been able to extend it to the case $d = 2$: Recurrence of planar Brownian motion yields extreme long-range dependence, and one of the steps in our proof, see (5.5), fails. The techniques we use for proving Theorems 1.2 and 1.3 allow us to establish (1.12) only when either the Lebesgue measure on $[0, 3T]$ is considered there, or when the limit in ϵ is replaced by the $\liminf$.

Theorem 1.4 *Let $\{W_t\}$ be a Brownian motion in $\mathbb{R}^d$, $d \ge 3$, and denote Lebesgue measure on $\mathbb{R}^1$ by $\mathcal{L}eb$. Then, for any $T \in (0, \infty)$ and $a > 1$,*

$$\lim_{\epsilon \to 0} \frac{\log \mathcal{L}eb\Big\{0 \le t \le T \,\Big|\, \mu_T(B(W_t, \epsilon)) \le a\epsilon^2/|\log \epsilon|\Big\}}{\log \epsilon} = 2/a \quad \text{a.s.} \tag{1.12}$$

These conclusions of remain true if $\mu_T(B(W_t, \epsilon))$ is replaced by $\mathcal{T}_\epsilon(t)$ or $\mathcal{S}_\epsilon^{T,T}(t)$.

Upper bounds on Hausdorff dimensions in Theorems 1.2 and 1.3 are proved in Section 2. Section 3 adapts the approach of [9, Section 3] to the computation of the Hausdorff and packing dimension of discrete limsup random fractals. Based on the general results of this section we complete the proof of Theorems 1.2 and 1.3 in Section 4. Finally, Theorem 1.4 is proved in Section 5.

Notations: throughout, we shall use τ, $\bar{\tau}$, $\mathcal{T}$, σ^{ξ}, $\bar{\sigma}^{\eta}$ and $\mathcal{S}^{\eta,\xi}$ to denote the corresponding random variables τ_r, $\bar{\tau}_r$, $\mathcal{T}_r$, σ_r^{ξ}, $\bar{\sigma}_r^{\eta}$ and $\mathcal{S}_r^{\eta,\xi}$ for $r = 1$, and omit the argument t when its exact value does not matter.

2 Upper bounds in Theorems 1.2 and 1.3

Of course there is no need for an upper bound on the stated packing dimensions of Theorems 1.2 and 1.3. Moreover, fixing $a \geq 1$, $d \geq 2$ and $T \in (0, \infty]$, all the sets considered there are contained in the set $\mathsf{BiFast}_{\leq a}$. Hence, it suffices to establish the upper bound on the Hausdorff dimension of the latter set, where by the monotonicity in T and Brownian scaling, we may and shall consider only the case of $T = 1$.

Turning to this task, we use the notations $\hat{h}(\epsilon) := \epsilon^2 / |\log \epsilon|$ and $\hat{\mathcal{T}}_\epsilon(t) := \mathcal{T}_\epsilon(t)/\hat{h}(\epsilon)$. Fixing $\delta > 0$ and $\epsilon_n = (1-\delta)^n$, we note that for any $\epsilon \in [\epsilon_n, \epsilon_{n-1}]$ and $t \in I\!R$

$$\frac{n-1}{n}(1-\delta)^2 \hat{\mathcal{T}}_{\epsilon_n}(t) \leq \hat{\mathcal{T}}_\epsilon(t) \leq \frac{n}{n-1}(1-\delta)^{-2} \hat{\mathcal{T}}_{\epsilon_{n-1}}(t) \ . \tag{2.1}$$

Therefore,

$$\mathsf{BiFast}_{\leq a} \subseteq D_a := \{t \in (0,1) \,\Big|\, \liminf_{n\to\infty} \hat{\mathcal{T}}_{\epsilon_n}(t) \leq (1-\delta)^{-2} a\} \tag{2.2}$$

Let $\rho_n = \delta^2 \hat{h}(\epsilon_n)/20$, $N_n = \lceil \rho_n^{-1} \rceil$, and $t_{j,n} = j\rho_n$, $\mathcal{V}_{n,j} = [t_{j,n} - \rho_n/2, t_{j,n} + \rho_n/2]$ for $j = 0, 1, \ldots, N_n$. By Lévy's uniform modulus of continuity, we have that a.s. for some finite $n_0 = n_0(\omega, \delta) \geq \delta^{-1}$ and all $n \geq n_0$,

$$\max_{j=0}^{N_n} \sup_{s,t \in \mathcal{V}_{n,j}} |W_t - W_s| < \delta \epsilon_n \ . \tag{2.3}$$

Thus, whenever $t \in \mathcal{V}_{n,j}$, both $\{W_s \,\big|\, s \in [t, t_{j,n}]\}$ and $B(W_{t_{j,n}}, \epsilon_{n+1})$ are contained in $B(W_t, \epsilon_n)$, implying that

$$\hat{\mathcal{T}}_{\epsilon_n}(t) \geq (1-\delta)^3 \hat{\mathcal{T}}_{\epsilon_{n+1}}(t_{j,n}) \; . \tag{2.4}$$

Consequently, with $\mathcal{A}_n$ denoting the set of j , $0 \leq j \leq N_n$, such that

$$\hat{\mathcal{T}}_{\epsilon_{n+1}}(t_{j,n}) \leq (1-\delta)^{-6} a \; , \tag{2.5}$$

clearly, $\cup_{n \geq m} \cup_{j \in \mathcal{A}_n} \mathcal{V}_{n,j}$ forms a cover of D_a of (2.2) by sets of maximal diameter ρ_m.

Combining [4, Theorem 4] with the Ciesielski-Taylor identity in law between $\mu_\infty(B(0,1))$, and the hitting time of the unit ball by a Brownian motion in $\mathbb{R}^{d-2}$ (see [1, Theorem 2]), it follows that for any $d \geq 1$,

$$\lim_{x \downarrow 0} x^{d/2-1} e^{1/(2x)} \mathbf{P}(\tau(0) \leq x) = 2^{2-d/2} / \Gamma(d/2) \; . \tag{2.6}$$

Similarly, the corresponding two-sided result of [4, Theorem 1] leads to

$$\lim_{x \downarrow 0} x^{d-1.5} e^{2/x} \mathbf{P}(\mathcal{T}(0) \leq x) = \sqrt{8\pi}\, \Gamma(d/2)^{-2} \; . \tag{2.7}$$

By Brownian scaling and (2.7), for any $\delta > 0$ and $n \geq n_1(\delta, a)$

$$\mathbf{P}(\hat{\mathcal{T}}_{\epsilon_{n+1}}(0) \leq (1-\delta)^{-6} a) = \mathbf{P}(\mathcal{T}(0) \leq \frac{(1-\delta)^{-6} a}{|\log \epsilon_{n+1}|}) \leq \epsilon_{n+1}^{2(1-\delta)^7/a} \; . \tag{2.8}$$

Thus, for some $c_1 = c_1(\delta, a)$ and all n,

$$\mathbb{E}|\mathcal{A}_n| \leq N_n \epsilon_{n+1}^{2(1-\delta)^7/a} \leq c_1 \epsilon_n^{2(1-\delta)^8/a-2} \; . \tag{2.9}$$

Since $\mathcal{V}_{n,j}$ have diameter ρ_n, it follows from (2.9) that for $\gamma = 1 - (1-\delta)^9/a > 0$ and some $c_2 = c_2(\delta, a) < \infty$,

$$\mathbb{E} \sum_{n=m}^{\infty} \sum_{j \in \mathcal{A}_n} |\mathcal{V}_{n,j}|^\gamma \leq c_2 \sum_{n=m}^{\infty} \epsilon_n^{2\delta(1-\delta)^8/a} < \infty \; .$$

Thus, $\sum_{n=m}^{\infty} \sum_{j \in \mathcal{A}_n} |\mathcal{V}_{n,j}|^\gamma$ is finite a.s. implying that $\dim(\mathsf{BiFast}_{\leq a}) \leq \dim(D_a) \leq \gamma$ a.s. Taking $\delta \downarrow 0$ completes the proof of the upper bound

$$\dim(\mathsf{BiFast}_{\leq a}) \leq 1 - 1/a \quad \text{a.s.} \tag{2.10}$$

□

We conclude this section with the following lemma, needed in Section 4.

Lemma 2.1 *For any $a > 0$ and any analytic set E with $\dim_{\mathrm{P}}(E) < 1/a$, we have that*

$$\mathbf{P}(\mathsf{BiFast}_{\le a} \cap E \neq \varnothing) = 0\,.$$

Proof. We use the same notations as in the last proof. By regularization (see [10]) it suffices to prove the lemma for sets E such that $\overline{\dim}_{\mathrm{M}}(E) < 1/a$. Then, by the definition of upper Minkowski dimension, for any δ small enough and all $n > n_0 = n_0(\delta)$,

$$\#\{j = 0, \ldots, N_n : \mathcal{V}_{n,j} \cap E \neq \varnothing\} \le \rho_n^{-(1-\delta)^{10}/a}\,. \tag{2.11}$$

Define

$$\mathcal{A}_n^E = \{j \in \mathcal{A}_n : \mathcal{V}_{n,j} \cap E \neq \varnothing\}\,.$$

Then, $\cup_{n \ge m} \cup_{j \in \mathcal{A}_n^E} \mathcal{V}_{n,j}$ form a cover of $D_a \cap E$. But, using (2.8) and (2.11), we have that for all δ small enough, uniformly in $n > n_1 = n_1(\delta)$,

$$\mathbb{E}|\mathcal{A}_n^E| \le \rho_n^{-(1-\delta)^{10}/a} \epsilon_{n+1}^{2(1-\delta)^7/a} \le \epsilon_n^{\delta/a}\,.$$

Hence, using only $m > n_0 \vee n_1$,

$$\mathbf{P}(\mathsf{BiFast}_{\le a} \cap E \neq \varnothing) \le \sum_{n=m}^{\infty} \epsilon_n^{\delta/a} \to_{m\to\infty} 0\,,$$

completing the proof of the lemma. $\square$

3 Discrete Limsup Random Fractals

Throughout, let us fix an integer $N \ge 1$. For every integer $n \ge 1$, let $\mathcal{D}_n$ denote the collection of all hyper-cubes of the form $\prod_{i=1}^N [k_i 2^{-n}, (k_i+1)2^{-n}] \subseteq I\!R^N$, where $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$ is any N–dimensional positive integer. In words, $\mathcal{D}_n$ denotes the totality of all N–dimensional dyadic hyper-cubes. For each integer $n \ge 1$, $\{Z_n(I);\ I \in \mathcal{D}_n\}$ denotes a collection of random variables, each taking values in $\{0, 1\}$. By a discrete **limsup random fractal**, we mean a random set of the form $A := \limsup_n A(n)$, where,

$$A(n) := \bigcup_{I \in \mathcal{D}_n : Z_n(I) = 1} I^o,$$

where I^o denotes the interior of I. Adapting the approach of [9, Section 3], we shall determine hitting probabilities for a discrete limsup random fractal A, under the following conditions on the random variables $\{Z_n(I); I \in \mathcal{D}_n\}$. These conditions are particularly well suited for dealing with thin points of Brownian motion.

Condition I: the index assumption. For $n \geq 1$, suppose that $p_n := \mathbb{E}[Z_n(I)]$ is independent of $I \in \mathcal{D}_n$ and that

$$\lim_{n\to\infty} \frac{1}{n} \log_2 p_n = -\gamma,$$

for some $\gamma > 0$, where $\log_2$ denotes the base 2 logarithm.

We shall refer to γ as the **index** of the limsup random fractal A.

Definition. We say that the family $Y = \{Y_n(I)\,;\, I \in \mathcal{D}_n\,,\, n = 1, 2, \ldots\}$ of random variables is **quasi-localized** if for any sequence $\mathcal{C}_i \subset \bigcup_n \mathcal{D}_n,\ i = 1, 2, \ldots$ with $\inf_{i\neq j} dist(\mathcal{C}_i\,, \mathcal{C}_j) > 0$, we have that the tail σ-algebras $\mathcal{F}_i = \bigcap_m \bigcup_{n\geq m} \sigma(Y_n(I) : I \in \mathcal{C}_i),\ i = 1, 2, \ldots$ are independent.

Remark: All results of this section apply when the assumption that the σ-algebras $\mathcal{F}_i$ are independent in the preceding definition is relaxed to the following (somewhat technical) condition:

$$\lim_{r\downarrow 0}\ \inf\{\mathbf{P}(\bigcup_{i=1}^{M_r} A_i)\ :\ \mathbf{P}(A_i) \geq \alpha,\ A_i \in \mathcal{F}_i,\ \inf_{i\neq j} dist(\mathcal{C}_i\,, \mathcal{C}_j) = r\} = 1,$$

for any $\alpha > 0$ and M_r such that $\lim_{r\to 0} \log M_r/|\log r| > \gamma$.

Condition II: quasi-local approximation. Suppose there exist a quasi-localized family of random variables, $Y_n(I) \in \{0, 1\}$, $n = 1, 2, \ldots; I \in \mathcal{D}_n$ such that

$$q_n := \sup_{I\in\mathcal{D}_n} \{\mathbf{P}(Y_n(I) \neq Z_n(I))/\mathbb{E}(Y_n(I))\} \to 0\,. \tag{3.1}$$

Condition III: a bound on second moments. Suppose that the family $Y_n(I) \in \{0,1\}$ of Condition II is such that with

$$f_K(n) := \max_{I \in \mathcal{D}_n} \#\Big\{ J \in \mathcal{D}_n : \ E(Y_n(I)Y_n(J)) > K\mathbb{E}[Y_n(I)]\mathbb{E}[Y_n(J)] \Big\},$$

we have

$$\lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log_2 f_K(n) = 0\,. \tag{3.2}$$

We are ready to state and prove the main result of this section.

Theorem 3.1 *Suppose $A = \limsup_n A(n)$ is a discrete limsup random fractal which satisfies Condition I with index γ, and Conditions II and III. Then for any analytic set $E \subset \mathbb{R}_+^N$,*

$$\mathbf{P}(A \cap E \neq \varnothing) = \begin{cases} 1, & \textit{if } \dim_{\mathrm{P}}(E) > \gamma \\ 0, & \textit{if } \dim_{\mathrm{P}}(E) < \gamma \end{cases}.$$

Proof. We adapt the proof of [9, Theorem 3.1] to the present setting. First, we show that $\dim_{\mathrm{P}}(E) < \gamma$ implies that $A \cap E = \varnothing$, a.s. By regularization (see [10]), it suffices to show that whenever $\overline{\dim}_{\mathrm{M}}(E) < \gamma$, then $A \cap E = \varnothing$, a.s. Fix an arbitrary but small $\eta > 0$ such that $\overline{\dim}_{\mathrm{M}}(E) < \gamma - \eta$. By the definition of upper Minkowski dimension, we can find $\theta \in (0, \gamma - \eta)$, such that for all $n \geq 1$,

$$\#\Big\{ I \in \mathcal{D}_n : \ I \cap E \neq \varnothing \Big\} \leq 2^{n\theta}. \tag{3.3}$$

On the other hand, by Condition I, for all n large enough,

$$p_n \leq 2^{-n(\gamma-\eta)}. \tag{3.4}$$

It follows from (3.3) and (3.4) that for each $n \geq 1$

$$\mathbf{P}\Big(E \cap A(n) \neq \varnothing\Big) \leq 2^{n\theta} \max_{I \in \mathcal{D}_n} \mathbf{P}(I \cap A(n) \neq \varnothing) = 2^{n\theta} p_n \leq 2^{-n(\gamma-\eta-\theta)}.$$

Since $\theta < \gamma - \eta$, the Borel-Cantelli lemma implies that there exists a random variable n_0, such that a.s., for all $n \geq n_0$, $E \cap A(n) = \varnothing$. This shows that $A \cap E = \varnothing$, a.s.

It remains to show that if $\dim_{\mathrm{P}}(E) > \gamma$, then $A \cap E \neq \varnothing$, a.s. Indeed, suppose $\dim_{\mathrm{P}}(E) > \gamma + 2\delta$ for some $\delta > 0$. By [5], we can find a closed $E_\star \subset E$, such that for all open sets V, whenever $E_\star \cap V \neq \varnothing$, then

$$\overline{\dim}_{\mathrm{M}}(E_\star \cap V) > \gamma + 2\delta. \tag{3.5}$$

It suffices to show that with probability one, $A \cap E_\star \neq \varnothing$. Define the open sets $B(n) := \cup_{k=n}^{\infty} A(k)$, $n \geq 1$. We claim that for all $n \geq 1$, the relatively open set $B(n) \cap E_\star$ is a.s. dense in (the complete metric space) $E_\star$. If so, Baire's category theorem (cf. [11]) implies that $E_\star \cap \bigcap_{n=1}^{\infty} B(n)$ is dense in $E_\star$ and in particular, nonempty. Since $A = \cap_n B(n)$, the result follows. Fix an open set V such that $V \cap E_\star \neq \varnothing$. It suffices to show that $A(n) \cap V \cap E_\star \neq \varnothing$ for infinitely many n, a.s. Indeed, this will imply that $B(n) \cap V \cap E_\star \neq \varnothing$ for all n a.s.; by letting V run over a countable base for the open sets, we will conclude that $B(n) \cap E_\star$ is a.s. dense in $E_\star$.

Thus fix an open set V such that $V \cap E_\star \neq \varnothing$. Define $T_n := \sum_{\mathcal{I}} Z_n(I)$, where the sum is taken over $\mathcal{I} = \{I \in \mathcal{D}_n : I \cap V \cap E_\star \neq \varnothing\}$. In words, T_n is the total number of hyper–cubes $I \in \mathcal{D}_n$ such that $I \cap V \cap E_\star \cap A(n) \neq \varnothing$. We need only show that almost surely $T_n > 0$ for infinitely many n.

Since $\overline{\dim}_{\mathrm{M}}(V \cap E_\star) > 0$, we can find constants $M_r \to \infty$ and $x_i \in V \cap E_\star$, $i = 1, \ldots, M = M_r$ such that $\min_{i \neq j} |x_i - x_j| \geq 4r$. Fix some $r > 0$. Let $E_i = B(x_i, r) \cap V \cap E_\star$, $i = 1, \ldots, M$ and $\mathcal{N}_n^i$ denote the total number of hyper-cubes $I \in \mathcal{D}_n$ such that $I \cap E_i \neq \varnothing$. Fixing $i = 1, \ldots, M$, by (3.5), $\overline{\dim}_{\mathrm{M}}(E_i) > \gamma + 2\delta$, hence $\mathcal{N}_n^i \geq 2^{n(\gamma+2\delta)}$ for infinitely many integers $n \geq 1$. In other words, $\#(\mathfrak{N}^i) = \infty$, where

$$\mathfrak{N}^i := \left\{ n \geq 1 : \; \mathcal{N}_n^i \geq 2^{n(\gamma+2\delta)} \right\}. \tag{3.6}$$

For each i, we set $T_n^i := \sum_{\mathcal{I}_i} Z_n(I)$ and $S_n^i := \sum_{\mathcal{I}_i} Y_n(I)$, where both sums are taken over $\mathcal{I}_i = \{I \in \mathcal{D}_n : I \cap E_i \neq \varnothing\}$. It follows from (3.1) that for all i and all n,

$$\mathbf{P}(S_n^i - T_n^i > \eta \mathbb{E}(S_n^i)) \leq \frac{1}{\eta \mathbb{E}(S_n^i)} \sum_{\substack{I \in \mathcal{D}_n: \\ I \cap E_i \neq \varnothing}} \mathbf{P}(Y_n(I) \neq Z_n(I)) \leq q_n / \eta \,.$$

Thus, choosing $n_{k,i} \in \mathfrak{N}^i$ such that $\sum_k q_{n_{k,i}} < \infty$ for $i = 1, \ldots, M$ and $\{n_k\} := \cup_i \{n_{k,i}\}$, by the Borel-Cantelli lemma, almost surely, $S_{n_k}^i - T_{n_k}^i \leq \eta \mathbb{E}(S_{n_k}^i)$ for all i and all large k. Hence for each $i = 1, \ldots, M$

$$\{T^i_{n_k} > 0 \text{ i.o.}\} \supseteq \{S^i_{n_k} > \eta \mathbb{E}(S^i_{n_k}) \text{ i.o.}\} \tag{3.7}$$

Note that with $A_i := \{S^i_{n_k} > \eta \mathbb{E}(S^i_{n_k}) \text{ i.o.}\}$ we have

$$\mathbf{P}(T_n > 0 \text{ i.o.}) \geq \mathbf{P}(\sum_{i=1}^M T^i_{n_k} > 0 \text{ i.o.}) \geq \mathbf{P}(\bigcup_{i=1}^M A_i) = 1 - \prod_{i=1}^M (1 - \mathbf{P}(A_i)),$$

where we used in the last equality the fact that the events $A_i \in \bigcap_m \bigcup_{n \geq m} \sigma(Y_n(I) : I \in \mathcal{I}_i)$ and $\inf_{i \neq j} dist(\mathcal{I}_i, \mathcal{I}_j) > r$, and hence are independent by the quasi-locality of the family $\{Y_n(I)\}$. We show below that

$$\mathbf{P}(A_i) = \mathbf{P}(S^i_{n_k} > \eta \mathbb{E}(S^i_{n_k}) \text{ i.o.}) \geq \alpha > 0 \tag{3.8}$$

for some $\alpha > 0$ independent of i and $r > 0$. Hence, taking $r \to 0$, so that $M = M_r \to \infty$ it follows that almost surely, $T_n > 0$ for infinitely many values of n, as needed to complete the proof of the theorem.

Thus it only remains to prove (3.8). We have

$$\mathbb{E}[(S^i_n)^2] = \sum_{\substack{I \in \mathcal{D}_n: \\ I \cap E_i \neq \varnothing}} \sum_{\substack{J \in \mathcal{D}_n: \\ J \cap E_i \neq \varnothing}} \mathbb{E}(Y_n(I) Y_n(J)).$$

By Condition III, we may and shall set $K < \infty$ large enough so that

$$\limsup_{n \to \infty} \log_2 f_K(n) \leq \delta \,. \tag{3.9}$$

For each $I \in \mathcal{D}_n$, let $\mathcal{B}_n(I)$ denote the collection of all $J \in \mathcal{D}_n$ such that

(i) $J \cap V \cap E_\star \neq \varnothing$, and

(ii) $\mathbb{E}(Y_n(I) Y_n(J)) > K \mathbb{E}[Y_n(I)] \mathbb{E}[Y_n(J)]$.

Then,

$$\mathbb{E}[(S^i_n)^2] \leq K \mathbb{E}[S^i_n]^2 + \sum_{\substack{I \in \mathcal{D}_n: \\ I \cap E_i \neq \varnothing}} \sum_{J \in \mathcal{B}_n(I)} \mathbb{E}(Y_n(I) Y_n(J)).$$

To handle the last sum, we note that by the definition of q_n in (3.1) we have that

$$\frac{p_n}{1+q_n} \le \mathbb{E}[Y_n(I)] \le \frac{p_n}{1-q_n} \tag{3.10}$$

Since all $Y_n(I)$'s are either 0 or 1, this shows that $\mathbb{E}(Y_n(I)Y_n(J)) \le \mathbb{E}[Y_n(I)] \le p_n/(1-q_n)$. Thus, in the notation of Condition III, we deduce that

$$\mathbb{E}[(S_n^i)^2] \le K\mathbb{E}[S_n^i]^2 + \mathcal{N}_n^i \frac{p_n}{1-q_n} \max_{I\in\mathcal{D}_n} \#\mathcal{B}_n(I) \le K\mathbb{E}[S_n^i]^2 + \mathcal{N}_n^i \frac{p_n}{1-q_n} f_K(n)\,.$$

By Conditions I and III, there exist $a_n \downarrow 0$ and $b_n \to 0$ such that $f_K(n) \le 2^{n(\delta+b_n)}$ and, for all n large,

$$2^{-n(\gamma+a_n)} \le p_n \le (1-q_n)2^{-n(\gamma-a_n)}\,.$$

Recall, using (3.10), that $\mathbb{E}[S_n^i] \ge \mathcal{N}_n^i\, p_n/(1+q_n) > 0$. Thus, using (3.6), (3.9) and [6, Inequality II, page 8], we see that for all $n \in \mathfrak{N}^i$ large enough and $\eta \in (0,1)$,

$$\frac{(1-\eta)^2}{\mathbf{P}(S_n^i > \eta\mathbb{E}(S_n^i))} \le \frac{\mathbb{E}[(S_n^i)^2]}{(\mathbb{E}S_n^i)^2} \le K + 2^{n(3a_n+b_n-\delta)} \le K+1\,.$$

Hence,

$$\mathbf{P}(S_n^i > \eta\mathbb{E}(S_n^i)) \ge \alpha \tag{3.11}$$

for some $\alpha = \alpha(\eta, K) > 0$, all $\eta \in (0,1)$, all $i = 1, \ldots, M$ and $n \in \mathfrak{N}^i$ large enough.

Recalling the $n_{k,i} \in \mathfrak{N}^i$ and $\{n_k\} := \cup_i\{n_{k,i}\}$ introduced above, an application of Fatou's lemma to (3.11) yields that for all i,

$$\mathbf{P}(S_{n_k}^i > \eta\mathbb{E}(S_{n_k}^i) \text{ i.o. }) \ge \liminf_{k\to\infty} \mathbf{P}(\, S_{n_{k,i}}^i > \eta\mathbb{E}(S_{n_{k,i}}^i)\,) \ge \alpha > 0\,. \tag{3.12}$$

which completes the proof of (3.8) and hence of our theorem. □

As in [9], we obtain the following useful corollary by a co–dimension argument.

Corollary 3.2 *Suppose A is a discrete limsup random fractal satisfying Condition I with index γ and Conditions II and III. Then, for any analytic set $E \subset \mathbb{R}_+^N$,*

$$\dim(E) - \gamma \le \dim(A \cap E) \le \dim_{\mathrm{P}}(E) - \gamma\,, \quad a.s. \tag{3.13}$$

In particular, $\dim(A) = N - \gamma$, *a.s.*

Remark. Whereas we do not need it here, one can easily check that [9, Theorem 3.2] also holds for discrete limsup random fractals satisfying Conditions I, II and III.

Proof. The right-hand inequality in (3.13) is verified by the following direct first-moment calculation involving only the index Condition I. By regularization, it suffices to prove that

$$\dim(A \cap E) \leq \overline{\dim}_{\mathrm{M}}(E) - \gamma \quad a.s. \tag{3.14}$$

Let $\mathcal{N}_n$ denote the total number of hyper-cubes $I \in \mathcal{D}_n$ such that $I \cap E \neq \varnothing$. Define $T_n := \sum_{\mathcal{I}} Z_n(I)$, where the sum is taken over $\mathcal{I} = \{I \in \mathcal{D}_n : I \cap E \neq \varnothing\}$. Then

$$\mathbb{E}(T_n) \leq \mathcal{N}_n p_n \leq 2^{n(\xi+\epsilon_n)} 2^{n(\epsilon_n - \gamma)}$$

where $\xi = \overline{\dim}_{\mathrm{M}}(E)$ and $\epsilon_n \to 0$. Thus $\mathbb{E}\sum_n T_n 2^{-n\theta} < \infty$ for any $\theta > \xi - \gamma$. Finally, for any n_0, the intersection $A \cap E$ has a cover consisting of T_n intervals in $\mathcal{D}_n$ for each $n \geq n_0$. By picking n_0 large, we see that the θ-dimensional Hausdorff measure of $A \cap E$ vanishes, whence (3.14) follows.

The left-hand inequality in (3.13) follows from Theorem 3.1 by the co–dimension argument of [9, Lemma 3.4]. □

Definition. An analytic set E is *γ-regular* if there exists a closed set $E_\star \subset E$ such that $\dim_{\mathrm{P}}(E_\star \cap V) = \gamma$ for any open set V that intersects $E_\star$.

Any analytic set $E \subset [0,1]^N$ with $\dim_{\mathrm{P}}(E) > \gamma$ is *γ-regular* . (This follows from the arguments of [5].) Also, $[0,1]^N$ is N-regular. The following corollary uses the machinery developed in [9] and in Theorem 3.1.

Corollary 3.3 *Suppose $A_m \in [0,1]^N$, $m \in \mathbb{N}$, are discrete limsup random fractal sets satisfying Condition I with index $\gamma_m \uparrow \gamma$, and Conditions II and III. Let $A_+ = \bigcap_{m=1}^\infty A_m$ and $B = A_+ \setminus (\cup_m \Gamma_m)$, where Γ_m are Borel subsets of $[0,1]^N$ such that, a.s., $\Gamma_m \cap \tilde{E} = \varnothing$ for all analytic sets $\tilde{E}$ satisfying $\dim_{\mathrm{P}}(\tilde{E}) \leq \gamma$. Then, for any γ-regular analytic set E, it holds that $B \cap E \neq \varnothing$, a.s.. Further, $\dim(B) = N - \gamma$ and $\dim_{\mathrm{P}}(B) = N$, a.s..*

Proof. Let $\Lambda_m(n) := \cup_{k=n}^\infty A_m(k)$. Since E is γ-regular, there exists a closed $E_\star \subset E$ such that $\dim_{\mathrm{P}}(E_\star \cap V) = \gamma$ for any open set V such that $E_\star \cap V \neq \varnothing$. For such V and all m we thus have that $\overline{\dim}_{\mathrm{M}}(E_\star \cap V) > \gamma_m$, implying as in the proof of Theorem 3.1 that $\Lambda_m(n) \cap E_\star$ is a.s. dense in the complete metric space $E_\star$. Consequently, by Baire's theorem it follows that $E_\star \cap (\cap_{n=1}^\infty \cap_{m=1}^\infty \Lambda_m(n))$ is dense in $E_\star$, a.s., and in particular is non-empty. Obviously, $\dim_{\mathrm{P}}(E_\star) = \gamma$, so by our assumptions $\Gamma_m \cap E_\star = \varnothing$, a.s.. It follows that $B \cap E \neq \varnothing$, a.s. as claimed.

Since any analytic set E with $\dim_{\mathrm{P}}(E) > \gamma$ is γ-regular, it thus follows form [9, Lemma 3.4] (with $E = [0,1]^N$) that a.s., $\dim(B) \geq N - \gamma$, while by Corollary 3.2, almost surely,

$$\dim(B) \leq \dim(A_+) \leq \dim(A_m) = N - \gamma_m \downarrow N - \gamma\,.$$

Turning to the last assertion of the corollary, fix $\alpha = N - \gamma$. We use two independent copies A_m, A'_m and Γ_m, Γ'_m. Construct a random closed set $\Upsilon = \Upsilon_\alpha$ as in [9, Remark after Lemma 3.4], of law $\mathbf{P}_\alpha$, independently of all other random sets considered here. That is, consider the natural tiling of the unit cube $[0,1]^N$ by 2^N closed cubes of side $1/2$, let Ξ_1 be a random subcollection of these cubes, where each cube has probability $2^{-\alpha}$ of belonging to Ξ_1, and these events are mutually independent. At the k-th stage, if Ξ_k is nonempty, tile each cube $Q \in \Xi_k$ by 2^N closed subcubes of side 2^{-k-1} (with disjoint interiors) and include each of these subcubes in Ξ_{k+1} with probability $2^{-\alpha}$, independently. Finally, let

$$\Upsilon = \bigcap_{k=1}^{\infty} \bigcup_{Q \in \Xi_k} Q\,.$$

Recall that a.s. $\dim_{\mathrm{P}}(\Upsilon) = \dim(\Upsilon) = \gamma$ (see [9, Lemma 3.5]), and thus by our assumptions,

$$\mathbf{P} \times \mathbf{P}_\alpha((\cup_m \Gamma_m) \cap \Upsilon \neq \varnothing) = \mathbf{P}' \times \mathbf{P}_\alpha((\cup_m \Gamma'_m) \cap \Upsilon \neq \varnothing) = 0\,. \tag{3.15}$$

Moreover, the closed set Υ is such that a.s., $\dim_{\mathrm{P}}(\Upsilon \cap V) = \gamma$ for any open set V such that $\Upsilon \cap V \neq \varnothing$.

Hence, taking $\Upsilon_\star = \Upsilon$ on the set of $\mathbf{P}_\alpha$ full measure with the above property, as in the proof of Theorem 3.1, $\{\Lambda'_m(n) \cap \Upsilon_\star,\ n, m \geq 1\} \cup \{\Lambda_m(n) \cap \Upsilon_\star,\ n, m \geq 1\}$ is a collection of open, dense subsets of the complete metric space $\Upsilon_\star$, $\mathbf{P} \times \mathbf{P}'$-almost surely. By Baire's theorem, one concludes that $A_+ \cap A'_+ \cap \Upsilon_\star \neq \varnothing$, $\mathbf{P} \times \mathbf{P}' \times \mathbf{P}_\alpha$ a.s.. Combined with (3.15), one concludes that

$$B \cap B' \cap \Upsilon \neq \varnothing\,, \quad \mathbf{P} \times \mathbf{P}' \times \mathbf{P}_\alpha \ \textit{a.s.}.$$

Since $B' \subset A'_m$ and $\gamma_m \uparrow \gamma$, it follows from Theorem 3.1 that $\mathbf{P}'(B' \cap E \neq \varnothing) = 0$ for any analytic set $E \subset I\!R_+^N$ such that $\dim_{\rm P}(E) < \gamma$. Considering $E = B \cap \Upsilon$, we see that

$$\mathbf{P} \times \mathbf{P}_\alpha(\dim_{\rm P}(B \cap \Upsilon) \geq \gamma) = 1 \ .$$

Applying [9, Lemma 3.5] for the analytic set $B \subset [0,1]^N$ such that $\dim_{\rm P}(B) \geq \dim(B) = \alpha$ almost surely, we see that

$$\dim_{\rm P}(B) \geq \alpha + \dim_{\rm P}(B \cap \Upsilon) \geq \alpha + \gamma = N$$

apart from a $\mathbf{P} \times \mathbf{P}_\alpha$-null set, as needed to complete the proof. $\square$

4 Lower bounds in Theorems 1.2 and 1.3

Let $\mathcal{D}_n$ denote the collection of dyadic intervals $\{[(i-1)2^{-n}, i2^{-n}]\}_{i=1}^{2^n}$ and $\hat{h}(\epsilon) = \epsilon^2/|\log \epsilon|$. By Brownian scaling we may and shall set $T = 1$ throughout this section. The next lemma is the key to our proof.

Lemma 4.1 *Fixing $a > 1$ and $d = 2$, almost surely, the set $\mathsf{Qscape}_{\leq a}$ contains a discrete limsup random fractal $A = A(a)$ (of dimension $N = 1$) that satisfies Conditions I, II and III with index $1/a$.*

Proof of Theorems 1.2 and 1.3: Fixing $a \geq 1$ let $A_m := A(a + 1/m)$ be the discrete limsup random fractals of Lemma 4.1. Then, for $d = 2$, almost surely,

$$A_+ := \bigcap_{m=1}^{\infty} A_m \subseteq \bigcap_{m=1}^{\infty} \mathsf{Qscape}_{\leq a+1/m} = \mathsf{Qscape}_{\leq a}$$

Considering the $I\!R^d$-valued Brownian motion as the first d coordinates of a Brownian motion in $I\!R^{d'}$, $d' > d$, it is easy to see that $\sigma_r^\xi(t)$, $\bar{\sigma}_r^\eta(t)$ and $\mathcal{S}_r^{\eta,\xi}(t)$ are decreasing in d, hence the sets $\mathsf{Qscape}_{\leq a}$ are increasing with d. Let $\Gamma_m := \mathsf{Thin}_{\leq a-1/(2m)}$ a Borel subset of $[0,1]$. By the set inclusions of (1.11) it follows that for all $d \geq 2$,

$$B := (A_+ \setminus \bigcup_{m=1}^{\infty} \Gamma_m) \subseteq \mathsf{Thin}_{\leq a} \setminus \bigcup_{m=1}^{\infty} \Gamma_m = \mathsf{Thin}_a \ .$$

By Lemma 2.1 and (1.11) we know that a.s., $\Gamma_m \cap \tilde{E} = \varnothing$ for any analytic set $\tilde{E}$ such that $\dim_{\mathrm{P}}(\tilde{E}) \le 1/a < 1/(a - 1/(2m))$. Since the discrete limsup random fractals A_m are of indices $\gamma_m \uparrow 1/a$, we have by Corollary 3.3 that for any $d \ge 2$, almost surely,

$$\dim(\mathsf{Thin}_a) \ge \dim(B) = 1 - 1/a \;,$$

$\dim_{\mathrm{P}}(\mathsf{Thin}_a) \ge \dim_{\mathrm{P}}(B) = 1$ and $\mathsf{Thin}_a \cap E \supseteq B \cap E \ne \varnothing$ if $E \subseteq I\!R$ is an analytic set such that $\dim_{\mathrm{P}}(E) > 1/a$.

To get the corresponding conclusions for BiFast_a we apply the same argument but with $\Gamma_m = \mathsf{BiFast}_{\le a-1/(2m)}$, whereas for Qscape_a we use $\Gamma_m = \mathsf{Qscape}_{\le a-1/(2m)}$. Finally, since $\mathsf{Qscape}_a \subseteq \mathsf{Qscape}_{\le a}$ we conclude the proof of our theorems via Lemma 2.1 and the set inclusions of (1.11). $\square$

Proof of Lemma 4.1: Take $\epsilon_n = n^3 2^{-n/2}$, $n = 1, 2, \ldots$ and $\beta_n = 1 + |\log \epsilon_n|^{-2}$. Let $A = A(a)$ be the discrete limsup random fractal corresponding to $N = 1$ and $Z_n(I) \in \{0, 1\}$ such that

$$Z_n(I) = 1 \quad \text{iff} \quad \mathcal{S}^{2,2}_{\epsilon_n \beta_n}(t) \le a\hat{h}(\epsilon_n) \,,$$

for $I = [t, t + 2^{-n}] \in \mathcal{D}_n$. By Lévy's uniform modulus of continuity, there exists an a.s. finite random variable $n_0(\omega)$, such that for all $n \ge n_0(\omega)$,

$$\sup\{|W_t - W_{t'}| \;:\; t, t' \in [0,1], |t - t'| \le 2^{-n}\} \le 2\sqrt{2^{-n}\log(2^n)} \le \epsilon_n(\beta_n - 1) \,,$$

so that $B(W_{t'}, \epsilon_n) \subseteq B(W_t, \epsilon_n \beta_n)$ and in particular the path has not escaped $B(W_t, \epsilon_n\beta_n)$ by time t'. Using these facts, we see that for all $n > n_0(\omega)$, if $I \in \mathcal{D}_n$ and $Z_n(I) = 1$, then $\mathcal{S}^{1,1}_{\epsilon_n}(t') \le \mathcal{S}^{2,2}_{\epsilon_n\beta_n}(t) \le a\hat{h}(\epsilon_n)$ for every $t' \in I$. Hence we have that $A \subseteq \mathsf{Qscape}_{\le a}$ a.s. Let $x_n = a|\log \epsilon_n|^{-1}\beta_n^{-2}$ and $\rho_n := 2(\epsilon_n\beta_n)^{-2}$. With $\rho_n \ge 2x_n$, we have by Brownian scaling and (2.7) that for some $C_0 = C_0(a) < \infty$

$$\begin{aligned} p_n &:= \mathbf{P}(Z_n(I) = 1) = \mathbf{P}(\mathcal{S}^{2,2}_{\epsilon_n\beta_n}(t) \le a\hat{h}(\epsilon_n)) \\ &= \mathbf{P}(\mathcal{S}^{\rho_n,\rho_n}(0) \le x_n) \le \mathbf{P}(\mathcal{T}(0) \le x_n) \le C_0 x_n^{-0.5} e^{-2/x_n} \;. \end{aligned} \tag{4.1}$$

We next prove the corresponding lower bound: for some $c_1 > 0$, and all n large enough,

$$p_n = \mathbf{P}(\mathcal{S}^{\rho_n,\rho_n}(0) \le x_n) \ge c_1 x_n^4 e^{-2/x_n} \;. \tag{4.2}$$

To see this, first note that by radial symmetry

$$\mathbf{P}^u(\inf_{s\in[0,t]}|W_s|>1):=g_t(|u|)$$

is a function of $|u|$, with $g_t(r)$ nonincreasing in $t>0$ and nondecreasing in $r>0$. Choosing $u=(|u|,0)$ and denoting $W_s=(W_s^{(1)},W_s^{(2)})$ we see that for $\gamma(t):=\sqrt{t}\log(e^2+t)$ and all $t>0$,

$$g_t(\gamma(t)+1)\geq \mathbf{P}(\inf_{s\in[0,t]}W_s^{(1)}>-\gamma(t))=1-2\mathbf{P}(W_t^{(1)}\leq-\gamma(t))\geq 1-1/\log(e^2+t)\,.$$

With $\phi:=\inf\{s\geq 0:|W_s|\geq\gamma(t)+1 \text{ or } |W_s|\leq 1\}$, it follows by the strong Markov property of Brownian motion (at the stopping time ϕ) that for all $r=|u|>1$ and $t>0$,

$$g_t(r)\geq \mathbf{P}^u(|W_\phi|\geq\gamma(t)+1)g_t(\gamma(t)+1)\geq\frac{\log r}{\log(\gamma(t)+1)}(1-\frac{1}{\log(e^2+t)})\,. \tag{4.3}$$

Note that for all $t\geq 2x$,

$$\mathbf{P}(\mathcal{S}^{t,t}(0)\leq 2x)\geq\mathbf{P}(\inf_{t\geq|s|\geq x}|\bar{W}_s|>1)=\Big[\mathbf{P}(\inf_{s\in[x,t]}|W_s|>1)\Big]^2 \tag{4.4}$$

where by (4.3),

$$\mathbf{P}(\inf_{s\in[x,t]}|W_s|>1)\geq\mathbb{E}(g_{t-x}(|W_x|);|W_x|\geq 1+x)\geq\frac{\log(1+x)}{2\log(1+\gamma(t))}\mathbf{P}(|W_x|\geq 1+x)\,. \tag{4.5}$$

Recall that $\mathbf{P}(|W_x|\geq 1+x)=\exp(-(1+x)^2/(2x))$. We apply the above with $x=x_n/2$ and $t=\rho_n$. Since $n^{-1}\log\rho_n\to\log 2$ and $nx_n\to c(a)\in(0,\infty)$, (4.2) follows.

Clearly, $p_n=\mathbb{E}(Z_n(I))$ is the same for all $I\in\mathcal{D}_n$. Moreover, with $e^{2/x_n}\epsilon_n^{2/a}\to 1$, $nx_n\to c(a)\in(0,\infty)$, and $\epsilon_n^{2/a}=n^{6/a}2^{-n/a}$ it follows from (4.1) and (4.2) that

$$\lim_{n\to\infty}n^{-1}\log_2 p_n=-1/a\,, \tag{4.6}$$

so that A satisfies Condition I with index $1/a$.

We next introduce the family of random variables $Y_n(I)\in\{0,1\}$ such that

$$Y_n(I)=1 \quad\text{iff}\quad \mathcal{S}^{2\xi_n,2\xi_n}_{\epsilon_n\beta_n}(t)\leq a\hat{h}(\epsilon_n)\,,$$

for $I=[t,t+2^{-n}]\in\mathcal{D}_n$ and $\xi_n=n^{-1}$. Since $Y_n(I)\in\sigma(W_s-W_t:|s-t|\leq 2\xi_n)$ for $I=[t,t+2^{-n}]\in\mathcal{D}_n$ and $\xi_n\downarrow 0$, we see that the family $Y_n(I)$ is quasi-localized.

It will be convenient to verify condition III before condition II. We note that the same proof will show that the family $Z_n(I)$ satisfies the second moment bound of condition III. The sole reason for introducing the family $Y_n(I)$ is to have quasi-locality. To verify condition III, let $t_n = 2\xi_n(\epsilon_n\beta_n)^{-2}$ and

$$\rho_{x,t}(\theta) := \mathbf{P}(\mathcal{S}^{t,t}(0) \le x, \mathcal{S}^{t,t}(2\theta) \le x)\,. \tag{4.7}$$

Note that when the distance between $I, J \in \mathcal{D}_n$ is $2\theta(\epsilon_n\beta_n)^2$ for some $\theta \in [0, t_n)$, then by Brownian scaling,

$$\mathbb{E}(Y_n(I)Y_n(J)) = \rho_{x_n,t_n}(\theta)\,, \tag{4.8}$$

while $\mathrm{Cov}(Y_n(I), Y_n(J)) = 0$ in case $\theta \ge t_n$. Since for any $\theta > 0$, $t \in (0,\infty)$,

$$\mathbf{P}^y(\mathcal{S}^{\theta,t}(\theta) \le x) = \mathbf{P}(\mathcal{S}^{t,\theta}(0) \le x)$$

is independent of the value of $W_0 = y$, it follows by the Markov property of Brownian motion that for any $t \ge \theta > 0$,

$$\begin{aligned} \mathbf{P}(\mathcal{S}^{t,t}(0) \le x, \mathcal{S}^{t,t}(2\theta) \le x) &\le \mathbf{P}(\mathcal{S}^{t,\theta}(0) \le x, \mathcal{S}^{\theta,t}(2\theta) \le x) \\ &= \mathbb{E}(\tilde{E}^{W_\theta}(\tilde{\mathcal{S}}^{\theta,t}(\theta) \le x); \mathcal{S}^{t,\theta}(0) \le x) = \mathbf{P}(\mathcal{S}^{t,\theta}(0) \le x)^2 \end{aligned} \tag{4.9}$$

where $\tilde{E}$ denotes expectation with respect to $\{\tilde{W}\}$, a Brownian motion independent of $\{W\}$ and $\tilde{\mathcal{S}}^{\theta,t}$ denote the random variables $\mathcal{S}^{\theta,t}$ corresponding to $\{\tilde{W}\}$. Note that

$$\mathbb{E}(Y_n(I)) = \mathbf{P}(\mathcal{S}^{t_n,t_n}(0) \le x_n). \tag{4.10}$$

We claim that for any $\theta \in [2^{2\delta n}, t_n]$

$$\mathbf{P}(\mathcal{S}^{t_n,t_n}(0) \le x_n \mid \mathcal{S}^{t_n,\theta}(0) \le x_n) \ge \delta\,. \tag{4.11}$$

Assuming this for the moment and using (4.7)-(4.11), we see that

$$\mathbb{E}(Y_n(I)Y_n(J)) \le K\mathbb{E}[Y_n(I)]\mathbb{E}[Y_n(J)] \tag{4.12}$$

for $K = K(\delta) = \delta^{-2}$ and all n large enough when the distance between I and J is at least $2^{-n}2^{4\delta n}$. For $K = K(\delta)$, thus,

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 f_K(n) \le 4\delta$$

which gives Condition III when considering $\delta \downarrow 0$.

We now prove (4.11). Let $\psi(\theta) := \sqrt{\theta}(\log\theta)^{-6}$. Note that for all $t \geq \theta \geq 1$,

$$\begin{aligned} \mathbf{P}(\mathcal{S}^{t_n,t_n}(0) \leq x \mid \mathcal{S}^{t_n,\theta}(0) \leq x) &\geq \mathbb{E}(g_{t_n}(|W_\theta|) \mid \mathcal{S}^{t_n,\theta}(0) \leq x) \\ &\geq g_{t_n}(\psi(\theta))\mathbf{P}(|W_\theta| > \psi(\theta) \mid \mathcal{S}^{t_n,\theta}(0) \leq x) \end{aligned} \tag{4.13}$$

Since $\tilde{\mathbf{P}}^{W_x}(|\tilde{W}_{\theta-x}| < \psi(\theta))$ is a nonincreasing function of $|W_x|$, by the strong Markov property of Brownian motion, it follows that for $t \geq \theta \geq x$,

$$\mathbf{P}(|W_\theta| < \psi(\theta) \mid \mathcal{S}^{t,\theta}(0) \leq x) \leq \frac{\mathbf{P}(|W_\theta| < \psi(\theta), \mathcal{T}(0) \leq x)}{\mathbf{P}(\mathcal{S}^{t,\theta}(0) \leq x)} \leq \frac{\mathbf{P}(|W_{\theta-x}| < \psi(\theta))\mathbf{P}(\mathcal{T}(0) \leq x)}{\mathbf{P}(\mathcal{S}^{t,t}(0) \leq x)}$$

Thus, by (4.1) and (4.2), for some $C_1, C_2 < \infty$, all n large enough and $\theta \in [2^{2\delta n}, t_n]$,

$$\mathbf{P}(|W_\theta| < \psi(\theta) \mid \mathcal{S}^{t_n,\theta}(0) \leq x_n) \leq C_1 x_n^{-4.5}\mathbf{P}(|W_1| < 2(\log\theta)^{-6}) \leq C_2 n^{-1}\,, \tag{4.14}$$

while $g_{t_n}(\psi(\theta)) \geq 1.5\delta$ by (4.3). This proves (4.11) and hence Condition III .

Moving on to check Condition II, we note that $Y_n(I) \geq Z_n(I)$ with

$$\mathbf{P}(Y_n(I) \neq Z_n(I)) \leq 2\mathbb{E}(Y_n(I))\mathbf{P}(\mathcal{S}^{t_n,\rho_n}(0) > x_n \mid \mathcal{S}^{t_n,t_n}(0) \leq x_n)\,.$$

Using the fact that $g \leq 1$, we have as in the derivation of (4.13) that

$$\begin{aligned} \mathbf{P}(\mathcal{S}^{t_n,\rho_n}(0) > x_n \mid \mathcal{S}^{t_n,t_n}(0) \leq x_n) &\leq 1 - \mathbb{E}(g_{\rho_n}(|W_{t_n}|) \mid \mathcal{S}^{t_n,t_n}(0) \leq x_n) \\ &\leq 1 - g_{\rho_n}(\psi(t_n)) + \mathbf{P}(|W_{t_n}| \leq \psi(t_n) \mid \mathcal{S}^{t_n,t_n}(0) \leq x_n). \end{aligned}$$

Now, $\mathbf{P}(|W_{t_n}| \leq \psi(t_n) \mid \mathcal{S}^{t_n,t_n}(0) \leq x_n) \to 0$ by (4.14), while by (4.3) we see that

$$g_{\rho_n}(\psi(t_n)) \sim \frac{\log t_n}{\log \rho_n} \to 1$$

thus establishing Condition II. $\square$

5 The coarse multifractal spectrum

By Brownian scaling we may and shall fix $T = 1$. Fixing $a > 1$ and $\delta > 0$, let $\epsilon_n = (1-\delta)^n$, $\rho_n = \delta^2 \hat{h}(\epsilon_n)/20$, $N_n = \lceil \rho_n^{-1} \rceil$, $t_{j,n} = j\rho_n$ be as in Section 2. Turning to the lower bounds in

(1.12), recall that by (2.1) and (2.4), we have that a.s. for some finite $n_0 = n_0(\omega,\delta)$ and all $n \geq n_0$, $\epsilon \in [\epsilon_n, \epsilon_{n-1}]$ and $|t - t_{j,n}| \leq \rho_n/2$,

$$(1-\delta)^6 \hat{\mathcal{T}}_{\epsilon_{n+1}}(t_{j,n}) \leq (1-\delta)^3 \hat{\mathcal{T}}_{\epsilon_n}(t) \leq \hat{\mathcal{T}}_{\epsilon}(t) \ .$$

For $j \in \{0,\ldots,N_n\}$ let $\mathbf{I}_j = 1$ when $j \in \mathcal{A}_n$ of (2.5) and $\mathbf{I}_j = 0$ otherwise. Thus, for any $\epsilon \in [\epsilon_n, \epsilon_{n-1}]$,

$$\mathcal{L}eb\Big\{0 \leq t \leq 1 \,\Big|\, \mathcal{T}_\epsilon(t) \leq a\hat{h}(\epsilon)\Big\} \leq \rho_n |\mathcal{A}_n| = \rho_n \sum_{j=0}^{N_n} \mathbf{I}_j \ . \tag{5.1}$$

Recall that by (2.8) for any $n \geq n_1(\delta, a)$ and all j,

$$\mathbf{P}(\mathbf{I}_j = 1) \leq \epsilon_{n+1}^{2(1-\delta)^7/a} := p_n$$

Fix an integer $\ell > 40a\delta^{-2}(1-\delta)^{-4} \geq 2$. Since $\{\mathbf{I}_j\}$ is a stationary sequence, for all n,

$$\mathbf{P}(\sum_{j=0}^{N_n} \mathbf{I}_j \geq 2\ell p_n N_n) \leq \mathbf{P}(\sum_{j=0}^{\ell N_n - 1} \mathbf{I}_j \geq 2\ell p_n N_n) \leq \ell \mathbf{P}(\sum_{j=0}^{N_n - 1} \mathbf{I}_{j\ell} \geq 2 p_n N_n)$$

With $\ell \rho_n > 2(1-\delta)^{-6} a\hat{h}(\epsilon_{n+1})$, it follows that $\{\mathbf{I}_{j\ell}\}$ are i.i.d. Bernoulli random variables. Hence, by standard tail estimates for Binomial(N_n, p_n) random variables, for some $C > 0$ and all n,

$$\mathbf{P}(\sum_{j=0}^{N_n - 1} \mathbf{I}_{j\ell} \geq 2 p_n N_n) \leq e^{-C p_n N_n}$$

As $\sum_n \exp(-C p_n N_n) < \infty$, it follows from (5.1) by the Borel-Cantelli lemma that a.s. for all n large enough and $\epsilon \in [\epsilon_n, \epsilon_{n-1}]$

$$\mathcal{L}eb\Big\{0 \leq t \leq 1 \,\Big|\, \mathcal{T}_\epsilon(t) \leq a\hat{h}(\epsilon)\Big\} \leq 3\ell p_n \ ,$$

so that

$$\liminf_{\epsilon \to 0} \frac{\log \mathcal{L}eb\Big\{0 \leq t \leq 1 \,\Big|\, \mathcal{T}_\epsilon(t) \leq a\hat{h}(\epsilon)\Big\}}{\log \epsilon} \geq 2(1-\delta)^7/a \quad \text{a.s.}$$

Recall that

$$\{\mathcal{S}_\epsilon^{1,1}(t) \leq a\hat{h}(\epsilon)\} \subset \{\mu_1^W(B(W_t,\epsilon)) \leq a\hat{h}(\epsilon)\} \subset \{\mathcal{T}_\epsilon(t) \leq a\hat{h}(\epsilon)\} \tag{5.2}$$

provided $t \wedge (1-t) \geq a\hat{h}(\epsilon)$. Therefore, taking $\delta \downarrow 0$ results with the lower bound $2/a$ for all sets considered in Theorem 1.4.

By (5.2) it suffices to prove the complementary upper bounds for the set

$$C(\epsilon,a)=\{t\in[0,1]:\mathcal{S}^{1,1}_{\epsilon}(t)\le a\hat{h}(\epsilon)\}\,.$$

To this end, recall that a.s. (2.3) holds for all $n\ge 2/\delta$ large enough, implying, with $\mathcal{S}:=\mathcal{S}^{\infty,\infty}$, that for all $|t-t_{j,n}|\le\rho_n/2$ and $\epsilon\in[\epsilon_{n+2},\epsilon_{n+1}]$,

$$\frac{\mathcal{S}_\epsilon(t)}{\hat{h}(\epsilon)}\le\frac{\mathcal{S}_{\epsilon_n}(t_{j,n})}{\hat{h}(\epsilon)}\le(1-\delta)^{-5}\frac{\mathcal{S}_{\epsilon_n}(t_{j,n})}{\hat{h}(\epsilon_n)}\,.$$

Taking hereafter $N_n=\lfloor\rho_n^{-1}-0.5\rfloor$, for $j\in\{1,\dots,N_n\}$ let $\mathbf{J}_j=1$ when $\mathcal{S}_{\epsilon_n}(t_{j,n})\le(1-\delta)^5a\hat{h}(\epsilon_n)$ and $\mathbf{J}_j=0$ otherwise. Then, for any $\epsilon\in[\epsilon_{n+2},\epsilon_{n+1}]$,

$$\mathcal{L}eb\Big(C(\epsilon,a)\Big)\ge\rho_n\sum_{j=1}^{N_n}\mathbf{J}_j:=\rho_nM_n\,. \tag{5.3}$$

Set $\eta_n:=\epsilon_n^{1+1/a}$ and $L_n=\sum_{j=1}^{N_n}\mathbf{I}_j$, with $\mathbf{I}_j=1$ when $\mathcal{S}^{\eta_n,\eta_n}_{\epsilon_n}(t_{j,n})\le(1-\delta)^5a\hat{h}(\epsilon_n)$ while $\mathbf{I}_j=0$ otherwise. It follows from (4.4) and (4.5) with $t=\xi_n=\eta_n\epsilon_n^{-2}$ and $2x=z_n:=(1-\delta)^5a/|\log\epsilon_n|$, that for some $c_1=c_1(a,\delta)>0$ and all n,

$$q_n:=\mathbf{P}(\mathbf{I}_j=1)=\mathbf{P}(\mathcal{S}^{\xi_n,\xi_n}(0)\le z_n)\ge c_1\epsilon_n^{2(1-\delta)^{-6}/a} \tag{5.4}$$

Note that $\mathbf{K}_j:=\mathbf{I}_j-\mathbf{J}_j$ are non-negative and $\xi_n\ge 2z_n>0$ for all $n\ge n_1(a,\delta)$. Therefore, by (2.7), for some $c_2,c_3,c_4(a,\delta)<\infty$ and all n,

$$\begin{aligned}r_n:=\mathbf{P}(\mathbf{K}_j=1)&=\mathbf{P}(\mathcal{S}^{\xi_n,\xi_n}(0)\le z_n,\mathcal{S}(0)>z_n)\le 2\mathbf{P}(\mathcal{T}(0)\le z_n,\inf_{s\ge\xi_n}|W_s|\le 1)\\&\le c_2\xi_n^{-1/2}\mathbf{P}(\mathcal{T}(0)\le z_n)\le c_3\xi_n^{-1/2}z_n^{-1.5}e^{-2/z_n}\le c_4\epsilon_n^{(a+3)/(2a)}\end{aligned} \tag{5.5}$$

(It is here that we used the fact that $d\ge 3$). Since $E(\mathbf{I}_j\mathbf{I}_k)-q_n^2\le q_n$ and the random variables $\mathbf{I}_j$, $\mathbf{I}_k$ are pairwise independent whenever $|j-k|\rho_n\ge 2\eta_n$, it follows that

$$\mathrm{Var}(L_n)=\sum_{j,k=1}^{N_n}[\mathbb{E}(\mathbf{I}_j\mathbf{I}_k)-q_n^2]\le N_nq_n(2\lceil 2\eta_n/\rho_n\rceil+1)\le 5N_n^2q_n\epsilon_n^{1+1/a}\le 5N_n^2q_n\epsilon_n^{(a+3)/(2a)}$$

where the last inequality used $a > 1$. Recalling that $M_n = \sum_{j=1}^{N_n} \mathbf{J}_j$, we see by (5.4) and (5.5) that for some $c_5 = c_5(a, \delta) < \infty$ and all n,

$$\begin{aligned} \mathbf{P}(M_n \le \frac{1}{3} N_n q_n) &\le \mathbf{P}(L_n \le \frac{2}{3} N_n q_n) + \mathbf{P}(\sum_{j=1}^{N_n} \mathbf{K}_j \ge \frac{1}{3} N_n q_n) \\ &\le \frac{9\mathrm{Var}(L_n)}{N_n^2 q_n^2} + \frac{3 N_n r_n}{N_n q_n} \le c_5 \epsilon_n^{\gamma(a,\delta)/a} , \end{aligned} \tag{5.6}$$

where $\gamma(a, \delta) := (a+3)/2 - 2(1-\delta)^{-6}$ is positive for all $\delta > 0$ small enough. For any such value of δ, by (5.4), (5.6) and the Borel-Cantelli lemma, we have that a.s.

$$\rho_n M_n \ge \frac{1}{3} \rho_n N_n q_n \ge \frac{1}{4} q_n \ge \epsilon_n^{2(1-\delta)^{-7}/a} ,$$

for all n large enough. It follows then that a.s.

$$\limsup_{n\to\infty} \frac{\log(\rho_n M_n)}{\log \epsilon_n} \le 2(1-\delta)^{-7}/a ,$$

which in view of (5.3) yields the upper bounds of (1.12) when considering $\delta \downarrow 0$. □

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