THE PROBABILITY OF SMALL GAUSSIAN ELLIPSOIDS AND ASSOCIATED CONDITIONAL MOMENTS

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ABSTRACT

The problem of computing the lower tail of a Gaussian norm is considered in this paper. Based on large deviations arguments, a bound on these tails is derived which is tighter than those obtained by other methods. Conditional exponential moment bounds are also derived and as an application, L^2 type Onsager-Machlup functionals for diffusions are computed.

KEYWORDS Gaussian norms. Onsager-Machlup. Large Deviations.

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I. Introduction

In this note, we consider the following problem: Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. Gaussian random variables. Consider the random variable $z = \sum_{i=1}^{\infty} x_i^2/a_i^2$, where $\{a_i\}_{i=1}^{\infty}$ is a sequence of given (deterministic) numbers satisfying $\sum_{i=1}^{\infty} \frac{1}{a_i^2} < \infty$. We are interested in computing the asymptotics $P(z \leq \epsilon)$ as $\epsilon \to 0$, and in computing the asymptotics of expectations of the form

$$E(\exp(\sum_{i=1}^{\infty} g_i(x_i))|z \leq \epsilon) \tag{1}$$

The first problem was considered by [4]. Using an approximation by finite sums and explicit computations for Gaussian random variables, they get upper and lower bounds on $P(z \leq \epsilon)$. Although their bounds are rather explicit, the ratio of upper to lower bound diverges badly (except in the particular case $a_i = i$, where they derive alternative bounds based on different methods), and therefore one may not use their results to compute (1).

Another approach to computing asymptotics of z could be via the theory of Large Deviations. Note that, since x_i are Gaussian and independent random variables, $\frac{1}{n^{1/2}}\underline{x} \stackrel{d}{=} \frac{1}{n}\sum_{i=1}^{n} \underline{x}^i$, where $\underline{x}$ denotes the (infinite) vector $x_1, x_2, \cdots$, $\stackrel{d}{=}$ denotes equality in distribution and $\underline{x}^i$ are independent copies of $\underline{x}$. Therefore, the asymptotics of $P(z \geq 1/\epsilon)$ may be read from the general Large Deviations theorem for vector valued Gaussian random variables coupled with the contraction principle. The situation considered here, however, is different in the sense that the random variable $\frac{1}{\epsilon}z$ converges to infinity a.s. as $\epsilon \to 0$, and standard Large Deviations results do not apply in the absence of an equilibrium point.

In spite of the above, the route taken in this article is still based on techniques inspired by Large Deviations theory. We modify the proof of *Cramer*'s theorem to get implicit upper and lower bounds whose ratio, as $\epsilon \to 0$, diverges relatively slowly. These bounds are then used to compute (1). The following corollary will be proved in section 3. Define

$$\mathbf{H}^l \stackrel{\triangle}{=} \{\psi(t) \in L^2(0,1)| \sum_{i=1}^{\infty} \ln(i) \left(\int_0^1 \psi(t)\cos((i-\frac{1}{2})\pi t)dt\right)^2 < \infty\}.$$

Then,

Corollary 1 *Let $\phi \in \mathbf{H}^l$ be a given deterministic function. Then*

$$E\left(\exp(\int_0^1 \phi_t dw_t)|\ ||w||_2 < \epsilon\right) \to_{\epsilon\to 0} 1 \tag{2}$$

where w_t denotes a standard Wiener process, and $||\cdot||_2$ denotes $\mathbf{L}^2$ norm.

This corollary may be used to show that, for a one dimensional diffusion which satisfies

$$dx_t = f(x_t)dt + dw_t,\ x_0 = 0 \tag{3}$$

with $f \in \mathbf{C}^2_\mathbf{b}$, one has that, for all ϕ such that $\dot{\phi} \in \mathbf{H}^l$, some $\alpha > 0$,

$$\frac{P(||x-\phi||_2 < \epsilon)}{P(||w||_2 < \epsilon)} \to_{\epsilon\to 0} \exp\left(-\frac{1}{2}\int_0^1 (\dot{\phi}_t - f(\phi_t))^2 dt - \frac{1}{2}\int_0^1 f'(\phi_t)dt\right) \tag{4}$$

An extension of the above holds also for the case of n-dimensional elliptic random fields, using the same methods as in [8] and [9]. In this case, however, one has to impose more regularity on ϕ due to the fact that the lower bound is not as tight as in the case of the Wiener process. For details, we refer to [6].

The limit in (4) is related to the *Onsager-Machlup* functional for diffusions, c.f. [5]. The result should be compared to [9], where a similar result for the supremum norm was obtained only after tedious computations, with $\mathbf{C}^\alpha$, some $\alpha > 0$, replacing $\mathbf{H}^l$. A tighter result, which does not rely on the bounds of Theorem 1 may be found in [7]. A related application to the computation of Onsager-Machlup functionals and estimators for stochastic PDE's will appear in [6].

The results of this paper can be somewhat extended in the case that the x_i's are not independent and not Gaussian, provided some moment bounds on their correlations and higher order moments hold. For details, c.f. the remark at the end of the next section. Also, by combining the results of this paper and the Berry-Esseen expansion, the bounds of Theorem 1 may be improved to yield the exact asymptotic expansion as $\epsilon \to 0$. The details will apear in [2].

We end this introduction with a notation. Throughout, for two positive functions (or sequences) f, g, $f \sim g$ will stand for

$$0 < \underline{\lim}\frac{f}{g} \le \overline{\lim}\frac{f}{g} < \infty.$$

II. Computation of Asymptotic Probabilities

Throughout, let $\{x_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. Gaussian random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, with zero mean and unit variance. Let $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive numbers. We assume that

$$\sum_{i=1}^{\infty} \frac{1}{a_i^2} < \infty. \tag{5}$$

Denote $z \triangleq \sum_{i=1}^{\infty} \frac{x_i^2}{a_i^2}$. From (5), $E(z) < \infty$ and therefore, z is a well defined random variable. Let now s_ϵ be any deterministic constant, which may be ϵ dependent. By the standard Chebycheff bound, one has

$$P(\frac{z}{\epsilon} < 1) \leq E(\exp(-s_\epsilon(\frac{z}{\epsilon} - 1)) = e^{s_\epsilon} \prod_{i=1}^{\infty} E(\exp(-\frac{s_\epsilon x_i^2}{\epsilon a_i^2})) = e^{s_\epsilon} \prod_{i=1}^{\infty} (1 + \frac{2s_\epsilon}{\epsilon a_i^2})^{-1/2} \triangleq UB(s_\epsilon) < \infty \tag{6}$$

which, written differently, yields

$$\log(UB) = s_\epsilon - \frac{1}{2} \sum_{i=1}^{\infty} \log(1 + \frac{2s_\epsilon}{\epsilon a_i^2}) \tag{7}$$

Obviously, any choice of s_ϵ in (7) leads to an upper bound for $P(z < \epsilon)$. To compute the complementary lower bound, we use a change of measure argument. Let s_ϵ be as before a deterministic constant, and define on $(\Omega, \mathcal{F})$ a probability measure by

$$d\eta_s = \frac{e^{-s_\epsilon z/\epsilon} dP}{E_P(e^{-s_\epsilon z/\epsilon})} \tag{8}$$

Note that under the measure η_s, $\{x_i\}$ is a sequence of independent Gaussian random variables with zero mean and variances $\sigma_i^2 = \frac{1}{1 + 2s_\epsilon/\epsilon a_i^2}$. Let $\delta(\epsilon) \in (0, 1/2)$ to be chosen later. Clearly, for any choice of $\delta(\epsilon)$,

$$\begin{aligned} P(z \leq \epsilon) &\geq \eta_s(\frac{z}{\epsilon} \in (1 - 2\delta(\epsilon), 1)) \exp(s_\epsilon(1 - 2\delta(\epsilon))) \int_{\mathbf{R}^\infty} e^{-s_\epsilon \frac{z}{\epsilon}} d\mu(\underline{x}) \\ &= \eta_s(\frac{z}{\epsilon} \in (1 - 2\delta(\epsilon), 1)) \exp(-2s_\epsilon \delta(\epsilon)) e^{s_\epsilon} \prod_{i=1}^{\infty} (1 + \frac{2s_\epsilon}{\epsilon a_i^2})^{-1/2} \triangleq LB(s_\epsilon) \end{aligned} \tag{9}$$

Choose now s_ϵ such that

$$E_{\eta_s}(\frac{z}{\epsilon}) = \frac{\int_{\mathbf{R}^\infty} \frac{z}{\epsilon} \exp(-s_\epsilon \frac{z}{\epsilon}) dP(\underline{x})}{\int_{\mathbf{R}^\infty} \exp(-s_\epsilon \frac{z}{\epsilon}) dP(\underline{x})} = 1 - \delta(\epsilon), \tag{10}$$

or, more explicitly,

$$\sum_{i=1}^{\infty} \frac{1}{a_i^2 + \frac{2s_\epsilon}{\epsilon}} = (1-\delta(\epsilon))\epsilon. \tag{11}$$

Such an s_ϵ exists for small enough ϵ since the left hand side of (10) is continuous in s_ϵ and assumes the values $0, \frac{1}{\epsilon}\sum_{i=1}^{\infty}\frac{1}{a_i^2}$ for $s_\epsilon = \infty, 0$, respectively. Under this choice of s_ϵ, we have:

Lemma 1

$$E_{\eta_s}\left(\frac{z}{\epsilon} - (1-\delta(\epsilon))\right)^2 = \frac{2}{\epsilon^2}\sum_{i=1}^{\infty}\left(\frac{E_{\eta_s}(x_i^2)}{a_i^2}\right)^2 \tag{12}$$

Proof Since $\sum_{i=1}^{\infty} E_{\eta_s}(x_i^2/\epsilon a_i^2) = (1-\delta(\epsilon))$, one obtains that

$$E_{\eta_s}\left(\frac{z}{\epsilon} - (1-\delta(\epsilon))\right)^2 = \sum_{i=1}^{\infty}\frac{E_{\eta_s}\left(x_i^2 - E_{\eta_s}(x_i^2)\right)^2}{\epsilon^2 a_i^4} = \sum_{i=1}^{\infty}\frac{2}{\epsilon^2}\left(\frac{E_{\eta_s}(x_i^2)}{a_i^2}\right)^2 \tag{13}$$

where the last equality used the fact that x_i is still Gaussian under η_s.

□

Since

$$E_{\eta_s}(x_i^2) = \frac{1}{1+\frac{2s_\epsilon}{\epsilon a_i^2}},$$

$E_{\eta_s}(x_i^2)/a_i^2$ is monotonically decreasing in i. Combining this fact with (12) and Chebycheff's bound, one obtains

$$\begin{aligned}
1-\eta_s\left(\frac{z}{\epsilon}\in(1-2\delta(\epsilon),1)\right) &\leq \frac{1}{\delta^2(\epsilon)}E_{\eta_s}\left(\frac{z}{\epsilon}-(1-\delta(\epsilon))\right)^2 \\
&\leq \frac{1}{\delta^2(\epsilon)}2\frac{1}{\epsilon}\sum_{i=1}^{\infty}\frac{E_{\eta_s}(x_i^2)}{a_i^2}\left(\frac{1}{a_1^2\epsilon(1+\frac{2s_\epsilon}{\epsilon a_1^2})}\right) \\
&= \frac{1}{\delta^2(\epsilon)}2(1-\delta(\epsilon))\left(\frac{1}{a_1^2\epsilon(1+\frac{2s_\epsilon}{\epsilon a_1^2})}\right) \\
&\leq \frac{1}{\delta^2(\epsilon)s_\epsilon}
\end{aligned} \tag{14}$$

Note that, with the choice of s_ϵ as in (11), the ratio of upper to lower bounds is bounded above for small enough ϵ by

$$\frac{UB(s_\epsilon)}{LB(s_\epsilon)} \leq \frac{\exp(2s_\epsilon\delta(\epsilon))}{(1-\frac{1}{\delta^2(\epsilon)s_\epsilon})} \tag{15}$$

and this will be used below in the computation of exponential moments.

We conclude this section with an explicit computation of the bounds for some special cases. Let $f(\cdot)$ be a non decreasing function of a real variable such that $a_i^2 = f(i)$. Since the s_ϵ stipulated by (11) tends to ∞ as $\epsilon \to 0$, it follows that

$$\frac{\frac{1}{\epsilon}\sum_{i=1}^{\infty}\frac{1}{a_i^2+\frac{2s_\epsilon}{\epsilon}}}{I_1(f,\epsilon)} \to_{\epsilon\to 0} 1 \tag{16}$$

where

$$I_1(f,\epsilon) \stackrel{\triangle}{=} \frac{1}{\epsilon}\int_1^\infty \frac{1}{f(x)+\frac{2s_\epsilon}{\epsilon}}dx$$

and, similarly,

$$\frac{\frac{1}{s_\epsilon}\log(UB)}{I_2(f,\epsilon)} \to_{\epsilon\to 0} 1 \tag{17}$$

where

$$I_2(f,\epsilon) \stackrel{\triangle}{=} 1 - \frac{1}{2s_\epsilon}\int_1^\infty \log(1+\frac{2s_\epsilon}{\epsilon f(x)})dx$$

One may now use asymptotic formulae for the integrals in hand to compute the upper bound and use then (15) to estimate the corresponding lower bound. In particular, consider the case $a_i = i^{\beta/2}$, $\beta > 1$, and take $f(x) = x^\beta$. Integrating by parts in the definition of I_2, one obtains that

$$\int_1^\infty \log(1+\frac{2s_\epsilon}{\epsilon x^\beta})dx = \frac{2\beta s_\epsilon}{\epsilon}\int_1^\infty \frac{1}{x^\beta+\frac{2s_\epsilon}{\epsilon}} = 2s_\epsilon\beta I_1 \tag{18}$$

On the other hand, by a direct computation,

$$\epsilon I_1(f) = t_\epsilon^{1/\beta-1}\int_{t_\epsilon^{-1/\beta}}^\infty \frac{dy}{1+y^\beta},$$

where $t_\epsilon = \frac{2s_\epsilon}{\epsilon}$. Let $\nu_\beta = \int_0^\infty \frac{dy}{1+y^\beta}$. By [3] $\nu_\beta = \frac{\pi/\beta}{sin(\pi/\beta)}$, and one concludes that as $\epsilon \to 0$ (and therefore $s_\epsilon \to \infty$),

$$\lim_{\epsilon\to 0}\epsilon^{\frac{1}{\beta-1}}s_\epsilon = \nu_\beta^{\frac{\beta}{\beta-1}}/2 \tag{19}$$

and

$$\lim_{\epsilon\to 0}\epsilon^{\frac{1}{\beta-1}}\log(UB) = -\frac{\beta-1}{2}\nu_\beta^{\frac{\beta}{\beta-1}}. \tag{20}$$

On the other hand, choosing $\delta(\epsilon)$ such that $\frac{\log(\delta(\epsilon))}{\log(\epsilon)} < \frac{1}{2(\beta-1)}$, one concludes that

$$\frac{UB}{LB} \le C\exp(\epsilon^{-\frac{1}{2(\beta-1)}}) \tag{21}$$

and therefore,

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{\beta-1}} \log(LB) = -\frac{\beta-1}{2}\nu_\beta^{\frac{\beta}{\beta-1}} \tag{22}$$

as well.

Remarks

1) That the same limit occurs in (20) and (22) is a large deviations statement. In this sense our bounds improve those of [4] (equations 4.5.1 and 4.5.2) whose ratio between the upper and lower bound is of the order of $\exp(\epsilon^{-\frac{1}{\beta-1}})$, which is worse than the ratio in (21).

2) There is one case in which precise computations can be made, namely the case $\beta = 2$. In that case, we may and will in the proof of Corollary 1 view the coefficients $a_i = i$ as the eigenvalues of the Wiener process' covariance function. From formula 4.3.5 in [1], one obtains after some straightforward manipulations

$$P(\sum_{i=1}^{\infty} \frac{x_i^2}{a_i^2} < \epsilon) = (1+o(1))\frac{4}{\sqrt{2\pi}}\exp(-\pi^2/8\epsilon). \tag{23}$$

The product in our upper bound (6) can also be evaluated explicitly to yield

$$UB(\epsilon) = \frac{(1+o(\epsilon))\pi}{\sqrt{\epsilon}}\exp(-\frac{\pi^2}{8\epsilon}) \tag{24}$$

This upper bound (up to a multiplying constant) was also obtained in [4] (formula 5.3.1) using a method specific for the case $\beta = 2$ (and different from their general treatment). Moreover, this technique produced a tight lower bound so that the ratio of their upper to lower diverges like $1/\epsilon$ which is better than the ratio in (21).

3) We note that the bounding techniques that we use can be applied also in the case of non-Gaussian random variables. In this case, the upper bound is of the form

$$P(\frac{z}{\epsilon} < 1) \leq e^{s_\epsilon} \prod_{i=1}^{\infty} E(\exp(-\frac{s_\epsilon x_i^2}{\epsilon a_i^2})) \triangleq UB_1 \tag{25}$$

whereas the lower bound is of the form

$$P(\frac{z}{\epsilon} < 1) \geq \eta(\frac{z}{\epsilon} \in (1-2\delta(\epsilon), 1))\exp(-2s_\epsilon\delta(\epsilon)))UB_1 \tag{26}$$

and the tightness of the bounds depends on the ability to obtain a counterpart of Lemma 1 in the non-Gaussian case. In the case that, as in the Gaussian case, one has that $E_\eta(x_i^4) \leq$

$c_4 E_\eta(x_i^2)^2$, the same technique as in the Gaussian case coupled with an appropriate choice of $\delta(\epsilon)$ yields similar bounds. The same remark applies also to the case where the random variables x_i are weakly dependent.

III. Expectation asymptotics and the Onsager-Machlup functional

In this section we turn our attention to the computation of expectations of the form (1). As mentioned in the introduction, the main motivation for doing that lies in computations related to the Onsager-Machlup functional presented in Corollary 1 above and its extensions to the random field situation. Therefore, throughout this section, we remain in the Gaussian case and keep the notations introduced in the previous sections. For the sake of simplicity, we assume throughout that $g_i(\cdot)$ is odd and has linear growth at most. We note that in some applications that we have in mind, this last condition is violated, however the same technique may be made to work. For details, c.f. [6].

By the antisymmetry of $g_i(\cdot)$ and Jensen's inequality, one has that

$$E(\exp(\sum_{i=1}^{\infty} g_i(x_i))|z \le \epsilon) \ge \exp(E\sum_{i=1}^{\infty} g_i(x_i)|z \le \epsilon) = 1 \tag{27}$$

On the other hand, note that for any $\gamma(\epsilon) > 1$, and s_ϵ as in (11),

$$\begin{aligned} E\left(\exp(\sum_{i=1}^{\infty} g_i(x_i))|z \le \epsilon\right) &\le E^{1/\gamma(\epsilon)}\left(\exp(\sum_{i=1}^{\infty} \gamma(\epsilon) g_i(x_i))|z \le \epsilon\right) \\ &\le \frac{E^{1/\gamma(\epsilon)}\left(\exp(\sum_{i=1}^{\infty} \gamma(\epsilon) g_i(x_i))\exp(-s_\epsilon(\frac{z}{\epsilon}-1)\right)}{P^{\frac{1}{\gamma(\epsilon)}}(\frac{z}{\epsilon} \le 1)} \end{aligned} \tag{28}$$

where the expectation in the numerator is finite due to the assumption of linear growth of $g_i(\cdot)$. Therefore, we obtain that

$$E(\exp(\sum_{i=1}^{\infty} g_i(x_i))|z \le \epsilon) \le \left(\frac{UB(s_\epsilon)}{LB(s_\epsilon)}\right)^{1/\gamma(\epsilon)} \prod_{i=1}^{\infty} \left(\frac{\int_{-\infty}^{\infty} \exp(\gamma(\epsilon) g_i(x_i) - \frac{s_\epsilon x_i^2}{\epsilon a_i^2} - \frac{x_i^2}{2}) dx_i}{\int_{-\infty}^{\infty} \exp(-\frac{s_\epsilon x_i^2}{\epsilon a_i^2} - \frac{x_i^2}{2}) dx_i}\right)^{1/\gamma(\epsilon)}. \tag{29}$$

Note that in (29), $UB(s_\epsilon)$ must be the particular upper bound obtained by the Chebycheff inequality as in (6), but any lower bound in hand could replace (9). Equation (29) is the basic upper bound on the exponential expectations (1). To obtain more explicit bounds, we assume in the sequel that $g_i(x) = b_i x$.

A substitution of $g_i(x_i) = b_i x_i$ in (29) yields the bound

$$E\left(\exp(\sum_{i=1}^{\infty} b_i x_i)|z \le \epsilon\right) \le \left(\frac{UB(s_\epsilon)}{LB(s_\epsilon)}\right)^{1/\gamma(\epsilon)} \exp\left(\sum_{i=1}^{\infty} \frac{\gamma(\epsilon) b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})}\right) \tag{30}$$

Thus, we have:

Theorem 1 *Let a_i be an increasing sequence such that $a_i \sim i^{\beta/2}$ as $i \to \infty$ for some $\beta > 1$. Assume that*

$$\sum_{i=1}^{\infty} i^{1/2} b_i^2 < \infty \tag{31}$$

Then

$$E\left(\exp(\sum_{i=1}^{\infty} b_i x_i)|z \le \epsilon\right) \to_{\epsilon \to 0} 1. \tag{32}$$

Moreover, when $\beta = 2$, instead of (31) it is enough to assume that $\sum_{i=1}^{\infty} ln(i) b_i^2 < \infty$.

Proof By (30) and (21),

$$E\left(\exp(\sum_{i=1}^{\infty} b_i(x_i)|z \le \epsilon\right) \le C^{1/\gamma(\epsilon)} \exp(\frac{\epsilon^{-\frac{1}{2(\beta-1)}}}{\gamma(\epsilon)} + \gamma(\epsilon) \sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})}) \tag{33}$$

Choose now

$$\gamma(\epsilon) = \sqrt{\frac{\epsilon^{-\frac{1}{2(\beta-1)}}}{\sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})}}}.$$

Clearly, for ϵ small enough, $\gamma(\epsilon) > 1$ (actually, $\gamma(\epsilon) \to \infty$ as $\epsilon \to 0$). Substituting in (33), one obtains that

$$E\left(\exp(\sum_{i=1}^{\infty} b_i(x_i)|z \le \epsilon\right) \le C^{1/\gamma(\epsilon)} \exp\left(2\sqrt{\epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})}}\right) \tag{34}$$

and, in view of (27) the claim will follow if we can prove that

$$\epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})} \to_{\epsilon \to 0} 0.$$

Define $i_0(\epsilon) = \inf\{i | \frac{2s_\epsilon}{\epsilon a_i^2} < 1\}$. Note that for $a_i \sim i^{\beta/2}$, $s_\epsilon \sim \epsilon^{-1/(\beta-1)}$ and therefore, $i_0(\epsilon) \sim \epsilon^{-1/(\beta-1)}$, i.e. $s_\epsilon \sim i_0(\epsilon)$ and $\epsilon \sim i_0(\epsilon)^{1-\beta}$. Now,

$$\epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})} = \epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i=1}^{i_0(\epsilon)} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon i^\beta})} + \epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i_0(\epsilon)+1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon i^\beta})} \tag{35}$$

Choose an increasing sequence η_i such that $\eta_i i^{-1/2} \to \infty$, $\eta_i i^{-1}$ decreases and $\sum_{i=1}^{\infty} \eta_i b_i^2 = M < \infty$. Consider the first term in the RHS of (35). One has

$$\epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i=1}^{i_0(\epsilon)} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon i^\beta})} \leq \epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i=1}^{i_0(\epsilon)} \frac{\epsilon b_i^2 i^\beta}{4s_\epsilon} \leq \epsilon^{\frac{2\beta-3}{2(\beta-1)}} \frac{i_0(\epsilon)^\beta}{s_\epsilon \eta_{i_0(\epsilon)}} \sum_{1}^{\infty} \eta_i b_i^2 \leq M c i_0(\epsilon)^{1/2} \eta_{i_0(\epsilon)}^{-1} \to_{\epsilon\to 0} 0 \tag{36}$$

where the second inequality follows from the fact that i^β/η_i is an increasing sequence. Returning now to the second term in the RHS of (35), one has

$$\epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i_0(\epsilon)+1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon i^\beta})} \leq \epsilon^{-\frac{1}{2(\beta-1)}} \sum_{i_0(\epsilon)}^{\infty} b_i^2 \leq \epsilon^{-\frac{1}{2(\beta-1)}} i_0(\epsilon)^{-1/2} \sum_{i=i_0(\epsilon)}^{\infty} i^{1/2} b_i^2 \leq c \sum_{i=i_0(\epsilon)}^{\infty} i^{1/2} b_i^2 \to_{\epsilon\to 0} 0 \tag{37}$$

which together with (36) and (35) yield (32) for general β. For the case of $\beta = 2$, using the tighter expressions $UB(s_\epsilon) \sim \frac{\exp(-\pi^2/8\epsilon)}{\sqrt{\epsilon}}$ from (24) and $LB(s_\epsilon) \sim \exp(-\pi^2/8\epsilon)$ from (23), we may modify (33) so that

$$E\left(\exp(\sum_{i=1}^{\infty} b_i(x_i) | z \leq \epsilon\right) \leq C^{1/\gamma(\epsilon)} \exp(\frac{-\ln(\epsilon)/2}{\gamma(\epsilon)} + \gamma(\epsilon) \sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})}). \tag{38}$$

By choosing $\gamma(\epsilon) = \sqrt{\frac{-\ln(\epsilon)/2}{\sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})}}}$, it remains to prove that

$$-\ln(\epsilon) \sum_{i=1}^{\infty} \frac{b_i^2}{2(1+\frac{2s_\epsilon}{\epsilon a_i^2})} \to_{\epsilon\to 0} 0.$$

Using a sequence η_i such that $\eta_i/\ln(i) \to \infty$, η_i/i decreases and $\sum_{i=1}^{\infty} \eta_i b_i^2 = M < \infty$, and mimicking the proof for the general β case, the limit in the RHS of the modified (36) and (37) becomes $c\ln(i_0(\epsilon))/\eta_{i_0(\epsilon)} \to_{\epsilon\to 0} 0$ and $c\sum_{i_0(\epsilon)}^{\infty} \ln(i) b_i^2 \to_{\epsilon\to 0} 0$, respectively.

$\square$

Proof of Corollary 1 Let $e_i(t)$ be the CONS in $L^2(0,1)$ which are the eigenfunctions of the Wiener process' covariance function, i.e. $e_i(t) = \sqrt{2}\sin(a_i t)$, $i \in \mathbf{N}$, where $a_i = (i-\frac{1}{2})\pi$. Using the Karhunen-Loeve expansion, represent the Wiener process as $w_t = \sum_{i=1}^{\infty} \frac{1}{a_i} x_i e_i(t)$ where $x_i = a_i \int_0^1 w_t e_i(t) dt$ are independent standard Gaussian random variables. Note that $||w||_2^2 = \sum_{i=1}^{\infty} x_i^2/a_i^2$. Next, we claim that for any deterministic $\phi \in L^2(0,1)$,

$$\int_0^1 \phi(t) dw(t) = \sum_{i=1}^{\infty} \frac{x_i}{a_i} \int_0^1 \phi(t) \dot{e}_i(t) dt. \tag{39}$$

Indeed,

$$\begin{aligned}
\int_0^1 \phi(t)dw_t &= \sum_{i=1}^{\infty} E(x_i \int_0^1 \phi(t)dw_t)x_i \\
&= \sum_{i=1}^{\infty} x_i a_i E(\int_0^1 w_t e_i(t)dt \int_0^1 \phi(t)dw_t) \\
&= \sum_{i=1}^{\infty} x_i a_i E\left((w(1)\int_0^1 e_i(t)dt - \int_0^1 \int_0^t e_i(s)dsdw_t)\int_0^1 \phi(t)dw_t\right) \\
&= -\sum_{i=1}^{\infty} x_i a_i \int_0^1 \phi(t) \int_0^t e_i(s)dsdt \\
&= \sum_{i=1}^{\infty} x_i a_i \int_0^1 \frac{\dot{e}_i(t)}{a_i^2}\phi(t)dt \\
&= \sum_{i=1}^{\infty} \frac{x_i}{a_i} \int_0^1 \dot{e}_i(t)\phi(t)dt \qquad (40)
\end{aligned}$$

where we have used the fact that $\int_0^t e_i(s)ds = -\dot{e}_i(t)/a_i^2$. Using now (39), Theorem 1 may be applied with $b_i = \int_0^1 \dot{e}_i(t)\phi(t)/a_i$, yielding Corollary 1.

□

References

[1] T.W Anderson and D.A. Darling. Asymptotic theory of certain goodness of fit criteria based on stochastic processes. *Annals of Math. Stat.*, 23:193–212, 1952.

[2] A. Dembo, E. Mayer-Wolf, and O. Zeitouni. Exact behaviour of Hilbertian seminorms. *In preparation*, 1991.

[3] I.S. Gradshtein and I.M. Rhyzhik. *Tables of integrals, series and products.* Academic Press, 1980.

[4] J. Hoffman-Jorgensen, L.A. Shepp, and R.M Dudley. On the lower tail of Gaussian seminorms. *Annals of Probability*, 7:319–342, 1979.

[5] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes.* North Holland, Amsterdam, 1981.

[6] E. Mayer-Wolf and O. Zeitouni. Onsager-Machlup functionals for non Gaussian, non trace class fields. *In preparation*, 1991.

[7] L.A. Shepp and O. Zeitouni. A note on conditional exponential moments and Onsager Machlup functionals. *preprint*, 1991.

[8] Y. Takahashi and S. Watanabe. The probability functional (Onsager-Machlup function) of diffusion processes. *Lecture Notes in Mathematics*, 851:433–463, 1981.

[9] O. Zeitouni. On the Onsager-Machlup functional of diffusions around non C^2 curves. *Annals of Probability*, 17:1037–1054, 1989.